$\mathcal{O}_\infty$ Realized on Bose Fock Space

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Abstract

We study the semigroup of Bogoliubov endomorphisms of the canonical commutation relations which give rise to representations of the Cuntz algebra $\mathcal{O}_\infty$ on Fock space and describe the corresponding Cuntz algebra generators in detail.

1 Introduction

The appearance of the Cuntz algebras $\mathcal{O}_d$ [1] is a generic feature of quantum field theory. This fact has been discovered, within the algebraic approach [2], by Doplicher and Roberts [3] who associated with each localized morphism ϱ of dimension d and obeying permutation group statistics a multiplet $\Psi_1, \ldots, \Psi_d$ of local field operators, acting on a Hilbert space which contains each superselection sector with some multiplicity, such that Cuntz' relations hold

$$\Psi_j^* \Psi_k = \delta_{jk} \mathbf{1}, \quad \sum_j \Psi_j \Psi_j^* = \mathbf{1} \tag{1.1}$$

and such that, for any local observable A,

$$\varrho(A) = \sum_j \Psi_j A \Psi_j^*. \tag{1.2}$$

Regarding the Ψ_j as an orthonormal basis for the closure $H(\varrho)$ of their linear span, one says that 'ϱ is implemented by the Hilbert space $H(\varrho)$ of isometries'.

These observations motivated a study of endomorphisms of the CAR algebra which can be implemented by Hilbert spaces of isometries on Fock space [4]. Here we present a similar analysis for endomorphisms of the canonical commutation relations (CCR) or, more precisely, of the Weyl relations. As in [4], we find it convenient to use Araki's 'selfdual' formulation [5, 6] which is briefly introduced in Sect. 2. We discuss a class of natural endomorphisms of the CCR algebra ('Bogoliubov endomorphisms') in Sect. 3. As a generalization of Shale's

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condition for automorphisms [7], we state a necessary and sufficient condition for Bogoliubov endomorphisms to be implementable in a fixed Fock representation. The derivation of this result is based on known criteria for quasi–equivalence of quasi–free states [5, 8, 6] and can, at one point, be reduced to the CAR case, by using an inequality due to Araki and Yamagami [9].

The topological semigroup of implementable endomorphisms is the subject of Sect. 4. It can be written as a product of a subgroup consisting of automorphisms which are close to the identity, and the sub–semigroup of endomorphisms which leave the given Fock state invariant. This decomposition enables us to determine the connected components of the semigroup. It also plays a role in Sect. 5 which is concerned with the construction of orthonormal bases for the Hilbert spaces $H(\varrho)$. These Hilbert spaces themselves carry the structure of symmetric Fock spaces and thus are, for genuine endomorphisms, infinite–dimensional. The C*–algebra generated by a single $H(\varrho)$ is $\mathcal{O}_\infty$. Implementers are constructed by an adaptation of Ruijsenaars' formulas for unitary implementers of automorphisms [10]. They can be written as products of certain isometries belonging to the commutant of the range of ϱ times a Wick ordered exponential of an expression which is bilinear in creation and annihilation operators. The connection with the aforementioned product decomposition is that, roughly speaking, the first factor carries the exponential term, whereas the second is responsible for the additional isometries. The proof of completeness of implementers can thereby be reduced to the case of endomorphisms which leave the given Fock state invariant.

2 Basic Notions

Let $\mathcal{K}^0$ be an infinite–dimensional complex linear space, equipped with a nondegenerate hermitian sesquilinear form γ and an antilinear involution $f \mapsto f^*$, such that

$$\gamma(f^*, g^*) = -\gamma(g, f), \qquad f, g \in \mathcal{K}^0.$$

(The reader who is unfamiliar with Araki's approach [5, 6] should think of $\mathcal{K}^0$ as being the complexification of the real linear space $\operatorname{Re}\mathcal{K}^0 \equiv \{f \in \mathcal{K}^0 \mid f^* = f\}$, together with its canonical conjugation. $-i\gamma$ should be viewed as the sesquilinear extension of a nondegenerate symplectic form on $\operatorname{Re}\mathcal{K}^0$. Hopefully, the reader will not be confused in the following by the appearance of too many stars with different meanings.) The *selfdual CCR algebra* $\mathcal{C}(\mathcal{K}^0, \gamma)$ [5, 6] over $(\mathcal{K}^0, \gamma)$ is the simple *–algebra which is generated by $\mathbf{1}$ and elements $f \in \mathcal{K}^0$, subject to the commutation relation

$$f^* g - g f^* = \gamma(f, g)\mathbf{1}, \quad f, g \in \mathcal{K}^0. \tag{2.1}$$

We henceforth assume the existence of a distinguished Fock state ω_{P_1}. Here P_1 is a *basis projection* of $\mathcal{K}^0$, i.e. a linear operator, defined on the whole of $\mathcal{K}^0$, which satisfies

$$\begin{aligned} P_1^2 = P_1, &\qquad \gamma(f, P_1 g) = \gamma(P_1 f, g), \\ P_1 f + P_1(f^*)^* = f, &\qquad \gamma(f, P_1 f) > 0 \text{ if } P_1 f \neq 0 \end{aligned} \tag{2.2}$$

for $f, g \in \mathcal{K}^0$. Let

$$P_2 \equiv \mathbf{1} - P_1, \quad C \equiv P_1 - P_2, \quad \langle f, g\rangle \equiv \gamma(f, Cg).$$

The positive definite inner product $\langle\ ,\ \rangle$ turns $\mathcal{K}^0$ into a pre–Hilbert space. We assume its completion $\mathcal{K}$ to be separable. By continuity, the involution $*$ extends to a conjugation on $\mathcal{K}$, P_1 and P_2 to orthogonal projections, C to a self–adjoint unitary, and γ to a nondegenerate hermitian form. These extensions will be denoted by the same symbols. Setting

$$\mathcal{K}_n \equiv P_n(\mathcal{K}), \qquad n = 1, 2,$$

we get a direct sum decomposition $\mathcal{K} = \mathcal{K}_1 \oplus \mathcal{K}_2$ which is orthogonal with respect to both γ and $\langle\ ,\ \rangle$. The following notations will frequently be used for $A \in \mathfrak{B}(\mathcal{K})$, where $\mathfrak{B}(\mathcal{K})$ is the algebra of bounded linear operators on $\mathcal{K}$:

$$\begin{aligned} A_{mn} &\equiv P_m A P_n, \quad m, n = 1, 2, \\ A^\dagger &\equiv C A^* C, \\ \overline{A} f &\equiv A(f^*)^*, \quad f \in \mathcal{K}. \end{aligned}$$

The components A_{mn} of A are regarded as operators from $\mathcal{K}_n$ to $\mathcal{K}_m$, and A will sometimes be written as a matrix $\left(\begin{smallmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{smallmatrix}\right)$. $A^\dagger$ is the adjoint of A relative to γ, whereas A^* is the Hilbert space adjoint. $\overline{A}$ may be viewed as the complex conjugate of A. Thus one has relations like

$$\overline{P_2} = P_1 = P_1^\dagger = P_1^*, \ \overline{C} = -C, \ A_{12}{}^\dagger = A^\dagger{}_{21} = -A_{12}{}^*, \ \overline{A_{11}} = \overline{A}_{22} \quad \text{etc.}$$

The *Fock state* ω_{P_1} is the unique state[1] which is annihilated by all $f \in \operatorname{ran} P_2$:

$$\omega_{P_1}(f^* f) = 0 \text{ if } P_1 f = 0.$$

(In the conventional setting mentioned above, ω_{P_1} is the Fock state corresponding to the complex structure iC on $\operatorname{Re}\mathcal{K}$.) Let $\mathcal{F}_s(\mathcal{K}_1)$ be the symmetric Fock space over $\mathcal{K}_1$ and let $\mathcal{D}$ be the dense subspace of algebraic tensors. A GNS representation π_{P_1} for ω_{P_1} is provided by

$$\pi_{P_1}(f) = a^*(P_1 f) + a\big(P_1(f^*)\big), \quad f \in \mathcal{K}$$

where $a^*(g)$ and $a(g)$, $g \in \mathcal{K}_1$, are the usual creation and annihilation operators on $\mathcal{D}$. The cyclic vector inducing the state ω_{P_1} is Ω_{P_1}, the Fock vacuum. The operators $\pi_{P_1}(a)$, $a \in \mathcal{C}(\mathcal{K}, \gamma)$, have invariant domain $\mathcal{D}$, are closable, and $\pi_{P_1}(a^*) \subset \pi_{P_1}(a)^*$. In particular, if $f \in \operatorname{Re}\mathcal{K}$, then $\pi_{P_1}(f)$ is essentially self–adjoint on $\mathcal{D}$, and the unitary *Weyl operator* $w(f)$ is defined as the exponential of the closure of $i\pi_{P_1}(f)$. Its vacuum expectation value is

$$\omega_{P_1}(w(f)) \equiv \langle \Omega_{P_1}, w(f)\Omega_{P_1} \rangle = e^{-\frac{1}{2}||P_1 f||^2},$$

and the *Weyl relations* hold

$$w(f)w(g) = e^{-\frac{1}{2}\gamma(f,g)} w(f+g), \quad f, g \in \operatorname{Re}\mathcal{K}.$$

The Weyl operators generate a simple C*–algebra $\mathcal{W}(\mathcal{K}, \gamma)$ which acts irreducibly on $\mathcal{F}_s(\mathcal{K}_1)$. If $\mathcal{H}$ is a subspace of $\mathcal{K}$ with $\mathcal{H} = \mathcal{H}^*$, then the C*–algebra generated by all $w(f)$ with $f \in \operatorname{Re}\mathcal{H}$ is denoted by $\mathcal{W}(\mathcal{H})$. If $\mathcal{H}_0$ is the orthogonal complement of $\mathcal{H}$ with respect to γ, then *duality* holds [11, 5, 6]:

$$\mathcal{W}(\mathcal{H})' = \mathcal{W}(\mathcal{H}_0)'' \tag{2.3}$$

(a prime denotes the commutant).

[1] A state ω over $\mathcal{C}(\mathcal{K}, \gamma)$ is a linear functional with $\omega(\mathbf{1}) = 1$ and $\omega(A^* A) \geq 0$, $A \in \mathcal{C}(\mathcal{K}, \gamma)$.

