Strictly pseudoconvex spin manifolds, Fefferman spaces and Lorentzian twistor spinors

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January 21, 1997

Abstract

We prove that there exist global solutions of the twistor equation on the Fefferman spaces of strictly pseudoconvex spin manifolds of arbitrary dimension and we study their properties.

Contents

1 Introduction

In the present paper we study a relation between CR-geometry and the Lorentzian twistor equation.
Besides the Dirac operator there is a second important conformally covariant differential operator acting on the spinor fields $\Gamma(S)$ of a semi-Riemannian spin manifold (M, g), the so-called *twistor operator* $\mathcal{D}$. The twistor operator is defined as the composition of the

spinor derivative ∇^S with the projection p onto the kernel of the Clifford multiplication μ

$$\mathcal{D} : \Gamma(S) \xrightarrow{\nabla^S} \Gamma(T^*M \otimes S) \overset{g}{\approx} \Gamma(TM \otimes S) \xrightarrow{p} \Gamma(\operatorname{Ker} \mu).$$

The elememts of the kernel of $\mathcal{D}$ are called *twistor spinors.* A spinor field φ is a twistor spinor if and only if it satisfies the *twistor equation*

$$\nabla^S_X \varphi + \frac{1}{n} X \cdot D\varphi = 0$$

for each vector field X, where D is the Dirac operator. Each twistor spinor φ defines a conformal vector field V_φ on M by

$$g(V_\varphi, X) = i^{k+1} \langle X \cdot \varphi, \varphi \rangle ,$$

where k is the index of g.

Twistor spinors were introduced by R.Penrose in General Relativity (see [Pen67], [PR86], [NW84]). In mathematics the twistor equation appeared as an integrability condition for the canonical almost complex structure of the twistor space of an oriented four-dimensional Riemannian manifold (see [AHS78]). In the second half of the 80th Lichnerowicz and Friedrich started the systematic investigation of twistor spinors on Riemannian spin manifolds from the view point of conformal differential geometry. Nowadays one has a lot of structure results and examples for manifolds with twistor spinors in the Riemannian setting (see [Lic88b], [Lic88a], [Lic89], [Fri89] [Lic90], [FP91], [BFGK91], [Hab90], [Hab92], [Hab93], [KR94], [KR95], [KR96b], [KR96a]). Crucial results were obtained by studying the properties of the conformal vector field V_φ of a twistor spinor φ. In particular, twistor operators turned out to be a usefull tool in proving sharp eigenvalue estimates for coupled Dirac operators on compact Riemannian manifolds (see eg [Bau94]).
In opposite to this, there is not much known about solutions of the twistor equation in the general Lorentzian setting. In 1991 Lewandowski studied local solutions of the twistor equation on 4-dimensional space-times, ([Lew91]). In particular, he proved that a 4-dimensional space-time admitting a twistor spinor φ without zeros and with twisting conformal vector field V_φ is locally conformal equivalent to a Fefferman space. On the other hand, on 4-dimensional Fefferman spaces there exist *local* solutions of the twistor equation.

The aim of the present paper is the generalisation of this result. Fefferman spaces were defined by Fefferman ([Fef76]) in case of strictly pseudoconvex hypersurfaces in $\mathbb{C}^n$, its definition was extended by Burns, Diederich, Shnider ([BDS77]), Farris ([Far86]) and Lee ([Lee86]) to general non-degenerate CR-manifolds. A Fefferman space is the total space of the S^1-principal bundle associated to the canonical bundle of a non-degenerate CR-manifold M equipped with a semi-Riemannian metric defined by means of the Webster connection which is given by an additional pseudo-hemitian structure on M.
By changing the topological type of the S^1-bundle defining the Fefferman space, we can

prove that there are *global* solutions of the twistor equation on the (modified) Fefferman spaces of strictly pseudoconvex spin manifolds of arbitrary dimension. These solutions have very special geometric properties which are only possible on Fefferman spaces. More exactly, we prove (see Theorem 1, Theorem 2):

Let $(M^{2n+1}, T_{10}, \theta)$ be a strictly pseudoconvex spin manifold and $(\sqrt{F}, h_\theta)$ its Fefferman space. Then, on the Lorentzian spin manifold $(\sqrt{F}, h_\theta)$ there exist a non-trivial twistor spinor ϕ such that

1. *The canonical vector field V_ϕ of ϕ is a regular isotropic Killing vector field.*
2. *$V_\phi \cdot \phi = 0$. In particular, ϕ is a pure or partially pure spinor field.*
3. *$\nabla_{V_\phi}\phi = i\, c\, \phi, \quad c = const \in \mathbb{R} \setminus \{0\}$.*
4. *ϕ has constant length.*

On the other hand, if (B, h) is a Lorentzian spin manifold with a non-trivial twistor spinor satisfying 1. - 3., then B is an S^1-principal bundle over a stricly pseudoconvex spin manifold (M, T_{10}, θ) and (B, h) is locally isometric to the Fefferman space $(\sqrt{F}, h_\theta)$ of (M, T_{10}, θ).

In particular, if $(M^{2n+1}, T_{10}, \theta)$ is a compact strictly pseudoconvex spin manifold of constant Webster scalar curvature, then the Fefferman space $(\sqrt{F}, h_\theta)$ of (M, T_{10}, θ) is a (2n+2)-dimensional non-Einsteinian Lorentzian spin manifold of constant scalar curvature R and the twistor spinor ϕ defines eigenspinors of the Dirac operator of $(\sqrt{F}, h_\theta)$ to the eigenvalues $\pm\frac{1}{2}\sqrt{\frac{2n+2}{2n+1}R}$ with constant length.

After some algebraic prelimeries in section 2 we introduce in section 3 the notion of Lorentzian twistor spinors and explain some of their basic properties. In order to define the (modified) Fefferman space we recall in section 4 the basic notions of pseudo-hermitian geometry. In particular, we explain the properties of the Webster connection of a non-degenerate pseudo-hermitian manifold, which are important for the spinor calculus on Fefferman spaces. In section 5 the Fefferman spaces are defined and in section 6 we derive a spinor calculus for Lorentzian metrics on S^1-principal bundles with isotropic fibre over strictly pseudoconvex spin manifolds. Finally, section 7 contains the proof of the Theorems 1 and 2 which state the properties of the solutions of the twistor equation on Fefferman spaces of strictly pseudoconvex spin manifolds.

2 Algebraic prelimeries

For concrete calculations we will use the following realization of the spinor representation. Let $\mathrm{Cliff}_{n,k}$ be the Clifford algebra of $(\mathbb{R}^n, -\langle\cdot,\cdot\rangle_k)$, where $\langle\cdot,\cdot\rangle_k$ is the scalar product

$$\langle x, y\rangle_k = -x_1y_1 - \ldots - x_ky_k + x_{k+1}y_{k+1} + \ldots + x_ny_n.$$

For the canonical basis $(e_1, \dots, e_n)$ of $\mathbb{R}^n$ one has the following relations in $\mathrm{Cliff}_{n,k}$:

$$e_i \cdot e_j + e_j \cdot e_i = -2\varepsilon_j \delta_{ij},$$

where

$$\varepsilon_j = \begin{cases} -1 & j \le k \\ \ \ 1 & j > k\,. \end{cases}$$

Denote

$$U = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad V = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad T = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \text{and}$$

$$\tau_j = \begin{cases} i & j \le k \\ 1 & j > k. \end{cases}$$

Then an isomorphism

$$\phi_{2m,k} : \mathrm{Cliff}^{\mathbb{C}}_{2m,k} \longrightarrow M(2^m; \mathbb{C})$$

is given by the Kronecker product

$$\begin{array}{lcll} \phi_{2m,k}(e_{2j-1}) & = & \tau_{2j-1} & E \otimes \ldots \otimes E \otimes U \otimes T \otimes \ldots \otimes T \\ \phi_{2m,k}(e_{2j}) & = & \tau_{2j} & E \otimes \ldots \otimes E \otimes V \otimes \underbrace{T \otimes \ldots \otimes T}_{j-1}\,. \end{array} \tag{1}$$

Let $\mathrm{Spin}_0(n,k) \subset \mathrm{Cliff}_{n,k}$ be the connected component of the identity of the spin group. The spinor representation is given by

$$x_{n,k} = \phi_{n,k}|_{\mathrm{Spin}_0(n,k)} : \mathrm{Spin}_0(n,k) \longrightarrow \mathrm{GL}(\mathbb{C}^{2^m}).$$

We denote this representation by $\Delta_{n,k}$. If $n = 2m$, $\Delta_{2m,k}$ splits into the sum $\Delta_{2m,k} = \Delta^+_{2m,k} \oplus \Delta^-_{2m,k}$, where $\Delta^{\pm}_{2m,k}$ are the eigenspaces of the endomorphism $\phi_{2m,k}(e_1 \cdot \ldots \cdot e_{2m})$ to the eigenvalue $\pm i^{m+k}$. Let us denote by $u(\varepsilon) \in \mathbb{C}^2$ the vector

$$u(\varepsilon) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -\varepsilon i \end{pmatrix} \qquad \varepsilon = \pm 1$$

and let

$$u(\varepsilon_1, \dots, \varepsilon_m) = u(\varepsilon_1) \otimes \ldots \otimes u(\varepsilon_m) \qquad \varepsilon_j = \pm 1. \tag{2}$$

Then $(u(\varepsilon_1, \dots, \varepsilon_m) \,|\, \prod_{j=1}^{m} \varepsilon_j = \pm 1)$ is an orthonormal basis of $\Delta^{\pm}_{2m,k}$ with respect to the standard scalar product of $\mathbb{C}^{2^m}$.

3 Lorentzian twistor spinors

Let $(M^{n,1}, g)$ be a connected space- and time oriented Lorentzian spin manifold with a fixed time orientation $\xi \in \Gamma(TM)$, $g(\xi, \xi) = -1$. We denote by S the spinor bundle of $(M^{n,1}, g)$, by $\nabla^S : \Gamma(S) \to \Gamma(TM^* \otimes S)$ the spinor derivative given by the Levi-Civita connection of $(M^{n,1}, g)$ and by $D : \Gamma(S) \to \Gamma(S)$ the Dirac operator on S.
On S there exists an indefinite scalar product $\langle \cdot, \cdot \rangle$ of index $\frac{1}{2} \dim S$ such that

$$\begin{aligned} \langle X \cdot \varphi, \psi \rangle &= \langle \varphi, X \cdot \psi \rangle && (3) \\ X \langle \varphi, \psi \rangle &= \langle \nabla^S_X \varphi, \psi \rangle + \langle \varphi, \nabla^S_X \psi \rangle && (4) \end{aligned}$$

for all vector fields X and all spinor fields $\varphi, \psi \in \Gamma(S)$. Furthermore, there is a positive definite scalar product $(\cdot, \cdot)_\xi$ on S depending on the time orientation ξ such that

$$\langle \varphi, \psi \rangle = (\xi \cdot \varphi, \psi)_\xi \tag{5}$$

for all $\varphi, \psi \in \Gamma(S)$ (see [Bau81], chap.1.5, 3.3.1.). Let $p : TM \otimes S \longrightarrow \mathrm{Ker}\,\mu$ denote the orthogonal projection onto the kernel of the Clifford multiplication μ (with respect to $\langle \cdot, \cdot \rangle$). p is given by

$$p(X \otimes \varphi) = X \otimes \varphi + \frac{1}{n} \sum_{k=1}^{n} \varepsilon_k s_k \otimes s_k \cdot X \cdot \varphi,$$

where $(s_1, \ldots, s_n)$ is a orthonormal basis of (M, g) and $\varepsilon_k = g(s_k, s_k) = \pm 1$.

Definition 1 *The twistor operator $\mathcal{D}$ of $(M^{n,1}, g)$ is the operator given by the composition of the spinor derivative with the projection p*

$$\mathcal{D} : \Gamma(S) \xrightarrow{\nabla^S} \Gamma(T^*M \otimes S) \overset{g}{\approx} \Gamma(TM \otimes S) \xrightarrow{p} \Gamma(Ker\,\mu).$$

Locally, we have

$$\mathcal{D}\varphi = \sum_{k=1}^{n} \varepsilon_k s_k \otimes (\nabla^S_{s_k} \varphi + \frac{1}{n} s_k \cdot D\varphi).$$

Definition 2 *A spinor field $\varphi \in \Gamma(S)$ is called a twistor spinor, if $\mathcal{D}\varphi = 0$.*

Let us first recall some properties of twistor spinors which are proved in the same way as in the Riemannian case.

Proposition 1 ([BFGK91], Th.1.2)
For a spinor field $\varphi \in \Gamma(S)$ the following conditions are equivalent:

1. *φ is a twistor spinor.*
2. *φ satisfies the so-called twistor equation*

$$\nabla^S_X \varphi + \frac{1}{n} X \cdot D\varphi = 0 \tag{6}$$

for all vector fields X.

3. *For all vector fields X and Y*
$$X \cdot \nabla^S_Y \varphi + Y \cdot \nabla^S_X \varphi = \frac{2}{n}\, g(X,Y)\, D\varphi \tag{7}$$
holds.

4. *There exists a spinor field $\psi \in \Gamma(S)$ such that*
$$\psi = g(X,X) X \cdot \nabla^S_X \varphi \tag{8}$$
for all vector fields X with $|g(X,X)| = 1$.

Proposition 2 ([BFGK91], Th.1.7)
The twistor operator is conformally covariant: Let $\tilde{g} = e^{2\sigma} g$ be a conformally equivalent metric to g and let $\tilde{D}$ be the twistor operator of $(M, \tilde{g})$. Then
$$\tilde{D}\tilde{\varphi} = e^{-\frac{1}{2}\sigma} D(\widetilde{e^{-\frac{1}{2}\sigma} \cdot \varphi}),$$
where $\sim : S \longrightarrow \tilde{S}$ denotes the canonical identification of the spinor bundles of (M,g) and $(M, \tilde{g})$.

Proposition 3 ([BFGK91] Cor.1.2)
The dimension of the space of twistor spinors is conformally invariant and bounded by
$$\dim Ker\mathcal{D} \leq 2^{[\frac{n}{2}]+1}.$$

Proposition 4 ([BFGK91] Cor.1.3)
Let $\varphi \in \Gamma(S)$ be a non-trivial twistor spinor and $x_0 \in M$. Then $\varphi(x_0) \neq 0$ or $D\varphi(x_0) \neq 0$.

