Pseudo-Riemannian T-duals of compact Riemannian reductive spaces

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Abstract

The aim of this paper is the construction of pseudo-Riemannian homogeneous spaces with special curvature properties such as Einstein spaces etc. using corresponding known compact Riemannian ones. This construction is based on the notion of a certain duality between compact and non-compact homogeneous spaces.

Contents

1 Introduction

The aim of this paper is to give a construction method for pseudo-Riemannian homogeneous spaces with special curvature properties such as Einstein spaces etc. using corresponding known compact Riemannian ones. This construction generalizes the duality appearing between compact and non-compact symmetric spaces. We start with a compact Riemannian reductive space G/H with left-invariant metric g and corresponding Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$. We fix a decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ of $\mathfrak{g}$ such that $[\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}$ holds. Now let $T : \mathfrak{g} \to \mathfrak{g}$ be an involutive automorphism. We denote by $\mathfrak{g}_+$ and $\mathfrak{g}_-$ the eigenspaces of T with eigenvalues 1 and -1, respectively. Then $\mathfrak{g}_+$ is a subalgebra and we have $[\mathfrak{g}_+, \mathfrak{g}_-] \subset \mathfrak{g}_-$, $[\mathfrak{g}_-, \mathfrak{g}_-] \subset \mathfrak{g}_+$. Conversely, any decomposition of $\mathfrak{g}$ of such a kind defines an involutive automorphism T. We now use this decomposition (or equivalently this automorphism) in order to assign to the Lie algebra $\mathfrak{g}$ a non-compact real form $\mathfrak{g}'$ of the complexified Lie algebra $\mathfrak{g}^{\mathbb{C}}$. We define

$$\mathfrak{g}' := \mathfrak{g}_+ \oplus i\mathfrak{g}_- \subset \mathfrak{g}^{\mathbb{C}}$$

Assume that $T(\mathfrak{h}) = \mathfrak{h}$ and $T(\mathfrak{m}) = \mathfrak{m}$. In particular, $\mathfrak{h}' = \mathfrak{h} \cap \mathfrak{g}_+ \oplus i(\mathfrak{h} \cap \mathfrak{g}_-)$ is a subalgebra of $\mathfrak{g}'$ and with $\mathfrak{m}' = \mathfrak{m} \cap \mathfrak{g}_+ \oplus i(\mathfrak{m} \cap \mathfrak{g}_-)$ we have $\mathfrak{g}' = \mathfrak{h}' \oplus \mathfrak{m}'$ and $[\mathfrak{h}', \mathfrak{m}'] \subset \mathfrak{m}'$. Now let g be a left-invariant metric on G/H. Then g corresponds to an $\mathrm{ad}(\mathfrak{h})$-invariant inner product $\langle \cdot\,, \cdot \rangle$ on $\mathfrak{m}$, i.e.

$$\langle \mathrm{ad}(Z)(Z_1), Z_2 \rangle + \langle Z_1, \mathrm{ad}(Z)(Z_2) \rangle = 0$$

holds for all $Z \in \mathfrak{h}$, $Z_1, Z_2 \in \mathfrak{m}$. Suppose that the involution $T|_{\mathfrak{m}} : \mathfrak{m} \to \mathfrak{m}$ is an isometry, i.e. $\mathfrak{m} \cap \mathfrak{g}_+ \perp \mathfrak{m} \cap \mathfrak{g}_-$. Then we can define by

$$\langle iX_1 + Y_1, iX_2 + Y_2 \rangle' := -\langle X_1, X_2 \rangle + \langle Y_1, Y_2 \rangle$$

for $X_1, X_2 \in \mathfrak{m} \cap \mathfrak{g}_-$, $Y_1, Y_2 \in \mathfrak{m} \cap \mathfrak{g}_+$ an $\mathrm{ad}'(\mathfrak{h}')$-invariant indefinit inner product $\langle \cdot\,, \cdot \rangle'$ on $\mathfrak{m}'$. t induces an involutive automorphism on $\mathfrak{so}(\mathfrak{m})$ by the decomposition

$$\begin{aligned}
\mathfrak{so}(\mathfrak{m})_+ &= \{A \in \mathfrak{so}(\mathfrak{m}) \mid A(\mathfrak{m} \cap \mathfrak{g}_+) \subset \mathfrak{g}_+ \,,\, A(\mathfrak{m} \cap \mathfrak{g}_-) \subset \mathfrak{g}_-\} \\
\mathfrak{so}(\mathfrak{m})_- &= \{B \in \mathfrak{so}(\mathfrak{m}) \mid B(\mathfrak{m} \cap \mathfrak{g}_-) \subset \mathfrak{g}_+ \,,\, B(\mathfrak{m} \cap \mathfrak{g}_+) \subset \mathfrak{g}_-\}
\end{aligned}$$

and $\mathfrak{so}(\mathfrak{m})'$ can be identified with $\mathfrak{so}(\mathfrak{m}')$.

If G' and H' are Lie groups with Lie algebras $\mathfrak{g}'$ and $\mathfrak{h}'$ and if g' is a left-invariant metric on G'/H' which corresponds to $\langle \cdot\,, \cdot \rangle'$ then we will say that $(G'/H', g')$ is T-dual to $(G/H, g)$. In Section 4 we will investigate curvature properties of T-dual spaces. We will prove that the curvature tensors R and R' of $(G/H, g)$ and $(G'/H', g')$ satisfy at the origins of G/H and G'/H'

$$\begin{aligned}
R'(iX_1, iX_2) = -R(X_1, X_2) &\in \mathfrak{so}(\mathfrak{m})_+ \subset \mathfrak{so}(\mathfrak{m})' \simeq \mathfrak{so}(\mathfrak{m}') \\
R'(iX, Y) = iR(X, Y) &\in i\mathfrak{so}(\mathfrak{m})_- \subset \mathfrak{so}(\mathfrak{m})' \simeq \mathfrak{so}(\mathfrak{m}') \\
R'(Y_1, Y_2) = R(Y_1, Y_2) &\in \mathfrak{so}(\mathfrak{m})_+ \subset \mathfrak{so}(\mathfrak{m})' \simeq \mathfrak{so}(\mathfrak{m}')
\end{aligned}$$

for all $X, X_1, X_2 \in \mathfrak{m} \cap \mathfrak{g}_-$ and $Y, Y_1, Y_2 \in \mathfrak{m} \cap \mathfrak{g}_+$. This implies that the curvature properties of $(G'/H', g')$ are very similar to those of $(G/H, g)$. For instance, if $(G/H, g)$ is an Einstein space then so is $(G'/H', g')$.

Let J be a left-invariant almost complex structure on $(G/H, g)$. Then J corresponds to a uniquely determined linear map $I : \mathfrak{m} \to \mathfrak{m}$ with properties $I^2 = -\mathrm{Id}$ and $I \circ \mathrm{Ad}(h) = \mathrm{Ad}(h) \circ I$ for all $h \in H$. Let $(G'/H', g')$ be T-dual to $(G/H, g)$ and H' be connected. In case $I(\mathfrak{m} \cap \mathfrak{g}_+) \subset \mathfrak{m} \cap \mathfrak{g}_+$ and $I(\mathfrak{m} \cap \mathfrak{g}_-) \subset \mathfrak{m} \cap \mathfrak{g}_-$ we can think of I as a map on $\mathfrak{m}'$. Then I induces an almost complex structure on G'/H'. If J is integrable, then so is J'. If g is hermitian or Kählerian then g' is hermitian or parallel. In case

$I(\mathfrak{m}\cap\mathfrak{g}_+) = \mathfrak{m}\cap\mathfrak{g}_-$ the map iI defined on $\mathfrak{m}'$ induces a bilagrangian structure on G'/H'. Again, if J is integrable, then so is J'. If g is hermitian or Kählerian then g' is hermitian or parallel. As an application twistor and reflector spaces of pseudo-Riemannian spheres and projective spaces are discussed in Section 6.

The last section deals with the original aim of this construction. In contrary to compact Riemannian manifolds only few pseudo-Riemannian ones with Killing spinors are known. Some of them were discussed in the context of $G^*_{2(2)}$- structures in [K2]. Here we construct by T-duality new homogeneous examples using the known compact Riemannian ones.

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2 Some facts on homogeneous spaces

We recall here some well-known facts concerning invariant connections on reductive spaces (see [KN1], [KN2]). Let G be a connected Lie group and H a closed subgroup of G. Let $\mathfrak{g}$ and $\mathfrak{h}$ denote the Lie algebras of G and H, respectively. Let $M = G/H$ be reductive, i.e. assume that there exists a subspace $\mathfrak{m}$ of $\mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{h}\oplus\mathfrak{m}$ and $\mathrm{Ad}(H)\mathfrak{m}\subset\mathfrak{m}$. If H is connected the last condition is equivalent to $[\mathfrak{h},\mathfrak{m}]\subset\mathfrak{m}$. We fix such a decomposition $\mathfrak{g} = \mathfrak{h}\oplus\mathfrak{m}$. Let $e\in G$ denote the unity and $x_0 = e\cdot H \in G/H$ the origin. The isotropy representation of G on the tangent space $T_{x_0}M$ of G/H in x_0 is defined by

$$\begin{array}{rcl} \rho_0 : H & \longrightarrow & GL(T_{x_0}M) \\ h & \longmapsto & dh_{x_0} \end{array}$$

We identify $T_{x_0}M$ with $\mathfrak{m}$ by

$$\begin{array}{rcl} \mathfrak{m} & \stackrel{\sim}{\longrightarrow} & T_{x_0}M \\ X & \longmapsto & \dfrac{d}{dt}\exp tX\cdot H|_{t=0} \end{array}$$

With this identification the isotropy representation of H becomes the adjoint representation on $\mathfrak{m}$. Furthermore, we identify the tangent bundle TM of M with the associated vector bundle $G\times_{\rho_0} T_{x_0} = G\times_{\mathrm{Ad}}\mathfrak{m}$ by

$$G\times_{\rho_0} T_{x_0} \ni [g,Y] \longmapsto dl_g(Y)\in TM,$$

where $l_g : G\to G$ denotes the left translation by g. Let now V be a finite dimensional real or complex vector space and $\rho : H\to GL(V)$ a representation. We consider the associated principal $GL(V)$-bundle $P = G\times_\rho GL(V)$ over $M = G/H$. Denote by $\mathrm{Id}\in GL(V)$ the identity and by u_0 the point $u_0 := [e,\mathrm{Id}]\in P$. G acts on P by left translations:

$$\begin{array}{rcl} G\times P & \longrightarrow & P \\ (g_0,[g,A]) & \longmapsto & L_{g_0}[g,A] := [g_0 g, A] \end{array}$$

where $g_0, g \in G, A \in GL(V)$.
The following result is due to Wang [KN1]. There is a one-to-one correspondence between the G-invariant connections in P and the $\mathbb{R}$-linear maps $\Lambda_{\mathfrak{m}} : \mathfrak{m} \to \mathfrak{gl}(V)$ satisfying

$$\Lambda_{\mathfrak{m}}(\mathrm{Ad}(h)X) = \rho(h)\Lambda_{\mathfrak{m}}(X)\rho(h)^{-1}$$

for all $X \in \mathfrak{m}$ and $h \in H$. The connection form ω of the G-invariant connection corresponding to $\Lambda_{\mathfrak{m}}$ is given by

$$\omega_{u_0}(\widetilde{X}) = \begin{cases} \Lambda_{\mathfrak{m}}(X) & \text{if} \quad X \in \mathfrak{m} \\ d\rho(X) & \text{if} \quad X \in \mathfrak{h}. \end{cases}$$

Here $\widetilde{X}$ denotes the fundamental vector field associated to X, i.e. $\widetilde{X}_{u_0} = \frac{d}{dt} L_{\exp tX} u_0|_{t=0}$.

