Clifford structures and spinor bundles

Thomas Friedrich
Institut für Reine Mathematik, Humboldt Universität
Ziegelstrasse 13 A, 10099 Berlin, Germany

Andrzej Trautman
Instytut Fizyki Teoretycznej, Uniwersytet Warszawski
Hoża 69, 00681 Warszawa, Poland

January 22, 1997

Abstract

It is shown that every bundle $\Sigma \to M$ of complex, irreducible and faithful modules over the Clifford bundle of an even-dimensional Riemannian space (M, g) with local model (V, h) is associated with a cpin ("Clifford") structure on M, this being an extension of the $\mathsf{SO}(h)$-bundle of orthonormal frames on M to the Clifford group $\mathsf{Cpin}(h) = (\mathbb{C}^\times \times \mathsf{Spin}(h))/\mathbb{Z}_2$. An explicit construction is given of the total space of the $\mathsf{Cpin}(h)$-bundle defining such a structure. A canonical line bundle on a cpin manifold, associated with the spinor norm homomorphism, is identified with a subbundle of $\mathsf{Hom}(\Sigma, \Sigma^*)$. The cpin structure restricts to a spin structure iff this line bundle is trivial.

1 Introduction

Spinor fields on Riemannian manifolds can be introduced in at least two ways. Let (V, h) be the local model of a Riemannian manifold (M, g). If such a manifold has a spin structure ("if M is spin"),

$$\begin{array}{ccccc} \mathsf{Spin}(h) & \longrightarrow & Q_0 & & \\ \big\downarrow{\scriptstyle \mathrm{Ad}} & & \big\downarrow{\scriptstyle \chi_0} & & \\ \mathsf{SO}(h) & \longrightarrow & P & \xrightarrow{\pi} & M, \end{array} \tag{1}$$

then, given a spinor representation of $\mathsf{Spin}(h)$ in a vector space S, one defines spinor fields as sections of the associated vector bundle $(Q_0 \times S)/\mathsf{Spin}(h) \to M$. Another definition [1] focuses on the vector bundle itself: it assumes the existence of a bundle $\Sigma \to M$ of modules over the bundle $\mathsf{Cl}(g)$ of Clifford algebras on M. The latter definition is more general in the sense that the fibres of Σ need not be isomorphic to a spinor space, carrying an irreducible representation of the Clifford algebra; for example, one can take for Σ the bundle $\wedge T^*M$ of exterior algebras. One easily sees that a bundle associated by a spinor representation with a spin structure is a bundle of modules over $\mathsf{Cl}(g)$ (see, e.g., Prop. 3.8 in [2]), but the converse is not true, even if $\mathsf{Cl}(g_x) \to \mathsf{End}\,\Sigma_x$ is the spinor representation for every $x \in M$ (Example 2 in Section 4 of this paper).

In this paper, we compare in some detail these two definitions and show that a bundle Σ of irreducible modules over $\mathsf{Cl}(g)$ is associated with a cpin structure on M, this being an extension

$$\begin{array}{ccccc} \mathsf{Cpin}(h) & \longrightarrow & Q & & \\ \big\downarrow{\scriptstyle \mathrm{Ad}} & & \big\downarrow{\scriptstyle \chi} & & \\ \mathsf{SO}(h) & \longrightarrow & P & \xrightarrow{\pi} & M, \end{array} \tag{2}$$

of the $\mathsf{SO}(h)$-bundle of orthonormal frames P to the (special) Clifford group [3]

$$\mathsf{Cpin}(h) = (\mathbb{C}^\times \times \mathsf{Spin}(h))/\mathbb{Z}_2. \tag{3}$$

In particular, if (M, g) is proper Riemannian, then the form h is definite and $\mathsf{Spin}^{\mathbb{C}}(h) = (\mathsf{U}_1 \times \mathsf{Spin}(h))/\mathbb{Z}_2$ is the maximal compact subgroup of $\mathsf{Cpin}(h)$: in this case every irreducible module over $\mathsf{Cl}(g)$ is associated with a spin$^{\mathbb{C}}$-structure on M; *cf.* Appendix D in [2].

