On Fillings by Holomorphic Discs of the Levels of a Morse Function

Klaus Mohnke*

Abstract

A filling by holomorphic discs of the level sets near the critical point is contructed. The so obtained family of discs filling the critical level surface near its conus point similar to a Bishop family is the uniform limit of the Bishop families filling the level surfaces consisting of two cups near their elliptic points and of the families of holomorphic discs filling the levels consisting of a totally real handle. In the case of a hypersurface of contact type near the critical point the regularity of the induced foliation of the pseudoconvex side is shown. This is used to construct smooth, regular extensions with Levi-flat levels of certain Morse functions on (smooth) boundaries of Stein surfaces.

1 Main Result

We wish to solve the following problem: Let Ω be a Stein surface with smooth boundary $\partial\Omega$. Let $f : \partial\Omega \longrightarrow \mathbb{R}$ be a smooth Morse function. Does there exist a smooth extension $\hat{f} : \Omega \longrightarrow \mathbb{R}$, such that all levels are Levi–flat?
This may be considered as a family–version of the question of finding a Levi–flat hypersurface in Ω bounded by a given surface in $\partial\Omega$ (see e.g. [2]).

In the present paper we construct this extension in the following situation

Theorem 1.1 *Let Ω be a Stein surface with smooth strongly pseudoconvex boundary $\partial\Omega \cong S^3$ diffeomorphic to the three-sphere. Consider a Morse function f on $\partial\Omega$ satifying the following properties:*
(i) $f^{-1}(t) \cong \coprod_i F_t^i$ or empty for all regular t, where each $F_t^i \cong S^2$ is a two-sphere.
(ii) $f^{-1}(t)$ admits at most complex points of elliptic type outside the critical points of f for all t.
(iii) The Hessian of f at all critical points restricted to their complex tangencies is a nondegenerate bilinear form on them.

Then there exists a smooth extension $\hat{f}$ of f to the whole space Ω, with no critical points and Levi–flat level sets. The corresponding foliation consits of analytic discs outside the complex and the critical points.

*Humboldt-Universität zu Berlin, SFB 288 der DFG, e-mail: mohnke@mathematik.hu-berlin.de

Remark 1.2 *(i) Conditions (ii) and (iii) of Theorem 1.1 imply that all critical points are of elliptic type (see Chapter 2 below for a definition).*
(ii) It is conceivable that one can drop condition (ii) allowing "admissible" hyperbolic points in the sense of Bedford and Klingenberg [3]. There are also efforts to generalize their result to general hyperbolic points. But, of course, one would have to carry out the corresponding analysis for critical points of hyperbolic type.

Proof. First step is to fill all levels with pairwise disjoint holomorphic discs, such that each point of $\overline{\Omega}$ is either a critical point or a complex point of a level surface or lies on (the closure) of one of these discs. For a regular t we can fill each of the spheres by Gromov's result: Bishop's construction in [4] gives two one parameter families of analytic discs in Ω whose boundaries foliate punctured neighbourhoods of the two elliptic points and converging to these on one of their ends. By a version of Gromov's compactness theorem the only obstruction for them to meet is cusp degeneration in the interior or at the boundary. The first implies the existence of a sphere of self intersection -1 which cannot occur because Ω is Stein, the second is excluded by more subtle arguments basically using the nice charcteristic foliation on the sphere induced by the complex tangencies of $\partial\Omega$. A uniqueness argument shows that they fit together to form a one parameter family filling up the sphere. In the end we obtain disjoint smooth Levi–flat 3-balls $\hat{F}_t^i$, foliated in the corresponding way, bounded by the components of the level t. There are no other holomorphic discs with boundary on the level than those of the foliation of the balls. Disjointness and uniqueness are both derived from positveness of intersections of complex curves in complex surfaces:
Assume $\hat{F}_t^i \cap \hat{F}_t^j \neq \emptyset$ for $i \neq j$. On account of $F_t^i \cap F_t^j = \emptyset$ and strong pseudoconvexity of Ω there is a pair of discs, one from each family, intersecting each other in the interior. But then each disc of the first family has to intersect each disc of the second in the interior which is impossible because the discs of the families converge to the corresponding elliptic point on the ends. Exactly the same argument shows that an analytic disc with boundary on F_t lies completely in $\hat{F}_t$ and therefore coincides with one of the discs of the foliation.

For filling a critical t we use a similar argument. Proposition 2.1 provides an analogue of the Bishop families at elliptic points for our elliptic cones. So, at a critical point of index 1 or 2 of elliptic type we obtain two families of analytic discs whose boundaries sweep out a punctured neighbourhood of the cone. These families serve as the substitute of the Bishop families with the remaining arguments applying exactly in the same way as above.

The so-obtained fillings $\hat{F}_t$ are disjoint for different t: Again we use positiveness of intersections of complex curves as above.

Now we extend f to a well-defined continuous function $\hat{f}$ on Ω symply setting

$$\hat{f}|_{\hat{F}_t} \equiv t.$$

In the second step we prove smoothness and regularity of $\hat{f}$. Let p be an interior point of Ω. The unique analytic disc of the above construction through p can be included in a smooth 2-parameter family of analytic discs parametrizing a neighbourhood of p. Thus all we have to show is that the vector field obtained by variation of the disc transverse to f does not vanish at p. Let $\hat{f}(p) = t_0$. Denote by $\{u^t : \mathbf{\Delta} \longrightarrow \Omega\}_t$ a variation of the disc with $\hat{f} \circ u^t \equiv t$. The normal bundle $N_\Omega(u^{t_0}(\mathbf{\Delta}))$ of the discs can be trivialized in such a way that the following diagram comutes:

$$\begin{array}{ccc} N_\Omega(u^{t_0}(\mathbf{\Delta})) & \longrightarrow & \mathbf{\Delta} \times \mathbb{C} \\ \uparrow & & \uparrow \\ N_{F_{t_0}}(u^{t_0}(\partial\mathbf{\Delta}) & \longrightarrow & \mathbf{S^1} \times \mathbb{R}. \end{array}$$

This is due to the fact that the deformation index of discs with boundary on F_{t_0} at $u^{t_0}(\mathbf{\Delta})$ is 1. We obtain a holomorphic map

$$\phi = \frac{\partial u^t}{\partial t}|_{t_0} : \mathbf{\Delta} \longrightarrow \mathbb{C}$$

for which one computes easily that the winding number $\mathrm{wind}(\phi(\partial\mathbf{\Delta}), 0) = 0$ an thus ϕ is nowhere zero and the nonvanishing of the variation follows.