Lemma 2.1. *For $f \in \mathcal{K}$, let $\mathcal{H}_f$ be the subspace spanned by f and f^*. Then the closure of $\pi_{P_1}(f)$ is affiliated with $\mathcal{W}(\mathcal{H}_f)''$.*

Proof. Let T be the closure of $\pi_{P_1}(f)$, with domain $D(T)$. We have to show that, for any $A \in \mathcal{W}(\mathcal{H}_f)'$

$$A(D(T)) \subset D(T), \qquad AT = TA \text{ on } D(T).$$

Now by virtue of the CCR (2.1), $\|T\phi\|^2 = \|T^*\phi\|^2 + \gamma(f,f)\|\phi\|^2$ for $\phi \in \mathcal{D}$. Hence, for a given Cauchy sequence $\phi_n \in \mathcal{D}$, $T\phi_n$ converges if and only if $T^*\phi_n$ does. This implies that

$$D(T) = D(T^*).$$

Let $f^\pm \in \operatorname{Re}\mathcal{H}_f$ be defined as $f^+ \equiv \frac{1}{2}(f + f^*)$, $f^- \equiv \frac{i}{2}(f - f^*)$, and let $T^\pm$ be the (self–adjoint) closure of $\pi_{P_1}(f^\pm)$. We claim that

$$D(T) = D(T^+) \cap D(T^-), \qquad T = T^+ - iT^- \text{ on } D(T).$$

For if $\phi \in D(T)$, then there exists a sequence $\phi_n \in \mathcal{D}$ converging to ϕ such that $\pi_{P_1}(f)\phi_n$ and $\pi_{P_1}(f^*)\phi_n$ converge. Thus ϕ belongs to the domain of the closure of $\pi_{P_1}(f^\pm)$. Conversely, if $\phi \in D(T^+) \cap D(T^-)$, then there exists a sequence $\phi_n \in \mathcal{D}$ converging to ϕ such that both $\pi_{P_1}(f + f^*)\phi_n$ and $\pi_{P_1}(f - f^*)\phi_n$ converge (cf. [10]). Therefore $\pi_{P_1}(f)\phi_n$ is also convergent, i.e. ϕ is contained in $D(T)$, and $T\phi = (T^+ - iT^-)\phi$.

Now if $A \in \mathcal{W}(\mathcal{H}_f)'$, then A commutes with the one–parameter unitary groups $w(tf^\pm) = \exp(itT^\pm)$. As a consequence, A leaves $D(T^\pm)$ invariant and commutes with $T^\pm$ on $D(T^\pm)$. It follows that $A(D(T)) \subset D(T)$ and $AT = TA$ on $D(T)$ as was to be shown. □

3 Implementability of Endomorphisms

Bogoliubov endomorphisms are the unital *–endomorphisms of $\mathcal{C}(\mathcal{K},\gamma)$ which map $\mathcal{K}$, viewed as a subspace of $\mathcal{C}(\mathcal{K},\gamma)$, into itself. They are completely determined by their restrictions to $\mathcal{K}$ which are called *Bogoliubov operators*. Hence $V \in \mathfrak{B}(\mathcal{K})$ is a Bogoliubov operator if and only if it commutes with complex conjugation and preserves the hermitian form γ[2]. Bogoliubov operators form a unital semigroup denoted by

$$\mathcal{S}(\mathcal{K},\gamma) \equiv \{V \in \mathfrak{B}(\mathcal{K}) \mid \overline{V} = V,\ V^\dagger V = \mathbf{1}\}.$$

Each $V \in \mathcal{S}(\mathcal{K},\gamma)$ extends to a unique Bogoliubov endomorphism of $\mathcal{C}(\mathcal{K},\gamma)$ and to a unique *–endomorphism of $\mathcal{W}(\mathcal{K},\gamma)$. By abuse of notation, both endomorphisms are denoted by ϱ_V, so that $\varrho_V(f) = Vf$, $f \in \mathcal{K}$, and $\varrho_V(w(g)) = w(Vg)$, $g \in \operatorname{Re}\mathcal{K}$.

The condition $V^\dagger V = \mathbf{1}$ entails that V is injective and V^* surjective; hence $\operatorname{ran} V$ is closed, and V is a semi–Fredholm operator [12]. We claim that the Fredholm index $-\operatorname{ind} V = \dim\ker V^\dagger$ cannot be odd, in contrast to the CAR case [4]. For let $f \in \ker V^\dagger$ such that $0 = \gamma(f,g) \equiv \langle f, Cg\rangle\ \forall g \in \ker V^\dagger$.

[2] We may disregard unbounded Bogoliubov operators V (defined on $\mathcal{K}^0$) since the topologies induced by the corresponding states $\omega_{P_1} \circ \varrho_V$ on $\mathcal{K}^0$ differ from the one induced by ω_{P_1}. Hence these states cannot be quasi–equivalent to ω_{P_1} (cf. [5, 6]), and ϱ_V cannot be implemented.

Then $f \in (C \ker V^\dagger)^\perp = (\ker V^*)^\perp = \operatorname{ran} V$, but $\operatorname{ran} V \cap \ker V^\dagger = \{0\}$ due to $V^\dagger V = \mathbf{1}$, so f has to vanish. This shows that the restriction of γ to $\ker V^\dagger$ stays nondegenerate. It follows that $\dim \ker V^\dagger$ cannot be odd (there is no nondegenerate symplectic form on an odd–dimensional space).

On the other hand, each even number (and ∞) occurs as $\dim\ker V^\dagger$ for some V. Hence we have an epimorphism of semigroups

$$\mathcal{S}(\mathcal{K},\gamma) \to \mathbb{N} \cup \{\infty\}, \quad V \mapsto -\frac{1}{2} \operatorname{ind} V = \frac{1}{2} \dim \ker V^\dagger$$

($0 \in \mathbb{N}$ by convention). Let

$$\mathcal{S}^n(\mathcal{K},\gamma) \equiv \{V \in \mathcal{S}(\mathcal{K},\gamma) \mid \operatorname{ind} V = -2n\}, \quad n \in \mathbb{N} \cup \{\infty\}.$$

$\mathcal{S}^0(\mathcal{K},\gamma)$ is the group of Bogoliubov automorphisms (isomorphic to the symplectic group of $\operatorname{Re}\mathcal{K}$). It acts on $\mathcal{S}(\mathcal{K},\gamma)$ by left multiplication. Analogous to the CAR case, the orbits under this action are the subsets $\mathcal{S}^n(\mathcal{K},\gamma)$, and the stabilizer of $V \in \mathcal{S}^n(\mathcal{K},\gamma)$ is isomorphic to the symplectic group $\mathrm{Sp}(n)$.

We are interested in endomorphisms ϱ_V which can be implemented by Hilbert spaces of isometries on $\mathcal{F}_s(\mathcal{K}_1)$. This means that there exist isometries Ψ_j on $\mathcal{F}_s(\mathcal{K}_1)$ which fulfill the Cuntz algebra relations (1.1) and implement ϱ_V according to (1.2)

$$\varrho_V(w(f)) = \sum_j \Psi_j w(f) \Psi_j^*, \quad f \in \operatorname{Re}\mathcal{K}.$$

As explained in [4], such isometries exist if and only if ϱ_V, viewed as a representation of $\mathcal{W}(\mathcal{K},\gamma)$ on $\mathcal{F}_s(\mathcal{K}_1)$, is quasi–equivalent to the defining (Fock) representation.

To study ϱ_V as a representation, for fixed $V \in \mathcal{S}(\mathcal{K},\gamma)$, let us decompose it into cyclic subrepresentations. Let $e_1, e_2, \ldots$ be an orthonormal basis in $\mathcal{K}_1 \cap \ker V^\dagger$ and let $\alpha = (\alpha_1, \ldots, \alpha_l)$ be a multi–index with $\alpha_j \leq \alpha_{j+1}$. Such α has the form

$$\alpha = (\underbrace{\alpha'_1, \ldots, \alpha'_1}_{l_1}, \underbrace{\alpha'_2, \ldots, \alpha'_2}_{l_2}, \ldots, \underbrace{\alpha'_r, \ldots, \alpha'_r}_{l_r}) \tag{3.1}$$

with $\alpha'_1 < \alpha'_2 < \cdots < \alpha'_r$ and $l_1 + \cdots + l_r = l$. Let

$$\begin{aligned}
\phi_\alpha &\equiv (l_1! \cdots l_r!)^{-\frac{1}{2}} a^*(e_{\alpha_1}) \cdots a^*(e_{\alpha_l}) \Omega_{P_1}, \\
\mathcal{F}_\alpha &\equiv \overline{\mathcal{W}(\operatorname{ran} V)\phi_\alpha}, \\
\pi_\alpha &\equiv \varrho_V|_{\mathcal{F}_\alpha}.
\end{aligned}$$

Lemma 3.1. *$\varrho_V = \oplus_\alpha \pi_\alpha$, where the sum extends over all multi–indices α as above, including $\alpha = 0$ ($\phi_0 \equiv \Omega_{P_1}$). Each $(\pi_\alpha, \mathcal{F}_\alpha, \phi_\alpha)$ is a GNS representation for $\omega_{P_1} \circ \varrho_V$ (regarded as a state over $\mathcal{W}(\mathcal{K},\gamma)$).*

Proof. By definition, the ϕ_α constitute an orthonormal basis for $\mathcal{F}_s(\mathcal{K}_1 \cap \ker V^\dagger)$, and $(\pi_\alpha, \mathcal{F}_\alpha, \phi_\alpha)$ is a cyclic representation of $\mathcal{W}(\mathcal{K},\gamma)$. Since the closures of $a^*(e_j)$ and $a(e_j)$ are affiliated with $\mathcal{W}(\ker V^\dagger)'' = \mathcal{W}(\operatorname{ran} V)'$ (see Lemma 2.1

and (2.3)), there holds for $f \in \mathrm{Re}\,\mathcal{K}$, with $N_\alpha \equiv (l_1! \cdots l_r!)^{-1}$

$$\begin{aligned}\langle \phi_\alpha, \pi_\alpha(w(f))\phi_\alpha\rangle &= N_\alpha \langle a^*(e_{\alpha_1})\cdots a^*(e_{\alpha_l})\Omega_{P_1}, w(Vf) a^*(e_{\alpha_1})\cdots a^*(e_{\alpha_l})\Omega_{P_1}\rangle \\ &= N_\alpha\langle \Omega_{P_1}, w(Vf)\,\underbrace{a(e_{\alpha_l})\cdots a(e_{\alpha_1})a^*(e_{\alpha_1})\cdots a^*(e_{\alpha_l})\Omega_{P_1}}_{N_\alpha^{-1}\Omega_{P_1}}\rangle \\ &= \langle \Omega_{P_1}, w(Vf)\Omega_{P_1}\rangle.\end{aligned}$$

This proves that $(\pi_\alpha, \mathcal{F}_\alpha, \phi_\alpha)$ is a GNS representation for $\omega_{P_1}\circ\varrho_V$. Similarly, one finds that $\langle\phi_\alpha, w(Vf)\phi_{\alpha'}\rangle = 0$ for $\alpha \neq \alpha'$, so the $\mathcal{F}_\alpha$ are mutually orthogonal.