2 Notation and preliminaries

We use a notation and terminology which are largely standard in differential geometry and spinor analysis [2, 4]. If S and S' are finite-dimensional complex vector spaces, then $\mathsf{Hom}(S, S')$ is the vector space of all complex-linear maps of S into S' and $\mathsf{End}\,S = \mathsf{Hom}(S, S)$ is an algebra over $\mathbb{C}$. We write $S^* = \mathsf{Hom}(S, \mathbb{C})$; if $f \in \mathsf{Hom}(S, S')$, then $f^* \in \mathsf{Hom}(S'^*, S^*)$ is defined by $\langle s, f^*(t')\rangle$ for every $s \in S$ and $t' \in S'^*$. A similar notation is used for real vector spaces.

A quadratic space is defined as a pair (V, h), where V is a real vector space of dimension m and $h : V \to \mathbb{R}$ is a positive-definite quadratic form. We denote by $\tilde{h}$ the symmetric linear isomorphism of V onto V^*, associated with h. The real Clifford algebra $\mathsf{Cl}(h)$ corresponding to (V, h) contains $\mathbb{R} \oplus V$ as a vector subspace and $v^2 = h(v)$ for every $v \in V$. The isometry $v \mapsto -v$ extends to an involution α of the algebra defining its $\mathbb{Z}_2$-grading: $\mathsf{Cl}(h) = \mathsf{Cl}^0(h) \oplus \mathsf{Cl}^1(h)$.

All manifolds and their maps are assumed to be smooth. If $\pi : E \to M$ is a fibre bundle over a manifold M, then $E_x = \pi^{-1}(x) \subset E$ is the fibre over $x \in M$; in particular, $T_x M \subset TM$ is the tangent vector space to M at x.

3 Spin spaces and the Clifford group

For simplicity, we restrict ourselves in this paper to *even*-dimensional spaces and proper Riemannian manifolds. It is not difficult to generalize our considerations to odd-dimensional spaces and to pseudo-Riemannian manifolds. If the dimension m of V is even, $m = 2n$, then $\mathsf{Cl}(h)$ is central simple and has one, up to complex equivalence, *spinor representation* in a *complex*, 2^n-dimensional vector space S (of spinors).

We define a *spin space* to be a triple (S, V, h), where S is a complex vector space of dimension 2^n and $h : V \to \mathbb{R}$ is a positive-definite quadratic form on the real, $2n$-dimensional vector subspace V of $\mathsf{End}\,S$ such that, for every $v \in V$, one has $v^2 = h(v)\,\mathrm{id}_S$. Given a spin space (S, V, h), one can identify $\mathsf{Cl}(h)$ with the subalgebra of $\mathsf{End}\,S$ generated, over the reals, by $V \subset \mathsf{End}\,S$. Over the complex numbers, the subspace V generates $\mathbb{C} \otimes \mathsf{Cl}(h) = \mathsf{End}\,S$; declaring the elements of V to be odd, one obtains a $\mathbb{Z}_2$-grading of the latter algebra, $\mathsf{End}\,S = \mathsf{End}^0 S \oplus \mathsf{End}^1 S$. An *isomorphism of spin spaces* (S, V, h) and (S', V', h') is defined as an isomorphism $\ell : S \to S'$ of complex vector spaces such that $\ell V \ell^{-1} = V'$.

Proposition 1 *To every isometry i of (V, h) onto (V', h') there corresponds a complex line*

$$L(i) = \{\ell \in \mathsf{Hom}(S, S') \mid i(v)\ell = \ell v \quad \text{for every} \quad v \in V\}.$$

Its non-zero elements are isomorphisms of (S, V, h) onto (S', V', h').

Proof. Indeed, the isometry i extends to an isomorphism of algebras, $\mathsf{Cl}(h) \to \mathsf{Cl}(h')$ and thus yields a faithful and irreducible representation of $\mathsf{Cl}(h)$ in S'; every isomorphism ℓ intertwining the representations of $\mathsf{Cl}(h)$ in S and S' belongs to $L(i)$. If $\ell, \ell' \in L(i)$ and $\ell \neq 0$, then $\ell^{-1}\ell'$ is in the commutant of $V \subset \mathsf{End}\,S$; therefore, it is a multiple of the identity.