Next let $p \in \partial\Omega$ be a noncritical real point of some level. All discs we constructed above have their boundaries in the totally real part of a level surface. So by the reflection principle they are smooth up to the boundary. An implicite function argument applied to the problem of finding discs with boundary in a level near the discs through p provides us again with a smooth two-parameter family of analytic discs, smooth up to the boundary, parametrizing a neighbourhood of p in $\overline{\Omega}$ (this is carried out in [9], see also [12]). This immediately shows smoothness of $\hat{f}$ at p. Regularity follows from regularity of f at p.

Pick a complex point p of some level surface F_t. Kenig and Webster show that $\hat{F}_t$ is smooth at p. The smoothness of $\hat{f}$ at p is a family version of that statement proved in [10] (see also [12]). Again the regularity is clear from that of f.

Finally, the property at critical points of index 1 or 2 is exactly the statement of Proposition 2.4. Smoothness at extremal points of f should follow from similar techniques used to derive that proposition and will be carried out elsewhere. □

Remark 1.3 *The construction of the filling of the critical levels can be derived alternatively using the fillings of the regular levels and a limiting argument exploiting pseudoconvexity and Gromov's compactness. So the new statement derived from the construction presented below is the smoothness and regularity of $\hat{f}$ at the critical points. However, the analogue of Bishop's result also provides a shortcut of the argument constructing the fillings of critical levels.*

2 Fillings of Cones

Let $M \subset \mathbb{C}^2$ be an oriented real hypersurface, $\xi \longrightarrow M$ its complex tangentials. Suppose $f : M \longrightarrow \mathbb{R}$ is a Morse function with exactly one critical point 0 with $f(0) = 0$ of index 1 or 2. Then $F_0 = f^{-1}(0)$ is a smooth (non-degenerate) cone. Assume moreover, that the restriction $\mathbf{Hess}f(0)|_{\xi_0}$ of the Hessian to the complex tangential at M in 0 is still a non-degenerate bilinear form. We call the critical point 0 *elliptic* if $\mathbf{Hess}f(0)|_{\xi_0}$ is definite, *hyperbolic* if it is indefinite. F_0 is called an *elliptic* or *hyperbolic cone*, respectively. This notion is justified by the following simple observation: Each of the two cups of the corresponding level near the elliptic cone admits a complex point. They are elliptic of different sign (see [1] for a definition). The pair cancels in 0 if one crosses the critical level. After the crossing each of the levels consists of a totally real handle. In the situation of a hyperbolic cone the cups are totally real while on each handle there is a pair of hyperbolic points of different sign which cancels in 0 while crossing the critical level.

Proposition 2.1 *Assume the critical point is elliptic. There exist two smooth one parameter families* $\{u_\tau : \mathbf{\Delta} \longrightarrow \mathbb{C}^2\}_{\tau \neq 0}$ *of disjoint, regularly embedded analytic discs with boundaries on* $F_0 \setminus \{0\}$ *and uniformly converging to the zero map as* τ *approaches* 0. *There are no other analytic discs with boundaries on* F_0 *near* 0. *The boundaries of the discs of the families foliate a punctured neighbourhood of* 0 *in* F_0.

Remark 2.2 *This result is an analogue of Bishop's construction in* [4] *foliating surfaces by boundaries of holomorphic discs near elliptic points. The smoothness of that filling at the conus point* 0 *is formulated in the next theorem which also contains the statement of the former.*

Proposition 2.3 *Let* $\Gamma_f := \{(z,w,t) \in M \times \mathbb{R} | f(z,w) = t\} \subset \mathbb{C}^2 \times \mathbb{R}$ *denote the graph of the Morse function. Assume again that the critical point* 0 *is elliptic. Then there exists a smooth hypersurface* $\hat{\Gamma}_f \subset \mathbb{C}^2 \times \mathbb{R}$ *whose boundary is a neighbourhood of* 0 *in* Γ_f, *such that* $pr_{\mathbb{R}}|_{\hat{\Gamma}_f}$ *is an everywhere regular smooth map and* $\hat{\Gamma}_f \cap \mathbb{C}^2 \times \{t\}$ *is Levi–flat for all paramters* t. *The corresponding leaves are analytic discs outside the elliptic points of the level surfaces and the critical point* 0. *There are no other holomorphic discs in* $\mathbb{C}^2$ *with boundary on a level surface of* f *near* 0. *In particular, the boundaries of the discs are disjoint and foliate a deleted neighbourhood of* 0 *in* M *outside the elliptic points of the level surfaces.*

Proposition 2.4 *Suppose that* ξ *is non-integrable in* 0. *Then near* 0 $pr_{\mathbb{C}^2}|_{\hat{\Gamma}_f}$ *is a diffeomorphism onto its image. Moreover, the function on the pseudoconvex side* $\hat{M}$ *of* M

$$\hat{f} = pr_{\mathbb{R}} \circ (pr_{\mathbb{C}^2}|_{\hat{\Gamma}_f})^{-1} : \hat{M} \longrightarrow \mathbb{R}$$

is a smooth extension of f *near* 0 *without critical points. Of course, all levels of* $\hat{f}$ *are Levi–flat. In that sense* $\hat{M}$ *is smoothly foliated at* 0.

Remark 2.5 *(i) The Bishop families, fibering neighbourhoods of the two elliptic points in the corresponding level surface (emerging or vanishing in the elliptic cone), converge to the two families in the cone, i.e. in particular they do not degenerate to a point when the critical level is approached.*
(ii) There are fillings by holomorphic discs of the handle levels, converging to the families filling the cone, with the following property: There is a radius R *such that for each radius* $r < R$ *and for all levels sufficiently close to the critical there is a member of the family having radius smaller than* r. *Furthermore, for all these levels there are two members of the family of radius bigger than* R *lying on two different sides of the small disc above in the family.*

Such results are of interest for constructions of fillings by holomorphic discs of families of surfaces on boundaries of strongly pseudoconvex symplectic 4–manifolds (see e.g. [5, 7, 6]).