Now we consider the vector bundle $E = G \times_\rho V$ over M. We identify sections of E with H-equivariant maps from G to V:

$$\begin{array}{rcl} \Gamma(G \times_\rho V) & \stackrel{\sim}{\longrightarrow} & C^\infty(G, V)^H \\ & & = \{\widetilde{\varphi} \in C^\infty(G, V) \mid \widetilde{\varphi}(g \cdot h) = \rho(h^{-1})\widetilde{\varphi}(g) \text{ for all } g \in G, h \in H\} \\ \varphi & \longmapsto & \widetilde{\varphi}, \end{array}$$

where $\varphi(gH) = [g, \widetilde{\varphi}(g)]$.

Now let Z be a G-invariant connection on P, ω the corresponding connection form and $\Lambda_{\mathfrak{m}} : \mathfrak{m} \to \mathfrak{gl}(V)$ the associated linear map. Z induces a covariant derivative $\nabla : \Gamma(E) \to \Gamma(T^*M \otimes E)$ on E. Using the dual representation $\mathrm{Ad}^* : H \to GL(\mathfrak{m}^*)$ of the adjoint representation we identify $T^*M \otimes E$ with $G \times_{\mathrm{Ad}^* \otimes \rho} m^* \otimes V$. Hence, if $\varphi \in \Gamma(E)$ is a section in E then $\nabla\varphi \in \Gamma(T^*M \otimes E)$ corresponds to an H-equivariant function

$$\widetilde{\nabla\varphi} : G \longrightarrow \mathfrak{m}^* \otimes V.$$

Let $X_1, \ldots, X_n$ be a basis of $\mathfrak{m}$ and $\omega^1, \ldots, \omega^n$ a dual basis of $\mathfrak{m}^*$. If $\varphi \in \Gamma(E)$ is a section in E, then the corresponding function $\widetilde{\nabla\varphi} : G \to \mathfrak{m}^* \otimes V$ of $\nabla\varphi$ equals

$$\widetilde{\nabla\varphi} = \sum_{i=1}^n (X_i(\widetilde{\varphi}) + \Lambda_{\mathfrak{m}}(X_i)\widetilde{\varphi}), \tag{1}$$

where $(\Lambda_{\mathfrak{m}}(X_i)\widetilde{\varphi})(g) = \Lambda_{\mathfrak{m}}(X_i)\widetilde{\varphi}(g)$ for all $g \in G$, and $X_i(\widetilde{\varphi})$ is the derivative of $\widetilde{\varphi}$ in direction of the left invariant vector field on G induced by $X_i \in \mathfrak{m} \subset \mathfrak{g}$.

Now let g be a left-invariant pseudo-Riemannian metric on G/H. g corresponds uniquely to an (indefinit) Ad-invariant inner product $\langle \cdot , \cdot \rangle$ on $\mathfrak{m}$, i.e. $\langle \mathrm{Ad}(h)X, \mathrm{Ad}(h)Y \rangle = \langle X, Y \rangle$ holds for any $h \in H$, $X, Y \in \mathfrak{m}$. Vice versa, any $\mathrm{Ad}(H)$-invariant inner product on $\mathfrak{m}$ induces a left-invariant metric g on M because of $TM = G \times_{\mathrm{Ad}} \mathfrak{m}$. If H is connected then $\langle \cdot , \cdot \rangle$ is $\mathrm{Ad}(H)$-invariant if and only if it is $\mathrm{ad}(\mathfrak{h})$-invariant, i.e. if and only if $\langle \mathrm{ad}(Z)Z_1, Z_2 \rangle + \langle Z_1, \mathrm{ad}(Z)Z_2 \rangle = 0$ holds for all $Z \in \mathfrak{h}$ and $Z_1, Z_2 \in \mathfrak{m}$. In the following $\mathcal{Z}$ denotes the Levi-Civita connection associated to g on $G \times_{\mathrm{Ad}} SO^+(\mathfrak{m})$

$\subset G \times_{\mathrm{Ad}} GL(\mathfrak{m})$, ω denotes the connection form and ∇ the associated covariant derivative in $G \times_{\mathrm{Ad}} \mathfrak{m}$. The corresponding map $\Lambda_{\mathfrak{m}} : \mathfrak{m} \to \mathfrak{gl}(\mathfrak{m})$ satisfies $\Lambda_{\mathfrak{m}}(\mathfrak{m}) \subset \mathfrak{so}(\mathfrak{m})$. It can be calculated in the following way:

$$\begin{aligned} \Lambda_{\mathfrak{m}} &: \mathfrak{m} \longrightarrow \mathfrak{so}(\mathfrak{m}) \\ \Lambda_{\mathfrak{m}}(X)Y &= \tfrac{1}{2}[X,Y]_{\mathfrak{m}} + U(X,Y); \qquad X, Y \in \mathfrak{m} \end{aligned}$$

where $[X,Y]_{\mathfrak{m}}$ is the $\mathfrak{m}$-component of $[X,Y]_{\mathfrak{m}} \in \mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ and $U : \mathfrak{m} \times \mathfrak{m} \to \mathfrak{m}$ is the symmetric map defined by

$$2\langle U(X,Y), Z\rangle = \langle X, [Z,Y]_{\mathfrak{m}}\rangle + \langle [Z,X]_{\mathfrak{m}}, Y\rangle \text{ for all } Z \in \mathfrak{m}.$$

3 The T-dual space

Let $\mathfrak{g}$ be a compact Lie algebra and $\mathfrak{h} \subset \mathfrak{g}$ a . Let $(\mathfrak{g}, \mathfrak{h})$ be reductive, i.e. assume that there exists an $\mathrm{ad}(\mathfrak{h})$-invariant complement $\mathfrak{m}$ of $\mathfrak{h}$ in $\mathfrak{g}$. We fix such a decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m} \qquad [\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}.$$

Now let $T : \mathfrak{g} \to \mathfrak{g}$ be an involutive automorphism. We denote by

$$\mathfrak{g}_+ := \{X \in \mathfrak{g} \mid TX = X\} \tag{2}$$
$$\mathfrak{g}_- := \{X \in \mathfrak{g} \mid TX = -X\} \tag{3}$$

the eigenspaces of T with eigenvalue 1 and -1, respectively. Then $\mathfrak{g}_+$ is a subalgebra and we have

$$[\mathfrak{g}_+, \mathfrak{g}_-] \subset \mathfrak{g}_-\,, \quad [\mathfrak{g}_-, \mathfrak{g}_-] \subset \mathfrak{g}_+. \tag{4}$$

Conversely, any decomposition of $\mathfrak{g}$ of such a kind defines an involutive automorphism T.

We now use this decomposition (or equivalently this automorphism) in order to assign to the Lie algebra $\mathfrak{g}$ a non-compact real form $\mathfrak{g}'$ of the complexified Lie algebra $\mathfrak{g}^{\mathbb{C}}$. We define

$$\mathfrak{g}' := \mathfrak{g}_+ \oplus i\mathfrak{g}_- \subset \mathfrak{g}^{\mathbb{C}}$$

Assume that $T(\mathfrak{h}) = \mathfrak{h}$ and $T(\mathfrak{m}) = \mathfrak{m}$. This is equivalent to $\mathfrak{h} = \mathfrak{h} \cap \mathfrak{g}_+ \oplus \mathfrak{h} \cap \mathfrak{g}_-$ and $\mathfrak{m} = \mathfrak{m} \cap \mathfrak{g}_+ \oplus \mathfrak{m} \cap \mathfrak{g}_-$, respectively. In particular, $\mathfrak{h}' = \mathfrak{h} \cap \mathfrak{g}_+ \oplus i(\mathfrak{h} \cap \mathfrak{g}_-)$ is a subalgebra of $\mathfrak{g}'$ and with $\mathfrak{m}' = \mathfrak{m} \cap \mathfrak{g}_+ \oplus i(\mathfrak{m} \cap \mathfrak{g}_-)$ we have $\mathfrak{g}' = \mathfrak{h}' \oplus \mathfrak{m}'$ and $[\mathfrak{h}', \mathfrak{m}'] \subset \mathfrak{m}'$. From (4) we obtain

Proposition 3.1 *The adjoint representations*

$$\begin{aligned} \mathrm{ad} &: \mathfrak{h} \longrightarrow \mathfrak{gl}(\mathfrak{m}) \\ \mathrm{ad}' &: \mathfrak{h}' \longrightarrow \mathfrak{gl}(\mathfrak{m}') \end{aligned}$$

satisfy

$$\begin{aligned}
&\operatorname{ad}'(iX_1)(iX_2) = -\operatorname{ad}(X_1)(X_2) \in \mathfrak{m}\cap\mathfrak{g}_+ = \mathfrak{m}'\cap\mathfrak{g}_+ \quad \text{for } X_1 \in \mathfrak{h}\cap\mathfrak{g}_-, X_2 \in \mathfrak{m}\cap\mathfrak{g}_-\\
&\operatorname{ad}'(iX)(Y) = i\operatorname{ad}(X)(Y) \in \mathfrak{m}'\cap i\mathfrak{g}_- = i(\mathfrak{m}\cap\mathfrak{g}_-) \quad \text{for } X \in \mathfrak{h}\cap\mathfrak{g}_-, Y \in \mathfrak{m}\cap\mathfrak{g}_+\\
&\operatorname{ad}'(Y)(iX) = i\operatorname{ad}(Y)(X) \in \mathfrak{m}'\cap i\mathfrak{g}_- = i(\mathfrak{m}\cap\mathfrak{g}_-) \quad \text{for } Y \in \mathfrak{h}\cap\mathfrak{g}_+, X \in \mathfrak{m}\cap\mathfrak{g}_-\\
&\operatorname{ad}'(Y_1)(Y_2) = -\operatorname{ad}(Y_1)(Y_2) \in \mathfrak{m}\cap\mathfrak{g}_+ = \mathfrak{m}'\cap\mathfrak{g}_+ \quad \text{for } Y_1 \in \mathfrak{h}\cap\mathfrak{g}_+, Y_2 \in \mathfrak{m}\cap\mathfrak{g}_+.
\end{aligned}$$

Now let $\langle\cdot,\cdot\rangle$ be an $\operatorname{ad}(\mathfrak{h})$-invariant inner product on $\mathfrak{m}$, i.e. assume that

$$\langle \operatorname{ad}(Z)(Z_1), Z_2\rangle + \langle Z_1, \operatorname{ad}(Z)(Z_2)\rangle = 0$$

holds for all $Z \in \mathfrak{h}$, $Z_1, Z_2 \in \mathfrak{m}$. Suppose that the involution $T : \mathfrak{m} \to \mathfrak{m}$ is an isometry, i.e. $\mathfrak{m}\cap\mathfrak{g}_+ \perp \mathfrak{m}\cap\mathfrak{g}_-$. Then we can define by

$$\langle iX_1 + Y_1, iX_2 + Y_2\rangle' := -\langle X_1, X_2\rangle + \langle Y_1, Y_2\rangle$$

for $X_1, X_2 \in \mathfrak{m}\cap\mathfrak{g}_-$, $Y_1, Y_2 \in \mathfrak{m}\cap\mathfrak{g}_+$ an indefinit inner product $\langle\cdot,\cdot\rangle'$ on $\mathfrak{m}'$. $\langle\cdot,\cdot\rangle'$ is $\operatorname{ad}'(\mathfrak{h}')$-invariant, i.e.