Let R be the scalar curvature and Ric the Ricci curvature of $(M^{n,1}, g)$. If $\dim M = n \geq 3$, K denotes the (2,0) -Schouten tensor
$$K(X,Y) = \frac{1}{n-2}\left\{ \frac{R}{2(n-1)} g - \text{ Ric} \right\}.$$
We always identify TM with TM^* using the metric g. For a $(2,0)$-tensor field B we denote by the same symbol B the corresponding $(1,1)$-tensor field $B : TM \longrightarrow TM$, $g(B(X), Y) = B(X,Y)$. Let C be the (2,1)-Schouten-Weyl tensor
$$C(X,Y) = (\nabla_X K)(Y) - (\nabla_Y K)(X).$$
Furthermore, let W be the (4,0)-Weyl tensor of (M,g) and let denote by the same symbol the corresponding (2,2)-tensor field $W : \Lambda^2 M \longrightarrow \Lambda^2 M$. Then we have

Proposition 5 ([BFGK91] Th.1.3, Th.1.5)
Let $\varphi \in \Gamma(S)$ be a twistor spinor and $\eta = Y \wedge Z \in \Lambda^2 M$ a two form. Then

$$\begin{aligned}
D^2\varphi &= \frac{1}{4}\frac{n}{n-1}R\varphi\,, && (9)\\
\nabla^S_X D\varphi &= \frac{n}{2}K(X)\cdot\varphi\,, && (10)\\
W(\eta)\cdot\varphi &= 0\,, && (11)\\
W(\eta)\cdot D\varphi &= n\,C(Y,Z)\cdot\varphi\,, && (12)\\
(\nabla_X W)(\eta)\cdot\varphi &= X\cdot C(Y,Z)\cdot\varphi + \frac{2}{n}(X \lrcorner W(\eta))\cdot D\varphi\,. && (13)
\end{aligned}$$

If the scalar curvature R of $(M^{n,1},g)$ is constant and non-zero, equation (9) shows that the spinor fields

$$\psi_\pm := \frac{1}{2}\varphi \pm \sqrt{\frac{n-1}{nR}}\,D\varphi$$

are formal eigenspinors of the Dirac operator D to the eigenvalue $\pm\frac{1}{2}\sqrt{\frac{nR}{n-1}}$.
A special class of twistor spinors are the so-called *Killing spinors* $\varphi \in \Gamma(S)$ defined by the condition

$$\nabla^S_X\varphi = \lambda\,X\cdot\varphi \qquad \text{for all} \quad X \in \Gamma(TM),$$

where λ is a constant complex number, called the *Killing number* of φ. Using the twistor equation and the properties (9) and (10) one obtains that for an Einstein space $(M^{n,1},g)$ with constant scalar curvature $R \neq 0$ the spinor fields $\psi_\pm$ are Killing spinors to the Killing number $\lambda = \mp\frac{1}{2}\sqrt{\frac{R}{n(n-1)}}$. Hence, on this class of Lorentzian manifolds each twistor spinor is the sum of two Killing spinors. Therefore, we are specially interested in non-Einsteinian Lorentzian manifolds which admit twistor spinors.
To each spinor field we associate a vector field in the following way.

Definition 3 *Let $\varphi \in \Gamma(S)$. The vector filed V_φ definied by*

$$g(V_\varphi, X) := -\langle X\cdot\varphi, \varphi\rangle\,, \qquad X \in \Gamma(TM)$$

is called the canonical vector field of φ.

Because of (1), V_φ is a real vector field. By Zero(φ) and Zero(X) we denote the zero sets of a spinor field φ or a vector field X.

Proposition 6 *1. For each spinor field $\varphi \in \Gamma(S)$ Zero$(\varphi)=$ Zero(V_φ).*

2. If n is even, $n \leq 6$ and $\varphi \in \Gamma(S^\pm)$ is a half spinor, then $V_\varphi\cdot\varphi = 0$. In particular, V_φ is an isotropic vector field.

Proof: Let $\varphi \in \Gamma(S)$. From (5) follows for the time orientation ξ

$$g(V_\varphi, \xi) \;=\; -\langle \xi \cdot \varphi, \varphi \rangle = -(\xi \cdot \xi \cdot \varphi, \varphi)_\xi = -(\varphi, \varphi)_\xi .$$

Since the scalar product $(\cdot,\cdot)_\xi$ is positive definite, this shows that $\mathrm{Zero}(V_\varphi) = \mathrm{Zero}(\varphi)$. The second statement is proved by a direct calculation using a basis representation of φ and V_φ and the formulas (1) and (2).

□

Remark: In the Riemannian case Proposition 6.1 is not true. There exist non-trivial spinor fields φ such that the canonical vector field V_φ is identically zero (see [KR95]). On the other hand, the zero set $\mathrm{Zero}(\varphi)$ of a Riemannian twistor spinor is discret ([BFGK91], Th.2.1). This is in the Lorentzian setting not the case.
We call a subset $A \subset M$ isotropic, if each differentiable curve in A is isotropic.

Proposition 7 *Let $\varphi \in \Gamma(S)$ be a twistor spinor. Then the zero set of φ is isotropic.*

Proof: Let $\gamma : I \longrightarrow \mathrm{Zero}(\varphi)$ be a curve in $\mathrm{Zero}(\varphi)$. Then $\varphi(\gamma(t)) \equiv 0$ and therefore $\nabla_{\dot\gamma(t)}\varphi \equiv 0$. From the twistor equation (6) it follows $\dot\gamma(t) \cdot D\varphi(\gamma(t)) \equiv 0$. Since by Proposition 4 $D\varphi(\gamma(t)) \neq 0$, $\dot\gamma(t)$ is isotropic for all $t \in I$.

□

Proposition 8 *Let $\varphi \in \Gamma(S)$ be a twistor spinor. Then V_φ is a conformal vector field and for the Lie derivative*

$$L_{V_\varphi} g = -\frac{4}{n}\, Re\langle \varphi, D\varphi \rangle\, g$$

holds.

Proof: Let $V := V_\varphi$. From the definition of V_φ it follows

$$\begin{aligned}
(L_V g)(X,Y) &= g(\nabla_X V, Y) + g(X, \nabla_Y V) \\
&= X(g(V,Y)) - g(V, \nabla_X Y) + Y(g(X,V)) - g(\nabla_Y X, V) \\
&= -X\langle Y \cdot \varphi, \varphi \rangle - Y\langle X \cdot \varphi, \varphi \rangle + \langle \nabla_X Y \cdot \varphi, \varphi \rangle + \langle \nabla_Y X \cdot \varphi, \varphi \rangle \\
&\overset{(4)}{=} -\langle \nabla_X Y \cdot \varphi, \varphi \rangle - \langle Y \cdot \nabla^S_X \varphi, \varphi \rangle - \langle Y \cdot \varphi, \nabla^S_X \varphi \rangle - \langle \nabla_Y X \cdot \varphi, \varphi \rangle \\
&\quad -\langle X \cdot \nabla^S_Y \varphi, \varphi \rangle - \langle X \cdot \varphi, \nabla^S_Y \varphi \rangle + \langle \nabla_X Y \cdot \varphi, \varphi \rangle + \langle \nabla_Y X \cdot \varphi, \varphi \rangle \\
&\overset{(3)}{=} -\langle Y \cdot \nabla^S_X \varphi + X \cdot \nabla^S_Y \varphi, \varphi \rangle - \langle \varphi, Y \cdot \nabla^S_X \varphi + X \cdot \nabla^S_Y \varphi \rangle .
\end{aligned}$$

Using (7) we obtain

$$(L_V g)\,(X,Y) = -\frac{4}{n} g(X,Y)\, \mathrm{Re}\langle \varphi, D\varphi \rangle .$$

□

From Proposition 8 follows that for each twistor spinor φ $\operatorname{div}(V_\varphi) = -2\,\mathrm{Re}\langle\varphi, D\varphi\rangle$. For the imaginary part of $\langle\varphi, D\varphi\rangle$ we have

Proposition 9 *Let $\varphi \in \Gamma(S)$ be a twistor spinor. Then the function $C_\varphi := Im\,\langle\varphi, D\varphi\rangle$ is constant on M.*

Proof: Because of (3) the function $\langle Y\cdot\psi, \psi\rangle$ is real for each vector field Y and each spinor field ψ. Furthermore,

$$\begin{aligned} X\langle D\varphi,\varphi\rangle &\overset{(4)}{=} \langle\nabla^S_X D\varphi,\varphi\rangle + \langle D\varphi,\nabla^S_X\varphi\rangle \\ &\overset{(6),(10)}{=} \frac{n}{2}\langle K(X)\cdot\varphi,\varphi\rangle - \frac{1}{n}\langle D\varphi, X\cdot D\varphi\rangle. \end{aligned}$$

Hence $X\langle D\varphi,\varphi\rangle$ is a real function. Therefore, $C_\varphi = \mathrm{Im}\langle\varphi, D\varphi\rangle$ is constant.

□

Let us denote by C the $(3,0)$-Schouten-Weyl tensor $C(X,Y,Z) = g(X, C(Y,Z))$.

Proposition 10 *Let $\varphi \in \Gamma(S)$ be a twistor spinor. Then*

1. *$V_\varphi \lrcorner\, C = 0$.*
2. *If $n = 4$, then $V_\varphi \lrcorner\, W = 0$.*

Proof: From (11) and (12) we obtain

$$\begin{aligned} C(V_\varphi, X, Y) &= g(V_\varphi, C(X,Y)) = -\langle C(X,Y)\cdot\varphi,\varphi\rangle \\ &= -\frac{1}{n}\langle W(X\wedge Y)\cdot\varphi,\varphi\rangle = \frac{1}{n}\langle\varphi, W(X\wedge Y)\cdot\varphi\rangle \;=\; 0. \end{aligned}$$

Let $\varphi = au(\varepsilon,1) + bu(-\varepsilon,-1) \in \Gamma(S^\varepsilon)$ be a half spinor on a 4-dimensional manifold. Then by a direct calculation using (1) and (2) we obtain

$$V_\varphi \;=\; (|a|^2+|b|^2)s_1 + (|a|^2-|b|^2)s_2 - 2\mathrm{Re}(i a\bar{b})s_3 - 2\varepsilon\mathrm{Re}(a\bar{b})s_4.$$

Hence,

$$\begin{aligned} W(V_\varphi, s_i, s_j, s_k) &= (|a|^2+|b|^2)W_{1ijk} + (|a|^2-|b|^2)W_{2ijk} \\ &\quad -2\mathrm{Re}(i a\bar{b})W_{3ijk} - 2\varepsilon\mathrm{Re}(a\bar{b})W_{4ijk}. \end{aligned} \tag{14}$$

On the other hand, from the basis representation of

$$0 = W(s_j\wedge s_k)\cdot\varphi = \sum_{r<l}\varepsilon_r\varepsilon_l W_{rljk}\, s_r\cdot s_l\cdot\varphi$$

result the equations

$$
\begin{aligned}
0 &= (W_{12jk} - \varepsilon i W_{34jk})a + (iW_{13jk} - \varepsilon W_{24jk} - \varepsilon W_{14jk} + iW_{14jk}) \cdot b && (15)\\
0 &= (-W_{12jk} + \varepsilon i W_{34jk})b + (-iW_{13jk} + \varepsilon W_{24jk} - \varepsilon W_{14jk} + iW_{23jk})a\,. && (16)
\end{aligned}
$$

Then looking at the real and imaginary part of the equations $(15)\bar{a} \pm (16)\bar{b}$ and $(15)\bar{b} \pm (16)\bar{a}$ one obtains $W(V_\varphi, s_i, s_j, s_k) = 0$.

□

Propositions 6 and 10 show that 4-dimensional Lorentzian spin manifolds with twistor spinor φ are of Petrov type N outside the zero zet of φ and W (see [ON95] chap. 5).

Examples of Lorentzian manifolds admitting twistor spinors:

1. *Minkowski space*
   A spinor field $\varphi \in C^\infty(\mathbb{R}^{n,1}, \Delta_{n,1})$ on the Minkowski space is a twistor spinor if and only if $\varphi(x) = u + x \cdot v$, where $u, v \in \Delta_{n,1}$ are constant spinors.

2. *Simply connected Lorentzian space forms*
   Let $M^{n,1}(K)$ be the simply connected Lorentzian space form of constant sectional curvature $K \neq 0$. On $M^{n,1}(K)$ there exist $2^{[\frac{n}{2}]}$ linearly independent Killing spinors to the Killing number $\mu \in \mathbb{C}$ with $\mu^2 = \frac{K}{4}$ (see [CGLS86]). Each Killing spinor is a twistor spinor.

3. *4-dimensional pp-manifolds*
   A 4-dimensional Lorentzian spin manifold $M^{4,1}$ admits a parallel spinor field if and only if $M^{4,1}$ is a *pp*-manifold. (see eg [Bau81] chap 4, [Gün88] chap 8, cor.4.4). Each parallel spinor field is a twistor spinor.

4. *4-dimensional Fefferman spaces*
   In [Lew91] and [Spi95] the following theorem was proved: Let φ be a twistor spinor on a 4-dimensional Lorentzian spin manifold $(M^{4,1}, g)$ such that $\mathrm{Zero}(\varphi) = \emptyset$ and $\mathrm{rot}(V_\varphi)$ is non-degenerate. Then $(M^{4,1}, g)$ is locally conformal equivalent to a Fefferman manifold. On each Fefferman space there exists a *local* solution of the twistor equation.

The aim of the present paper is the generalisation of this result of J. Lewandowski. We prove that there are *global* solutions of the twistor equation on the (modified) Fefferman spaces of strictly preudoconvex spin manifolds of arbitrary dimension and study their properties.

4 Pseudo-hermitian geometry

Before we are able to define the Fefferman spaces we need some notions from pseudo-hermitian geometry. Let M^{2n+1} be a connected manifold of odd dimension $2n+1$.
A *complex CR-structure* on M is a complex subbundle T_{10} of $TM^{\mathbb{C}}$ such that

1. $\dim_{\mathbb{C}} T_{10} = n$,
2. $T_{10} \cap \overline{T_{10}} = \{0\}$,
3. $[\Gamma(T_{10}), \Gamma(T_{10})] \subset \Gamma(T_{10})$ (integrability condition).

A *real CR-structure* on M is a pair (H, J), where

1. $H \subset TM$ is a real $2n$-dimensional subbundle,
2. $J : H \longrightarrow H$ is an almost complex structure on H : $J^2 = -\mathrm{id}$,
3. If $X, Y \in \Gamma(H)$, then $[JX, Y] + [X, JY] \in \Gamma(H)$ and $N_J(X,Y) := J([JX,Y] + [X,JY]) - [JX,JY] + [X,Y] \equiv 0$ (integrability condition).