It remains to show that $\oplus_\alpha \mathcal{F}_\alpha = \mathcal{F}_s(\mathcal{K}_1)$. We claim that $\mathcal{F}_0$ equals $\mathcal{F}_s(\overline{\mathrm{ran}\,P_1V})$, the symmetric Fock space over the closure of $\mathrm{ran}\,P_1V$. The inclusion $\mathcal{F}_0 \subset \mathcal{F}_s(\overline{\mathrm{ran}\,P_1V})$ holds because vectors of the form $w(Vf)\Omega_{P_1} = \exp i\big(a^*(P_1Vf) + a(P_1Vf)\big)\Omega_{P_1} \in \mathcal{F}_s(\overline{\mathrm{ran}\,P_1V})$ are total in $\mathcal{F}_0$. The converse inclusion may be proved inductively. Assume that $a^*(g_1)\cdots a^*(g_m)\Omega_{P_1}$ is contained in $\mathcal{F}_0$ for all $m \leq n$, $g_1,\ldots,g_m \in \mathrm{ran}\,P_1V$. Then, for $f \in V(\mathrm{Re}\,\mathcal{K})$ and $g_1,\ldots,g_n \in \mathrm{ran}\,P_1V$, $\frac{1}{i}\frac{w(tf)-\mathbf{1}}{t}a^*(g_1)\cdots a^*(g_n)\Omega_{P_1}$ has a limit $a^*(P_1f)a^*(g_1)\cdots a^*(g_n)\Omega_{P_1} + a(P_1f)a^*(g_1)\cdots a^*(g_n)\Omega_{P_1}$ in $\mathcal{F}_0$ as $t \searrow 0$. By assumption, the second term lies in $\mathcal{F}_0$, and so does the first. Since each $g \in \mathrm{ran}\,P_1V$ is a linear combination of such P_1f, it follows that $a^*(g_1)\cdots a^*(g_{n+1})\Omega_{P_1}$ is contained in $\mathcal{F}_0$ for arbitrary $g_j \in \mathrm{ran}\,P_1V$, and, by induction, for arbitrary $n \in \mathbb{N}$. But such vectors span a dense subspace in $\mathcal{F}_s(\overline{\mathrm{ran}\,P_1V})$, so $\mathcal{F}_0 = \mathcal{F}_s(\overline{\mathrm{ran}\,P_1V})$ as claimed.

Finally, $\mathcal{K}_1 \cap \ker V^\dagger$ equals $\ker V^*P_1$, where V^*P_1 is regarded as an operator from $\mathcal{K}_1$ to $\mathcal{K}$. Thus we have $\mathcal{K}_1 = \overline{\mathrm{ran}\,P_1V} \oplus (\mathcal{K}_1 \cap \ker V^\dagger)$ and $\mathcal{F}_s(\mathcal{K}_1) \cong \mathcal{F}_0 \otimes \mathcal{F}_s(\mathcal{K}_1 \cap \ker V^\dagger)$. Under this isomorphism, $\mathcal{F}_\alpha$ is identified with $\mathcal{F}_0 \otimes (\mathbb{C}\phi_\alpha)$. Since the ϕ_α form an orthonormal basis for $\mathcal{F}_s(\mathcal{K}_1 \cap \ker V^\dagger)$, the desired result $\oplus_\alpha\mathcal{F}_\alpha = \mathcal{F}_s(\mathcal{K}_1)$ follows. □

As a consequence, the representation ϱ_V is quasi–equivalent to the GNS representation associated with the quasi–free state $\omega_{P_1}\circ\varrho_V$. So ϱ_V is implementable if and only if $\omega_{P_1}\circ\varrho_V$ and ω_{P_1} (i.e. their GNS representations) are quasi–equivalent. Now the two–point function of $\omega_{P_1}\circ\varrho_V$ (as a state over $\mathcal{C}(\mathcal{K},\gamma)$) is given by

$$\omega_{P_1}\circ\varrho_V(f^*g) = \gamma(f, Sg) = \langle f, \tilde{S}g\rangle, \quad f, g \in \mathcal{K},$$

with

$$S \equiv V^\dagger P_1 V, \qquad \tilde{S} \equiv V^* P_1 V.$$

The latter operators contain valuable information about $\omega_{P_1}\circ\varrho_V$. For example, it can be shown (cf. [13]) that $\omega_{P_1}\circ\varrho_V$ is a *pure* state over $\mathcal{W}(\mathcal{K},\gamma)$ if and only if S is a basis projection, that is, if and only if S is idempotent (the remaining conditions in (2.2) are automatically fulfilled). This is further equivalent to $[P_1, VV^\dagger] = 0$, by the following chain of equivalences:

$$\begin{array}{lll} S^2 = S & \Leftrightarrow \quad 0 = S\overline{S} & \text{(since } \overline{S} = \mathbf{1} - S) \\ & \Leftrightarrow \quad 0 = V^*P_1VCV^*P_2V & \\ & \Leftrightarrow \quad 0 = P_1VCV^*P_2 & \text{(since } \mathrm{ran}\,V^*P_2V = \mathrm{ran}\,V^*P_2 \\ & & \ \text{and } \ker V^*P_1V = \ker P_1V) \\ & \Leftrightarrow \quad 0 = P_1VV^\dagger P_2 & \\ & \Leftrightarrow \quad 0 = [P_1, VV^\dagger]. & \end{array}$$

On the other hand, the criterion for quasi–equivalence of quasi–free states, in the form given by Araki and Yamagami [6], states that $\omega_{P_1} \circ \varrho_V$ is quasi–equivalent to ω_{P_1} if and only if $P_1 - \tilde{S}^{\frac{1}{2}}$ is a Hilbert–Schmidt operator on $\mathcal{K}$. This condition can be simplified in the present context, as the following result shows.

Theorem 3.2. *Let a Bogoliubov operator $V \in \mathcal{S}(\mathcal{K}, \gamma)$ be given. Then there exists a Hilbert space of isometries $H(\varrho_V)$ which implements the endomorphism ϱ_V in the Fock representation determined by the basis projection P_1 if and only if $[P_1, V]$ (or, equivalently, V_{12}) is a Hilbert–Schmidt operator. The dimension of $H(\varrho_V)$ is 1 if $\operatorname{ind} V = 0$, otherwise ∞.*

Proof. First note that $[P_1, V] = V_{12} - V_{21} = V_{12} - \overline{V_{12}}$ is Hilbert–Schmidt (HS) if and only if V_{12} is.

By the preceding discussion, ϱ_V is implementable if and only if $P_1 - \tilde{S}^{\frac{1}{2}}$ is HS. In this case, $P_2(P_1 - \tilde{S}^{\frac{1}{2}})^2 P_2 = P_2 \tilde{S} P_2 = {V_{12}}^* V_{12}$ is of trace class, hence V_{12} is HS.

Conversely, assume V_{12} to be HS. Let $V = V'|V|$ be the polar decomposition of V. Then $|V| = \overline{|V|}$ is a bounded bijection with a bounded inverse, and $|V| - \mathbf{1} = (|V|^2 - \mathbf{1})(|V| + \mathbf{1})^{-1} = (V^* - V^\dagger)V(|V| + \mathbf{1})^{-1} = 2({V_{12}}^* + {V_{21}}^*)V(|V| + \mathbf{1})^{-1}$ is HS. Thus, by a corollary [6] of an inequality of Araki and Yamagami [9], $(|V|A|V|)^{\frac{1}{2}} - A^{\frac{1}{2}}$ is HS for any positive $A \in \mathfrak{B}(\mathcal{K})$. Applying this to $A = V'^* P_1 V'$, we get that

$$\tilde{S}^{\frac{1}{2}} - (V'^* P_1 V')^{\frac{1}{2}} \text{ is HS.} \tag{3.2}$$

Now V' is an isometry with $\overline{V'} = V'$, i.e. a CAR Bogoliubov operator [4]. Since $[P_1, V]$ and $[P_1, |V|^{-1}] = |V|^{-1}\big[|V|, P_1\big]|V|^{-1} = |V|^{-1}\big[|V| - \mathbf{1}, P_1\big]|V|^{-1}$ are HS, the same holds true for $[P_1, V'] = [P_1, V|V|^{-1}]$. So V' fulfills the implementability condition for CAR Bogoliubov operators derived in [4], and, as shown there, this forces $P_1 - (V'^* P_1 V')^{\frac{1}{2}}$ to be HS. This, together with (3.2), implies that $P_1 - \tilde{S}^{\frac{1}{2}}$ is HS as claimed.

It remains to prove the statement about $\dim H(\varrho_V)$. Let $\tilde{\varrho}_V$ be the normal extension of ϱ_V to $\mathfrak{B}(\mathcal{F}_s(\mathcal{K}_1))$. Then $\mathfrak{B}(H(\varrho_V)) \cong \tilde{\varrho}_V(\mathfrak{B}(\mathcal{F}_s(\mathcal{K}_1)))' = \varrho_V(\mathcal{W}(\mathcal{K}, \gamma))' = \mathcal{W}(\operatorname{ran} V)' = \mathcal{W}(\ker V^\dagger)''$. The latter (and hence $H(\varrho_V)$) is one–dimensional if $\ker V^\dagger = \{0\}$ and infinite–dimensional if $\ker V^\dagger \neq \{0\}$. □

Remark. Shale's original result [7] asserts that a Bogoliubov automorphism ϱ_V, $V \in \mathcal{S}^0(\mathcal{K}, \gamma)$, is implementable if and only if $|V| - \mathbf{1}$ is HS. This condition is equivalent to $[P_1, V]$ being HS, not only for $V \in \mathcal{S}^0(\mathcal{K}, \gamma)$, but for all $V \in \mathcal{S}(\mathcal{K}, \gamma)$ with $-\operatorname{ind} V < \infty$. However, the two conditions are *not* equivalent for $V \in \mathcal{S}^\infty(\mathcal{K}, \gamma)$, as the following example shows. Let $\mathcal{K}_1 = \mathcal{H} \oplus \mathcal{H}'$ be a decomposition into infinite–dimensional subspaces. Choose an operator V_{12} from $\mathcal{K}_2$ to $\mathcal{H}$ with $\operatorname{tr}|V_{12}|^4 < \infty$, but $\operatorname{tr}|V_{12}|^2 = \infty$. Let $V_{21} \equiv \overline{V_{12}}$ and $|V_{11}| \equiv (P_1 + |V_{21}|^2)^{\frac{1}{2}}$. Choose an isometry v_{11} from $\mathcal{K}_1$ to $\mathcal{H}'$ and set $V_{11} \equiv v_{11}|V_{11}|$, $V_{22} \equiv \overline{V_{11}}$. These components define a Bogoliubov operator $V \in \mathcal{S}^\infty(\mathcal{K}, \gamma)$ (cf. (3.4a)–(3.4d) below) which violates the condition of Theorem 3.2. But it fulfills Shale's condition since $|V|^2 - \mathbf{1} = 2(|V_{12}|^2 + |V_{21}|^2)$ is HS and since $|V| - \mathbf{1} = (|V|^2 - \mathbf{1})(|V| + \mathbf{1})^{-1}$.