$$\langle \operatorname{ad}'(iX+Y)(iX_1+Y_1), iX_2+Y_2\rangle' + \langle iX_1+Y_1, \operatorname{ad}'(iX+Y)(iX_2+Y_2)\rangle' = 0$$

holds for all $X \in \mathfrak{h}\cap\mathfrak{g}_-$, $Y \in \mathfrak{h}\cap\mathfrak{g}_+$, $X_1, X_2 \in \mathfrak{m}\cap\mathfrak{g}_-$, $Y_1, Y_2 \in \mathfrak{m}\cap\mathfrak{g}_+$ since we obtain from Proposition 3.1

$$\begin{aligned}
&\langle \operatorname{ad}'(iX+Y)(iX_1+Y_1), iX_2+Y_2\rangle'\\
&= \langle -\operatorname{ad}(X)(X_1) + i\operatorname{ad}(Y)(X_1) + i\operatorname{ad}(X)(Y_1) + \operatorname{ad}(Y)(Y_1), iX_2+Y_2\rangle'\\
&= -\langle \operatorname{ad}(Y)(X_1) + \operatorname{ad}(X)(Y_1), X_2\rangle + \langle -\operatorname{ad}(X)(X_1) + \operatorname{ad}(Y)(Y_1), Y_2\rangle\\
&= -\langle \operatorname{ad}(Y)(X_1), X_2\rangle - \langle \operatorname{ad}(X)(Y_1), X_2\rangle - \langle \operatorname{ad}(X)(X_1), Y_2\rangle + \langle \operatorname{ad}(Y)(Y_1), Y_2\rangle\\
&= \langle X_1 \operatorname{ad}(Y)(X_2)\rangle + \langle Y_1, \operatorname{ad}(X)(X_2)\rangle + \langle X_1, \operatorname{ad}(X)(Y_2)\rangle - \langle Y_1, \operatorname{ad}(Y)(Y_2)\rangle\\
&= -(-\langle X_1, \operatorname{ad}(Y)(X_2)\rangle - \langle X_1, \operatorname{ad}(X)(Y_2)\rangle - \langle Y_1, \operatorname{ad}(X)(X_2)\rangle + \langle Y_1, \operatorname{ad}(Y)(Y_2)\rangle)\\
&= -(i\langle X_1, i\operatorname{ad}(Y)(X_2)\rangle' + i\langle X_1, i\operatorname{ad}(X)(Y_2)\rangle'\\
&\qquad + \langle Y_1, \operatorname{ad}'(iX)(iX_2)\rangle' + \langle Y_1, \operatorname{ad}'(Y)(Y_2)\rangle')\\
&= -(\langle iX_1, \operatorname{ad}'(Y)(iX_2) + \operatorname{ad}'(iX)(Y_2)\rangle' + \langle Y_1, \operatorname{ad}'(iX)(iX_2) + \operatorname{ad}'(Y)(Y_2)\rangle')\\
&= -\langle iX_1 + Y_1, \operatorname{ad}'(iX+Y)(iX_2+Y_2)\rangle'.
\end{aligned}$$

Definition 3.1 *If* $(\mathfrak{g}', \mathfrak{h}', \mathfrak{m}', \langle\cdot,\cdot\rangle')$ *arises from* $(\mathfrak{g}, \mathfrak{h}, \mathfrak{m}, \langle\cdot,\cdot\rangle)$ *in the above described manner, then* $(\mathfrak{g}', \mathfrak{h}', \mathfrak{m}', \langle\cdot,\cdot\rangle')$ *is called* T*-dual to* $(\mathfrak{g}, \mathfrak{h}, \mathfrak{m}, \langle\cdot,\cdot\rangle)$.

Definition 3.2 *We will say that* $(G/H, g)$ *is a reductive homogeneous space associated to* $(\mathfrak{g}, \mathfrak{h}, \mathfrak{m}, \langle\cdot,\cdot\rangle)$ *if* G *and* $H \subset G$ *are Lie groups with Lie algebras* $\mathfrak{g}$ *and* $\mathfrak{h}$ *and* g *is the left invariant metric on* G/H *defined by* $\langle\cdot,\cdot\rangle$.

Definition 3.3 *Let* $(G/H, g)$ *and* $(G'/H', g')$ *be reductive homogeneous spaces associated to* $(\mathfrak{g}, \mathfrak{h}, \mathfrak{m}, \langle\cdot,\cdot\rangle)$ *and* $(\mathfrak{g}', \mathfrak{h}', \mathfrak{m}', \langle\cdot,\cdot\rangle')$*, respectively. Let* G *be compact. Then* $(G/H, g)$ *and* $(G'/H', g')$ *are called* T*-dual if*

1. *T is an involutive automorphism of* $\mathfrak{g}$.

2. $(\mathfrak{g}, \mathfrak{h}, \mathfrak{m}, \langle\cdot\,,\cdot\rangle)$ *and* $(\mathfrak{g}', \mathfrak{h}', \mathfrak{m}', \langle\cdot\,,\cdot\rangle')$ *are T-dual.*

Now we define an involutive automorphism T^* on $\mathfrak{gl}(\mathfrak{m}) = \mathrm{End}(\mathfrak{m})$ associated to T by the decomposition

$$\begin{aligned}\mathrm{End}(\mathfrak{m})_+ &= \{A \in \mathrm{End}(\mathfrak{m}) \mid A(\mathfrak{m} \cap \mathfrak{g}_+) \subset \mathfrak{g}_+\,,\ A(\mathfrak{m} \cap \mathfrak{g}_-) \subset \mathfrak{g}_-\}\\ \mathrm{End}(\mathfrak{m})_- &= \{B \in \mathrm{End}(\mathfrak{m}) \mid B(\mathfrak{m} \cap \mathfrak{g}_-) \subset \mathfrak{g}_+\,,\ B(\mathfrak{m} \cap \mathfrak{g}_+) \subset \mathfrak{g}_-\}\end{aligned}$$

The Lie algebra $\mathrm{End}(\mathfrak{m})_+ + i\,\mathrm{End}(\mathfrak{m})_- \subset \mathrm{End}(\mathfrak{m})^{\mathbb{C}}$ is isomorphic to $\mathrm{End}(\mathfrak{m}')$. In fact, for $A \in \mathrm{End}(\mathfrak{m})_+$, $B \in \mathrm{End}(\mathfrak{m})_-$, $X \in \mathfrak{m} \cap \mathfrak{g}_-$, $Y \in \mathfrak{m} \cap \mathfrak{g}_+$ we set

$$A(iX) := iA(X) \in \mathfrak{m}' \cap i\mathfrak{g}_- = i(\mathfrak{m} \cap \mathfrak{g}_-) \tag{5}$$

$$A(Y) := A(Y) \in \mathfrak{m}' \cap \mathfrak{g}_+ = \mathfrak{m} \cap \mathfrak{g}_+ \tag{6}$$

$$(iB)(iX) := -B(X) \in \mathfrak{m}' \cap \mathfrak{g}_+ = \mathfrak{m} \cap \mathfrak{g}_+ \tag{7}$$

$$(iB)(Y) := iB(Y) \in \mathfrak{m}' \cap i\mathfrak{g}_- = i(\mathfrak{m} \cap \mathfrak{g}_-) \tag{8}$$

T^* can be restricted to $\mathfrak{so}(\mathfrak{m}) \subset \mathfrak{gl}(\mathfrak{m})$, i.e.

$$T^*(\mathfrak{so}(\mathfrak{m})) = \mathfrak{so}(\mathfrak{m}).$$

On $\mathfrak{so}(\mathfrak{m})$ the automorphism T^* is an isometry with respect to the Killing form. With the above described isomorphism $\mathfrak{so}(\mathfrak{m})_+ + i\mathfrak{so}(\mathfrak{m})_-$ becomes $\mathfrak{so}(\mathfrak{m}')$. For instance, we obtain for $X \in \mathfrak{h} \cap \mathfrak{g}_-$ and $Y \in \mathfrak{h} \cap \mathfrak{g}_+$

$$\begin{aligned}\mathrm{ad}'(iX) = i\,\mathrm{ad}(X) &\in i\mathfrak{so}(\mathfrak{m})_- \subset \mathfrak{so}(\mathfrak{m})' \simeq \mathfrak{so}(\mathfrak{m}')\\ \mathrm{ad}'(Y) = \mathrm{ad}(Y) &\in \mathfrak{so}(\mathfrak{m})_+ \subset \mathfrak{so}(\mathfrak{m})' \simeq \mathfrak{so}(\mathfrak{m}')\end{aligned} \tag{9}$$

4 The curvature of T-dual reductive spaces

Let $(G/H, g)$ be a compact reductive homogeneous space and $(G'/H', g')$ a reductive space T-dual to $(G/H, g)$. Let $\mathcal{Z}$ and $\mathcal{Z}'$ be the Levi-Civita connections of g and g', respectively.

Proposition 4.1 *The linear maps*

$$\begin{aligned}\Lambda_{\mathfrak{m}} &: \mathfrak{m} \longrightarrow \mathfrak{so}(\mathfrak{m})\\ \Lambda'_{\mathfrak{m}'} &: \mathfrak{m}' \longrightarrow \mathfrak{so}(\mathfrak{m}')\end{aligned}$$

associated to $\mathcal{Z}$ and $\mathcal{Z}'$ satisfy

$$\begin{aligned}\Lambda'_{\mathfrak{m}'}(iX) = i\Lambda_{\mathfrak{m}}(X) &\in i\mathfrak{so}(\mathfrak{m})_- \subset \mathfrak{so}(\mathfrak{m})' \simeq \mathfrak{so}(\mathfrak{m}')\\ &\quad \text{if } X \in \mathfrak{m} \cap \mathfrak{g}_-\\ \Lambda'_{\mathfrak{m}'}(Y) = \Lambda_{\mathfrak{m}}(Y) &\in \mathfrak{so}(\mathfrak{m})_+ \subset \mathfrak{so}(\mathfrak{m})' \simeq \mathfrak{so}(\mathfrak{m}')\\ &\quad \text{if } Y \in \mathfrak{m} \cap \mathfrak{g}_+.\end{aligned}$$

Proof. First we compare the bilinear symmetric maps

$$U : \mathfrak{m} \times \mathfrak{m} \to \mathfrak{m}\,, \qquad U' : \mathfrak{m}' \times \mathfrak{m}' \to \mathfrak{m}'$$

defined by

$$\begin{aligned}
2\langle U(Z_1, Z_2), Z\rangle &= \langle [Z, Z_1]_{\mathfrak{m}}, Z_2\rangle + \langle Z_1, [Z, Z_2]_{\mathfrak{m}}\rangle & (10)\\
&\quad \text{for all } Z, Z_1, Z_2 \in \mathfrak{m}\\
2\langle U'(Z_1', Z_2'), Z'\rangle' &= \langle [Z', Z_1']_{\mathfrak{m}'}, Z_2'\rangle' + \langle Z_1', [Z', Z_2']_{\mathfrak{m}'}\rangle' & (11)\\
&\quad \text{for all } Z', Z_1', Z_2' \in \mathfrak{m}'\,.
\end{aligned}$$