Obviously the complex and real CR-structure correspond to each other: If $T_{10} \subset TM^{\mathbb{C}}$ is a complex CR-structure, then $H := \mathrm{Re}\ (T_{10} \oplus \overline{T_{10}}))$, $J(U+\bar{U}) := i(U-\bar{U})$ defines a real CR-structure. If (H, J) is a real CR-structure, then the eigenspace of the complex extension of J on $H^{\mathbb{C}}$ to the eigenvalue i is a complex CR-structure. CR-structures one has for example on real hypersurfaces of complex manifolds, on Sasaki-manifolds or on Heisenberg manifolds.
A *CR-manifold* is an odd-dimensional manifold equipped with a (real or complex) CR-structure. Let (M, T_{10}) be a CR-manifold. The hermitian form on T_{10}

$$L : T_{10} \times T_{10} \longrightarrow E := TM^{\mathbb{C}}/_{T_{10} \oplus \overline{T_{10}}}$$

$$L(U,V) := i[U, \bar{V}]_E ,$$

where X_E denotes the projection of $X \in TM^{\mathbb{C}}$ onto E, is called the *Levi-form* of (M, T_{10}). The CR-manifold is called *non-degenerate*, if its Levi-form L is non-degenerate. An nowhere vanishing 1-form $\theta \in \Omega^1(M)$ is called a *pseudo-hermitian structure* on (M, T_{10}), if $\theta|_H \equiv 0$. (M, T_{10}, θ) is called a *pseudo-hermitian manifold.* There exists a pseudo-hermitian structure θ on (M, T_{10}) if and only if M is orientable. Two pseudo-hermitian structures $\theta, \tilde{\theta}$ differs by a real nowhere vanishing function $f \in C^{\infty}(M)$: $\tilde{\theta} = f \cdot \theta$. If M is oriented, we always choose θ such that a basis of the form $(X_1, JX_1, \ldots, X_n, JX_n, T)$ is positive oriented on M.
Let (M, T_{10}, θ) be a pseudo-hermitian manifold. The hermitian form $L_\theta : T_{10} \times T_{10} \longrightarrow \mathbb{C}$

$$L_\theta(U,V) := -id\theta(U, \bar{V})$$

is called the *Levi-form* of (M, T_{10}, θ). Obviously, we have $\theta(L(U,V)) = L_\theta(U,V)$. The pseudo-hermitian manifold (M, T_{10}, θ) is called *strictly pseudoconvex*, if the Levi-form L_θ is positive definite. If the pseudo-hermitian manifold (M, T_{10}, θ) is non-degenerate, then the pseudo-hermitian structure θ is a contact form. We denote by $T \in \Gamma(TM)$

the *characteristic vector field* of this contact form, e.g. the vector field uniquely defined by

$$\theta(T) \equiv 1 \qquad \text{and} \qquad T \lrcorner\, d\theta \equiv 0.$$

From now on we always suppose, that (M, T_{10}, θ) is non-degenerate.
On CR-manifolds we consider the following spaces of forms:

$$\begin{aligned} \Lambda^{q,0}M &:= \{\omega \in \Lambda^q M^{\mathbb{C}} \mid V \lrcorner\, \omega = 0 \quad \forall V \in \overline{T_{10}}\} \\ \Lambda^{0,q}M &:= \{\omega \in \Lambda^q M^{\mathbb{C}} \mid V \lrcorner\, \omega = 0 \quad \forall V \in T_{10}\} \\ \Lambda^{p,q}M &:= \operatorname{span}\{\omega \wedge \sigma \mid \omega \in \Lambda^{p,0}M,\ \sigma \in \Lambda^{0,q}M\}. \end{aligned}$$

Obviously,

$$\begin{aligned} \Lambda^{1,0}M &= \{\omega \in \Lambda^1 M^{\mathbb{C}} \mid \omega(JX) = i\omega(X) \quad \forall X \in H\} \\ \Lambda^{0,1}M &= \{\omega \in \Lambda^1 M^{\mathbb{C}} \mid \omega(JX) = -i\omega(X) \quad \forall X \in H\} \\ \Lambda^{1,1}M &= \{\omega \in \Lambda^2 M^{\mathbb{C}} \mid \omega(JX, JY) = \omega(X, Y) \quad \forall X, Y \in H\} \\ &= \{\omega \in \Lambda^2 M^{\mathbb{C}} \mid \omega(Z, W) = \omega(\bar{Z}, \bar{W}) = 0 \quad \forall Z, W \in T_{10}\}. \end{aligned}$$

Let (M, T_{10}, θ) be a non-degenerate pseudo-hermitian manifold and let T be the characteristic vector field of θ. Then we define

$$\Lambda^{p,q}_\theta M := \{\omega \in \Lambda^{p,q}M \mid T \lrcorner\, \omega = 0\}$$

and have

$$\Lambda^{p,q}M = \Lambda^{p,q}_\theta M \oplus (\Lambda^{p-1,q}_\theta \wedge \mathbb{C}\theta) \oplus (\Lambda^{p,q-1}M \wedge \mathbb{C}\theta).$$

For example, by definition, $d\theta \in \Lambda^{1,1}_\theta M$.

Now, let us extend the Levi-form of (M, T_{10}, θ) to $TM^{\mathbb{C}}$ by

$$\begin{aligned} L_\theta(\bar{U}, \bar{V}) &:= \overline{L_\theta(U, V)} = L_\theta(V, U) \\ L_\theta(U, \bar{V}) &:= 0 \qquad\qquad\qquad\qquad U, V \in T_{10} \\ L_\theta(T, \cdot) &:= 0. \end{aligned}$$

Proposition 11 *Let $L_\theta : TM^{\mathbb{C}} \times TM^{\mathbb{C}} \longrightarrow \mathbb{C}$ be the Levi-form of (M, T_{10}, θ) and let T be the characteristic vector field of θ. Then*

$$[T, Z] \in \Gamma(T_{10} \oplus \overline{T_{10}}) \qquad \textit{if } Z \in \Gamma(T_{10}) \textit{ or } Z \in \Gamma(\overline{T_{10}}), \tag{17}$$

$$L_\theta([T, U], V) + L_\theta(U, [T, V]) = T(L_\theta(U, V)) \qquad \forall\, U, V \in \Gamma(T_{10}), \tag{18}$$

$$L_\theta([T, \bar{U}], V) = L_\theta([T, \bar{V}], U) \qquad \forall\, U, V \in \Gamma(T_{10}), \tag{19}$$

$$L_\theta([T, U], \bar{V}) = L_\theta([T, V], \bar{U}) \qquad \forall\, U, V \in \Gamma(T_{10}), \tag{20}$$

Proof: Let $Z \in \Gamma(T_{10})$ or $Z \in \Gamma(\overline{T_{10}})$. Then, by definition of θ and T

$$\theta([T,Z]) = -d\theta(T,Z) - T(\theta(Z)) + Z(\theta(T)) = 0.$$

Hence, $[T,Z] \in \Gamma(H^{\mathbb{C}}) = \Gamma(T_{10} \oplus \overline{T_{10}})$. By definition of L_θ and since $d\theta$ is an (1,1)-form, we obtain for $U, V \in \Gamma(T_{10})$

$$\begin{aligned} L_\theta([T,U],V) + L_\theta(U,[T,V]) &= -i(d\theta([T,U],\bar{V}) + d\theta(U,[T,\bar{V}])) \\ &\overset{(17)}{=} i(\theta([[T,U],\bar{V}]) + \theta([U,[T,\bar{V}]])). \end{aligned}$$

Then, the Bianchi-identity gives

$$\begin{aligned} L_\theta([T,U],V) + L_\theta(U,[T,V]) &= -i\theta([[U,\bar{V}],T]) \\ &= id\theta([U,\bar{V}],T) + iT(\theta([U,\bar{V}])) - i[U,\bar{V}](\theta(T)) \\ &= -iT(d\theta(U,\bar{V})) \;=\; T(L_\theta(U,V)). \end{aligned}$$

Since T_{10} is integrable, we have $[\bar{V},\bar{U}] \in \Gamma(\overline{T_{10}})$ and because of (17) $[[\bar{V},\bar{U}],T] \in \Gamma(H^{\mathbb{C}})$. Hence,

$$\begin{aligned} L_\theta([T,\bar{U}],V) - L_\theta([T,\bar{V}],U) &= -id\theta([T,\bar{U}],\bar{V}) + id\theta([T,\bar{V}],\bar{U}) \\ &= i\theta([[T,\bar{U}],\bar{V}]) - i\theta([[T,\bar{V}],\bar{U}]) \\ &= i\theta([[\bar{V},\bar{U}],T]) \;=\; 0. \end{aligned}$$

Finally,

$$L_\theta([T,U],\bar{V}) = \overline{L_\theta([T,\bar{U}],V)} \overset{(19)}{=} \overline{L_\theta([T,\bar{V}],U)} = L_\theta([T,V],\bar{U}).$$

□

If we consider the Levi-form L_θ as a bilinear form on the real tangent bundle, we obtain a symmetric bilinear form on TM which is non-degenerate on H.

Proposition 12 *Let $(M^{2n+1}, T_{10}, \theta)$ be a non-degenerate pseudo-hermitian manifold and (H,J) the real CR-structure, defined by T_{10}. Let X and Y be two vector fields in H. Then the Levi-form $L_\theta : TM \times TM \longrightarrow \mathbb{R}$ satisfies*

$$L_\theta(X,Y) = d\theta(X,JY)\,, \tag{21}$$

$$L_\theta(JX,JY) = L_\theta(X,Y) \quad \textit{and} \quad L_\theta(JX,Y) + L_\theta(X,JY) = 0\,, \tag{22}$$

$$L_\theta([T,X],Y) - L_\theta([T,Y],X) \;=\; L_\theta([T,JX],JY) - L_\theta([T,JY],JX)\,. \tag{23}$$

Proof: By definition of H there are vector fields $U, V \in \Gamma(T_{10})$ such that $X = U + \bar{U}$, $Y = V + \bar{V}$. Therefore,

$$L_\theta(X,Y) = L_\theta(U+\bar{U}, V+\bar{V}) = L_\theta(U,V) + L_\theta(\bar{U},\bar{V}) = \; 2\mathrm{Re}\, L_\theta(U,V).$$

Hence, $L_\theta|_{TM\times TM}$ is real, symmetric and on H non-degenerate. If $X=U+\bar{U}$, then $JX=i(U-\bar{U})$. Therefore,

$$\begin{aligned} d\theta(X,JY) &= d\theta(U+\bar{U},i(V-\bar{V})) = id\theta(\bar{U},V)-id\theta(U,\bar{V}) \\ &= -2\mathrm{Re}\,(id\theta(U,\bar{V})) = 2\mathrm{Re}\,(L_\theta(U,V)) = L_\theta(X,Y). \end{aligned}$$

Furthermore,

$$L_\theta(JX,JY)=d\theta(JX,J^2Y)=-d\theta(JX,Y)=L_\theta(Y,X)=L_\theta(X,Y).$$

The second formula of (22) follows since $J^2=-id$. Formula (23) is a direct corollary of (19) and (20).

□

On non-degenerate pseudo-hermitian manifolds there exists a special covariant derivative, the so-called *Webster connection*.

Proposition 13 *Let (M,T_{10},θ) be a non-degenerate pseudo-hermitian manifold and let T be the characteristic vector field of θ. Then there exists an uniquely determined covariant derivative $\nabla^W:\Gamma(T_{10})\longrightarrow\Gamma(T^*M^{\mathbb{C}}\otimes T_{10})$ on T_{10} such that*

1. *∇^W is metric with respect to L_θ :*
$$X(L_\theta(U,V))=L_\theta(\nabla^W_X U,V)+L_\theta(U,\nabla^W_{\bar{X}}V) \qquad U,V\in\Gamma(T_{10}),\ X\in\Gamma(TM^{\mathbb{C}}) \quad (24)$$
2. $$\nabla^W_T U=pr_{10}[T,U], \quad (25)$$
3. $$\nabla^W_{\bar{V}} U=pr_{10}[\bar{V},U], \quad (26)$$

where pr_{10} denotes the projection on T_{10} . Furthermore, ∇^W satisfies

$$\nabla^W_U V-\nabla^W_V U=[U,V], \qquad U,V\in\Gamma(T_{10}). \quad (27)$$

Remark: This covariant derivative was introduced by Tanaka [Tan75] for strictly pseudoconvex manifolds and by Webster [Web78] in the pseudo-hermitian case.

Proof: Let ∇ be a covariant derivative satisfying (24)-(26). Then for $U,V,Z\in\Gamma(T_{10})$

$$L_\theta(\nabla_Z U,V)=Z(L_\theta(U,V))-L_\theta(U,\nabla_{\bar{Z}}V)=Z(L_\theta(U,V))-L_\theta(U,[\bar{Z},V]). \quad (28)$$

Hence, ∇ is uniquely determined by (25), (26), (28). On the other hand it is easy to check, that (25), (26), (28) define a metric covariant derivative on T_{10}. If $U,V,Z\in\Gamma(T_{10})$, then

$$\begin{aligned} &L_\theta(U,[\bar{V},Z])-L_\theta(V,[\bar{U},Z])=-id\theta(U,[V,\bar{Z}])+id\theta(V,[U,\bar{Z}])= \\ &\quad =-iU(\theta([V,\bar{Z}]))+i\theta([U,[V,\bar{Z}]])+iV(\theta([U,\bar{Z}]))-i\theta([V,[U,\bar{Z}]]) \\ &\quad =iU(d\theta(V,\bar{Z}))-iV(d\theta(U,\bar{Z}))+i\theta([[U,V],\bar{Z}]) \\ &\quad =-U(L_\theta(V,Z))+V(L_\theta(U,Z))-id\theta([U,V],\bar{Z}) \\ &\quad =-U(L_\theta(V,Z))+V(L_\theta(U,Z))+L_\theta([U,V],Z). \end{aligned}$$

Hence,

$$\begin{aligned} L_\theta(\nabla^W_U V - \nabla^W_V U, Z) &\overset{(28)}{=} U(L_\theta(V,Z)) - V(L_\theta(U,Z)) - L_\theta(V,[\bar U, Z]) + L_\theta(U,[\bar V, Z]) \\ &= L_\theta([U,V],Z). \end{aligned}$$

□

Now, we extend the Webster connection to $TM^{\mathbb{C}}$ by

$$\nabla^W \bar U := \overline{\nabla^W U} \qquad \text{and} \qquad \nabla^W T := 0.$$

Proposition 14 *The torsion* Tor^W *of the Webster connection* $\nabla^W : \Gamma(TM^{\mathbb{C}}) \longrightarrow \Gamma(T^*M^{\mathbb{C}} \otimes TM^{\mathbb{C}})$ *satisfies*

$$\begin{aligned} Tor^W(U,V) &= Tor^W(\bar U, \bar V) = 0\,, & (29)\\ Tor^W(U,\bar V) &= iL_\theta(U,V)\,T\,, & (30)\\ Tor^W(T,U) &= -pr_{01}[T,U]\,, & (31)\\ Tor^W(T,\bar U) &= -pr_{10}[T,\bar U]\,, & (32) \end{aligned}$$

where pr_{01} *denotes the projection onto* $\overline{T_{10}}$, p_{10} *the projection onto* T_{10} *and* $U,V \in \Gamma(T_{10})$.