Outline of the Proof. First consider the flat case, i.e. $M = \mathbb{C} \times \mathbb{R}$. The curve of the elliptic points of the levels is described by $\{(z(u), u)\}_u$ for a function $z = O(|u|^2)$. For small parameters u, t $\gamma_{u,t} := \{z | f(z, u) = t\}$ is a simple closed curve around $z(u)$, i.e. the compact component $E_{u,t}$ of $\mathbb{C} \setminus \gamma_{u,t}$ is an analytic disc and we obtain a family of fillings

$$\hat{F}_t^0 := \cup_u E_{u,t} \subset \mathbb{C} \times \mathbb{R}.$$

In Chapter 3 we choose a family of coordinates for each level of the Morse function such that the manifold one gets piecing together all transformed levels is of arbitrary high tangency to $\mathbb{C} \times \mathbb{R}$ at zero and each smooth level surface is of high tangency to $\mathbb{C} \times \mathbb{R}$ at its elliptic points. We consider f as a smooth function on $\mathbb{C}^2$ and model the filling over the flat filling above. More precisely, for each k we find coordinates and a $\mathbf{C}^k$–transformation

$$\mathcal{T}_k : \cup_t \hat{F}_t^0 \times \{t\} \subset \mathbb{C} \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{C}^2 \times \mathbb{R}$$

holomorphic in the first component and level preserving, with $\mathcal{T}_k(F_t^0 \times \{t\}) \subset F_t \times \{t\}$. In Chapter 5 we solve the corresponding functional introduced in Chapter 4 as usual normalizing $\mathcal{T}_k$ in z–direction and using the Hilbert transform $\mathcal{H}_{u,t}$ on $\gamma_{u,t}$. Finally, in Chapter 6 the transformation is constructed and uniqueness is shown by standard arguments. The problem remains, that the size of $M_k \subset \Gamma_f$, $0 \in M_k$, $\mathbf{C}^\mathbf{k}$– foliated by the families of analytic discs may shrink to zero for $k \to \infty$. This can be overcome by the reflection principle: An analytic disc $\mathbf{C}^k$ up to the boundary lying on $F_t \setminus \{\text{elliptic and critical points}\}$, $k \geq 1$, is smooth up to the boundary (see e.g. [8]). An implicite function argument for the space of such discs like in [9] shows smoothness of $\hat{\Gamma}_f$ outside the elliptic and the critical points. On the other hand $\hat{\Gamma}_f$ is smooth at the elliptic points by [10]. So, Theorems 1.1 and 1.3 follow.

Consider the slightly different model $M = \{(z, u + \mathrm{i}f(z, u) | z \in \mathbb{C}, u \in \mathbb{R}\} \subset \mathbb{C}^2$ with the same function f. Choosing the right orientation M is strongly plurisubharmonic and Theorem 1.4 is true for the corresponding filling. Thus the statement boils down to the invertability of $pr_{\mathbb{C}^2} \mathcal{T}_k$ at 0. $\square$

The proof uses the scheme of Kenig and Webster [8], which is a refinement of Bishop's arguments [4]. The main difficulty lies in handling a degenerating family of real surfaces and obtaining uniform estimates for the resulting families of holomorphic discs. A more detailed discussion can be found in [12].

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3 A Flattening of the Levels of the Morse Function

Let f be given as a smooth function in a neighbourhood of M. A complex linear transformation at 0 and the implicite function theorem gives M as the zero set of

$$v + k(z, u) = 0,$$

where $k(z,w) = O(|z,u|^2)$. Then we change f (simply setting $v = k(z,u)$), and with a complex linear transformation of the z coordinate we are left with

$$f = \epsilon_1 u^2 - q(z+\lambda u) + h(z,u), \quad \lambda \in \mathbb{C}, \epsilon_1 = \pm 1$$

q is a quadratic bilinear form

$$q(z) = \gamma(z^2 + \bar{z}^2) + \epsilon |z|^2 \qquad \gamma \in \mathbb{R}, \epsilon \in \{-1; 0; 1\}.$$

This simply uses the nondegeneracy of the retriction of the Hessian of f at 0 to ξ_0. q is determined by $\mathbf{Hess} f(0)|_{\xi_0}$ and λ given by quadratic completion. We change coordinate via $z \mapsto z - \lambda w$, again eliminate the v's in the quadratic expressions to obtain

$$f = \epsilon_1 u^2 - q(z) + h(z,u)$$

with $h(z,u) = O(|z,u|^3)$. Thus the Hessian of f at 0 is given by $\begin{pmatrix} \epsilon_1 & 0 \\ 0 & -q \end{pmatrix}$.

Without loss of generality assume that $\epsilon_1 = 1$ (just interchanging $f \mapsto -f$ and acoordingly the index). By the index assumption $\epsilon = 1$ or q is indefinite. Moreover, without loss of generality $\gamma \geq 0$ (changing $z \mapsto \mathrm{i}z$). The nondegeneracy assumption gives $\gamma \neq \frac{1}{2}$ and the cone is *elliptic* if $0 \leq \gamma < \frac{1}{2}$ and $\epsilon = 1$ or else *hyperbolic*. If we interpret that definition geometrically we see that in the elliptic case the complex tangency on M intersects the tangent cone in the cone point only while in the hyperbolic case it intersects it in a 1-dimensional cone. The complex invariant of the cone in either case is thus $\epsilon_1 \frac{\epsilon + 2\gamma}{\epsilon - 2\gamma}$ and the cone is elliptic if this is positive and hyperbolic in the negative case.

The elliptic points of the two cups in the flat model $F_t^{0\pm}$ at level t ($t > 0$) for $|t|, |u|$ small are given by the two small solutions $(z^\pm(t) = z(u^\pm), u^\pm(t))$ of $f(z(u), u) = t$. Secondly, there are, of course, no points on the level surface F_t^0 having a u–coordinate with $u_0^-(t) < u < u_0^+(t)$. For $u \notin [u_0^-(t), u_0^+(t)]$ and for $|t|$ and $|u|$ small $\gamma_{u,t}$ is a simple closed curve about $z(u)$ allowing a parametrization by the polar angle with respect to $z(u)$

$$\gamma_{u,t} : \theta \mapsto z(u) + \zeta_{u,t}(\theta)$$

Moreover, it can well be approximated by an ellipse as is shown in the next

Lemma 3.1 *Assume (M, f) is of class $\mathbf{C}^{k,\alpha}$ for $k \in \mathbb{N}, 0 < \alpha < 1$. Let*

$$w_{u,t}(\theta) = \zeta_{u,t}(\theta) - \rho_{u,t}\zeta_u^0(\theta),$$

with $\rho_{u,t} = \sqrt{f(z(u),u) - t}$ being the radius parameter and the ellipse $\zeta_u^0(\theta)$ given by $Q(\zeta, u) := f_{z\bar{z}}(z(u),u)|\zeta|^2 + \Re\{f_{zz}(z(u),u)\zeta^2\} = 1$ and parametrized by the polar angle:

$$\zeta_u^0(\theta) := \Big(\frac{cos(\theta(u)+\theta)}{\sqrt{1+2\alpha(u)}} + i\frac{sin(\theta(u)+\theta)}{\sqrt{1-2\alpha(u)}}\Big)$$

for $f(z(u),u)-t>0$ ($\alpha(u)$ describing the form and $\theta(u)$ the rotation of the ellipse). Then

$$\begin{array}{lrcl} \text{(i)} & \left\|\frac{w}{\rho_{u,t}}\right\|_{\mathbf{C}^{k,\alpha}(\theta)} & = & O(\rho_{u,t}) \\ \text{(ii)} & \left\|D_u^r D_t^s(\frac{w}{\rho_{u,t}})\right\|_{\mathbf{C}^{j,\alpha\theta)}} & = & O(1). \end{array}$$

for $2r+2s+j+\beta \leq k+\alpha-2$, introducing

$$\begin{aligned} D_u &= \rho_{u,t}\partial_u \\ D_t &= \rho_{u,t}\partial_t. \end{aligned}$$