In the following we denote by X, X_1, X_2 vectors in $\mathfrak{m} \cap \mathfrak{g}_-$ and by Y, Y_1, Y_2, Y_3 vectors in $\mathfrak{m} \cap \mathfrak{g}_+$. From (4) and (10) we deduce

$$\begin{aligned}
U(X_1, X_2) &\in \mathfrak{m} \cap \mathfrak{g}_+\\
U(X, Y) &\in \mathfrak{m} \cap \mathfrak{g}_-\\
U(Y_1, Y_2) &\in \mathfrak{m} \cap \mathfrak{g}_+ & (12)
\end{aligned}$$

and from (4) and (11)

$$\begin{aligned}
U'(iX_1, iX_2) &\in \mathfrak{m}' \cap \mathfrak{g}_+ = \mathfrak{m} \cap \mathfrak{g}_+\\
U'(iX, Y) &\in \mathfrak{m}' \cap i\mathfrak{g}_- = i(\mathfrak{m} \cap \mathfrak{g}_-)\\
U'(Y_1, Y_2) &\in \mathfrak{m}' \cap \mathfrak{g}_+ = \mathfrak{m} \cap \mathfrak{g}_+\,. & (13)
\end{aligned}$$

We have by definition

$$\begin{aligned}
2\langle U'(iX_1, iX_2), Y\rangle' &= \langle [Y, iX_1]_{\mathfrak{m}'}, iX_2\rangle' + \langle iX_1, [Y, iX_2]_{\mathfrak{m}'}\rangle'\\
&= \langle i[Y, X_1]_{\mathfrak{m}}, iX_2\rangle' + \langle iX_1, i[Y, X_2]_{\mathfrak{m}}\rangle'\\
&= -\langle [Y, X_1]_{\mathfrak{m}}, X_2\rangle - \langle X_1, [Y, X_2]_{\mathfrak{m}}\rangle\\
&= -2\langle U(X_1, X_2), Y\rangle\,.
\end{aligned}$$

On the other hand (13) implies

$$2\langle U'(iX_1, iX_2), Y\rangle' = 2\langle U'(iX_1, iX_2), Y\rangle$$

and by (12) and (13) we get

$$U'(iX_1, iX_2) = -U(X_1, X_2)\,.$$

In the same way we obtain from

$$\begin{aligned}
2\langle U'(iX, Y), iX_1\rangle' &= \langle [iX_1, iX]_{\mathfrak{m}'}, Y\rangle' + \langle iX, [iX_1, Y]_{\mathfrak{m}'}\rangle'\\
&= \langle -[X_1, X]_{\mathfrak{m}}, Y\rangle' + \langle iX, i[X_1, Y]_{\mathfrak{m}}\rangle'\\
&= -\langle [X_1, X]_{\mathfrak{m}}, Y\rangle - \langle X, [X_1, Y]_{\mathfrak{m}}\rangle\\
&= -2\langle U(X, Y), X_1\rangle
\end{aligned}$$

and (12) and (13) the equation

$$U'(iX, Y) = iU(X, Y)\,.$$

Finally, using (12) and (13) we get from

$$\begin{aligned} 2\langle U'(Y_1, Y_2), Y_3\rangle' &= \langle [Y_3, Y_1]_{\mathfrak{m}'}, Y_2\rangle' + \langle Y_1, [Y_3, Y_2]_{\mathfrak{m}'}\rangle' \\ &= \langle [Y_3, Y_1]_{\mathfrak{m}}, Y\rangle + \langle Y_1, [Y_3, Y_2]_{\mathfrak{m}}\rangle \\ &= 2\langle U(Y_1, Y_2), Y_3\rangle \end{aligned}$$

the equation

$$U'(Y_1, Y_2) = U(Y_1, Y_2)\,.$$

The statement now follows from

$$\begin{aligned} \Lambda_{\mathfrak{m}}(Z_1)(Z_2) &= \frac{1}{2}[Z_1, Z_2]_{\mathfrak{m}} + U(Z_1, Z_2) \\ \Lambda'_{\mathfrak{m}'}(Z'_1)(Z'_2) &= \frac{1}{2}[Z'_1, Z'_2]_{\mathfrak{m}'} + U'(Z'_1, Z'_2) \end{aligned}$$

and (4). □

Proposition 4.2 *The curvature tensors R and R' of $(G/H, g)$ and $(G'/H', g')$ satisfy at the origins*

$$\begin{aligned} R'(iX_1, iX_2) = -R(X_1, X_2) &\in \mathfrak{so}(\mathfrak{m})_+ \subset \mathfrak{so}(\mathfrak{m})' \simeq \mathfrak{so}(\mathfrak{m}') \\ & \quad \textit{if } \; X_1, X_2 \in \mathfrak{m} \cap \mathfrak{g}_- \\ R'(iX, Y) = iR(X, Y) &\in i\mathfrak{so}(\mathfrak{m})_- \subset \mathfrak{so}(\mathfrak{m})' \simeq \mathfrak{so}(\mathfrak{m}') \\ & \quad \textit{if } \; X \in \mathfrak{m} \cap \mathfrak{g}_-\,,\; Y \in \mathfrak{m} \cap \mathfrak{g}_+ \\ R'(Y_1, Y_2) = R(Y_1, Y_2) &\in \mathfrak{so}(\mathfrak{m})_+ \subset \mathfrak{so}(\mathfrak{m})' \simeq \mathfrak{so}(\mathfrak{m}') \\ & \quad \textit{if } \; Y_1, Y_2 \in \mathfrak{m} \cap \mathfrak{g}_+. \end{aligned}$$

Proof. We know the following formulas for R and R' at the origins of $(G/H, g)$ and $(G'/H', g')$ (see [KN2]). If $Z_1, Z_2 \in \mathfrak{m}$, $Z'_1, Z'_2 \in \mathfrak{m}'$ then

$$\begin{aligned} R(Z_1, Z_2) &= [\Lambda_{\mathfrak{m}}(Z_1), \Lambda_{\mathfrak{m}}(Z_2)] - \Lambda_{\mathfrak{m}}([Z_1, Z_2]_{\mathfrak{m}}) - \mathrm{ad}([Z_1, Z_2]_{\mathfrak{h}}) \\ R'(Z'_1, Z'_2) &= [\Lambda'_{\mathfrak{m}'}(Z'_1), \Lambda'_{\mathfrak{m}'}(Z'_2)] - \Lambda'_{\mathfrak{m}'}([Z'_1, Z'_2]_{\mathfrak{m}'}) - \mathrm{ad}'([Z'_1, Z'_2]_{\mathfrak{h}'}) \end{aligned}$$

In the following we assume that $X, X_1, \ldots, X_4$ are vectors in $\mathfrak{m} \cap \mathfrak{g}_-$ and $Y, Y_1, \ldots, Y_4$ are vectors in $\mathfrak{m} \cap \mathfrak{g}_+$. We have to prove

1. $R'(iX_1, iX_2, Y_1, Y_2) = -R(X_1, X_2, Y_1, Y_2)$
2. $R'(iX_1, Y_1, iX_2, Y_2) = -R(X_1, Y_1, X_2, Y_2)$
3. $R'(iX_1, iX_2, iX_3, iX_4) = R(X_1, X_2, X_3, X_4)$

4. $R'(Y_1,Y_2,Y_3,Y_4)=R(Y_1,Y_2,Y_3,Y_4)$

5. $R'(iX,Y_1,Y_2,Y_3)=R(X,Y_1,Y_2,Y_3)=0$

6. $R'(iX_1,iX_2,iX_3,Y)=R(X_1,X_2,X_3,Y)=0$

We have

$$\begin{aligned}
R'(iX_1,iX_2)Y &= (\Lambda'_{\mathfrak{m}'}(iX_1)\Lambda'_{\mathfrak{m}'}(iX_2)-\Lambda'_{\mathfrak{m}'}(iX_2)\Lambda'_{\mathfrak{m}'}(iX_1))(Y)\\
&\quad -\Lambda'_{\mathfrak{m}'}([iX_1,iX_2]_{\mathfrak{m}'})(Y)-\mathrm{ad}'([iX_1,iX_2]_{\mathfrak{h}'})(Y)\\
&= \Lambda'_{\mathfrak{m}'}(iX_1)(i\Lambda_{\mathfrak{m}}(X_2)(Y))-\Lambda'_{\mathfrak{m}'}(iX_2)(i\Lambda_{\mathfrak{m}}(X_1)(Y))\\
&\quad +\Lambda_{\mathfrak{m}}([X_1,X_2]_{\mathfrak{m}})(Y)+\mathrm{ad}([X_1,X_2]_{\mathfrak{h}})(Y)\\
&= -\Lambda_{\mathfrak{m}}(X_1)(\Lambda_{\mathfrak{m}}(X_2)(Y))+\Lambda_{\mathfrak{m}}(X_2)(\Lambda_{\mathfrak{m}}(X_1)(Y))\\
&\quad +\Lambda_{\mathfrak{m}}([X_1,X_2]_{\mathfrak{m}})(Y)+\mathrm{ad}([X_1,X_2]_{\mathfrak{h}})(Y)\\
&= -[\Lambda_{\mathfrak{m}}(X_1),\Lambda_{\mathfrak{m}}(X_2)](Y)+\Lambda_{\mathfrak{m}}([X_1,X_2]_{\mathfrak{m}})(Y)+\mathrm{ad}([X_1,X_2]_{\mathfrak{h}})(Y)\\
&= -R(X_1,X_2)Y \ \in\ \mathfrak{m}\cap\mathfrak{g}_+
\end{aligned}$$

since according to Propositions 3.1 and 4.1 each summand is in $\mathfrak{m}\cap\mathfrak{g}_+$. This proves 1. and 6. Furthermore, we have

$$\begin{aligned}
R'(iX_1,Y)(iX_2) &= \Lambda'_{\mathfrak{m}'}(iX_1)(\Lambda'_{\mathfrak{m}'}(Y)(iX_2))-\Lambda'_{\mathfrak{m}'}(Y)(\Lambda'_{\mathfrak{m}'}(iX_1)(iX_2))\\
&\quad -\Lambda'_{\mathfrak{m}'}([iX_1,Y]_{\mathfrak{m}'})(iX_2)-\mathrm{ad}'([iX_1,Y]_{\mathfrak{h}'})(iX_2)\\
&= \Lambda'_{\mathfrak{m}'}(iX_1)(i\Lambda_{\mathfrak{m}}(Y)(iX_2))-\Lambda'_{\mathfrak{m}'}(Y)(-\Lambda_{\mathfrak{m}}(X_1)(X_2))\\
&\quad -\Lambda'_{\mathfrak{m}'}(i[X_1,Y]_{\mathfrak{m}'})(iX_2)-\mathrm{ad}'(i[X_1,Y]_{\mathfrak{h}})(iX_2)\\
&= -\Lambda_{\mathfrak{m}}(X_1)(\Lambda_{\mathfrak{m}}(Y)(X_2))+\Lambda_{\mathfrak{m}}(Y)(\Lambda_{\mathfrak{m}}(X_1)(X_2))\\
&\quad +\Lambda_{\mathfrak{m}}([X_1,Y]_{\mathfrak{m}})(X_2)+\mathrm{ad}([X_1,Y]_{\mathfrak{h}})(X_2)\\
&= -[\Lambda_{\mathfrak{m}}(X_1),\Lambda_{\mathfrak{m}}(Y)](X_2)+\Lambda_{\mathfrak{m}}([X_1,Y]_{\mathfrak{m}})(X_2)+\mathrm{ad}([X_1,Y]_{\mathfrak{h}})(X_2)\\
&= -R(X_1,Y)X_2 \ \in\ \mathfrak{m}\cap\mathfrak{g}_+
\end{aligned}$$