Proof: By definition of ∇^W on $TM^{\mathbb{C}}$ follows $\overline{\mathrm{Tor}^W(X,Y)} = \mathrm{Tor}^W(\bar X, \bar Y)$ for all $X,Y \in \Gamma(TM^{\mathbb{C}})$. Then (29) follows from (27). Furthermore,

$$\begin{aligned} \mathrm{Tor}^W(U,\bar V) &= \nabla^W_U \bar V - \nabla^W_{\bar V} U - [U,\bar V] = \overline{\nabla^W_{\bar U} V} - \mathrm{pr}_{10}[\bar V, U] + [\bar V, U] \\ &= \overline{\mathrm{pr}_{10}[\bar U, V]} + \mathrm{pr}_{01}[\bar V, U] + \theta([\bar V, U]) \cdot T \\ &= \theta([\bar V, U]) \cdot T = d\theta(U,\bar V) \cdot T = iL_\theta(U,V) \cdot T\,, \\ \mathrm{Tor}^W(T,U) &= \nabla^W_T U - \nabla^W_U T - [T,U] = \mathrm{pr}_{10}[T,U] - [T,U] = -\mathrm{pr}_{01}[T,U]\,. \end{aligned}$$

$\mathrm{Tor}^W(T,\bar U) = -\mathrm{pr}_{10}[T,\bar U]$ follows, since $\overline{\mathrm{pr}_{10}[X,Y]} = \mathrm{pr}_{01}[\bar X, \bar Y]$ for all $X,Y \in TM^{\mathbb{C}}$.

□

Let (M,T_{10},θ) be a non-degenerate pseudo-hermitian manifold and let (p,q) be the signature of (T_{10},L_θ). Then $g_\theta := L_\theta + \theta\circ\theta$ defines a metric of signature $(2p,2q{+}1)$ on M.

Proposition 15 *Let (M, T_{10}, θ) be a non-degenerate pseudo-hermitian manifold. Then the Webster connection $\nabla^W : \Gamma(TM) \longrightarrow \Gamma(T^*M \otimes TM)$ considered on the real tangent bundle is metric with respect to g_θ and the torsion of ∇^W is given by*

$$\begin{aligned} Tor^W(X,Y) &= L_\theta(JX,Y)\cdot T \qquad && \textit{for } X,Y \in \Gamma(H), && (33)\\ Tor^W(T,X) &= -\frac{1}{2}\{[T,X] + J[T,JX]\} && \textit{for } X \in \Gamma(H). && (34) \end{aligned}$$

Furthermore, on $\Gamma(H)$

$$\nabla^W \circ J = J \circ \nabla^W \tag{35}$$

holds.

Proof: ∇^W is metric with respect to g_θ, since ∇^W is metric on (T_{10}, L_θ) and $\nabla^W T \equiv 0$. If $Z \in H^{\mathbb{C}}$, then $\mathrm{pr}_{01}(Z) = \frac{1}{2}(Z + iJZ)$ and $\mathrm{pr}_{10}(Z) = \frac{1}{2}(Z - iJZ)$. Now, let $X = U + \bar{U}$, $Y = V + \bar{V} \in \Gamma(H)$, $U, V \in \Gamma(T_{10})$. Then from (29) - (31) follow

$$\begin{aligned} \mathrm{Tor}^W(T,X) &= \mathrm{Tor}^W(T, U+\bar{U}) \;=\; -\mathrm{pr}_{01}[T,U] - \mathrm{pr}_{10}[T,\bar{U}]\\ &= -\frac{1}{2}\left([T,U] + iJ[T,U] + [T,\bar{U}] - iJ[T,\bar{U}]\right)\\ &= -\frac{1}{2}\left([T,U+\bar{U}] + J[T, iU - i\bar{U}]\right)\\ &= -\frac{1}{2}([T,X] + J[T,JX])\,,\\ \mathrm{Tor}^W(X,Y) &= \mathrm{Tor}^W(\bar{U},V) + \mathrm{Tor}^W(U,\bar{V}) \;=\; -i\,\overline{L_\theta(U,V)}\cdot T + iL_\theta(U,V)\cdot T\\ &= (-iL_\theta(\bar{U}, V+\bar{V}) + iL_\theta(U, V+\bar{V}))\,T\\ &= L_\theta(JX,Y)\cdot T. \end{aligned}$$

Furthermore,

$$\begin{aligned} \nabla^W JX &= \nabla^W(i(U-\bar{U})) \;=\; i(\nabla^W U - \overline{\nabla^W U})\\ &= J(\nabla^W U + \overline{\nabla^W U}) \;=\; J\nabla^W X. \end{aligned}$$

$\square$

Now, let $R^{\nabla^W} \in \Gamma(\Lambda^2 M^{\mathbb{C}} \otimes \mathrm{End}(TM^{\mathbb{C}}, TM^{\mathbb{C}}))$ be the curvature operator of ∇^W

$$R^{\nabla^W}(X,Y) = [\nabla^W_X, \nabla^W_Y] - \nabla^W_{[X,Y]}.$$

Then the (4,0)-curvature tensor $\mathcal{R}^W$

$$\mathcal{R}^W(X,Y,Z,V) := g_\theta(R^{\nabla^W}(X,Y)Z, \bar{W}), \qquad X,Y,Z,W \in TM^{\mathbb{C}}$$

has the following properties

Proposition 16 *Let* $X, Y, Z, V \in TM^{\mathbb{C}}$, $A, B, C, D \in T_{10}$. *Then*

1) $\mathcal{R}^W(X,Y,Z,V) = -\mathcal{R}^W(Y,X,Z,V) = -\mathcal{R}^W(X,Y,V,Z)$

2) $\overline{\mathcal{R}^W(X,Y,Z,V)} = \mathcal{R}^W(\bar{X},\bar{Y},\bar{Z},\bar{V})$

3) $\mathcal{R}^W(A,\bar{B},C,\bar{D}) = \mathcal{R}^W(C,\bar{B},A,\bar{D})$

4) $\mathcal{R}^W(A,B,\cdot,\cdot) = 0$

Proof: This follows by a direct calculation from the definition of ∇^W on $TM^{\mathbb{C}}$. □

Let $\omega \in \Lambda^2 M^{\mathbb{C}}$ be a complex 2-form and $\tilde{\omega} : T_{10} \longrightarrow T_{10}$ the uniquely determined $\mathbb{C}$-linear map with $\omega(U,\bar{V}) = L_\theta(\tilde{\omega}U, V)$, $U, V \in T_{10}$. Then the θ-trace of ω is defined by $\mathrm{Tr}_\theta\omega := \mathrm{Tr}(\tilde{\omega})$. If $(Z_1,\ldots,Z_n)$ is an orthonormal basis of (T_{10}, L_θ), $\varepsilon_k = L_\theta(Z_k, Z_k)$, then

$$\mathrm{Tr}_\theta\omega = \sum_{\alpha=1}^{n} \varepsilon_\alpha\, \omega(Z_\alpha, \bar{Z}_\alpha).$$

The (2,0)-tensor field

$$\mathrm{Ric}^W := \mathrm{Tr}_\theta^{(3,4)}\mathcal{R}^W = \sum_{\alpha=1}^{n} \varepsilon_\alpha \mathcal{R}^W(\cdot,\cdot,Z_\alpha,\bar{Z}_\alpha)$$

is called the *Webster-Ricci-tensor*, the function

$$R^W := \mathrm{Tr}_\theta \mathrm{Ric}^W$$

is the *Webster scalar curvature*. Proposition 16 shows that

1. $\mathrm{Ric}^W \in \Lambda^{1,1}M$,
2. $\mathrm{Ric}^W(X,Y) \in i\mathbb{R}, \qquad X, Y \in TM$,
3. R^W is a real function.

Geometric properties of pseudo-hermitian manifolds $(M^{2n+1}, T_{10}, \theta, \nabla^W)$ were studied for example in [Web78], [Lee88], [JL87], [BGS84], ([Gre85] .

5 Fefferman spaces

Let (M^{2n+1}, T_{10}) be a CR-manifold. The complex line bundle $K := \Lambda^{n+1,0}M$ of $(n+1,0)$-forms is called the *canonical bundle* of (M^{2n+1}, T_{10}). $\mathbb{R}^+$ acts on $K^* = K\backslash\{0\}$ by multiplication. Let $F := K^*/_{\mathbb{R}^+}$. Then (F, π, M) is the S^1-principal bundle over M associated to K. We call (F,π,M) the *canonical* S^1*-bundle* of (M, T_{10}). Now, let (M,T_{10},θ) be a non-degenerate pseudo-hermitian manifold and $\nabla^W : \Gamma(T_{10}) \longrightarrow \Gamma(T^*M^{\mathbb{C}} \otimes T_{10})$ its Webster-connection. ∇^W allows us to define a connection A^W on the canonical S^1-bundle F in the following way: Let $s = (Z_1,\ldots,Z_n)$ be a local orthonormal basis of (T_{10},L_θ) over $U \subset M$ and let us denote by $\omega_s := (\omega_{\alpha\beta})$ the matrix of connection forms of ∇^W with respect to s

$$\nabla^W Z_\alpha = \sum_\beta \omega_{\alpha\beta} Z_\beta .$$

$(Z_1,\ldots,Z_n,\bar{Z}_1,\ldots,\bar{Z}_n,T)$ is a local basis of $TM^{\mathbb{C}}$ over U. Let $(\theta^1,\ldots,\theta^n,\bar{\theta}^1,\ldots,\bar{\theta}^n,\theta)$ be the corresponding dual basis. Then

$$\hat{\tau}_s := \theta \wedge \theta^1 \wedge \ldots \wedge \theta^n : U \longrightarrow K$$

is a local section in K. We denote by $\tau_s := [\hat{\tau}_s]$ the corresponding local section in $F = K^*/_{\mathbb{R}^+}$. The Webster connection ∇^W defines in the standard way a covariant derivative ∇^K in the canonical line bundle K such that

$$\nabla^K \hat{\tau}_s = -\sum_\alpha \omega_{\alpha\alpha} \cdot \hat{\tau}_s = -\operatorname{Tr} \omega_s \cdot \hat{\tau}_s \,.$$

Since ∇^W is metric with respect to L_θ, the trace $\operatorname{Tr}\omega_s$ is purely imaginary. Hence ∇^K is induced by a connection A^W on the associated S^1-principal bundle $(F,\pi,M;S^1)$ with the local connection forms

$$\tau_s^* A^W = -\operatorname{Tr}\omega_s \,.$$

Let Ω^W be the curvature form of the connection A^W on F. Since Ω^W is tensionell and right-invariant, it can be considered as 2-form on M with values in $i\mathbb{R}$. Over $U \subset M$

$$\Omega^W = dA^{\tau_s} = -\operatorname{Tr} d\omega_s . \tag{36}$$

holds. On the other hand,

$$\begin{aligned} \operatorname{Ric}^W(X,Y) &= \sum_\alpha \varepsilon_\alpha L_\theta(([\nabla^W_X,\nabla^W_Y] - \nabla^W_{[X,Y]})Z_\alpha,\bar{Z}_\alpha) \\ &= \Big(\sum_\alpha d\omega_{\alpha\alpha} - \sum_{\alpha,\beta} \omega_{\alpha\beta}\wedge\omega_{\beta\alpha}\Big)(X,Y). \end{aligned}$$

Hence,

$$\operatorname{Ric}^W = \operatorname{Tr} d\omega_s - \operatorname{Tr}(\omega_s \wedge \omega_s) = \operatorname{Tr} d\omega_s .$$

From (36) it follows

$$\Omega^W = -\operatorname{Ric}^W \tag{37}$$

The connection A^W on the canonical S^1-bundle (F,π,M) is called the *Webster-connection on F*. Two connections on an S^1-principal bundle over M differ by an 1-form on M with values in $i\mathbb{R}$. The connection

$$A_\theta := A^W - \frac{i}{2(n+1)} R^W \theta$$

on the canonical S^1-bundle (F, π, M) is called the *Fefferman connection on* F.
Let us consider the following right-invariant metric on F:

$$h_\theta := \pi^* L_\theta - i \frac{4}{n+2} \pi^* \theta \circ A_\theta,$$

where $\circ$ denotes the symmetric tensor product. h_θ is called the *Fefferman metric on* F. If (T_{10}, L_θ) is of signature (p,q), then h_θ has signature $(2p+1, 2q+1)$. In particular, if (M, T_{10}, θ) is strictly pseudoconvex, h_θ is a Lorentzian metric. The semi-Riemannian manifold (F, h_θ) is called the *Fefferman space of* (M, T_{10}, θ). The fibres of the canonical S^1-bundle F are isotropic submanifolds of (F, h_θ). From the special choise of the Fefferman connection A_θ in the definition of h_θ results that the conformal class $[h_\theta]$ of the metric h_θ is an invariant of the oriented CR-manifold (M, T_{10}), e.g. if $\tilde{\theta} = f \cdot \theta$, $f > 0$, is a further pseudo-hermitian structure on (M, T_{10}), then $h_{\tilde{\theta}} = f \cdot h_\theta$ (see [Lee86], Th. 5.17.). We remark that Fefferman spaces are never Einsteinian.
Fefferman spaces were first defined by Fefferman ([Fef76]) in case of strictly pseudoconvex hypersurfaces in $\mathbb{C}^n$, its definition was extended by Burns, Diederich, Shnider ([BDS77]), Farris ([Far86]), and Lee ([Lee86]) to general CR-manifolds. Sparling ([Spa85]), Lee ([Lee86]), Graham ([Gra87]) and Koch ([Koc88]) studied geometric properties of Fefferman spaces.

In the following we always assume that (M, T_{10}, θ) is strictly pseudoconvex. In order to find global solutions of the Lorentzian twistor equation on Fefferman spaces it is necessary to change the topological type of the canonical S^1-bundle.