Let $V \in \mathcal{S}(\mathcal{K}, \gamma)$ with V_{12} compact. Due to stability under compact perturbations [12], V_{11} and $V_{22} = \overline{V_{11}}$ are semi–Fredholm with

$$\operatorname{ind} V_{11} = \operatorname{ind} V_{22} = \frac{1}{2} \operatorname{ind} V. \tag{3.3}$$

We will occasionally use the relation $V^\dagger V = \mathbf{1}$ componentwise:

$$V_{11}{}^* V_{11} - V_{21}{}^* V_{21} = P_1, \tag{3.4a}$$
$$V_{22}{}^* V_{22} - V_{12}{}^* V_{12} = P_2, \tag{3.4b}$$
$$V_{11}{}^* V_{12} - V_{21}{}^* V_{22} = 0, \tag{3.4c}$$
$$V_{22}{}^* V_{21} - V_{12}{}^* V_{11} = 0. \tag{3.4d}$$

Since V_{11} is injective by (3.4a) and has closed range, we may define a bounded operator $V_{11}{}^{-1}$ as the inverse of V_{11} on $\operatorname{ran} V_{11}$ and as zero on $\ker V_{11}{}^*$ (the same applies to V_{22}). These operators will be needed later. Note that $\dim\ker V_{11}{}^* = \frac{1}{2}\operatorname{ind} V$.

4 On the Semigroup of Implementable Endomorphisms

According to Theorem 3.2, the semigroup of implementable Bogoliubov endomorphisms is isomorphic to the following semigroup of Bogoliubov operators:

$$\mathcal{S}_{P_1}(\mathcal{K}, \gamma) \equiv \{V \in \mathcal{S}(\mathcal{K}, \gamma) \mid V_{12} \text{ is Hilbert–Schmidt}\}.$$

$\mathcal{S}_{P_1}(\mathcal{K}, \gamma)$ is a topological semigroup with respect to the metric $d_{P_1}(V, V') \equiv \|V - V'\| + \|V_{12} - V'_{12}\|_{\mathrm{HS}}$, where $\|\ \|_{\mathrm{HS}}$ denotes Hilbert–Schmidt norm. It contains the closed sub–semigroup of diagonal Bogoliubov operators

$$\mathcal{S}_{\mathrm{diag}}(\mathcal{K}, \gamma) = \{V \in \mathcal{S}(\mathcal{K}, \gamma) \mid [P_1, V] = 0\}$$

which is isomorphic to the semigroup of isometries of the Hilbert space $\mathcal{K}_1$, via the map $V \mapsto V_{11}$. The Fredholm index yields a decomposition

$$\mathcal{S}_{P_1}(\mathcal{K}, \gamma) = \bigcup_{n \in \mathbb{N} \cup \{\infty\}} \mathcal{S}^n_{P_1}(\mathcal{K}, \gamma), \qquad \mathcal{S}^n_{P_1}(\mathcal{K}, \gamma) \equiv \mathcal{S}_{P_1}(\mathcal{K}, \gamma) \cap \mathcal{S}^n(\mathcal{K}, \gamma).$$

The group $\mathcal{S}^0_{P_1}(\mathcal{K}, \gamma)$ is usually called the *restricted symplectic group* [7, 15]. It has a natural normal subgroup

$$\mathcal{S}_{\mathrm{HS}}(\mathcal{K}, \gamma) \equiv \{V \in \mathcal{S}(\mathcal{K}, \gamma) \mid V - \mathbf{1} \text{ is Hilbert–Schmidt}\} \subset \mathcal{S}^0_{P_1}(\mathcal{K}, \gamma).$$

We will eventually show that each $V \in \mathcal{S}_{P_1}(\mathcal{K}, \gamma)$ can be written as a product $V = UW$ with $U \in \mathcal{S}_{\mathrm{HS}}(\mathcal{K}, \gamma)$ and $W \in \mathcal{S}_{\mathrm{diag}}(\mathcal{K}, \gamma)$. Assume that such U and W exist. Then $P_V \equiv U P_1 U^\dagger$ is a basis projection such that

$$P_1 - P_V \text{ is Hilbert–Schmidt}, \qquad V^\dagger P_V V = P_1, \tag{4.1}$$

so the corresponding Fock state ω_{P_V} is unitarily equivalent to ω_{P_1} and fulfills $\omega_{P_V} \circ \varrho_V = \omega_{P_1}$. In order to construct such basis projections, let us investigate the set $\mathcal{P}_{P_1}$ of basis projections of $\mathcal{K}$ which differ from P_1 only by a Hilbert–Schmidt operator. Let $\mathcal{E}_{P_1}$ be the infinite–dimensional analogue of the open unit disk [14, 15], consisting of all Hilbert–Schmidt operators Z from $\mathcal{K}_1$ to $\mathcal{K}_2$ which are *symmetric* in the sense that

$$Z = \overline{Z^*} \tag{4.2}$$

and have norm less than 1 (the latter condition is equivalent to $P_1 + Z^\dagger Z$ being positive definite on $\mathcal{K}_1$, since $Z^\dagger = -Z^*$ and Z is compact). Then the following is more or less well–known (cf. [15]).

Proposition 4.1. *$P \mapsto P_{21}{P_{11}}^{-1}$ defines a bijection from $\mathcal{P}_{P_1}$ onto $\mathcal{E}_{P_1}$, with inverse given by*

$$Z \mapsto P_Z \equiv (P_1 + Z)(P_1 + Z^\dagger Z)^{-1}(P_1 + Z^\dagger). \tag{4.3}$$

The restricted symplectic group $\mathcal{S}^0_{P_1}(\mathcal{K}, \gamma)$ acts transitively on either set, in a way compatible with the above bijection, through the formulas

$$P \mapsto UPU^\dagger \tag{4.4}$$

$$Z \mapsto (U_{21} + U_{22}Z)(U_{11} + U_{12}Z)^{-1}. \tag{4.4$'$}$$

The restrictions of these actions to the subgroup $\mathcal{S}_{\mathrm{HS}}(\mathcal{K}, \gamma)$ remain transitive, as follows from the fact that, for $Z \in \mathcal{E}_{P_1}$,

$$U_Z \equiv (P_1 + Z)(P_1 + Z^\dagger Z)^{-\frac{1}{2}} + (P_2 - Z^\dagger)(P_2 + ZZ^\dagger)^{-\frac{1}{2}} \tag{4.5}$$

lies in $\mathcal{S}_{\mathrm{HS}}(\mathcal{K}, \gamma)$ and fulfills $U_Z P_1 U_Z^\dagger = P_Z$ (equivalently, under the action (4.4$'$), U_Z takes $0 \in \mathcal{E}_{P_1}$ to Z).

Proof. Having made $\mathcal{K}$ into a Hilbert space, the conditions on P to be a basis projection (2.2) may be rewritten as

$$P = P^\dagger = \mathbf{1} - \overline{P} = P^2, \qquad CP \text{ is positive definite on } \operatorname{ran} P; \tag{4.6}$$

or, in components:

$$P_{11} = {P_{11}}^* = P_1 - \overline{P_{22}}, \tag{4.7a}$$

$$P_{22} = {P_{22}}^* = P_2 - \overline{P_{11}}, \tag{4.7b}$$

$$P_{21} = \overline{P_{21}}^* = -{P_{12}}^*, \tag{4.7c}$$

$${P_{11}}^2 - P_{11} = {P_{21}}^* P_{21}, \tag{4.7d}$$

$${P_{22}}^2 - P_{22} = {P_{12}}^* P_{12}, \tag{4.7e}$$

$$(P_1 - P_{11})P_{12} = P_{12}P_{22}, \tag{4.7f}$$

$$(P_2 - P_{22})P_{21} = P_{21}P_{11}, \tag{4.7g}$$

$$\begin{pmatrix} P_{11} & P_{12} \\ -P_{21} & -P_{22} \end{pmatrix} \text{ is positive definite on } \operatorname{ran} P. \tag{4.7h}$$

Moreover, $P_1 - P$ is Hilbert–Schmidt if and only if $P_2 P$ is.

Now let $P \in \mathcal{P}_{P_1}$. Then $P_{22} \leq 0$ by (4.7h), hence, by (4.7a),

$$P_{11} = P_1 - \overline{P_{22}} \geq P_1$$

has a bounded inverse. Thus $Z \equiv P_{21}{P_{11}}^{-1}$ is a well–defined Hilbert–Schmidt operator. By (4.7a)–(4.7c) and (4.7g),

$$\begin{aligned} Z - \overline{Z^*} &= P_{21}{P_{11}}^{-1} - \overline{{P_{11}}^{-1}{P_{21}}^*} \\ &= \overline{{P_{11}}^{-1}}\big((P_2 - P_{22})P_{21} - P_{21}P_{11}\big){P_{11}}^{-1} \\ &= 0, \end{aligned}$$

so Z is symmetric in the sense of (4.2). Furthermore, by (4.7d),

$$\begin{aligned} P_1 - Z^*Z &= P_1 - {P_{11}}^{-1}{P_{21}}^*P_{21}{P_{11}}^{-1} \\ &= P_1 - {P_{11}}^{-1}({P_{11}}^2 - P_{11}){P_{11}}^{-1} \\ &= {P_{11}}^{-1} \end{aligned} \tag{4.8}$$

is positive definite on $\mathcal{K}_1$, which proves $Z \in \mathcal{E}_{P_1}$.

Next let $Z \in \mathcal{E}_{P_1}$ and let P_Z be given by (4.3). We associate with Z an operator

$$Y \equiv (P_1 + Z^\dagger Z)^{-1} = (P_1 - Z^*Z)^{-1} \tag{4.9}$$

which is bounded by assumption. Then $P_Z = P_Z^\dagger = P_Z^2$ since $(P_1+Z^\dagger)(P_1+Z) = Y^{-1}$. To prove that $P_Z + \overline{P_Z} = \mathbf{1}$ holds, note that $ZY^{-1} = \overline{Y}^{-1}Z$ and therefore $\overline{Y}Z = ZY$, $YZ^\dagger = Z^\dagger\overline{Y}$. It follows that

$$\begin{aligned} P_Z + \overline{P_Z} &= (P_1 + Z)Y(P_1 + Z^\dagger) + (P_2 - Z^\dagger)\overline{Y}(P_2 - Z) \\ &= Y + ZY + YZ^\dagger + ZZ^\dagger\overline{Y} + \overline{Y} - YZ^\dagger - ZY + Z^\dagger ZY \\ &= Y^{-1}Y + \overline{Y}^{-1}\overline{Y} \\ &= P_1 + P_2 \\ &= \mathbf{1}. \end{aligned}$$

Since $P_2 P_Z$ is clearly HS and since

$$CP_Z = (P_1 - Z)Y(P_1 - Z^*) \tag{4.10}$$

is positive definite on ran $P_Z = \mathrm{ran}(P_1 + Z)$, we get that $P_Z \in \mathcal{P}_{P_1}$ as desired.