which induces 2. In the same way we obtain from

$$\begin{aligned}
R'(iX_1,iX_2)iX_3 &= \Lambda'_{\mathfrak{m}'}(iX_1)(\Lambda'_{\mathfrak{m}'}(iX_2)(iX_3))-\Lambda'_{\mathfrak{m}'}(iX_2)\Lambda'_{\mathfrak{m}'}(iX_1))(iX_3))\\
&\quad -\Lambda'_{\mathfrak{m}'}([iX_1,iX_2]_{\mathfrak{m}'})(iX_3)-\mathrm{ad}'([iX_1,iX_2]_{\mathfrak{h}'})(iX_3)\\
&= \Lambda'_{\mathfrak{m}'}(iX_1)(-\Lambda_{\mathfrak{m}}(X_2)(X_3))-\Lambda'_{\mathfrak{m}'}(iX_2)(-\Lambda_{\mathfrak{m}}(X_1)(X_3))\\
&\quad -\Lambda'_{\mathfrak{m}'}(-[X_1,X_2]_{\mathfrak{m}'})(iX_3)-\mathrm{ad}'(-[X_1,X_2]_{\mathfrak{h}'})(iX_3)\\
&= -i\Lambda_{\mathfrak{m}}(X_1)(\Lambda_{\mathfrak{m}}(X_2)(X_3))+i\Lambda_{\mathfrak{m}}(X_2)(\Lambda_{\mathfrak{m}}(X_1)(X_3))\\
&\quad +i\Lambda_{\mathfrak{m}}([X_1,X_2]_{\mathfrak{m}})(X_3)+i\,\mathrm{ad}([X_1,X_2]_{\mathfrak{h}})(X_3)\\
&= -iR(X_1,X_2)X_3 \ \in\ i(\mathfrak{m}\cap\mathfrak{g}_-)
\end{aligned}$$

assertion 3. and from

$$R'(Y_1,Y_2)Y_3 \;=\; \Lambda'_{\mathfrak{m}'}(Y_1)(\Lambda'_{\mathfrak{m}'}(Y_2)(Y_3))-\Lambda'_{\mathfrak{m}'}(Y_2)\Lambda'_{\mathfrak{m}'}(Y_1))(Y_3))$$

$$
\begin{aligned}
&\quad -\Lambda'_{\mathfrak{m}'}([Y_1,Y_2]_{\mathfrak{m}'})(Y_3) - \mathrm{ad}'([Y_1,Y_2]_{\mathfrak{h}'})(Y_3)\\
&= \Lambda_{\mathfrak{m}}(Y_1)(\Lambda_{\mathfrak{m}}(Y_2)(Y_3)) - \Lambda_{\mathfrak{m}}(Y_2)(\Lambda_{\mathfrak{m}}(Y_1)(Y_3))\\
&\quad -\Lambda_{\mathfrak{m}}([Y_1,Y_2]_{\mathfrak{m}})(Y_3) - \mathrm{ad}([Y_1,Y_2]_{\mathfrak{h}})(Y_3)\\
&= R(Y_1,Y_2)Y_3 \ \in \ \mathfrak{m}\cap\mathfrak{g}_+
\end{aligned}
$$

assertions 4. and 5. □

Corollary 4.1 *The curvature tensors* Ric *and* Ric$'$ *of* $(G/H,g)$ *and* $(G'/H',g')$ *satisfy at the origins*

1. $\mathrm{Ric}'(iX_1,iX_2) = -\mathrm{Ric}(X_1,X_2)$
2. $\mathrm{Ric}'(iX,Y) = \mathrm{Ric}(X,Y) = 0$
3. $\mathrm{Ric}'(Y_1,Y_2) = \mathrm{Ric}(Y_1,Y_2)$

for all $X,X_1,X_2 \in \mathfrak{m}\cap\mathfrak{g}_-$ *and all* $Y,Y_1,Y_2 \in \mathfrak{m}\cap\mathfrak{g}_+$.

Proof. Let $b_1,\ldots,b_k$ be a basis of $\mathfrak{m}\cap\mathfrak{g}_-$ and $b_{k+1},\ldots,b_n$ a basis of $\mathfrak{m}\cap\mathfrak{g}_+$. Then $b_1,\ldots,b_n$ is a basis of $\mathfrak{m}$ and $ib_1,\ldots,ib_k,b_{k+1},\ldots,b_n$ a basis of $\mathfrak{m}'$ and we have

$$
\begin{aligned}
\mathrm{Ric}'(iX_1,iX_2) &= -\sum_{\alpha=1}^{k} R'(iX_1,ib_\alpha,ib_\alpha,iX_2) + \sum_{\beta=k+1}^{n} R'(iX_1,b_\beta,b_\beta,iX_2)\\
&= -\sum_{\alpha=1}^{k} R(X_1,b_\alpha,b_\alpha,X_2) - \sum_{\beta=k+1}^{n} R(X_1,b_\alpha,b_\alpha,X_2)\\
&= -\mathrm{Ric}(X_1,X_2)
\end{aligned}
$$

for $X_1,X_2 \in \mathfrak{m}\cap\mathfrak{g}_-$ and

$$
\begin{aligned}
\mathrm{Ric}'(Y_1,Y_2) &= -\sum_{\alpha=1}^{k} R'(Y_1,ib_\alpha,ib_\alpha,Y_2) + \sum_{\beta=k+1}^{n} R'(Y_1,b_\beta,b_\beta,Y_2)\\
&= \sum_{j=1}^{n} R(Y_1,b_j,b_j,Y_2)\\
&= \mathrm{Ric}(Y_1,Y_2)
\end{aligned}
$$

for $Y_1,Y_2 \in \mathfrak{m}\cap\mathfrak{g}_+$. Assertion 3. is obvious. □

Corollary 4.2 *If g is an Einstein metric on G/H then g' is also an Einstein metric.*

Corollary 4.3 *The scalar curvatures τ and τ' of $(G/H,g)$ and $(G'/H',g')$ are equal:*

$$\tau = \tau'.$$

Next we compare the Lie algebras of the holonomy groups $\Psi(u_0)$ and $\Psi'(u_0')$ of $(G/H,g)$ and $(G'/H',g')$ at the origins. We define for $Z_1,Z_2\in\mathfrak{m}$, $Z_1',Z_2'\in\mathfrak{m}'$

$$\begin{aligned}
m_0(Z_1,Z_2) &:= [\Lambda_{\mathfrak{m}}(Z_1),\Lambda_{\mathfrak{m}}(Z_2)]-\Lambda_{\mathfrak{m}}([Z_1,Z_2]_{\mathfrak{m}})-\mathrm{ad}([Z_1,Z_2]_{\mathfrak{h}})\in\mathfrak{so}(\mathfrak{m})\\
m_0'(Z_1',Z_2') &:= [\Lambda'_{\mathfrak{m}'}(Z_1'),\Lambda'_{\mathfrak{m}'}(Z_2')]-\Lambda'_{\mathfrak{m}'}([Z_1',Z_2']_{\mathfrak{m}'})-\mathrm{ad}'([Z_1',Z_2']_{\mathfrak{h}'})\in\mathfrak{so}(\mathfrak{m}')
\end{aligned}$$

and

$$\begin{aligned}
\mathfrak{m}_0 &:= \mathrm{span}\{m_0(Z_1,Z_2)\mid Z_1,Z_2\in\mathfrak{m}\}\\
\mathfrak{m}_0' &:= \mathrm{span}\{m_0'(Z_1',Z_2')\mid Z_1',Z_2'\in\mathfrak{m}'\}.
\end{aligned}$$

The Lie algebras $\underline{\psi}(u_0)$ and $\underline{\psi}'(u_0')$ of $\Psi(u_0)$ and $\Psi'(u_0')$ are given by

$$\begin{aligned}
\underline{\psi}(u_0) &= \mathrm{span}\{\mathfrak{m}_0+[\Lambda_{\mathfrak{m}}(\mathfrak{m}),\mathfrak{m}_0]+[\Lambda_{\mathfrak{m}}(\mathfrak{m}),[\Lambda_{\mathfrak{m}}(\mathfrak{m}),\mathfrak{m}_0]]+\ldots\}\\
\underline{\psi}'(u_0') &= \mathrm{span}\{\mathfrak{m}_0'+[\Lambda'_{\mathfrak{m}'}(\mathfrak{m}'),\mathfrak{m}_0']+[\Lambda'_{\mathfrak{m}'}(\mathfrak{m}'),[\Lambda_{\mathfrak{m}'}(\mathfrak{m}'),\mathfrak{m}_0']]+\ldots\}
\end{aligned}$$

(see [KN2], X.4.). We will need the following

Lemma 4.1 *Let $U,V\subset\mathfrak{g}$ be subspaces and assume that $T(U)\subset U$ and $T(V)\subset V$. Then we have*

1. $T(\mathrm{span}[U,V])=\mathrm{span}[U,V]$
2. $\mathrm{span}\{[U\cap\mathfrak{g}_++i(U\cap\mathfrak{g}_-)\,,\,V\cap\mathfrak{g}_++i(V\cap\mathfrak{g}_-)]\}=\mathrm{span}[U,V]\cap\mathfrak{g}_++i(\mathrm{span}[U,V]\cap\mathfrak{g}_-)$.

□

Theorem 4.1 *1. $T^*(\underline{\psi}(u_0))=\underline{\psi}(u_0)$*

2. $\underline{\psi}(u_0)$ and $\underline{\psi}'(u_0')$ are T^*-dual.

Proof. Let X_1,X_2 be vectors in $\mathfrak{m}\cap\mathfrak{g}_-$ and Y_1,Y_2 vectors in $\mathfrak{m}\cap\mathfrak{g}_+$. Then

$$m_0(X_1+Y_1,X_2+Y_2) = \underbrace{m_0(X_1,X_2)}_{\in\,\mathfrak{so}(\mathfrak{m})_+}+\underbrace{m_0(X_1,Y_2)}_{\in\,\mathfrak{so}(\mathfrak{m})_-}+\underbrace{m_0(Y_1,X_2)}_{\in\,\mathfrak{so}(\mathfrak{m})_-}+\underbrace{m_0(Y_1,Y_2)}_{\in\,\mathfrak{so}(\mathfrak{m})_+}$$

Using the identification $\mathfrak{so}(\mathfrak{m}')=\mathfrak{so}(\mathfrak{m})'$ we obtain from Proposition 4.1 and Equation (9)

$$\begin{aligned}
m_0'(iX_1+Y_1,iX_2+Y_2) &= \underbrace{-m_0(X_1,X_2)}_{\in\,\mathfrak{so}(\mathfrak{m})_+}\underbrace{+im_0(X_1,Y_2)}_{\in\,i\mathfrak{so}(\mathfrak{m})_-}\underbrace{+im_0(Y_1,X_2)}_{\in\,i\mathfrak{so}(\mathfrak{m})_-}\\
&\quad\underbrace{+m_0(Y_1,Y_2)}_{\in\,\mathfrak{so}(\mathfrak{m})_+}\quad .
\end{aligned}$$

This implies $T^*(\mathfrak{m}_0)=\mathfrak{m}_0$ and $\mathfrak{m}_0'=\mathfrak{m}_0\cap\mathfrak{so}(\mathfrak{m})_++i(\mathfrak{m}_0\cap\mathfrak{so}(\mathfrak{m})_-)$. On the other hand we have

$$\Lambda_{\mathfrak{m}}(\mathfrak{m}) = \underbrace{\Lambda_{\mathfrak{m}}(\mathfrak{m}\cap\mathfrak{g}_+)}_{\subset\,\mathfrak{so}(\mathfrak{m})_+}+\underbrace{\Lambda_{\mathfrak{m}}(\mathfrak{m}\cap\mathfrak{g}_-)}_{\subset\,\mathfrak{so}(\mathfrak{m})_-}.$$