Proof. One uses a fixed point argument and the above form of f to show (i) and then differentiates the corresponding equation to derive (ii). □

Proposition 3.2 *Fix any $l \geq 2$. We may without loss of generality suppose, that*

$$k(z,u) = O(|z,u|^l),$$

i.e., in particular, there is a Morse function $f' : M' \longrightarrow \mathbb{R}$ on a smooth hypersurface $M' \subset \mathbb{C}^2$, and a family of biholomorphisms Φ_t of a neighbourhood of $0 \in \mathbb{C}^2$, such that $\Phi(z,w) := \Phi_{f(z,u)}(z,w)$ maps M to M', $f = \Phi^* f'$ (after possibly shrinking M to a smaller neighbourhood of 0) and $k'(z,u) = O(|z,u|^l)$.

Proof. The new coordinates (z', w') are expressed via

$$\begin{aligned} z' &= z \\ w' &= w + B(t,z,w) \\ B(t,z,w) &= \sum_{l=2}^{\infty} B_i(t,z,w) \\ B_l(z,w) &= \sum_{2i+j+k=l} b_{ijk} t^i z^j w^k. \end{aligned}$$

It is convenient to assume $\Re(B_l(t,0,u)) = 0$ for all $l \in \mathbb{N}$ and $t,u \in \mathbb{R}$, i.e. $b_{i0(l-2i)} \in \mathrm{i}\mathbb{R}$. The space $\mathcal{B}_{li}$ of such homogeneous polynomials divisible exactly by t^i has real dimension $2(l-2i)+1$. We proceed by induction. Suppose $k(z,u) = O(|z,u|^l)$ for an $l \geq 2$. On account of $v + k(z,u) = 0$ and $b_{i0(l-2i)} \in \mathrm{i}\mathbb{R}$ the equation to solve is equivalent to

$$(\Im(B_l(t,z,u)) - k(z,u)))|_{F_t} = O(|z,u|^{l+1}).$$

k is a given formal power series starting with homogeneous polynomials of degree l

$$\begin{aligned} k(z,u) &= \sum_{i=l}^{\infty} k_i(z,u) \\ k_i(z,u) &= \sum_{p+q=0}^{i} k_{ipq} z^p \bar{z}^q u^{i-p-q} \end{aligned}$$

with $k_{ipq} = \overline{k_{iqp}}$. But, on the other hand $u^2 - q(z) = t + O(|z,u|^3)$ on F_t so without loss of generality we may suppose $k_{lipq} = 0$ for all i,p,q with $l - 2i - p - q \geq 2$, introducing terms containing the level parameter t:

$$k_l(t,z,u) = \sum_{2i+p+q=0}^{l} k_{lipq} t^i z^p \bar{z}^q u^{l-2i-p-q}$$

with $k_{lipq} = \overline{k_{liqp}}$. The real dimension of the space $\mathcal{K}_{li}$ of such homogeneous polynomials k_l containing exactly t^i is $2(l-2i)+1$. So, it suffices to show that the homomorphism of vector spaces

$$B \in \mathcal{B}_{li} \mapsto \Im(B(t,z,u)) \mathrm{mod}(u^2 - q(z) - t) \in \mathcal{K}_{li}$$

has trivial kernel. Now

$$\Im(B(t,z,u)) \mathrm{mod}(u^2 - q(z) - t) = 0 \in \mathcal{K}_{li}$$

iff $\Im(B(t,z,u))|_{\{z|u^2-q(z)=t\}} = 0$. But $B(t,.,u)$ is a holomorphic function on $\hat{E}^0_{u,t} = \{z|u^2 - q(z) - t \geq 0\}$ for all elements $\{u|u^2 - t \geq 0\}$ and $\Re(B(t,0,u)) = 0$. It follows that $B(t,z,u) = 0$ for all t, $u \in \{u|u^2 - t \geq 0\}$ and $z \in \hat{E}^0_{u,t}$ and thus for all $z \in \mathbb{C}$ and so $B \equiv= 0$. We form a new real hypersurface M' which will serve as the flattened model given by $M' := \cup_t \Phi_t(F_t)$. Although M, f (and F_t likewise) have changed we will still denote them by M, f and F_t, respectively. □

Now we bring the situation into its final form. The problem is, that for $t > 0$ the coordinates near the elliptic points of the cups are not in normal form: in order to model their fillings over the flat fillings the elliptic points have to lie over those of the flat cups $(z^\pm, u^\pm)$, i.e., to be of the form $(z^\pm, u^\pm + \mathrm{i}k(z^\pm, u^\pm))$ with $k(z,u)|F_t = O(|z - z^\pm|^n)$ near $(z^\pm, u^\pm)$ for a fixed $n \geq 3$. This is achieved by the next

Lemma 3.3 *Assume $k = O(|z,u|^l)$. For any $n \leq l/2$ there exists a Morse function $f' : M' \longrightarrow \mathbb{R}$ on a hypersurface of class $\mathbf{C}^{l/2-n+1}$ together with a diffeomorphism $\Phi : M \longrightarrow M'$, such that $f = \Phi^* f'$, $(M', f'), \Phi$ are smooth outside the zero level, $\Phi|_{F_t}$ is the restriction of a biholomorphism of a neighbourhood of F_t, $\Phi|_{F_t} = Id$ for all $t \leq 0$ and $k'(z,u) = O(|z - z(u)|^n)$.*

Proof. Assume we achieved a transformation as desired for an n with $n < l/2$ such that the hypersurface is of class $\mathbf{C}^{l/2-n+1}$. Simply one has to eliminate the lowest order terms of the formal Taylor expansion in z–direction of k at $(z(u), u)$:

$$k(z,u)=\sum_{i+j=n}^{\infty}k_{ij}(u)\zeta^i\overline{\zeta}^j$$

$k_{ij}=\overline{k_{ji}}$ and $\zeta=z-z(u)$. This is achieved by a family of transformations of the form

$$\begin{aligned}\Phi_t^{\pm}(\zeta,\omega) &= (\zeta,\omega+B^{\pm}(t,\zeta,\omega)),\\ B^{\pm}(t,\zeta,\omega) &= \sum_{i+2j=n}b_{ij}^{\pm}(t)\zeta^i\omega^j,\end{aligned}$$