Proposition 17 *Let $(M^{2n+1}, T_{10}, \theta)$ be a strictly pseudoconvex spin manifold. Then each spinor structure of the Riemann manifold (M, g_θ) defines a square root $\sqrt{F}$ of the canonical S^1-bundle F. (e.g. $\sqrt{F}$ is an S^1-bundle over M such that the associated line bundle $L := \sqrt{F} \times_{S^1} \mathbb{C}$ satisfies $L \otimes L = K$).*

Proof: Let

$$\begin{array}{ccccc} U(n) & \longrightarrow & SO(2n) & \longrightarrow & SO(2n+1) \\ A + iB & \longmapsto & \mathcal{A} := \begin{pmatrix} A & B \\ -B & A \end{pmatrix} & \longmapsto & \begin{pmatrix} \mathcal{A} & 0 \\ 0 & 1 \end{pmatrix} \end{array}$$

be the canonical embedding of $U(n)$ in $SO(2n+1)$. The bundle P_M of $SO(2n+1)$-frames of (M, g_θ) reduces to the group $U(n)$:

$$P_H := \{(X_1, JX_1, \ldots, X_n, JX_n, T) \mid (X_1, JX_1, \ldots, X_n, JX_n) \text{ on-basis of } (H, L_\theta)\}$$

is an $U(n)$-reduction of P_M. Let (Q_M, f_M) be a spinor structure of (M, g_θ) and let us denote by (Q_H, f_H) the reduced spinor structure

$$Q_H := f_M^{-1}(P_H), \qquad f_H := f_M \mid_{Q_H} .$$

Now, the proof of Proposition 17 is a repetition of Hitchin's proof of the fact that each spinor structure on a Kähler manifold defines a square root of the canonical bundle (see [Hit74]). Since we need some notation later on, we repeat the idea of the proof. Let $\ell : U(n) \longrightarrow \mathrm{Spin}(2n)^{\mathbb{C}} = \mathrm{Spin}(2n) \times_{\mathbb{Z}_2} S^1$ be defined by

$$\ell(A) = \prod_{k=1}^{n} \Big(\cos\frac{\theta_k}{2} + \sin\frac{\theta_k}{2} \cdot f_k \cdot J_0(f_k) \Big) \times e^{\frac{i}{2}\sum_{k=1}^{n}\theta_k}, \tag{38}$$

where $(f_1, \dots, f_n)$ is an unitary basis of $\mathbb{C}^n$ such that $Af_k = e^{i\theta_k} f_k$ and $J_0 : \mathbb{C}^n \to \mathbb{C}^n$ is the standard complex structure of $\mathbb{C}^n$. Then we have the following commutative diagram

$$\begin{tikzcd}
S^1 \arrow[r, "j_2"'] \arrow[d, "z \mapsto z^2"] & \mathrm{Spin}(2n)^{\mathbb{C}} & \mathrm{Spin}(2n) \arrow[l, "j_1"'] \arrow[d, "\lambda"] \\
S^1 & U(n) \arrow[l, "\det"'] \arrow[u, "\ell"] \arrow[r, "i"] & \mathrm{SO}(2n)
\end{tikzcd}$$

with the diagonal arrow $\lambda \circ pr_1 : \mathrm{Spin}(2n)^{\mathbb{C}} \to \mathrm{SO}(2n)$,

where i, j_1, j_2 denote the canonical embeddings and $\lambda : \mathrm{Spin}(2n) \to SO(2n)$ is the universal covering of $SO(2n)$. Hence, for each $A \in U(n)$ and each square root of $\det(A)$ one has

$$\lambda^{-1}(A) := j_1\lambda^{-1}(i(A)) = \pm\ell(A)\,\mathrm{Det}(A)^{-\frac{1}{2}}.$$

Now, let $\{(U_{\alpha\beta}\,,\, g_{\alpha\beta} \,:\, U_{\alpha\beta} \to \lambda^{-1}(U(n)))\}_{\alpha,\beta}$ are the cocycles defining the reduced spinor structure (Q_H, f_H). Then on $U_{\alpha\beta}$ we choose a square root $h_{\alpha\beta} : U_{\alpha\beta} \to S^1$ of the determinant of $\lambda(g_{\alpha\beta})^{-1}$ such that

$$h^2_{\alpha\beta} = \mathrm{Det}(\lambda(g_{\alpha\beta}))^{-1} \qquad \text{and} \qquad g_{\alpha\beta} = \ell(\lambda(g_{\alpha\beta})) \cdot h_{\alpha\beta}. \tag{39}$$

$\{(U_{\alpha\beta}, h_{\alpha\beta})\}_{\alpha\beta}$ are cocyles defining a square root $(\sqrt{F}, \pi, M)$ of the canonical S^1-bundle (F, π, M).

$\square$

Let $(\sqrt{F}, \pi, M)$ be the square root of the canonical S^1-bundle defined by the spinor structure of (M, g_θ). Then the Webster connection A^W on F defines a corresponding connection $A^{\sqrt{W}}$ on $\sqrt{F}$: Let $\{\tilde{s}_\alpha : U_\alpha \to Q_H\}$ be a covering of Q_H by local sections with the transition functions $g_{\alpha\beta}$; $\tilde{s}_\alpha = \tilde{s}_\beta \cdot g_{\alpha\beta}$. Let $s_\alpha = f_H(\tilde{s}_\alpha) \in P_H$ and denote by $\sqrt{\tau_{s_\alpha}} : U_\alpha \to \sqrt{F}$ the local sections in $\sqrt{F}$ with transition functions $h_{\alpha\beta}$

$$\sqrt{\tau_{s_\alpha}} = \sqrt{\tau_{s_\beta}} \cdot h_{\alpha\beta},$$

defined by (39). Then the local connection forms of $A^{\sqrt{W}}$ are given by

$$\sqrt{\tau_{s_\alpha}}^{\,*} A^{\sqrt{W}} = \frac{1}{2}\tau^*_{s_\alpha} A^W = -\frac{1}{2}\,\mathrm{Tr}\,\omega_{s_\alpha} \tag{40}$$

and the curvature of $A^{\sqrt{W}}$ is

$$\Omega^{\sqrt{W}} = \frac{1}{2}\Omega^W = -\frac{1}{2}\mathrm{Ric}^W. \tag{41}$$

The connection $A_\theta^{\surd}$ on $\sqrt{F}$ defined by

$$A_\theta^{\surd} := A^{\sqrt{W}} - \frac{i}{4(n+1)} R^W \cdot \theta$$

is called the *Fefferman connection on* $\sqrt{F}$ and the Lorentzian metric

$$h_\theta := \pi^* L_\theta - i\frac{8}{n+2}\pi^*\theta \circ A_\theta^{\surd}$$

is the *Fefferman metric on* $\sqrt{F}$. As we will see in the next section, the spinor structure (Q_M, f_M) of (M, g_θ) defines a canonical spinor structure on $(\sqrt{F}, h_\theta)$.

Definition 4 *The Lorentzian spin manifold* $(\sqrt{F}, h_\theta)$ *is called the Fefferman space of the strictly pseudoconvex spin manifold* $(M, T_{10}, \theta, (Q_M, f_M))$.

6 Spinor calculus for S^1-bundles with isotropic fibre over strictly pseudoconvex spin manifolds

Let $(M^{2n+1}, T_{10}, \theta)$ be a strictly pseudoconvex manifold and let (Q_M, f_M) be a spinor structure of (M, g_θ). Furthermore, consider an S^1-principle bundle $(B, \pi, M; S^1)$ over M, a connection A on B and a constant $c \in \mathbb{R}\backslash\{0\}$. Then

$$h := h_{A,c} := \pi^* L_\theta - ic\,\pi^*\theta \circ A$$

is a Lorentzian metric on B. In this section we want to derive a suitable spinor calculus for the Lorentzian manifold (B, h).
Let $N \in \Gamma(TB)$ be the fundamental vector field on B defined by the element $\frac{2}{c}i \in i\mathbb{R}$ of the Lie algebra $i\mathbb{R}$ of S^1

$$N(b) = \widetilde{\frac{2}{c}i}\,(b) := \frac{d}{dt}\left(b \cdot e^{\frac{2}{c}it}\right)|_{t=0}.$$

Denote by $T^* \in \Gamma(TB)$ the A-horizontal lift of the characteristic vector field T of θ. Then N and T^* are global isotropic vector fields on B such that $h(N, T^*) = 1$. Consider the global vector fields

$$s_1 = \frac{1}{\sqrt{2}}(N - T^*) \qquad \text{and} \qquad s_2 = \frac{1}{\sqrt{2}}(N + T^*). \tag{42}$$

Then

$$h(s_1,s_1)=-1, \qquad h(s_2,s_2)=1, \qquad h(s_1,s_2)=0.$$

Let the time orientation of (B,h) be given by s_1 and the space orientation by the vectors $(s_2, X_1^*, JX_1^*, \ldots, X_n^*, JX_n^*))$, where $(X_1, JX_1, \ldots, X_n, JX_n) \in P_H$, and X^* denotes the A-horizontal lift of a vector field X on M. Now, let (Q_H, f_H) be the reduced spinor structure of (M, g_θ) defined in the previous section. Denote by

$$S_H := Q_H \times_{\lambda^{-1}(U(n))} \Delta_{2n,0}$$

the corresponding spinor bundle of (H, L_θ). Obviously, the bundle

$$\hat{P}_B := \{(s_1, s_2, X_1^*, JX_1^*, \ldots, X_n^*, JX_n^*) \mid (X_1, JX_1, \ldots, X_n, JX_n) \text{ on-basis of } (H, L_\theta)\}$$

is an $U(n)$-reduction of the frame bundle P_B of (B,h) with respect to the embedding

$$\begin{array}{ccc} U(n) & \longrightarrow & SO_0(2n+2,1) \\ A+iB & \longmapsto & \left(\begin{array}{cc|cc} 1 & & & \\ & 1 & & \\ \hline & & A & B \\ & & -B & A \end{array}\right). \end{array}$$

Since $\hat{P}_B \approx \pi^* P_H$ we have

$$P_B \;\approx\; \pi^* P_H \times_{U(n)} SO_0(2n+2,1).$$

Therefore,

$$Q_B := \pi^* Q_H \times_{\lambda^{-1}(U(n))} \mathrm{Spin}_0(2n+2,1)\,, \qquad f_B := [f_H, \lambda]$$

is a spinor structure of the Lorentzian manifold (B,h). The corresponding spinor bundle S on (B,h) is given by

$$S \;=\; \pi^* Q_H \times_{\lambda^{-1}(U(n))} \Delta_{2n+2,1}. \tag{43}$$

Proposition 18 *Let S_H be the spinor bundle of (H, L_θ) over M. Then the spinor bundle S of (B,h) can be identified with the sum*

$$S \approx \pi^* S_H \oplus \pi^* S_H,$$

where the Clifford multiplication is given by

$$\begin{aligned} s_1 \cdot (\varphi, \psi) &= (-\psi, -\varphi) && (44)\\ s_2 \cdot (\varphi, \psi) &= (-\psi, \varphi) && (45)\\ X^* \cdot (\varphi, \psi) &= (-X \cdot \varphi, X \cdot \psi), \qquad X \in H. && (46) \end{aligned}$$

In particular,

$$N \cdot (\varphi, \psi) = (-\sqrt{2}\,\psi, 0) \tag{47}$$
$$T^* \cdot (\varphi, \psi) = (0, \sqrt{2}\,\varphi). \tag{48}$$

Furthermore, the positive and negative parts of S are

$$S^+ = \pi^* S_H^+ \oplus \pi^* S_H^-, \qquad S^- = \pi^* S_H^- \oplus \pi^* S_H^+. \tag{49}$$

The indefinite scalar product $\langle \cdot, \cdot \rangle$ in S is given by

$$\langle (\varphi, \psi), (\hat{\varphi}, \hat{\psi}) \rangle = -(\psi, \hat{\varphi})_{S_H} - (\varphi, \hat{\psi})_{S_H}, \tag{50}$$

where $(\cdot, \cdot)_{S_H}$ is the usual positive definite scalar product in S_H.

Proof: By definition of the spinor bundle S (see (43)) we have only to check, how the $\mathrm{Spin}(2n)$-modul $\Delta_{2n+2,1}$ decomposes into $\mathrm{Spin}(2n)$-representations. Let the embedding $i : \mathbb{R}^{2n} \to \mathbb{R}^{2n+2,1}$ be given by $i(x) = (0, 0, x)$ and let $\mathrm{Spin}(2n) \hookrightarrow \mathrm{Spin}_0(2n+2, 1)$ be the corresponding embedding of the spin groups. Consider the following isomorphisms of the representation spaces

$$\begin{array}{rccccc} \chi : & \Delta_{2n+2,1} & \longrightarrow & \Delta_{2n,0} & \oplus & \Delta_{2n,0} \\ & u \otimes u(1) + v \otimes u(-1) & \longmapsto & (u & , & v) \end{array}$$

where we use the notation of section 2. Then formula (1) shows that

$$\begin{aligned} \chi\,(e_1 \cdot (u \otimes u(1) + v \otimes u(-1))) &= (-u, -v) \\ \chi\,(e_2 \cdot (u \otimes u(1) + v \otimes u(-1))) &= (u, -v) \\ \chi\,(e_k \cdot (u \otimes u(1) + v \otimes u(-1))) &= (-e_{k-2} \cdot u, e_{k-2} \cdot v), \qquad k > 2. \end{aligned}$$

Therefore, χ is an isomorphism of the $\mathrm{Spin}(2n)$-representations and (44)-(46) and because of (42) also the formulas (47), (48) are valid. Let $\omega_{2n+2} = e_1 \cdot \ldots \cdot e_{2n+2}$ be the volume element of $\mathrm{Cliff}_{2n+2,1}$ and $\omega_{2n} = e_1 \cdots e_{2n}$ the volume element of $\mathrm{Cliff}_{2n,0}$. Then using the identification χ we obtain

$$\omega_{2n+2} \cdot (u, v) = (-\omega_{2n} \cdot u\,,\, \omega_{2n} \cdot v).$$

According to the definition of $S^{\pm}$ this shows (49). Because of (5) the scalar product satisfies

$$\begin{aligned} \langle (\varphi, \psi), (\hat{\varphi}, \hat{\psi}) \rangle &= (s_1 \cdot (\varphi, \psi), (\hat{\varphi}, \hat{\psi}))_{s_1} \\ &= ((-\psi, -\varphi), (\hat{\varphi}, \hat{\psi}))_{s_1} \\ &= -(\psi, \hat{\varphi})_{S_H} - (\varphi, \hat{\psi})_{S_H}. \end{aligned}$$

$\square$

In order to describe the spinor derivative in the spinor bundle S of B we need the connection forms of the Levi-Civita connection of (B,h). Let X,Y,Z be local vector fields on (B,h) of constant length and constant scalar products with each other. Then the Levi-Civita connection ∇ of (B,h) satisfies

$$h(\nabla_X Y, Z) = \frac{1}{2}\{h([X,Y],Z) + h([Z,Y],X) + h([Z,X],Y)\}. \tag{51}$$