Hence, $T^*(\Lambda_{\mathfrak{m}}(\mathfrak{m})) = \Lambda_{\mathfrak{m}}(\mathfrak{m})$ and again by Proposition 4.1

$$\begin{aligned}
\Lambda'_{\mathfrak{m}'}(\mathfrak{m}') &= \Lambda'_{\mathfrak{m}'}(\mathfrak{m}' \cap \mathfrak{g}_+ + \mathfrak{m}' \cap i\mathfrak{g}_-) \\
&= \Lambda_{\mathfrak{m}}(\mathfrak{m} \cap \mathfrak{g}_+) + i\Lambda_{\mathfrak{m}}(\mathfrak{m} \cap \mathfrak{g}_-)\,.
\end{aligned}$$

Lemma 4.1 now implies

$$\begin{aligned}
&\operatorname{span}[\Lambda_{\mathfrak{m}}(\mathfrak{m}), \mathfrak{m}_0] \cap \mathfrak{so}(\mathfrak{m})_+ + i(\operatorname{span}[\Lambda_{\mathfrak{m}}(\mathfrak{m}), \mathfrak{m}_0] \cap \mathfrak{so}(\mathfrak{m})_-) \\
&\quad = \operatorname{span}[\Lambda_{\mathfrak{m}}(\mathfrak{m}) \cap \mathfrak{so}(\mathfrak{m})_+ + i(\Lambda_{\mathfrak{m}}(\mathfrak{m}) \cap \mathfrak{so}(\mathfrak{m})_-)\,,\ \mathfrak{m}_0 \cap \mathfrak{so}(\mathfrak{m})_+ + i(\mathfrak{m}_0 \cap \mathfrak{so}(\mathfrak{m})_-)] \\
&\quad = \operatorname{span}[\Lambda'_{\mathfrak{m}'}(\mathfrak{m}'), \mathfrak{m}'_0] \\
&\operatorname{span}[\Lambda_{\mathfrak{m}}(\mathfrak{m}), [\Lambda_{\mathfrak{m}}(\mathfrak{m}), \mathfrak{m}_0]] \cap \mathfrak{so}(\mathfrak{m})_+ + i(\operatorname{span}[\Lambda_{\mathfrak{m}}(\mathfrak{m}), [\Lambda_{\mathfrak{m}}(\mathfrak{m}), \mathfrak{m}_0]] \cap \mathfrak{so}(\mathfrak{m})_-) \\
&\quad = \operatorname{span}[\Lambda_{\mathfrak{m}}(\mathfrak{m}), \operatorname{span}[\Lambda_{\mathfrak{m}}(\mathfrak{m}), \mathfrak{m}_0]] \cap \mathfrak{so}(\mathfrak{m})_+ \\
&\qquad\qquad + i(\operatorname{span}[\Lambda_{\mathfrak{m}}(\mathfrak{m}), \operatorname{span}[\Lambda_{\mathfrak{m}}(\mathfrak{m}), \mathfrak{m}_0]] \cap \mathfrak{so}(\mathfrak{m})_-) \\
&\quad = \operatorname{span}[\Lambda_{\mathfrak{m}}(\mathfrak{m}) \cap \mathfrak{so}(\mathfrak{m})_+ + i(\Lambda_{\mathfrak{m}}(\mathfrak{m}) \cap \mathfrak{so}(\mathfrak{m})_-)\,, \\
&\qquad\qquad \operatorname{span}[\Lambda_{\mathfrak{m}}(\mathfrak{m}), \mathfrak{m}_0] \cap \mathfrak{so}(\mathfrak{m})_+ + i(\operatorname{span}[\Lambda_{\mathfrak{m}}(\mathfrak{m}), \mathfrak{m}_0] \cap \mathfrak{so}(\mathfrak{m})_-)] \\
&\quad = \operatorname{span}[\Lambda'_{\mathfrak{m}'}(\mathfrak{m}'), \operatorname{span}[\Lambda'_{\mathfrak{m}'}(\mathfrak{m}'), \mathfrak{m}'_0]] \\
&\quad = \operatorname{span}[\Lambda'_{\mathfrak{m}'}(\mathfrak{m}'), [\Lambda'_{\mathfrak{m}'}(\mathfrak{m}'), \mathfrak{m}'_0]] \\
&\quad \ldots \\
&\operatorname{span}[\Lambda_{\mathfrak{m}}(\mathfrak{m}), [\Lambda_{\mathfrak{m}}(\mathfrak{m}), \ldots [\Lambda_{\mathfrak{m}}(\mathfrak{m}), \mathfrak{m}_0]]] \cap \mathfrak{so}(\mathfrak{m})_+ \\
&\qquad + i(\operatorname{span}[\Lambda_{\mathfrak{m}}(\mathfrak{m}), [\Lambda_{\mathfrak{m}}(\mathfrak{m}), \ldots [\Lambda_{\mathfrak{m}}(\mathfrak{m}), \mathfrak{m}_0]]] \cap \mathfrak{so}(\mathfrak{m})_-) \\
&\quad = \operatorname{span}[\Lambda'_{\mathfrak{m}'}(\mathfrak{m}'), [\Lambda'_{\mathfrak{m}'}(\mathfrak{m}'), \ldots [\Lambda'_{\mathfrak{m}'}(\mathfrak{m}'), \mathfrak{m}'_0]]]\,.
\end{aligned}$$

Consequently, $\underline{\psi}'(u'_0) = \underline{\psi}(u_0) \cap \mathfrak{so}(\mathfrak{m})_+ + i(\underline{\psi}(u_0) \cap \mathfrak{so}(\mathfrak{m})_-)$. □

5 Invariant almost complex and almost bilagrangian structures

Let $(G/H, g)$ be a reductive homogeneous space associated to $(\mathfrak{g}, \mathfrak{h}, \mathfrak{m}, \langle\cdot\,,\cdot\rangle)$ and J a G-invariant almost complex structure on G/H. J corresponds to a uniquely determined linear map $I : \mathfrak{m} \to \mathfrak{m}$ with properties

(a) $I^2 = -\operatorname{Id}$

(b) $I \circ \operatorname{Ad}(h) = \operatorname{Ad}(h) \circ I$ for all $h \in H$.

If H is connected then (b) can be replaced by

(b') $I \circ \operatorname{ad}(Z) = \operatorname{ad}(Z) \circ I$ for all $Z \in \mathfrak{h}$.

J is integrable if and only if

$$[IZ_1, IZ_2]_{\mathfrak{m}} - [Z_1, Z_2]_{\mathfrak{m}} - I[Z_1, IZ_2]_{\mathfrak{m}} - I[IZ_1, Z_2]_{\mathfrak{m}} = 0 \tag{14}$$

holds for all $Z_1, Z_2 \in \mathfrak{m}$. The metric g is hermitian with respect to J if and only if

$$\langle IZ_1, IZ_2\rangle = \langle Z_1, Z_2\rangle$$

holds for all $Z_1, Z_2 \in \mathfrak{m}$. A G-invariant hermitian metric g is Kählerian if and only if J is parallel with respect to g, i.e. if the linear map $I : \mathfrak{m} \to \mathfrak{m}$ associated to J and the map $\Lambda_{\mathfrak{m}} : \mathfrak{m} \to \mathfrak{so}(\mathfrak{m})$ describing the Levi-Civita connection satisfy

$$\Lambda_{\mathfrak{m}}(Z) \circ I = I \circ \Lambda_{\mathfrak{m}}(Z)$$

for all $Z \in \mathfrak{m}$. Let $(\mathfrak{g}', \mathfrak{h}', \mathfrak{m}', \langle\cdot\,,\cdot\rangle')$ be T-dual to $(\mathfrak{g}, \mathfrak{h}, \mathfrak{m}, \langle\cdot\,,\cdot\rangle)$ and assume that I is contained in $\mathrm{End}(\mathfrak{m})_+$, i.e.

$$\begin{aligned} I(\mathfrak{m}\cap\mathfrak{g}_+) &\subset \mathfrak{m}\cap\mathfrak{g}_+ \\ I(\mathfrak{m}\cap\mathfrak{g}_-) &\subset \mathfrak{m}\cap\mathfrak{g}_-. \end{aligned}$$

We consider I as a map $I \in \mathrm{End}(\mathfrak{m})_+ \subset \mathrm{End}(\mathfrak{m})'$ and identify $\mathrm{End}(\mathfrak{m})'$ with $\mathrm{End}(\mathfrak{m}')$ (see (5)). We denote the image of I under this identification again by I. Furthermore, with this identification the maps $i\,\mathrm{ad}(X) \in i\,\mathrm{End}(\mathfrak{m})_-$ and $\mathrm{ad}(Y) \in \mathrm{End}(\mathfrak{m})_+$ for $X \in \mathfrak{h}\cap\mathfrak{g}_-$ and $Y \in \mathfrak{h}\cap\mathfrak{g}_+$ become $\mathrm{ad}'(iX) \in \mathrm{End}(\mathfrak{m}')$ and $\mathrm{ad}'(Y) \in \mathrm{End}(\mathfrak{m}')$. From condition (b') we get

$$\begin{aligned} iI\circ\mathrm{ad}(X) = i\,\mathrm{ad}(X)\circ I \qquad &\text{for } X \in \mathfrak{h}\cap\mathfrak{g}_- \\ I\circ\mathrm{ad}(Y) = \mathrm{ad}(Y)\circ I \qquad &\text{for } Y \in \mathfrak{h}\cap\mathfrak{g}_+. \end{aligned}$$

We obtain for $I \in \mathrm{End}(\mathfrak{m}')$

$$\begin{aligned} I\circ\mathrm{ad}'(iX) &= \mathrm{ad}'(iX)\circ I \\ I\circ\mathrm{ad}'(Y) &= \mathrm{ad}'(Y)\circ I. \end{aligned}$$

Let now $(G'/H', g')$ be a homogeneous space associated to $(\mathfrak{g}', \mathfrak{h}', \mathfrak{m}', \langle\cdot\,,\cdot\rangle')$ and assume that H' is connected. Then $I \in \mathrm{End}(\mathfrak{m}')$ defines an almost complex structure J' on G'/H'. If $I \in \mathrm{End}(\mathfrak{m})_+$ satisfies the condition (14) then we get from (5)

$$[IZ_1', IZ_2']_{\mathfrak{m}'} - [Z_1', Z_2']_{\mathfrak{m}'} - I[Z_1', IZ_2']_{\mathfrak{m}'} - I[IZ_1', Z_2']_{\mathfrak{m}'} = 0$$

for all $Z_1', Z_2' \in \mathfrak{m}'$. Thus, if J is integrable so is J'. Furthermore, we have

$$\begin{aligned} \langle I(iX_1+Y_1), I(iX_2+Y_2)\rangle' &= \langle iIX_1 + IY_1, iIX_2 + IY_2\rangle' \\ &= -\langle IX_1, IX_2\rangle + \langle IY_1, IY_2\rangle \end{aligned}$$

for all $X_1, X_2 \in \mathfrak{m}\cap\mathfrak{g}_-$, $Y_1, Y_2 \in \mathfrak{m}\cap\mathfrak{g}_+$. Hence, if g is hermitian with respect to J then we obtain

$$\begin{aligned} \langle I(iX_1+Y_1), I(iX_2+Y_2)\rangle' &= -\langle X_1, X_2\rangle + \langle Y_1, Y_2\rangle \\ &= \langle iX_1+Y_1, iX_2+Y_2\rangle'. \end{aligned}$$

Consequently, in this case g' is hermitian with respect to J'. Let now the almost complex structure J on G/H be parallel, i.e. assume

$$\Lambda_{\mathfrak{m}}(Z) \circ I = I \circ \Lambda_{\mathfrak{m}}(Z)$$

for all $Z \in \mathfrak{m}$. Then for all $X \in \mathfrak{m} \cap \mathfrak{g}_-$ the maps $\Lambda_{\mathfrak{m}}(X) \circ I$ and $I \circ \Lambda_{\mathfrak{m}}(X)$ are contained in $\mathrm{End}(\mathfrak{m})_-$ and satisfy

$$i\Lambda_{\mathfrak{m}}(X) \circ I = iI \circ \Lambda_{\mathfrak{m}}(X) \in i\,\mathrm{End}(\mathfrak{m})_- \subset \mathrm{End}(\mathfrak{m})'.$$

Identifying $\mathrm{End}(\mathfrak{m})'$ with $\mathrm{End}(\mathfrak{m}')$ we obtain from Proposition 4.1

$$\Lambda'_{\mathfrak{m}'}(iX) \circ I = I \circ \Lambda'_{\mathfrak{m}'}(iX).$$

Analogously, we have for $Y \in \mathfrak{m} \cap \mathfrak{g}_+$

$$\Lambda_{\mathfrak{m}}(Y) \circ I\,,\; I \circ \Lambda_{\mathfrak{m}}(Y) \in \mathrm{End}(\mathfrak{m})_+ \subset \mathrm{End}(\mathfrak{m})' \cong \mathrm{End}(\mathfrak{m}')$$

and

$$\Lambda'_{\mathfrak{m}'}(Y) \circ I = I \circ \Lambda'_{\mathfrak{m}'}(Y).$$

Consequently, J' is parallel, too.