To show that these two maps are mutually inverse, let first $Z \in \mathcal{E}_{P_1}$. Then $(P_Z)_{21}{(P_Z)_{11}}^{-1} = ZYY^{-1} = Z$. Conversely, let $P \in \mathcal{P}_{P_1}$ be given and set $Z \equiv P_{21}{P_{11}}^{-1}$. Then $ZP_{11} = P_{21}$ and $P_{11}Z^\dagger = {P_{21}}^\dagger = P_{12}$. By (4.8) and (4.9), $Y = P_{11}$, hence $P_{11}Z^\dagger = Z^\dagger\overline{P_{11}}$. Thus we get

$$\begin{aligned} P - P_Z &= P - (P_1 + Z)P_{11}(P_1 + Z^\dagger) \\ &= P - P_{11} - ZP_{11} - P_{11}Z^\dagger - ZP_{11}Z^\dagger \\ &= P - P_{11} - P_{21} - P_{12} - ZZ^\dagger\overline{P_{11}} \\ &= P_{22} - ZZ^\dagger\overline{P_{11}} \\ &= P_2 - (P_2 + ZZ^\dagger)\overline{P_{11}} \text{ (by (4.7b))} \\ &= 0. \end{aligned}$$

It remains to prove the statements about the group actions. It is fairly obvious that $\mathcal{S}^0_{P_1}(\mathcal{K},\gamma)$ acts on $\mathcal{P}_{P_1}$ via (4.4). The proof that U_Z is a Bogoliubov operator which takes P_1 to P_Z is also straightforward. To show that $U_Z \in \mathcal{S}_{\mathrm{HS}}(\mathcal{K},\gamma)$, let Y be given by (4.9). Then

$$Y^{\frac{1}{2}} - P_1 = Y^{\frac{1}{2}}(P_1 - Y^{-1})(P_1 + Y^{-\frac{1}{2}})^{-1} = Y^{\frac{1}{2}} Z^* Z (P_1 + Y^{-\frac{1}{2}})^{-1}$$

is of trace class. Therefore $(U_Z - \mathbf{1})P_1 = (P_1 + Z)Y^{\frac{1}{2}} - P_1 = Y^{\frac{1}{2}} - P_1 + ZY^{\frac{1}{2}}$ is HS, which implies $U_Z \in \mathcal{S}_{\mathrm{HS}}(\mathcal{K},\gamma)$.

Finally we have to show that the action (4.4) on $\mathcal{P}_{P_1}$ carries over to an action (4.4′) on $\mathcal{E}_{P_1}$. Thus, for given $Z \in \mathcal{E}_{P_1}$ and $U \in \mathcal{S}^0_{P_1}(\mathcal{K},\gamma)$, we have to compute the operator $Z' = P'_{21}{P'_{11}}^{-1}$ which corresponds to $P' = UP_ZU^\dagger$. By definition,

$$\begin{aligned} P'_{21} &= (U_{21} + U_{22}Z)Y(U_{11} + U_{12}Z)^*, \\ P'_{11} &= (U_{11} + U_{12}Z)Y(U_{11} + U_{12}Z)^*. \end{aligned} \tag{4.11}$$

Suppose that $(U_{11} + U_{12}Z)f = 0$ for some $f \in \mathcal{K}_1$. Then $\|f\| = \|{U_{11}}^{-1}U_{12}Zf\|$. Since $\|{U_{11}}^{-1}U_{12}\|^2 = \|{U_{12}}^*{U_{11}}^{-1*}{U_{11}}^{-1}U_{12}\| = \|{U_{12}}^*(P_1 + U_{12}{U_{12}}^*)^{-1}U_{12}\| = \|U_{12}\|^2/(1 + \|U_{12}\|^2) < 1$ and $\|Z\| < 1$, it follows that $f = 0$. Hence $U_{11} + U_{12}Z$ is injective, and, as a Fredholm operator with vanishing index (3.3), it has a bounded inverse. So we get from (4.11) that $Z' = P'_{21}{P'_{11}}^{-1} = (U_{21} + U_{22}Z)(U_{11} + U_{12}Z)^{-1}$ as claimed. □

The following construction will enable us to assign, in an unambiguous way, to each Bogoliubov operator $V \in \mathcal{S}_{P_1}(\mathcal{K},\gamma)$ a basis projection P_V such that (4.1) holds.

Lemma 4.2. *Let $\mathcal{H} \subset \mathcal{K}$ be a closed *–invariant subspace such that $\gamma|_{\mathcal{H}\times\mathcal{H}}$ is nondegenerate and such that $[P_1, E]$ is Hilbert–Schmidt where E is the orthogonal projection onto $\mathcal{H}$. Let $A \equiv ECE$ be the self–adjoint operator, invertible on $\mathcal{H}$, such that $\gamma(f,g) = \langle f, Ag\rangle$, $f,g \in \mathcal{H}$, and let $A_\pm$ be the unique positive operators such that $A = A_+ - A_-$ and $A_+A_- = 0$. Further let A^{-1} be defined as the inverse of A on $\mathcal{H}$ and as zero on $\mathcal{H}^\perp$, and similarly for $A_\pm^{-1}$. Then $A^{-1}C$ is the γ–orthogonal projection onto $\mathcal{H}$, $P_+ \equiv A_+^{-1}C$ is a basis projection of $\mathcal{H}$, and P_2P_+ is Hilbert–Schmidt. Moreover, $P_+ = P_1E$ if and only if $[P_1, E] = 0$.*

Proof. Let $E' \equiv \mathbf{1} - E$. Since ECE' and $E'CE$ are compact by assumption, $C - ECE' - E'CE = A + E'CE'$ is a Fredholm operator on $\mathcal{K}$ with vanishing index. Hence A is Fredholm on $\mathcal{H}$ with $\operatorname{ind} A = 0$. A is injective since γ is nondegenerate on $\mathcal{H}$. It is therefore a bounded bijection on $\mathcal{H}$ with a bounded inverse (the same holds true for $A_\pm$ as operators on $\operatorname{ran} A_\pm$). Thus $Q \equiv A^{-1}C$ is well–defined. It fulfills $Q^2 = A^{-1}(ECE)A^{-1}C = Q$ and $Q^\dagger = C(CA^{-1})C = Q$. So Q is a projection, self–adjoint with respect to γ. Since its range equals $\operatorname{ran} A^{-1} = \mathcal{H}$, it is the γ–orthogonal projection onto $\mathcal{H}$.

By a similar argument, P_+ is also a γ–orthogonal projection. It is straightforward to see that $P_+ = P_1E$ if and only if $[P_1, E] = 0$. To show that P_+ is actually a basis projection of $\mathcal{H}$ (cf. (4.6)), note that $\overline{A_+} = A_-$ because of $\overline{A} = -A$ (and uniqueness of $A_\pm$). This implies $P_+ + \overline{P_+} = A_+^{-1}C - A_-^{-1}C = A^{-1}C = \mathbf{1}_{\mathcal{H}}$. Positive definiteness of CP_+ on $\operatorname{ran} P_+$ follows from $\langle f, CP_+f\rangle = \|A_+^{-1/2}Cf\|^2$.

To prove that $P_2 P_+$ is HS, let $D \equiv EP_1E - A_+$. Since $EP_1E - EP_2E = A = A_+ - A_-$, we have $D = \overline{D}$. We claim that D is of trace class. Since ECE' is HS,

$$\begin{aligned} ECE'CE &= EC(\mathbf{1} - E)CE \\ &= E - (ECE)^2 \\ &= E - A^2 \\ &= (E + |A|)(E - |A|) \end{aligned}$$

is of trace class. Since $E + |A|$ has a bounded inverse (as an operator on $\mathcal{H}$) and since $|A| = A_+ + A_-$, it follows that $E - |A| = EP_1E + EP_2E - A_+ - A_- = D + \overline{D} = 2D$ is of trace class as claimed. As a consequence, $A_+P_2 = (EP_1E - D)P_2$ is HS (P_1EP_2 is HS by assumption). By boundedness of A_+^{-1}, $P_+P_2 = -A_+^{-2}(A_+P_2)$ and $P_2P_+ = (P_+P_2)^\dagger$ are also HS. This completes the proof. □

Now let $V \in \mathcal{S}_{P_1}(\mathcal{K}, \gamma)$. We already showed in Section 3 that the restriction of γ to $\ker V^\dagger$ is nondegenerate. We also showed in the proof of Theorem 3.2 that $[P_1, V']$ is Hilbert–Schmidt where V' is the isometry arising from polar decomposition of V. Hence $[P_1, E]$ is Hilbert–Schmidt where $E = C(\mathbf{1} - V'V'^*)C$ is the orthogonal projection onto $\ker V^\dagger$. Thus Lemma 4.2 applies to $\mathcal{H} = \ker V^\dagger$.

Definition 4.3. *For $V \in \mathcal{S}_{P_1}(\mathcal{K}, \gamma)$, let P_{V+} be the basis projection of* $\ker V^\dagger$ *given by Lemma 4.2, and set*

$$P_V \equiv VP_1V^\dagger + P_{V+} \quad \in \mathcal{P}_{P_1},$$

$$Z_V \equiv (P_V)_{21}(P_V)_{11}{}^{-1} \in \mathcal{E}_{P_1}$$

(cf. Proposition 4.1). Further let $U_V \in \mathcal{S}_{\mathrm{HS}}(\mathcal{K}, \gamma)$ be the Bogoliubov operator associated with Z_V according to (4.5)*, and define $W_V \equiv U_V^\dagger V \in \mathcal{S}_{\mathrm{diag}}(\mathcal{K}, \gamma)$.*

P_V clearly is a basis projection which satisfies (4.1). Actually, any basis projection P fulfilling $V^\dagger PV = P_1$ or, equivalently, $PV = VP_1$, is of the form $P = VP_1V^\dagger + P'$ where P' is some basis projection of $\ker V^\dagger$. What had to be proved above is that P' can be chosen such that P_2P' is Hilbert–Schmidt, in the case $\dim \ker V^\dagger = \infty$. In fact, any such choice would suffice for what follows.