For a vector $Z \in T_bB$ we denote by Z^h the projection on the horizontal tangent space and by Z^v the projection on the vertical tangent space. If $X \in T_{\pi(b)}M$, then $X^* \in T_bB$ denotes the horizontal lift of X. Let $\Omega^A \in \Omega^2(M; i\mathbb{R})$ be the curvature form of the connection A. From the connection theory in principle bundles follows for vector fields X,Y on M

$$[X^*, N] \;=\; 0\,, \tag{52}$$

$$[X^*, Y^*]^v \;=\; i\,\frac{c}{2}\,\Omega^A(X,Y)\cdot N\,, \tag{53}$$

$$[X^*, Y^*]^h \;=\; [X,Y]^*\,. \tag{54}$$

Now, let $X,Y \in \Gamma(H)$. Since $[T,X] \in \Gamma(H)$ and

$$[X,Y] = \mathrm{pr}_H[X,Y] + \theta([X,Y])\cdot T = \mathrm{pr}_H[X,Y] - d\theta(X,Y)\cdot T$$

we obtain from (53) and (54)

$$[T^*, X^*] \;=\; [T,X]^* + i\,\frac{c}{2}\,\Omega^A(T,X)\cdot N\,, \tag{55}$$

$$[X^*, Y^*] \;=\; \mathrm{pr}_H[X,Y]^* - d\theta(X,Y)\cdot T^* + i\,\frac{c}{2}\,\Omega^A(X,Y)\cdot N. \tag{56}$$

Proposition 19 *Let $X,Y,Z \in \Gamma(H)$ be vector fields of constant lenght and constant L_θ-scalar products with each other. Then*

$$\begin{aligned}
h(\nabla_{X^*}Y^*, Z^*) &= L_\theta(\nabla^W_X Y, Z)\\
h(\nabla_N Y^*, Z^*) &= \frac{1}{2}d\theta(Y,Z)\\
h(\nabla_{T^*}Y^*, Z^*) &= \frac{1}{2}\{L_\theta([T,Y],Z) - L_\theta([T,Z],Y) - i\frac{c}{2}\Omega^A(Y,Z)\}\\
h(\nabla_{X^*}Y^*, N) &= -\frac{1}{2}d\theta(X,Y)\\
h(\nabla_{X^*}Y^*, T^*) &= \frac{1}{2}\{L_\theta([T,X],Y) + L_\theta([T,Y],X) + i\frac{c}{2}\Omega^A(X,Y)\}\\
h(\nabla_{T^*}T^*, Z^*) &= -i\,\frac{c}{2}\,\Omega^A(T,Z)\\
h(\nabla N, T^*) &= h(\nabla T^*, T^*) \;=\; h(\nabla N^*, N^*) \;=\; 0\\
h(\nabla_N N, Z^*) &= h(\nabla_N T^*, Z^*) \;=\; h(\nabla_{T^*}N, Z^*) \;=\; 0.
\end{aligned}$$

Proof: From (51) and (56) it follows

$$\begin{aligned} 2\,h(\nabla_{X^*}Y^*,Z^*) &= h(\,\mathrm{pr}_H[X,Y]^*,Z^*)+h(\,\mathrm{pr}_H[Z,Y]^*,X^*)+h(\,\mathrm{pr}_H[Z,X]^*,Y^*) \\ &= L_\theta([X,Y],Z)+L_\theta([Z,Y],X)+L_\theta([Z,X],Y). \end{aligned}$$

According to (33) $\mathrm{Tor}^W(X,Y)=L_\theta(JX,Y)\cdot T$. Hence,

$$\begin{aligned} L_\theta([X,Y],Z) &= L_\theta(\nabla^W_X Y-\nabla^W_Y X-\mathrm{Tor}^W(X,Y),Z) \\ &= L_\theta(\nabla^W_X Y-\nabla^W_Y X,Z). \end{aligned}$$

Therefore, using that ∇^W is metric with respect to L_θ we obtain

$$h(\nabla_{X^*}Y^*,Z^*)=L_\theta(\nabla^W_X Y,Z).$$

The other formulas follow immediately from the definition of h and (51), (52), (55) and (56).

□

By definition the spinor derivative on S is given by the following formula:
Let $\tilde{s}:U\longrightarrow Q_H$ be a local section in Q_H and $s=(X_1,\ldots,X_{2n})=f_H(\tilde{s})\in P_H$ the corresponding orthonormal basis in (H,L_θ). Consider a local spinor field $\phi=[\,\tilde{s},u\,]$ in S. Then

$$\begin{aligned} \nabla^S\phi &= [\,\tilde{s},du\,]-\frac{1}{2}h(\nabla s_1,s_2)\,s_1\cdot s_2\cdot\phi-\frac{1}{2}\sum_{k=1}^{2n}h(\nabla s_1,X_k^*)\,s_1\cdot X_k^*\cdot\phi \\ &\quad +\frac{1}{2}\sum_{k=1}^{2n}h(\nabla s_2,X_k^*)\,s_2\cdot X_k^*\cdot\phi+\frac{1}{2}\sum_{k<l}h(\nabla X_k^*,X_l^*)\,X_k^*\cdot X_l^*\cdot\phi. \end{aligned}$$

Using the definition of s_1 and s_2 (see (42)) and Proposition 19 we obtain $h(\nabla s_1,s_2)=0$. Furthermore, if we denote by $a_k(Z)$ the vector field

$$a_k(Z):=h(\nabla_Z s_2,X_k^*)\,s_2-h(\nabla_Z s_1,X_k^*)\,s_1,$$

from Proposition 19 results

$$\begin{aligned} a_k(N) &= 0 \\ a_k(T^*) &= -i\,\frac{c}{2}\,\Omega^A(T,X_k)\cdot N \\ a_k(X_j^*) &= \frac{1}{2}\,d\theta(X_j,X_k)\,T^*-\frac{1}{2}\{L_\theta([T,X_j],X_k)+ \\ &\quad +L_\theta([T,X_k],X_j)+i\,\frac{c}{2}\,\Omega^A(X_j,X_k)\}N. \end{aligned}$$

These formulas and Proposition 19 give the following formulas for the spinor derivative in the spinor bundle S of (B,h):

Proposition 20 *Let $\tilde{s} : U \longrightarrow Q_H$ be a local section in Q_H, $s = f_H(\tilde{s}) = (X_1, \ldots, X_{2n})$ and let $\phi = [\,\tilde{s}, u\,]$ be a local section in S. Then for the spinor derivative of ϕ holds:*

$$\begin{aligned}
\nabla^S_N \phi &= [\,\tilde{s}, N(u)\,] + \frac{1}{4}\, d\theta^* \cdot \phi \\
\nabla^S_{T^*} \phi &= [\,\tilde{s}, T^*(u)\,] + i\,\frac{c}{2}\,(T \,\lrcorner\, \Omega^A)^* \cdot N \cdot \phi - i\,\frac{c}{8}\,(\Omega^A_\theta)^* \cdot \phi \\
&\quad + \frac{1}{4} \sum_{k<l} \{L_\theta([T, X_k], X_l) - L_\theta([T, X_l], X_k)\} X_k^* \cdot X_l^* \cdot \phi \\
\nabla^S_{X^*} \phi &= [\,\tilde{s}, X^*(u)\,] - \frac{1}{4}(X \,\lrcorner\, d\theta)^* \cdot T^* \cdot \phi + i\,\frac{c}{8}\,(X \,\lrcorner\, \Omega^A)^*_\theta \cdot N \cdot \phi \\
&\quad + \frac{1}{4} \sum_{k=1}^{n} \{L_\theta([T, X], X_k) + L_\theta([T, X_k], X)\} X_k^* \cdot N \cdot \phi \\
&\quad + \frac{1}{2} \sum_{k<l} L_\theta(\nabla^W_X X_k, X_l)\, X_k^* \cdot X_l^* \cdot \phi,
\end{aligned}$$

where σ_θ denotes the projection of a form $\sigma \in \Lambda^p M$ onto $\Lambda^p_\theta M$, σ^*_θ is its horizontal lift on B and the vector field X belongs to the set $\{X_1, \ldots, X_{2n}\}$.

Proposition 21 *Let $(X_1, \ldots, X_{2n})$ be a local orthonormal basis of (H, L_θ) with $X_{2\alpha} = J(X_{2\alpha-1})$. Denote by $\sigma^1, \ldots, \sigma^{2n}$ the dual basis of $(X_1, \ldots, X_{2n})$ and by $s = (Z_1, \ldots, Z_n)$, $Z_\alpha = \frac{1}{\sqrt{2}}(X_{2\alpha-1} - iX_{2\alpha})$, the corresponding local unitary basis of (T_{10}, L_θ). Consider the 2-forms*

$$\begin{aligned}
b_s &:= \sum_{k<l} \{L_\theta([T, X_k], X_l) - L_\theta([T, X_l], X_k)\}\, \sigma^k \wedge \sigma^l, \\
d_s(X) &:= \sum_{k<l} L_\theta(\nabla^W_X X_k, X_l)\, \sigma^k \wedge \sigma^l\,, \qquad X \in H.
\end{aligned}$$

Then

1) $b_s \in \Lambda^{1,1}_\theta(M)$ *and* $Tr_\theta\, b_s = 2\, Tr\, \omega_s(T)$

2) $d_s(X) \in \Lambda^{1,1}_\theta(M)$ *and* $Tr_\theta\, d_s(X) = Tr\, \omega_s(X)$,

where ω_s is the matrix of connection forms of the Webster connection ∇^W with respect to $s = (Z_1, \ldots, Z_n)$.

Proof: A 2-form σ belongs to $\Lambda^{1,1} M$ iff $\sigma(JX, JY) = \sigma(X, Y)$ for all $X, Y \in H$. From formula (23) of Proposition 12 follows for $X, Y \in \{X_1, \ldots, X_{2n}\}$

$$\begin{aligned}
b_s(JX, JY) &= L_\theta([T, JX], JY) - L_\theta([T, JY], JX) \\
&\overset{(23)}{=} L_\theta([T, X], Y) - L_\theta([T, Y], X) \\
&= b_s(X, Y).
\end{aligned}$$

Therefore, $b_s \in \Lambda^{1,1}_\theta M$. Furthermore,

$$\begin{aligned}
\mathrm{Tr}_\theta\, b_s &= i \sum_{\alpha=1}^{n} b_s(X_{2\alpha-1}, X_{2\alpha}) \\
&= i \sum_{\alpha=1}^{n} \{ L_\theta([T, X_{2\alpha-1}], X_{2\alpha}) - L_\theta([T, X_{2\alpha}], X_{2\alpha-1}) \}.
\end{aligned}$$

Inserting

$$X_{2\alpha-1} = \frac{1}{\sqrt{2}}(Z_\alpha + \bar{Z}_\alpha), \quad X_{2\alpha} = \frac{i}{\sqrt{2}}(Z_\alpha - \bar{Z}_\alpha)$$

one obtains

$$\begin{aligned}
\mathrm{Tr}_\theta\, b_s &= \sum_{\alpha=1}^{n} \{ L_\theta([T, Z_\alpha], Z_\alpha) - L_\theta([T, \bar{Z}_\alpha], \bar{Z}_\alpha) \} \\
&= 2i \sum_{\alpha=1}^{n} \mathrm{Im}\, \{ L_\theta(\mathrm{pr}_{10}[T, Z_\alpha], Z_\alpha) \} \\
&= 2i \sum_{\alpha=1}^{n} \mathrm{Im}\, L_\theta(\nabla^W_T Z_\alpha, Z_\alpha)) \\
&= 2i\, \mathrm{Im}\, (\, \mathrm{Tr}\, \omega_s(T))\, .
\end{aligned}$$

Since ∇^W is metric with respect to L_θ, we have $\omega_{\alpha\beta} + \overline{\omega_{\beta\alpha}} = 0$. Hence, $\omega_{\alpha\alpha}(T)$ is imaginary. Therefore, $\mathrm{Tr}_\theta\, b_s = 2\, \mathrm{Tr}\, w_s(T)$.
According to formula (35) of Proposition 15 and formula (22) of Proposition 12 we have for $Y, Z \in \{X_1, \ldots, X_{2n}\}$ and $X \in H$

$$\begin{aligned}
d_s(X)(JY, JZ) &= L_\theta(\nabla^W_X JY, JZ) \;=\; L_\theta(J\nabla^W_X Y, JZ) \\
&= L_\theta(\nabla^W_X Y, Z) \;=\; d_s(X)(Y, Z).
\end{aligned}$$

This shows that $d_s(X) \in \Lambda^{1,1}_\theta(M)$. Furthermore,

$$\begin{aligned}
\mathrm{Tr}_\theta\, d_s(X) &= i \sum_{\alpha=1}^{n} L_\theta(\nabla^W_X X_{2\alpha-1}, X_{2\alpha}) \\
&= \frac{1}{2} \sum_{\alpha=1}^{n} \{ L_\theta(\nabla^W_X Z_\alpha, Z_\alpha) - L_\theta(\nabla^W_X \bar{Z}_\alpha, \bar{Z}_\alpha) \} \\
&= i\, \mathrm{Im}\, \mathrm{Tr}\, \omega_s(X) \\
&= \mathrm{Tr}\, \omega_s(X).
\end{aligned}$$

□

Next we proof a property of the spinor bundle S_H of (H, L_θ), which is very similar to the properties of the spinor bundle of Kähler manifolds (see [Kir86]).

Proposition 22 *Let $(M^{2n+1}, T_{10}, \theta)$ be a strictly pseudoconvex spin manifold and $(\sqrt{F}, h_\theta)$ its Fefferman space. Then the spinor bundle S_H of (H, L_θ) has the following properties:*

1. *S_H decomposes into $n+1$ subbundles*
$$S_H = \bigoplus_{r=0}^{n} S_{(-n+2r)i},$$
where S_{ki} is the eigenspace of the endomorphism $d\theta\cdot : S_H \to S_H$ to the eigenvalue ki. The dimension of S_{ki} is $\left(\begin{array}{c} n \\ \frac{n+k}{2} \end{array}\right)$. In particular, there are two 1-dimensional subbundles $S_{\varepsilon ni}$, $\varepsilon = \pm 1$, of S_H satisfying $d\theta \cdot |_{S_{\varepsilon ni}} = \varepsilon ni \cdot Id_{S_{\varepsilon ni}}$.