Now let us assume that I is contained in $\mathrm{End}(\mathfrak{m})_-$. We will see that in this case J leads to an almost bilagrangian structure on the T-dual space. However, let us first recall the notion of bilagrangian structures.

Definition 5.1 *Let V be a real vector space of even dimension $2m$ and $\langle \cdot\,, \cdot \rangle$ a pseudo-Euclidean inner product of index m on V. A linear operator $J : V \to V$ is called bilagrangian structure on V if*

1. *J is involutive, i.e. $J^2 = \mathrm{Id}$.*
2. *$\langle JX, JY \rangle = -\langle X, Y \rangle$ for all $X, Y \in V$.*

In particular, 1. and 2. imply that each of the eigenvalues ± 1 of J has multiplicity m.

Definition 5.2 *Let $(N^{2m,m}, g)$ be a neutral manifold, i.e. a manifold of even dimension $2m$ with metric of index m. A smooth (1,1)-tensor field J on N is called almost bilagrangian structure if $J_x : T_xN \to T_xN$ is a bilagrangian structure of T_xN for all $x \in N$. An almost bilagrangian structure J is called integrable (or bilagrangian structure) if the distributions*

$$\begin{aligned} V_+ &= \{X \in TN \mid JX = X\} \\ V_- &= \{X \in TN \mid JX = -X\} \end{aligned}$$

are involutive.

Suppose now that $I \in \mathrm{End}(\mathfrak{m})_-$, i.e. $I(\mathfrak{m}\cap\mathfrak{g}_+) \subset \mathfrak{m}\cap\mathfrak{g}_-$ and $I(\mathfrak{m}\cap\mathfrak{g}_-) \subset \mathfrak{m}\cap\mathfrak{g}_+$. Since I is an isomorphism we obtain

$$\dim \mathfrak{m}\cap\mathfrak{g}_+ = \dim \mathfrak{m}\cap\mathfrak{g}_- \,.$$

In particular, $\dim\mathfrak{m} = \dim\mathfrak{m}' = 2m$ is even and the index of $\langle\cdot\,,\cdot\rangle'$ equals m. Now we consider $iI \in \mathrm{End}(\mathfrak{m})' = \mathrm{End}(\mathfrak{m}')$. Obviously, $(iI)^2 = \mathrm{Id}$ holds. Let $b_1,\dots,b_m$ be a basis of $\mathfrak{m}\cap\mathfrak{g}_-$. Then $Ib_1,\dots,Ib_m$ is a basis of $\mathfrak{m}\cap\mathfrak{g}_+$ and $ib_1,\dots,ib_m,Ib_1,\dots,Ib_m$ a basis of $\mathfrak{m}'$. Now we have

$$\begin{aligned}(iI)(ib_\alpha + Ib_\alpha) &= -Ib_\alpha - ib_\alpha\\ (iI)(ib_\alpha - Ib_\alpha) &= -Ib_\alpha + ib_\alpha \quad (1\le\alpha\le m)\,.\end{aligned}$$

Hence, the eigenspace of (iI) of eigenvalue 1 equals $\mathrm{span}\{ib_\alpha - Ib_\alpha \mid \alpha = 1,\dots,m\}$ and the one of eigenvalue -1 equals $\mathrm{span}\{ib_\alpha + Ib_\alpha \mid \alpha = 1,\dots,m\}$. In particular, both eigenspaces have dimension m. Using the usual identifications we get from condition (b')

$$\begin{aligned}iI\circ\mathrm{ad}'(iX) &= iI\circ i\,\mathrm{ad}(X) = -I\circ\mathrm{ad}(X) = -\mathrm{ad}(X)\circ I\\ &= i\,\mathrm{ad}(X)\circ(iI) = \mathrm{ad}'(iX)\circ(iI)\\ iI\circ\mathrm{ad}'(Y) &= iI\circ\mathrm{ad}(Y) = i\,\mathrm{ad}(Y)\circ I = \mathrm{ad}(Y)\circ(iI) = \mathrm{ad}'(Y)\circ(iI)\end{aligned}$$

for $X\in\mathfrak{h}\cap\mathfrak{g}_-$ and $Y\in\mathfrak{h}\cap\mathfrak{g}_+$. Consequently, iI is $\mathrm{ad}'(\mathfrak{h}')$-invariant.

If g is hermitian with respect to J then we have

$$\begin{aligned}\langle(iI)(iX_1+Y_1),(iI)(iX_2+Y_2)\rangle' &= \langle -IX_1 + iIY_1, -IX_2 + iIY_2\rangle'\\ &= \langle -IX_1,-IX_2\rangle - \langle IY_1, IY_2\rangle\\ &= \langle X_1,X_2\rangle - \langle Y_1,Y_2\rangle\\ &= -\langle iX_1+Y_1, iX_2+Y_2\rangle'\end{aligned}$$

for all $X_1,X_2\in\mathfrak{m}\cap\mathfrak{g}_-$ and $Y_1,Y_2\in\mathfrak{m}\cap\mathfrak{g}_+$.

Let now $(G'/H',g')$ be a homogeneous space associated to $(\mathfrak{g}',\mathfrak{h}',\mathfrak{m}',\langle\cdot\,,\cdot\rangle')$ and assume that H' is connected. If g is hermitian with respect to J then $iI\in\mathrm{End}(\mathfrak{m}')$ defines an almost bilagrangian structure J' on G'/H'. Next we will verify assuming $I\in\mathrm{End}(\mathfrak{m})_-$ satisfies the condition (14) that this bilagrangian structure is integrable. We have to prove that the distributions consisting of the $(+1)$ and (-1) eigenvectors of J' are involutive. However, this follows from

$$\begin{aligned}[ib_\alpha \pm Ib_\alpha, ib_\beta\pm Ib_\beta] &= -[b_\alpha,b_\beta] \pm i[b_\alpha,Ib_\beta] \pm i[Ib_\alpha,b_\beta] + [Ib_\alpha,Ib_\beta]\\ &= \pm i([b_\alpha,Ib_\beta]+[Ib_\alpha,b_\beta]) + ([Ib_\alpha,Ib_\beta]-[b_\alpha,b_\beta])\\ &= \pm i([b_\alpha,Ib_\beta]+[Ib_\alpha,b_\beta]) + I([b_\alpha,Ib_\beta]+[Ib_\alpha,b_\beta])\,.\end{aligned}$$

6 Example: Twistor and reflector spaces

As an application we now describe Kählerian and parallel bilagrangian structures on twistor and reflector spaces of 4-dimensional pseudo-Riemannian spheres and complex projective spaces. Let $(E^{4,k}, \langle\cdot\,,\cdot\rangle, \mathcal{O})$ be a 4-dimensional oriented pseudo-Euclidean vector space of index k.

Then we set in case $k=0$

$$J^-(E^4) = \{J : E^4 \to E^4 \text{ linear} \mid \quad \begin{array}{ll} (1) & J^2 = -\operatorname{Id} \\ (2) & J \in SO(E^4) \\ (3) & \Omega^J \wedge \Omega^J = -\mathcal{O} \quad \}, \end{array} \tag{15}$$

where $\Omega^J := \langle J\cdot, \cdot\rangle$.

If $k=4$, then we define $E^4 := (E^{4,4}, -\langle\cdot\,,\cdot\rangle, \mathcal{O})$ and $J^-(E^{4,4}) = J^-(E^4)$.

If $k=2$ we define

$$J^-(E^{4,2}) = \{J : E^{4,2} \to E^{4,2} \text{ linear} \mid \quad \begin{array}{ll} (1) & J^2 = -\operatorname{Id} \\ (2) & J \in SO(E^{4,2}) \\ (3) & \Omega^J \wedge \Omega^J = -\mathcal{O} \quad \}, \end{array} \tag{16}$$

and in case there is given a space and time orientation on $E^{4,2}$

$$J^{-+}(E^{4,2}) = \{J \in J^-(E^{4,2}) \mid \quad \begin{array}{l} (X, JX) \text{ is positively time oriented} \\ \text{for any time-like vector } X\}. \end{array}$$

Furthermore, we define in case $k=2$

$$S^-(E^{4,2}) = \{J : E^{4,2} \to E^{4,2} \text{ linear} \mid \quad \begin{array}{ll} (1) & J^2 = \operatorname{Id} \\ (2) & \langle JX, JY\rangle = -\langle X, Y\rangle \text{ for all } X, Y \in E^{4,2} \\ (3) & (X, JX, Y, JY) = -\mathcal{O} \\ & \text{for all space-like vectors } X, Y \\ & \text{with } Y \in \{X, JX\}^\perp \qquad \}. \end{array} \tag{17}$$

Choosing an oriented pseudo-orthonormal basis we identify $E^{4,k}$ with the standard Euclidean space $\mathbb{R}^{4,k}$. In case $k=2$ we also use the isometry of $E^{4,2}$ and $(\mathbb{R}^{4,2}_{\mathrm{neut}}, g^{\mathrm{neut}}_{4,2} = \operatorname{diag}(1,-1,1,-1))$. We denote by $SO(4,2)^{\mathrm{neut}}$ the group $SO(\mathbb{R}^{4,2}_{\mathrm{neut}})$. $SO(4,2)^{\mathrm{neut}}$ is conjugate to $SO(4,2)$ in $GL(4,\mathbb{R})$. Let $SO^+(4,2)^{\mathrm{neut}}$ be the connected component of the unity of this group.

Notation By D_{ij} we denote a quadratic matrix (of suitable dimension) whose (i,j)-entry is 1 and all of whose other entries are 0. We set $E_{ij} = -D_{ij} + D_{ji}$ and $A_{ij} = D_{ij} + D_{ji}$.