The condition $V^\dagger P_V V = P_1$ translates into the condition

$$Z_V V_{11} = V_{21} \tag{4.12}$$

for Z_V. Again, each $Z \in \mathcal{E}_{P_1}$ fulfilling (4.12) would do, but we prefer to have a definite choice. It follows from symmetry (4.2) that any Z which solves (4.12) must have the form

$$Z = V_{21}V_{11}{}^{-1} + V_{22}{}^{-1*}V_{12}{}^*P_{\ker V_{11}{}^*} + Z' \tag{4.13}$$

where $P_\mathcal{H}$ denotes the orthogonal projection onto some closed subspace $\mathcal{H} \subset \mathcal{K}$, $V_{11}{}^{-1}$ and $V_{22}{}^{-1}$ have been defined below (3.4), and Z' is a symmetric Hilbert–Schmidt operator from $\ker V_{11}{}^*$ to $\ker V_{22}{}^*$. The freedom in the choice of Z' corresponds to the freedom in the choice of P'. Note that Z can be written, with respect to the decompositions $\mathcal{K}_1 = \operatorname{ran} V_{11} \oplus \ker V_{11}{}^*$, $\mathcal{K}_2 = \operatorname{ran} V_{22} \oplus \ker V_{22}{}^*$, as

$$Z = \begin{pmatrix} P_{\operatorname{ran} V_{22}} V_{21} V_{11}{}^{-1} & V_{22}{}^{-1*} V_{12}{}^* P_{\ker V_{11}{}^*} \\ P_{\ker V_{22}{}^*} V_{21} V_{11}{}^{-1} & Z' \end{pmatrix}. \tag{4.13$'$}$$

The Hilbert–Schmidt norm of Z is minimized by choosing $Z' = 0$, but there are examples in which this choice violates the condition $\|Z\| < 1$, i.e. it does not always define an element of $\mathcal{E}_{P_1}$. This is in contrast to the CAR case where the choice analogous to $Z' = 0$ appears to be natural [4]. As we shall see in Section 5, Z_V describes the values of implementers on the Fock vacuum.

The operators U_V and W_V constitute the product decomposition of V that was announced earlier. W_V is diagonal because $P_1 W_V = P_1 U_V^\dagger V = U_V^\dagger P_V V = U_V^\dagger V P_1 = W_V P_1$. Explicitly, one computes that

$$W_V = \begin{pmatrix} (P_1 + Z_V^\dagger Z_V)^{\frac{1}{2}} V_{11} & 0 \\ 0 & (P_2 + Z_V Z_V^\dagger)^{\frac{1}{2}} V_{22} \end{pmatrix}$$

with respect to the decomposition $\mathcal{K} = \mathcal{K}_1 \oplus \mathcal{K}_2$. Let us summarize the properties of these operators.

Proposition 4.4. *Definition 4.3 establishes a decomposition of $V \in \mathcal{S}_{P_1}(\mathcal{K}, \gamma)$*

$$V = U_V W_V$$

where $U_V \in \mathcal{S}_{\mathrm{HS}}(\mathcal{K}, \gamma)$ and $W_V \in \mathcal{S}_{\mathrm{diag}}(\mathcal{K}, \gamma)$ have the properties

$$\begin{aligned} &\operatorname{ind} U_V = 0, && Z_{U_V} = Z_V, && P_{U_V} = P_V; \\ &\operatorname{ind} W_V = \operatorname{ind} V, && Z_{W_V} = 0, && P_{W_V} = P_1. \end{aligned}$$

In particular, if $V \in \mathcal{S}^0_{P_1}(\mathcal{K}, \gamma)$, then

$$U_V = \begin{pmatrix} |V_{11}{}^*| & V_{12} v_{22}{}^* \\ V_{21} v_{11}{}^* & |V_{22}{}^*| \end{pmatrix}, \qquad W_V = \begin{pmatrix} v_{11} & 0 \\ 0 & v_{22} \end{pmatrix}$$

where $v_{11} \equiv V_{11}|V_{11}|^{-1}$ and $v_{22} = \overline{v_{11}}$ are the unitary parts of V_{11} and V_{22}; whereas if $V \in \mathcal{S}_{\mathrm{diag}}(\mathcal{K}, \gamma)$, then $U_V = \mathbf{1}$ and $W_V = V$.

Remark. The product decomposition described above is the generalization to the infinite–dimensional case of a construction given by Maaß [16]. The exact analogue of the construction given in [4] in the fermionic case would be to define $W' \in \mathcal{S}_{\mathrm{diag}}(\mathcal{K}, \gamma)$ through $W'_{11} \equiv V_{11}|V_{11}|^{-1}$ (the isometric part of V_{11}), to choose a Bogoliubov operator u' from $\ker W'^\dagger$ to $\ker V^\dagger$ such that $u' P_1 = P_V u'$, and to set $U' \equiv V W'^\dagger + u' \in \mathcal{S}^0_{P_1}(\mathcal{K}, \gamma)$. Then U' and W' would also have the properties listed in Proposition 4.4, with the exception that $U' - \mathbf{1}$ is not necessarily Hilbert–Schmidt. On the other hand, this choice has the merit that the definition of W' is completely canonical (independent of the choice of Z).

Though it was not shown in [4], it holds true also in the CAR case that each implementable Bogoliubov operator can be written as a product of two factors where the first differs from $\mathbf{1}$ only by a Hilbert–Schmidt part, and the second is diagonal.

Corollary 4.5. $\mathcal{S}_{P_1}(\mathcal{K}, \gamma) = \mathcal{S}_{\mathrm{HS}}(\mathcal{K}, \gamma) \cdot \mathcal{S}_{\mathrm{diag}}(\mathcal{K}, \gamma)$. *The orbits of the action of $\mathcal{S}^0_{P_1}(\mathcal{K}, \gamma)$ on $\mathcal{S}_{P_1}(\mathcal{K}, \gamma)$ are the subsets $\mathcal{S}^n_{P_1}(\mathcal{K}, \gamma)$, $n \in \mathbb{N} \cup \{\infty\}$. They coincide with the connected components of $\mathcal{S}_{P_1}(\mathcal{K}, \gamma)$.*

5 Normal Form of Cuntz Algebra Generators

The first step in the construction of implementers consists in a generalization of the definition of 'bilinear Hamiltonians' [5] from the finite rank case to the case of bounded operators. If H is a finite rank operator on $\mathcal{K}$ such that $H^* = \overline{H} = -H$, then e^{HC} belongs to $\mathcal{S}_{\mathrm{HS}}(\mathcal{K}, \gamma)$. Expanding $H = \sum f_j \langle g_j, . \rangle$, one obtains a skew–adjoint element $b_0(H) \equiv \sum f_j g_j^*$ of $\mathcal{C}(\mathcal{K}, \gamma)$ which is a linear function of H, independent of the choice of $f_j, g_j \in \mathcal{K}$. Then $\pi_{P_1}\big(b_0(H)\big)$ is essentially skew–adjoint on $\mathcal{D}$, and, if $b(H)$ denotes its closure, $\exp\big(\frac{1}{2} b(H)\big)$ is a unitary which implements the automorphism induced by e^{HC} [5, 6].

Using Wick ordering, the definition of bilinear Hamiltonians can be extended to arbitrary bounded operators H which are symmetric in the sense of (4.2)[3]:

$$H_{11} = \overline{H_{22}}^*, \qquad H_{12} = \overline{H_{12}}^*, \qquad H_{21} = \overline{H_{21}}^*. \tag{5.1}$$

Without loss of generality, we henceforth assume that $\mathcal{K}_1 = L^2(\mathbb{R}^d)$. Then let $\mathfrak{S} \subset \mathcal{F}_s(\mathcal{K}_1)$ be the dense subspace consisting of finite particle vectors ϕ with n–particle wave functions $\phi^{(n)}$ in the Schwartz space $\mathfrak{S}(\mathbb{R}^{dn})$. The unsmeared annihilation operator $a(p)$ with (invariant) domain $\mathfrak{S}$ is defined as usual

$$(a(p)\phi)^{(n)}(p_1, \ldots, p_n) \equiv \sqrt{n+1}\, \phi^{(n+1)}(p, p_1, \ldots, p_n).$$

Let $a^*(p)$ be its quadratic form adjoint on $\mathfrak{S} \times \mathfrak{S}$. Then Wick ordered monomials $a^*(q_1) \cdots a^*(q_m) a(p_1) \cdots a(p_n)$ make sense as quadratic forms on $\mathfrak{S} \times \mathfrak{S}$ [17, 18], and, for $\phi, \phi' \in \mathfrak{S}$,

$$\langle \phi, a^*(q_1) \cdots a^*(q_m) a(p_1) \cdots a(p_n) \phi' \rangle \equiv \langle a(q_1) \cdots a(q_m)\phi, a(p_1) \cdots a(p_n)\phi' \rangle$$

is a Schwartz function to which tempered distributions may be applied. In particular, the distributions $H_{jk}(p,q)$, $j,k = 1,2$, given by

$$\begin{aligned}
\langle f, H_{11} g \rangle &= \int \overline{f(p)} H_{11}(p,q) g(q)\, dp\, dq, \\
\langle f, H_{12} g^* \rangle &= \int \overline{f(p)} H_{12}(p,q) \overline{g(q)}\, dp\, dq, \\
\langle f^*, H_{21} g \rangle &= \int f(p) H_{21}(p,q) g(q)\, dp\, dq, \\
\langle f^*, H_{22} g^* \rangle &= \int f(p) H_{22}(p,q) \overline{g(q)}\, dp\, dq
\end{aligned}$$

for $f, g \in \mathfrak{S}(\mathbb{R}^d) \subset \mathcal{K}_1$, give rise to the following quadratic forms on $\mathfrak{S} \times \mathfrak{S}$:

$$\begin{aligned}
H_{11} a^* a &\equiv \int a(p)^* H_{11}(p,q) a(q)\, dp\, dq \\
H_{12} a^* a^* &\equiv \int a(p)^* H_{12}(p,q) a(q)^*\, dp\, dq \\
H_{21} a a &\equiv \int a(p) H_{21}(p,q) a(q)\, dp\, dq \\
{:}H_{22} a a^*{:} &\equiv \int a(q)^* H_{22}(p,q) a(p)\, dp\, dq = H_{11} a^* a.
\end{aligned}$$

[3]The bilinear Hamiltonian corresponding to an antisymmetric operator ($H = -\overline{H^*}$) vanishes.