2. *If $\sigma \in \Lambda^{1,1}_\theta M$, then*
$$\sigma \cdot |_{S_{\varepsilon ni}} = \varepsilon \cdot Tr_\theta(\sigma) \cdot Id_{S_{\varepsilon ni}}.$$

3. *The induced bundles $\pi^* S_{n\varepsilon i}$ on the Fefferman space $\sqrt{F}$ are trivial. A global section $\psi_\varepsilon \in \Gamma(\pi^* S_{n\varepsilon i})$ is given in the following way:*
Let $\tilde{s} : U \longrightarrow Q_H$ be a local section in Q_H, s the local unitary basis in (T_{10}, L_θ), corresponding to $f_H(\tilde{s}) : U \longrightarrow P_H$. Furthermore, let $\sqrt{\tau_s} : U \longrightarrow \sqrt{F}$ be the local section in $\sqrt{F}$ defined by $\tilde{s}$ and let $\varphi_s : \sqrt{F}|_U \longrightarrow S^1$ be given by $p = \sqrt{\tau_s(\pi(p))} \cdot \varphi_s(p)$. Then
$$\psi_\varepsilon(p) := [\, \tilde{s}(\pi(p)), \varphi_s(p)^{-\varepsilon} u(\varepsilon, \cdots, \varepsilon)\,]$$
defines a global section in $\Gamma(\pi^* S_{n\varepsilon i})$.

Proof: $\Delta^{\pm}_{2n,0}$ is a $U(n)$-representation, where $U(n)$ acts by

$$U(n) \xrightarrow{\ell} \mathrm{Spin}^{\mathbb{C}}(2n) \xrightarrow{\Phi_{2n,0}} \mathrm{GL}(\Delta^{\pm}_{2n,0}).$$

The element $\Omega_0 = e_1 \cdot e_2 + \cdots + e_{2n-1} \cdot e_{2n} \in \mathrm{Cliff}^{\mathbb{C}}_{2n,0}$ acts on $\Delta^{\pm}_{2n,0}$ by

$$\Omega_0 \cdot u(\varepsilon_1, \ldots, \varepsilon_n) = i\,(\sum_{k=1}^{n} \varepsilon_k)\, u(\varepsilon_1, \ldots, \varepsilon_n).$$

Hence, $\Delta^{\pm}_{2n,0}$ decomposes into $U(n)$-invariant eigenspaces $E_{\mu_r}(\Omega_0)$ of Ω_0 to the eigenvalues $\mu_r = (-n+2r)i$, $r = 0, \ldots, n$. In particular, the eigenspace to the eigenvalue εni, $\varepsilon = \pm 1$, is 1-dimensional and given by

$$E_{in\varepsilon}(\Omega_0) = \mathbb{C} \cdot u(\varepsilon, \ldots, \varepsilon).$$

By definition of ℓ (see (38)) we obtain for $A = \mathrm{diag}(e^{i\theta_1}, \ldots e^{i\theta n})$

$$\ell(A) u(\varepsilon, \ldots, \varepsilon) = \left\{ \begin{array}{cc} u(\varepsilon, \ldots, \varepsilon) & \varepsilon = -1 \\ \mathrm{Det} A \cdot u(\varepsilon, \ldots, \varepsilon) & \varepsilon = 1. \end{array} \right. \tag{57}$$

Hence, E_{-ni} is the trivial $U(n)$-representation and E_{ni} is isomorphic to the $U(n)$-representation $\Lambda^n(\mathbb{C}^n)$. Since the subspaces $E_{\mu_r}(\Omega_0)$ of $\Delta^{\pm}_{2n,0}$ are invariant under the action of $\lambda^{-1}(U(n))$ we obtain the decomposition

$$S_H = \bigoplus_{r=0}^{n} S_{\mu_r},$$

where $S_{\mu_r} := Q_H \times_{\lambda^{-1}(U(n))} E_{\mu_r}(\Omega_0)$.
If $\tilde{s} : U \longrightarrow Q_H$ is a local section in Q_H, $d\theta$ acts on S_H by

$$d\theta \cdot [\,\tilde{s}\,,\, v\,] = [\,\tilde{s}\,,\, \Omega_0 \cdot v\,].$$

Therefore, S_{μ_r} is the eigenspace of $d\theta\cdot$ to the eigenvalue μ_r.
Now, let $\eta = [\,q\,,\, u(\varepsilon,\dots,\varepsilon)\,] \in S_{\varepsilon ni}\,,\ \varepsilon = \pm 1$. Denote $f_H(q) = (X_1,\dots,X_{2n}) \in P_H\,,$ $X_{2\alpha} = JX_{2\alpha-1}$ and $s = (Z_1,\dots,Z_n)$ the corresponding unitary basis in (T_{10}, L_θ) with $Z_\alpha = \frac{1}{\sqrt{2}}(X_{2\alpha-1} - iJX_{2\alpha-1})$. Let $(\theta^1,\dots,\theta^n)$ be the dual basis of $(Z_1,\dots,Z_n)$ and $(\sigma^1,\dots,\sigma^{2n})$ the dual basis of $(X_1,\dots,X_{2n})$. If $\sigma \in \Lambda^{1,1}_\theta M$ is a form of type (1,1), then

$$\begin{aligned}
\sigma &= \sum_{\alpha,\beta=1}^{n} \sigma_{\alpha\beta}\,\theta^\alpha \wedge \overline{\theta^\beta}\\
&= \frac{1}{2}\sum_{\alpha\neq\beta} \sigma_{\alpha\beta}\,(\sigma^{2\alpha-1}\wedge\sigma^{2\beta-1} + \sigma^{2\alpha}\wedge\sigma^{2\beta}) + \frac{i}{2}\sum_{\alpha,\beta}\sigma_{\alpha\beta}\,(\sigma^{2\alpha}\wedge\sigma^{2\beta-1} - \sigma^{2\alpha-1}\wedge\sigma^{2\beta}).
\end{aligned}$$

Hence,

$$\begin{aligned}
\sigma\cdot\eta = [\,q\,,\quad & \tfrac{1}{2}\sum_{\alpha\neq\beta}\sigma_{\alpha\beta}\,(e_{2\alpha-1}\cdot e_{2\beta-1} + e_{2\alpha}\cdot e_{2\beta})\cdot u(\varepsilon,\dots,\varepsilon)\\
& + \tfrac{i}{2}\sum_{\alpha,\beta}\sigma_{\alpha\beta}\,(e_{2\alpha}\cdot e_{2\beta-1} - e_{2\alpha-1}\cdot e_{2\beta})\cdot u(\varepsilon,\dots,\varepsilon)\,]
\end{aligned}$$

where $\sigma_{\alpha\beta} = \sigma(Z_\alpha, \bar{Z}_\beta)$. Using formula (1) we obtain

$$\begin{aligned}
(e_{2\alpha-1}\cdot e_{2\beta-1} + e_{2\alpha}\cdot e_{2\beta})\cdot u(\varepsilon,\dots,\varepsilon) &= 0 && \alpha\neq\beta\\
(e_{2\alpha}\cdot e_{2\beta-1} - e_{2\alpha-1}\cdot e_{2\beta})\cdot u(\varepsilon,\dots,\varepsilon) &= 0 && \alpha\neq\beta\\
(e_{2\alpha}\cdot e_{2\alpha-1} - e_{2\alpha-1}\cdot e_{2\alpha})\cdot u(\varepsilon,\dots,\varepsilon) &= -2\varepsilon i\,u(\varepsilon,\dots,\varepsilon).
\end{aligned}$$

Therefore,

$$\begin{aligned}
\sigma\cdot\eta &= [\,q\,,\,\varepsilon\sum_\alpha \sigma(Z_\alpha,\bar{Z}_\alpha)\cdot u(\varepsilon,\dots,\varepsilon)\,]\\
&= \varepsilon\cdot \mathrm{Tr}_\theta\,\sigma\cdot\eta.
\end{aligned}$$

Now, let us consider the section $\psi_\varepsilon \in \Gamma(\pi^* S_{\varepsilon ni})$ defined by

$$\psi_\varepsilon(p) := [\,\tilde{s}(\pi(p))\,,\,\varphi_s(p)^{-\varepsilon} u(\varepsilon,\dots,\varepsilon)\,].$$

Let $\tilde{s}, \tilde{\hat{s}} : U \longrightarrow Q_H$ be two local sections, $\tilde{s} = \tilde{\hat{s}} \cdot g$ and let $h : U \longrightarrow S^1$ be the function defined by (39):

$$\ell(\lambda(g)) \cdot h = g, \quad h^2 = \mathrm{Det}(\lambda(g))^{-1}. \tag{58}$$

Then $\varphi_{\hat{s}}(p) = \varphi_s(p) \cdot h(\pi(p))$ and

$$\begin{aligned}
\psi_\varepsilon(p) &= [\,\tilde{\hat{s}} \cdot g \,,\, \varphi_s(p)^{-\varepsilon} u(\varepsilon, \ldots, \varepsilon)\,] \\
&= [\,\tilde{\hat{s}} \,,\, \varphi_s(p)^{-\varepsilon}\, g \cdot u(\varepsilon, \ldots, \varepsilon)\,] \\
&= [\,\tilde{\hat{s}} \,,\, \varphi_{\hat{s}}(p)^{-\varepsilon}\, h^\varepsilon\, g \cdot u(\varepsilon, \ldots, \varepsilon)\,] \\
&\overset{(58)}{=} [\,\tilde{\hat{s}} \,,\, \varphi_{\hat{s}}(p)^{-\varepsilon} h^{\varepsilon+1} \ell(\lambda(g))\, u(\varepsilon, \ldots, \varepsilon)\,] \\
&\overset{(57),(58)}{=} [\,\tilde{\hat{s}} \,,\, \varphi_{\hat{s}}(p)^{-\varepsilon}\, u(\varepsilon, \ldots, \varepsilon)\,].
\end{aligned}$$

Hence, ψ_ε is a global section in the bundle $\pi^* S_{\varepsilon n i}$ on $\sqrt{F}$.

□

7 Twistor spinors on Fefferman spaces

Let $(M^{2n+1}, T_{10}, \theta)$ be a strictly pseudoconvex spin manifold, $(\sqrt{F}, \pi, M)$ the square root of the canonical S^1-bundle corresponding to the spinor structure of (M, g_θ) and h_θ the Fefferman metric on $\sqrt{F}$. Denote by $\psi_\varepsilon \in \Gamma(\pi^* S_H)$ the global sections in the bundles $\pi^* S_{\varepsilon n i}$ over $\sqrt{F}$ defined in Proposition 22. Now, we are able to solve the twistor equation on the Lorentzian spin manifold $(\sqrt{F}, h_\theta)$.

Theorem 1 *Let $S := \pi^* S_H \oplus \pi^* S_H$ be the spinor bundle of $(\sqrt{F}, h_\theta)$. Then the spinor fields $\phi_\varepsilon := (\psi_\varepsilon, 0) \in \Gamma(S)$, $\varepsilon = \pm 1$, are solutions of the twistor equation on $(\sqrt{F}, h_\theta)$ with the following properties:*

1. *The canonical vector field V_{ϕ_ε} of ϕ_ε is a regular isotropic Killing vector field.*
2. $V_{\phi_\varepsilon} \cdot \phi_\varepsilon = 0$.
3. $\nabla^S_{V_{\phi_\varepsilon}} \phi_\varepsilon = -\frac{1}{\sqrt{2}}\, \varepsilon\, i\, \phi_\varepsilon$.
4. $\|\phi_\varepsilon\|_\xi \equiv 1$.

Remark: If n is even, then ϕ_1 and ϕ_{-1} are linearly independent spinor fields in S^+. If n is odd then $\phi_1 \in \Gamma(S^+)$ and $\phi_{-1} \in \Gamma(S^-)$ (see Proposition 18). The second property of Theorem 1 shows that ϕ_ε is a pure or partially pure spinor field (see [TT94]). A vector field is called *regular*, if all of its integral curves are closed and of the same shortest period.

Proof of Theorem 1: We use the formulas for the spinor derivative in S given in Proposition 20 for the Fefferman connection $A = A_\theta^{\surd}$ and the constant $c = \frac{8}{n+2}$. Let $\tilde{s} : U \longrightarrow Q$ be a local section and $\varphi_s : \sqrt{F}|_U \longrightarrow S^1$ the corresponding transition function in $\sqrt{F}$ (see Proposition 22). Then for the fundamental vector field N on $\sqrt{F}$

$$N(\varphi_s) = \frac{n+2}{4} i \, \varphi_s \tag{59}$$

holds. If Y^* is an $A_\theta^{\surd}$-horizontal lift of a vector field Y on M, we obtain using standard formulas from connection theory

$$\begin{aligned} Y^*(\varphi_s) &= -\varphi_s \cdot \sqrt{\tau_s}^* A_\theta^{\surd}(Y) \\ &= \frac{1}{2} \varphi_s \{ \operatorname{Tr} \omega_s(Y) + \frac{i}{2(n+1)} R^W \theta(Y) \}, \end{aligned} \tag{60}$$

where ω_s is the matrix of connection forms of the Webster connection with respect to the unitary basis s in (T_{10}, L_θ) corresponding to $f_H(\tilde{s})$. According to Proposition 18 we have $N \cdot \phi_\varepsilon = 0$. Therefore, from Proposition 20 and (59), (60) result

$$\begin{aligned} \nabla^S_N \phi_\varepsilon &= \Big(-\varepsilon \frac{n+2}{4} i \, \psi_\varepsilon + \frac{1}{4} d\theta \cdot \psi_\varepsilon \, , \, 0 \Big) \\ \nabla^S_{T^*} \phi_\varepsilon &= \Big(-\frac{1}{2} \varepsilon \{ \operatorname{Tr} \omega_s(T) + \frac{i}{2(n+1)} R^W \} \psi_\varepsilon + \frac{1}{4} b_s \cdot \psi_\varepsilon - i \frac{1}{n+2} \Omega_\theta^{A_\theta^{\surd}} \cdot \psi_\varepsilon \, , \, 0 \Big) \\ \nabla^S_{X^*} \phi_\varepsilon &= \Big(-\frac{1}{2} \varepsilon \operatorname{Tr} \omega_s(X) \, \psi_\varepsilon + \frac{1}{2} d_s(X) \cdot \psi_\varepsilon \, , \, 0 \Big) - \frac{1}{4} (X \,\lrcorner\, d\theta)^* \cdot T^* \cdot \phi_\varepsilon \, , \end{aligned}$$

where b_s and $d_s(X)$ are the $\Lambda^{1,1}$-forms defined in Proposition 21. Since ψ_ε is a section in $\pi^* S_{\varepsilon ni}$, b_s and $d_s(X)$ act on ψ_ε by multiplication with $\varepsilon \operatorname{Tr}_\theta b_s$ and $\varepsilon \operatorname{Tr}_\theta d_s(X)$, respectively (Proposition 22). Hence, according to Proposition 21,

$$\begin{aligned} \nabla^S_{T^*} \phi_\varepsilon &= \Big(-i \frac{1}{n+2} \Omega_\theta^{A_\theta^{\surd}} \cdot \psi_\varepsilon - \varepsilon \frac{i}{4(n+1)} R^W \psi_\varepsilon \, , \, 0 \Big) \\ \nabla^S_{X^*} \phi_\varepsilon &= -\frac{1}{4} (X \,\lrcorner\, d\theta)^* \cdot T^* \cdot \phi_\varepsilon . \end{aligned}$$