$t=f(z(u),u)$, $\Re\mathrm{e}B^{\pm}(0,u)=0$ for all $u\in\mathbb{R}$, i.e., $\Re\mathrm{e}b_{0j}^{\pm}=0$, for which we have to solve

$$\begin{aligned}\Im\mathrm{m}B^{\pm}(t,\zeta,\omega)|_{F_t^{\pm}}\Im\mathrm{m}\bmod O(|\zeta|^{n+1}) &= B^{\pm}(t,\zeta^{\pm},\omega^{\pm})|_{F_t^{\pm}}\\ &= \sum_{i+j=n}k_{ij}(u)(\zeta)^i\overline{\zeta}^j|_{F_t^{\pm}}\bmod O(|\zeta|^{n+1})\\ &= \sum_{i+j=n}k_{ij}(u^{\pm})(\zeta^{\pm})^i\overline{\zeta^{\pm}}^j|_{F_t^{\pm}}\bmod O(|\zeta^{\pm}|^{n+1}).(1)\end{aligned}$$

The second equality is derived from $(u-u^{\pm})|_{F_t^{\pm}}=O(|\zeta^{\pm}|^2)$ and accordingly $(\zeta-\zeta^{\pm})|_{F_t^{\pm}}=(z(u)-z^{\pm})|_{F_t^{\pm}}=O(|\zeta^{\pm}|^2)=O(|\zeta|^2)$. Here we introduced $(\zeta^{\pm},\omega^{\pm})=(z,w)-(z^{\pm},u^{\pm})$. On account of $(u-u^{\pm})|_{F_t^{\pm}}=\frac{Q(\zeta^{\pm},u^{\pm})}{P'(u^{\pm})}+O(|\zeta^{\pm}|^3)$, $\Re\mathrm{e}b_{0j}=0$ and induction hypotheses (1) is equivalent to solving

$$\Im\mathrm{m}B(t,\zeta^{\pm},\frac{Q(\zeta^{\pm},u^{\pm})}{P'(u^{\pm})})=\sum_{i+j=n}k_{ij}(u^{\pm})(\zeta^{\pm})^i\overline{\zeta^{\pm}}^j$$

which is possible because the map $B\mapsto\Im\mathrm{m}B(t,\zeta^{\pm},\frac{Q(\zeta^{\pm},u^{\pm})}{P'(u^{\pm})})$ is an isomorphism. Again that is deduced using the ellipse form of $\{\zeta\big|Q(\zeta,u^{\pm})=P'(u^{\pm})(u-u^{\pm})\}$ and an analogous argument as for the proof of Proposition 3.2: the corresponding real vector spaces both have dimension $2n+1$ (remember that $t>0$ is fixed). Assume the homomorphism $B(t,.,.)\mapsto\Im\mathrm{m}B(t,\zeta^{\pm},\frac{Q(\zeta^{\pm},u^{\pm})}{P'(u^{\pm})})$ has a non-trivial kernel. The ellipse form and the normalizing condition $\Re\mathrm{e}B(t,0,\omega^{\pm})=0$ guarantee that then $\Im\mathrm{m}B(t,\zeta^{\pm},u-u^{\pm})=0$ for all $u>u^+$ or $u<u^-$, respectively. Hence $B(t,\zeta^{\pm},u-u^{\pm})=0$. We conclude $B\equiv 0$.

From $u^{\pm}=O(\sqrt{t})$ and $k_{ij}(u)=O(|u|^{l-n})$ one deduces by coefficient comparison, that $B(t,\zeta,\omega)=O(\sqrt{t}^{l-2n})$ in the space of homogeneous polynomials of degree n. Thus we conclude that the transformations are defined on a neighbourhood of $F_t^{\pm}\cap\mathcal{U}$ for a common neighbourhood $\mathcal{U}$ of 0 and of the desired regularity. □

4 The Family of Functionals of the Problem

As in the case of a single surface we are going to model our filling over the family of (trivial) fillings $\{\{E_{u,t}\}_u\}_t$ of the flat model. We are seeking a smooth family of continuous maps

$(z', w') = T_t(z, u)$ which are holomorphic in the complex component in the interior of $\hat{F}_t^0$ and map F_t^0 to F_t. In order to rule out ambiguity of the solution of the problem, coming from Möbius transformations of the disc and reparameterization, we normalize T_t in such a way that

$$\begin{aligned} z'(z(u), u) &= z(u), \\ \frac{\partial z'}{\partial z}(z(u), u) &> 0, \\ \Re\mathrm{e}(w'(z(u), u)) &= 0, \end{aligned}$$

i.e., the T_t are of the form

$$\begin{aligned} T_t &: \hat{F}_t^0 \longrightarrow \mathbb{C}^2, \\ T_t(z, u) &= (z(u) + \zeta + \zeta A_t(\zeta, u), w + B_t(\zeta, u)), \\ T_t(\gamma_{u,t}, u) &\subset F_t, \end{aligned}$$

where $\zeta = z - z(u)$, A_t, B_t holomorphic in z and $\Re\mathrm{e}(A_t(0, u)) = 0, \Im\mathrm{m}(B_t(0, u)) = 0$. The last condition is equivalent to the following two equations

$$\Big(\Im\mathrm{m}(B_t(\zeta, u)) + k\Big(z(u) + \zeta(1 + A_t(\zeta, u)), u + \Re\mathrm{e}(B_t(\zeta, u))\Big)\Big)\Big|_{\gamma_{u,t}} = 0, \tag{2}$$

$$f(z(u) + \zeta(1 + A_t(\zeta, u)), u + \Re\mathrm{e}(B_t(\zeta, u)))|_{\gamma_{u,t}} - t = 0 \tag{3}$$

for all $z \in \gamma_{u,t}$ which have to be satisfied for $|u|, |t|$ sufficiently small. Now we apply the Hilbert transform $\mathcal{H}_{u,t}$ on $\gamma_{u,t} = \partial E_{u,t}$. Under the above assumptions on A_t and B_t we have

$$\begin{aligned} \Re\mathrm{e}(B_t(\zeta, u))|_{\gamma_{u,t}} &= -\mathcal{H}_{u,t}[\Im\mathrm{m}(B_t(\zeta, u))|_{\gamma_{u,t}}] \qquad \text{and} \\ A_t(\zeta, u)|_{\gamma_{u,t}} &= \phi + \mathrm{i}\mathcal{H}_{u,t}[\phi] \qquad \text{with} \\ \phi &:= \Re\mathrm{e}(A_t(\zeta, u))|_{\gamma_{u,t}}. \end{aligned}$$