The (special) *Clifford group*[1] $\mathsf{Cpin}(h)$ is defined here as the group of all even automorphisms of the spin space (S, V, h),

$$\mathsf{Cpin}(h) = \{a \in \mathsf{End}^0 S \mid a \text{ is invertible and } aVa^{-1} = V\} \subset \mathbb{C} \otimes \mathsf{Cl}^0(h).$$

The spin group $\mathsf{Spin}(h)$ is the subgroup of $\mathsf{Cpin}(h)$ consisting of the Clifford products of elements of all sequences of an even number of unit vectors; $\mathsf{Spin}(h) \subset \mathsf{Cl}^0(h)$. Let $\mathbb{C}^\times = \mathbb{C} \setminus \{0\}$ be the multiplicative group of complex numbers. There is a surjective homomorphism of groups, $\mathbb{C}^\times \times \mathsf{Spin}(h) \to \mathsf{Cpin}(h)$, given by $(\mu, a) \mapsto \mu a$; since its kernel is isomorphic to $\mathbb{Z}_2$, one has the isomorphism (3).

The adjoint representation of the Clifford group in V gives rise to the exact sequence

$$1 \to \mathbb{C}^\times \to \mathsf{Cpin}(h) \xrightarrow{\mathrm{Ad}} \mathsf{SO}(h) \to 1.$$

4 Clifford structures, bundles and modules

Consider now an oriented $2n$-dimensional manifold M with a metric tensor field g, having an oriented quadratic space (V, h) as a local model; the quadratic spaces (T_xM, g_x) and (V, h) are isometric for every $x \in M$. One defines the $\mathsf{SO}(h)$-bundle $P \to M$ of orthonormal frames of coherent orientation on M by putting

$$P = \{p : V \to T_xM \mid p \text{ is an orientation preserving isometry, } x \in M\}.$$

A "Clifford" or cpin structure on M is given by the maps (2) where $\pi \circ \chi : Q \to M$ is a $\mathsf{Cpin}(h)$-bundle and $\chi(qa) = \chi(q) \circ \mathrm{Ad}(a)$ holds for every $q \in Q$ and $a \in \mathsf{Cpin}(h)$. Since $\mathsf{Spin}(h)$ is a subgroup of $\mathsf{Cpin}(h)$, if M has a spin structure, then it has a cpin structure, but not conversely: there are spin$^{\mathbb{C}}$-manifolds that are not spin. If (M, g) is almost Hermitean, i.e. if it has an orthogonal almost complex structure, then it has a spin$^{\mathbb{C}}$ structure and, therefore, a cpin structure.

With every Riemannian manifold (M, g) there is associated the *Clifford bundle* $\mathsf{Cl}(g) \to M$. Its total space is defined as

$$\mathsf{Cl}(g) = \bigcup_{x \in M} \mathsf{Cl}(g_x)$$

with a natural manifold structure. A *bundle of Clifford modules* over $\mathsf{Cl}(g)$ is a vector bundle $\Sigma \to M$ together with a *Clifford morphism* defined here as a linear injective bundle map

$$f : TM \to \mathsf{End}\,\Sigma$$

having the Clifford property: for every $v \in T_xM$ one has $f(v)^2 = g_x(v)\mathrm{id}_{\Sigma_x}$ [2, 6]. We identify TM with its image in $\mathsf{End}\,\Sigma$. It follows from the universality of Clifford algebras that f extends to a "representation morphism", i.e. to a bundle map $F : \mathsf{Cl}(g) \to \mathsf{End}\,\Sigma$ such that $F_x \stackrel{\mathrm{def}}{=} F \mid \mathsf{Cl}(g_x)$ is a representation

$$F_x : \mathsf{Cl}(g_x) \to \mathsf{End}\,\Sigma_x \tag{4}$$

for every $x \in M$. In other words, the vector space Σ_x is a left module over the algebra $\mathsf{Cl}(g_x)$. The following examples are well-known:

Example 1. The bundle of *exterior algebras* on M. Put $\Sigma = \wedge T^*M$ and define f by $f(v)\sigma = v \lrcorner \sigma + \tilde{g}_x(v) \wedge \sigma$ for $v \in T_xM$ and $\sigma \in \Sigma_x$.