Furthermore, ψ_ε is an eigenspinor of the action of $d\theta$ on S_H to the eigenvalue $\varepsilon n i$. Therefore,

$$\nabla^S_N \phi_\varepsilon = -\frac{\varepsilon}{2} i \, \phi_\varepsilon \, . \tag{61}$$

Because of

$$\Omega_\theta^{A_\theta^{\surd}} = -\frac{1}{2} \operatorname{Ric}_\theta^W - \frac{i}{4(n+1)} d(R^W \theta)_\theta = -\frac{1}{2} \operatorname{Ric}_\theta^W - \frac{i}{4(n+1)} R^W d\theta,$$

the curvature $\Omega_\theta^{A_\theta^\surd}$ of the Fefferman connection is a form of type (1,1). Hence,

$$\begin{aligned}
\Omega_\theta^{A_\theta^\surd} \cdot \psi_\varepsilon &= \varepsilon \ \mathrm{Tr}_\theta(\Omega^{A_\theta^\surd}) \, \psi_\varepsilon \\
&= (-\frac{1}{2} \varepsilon \, R^W - \frac{i\varepsilon}{4(n+1)} R^W \, in) \, \psi_\varepsilon \\
&= -\varepsilon \frac{n+2}{4(n+1)} R^W \, \psi_\varepsilon .
\end{aligned}$$

Therefore, we obtain

$$\nabla^S_{T^*} \phi_\varepsilon = 0. \tag{62}$$

According to Proposition 18, $T^* \cdot \phi_\varepsilon = (0, \sqrt{2}\psi_\varepsilon)$. If $X \in \{X_1, \ldots, X_{2n}\}$, the 1-form $X \ \lrcorner \ d\theta$ acts on the spinor bundle by Clifford multiplication with $J(X)$. Hence, we have

$$\nabla^S_{X^*} \phi_\varepsilon = \left(0 \, , \, -\frac{\sqrt{2}}{4} J(X) \cdot \psi_\varepsilon \right) . \tag{63}$$

Now, using $s_1 = \frac{1}{\sqrt{2}}(N - T^*)$, $s_2 = \frac{1}{\sqrt{2}}(N + T^*)$, we obtain

$$-s_1 \cdot \nabla^S_{s_1} \phi_\varepsilon \ = \ s_2 \cdot \nabla^S_{s_2} \phi_\varepsilon \ = \ X^* \cdot \nabla^S_{X^*} \phi_\varepsilon \ = \ \left(0, -\frac{1}{2\sqrt{2}} \, \varepsilon \, i \, \psi_\varepsilon \right) ,$$

where $X \in \{X_1, \ldots, X_{2n}\}$. This shows, that ϕ_ε is a twistor spinor (see Proposition 1). From Proposition 18 it follows

$$(\phi_\varepsilon, \phi_\varepsilon)_\xi \ = \ \langle s_1 \cdot \phi_\varepsilon, \phi_\varepsilon \rangle \ = \ \langle \, (0, -\psi_\varepsilon), (\psi_\varepsilon, 0) \, \rangle \ = \ (\psi_\varepsilon, \psi_\varepsilon)_{S_H} \ = 1 \, .$$

Furthermore, we obtain for the canonical vector field V_{ϕ_ε}

$$\begin{aligned}
V_{\phi_\varepsilon} \ &= \ \langle s_1 \cdot \phi_\varepsilon, \phi_\varepsilon \rangle \, s_1 - \langle s_2 \cdot \phi_\varepsilon, \phi_\varepsilon \rangle \, s_2 - \sum_{k=1}^{2n} \langle X_k^* \cdot \phi_\varepsilon, \phi_\varepsilon \rangle X_k^* \\
&= \ s_1 + s_2 \ = \ \sqrt{2} \, N.
\end{aligned}$$

Therefore, V_{ϕ_ε} is regular and isotropic and satisfies $V_{\phi_\varepsilon} \cdot \phi_\varepsilon = 0$. Because of (61) we have

$$\nabla^S_{V_{\phi_\varepsilon}} \phi_\varepsilon = -\frac{1}{\sqrt{2}} \, \varepsilon \, i \, \phi_\varepsilon .$$

It remains to show, that the vertical vector field N is a Killing vector field. This follows directly from the formulas of Proposition 19:

$$L_N h_\theta(Y, Z) = h_\theta(\nabla_Y N, Z) + h_\theta(Y, \nabla_Z N) = 0$$

for all vector fields Y and Z on $\sqrt{F}$.

□

Conversely, we have

Theorem 2 *Let (B^{2n+2}, h) be a Lorentzian spin manifold and let $\varphi \in \Gamma(S)$ be a non-trivial twistor spinor on (B, h) such that*

1. *The canonical vector field V_φ of φ is a regular isotropic Killing vector field.*
2. $V_\varphi \cdot \varphi = 0\,.$
3. $\nabla^S_{V_\varphi} \varphi = i\,c\,\varphi\,, \qquad c = const \in \mathbb{R}\backslash\{0\}.$

Then B is an S^1-principal bundle over a strictly pseudoconvex spin manifold $(M^{2n+1}, T_{10}, \theta)$ and (B, h) is locally isometric to the Fefferman space $(\sqrt{F}, h_\theta)$ of (M, T_{10}, θ).

Proof: Since V_φ is regular, it defines an S^1-action on B

$$\begin{array}{rcl} B \times S^1 & \longrightarrow & B \\ (p, e^{it}) & \longmapsto & \gamma^V_{t \cdot \frac{L}{2\pi}}(p) \end{array}$$

where $\gamma^V_t(p)$ is the integral curve of $V = V_\varphi$ through p and L is the period of the integral curves. Then $M := B/_{S^1}$ is an $2n+1$-dimensional manifold and V is the fundamental vector field defined by the element $\frac{2\pi}{L}i$ of the Lie algebra $i\mathbb{R}$ of S^1 in the S^1-principal bundle $(B, \pi, M; S^1)$. Now we use Sparling's characterization of Fefferman spaces, proved by Graham in [Gra87]. Let W denote the (4,0)-Weyl tensor, C the (3,0)-Schouten-Weyl tensor and K the (2,0)-Schouten tensor of (B, h). Graham proved:
If V is an isotropic Killing vector field such that

$$V \,\lrcorner\, W = 0 \tag{64}$$
$$V \,\lrcorner\, C = 0 \tag{65}$$
$$K(V, V) = \text{const} < 0, \tag{66}$$

then there exists a pseudo-hermitian structure (T_{10}, θ) on M such that (B, h) is locally isometric to the Fefferman space (F, h_θ) of (M, T_{10}, θ). The local isometry is given by S^1-equivariant bundle maps $\phi_U : B|_U \longrightarrow F|_U$.
We first prove that $V = V_\varphi$ satisfies (64)-(66). Property (65) is valid for each twistor spinor (see Proposition 10). Using $W(X \wedge Y) \cdot \varphi = 0$ (see (11) of Proposition 5) and the assumption $V_\varphi \cdot \varphi = 0$ we obtain

$$\begin{aligned} 0 &= \{W(X \wedge Y) \cdot V - V \cdot W(X \wedge Y)\} \cdot \varphi \\ &= 2\{V \,\lrcorner\, W(X \wedge Y)\} \cdot \varphi \\ &= 2\,W(X, Y, V) \cdot \varphi \end{aligned}$$

for all vector fields X and Y on B. Since V_φ is a nontrivial isotropic Killing field, it has no zeros. Hence, by Proposition 6, the twistor spinor φ has no zeros and therefore, the vector field $W(X,Y,V)$ must be isotropic for all vector fields X, Y on B. Because of

$$W(X,Y,V,V) = h(W(X,Y,V),V) = 0\,,$$

$W(X,Y,V)$ is orthogonal to the isotropic vector field V. Since (B,h) has Lorentzian signature, it follows that there is a 2-form λ on B such that

$$W(X,Y,V) = \lambda(X,Y)\,V \qquad \text{for all } X,Y \in \Gamma(TB). \tag{67}$$

Now, we use formula (12) of Proposition 5 to obtain

$$\begin{aligned} 0 &= V \cdot W(X \wedge Y) \cdot D\varphi - n\,\{V \cdot C(X,Y) + C(X,Y) \cdot V\} \cdot \varphi \\ &= V \cdot W(X \wedge Y) \cdot D\varphi + 2n\,C(V,X,Y)\,\varphi. \end{aligned}$$

Because of $V \lrcorner\ C = 0$ it results

$$V \cdot W(X \wedge Y) \cdot D\varphi = 0. \tag{68}$$

From the twistor equation (6) and the assumption $\nabla^S_V \varphi = i\,c\,\varphi$ it follows

$$\begin{aligned} W(X \wedge Y) \cdot V \cdot D\varphi &= -n\; W(X \wedge Y) \cdot \nabla^S_V \varphi \\ &= -nic\; W(X \wedge Y) \cdot \varphi \\ &\overset{(11)}{=} 0\,. \end{aligned} \tag{69}$$

Then (67), (68) and (69) give

$$\begin{aligned} 0 &= W(X \wedge Y) \cdot V \cdot D\varphi - V \cdot W(X \wedge Y) \cdot D\varphi \\ &= 2\; W(X,Y,V) \cdot D\varphi \\ &= 2\lambda(X,Y)\,V \cdot D\varphi \\ &\overset{(6)}{=} -2n\;\lambda(X,Y)\,\nabla^S_V \varphi \\ &= -2nci\,\lambda(X,Y)\,\varphi. \end{aligned}$$

Therefore, $\lambda \equiv 0$ and $V \lrcorner\ W = 0$. Using formula (10) of Proposition 5 we obtain

$$V \cdot \nabla^S_V D\varphi \;=\; \frac{n}{2}\,\{V \cdot K(V) + K(V) \cdot V\} \cdot \varphi \;=\; -n\,K(V,V)\varphi.$$

Since V is an isotropic Killing field, it satisfies $\nabla_V V = 0$. It follows

$$\nabla^S_V(V \cdot D\varphi) \;=\; \nabla_V V \cdot D\varphi + V \cdot \nabla^S_V D\varphi \;=\; -n\,K(V,V)\,\varphi$$

and from the twistor equation

$$\nabla^S_V \nabla^S_V \varphi = K(V,V)\,\varphi.$$

Using $\nabla^S_V \varphi = ic\varphi$ we obtain $K(V,V) = -c^2$. Therefore, the canonical vector field V_φ of the twistor spinor φ satisfies the conditions of Sparling's characterization theorem for Fefferman metrics. Now, we proceed as in Graham's proof of that theorem. Since $V_{\alpha\varphi} = |\alpha|^2 V_\varphi$ we can normalize φ in such a way that $K(V_\varphi, V_\varphi) = -\frac{1}{4}$. Then, let $\tilde{T}$ be the vector field on B defined by

$$h(\tilde{T}, X) = -4\, K(X, V_\varphi)\,, \qquad X \in \Gamma(TB).$$

$\tilde{T}$ is isotropic and $h(\tilde{T}, V_\varphi) = 1$. Then we can use V_φ and $\tilde{T}$ to reduce the spinor structure of the Lorentzian manifold (B,h) to the group $\mathrm{Spin}(2n)$. This reduced spinor structure projects to a spinor structure of (H, L_θ), where θ is the projection of the 1-form $\tilde{\theta} \in \Omega^1(B)$ dual to V_φ and $H \subset TM$ is the projection of the subbundle $\tilde{H} = \mathrm{span}(\tilde{T}, V_\varphi)^\perp \subset TB$ onto M. $J : H \to H$ is given by projection of the map

$$\begin{array}{rcl} \tilde{J} : TB & \longrightarrow & TB \\ X & \longmapsto & 2\,\nabla_X V_\varphi\,, \end{array}$$

which acts on $\tilde{H}$ with $\tilde{J}^2 = -id$. Then in [Gra87] is proved that (M,H,J,θ) in fact is a strictly pseudoconvex manifold which we equip with the spinor structure arising from that of (H, L_θ) by enlarging the structure group. In the same way as in [Gra87] it follows that (B,h) is locally isometric to the Fefferman space $(\sqrt{F}, h_\theta)$, where the isometries are given by S^1-bundle maps $\sqrt{F}|_U \longrightarrow B|_U$.

$\square$

Remark: Jerison and Lee studied the Yamabe problem on CR-manifolds (see [JL87]). They proved that there is a numerical CR-invariant $\lambda(M)$ associated with every compact oriented strictly pseudoconvex manifold M^{2n+1}, which is always less than or equal to the value corresponding to the sphere S^{2n+1} in $\mathbb{C}^n$ with its standard CR-structure. If $\lambda(M)$ is strictly less than $\lambda(S^{2n+1})$, then M admits a pseudo-hermitian structure θ with constant Webster scalar curvature $R^W = \lambda(M)$. Furthermore, one knows that the scalar curvature R of the Fefferman metric h_θ is a constant positive multiple of the lift of the Webster scalar curvature R^W to the Fefferman space (see [Lee86]) . Now, let (M^{2n+1}, T_{10}) be a compact strictly pseudoconvex spin manifold with $0 \neq \lambda(M) < \lambda(S^{2n+1})$. Choose a pseudo-hermitian structure θ on (M, T_{10}) such that the Webster scalar curvature R^W is constant (and non-zero since $\lambda(M) \neq 0$).
Let ϕ_ε, $\varepsilon = \pm 1$, be the twistor spinors on $(\sqrt{F}, h_\theta)$, defined in Theorem 1. Then according to the remark following Proposition 5 the spinor fields

$$\eta_{\varepsilon,\pm} := \frac{1}{2}\,\phi_\varepsilon \pm \sqrt{\frac{2n+1}{(2n+2)R}}\; D\phi_\varepsilon$$

are eigenspinors of the Dirac operator of the Lorentzian spin manifold $(\sqrt{F}, h_\theta)$ to the eigenvalue $\pm\frac{1}{2}\sqrt{\frac{2n+2}{(2n+1)}R}$. The length the spinor fields $\eta_{\varepsilon,\pm}$ is constant with respect to the indefinite scalar product $\langle\cdot,\cdot\rangle$ as well as to the positive definite scalar product $(\cdot,\cdot)_\xi$.

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