We set $\mathcal{A}_{u,t}[\phi] := \phi + \mathrm{i}\mathcal{H}_{u,t}[\phi]$. Take the parametrization $\zeta_{u,t}(\theta)$ of $\zeta|_{\gamma_{u,t}}$ and consider $\phi = \phi(\theta)$ as a function of θ. Then, applying the Hilbert transform to (2), we obtain

$$\Re\mathrm{e}(B_t(\zeta_{u,t}, u)) - \mathcal{H}_{u,t}[k(z(u) + \zeta_{u,t}(1 + \mathcal{A}_{u,t}[\phi]), u + \Re\mathrm{e}(B_t(\zeta_{u,t}, u)))] = 0.$$

It will turn out later that for $|t|, |u|, \|\phi\|_{j,\alpha}$ small this equation in $\Re\mathrm{e}(B_t(\zeta_{u,t}, u))$ has a unique small solution $\mathcal{K}_{u,t}[\phi]$. Inserting this into (3) gives

$$\mathcal{F}_t(\phi, u) := f(z(u) + \zeta_{u,t}(1 + \mathcal{A}_{u,t}[\phi]), u + \mathcal{K}_{u,t}[\phi]) - t = 0. \tag{4}$$

We are looking for solutions $\phi_{u,t} \in \mathbf{C}^{j,\alpha}(\theta)$ with $j \in \mathbb{N}$, $0 < \alpha < 1$.

We consider the following linear operator on the Hölder space $\mathbf{C}^{j,\alpha}$ which will be a sufficient approximation of the functional:

$$\mathcal{L}_{u,t}[\Phi][\phi] := 2\Re\{f_z(z(u) + \zeta_{u,t}, u + \mathcal{K}_{u,t}[\Phi])\zeta_{u,t}\mathcal{A}_{u,t}[\phi]\},$$

The main part of this for u small will turn out to be

$$\mathcal{L}^0_{u,t}[\phi] := \rho^2_{u,t} 2\Re\{Q_z(\zeta^0_{u,t}, u)\zeta^0_{u,t}\mathcal{A}^0_{u,t}[\phi]\}.$$

$\mathcal{A}^0_{u,t}[\phi] = \phi + \mathrm{i}\mathcal{H}^0_u[\phi]$ and $\mathcal{H}^0_u$ is the Hilbert transform on γ^0_u, in particular, depends only on $\alpha(u)$ and $\theta(u)$ but not on the radius $\rho_{u,t}$, i.e.

$$\mathcal{L}^0_{u,t} = \rho^2_{u,t}\mathcal{L}^0_u.$$

Lemma 4.1 *$\mathcal{L}^0_u$ is a bounded invertible operator on all spaces $\mathbf{C}^{j,\alpha}$ $(j \geq 1,\ 0 < \alpha < 1)$ uniform in u.*

Proof. The proof is via conformal mappings reduced to the round circle ($\alpha(u) = 0$), which is a well known result of Privalov. (see Lemma 1.2. in [8]). Uniformity in u is derived from $\alpha(0) = \gamma < \frac{1}{2}$. □

5 The Solution of the Functional Equations

First we recall two results from [8] on the Hilbert transforms on $\gamma_{u,t}$ with small parameters.

Proposition 5.1 ([8]) *Fix $j \geq 1, 0 < \alpha < 1$. For a sufficiently small $N_{j,\alpha,l,n}$*

$$\begin{array}{llcllcl} (i) & \text{for } j+\alpha & \leq & \frac{l}{2}-n-1 & \|\mathcal{H}_{u,t}\|_{j,\alpha} & = & O(1), \\ & & & & \|\mathcal{H}_{u,t} - \mathcal{H}^0_u\|_{j,\alpha} & = & O(\rho_{u,t}) \\ (ii) & \text{for } r+s+j+\alpha & \leq & \frac{l}{2}-n-3 & \|D^r_u D^s_t \mathcal{H}_{u,t}\|_{j,\alpha} & = & O(1), \end{array}$$

for all $|t|, |u| < N_{j,\alpha,l,n}$. $\mathcal{H}^0_u$ denotes the Hilbert transformation on γ^0_u.

Proof. Using Lemma 3.1 the proof goes very much along the lines of the proof of Theorem 2.5 in [8]. □

The next proposition establishes the existence of the operator $\mathcal{K}_{u,t}$ and shows its behaviour near the singulartity $\rho_{u,t} = 0$:

Proposition 5.2 *For a sufficiently small $N_{j,\alpha,l,n}$ and for all u,t and $\phi \in \mathbf{C}^{j,\alpha}(\theta)$ with $|t|,|u| < N_{j,\alpha,l,n}$ and $\|\phi\|_{j,\alpha} < 1$ there exist unique $\mathcal{K}_{u,t}[\phi]$ solving the equation*

$$\mathcal{K}_{u,t}[\phi] = \mathcal{H}_{u,t}[k(z(u) + \zeta_{u,t}(1 + \mathcal{A}_u[\phi]), u + \mathcal{K}_{u,t}[\phi])].$$

and satisfying the following estimates for $j+\alpha \le \frac{l}{2} - n - 1$ and $j + \alpha + 2r + 2s \le \frac{l}{2} - n - 3$, $i + j + \alpha + r + s \le \frac{l}{2} - n$, respectively

$$\begin{array}{llcl} (i) & \|\mathcal{K}_{u,t}[\phi_1] - \mathcal{K}_{u,t}[\phi_2]\|_{j,\alpha} & \le & C(\rho_{u,t}(1 + \|\phi_1\|_{j,\alpha} + \|\phi_2\|_{j,\alpha}))^{n-1}\|\phi_1 - \phi_2\|_{j,\alpha} \text{ and} \\ (ii) & \|D_u^r D_t^s \partial_\phi^i \mathcal{K}_{u,t}[\phi]\|_{j,\alpha} & \le & C((1 + \|\phi\|_{j,\alpha}))^{n-i-s-r} \rho_{u,t}^{n-s-r} \end{array}$$