Example 2. Let (M, g) be an *almost Hermitean space* and let J be the associated orthogonal almost complex structure. Define $N = \{n \in \mathbb{C} \otimes TM \mid J(n) = \sqrt{-1}n\}$

[1] This name was introduced by Chevalley [3]. R. Lipschitz was the first to consider groups associated with Clifford algebras; see [5] and the references given there.

and put $\Sigma = \wedge N$. The map f given, for every $n \in N$ and $\sigma \in \Sigma$, by $f(n + \bar{n})\sigma = \sqrt{2}(\tilde{g}(\bar{n}) \lrcorner \sigma + n \wedge \sigma)$ is a Clifford morphism.

Example 3. *Bundles of spinors.* Let (2) be a cpin structure on M and let $\Sigma = (Q \times S)/\mathsf{Cpin}(h)$ be the associated bundle of spinors. Every element of Σ is of the form $[(q, s)]$, where $q \in Q$, $s \in S$ and $[(q, s)] = [(q', s')]$ iff there is $a \in \mathsf{Cpin}(h)$ such that $q' = qa$ and $s' = a^{-1}s$. In this case, the Clifford morphism is given by $f(v)[(q, s)] = [(q, (\chi(q)^{-1}v)s)]$, where $v \in T_xM$, $q \in Q_x$ and $s \in S$; note that $\chi(q) : V \to T_xM$ is an isometry; one easily checks that the above definition is correct.

5 Spinor bundles

The main result of this paper is the proof of the converse of the statement appearing in Example 3. We say that a bundle of Clifford modules $\Sigma \to M$ over $\mathsf{Cl}(g)$ is a *spinor bundle* if the representation (4) is a spinor representation for every $x \in M$. For example, the bundle of Clifford modules described in Example 2 is a spinor bundle.

Proposition 2 *Every spinor bundle $\Sigma \to M$ on an even-dimensional Riemannian manifold (M, g) with local model (V, h) is isomorphic to the vector bundle $(Q \times S)/\mathsf{Cpin}(h) \to M$ associated with a cpin structure (2) on that manifold.*

Proof. Given a spinor bundle $\Sigma \to M$, one constructs the total space Q of the cpin structure by taking as the fibre Q_x the set of all isomorphisms of the spin space (S, V, h) onto the spin space (Σ_x, T_xM, g_x). The map $\chi : Q_x \to P_x$ is given by $\chi(q) : V \to T_xM$, $\chi(q)v = qvq^{-1}$. If q and $q' \in Q_x$, then $q^{-1}q' \in \mathsf{Cpin}(h)$; the group $\mathsf{Cpin}(h)$ acts freely and transitively on Q_x and $\chi(qa) = \chi(q) \circ \mathrm{Ad}(a)$. It remains to check that the associated bundle of spinors $(Q \times S)/\mathsf{Cpin}(h) \to M$ is isomorphic to $\Sigma \to M$: such an isomorphism is given by $[(q, s)] \mapsto q(s)$, where $q \in Q$ and $s \in S$.

6 A line bundle associated with spinor bundles

The reflection $v \mapsto \alpha(v) = -v$, is an isometry of (V, h) onto itself; if (S, V, h) is a spin space and the dimension of V is $2n$, then the line $L(\alpha)$, defined in Section 3, is spanned by $e_{2n+1} = e_1 \dots e_{2n} \in \mathsf{GL}(S)$, where $(e_1, \dots, e_{2n})$ is an orthonormal frame in (V, h).

If $v \in V \subset \mathsf{End}\, S$, then $v^* \in \mathsf{End}\, S^*$ and $(v^*)^2 = (v^2)^* = h(v)\mathrm{id}_{S^*}$. Therefore, defining

$$V^\bullet = \{v^* \mid v \in V\} \quad \text{and} \quad h^\bullet(v^*) = h(v),$$

we obtain a spin space $(S^*, V^\bullet, h^\bullet)$ isomorphic to (S, V, h). The map $\beta : V \to V^\bullet$ given by $\beta(v) = (-1)^n v^*$ is an isometry; therefore, there is the line $L \overset{\text{def}}{=} L(\beta) \subset \mathsf{Hom}(S, S^*)$; its elements are either symmetric or skew: if $\ell \in L$, then $\ell^* = (-1)^{\frac{1}{2}n(n+1)}$. (The factor $(-1)^n$ in the definition of β is chosen so that $e_\tau^* = (-1)^n \ell e_\tau \ell^{-1}$ holds not only for $\tau = 1, \dots, 2n$, but also for $\tau = 2n + 1$; see §101 in [7]).