Wick ordering of $H_{22}aa^*$ is necessary to make this expression well–defined. The last equality follows from symmetry of H:

$$H_{11}(p,q) = H_{22}(q,p), \quad H_{12}(p,q) = H_{12}(q,p), \quad H_{21}(p,q) = H_{21}(q,p).$$

We next define $:b(H):$ and its Wick ordered powers as quadratic forms on $\mathfrak{S} \times \mathfrak{S}$:

$$\begin{aligned}
:b(H):\, &\equiv H_{12}a^*a^* + 2H_{11}a^*a + H_{21}aa, \\
:b(H)^l:\, &\equiv l! \sum_{\substack{l_1,l_2,l_3=0 \\ l_1+l_2+l_3=l}}^{l} \frac{2^{l_2}}{l_1!l_2!l_3!} H_{l_1,l_2,l_3}, \qquad l \in \mathbb{N},
\end{aligned}$$

$$\begin{aligned}
\text{with } H_{l_1,l_2,l_3} \equiv \int & H_{12}(p_1,q_1)\cdots H_{12}(p_{l_1},q_{l_1}) H_{11}(p'_1,q'_1)\cdots H_{11}(p'_{l_2},q'_{l_2}) \\
&\cdot H_{21}(p''_1,q''_1)\cdots H_{21}(p''_{l_3},q''_{l_3}) a^*(p_1)\cdots a^*(p_{l_1}) a^*(q_1)\cdots a^*(q_{l_1}) \\
&\cdot a^*(p'_1)\cdots a^*(p'_{l_2}) a(q'_1)\cdots a(q'_{l_2}) a(p''_1)\cdots a(p''_{l_3}) a(q''_1)\cdots a(q''_{l_3}) \\
&\cdot dp_1\,dq_1\ldots dp_{l_1}\,dq_{l_1}\,dp'_1\,dq'_1\ldots dp'_{l_2}\,dq'_{l_2}\,dp''_1\,dq''_1\ldots dp''_{l_3}\,dq''_{l_3}.
\end{aligned}$$

The Wick ordered exponential of $\frac{1}{2}b(H)$ is also well-defined on $\mathfrak{S} \times \mathfrak{S}$, since only a finite number of terms contributes when applied to vectors from $\mathfrak{S}$:

$$:\exp\left(\frac{1}{2}b(H)\right):\, \equiv \sum_{l=0}^{\infty} \frac{1}{l!2^l} :b(H)^l: .$$

The important point is that these quadratic forms are actually the forms of uniquely determined linear operators, defined on the dense subspace $\mathcal{D}$ and mapping $\mathcal{D}$ into the domain of (the closure of) any creation or annihilation operator, provided that [10]

$$\|H_{12}\| < 1, \qquad H_{12} \text{ is Hilbert–Schmidt.} \tag{5.2}$$

These operators will be denoted by the same symbols as the quadratic forms.

Lemma 5.1. *Let $H \in \mathfrak{B}(\mathcal{K})$ satisfy* (5.1) *and* (5.2). *Then the following commutation relations hold on $\mathcal{D}$, for $f \in \mathcal{K}_1$:*

$$\begin{aligned}
[H_{l_1,l_2,l_3}, a(f)^*] &= l_2 a(H_{11}f)^* H_{l_1,l_2-1,l_3} + 2l_3 H_{l_1,l_2,l_3-1} a\big((H_{21}f)^*\big), \\
[a(f), H_{l_1,l_2,l_3}] &= 2l_1 a(H_{12}f^*)^* H_{l_1-1,l_2,l_3} + l_2 H_{l_1,l_2-1,l_3} a(H_{11}{}^*f),
\end{aligned}$$

implying that

$$\begin{aligned}
&\left[:\exp\left(\frac{1}{2}b(H)\right):, a(f)^*\right] \\
&\qquad = a(H_{11}f)^* :\exp\left(\frac{1}{2}b(H)\right): + :\exp\left(\frac{1}{2}b(H)\right): a\big((H_{21}f)^*\big), \\
&\left[a(f),\, :\exp\left(\frac{1}{2}b(H)\right):\right] \\
&\qquad = a(H_{12}f^*)^* :\exp\left(\frac{1}{2}b(H)\right): + :\exp\left(\frac{1}{2}b(H)\right): a(H_{11}{}^*f).
\end{aligned}$$

Proof. Compute as in [10, 4]. □

For given $V \in \mathcal{S}_{P_1}(\mathcal{K}, \gamma)$, we are now looking for symmetric bounded operators H which satisfy (5.2) and the following intertwiner relation on $\mathcal{D}$

$$:\!\exp\left(\frac{1}{2}b(H)\right)\!: \pi_{P_1}(f) = \pi_{P_1}(Vf) :\!\exp\left(\frac{1}{2}b(H)\right)\!:, \quad f \in \mathcal{K} \tag{5.3}$$

(taking the closure of $\pi_{P_1}(Vf)$ is tacitly assumed here). This problem turns out to be equivalent to the determination of the operators Z done in (4.12), (4.13).

Lemma 5.2. *Each $Z \in \mathcal{E}_{P_1}$ fulfilling* (4.12) *gives rise to a unique solution H of the above problem through the formula*

$$H = \begin{pmatrix} V_{11} - P_1 + Z^\dagger V_{21} & Z^\dagger \\ ({V_{22}}^* + {V_{12}}^* Z^\dagger) V_{21} & {V_{22}}^* - P_2 + {V_{12}}^* Z^\dagger \end{pmatrix},$$

and each solution arises in this way.

Proof. Let us abbreviate $\eta_H \equiv\, :\!\exp(\frac{1}{2}b(H))\!:$. Choosing $f \in \mathcal{K}_2$ resp. $f \in \mathcal{K}_1$ and inserting the definition of π_{P_1}, one finds that (5.3) is equivalent to

$$\eta_H a(g) = \big(a(V_{11}g) + a^*(V_{12}g^*)\big)\eta_H, \quad \eta_H a^*(g) = \big(a^*(V_{11}g) + a(V_{12}g^*)\big)\eta_H$$

for $g \in \mathcal{K}_1$. Using the commutation relations from Lemma 5.1, these equations may be brought into Wick ordered form:

$$0 = a^*\big((V_{12} + H_{12}V_{22})g^*\big)\eta_H + \eta_H a\Big(\big((P_1 + {H_{11}}^*)V_{11} - P_1\big)g\Big),$$
$$0 = a^*\big((P_1 + H_{11} - V_{11} - H_{12}V_{21})g\big)\eta_H + \eta_H a\Big(\big(\overline{H_{21}} - (P_1 + {H_{11}}^*)V_{12}\big)g^*\Big).$$

As in the CAR case [4], these equations hold for all $g \in \mathcal{K}_1$ if and only if

$$0 = V_{12} + H_{12}V_{22}, \tag{5.4a}$$
$$0 = P_1 + H_{11} - V_{11} - H_{12}V_{21}, \tag{5.4b}$$
$$0 = H_{21} - (P_2 + H_{22})V_{21}, \tag{5.4c}$$
$$0 = P_2 - (P_2 + H_{22})V_{22} \tag{5.4d}$$

(we applied complex conjugation and used $\overline{{H_{11}}^*} = H_{22}$).

Now assume that H solves the above problem. It is then obvious from (5.1), (5.2) and (5.4a) that $Z \equiv {H_{12}}^\dagger$ belongs to $\mathcal{E}_{P_1}$ and fulfills (4.12).

Conversely, let $Z \in \mathcal{E}_{P_1}$ satisfy (4.12). If there exists a solution H with $H_{12} = Z^\dagger$, then H_{11} is fixed by (5.4b), H_{22} must equal $\overline{{H_{11}}^*}$, and H_{21} is determined by (5.4c). Thus there can be at most one solution corresponding to Z, and it is necessarily of the form stated in the proposition.

It remains to prove that the so–defined H has all desired properties, i.e. that H_{21} is symmetric and that (5.4d) holds, the rest being clear by construction. The first claim follows from (3.4d):

$$H_{21} - \overline{{H_{21}}^*} = ({V_{22}}^* + {V_{12}}^* Z^\dagger)V_{21} - {V_{12}}^*(V_{11} + Z^\dagger V_{21}) = 0,$$

and the second from (4.12) and (3.4b):

$$(P_2 + H_{22})V_{22} = ({V_{22}}^* - {V_{12}}^* \overline{Z})V_{22} = {V_{22}}^* V_{22} - {V_{12}}^* V_{12} = P_2.$$

□

Inserting the formula (4.13) for Z, one obtains

$$\begin{aligned}
H_{11} &= V_{11}{}^{-1*} - P_1 - P_{\ker V_{11}{}^*} V_{12} V_{22}{}^{-1} V_{21} + Z'^{\dagger} V_{21}, \\
H_{12} &= -V_{12} V_{22}{}^{-1} - V_{11}{}^{-1*} V_{21}{}^* P_{\ker V_{22}{}^*} + Z'^{\dagger}, \\
H_{21} &= (V_{22}{}^{-1} - V_{12}{}^* V_{11}{}^{-1*} V_{21}{}^* P_{\ker V_{22}{}^*}) V_{21} + V_{12}{}^* Z'^{\dagger} V_{21}, \\
H_{22} &= V_{22}{}^{-1} - P_2 - V_{12}{}^* V_{11}{}^{-1*} V_{21}{}^* P_{\ker V_{22}{}^*} + V_{12}{}^* Z'^{\dagger}.
\end{aligned}$$

H corresponds to Ruijsenaars' operator Λ [10]. If one compares the above formula for H with Ruijsenaars' formula for Λ in the case of automorphisms ($\ker V_{jj}{}^* = \{0\},\ j = 1,2,\ Z' = 0$), one finds that the off–diagonal components carry opposite signs. This is due to the fact that Ruijsenaars actually constructs implementers for the transformation induced by CVC rather than V, cf. (3.27) and (3.29) in [10].