Proof. Remember that $k = O(|z,u|^l)$ and $k(z,u) = O(|z - z(u)|^n)$ with $l > n > 2$. $\mathcal{A}_{u,t} = \mathbf{Id} + \mathcal{H}_{u,t}$ and with the estimates for the Hilbert transform

$$\|\mathcal{H}_{u,t}[k(z(u) + \zeta_{u,t}(1 + \mathcal{A}_{u,t}[\phi]), u + \psi)]\|_{j,\alpha} \le C(\rho_{u,t}(1 + \|\phi\|_{j,\alpha}) + |u|\|\psi\|_{j,\alpha} + \|\psi\|_{j,\alpha}^2)^n,$$

$$\begin{aligned} &\|\mathcal{H}_{u,t}[k(z(u) + \zeta_{u,t}(1 + \mathcal{A}_{u,t}[\phi]), u + \psi_1)] - \mathcal{H}_{u,t}[k(z(u) + \zeta_{u,t}(1 + \mathcal{A}_{u,t}[\phi]), u + \psi_2)]\|_{j,\alpha} \\ &\quad \le \; C(\rho_{u,t}(1 + \|\phi\|_{j,\alpha}) + |u|(\|\psi_1\|_{j,\alpha} + \|\psi_2\|_{j,\alpha}) + (\|\psi_1\|_{j,\alpha} + \|\psi_2\|_{j,\alpha})^2)^n \|\psi_1 - \psi_2\|_{j,\alpha} \end{aligned}$$

$$\begin{aligned} &\|\mathcal{H}_{u,t}[k(z(u) + \zeta_{u,t}(1 + \mathcal{A}_{u,t}[\phi_1]), u + \psi)] - \mathcal{H}_{u,t}[k(z(u) + \zeta_{u,t}(1 + \mathcal{A}_{u,t}[\phi_2]), u + \psi)]\|_{j,\alpha} \\ &\quad \le \; C(\rho_{u,t}(1 + \|\phi_1\|_{j,\alpha} + \|\phi_2\|_{j,\alpha}) + |u|\|\psi\|_{j,\alpha} + \|\psi\|_{j,\alpha}^2)^{n-1}\|\phi_1 - \phi_2\|_{j,\alpha}. \end{aligned}$$

The constants depend on α, j, l, n only. Thus for $N_{j,\alpha,l}$ small enough

$$\psi \mapsto \mathcal{H}_{u,t}[k(z(u) + \zeta_{u,t}(1 + \mathcal{A}_{u,t}[\phi]), u + \psi)]$$

maps the ball $B_{\rho_{u,t}} \subset \mathbf{C}^{j,\alpha}(\theta)$, into itself due to the first and is a contraction according to the second inequality. The contraction property (i) follows from the third estimate. For (ii) differentiate the fixed point equation defining $\mathcal{K}_{u,t}[\phi]$ with respect to u and t or ϕ, respectively, using

$$\begin{aligned} &\|D_u^r D_t^s \partial_\phi^i \mathcal{H}_{u,t}[k(z(u) + \zeta_{u,t}(1 + \mathcal{A}_{u,t}[\phi]), u + \psi)]\|_{j,\alpha} \\ &\quad \le \; C \max_{\substack{a \le r \\ b \le s}} \{\|D_u^a D_t^b \partial_\phi^i \mathcal{H}_{u,t}[k(z(u) + \zeta_{u,t}(1 + \mathcal{A}_{u,t}[\phi]), u + \psi)]\|_{j,\alpha}\} \\ &\quad \le \; C(\rho_{u,t}(1 + \|\phi\|_{j,\alpha}) + |u|\|\psi\|_{j,\alpha} + \|\psi\|_{j,\alpha}^2)^{n-i-r-s} \rho_{u,t}^i \end{aligned}$$

and proceed by induction. □

As in the previous sections we will establish an iteration scheme to apply Banach's fixed point theorem.

Lemma 5.3 *Suppose $\|\phi\|_{j,\alpha} \leq 1$. There exists an $N_{j,\alpha,l,n}$ such that for all parameters with $|t|, |u| < N_{j,\alpha,l,n}$ the inverse of $\mathcal{L}_{u,t}$ exists and satisfies*

$$\|\mathcal{L}_{u,t}[\phi]^{-1}\|_{j,\alpha} \leq \frac{C_{j,\alpha,l,n}}{\rho_{u,t}^2}.$$

Proof. For the difference we have

$$\|(\mathcal{L}_{u,t}[\phi] - \mathcal{L}_{u,t}^0)[\psi]\|_{j,\alpha} \leq C\rho_{u,t}^3\|\psi\|_{j,\alpha}$$

using Lemma 3.1, Theorem 5.1 and Proposition 5.2 □

We consider the remainder

$$\mathcal{R}_t(\phi, u) = \mathcal{F}_t(\phi, u) - \mathcal{L}_{u,t}[\phi][\phi].$$

Lemma 5.4 *Suppose $\|\phi\|_{j,\alpha} \leq 1$. For $N_{j,\alpha,l,n}$ sufficiently small, $j + \alpha \leq \frac{l}{2} - n - 3$ and $|u|, |t| < N_{j,\alpha,l,n}$*

$$\begin{array}{lrcl} \text{(i)} & \|\mathcal{R}_t(\phi, u)\|_{j,\alpha} & \leq & C(\rho_{u,t}^n + \rho_{u,t}^2\|\phi\|_{j,\alpha}^2), \\ \text{(ii)} & \|\mathcal{R}_t(\phi_1, u) - \mathcal{R}_t(\phi_2, u)\|_{j,\alpha} & \leq & C\|\phi_1 - \phi_2\|_{j,\alpha}\Big(\rho_{u,t}^2(\|\phi_1\|_{j,\alpha} + \|\phi_2\|_{j,\alpha}) \\ & & + & (\rho_{u,t}(1 + \|\phi_1\|_{j,\alpha} + \|\phi_2\|_{j,\alpha}))^{n-1}\Big) \end{array}$$

Proof. $\mathcal{R}$ is the sum of the absolute and the remaining term in the first order Taylor expansion in $\tau = 0$ at $\tau = 1$

$$f(z(u) + \zeta_{u,t}(1 + \mathcal{A}_{u,t}[\tau\phi]), u + \mathcal{K}_{u,t}[\phi]) - t$$

These terms are computed and estimated as in the proof of Lemma 3.2 of [8]. □

Now we set $\mathcal{T}_{u,t}[\phi] = -\mathcal{L}_{u,t}^{-1}[\mathcal{R}_t(\phi, u)]$ and have to solve

$$\phi_{u,t} = \mathcal{T}_{u,t}[\phi_{u,t}].$$

By the above for $\|\phi\|_{j,\alpha}, \|\phi_1\|_{j,\alpha}, \|\phi_2\|_{j,\alpha} \leq 1$ we have

$$\begin{array}{rcl} \|\mathcal{T}_{u,t}[\phi]\|_{j,\alpha} & \leq & C((|u| + \rho_{u,t}^2)\rho_{u,t}^{n-2} + \|\phi\|_{j,\alpha}^2) \\ \|\mathcal{T}_{u,t}[\phi_1] - \mathcal{T}_{u,t}[\phi_2]\|_{j,\alpha} & & \\ & \leq & C\left(\|\phi_1\|_{j,\alpha} + \|\phi_2\|_{j,\alpha} + (|u| + \rho_{u,t}^2)\rho_{u,t}^{n-3}\right)\|\phi_1 - \phi_2\|_{j,\alpha}. \end{array}$$

Now fix $l, n \in \mathbb{N}$ as above. Provided $|t|, |u| < N_{j,\alpha,l,n}$, for a sufficiently small $N_{j,\alpha,l,n}$, $\mathcal{T}_{u,t}$ is a contraction mapping on $B_{\rho_{u,t}} \subset \mathbf{C}^{j,\alpha}(\theta)$ for $|u|, |t| < N_{j,\alpha,l,n}$.