The group $\mathsf{Cpin}(h)$ has a one-dimensional representation in L and acts transitively on the space $L^\times = L \setminus \{0\}$: the element $a \in \mathsf{Cpin}(h)$ sends $\ell \in L$ to $a^*\ell a = \mathrm{N}(a)\ell$, where $\mathrm{N} : \mathsf{Cpin}(h) \to \mathbb{C}^\times$ is the *spinor norm homomorphism.* The group $\mathsf{Spin}(h)$ is now seen to coincide with the kernel of N. Consider now a manifold with a cpin structure (2). The representation of $\mathsf{Cpin}(h)$ in L defines the complex line bundle

$$\Lambda = (Q \times L)/\mathsf{Cpin}(h) \to M. \tag{5}$$

Proposition 3 *Let (M, g) be a Riemannian cpin manifold with a local model (V, h). This manifold is spin if, and only if, the line bundle (5) is trivial.*

Proof. Assume first that (5) is trivial. A nowhere vanishing section of this bundle corresponds to a map$\lambda : Q \to L^\times$ such that $\lambda(qa) = \mathrm{N}(a^{-1})\lambda(q)$, where $q \in Q$ and $a \in \mathsf{Cpin}(h)$. Let $\ell_0 \in L^\times$. The set $Q_0 = \{q \in Q \,|\, \lambda(q) = \ell_0\}$ is the total space of a spin structure on M. Conversely, given a reduction Q_0 of Q to $\mathsf{Spin}(h)$, one constructs a nowhere vanishing section of (5) by defining the equivariant map λ so that $\lambda(q) = \ell_0$ for every $q \in Q_0$.

The group $\mathsf{Cpin}(h)$ can be interpreted as the *conformal spin group* [8, 9]. Defining $\mathsf{CO}(h) = (\mathbb{C}^\times \times \mathsf{SO}(h))/\mathbb{Z}_2$, one has the exact sequence

$$1 \to \mathbb{Z}_2 \to \mathsf{Cpin}(h) \xrightarrow{\rho} \mathsf{CO}(h) \to 1,$$

where ρ is given by $\rho(a)v = \mathrm{N}(a)ava^{-1}$ for $a \in \mathsf{Cpin}(h)$ and $v \in V$.

Acknowledgments

Work on this paper was started in November 1996 in Berlin, during the second author's visit to the Institut für Reine Mathematik of Humboldt University. That visit was financed by the Sonderforschungsbereich 288 *Differentialgeometrie und Quantenphysik*. This research was also supported in part with funds provided by the Foundation for Polish-German cooperation.

References

[1] G. Karrer, Einführung von Spinoren auf Riemannschen Mannigfaltigkeiten, *Ann. Academiae Sci. Fennicae* Ser. A, Math. 336/**5** (1963) 1–16.

[2] H. Blaine Lawson, Jr. and M.-L. Michelsohn, *Spin Geometry*, Princeton University Press, Princeton, 1989.

[3] C. Chevalley, *The Algebraic Theory of Spinors*, Columbia University Press, New York, 1954.

[4] S. Kobayashi and K. Nomizu, *Foundations of Differential Geometry*, vol. I and 2, Interscience, New York, 1963 and 1969.

[5] R. Lipschitz, Correspondence, from an ultramundane correspondent, *Ann. of Math.* **69** (1959) 247–251.

[6] J. J. Duistermaat, *The Heat Kernel Lefschetz Fixed Point Formula for the Spin-c Dirac Operator*, Birkhäuser, Basel, 1996.

[7] É. Cartan, *Leçons sur la théorie des spineurs*, vols 1 and 2, Exposés de Géométrie, Hermann, Paris, 1938.

[8] F. Reese Harvey, *Spinors and Calibrations*, Academic Press, Boston, 1990.

[9] I. Robinson and A. Trautman, The conformal geometry of complex quadrics and the fractional-linear form of Möbius transformations, *J. Math. Phys.* **34** (1993) 5391–5406.