Note that $:\!\exp(\frac{1}{2}b(H))\!: \Omega_{P_1} = \exp(\frac{1}{2}H_{12}a^*a^*)\Omega_{P_1}$. By Ruijsenaars' computation [10] (see also [15]), the norm of such vectors is

$$\left\| :\!\exp\left(\frac{1}{2}b(H)\right)\!: \Omega_{P_1} \right\| = \left(\det(P_1 + H_{12}H_{12}{}^{\dagger})\right)^{-1/4}.$$

Definition 5.3. *Let $V \in \mathcal{S}_{P_1}(\mathcal{K},\gamma)$, and let P_V, Z_V and H_V be the operators associated with V according to Definition 4.3 and Lemma 5.2. Choose a γ–orthonormal basis $f_1, f_2, \ldots$ in $P_V(\ker V^{\dagger})$, i.e. a basis such that $\gamma(f_j, f_k) = \delta_{jk}$ (this is possible because the restriction of γ to $P_V(\ker V^{\dagger})$ is positive definite). Let ψ_j be the isometry obtained by polar decomposition of the closure of $\pi_{P_1}(f_j)$. Then define operators $\Psi_\alpha(V)$ on $\mathcal{D}$, for any multi–index $\alpha = (\alpha_1, \ldots, \alpha_l)$ with $\alpha_j \le \alpha_{j+1}$ (or $\alpha = 0$) as in (3.1), as*

$$\Psi_\alpha(V) \equiv \left(\det(P_1 + Z_V^{\dagger} Z_V)\right)^{\frac{1}{4}} \psi_{\alpha_1} \cdots \psi_{\alpha_l} :\!\exp\left(\frac{1}{2}b(H_V)\right)\!:. \tag{5.5}$$

Theorem 5.4. *The $\Psi_\alpha(V)$ extend continuously to isometries (denoted by the same symbols) on the symmetric Fock space $\mathcal{F}_s(\mathcal{K}_1)$ such that*

$$\Psi_\alpha(V)^* \Psi_\beta(V) = \delta_{\alpha\beta}\mathbf{1}, \quad \sum_\alpha \Psi_\alpha(V)\Psi_\alpha(V)^* = \mathbf{1} \tag{5.6}$$

and, for any element w of the Weyl algebra $\mathcal{W}(\mathcal{K},\gamma)$,

$$\varrho_V(w) = \sum_\alpha \Psi_\alpha(V) w \Psi_\alpha(V)^*. \tag{5.7}$$

Proof. By (2.1) we have $\pi_{P_1}(f_j)^*\pi_{P_1}(f_j) = \mathbf{1} + \pi_{P_1}(f_j)\pi_{P_1}(f_j)^*$ on $\mathcal{D}$, so the closure of $\pi_{P_1}(f_j)$ is injective, and ψ_j is isometric. It is also easy to see, using (5.3), the CCR and $\|\Psi_\alpha(V)\Omega_{P_1}\| = 1$, that

$$\begin{aligned}
&\langle \Psi_\alpha(V)\pi_{P_1}(g_1 \cdots g_m)\Omega_{P_1}, \Psi_\alpha(V)\pi_{P_1}(h_1 \cdots h_n)\Omega_{P_1} \rangle \\
&\qquad = \langle \pi_{P_1}(g_1 \cdots g_m)\Omega_{P_1}, \pi_{P_1}(h_1 \cdots h_n)\Omega_{P_1} \rangle.
\end{aligned}$$

Hence $\Psi_\alpha(V)$ is isometric on $\mathcal{D}$ and has a continuous extension to an isometry on $\mathcal{F}_s(\mathcal{K}_1)$.

Let $\mathcal{H}_j \equiv \operatorname{span}(f_j, f_j^*)$, so that $\psi_j \in \mathcal{W}(\mathcal{H}_j)''$ by virtue of Lemma 2.1. Since $\mathcal{H}_j \subset \ker V^\dagger$, there holds $\mathcal{W}(\mathcal{H}_j) \subset \mathcal{W}(\operatorname{ran} V)'$ by duality (2.3). Now let $f \in \operatorname{Re}\mathcal{K}$ and $\phi \in \mathcal{D}$. Since ϕ is an entire analytic vector for $\pi_{P_1}(f)$ [5], since $\mathcal{D}$ is invariant under $\pi_{P_1}(f)$, and since $\overline{\pi_{P_1}(Vf)}$ is affiliated with $\mathcal{W}(\operatorname{ran} V)$ by Lemma 2.1 (the bar denotes closure), it follows from (5.3) that

$$\begin{aligned}
\Psi_\alpha(V)w(f)\phi &= \sum_{n=0}^{\infty} \frac{i^n}{n!}\Psi_\alpha(V)(\pi_{P_1}(f))^n\phi \\
&= \sum_{n=0}^{\infty} \frac{i^n}{n!}\psi_{\alpha_1}\cdots\psi_{\alpha_l}\left(\overline{\pi_{P_1}(Vf)}\right)^n\Psi_0(V)\phi \\
&= \sum_{n=0}^{\infty} \frac{i^n}{n!}\left(\overline{\pi_{P_1}(Vf)}\right)^n\Psi_\alpha(V)\phi \\
&= w(Vf)\Psi_\alpha(V)\phi.
\end{aligned}$$

By continuity, this entails

$$\Psi_\alpha(V)w = \varrho_V(w)\Psi_\alpha(V), \qquad w \in \mathcal{W}(\mathcal{K},\gamma). \tag{5.8}$$

We next claim that

$$\psi_j^*\Psi_0(V) = 0 \tag{5.9}$$

or, equivalently, that $\pi_{P_1}(f_j)^*\Psi_0(V) = 0$. To see this, apply Lemma 5.1 and write $\pi_{P_1}(f_j)^*\Psi_0(V)$ in Wick ordered form:

$$\pi_{P_1}(f_j)^*\Psi_0(V) = a\big((P_1 + H_{12})f_j^*\big)^*\Psi_0(V) + \Psi_0(V)a\big((P_1 + {H_{11}}^*)f_j\big)$$

on $\mathcal{D}$, with $H \equiv H_V$. Then (5.9) holds if and only if

$$(P_1 + H_{12})f_j^* = 0, \qquad (P_1 + {H_{11}}^*)f_j = 0. \tag{5.9$'$}$$

Now $f_j \in \operatorname{ran} P_V$ is equivalent to $f_j^* \in \ker P_V = \ker CP_V = \ker(P_1 + H_{12})$ (we used (4.10)). This proves the first equation in (5.9$'$). It also shows that ${H_{12}}^* f_j = -f_j$. Hence by Lemma 5.2,

$$(P_1 + {H_{11}}^*)f_j = ({V_{11}}^* + {V_{21}}^*{H_{12}}^*)f_j = ({V_{11}}^* - {V_{21}}^*)f_j = P_1V^\dagger f_j = 0$$

which proves the second equation in (5.9$'$) and therefore (5.9).

The orthogonality relation $\Psi_\alpha(V)^*\Psi_\beta(V) = 0$ ($\alpha \neq \beta$) now follows from (5.9) and from $\mathcal{W}(\mathcal{H}_j) \subset \mathcal{W}(\mathcal{H}_k)'$ ($j \neq k$) which in turn is a consequence of $\gamma(\mathcal{H}_j, \mathcal{H}_k) = 0$ and (2.3).

The proof of the completeness relation $\sum \Psi_\alpha(V)\Psi_\alpha(V)^* = \mathbf{1}$ is facilitated by invoking the product decomposition $V = U_V W_V$ from Proposition 4.4. Set $e_j \equiv U_V^\dagger f_j$ to obtain a γ–orthonormal basis $e_1, e_2, \ldots$ in $P_1(\ker W_V^\dagger) = \mathcal{K}_1 \cap \ker W_V^\dagger$. Let ψ_j' be the isometric part of $a(e_j)^*$. An application of Definition 5.3 to W_V yields implementers $\Psi_\alpha(W_V) = \psi_{\alpha_1}'\cdots\psi_{\alpha_l}'\Psi_0(W_V)$ for W_V. $Z_{W_V} = 0$ entails that

$$\Psi_\alpha(W_V)\Omega_{P_1} = \psi_{\alpha_1}'\cdots\psi_{\alpha_l}'\Omega_{P_1}.$$

One computes, using the CCR, that $\psi_{\alpha_1}'\cdots\psi_{\alpha_l}'\Omega_{P_1} = \phi_\alpha'$, where the ϕ_α' are the cyclic vectors associated with the pure state $\omega_{P_1}\circ\varrho_{W_V} = \omega_{P_1}$ as in Lemma 3.1.

Let $\mathcal{F}'_\alpha$ be the closure of $\mathcal{W}(\operatorname{ran} W_V)\phi'_\alpha$. Since the $\mathcal{F}'_\alpha$ are irreducible subspaces for $\mathcal{W}(\operatorname{ran} W_V)$ by Lemma 3.1, they must coincide with the irreducible subspaces $\operatorname{ran}\Psi_\alpha(W_V)$. $\oplus\mathcal{F}'_\alpha = \mathcal{F}_s(\mathcal{K}_1)$ then implies completeness of the $\Psi_\alpha(W_V)$.

The proof will be completed by showing that

$$\Psi_\alpha(V) = \Psi(U_V)\Psi_\alpha(W_V) \tag{5.10}$$

holds where $\Psi(U_V)$ is the unitary implementer for U_V given by Definition 5.3. It suffices to show that (5.10) holds on Ω_{P_1} since any bounded operator fulfilling (5.8) is already determined by its value on Ω_{P_1}. Because of $Z_{U_V} = Z_V$ we have

$$\Psi_0(V)\Omega_{P_1} = \Psi(U_V)\Omega_{P_1}, \tag{5.11}$$

so it remains to show that $\psi_{\alpha_1}\cdots\psi_{\alpha_l}\Psi(U_V)\Omega_{P_1} = \Psi(U_V)\psi'_{\alpha_1}\cdots\psi'_{\alpha_l}\Omega_{P_1}$. We claim that

$$\psi_j\Psi(U_V) = \Psi(U_V)\psi'_j.$$

For let T (resp. T') be the closure of $\pi_{P_1}(f_j)$ (resp. $\pi_{P_1}(e_j)$), and let $T^\pm$ (resp. $T'^\pm$) be the corresponding self–adjoint operators as in the proof of Lemma 2.1, so that

$$D(T) = D(T^+)\cap D(T^-), \qquad T = T^+ - iT^-,$$

and similar for T'. Then there holds

$$\Psi(U_V)\exp(itT'^\pm)\Psi(U_V)^* = \exp(itT^\pm), \qquad t\in\mathbb{R}.$$

Therefore $\Psi(U_V)$ maps $D(T'^\pm)$ onto $D(T^\pm)$, and one has $\Psi(U_V)T'^\pm\Psi(U_V)^* = T^\pm$. Consequently, $\Psi(U_V)\big(D(T')\big) = D(T)$ and $\Psi(U_V)T'\Psi(U_V)^* = T$. This implies that $\Psi(U_V)\psi'_j\Psi(U_V)^* = \psi_j$ as claimed. The proof is complete since (5.6) and (5.8) together imply (5.7). □

Corollary 5.5. *There is a unitary isomorphism from $H(\varrho_V)$, the Hilbert space generated by the $\Psi_\alpha(V)$, onto the symmetric Fock space $\mathcal{F}_s(P_V(\ker V^\dagger))$ over $P_V(\ker V^\dagger)$, which maps $\Psi_\alpha(V)$ to $(l_1!\cdots l_r!)^{-\frac{1}{2}}a^*(f_{\alpha_1})\cdots a^*(f_{\alpha_l})\Omega$, where the notation is as in* (3.1)*, and $a^*(f_j)$ and Ω are now creation operators and the Fock vacuum in $\mathcal{F}_s(P_V(\ker V^\dagger))$.*

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