Differentiating the equation and using Lemma 3.1, Theorem 5.1 and Proposition 5.2 we obtain

Proposition 5.5 *For $|u|, |t| < N_{j,\alpha,l,n}$ there exists a unique $\phi_{u,t}$ with*

$$\|D_u^r D_t^s \phi_{u,t}\|_{j,\alpha} \leq C\rho_{u,t}^{n-2-r-s}.$$

provided $j + \alpha + 2r + 2s \leq \frac{l}{2} - n - 3$. $\square$

6 Construction of the Family of Analytic Discs

Now we are able to construct the family via the Cauchy integral formula and to give the first estimates. Let

$$\begin{aligned} A_{u,t} &= \phi_{u,t} + \mathrm{i}\mathcal{H}_{u,t}[\phi_{u,t}] \\ B_{u,t} &= \mathcal{K}_{u,t}[\phi_{u,t}] - \mathrm{i}k(z(u) + \zeta_{u,t}A_{u,t}, u + \mathcal{K}_{u,t}[\phi_{u,t}]). \end{aligned}$$

For $\zeta \in \mathbf{C}$ we define

$$\begin{aligned} A_{u,t}(\zeta) &= C_{u,t}[A_{u,t}](\zeta) := \frac{1}{2\pi\mathrm{i}}\int_{\gamma_{u,t}} \frac{A_{u,t}(\theta)}{z-\zeta}dz \\ B_{u,t}(\zeta) &= C_{u,t}[B_{u,t}](\zeta) \\ T_t(\zeta, u) &= (z(u) + \zeta(1 + A_{u,t}(\zeta)), u + B_t(\zeta, u)), \end{aligned}$$

It follows immediately that T_t is continous up to the boundary F_t and solves our problem. Moreover, we can prove the following

Lemma 6.1 (Lemma 4.1 in [8]) *For $|u|, |t| < N_{j,\alpha,l,n}$, $N_{j,\alpha,l,n}$ sufficiently small*

$$\begin{aligned} \partial_t^r \partial_u^s \partial_z^i A(z,u) &= O\left(\rho_{u,t}^{n-2-2s-2r-i}\right) \\ \partial_t^r \partial_u^s \partial_z^i B(z,u) &= O\left(\rho_{u,t}^{n-2s-2r-i}\right). \end{aligned}$$

for all $i + 2r + 2s \leq \frac{l}{2} - n - 3$. $\square$

Finally, we transform the family of holomorphic discs constructed in the flattened situation back to the original situation. For the levels $t \leq 0$ the transformations Φ_t are of the form $\mathrm{Id} + P_t$ where P_t is a family of polynomials vanishing at 0, so the inverses will do the job. Unfortunately, for $t > 0$ only the restrictions to F_t^+ or F_t^- are of the form $\mathrm{Id} + P_t^\pm$. Here $P_t^\pm$ are families of polynimoals with $P_0^\pm = P_0$. But $\hat{F}_t^\pm$ are disjoint near the origin. Thus it is correct to define

$$\mathcal{T}(z,u,t) := (\Phi_t)^{-1}(T_t(z,u)).$$

$\mathcal{T}$ is holomorphic in z.

We are going to prove the uniqueness property (iii) of Proposition 2.3. Our proof is in the spirit of [2].

Translating the levels of the Morse function in v–direction by the value of $\pm f$ we may suppose without loss of generality that M is strongly pseudoconvex near 0 (see 5 below). Pick a ball $\mathcal{U}$ around the origin such that $\mathcal{U} \cap M$ is strongly pseudoconvex. On account of Levi–flatness of $\hat{f}^{-1}(s)$ we see that

$$\mathcal{U} \cap \{(z, w)|v + k(z, u) \geq 0\} \cap \{(z, w)|\hat{f} \geq t\}$$

and

$$\mathcal{U} \cap \{(z, w)|v + k(z, u) \geq 0\} \cap \{(z, w)|\hat{f} \leq t\}$$

are pseudoconvex and

$$\overline{\phi(\boldsymbol{\Delta})} \subset\subset \mathcal{U} \cap \{(z, w)|v + k(z, u) \geq 0\} \cap \hat{f}^{-1}(t).$$

(One has to be careful at the level $t = 0$ on account of the singularity 0. Anyway, the statement about pseudoconvexity is true for the regular levels. Thus we may conclude the pseudoconvexity at $t = 0$ by the general fact, that the intersection of pseudoconvex domains is pseudoconvex.) Hence $\phi(\boldsymbol{\Delta})$ is a leaf of the Levi-foliation of $\hat{F}_t$ which consists exactly of the constructed discs.

Take the complex coordinate Φ_0 and note that on account of $k_0|F_0(z, u) = O(|z, u|^l)$ for $l \geq 3$ it follows that k_0 has to be of the form

$$k_0(z, u) = \lambda(u^2 - q(z) + h(z, u)) + O(|z, u|^3), \tag{5}$$

$\lambda \in \mathbb{R}$.

Suppose now, that the distribution of complex tangencies at M $\xi \longrightarrow M$ is completely non-integrable near 0, i.e. the commutator at 0 $[\xi, \xi]_0 = T_0M$. Then $\lambda \neq 0$ and without loss of generality (changing $f \mapsto -f$) we assume $\lambda = 1$. The graph $\Gamma_f^0 := \{(z, w)|v = f(z, u)\}$ is clearly tangent up to second order to M at 0 and $pr_{\mathbb{C}^2}\mathcal{T}$ can be considered as a map from Γ_f^0. Its Jacobian at zero is the identity which follows from the computations in Lemma 5.1:

$$T_t(z, u) = (z, u) + O(|u|^2 + \rho_{u,t}^{n-1})$$

for $|u|, |t|$ small.

$\mathcal{T}(z, w, t) := \Phi_t^{-1}(T_t(z, u))$ and the coordinate transforms were of the form

$$\Phi_t(\zeta, \omega) = (\zeta, \omega - \mathrm{i}t) + O(|\zeta, \omega, t|^2)$$

What follows is that

$$\Phi_t^{-1}(\zeta,\omega) = (\zeta, \omega + \mathrm{i}t) + O(|\zeta,\omega,t|^2)$$

which completes the proof of Proposition 2.4. □

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