In the following let $J_0^+, J_0^-, J_1^+, J_1^- : \mathbb{R}^4 \to \mathbb{R}^4$ be those linear maps that are given with respect to the standard basis by the matrices

$$\begin{array}{ll} J_0^+ = E_{12}+E_{34} & J_0^- = E_{12}-E_{34} \\ J_1^+ = A_{12}+A_{34} & J_1^- = -A_{12}+A_{34}\,. \end{array}$$

The standard embeddings $U(2) \hookrightarrow SO(4)$ and $U(2,1) \hookrightarrow SO^+(4,2)$ are given by

$$\begin{aligned} U(2) &= \{A \in SO(4) \mid AJ_0^+ = J_0^+ A\} \\ U(2,1) &= \{A \in SO^+(4,2) \mid AJ_0^+ = J_0^+ A\} \\ &= \{A \in SO(4,2) \mid AJ_0^+ = J_0^+ A\}. \end{aligned}$$

Proposition 6.1 *The twistor and reflector spaces of $\mathbb{R}^{4,k}$ are the following reductive spaces:*
$J^-(\mathbb{R}^4) = SO(4)/G_0^-$
$J^-(\mathbb{R}^{4,2}) = SO(4,2)/(G_0^-)'$
$J^{-+}(\mathbb{R}^{4,2}) = SO^+(4,2)/(G_0^-)'$
$S^-(\mathbb{R}^{4,2}_{\text{neut}}) = SO^(4,2)^{\text{neut}}/G_1^- = SO^+(4,2)^{\text{neut}}/(G_1^-)^+$
where
$G_0^- = \{A \in SO(4) \mid AJ_0^- = J_0^- A\}$
$(G_0^-)' = \{A \in SO(4,2) \mid AJ_0^- = J_0^- A\}$
$G_1^- = \{A \in SO(4,2)^{\text{neut}} \mid AJ_1^- = J_1^- A\}$
$(G_1^-)^+ = \{A \in SO^+(4,2)^{\text{neut}} \mid AJ_1^- = J_1^- A\}$
$J^-(\mathbb{R}^4)$ is diffeomorphic to the 2-sphere, $J^-(\mathbb{R}^{4,2})$ is a two-sheet hyperboloid, $J^{-+}(\mathbb{R}^{4,2})$ is one sheet of $J^-(\mathbb{R}^{4,2})$, and $S^-(\mathbb{R}^{4,2}_{\text{neut}})$ is a one-sheet hyperboloid.

Proof. Obviously the indicated groups act on the twistor and reflector spaces by conjugation. We have to verify, that these actions are transitiv. First we prove that $SO(4)$ acts transitively on $J^-(\mathbb{R}^4)$. Indeed, let J be an arbitrary element of $J^-(\mathbb{R}^4)$. We choose orthonormal vectors X, Y such that $Y \perp \{X, JX\}$. Then $X, JX, Y, -JY$ is a positively oriented orthonormal basis and J is given with respect to this basis by the matrix $E_{12} - E_{34} \in J^-(\mathbb{R}^4)$.

In the same way we prove that $SO(4,2)$ acts transitively on $J^-(\mathbb{R}^{4,2})$. Let $J \in J^-(\mathbb{R}^{4,2})$ be arbitrarily choosen. Let X be a time-like unit vector. Then $J(X)$ is time-like, too. Now choose a space-like unit vector Z satisfying $Y \perp \{X, JX\}$. Then $X, JX, Y, -JY$ is a positively oriented pseudo-orthonormal basis and J is given with respect to this basis by the matrix $E_{12} - E_{34}$.

Clearly, $SO^+(4,2)$ acts transitively on $J^{-+}(\mathbb{R}^{4,2})$. Furthermore, one checks directly that a matrix in $SO(4,2)$ which commutes with J_0^- must be in $SO^+(4,2)$. Hence, the same isotropy group occurs as in the previous case.

Finally, $SO^+(4,2)^{\text{neut}}$ acts transitively on $S^-(\mathbb{R}^{4,2}_{\text{neut}})$. Let $J \in S^-(\mathbb{R}^{4,2}_{\text{neut}})$ be arbitrarily choosen. Now fix a space-like unit vector X and choose a space-like unit vector Y such that $Y \perp \{X, JX\}$ and (X, Y) is positively space oriented. $\square$

Denote by D the matrix $\operatorname{diag}(1,1,-1,1)$. Then the isotropy groups of J_0^- are equal to

$$G_0^- = \{A \in SO(4) \mid AJ_0^- = J_0^- A\}$$

$$
\begin{aligned}
&= \{A \in SO(4) \mid ADJ_0^+D^{-1}A^{-1} = DJ_0^+D^{-1}\} \\
&= \{A \in SO(4) \mid (D^{-1}AD)J_0^+(D^{-1}A^{-1}D) = J_0^+\} \\
&= \{A \in SO(4) \mid D^{-1}AD \in U(2)\} \\
&= D\,U(2)\,D^{-1} \\
(G_0^-)' &= \{A \in SO(4,2) \mid AJ_0^-A^{-1} = J_0^-\} \\
&= D\,U(2,1)\,D^{-1}.
\end{aligned}
$$

Now let F be the matrix

$$
F = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 1 \end{pmatrix}.
$$

Then we have $J_1^- = F\,\mathrm{diag}(1,1,-1,-1)F^{-1}$. Consequently, $A \in GL(4,\mathbb{R})$ commutes with J_1^- if and only if $AF\,\mathrm{diag}(1,1,-1,-1)F^{-1} = F\,\mathrm{diag}(1,1,-1,-1)F^{-1}A$, i.e. $F^{-1}AF\,\mathrm{diag}(1,1,-1,-1) = \mathrm{diag}(1,1,-1,-1)F^{-1}AF$. Thus

$$
A = F \begin{pmatrix} M_1 & 0 \\ 0 & M_2 \end{pmatrix} F^{-1}, \quad M_1, M_2 \in GL(2,\mathbb{R}).
$$

If now $A \in SO(4,2)^{\mathrm{neut}}$ then we have

$$
F \begin{pmatrix} M_1 & 0 \\ 0 & M_2 \end{pmatrix} F^{-1}\,\mathrm{diag}(1,-1,1,-1)(F^{-1})^t \begin{pmatrix} M_1^t & 0 \\ 0 & M_2^t \end{pmatrix} F^t = \mathrm{diag}(1,-1,1,-1)
$$

or, equivalently,

$$
\begin{pmatrix} M_1 & 0 \\ 0 & M_2 \end{pmatrix} \begin{pmatrix} 0 & -E \\ -E & 0 \end{pmatrix} \begin{pmatrix} M_1^t & 0 \\ 0 & M_2^t \end{pmatrix} \begin{pmatrix} 0 & -E \\ -E & 0 \end{pmatrix}, \quad \text{where } E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.
$$

Hence, $M_2 = (M_1^t)^{-1}$ and we conclude $G_1^- \cong GL(2,\mathbb{R})$, $(G_1^-)^+ \cong GL^+(2,\mathbb{R})$ (see also [JR]).

We denote by $\mathfrak{g}_0^-$ and $\mathfrak{g}_1^-$ the Lie algebras of G_0^- and G_1^-, respectively. Next we prove that the various twistor and reflector spaces arise by duality from $J^-(\mathbb{R}^4)$. One can check this directly, but it is more conveniently to use the following facts.

Lemma 6.1 *Let $\mathfrak{g}$ be a Lie algebra, $Z_0 \in \mathfrak{g}$ a fixed element and $\mathfrak{h}$ the subalgebra of $\mathfrak{g}$ defined by*

$$
\mathfrak{h} = \{Z \in \mathfrak{g} \mid [Z, Z_0] = 0\}.
$$

If now $T : \mathfrak{g} \to \mathfrak{g}$ is an involutive automorphism, then in case $Z_0 \in \mathfrak{g}_+$ the subalgebra $\mathfrak{h}'$ equals

$$
\mathfrak{h}' = \{Z' \in \mathfrak{g}' \mid [Z', Z_0] = 0\}
$$

and in case $Z_0 \in \mathfrak{g}_-$ we have

$$
\mathfrak{h}' = \{Z' \in \mathfrak{g}' \mid [Z', iZ_0] = 0\}.
$$

$\square$

Lemma 6.2 *Let $\langle\cdot,\cdot\rangle$ be an inner product on the Lie algebra $\mathfrak{g}$ and $T:\mathfrak{g}\to\mathfrak{g}$ an isometric involutive automorphism. Let $(\mathfrak{g}',\langle\cdot,\cdot\rangle')$ be a T-dual of $(\mathfrak{g},\langle\cdot,\cdot\rangle)$. Furthermore let $\mathfrak{h}\subset\mathfrak{g}$ be a subalgebra such that $T(\mathfrak{h})=\mathfrak{h}$. Define $\mathfrak{m}=\mathfrak{h}^\perp$. Then we have $T(\mathfrak{m})=\mathfrak{m}$ and $\mathfrak{m}'=(\mathfrak{h}')^\perp$.*

$\square$

Now we define involutive automorphisms t_1 and t_2 of $\mathfrak{g}$ by the following decompositions.

$$\begin{aligned} t_1\ :\quad \mathfrak{so}(4)_+ &= \mathrm{span}\{E_{12},E_{34}\}\\ \mathfrak{so}(4)_- &= \mathrm{span}\{E_{13},E_{14},E_{23},E_{24}\}. \end{aligned}\tag{18}$$

$$\begin{aligned} t_2\ :\quad \mathfrak{so}(4)_+ &= \mathrm{span}\{E_{13},E_{24}\}\\ \mathfrak{so}(4)_- &= \mathrm{span}\{E_{12},E_{14},E_{23},E_{34}\}. \end{aligned}\tag{19}$$

We consider the reductive spaces $J^-(\mathbb{R}^4)$, , $J^-(\mathbb{R}^{4,2})$, $J^{-+}(\mathbb{R}^{4,2})$, $S^-(\mathbb{R}^{4,2}_{\mathrm{neut}})$ together with their homogeneous standard metrics.

Proposition 6.2 *$J^-(\mathbb{R}^{4,2})=SO(4,2)/(G_0^-)'$ and $J^{-+}(\mathbb{R}^{4,2})=SO^+(4,2)/(G_0^-)'$ are t_1-dual to $J^-(\mathbb{R}^4)=SO(4)/G_0^-$. $S^-(\mathbb{R}^{4,2}_{\mathrm{neut}})=SO^+(4,2)^{\mathrm{neut}}/(G_1^-)^+$ is t_2-dual to $J^-(\mathbb{R}^4)$.*

Proof. The homogeneous standard metrics are induced by the negative of the Killing form. In case of t_1 we use the identification $\mathfrak{so}(4)'\cong\mathfrak{so}(4,2)$. We have $t_1(\mathfrak{g}_0^-)=\mathfrak{g}_0^-$ and identify the dual subalgebra $(\mathfrak{g}_0^-)'$ with the corresponding subalgebra of $\mathfrak{so}(4,2)$. We denote by $-\kappa$ and $-\kappa'$ the negatives of the Killing forms of $\mathfrak{so}(4)$ and $\mathfrak{so}(4,2)$. Then $(\mathfrak{so}(4),-\kappa)$ and $(\mathfrak{so}(4,2),-\kappa')$ are t_1-dual. We define $\mathfrak{m}:=(\mathfrak{g}_0^-)^\perp$, $\mathfrak{m}':=(\mathfrak{g}_0^-)'^\perp$ and apply Lemma (6.2). We conclude that $(\mathfrak{so}(4,2),(\mathfrak{g}_0^-)',\mathfrak{m}',-\kappa'|_{\mathfrak{m}'})$ is t_1-dual to $(\mathfrak{so}(4),\mathfrak{g}_0^-,\mathfrak{m},-\kappa|_{\mathfrak{m}})$. It remains to prove that $(\mathfrak{g}_0^-)'$ is the Lie algebra of $(G_0^-)'$. However, this follows from Lemma (6.1) and $J_0^-\in\mathfrak{so}(4)_+$.

In case of t_2 the Lie algebra $\mathfrak{so}(4)'=\mathfrak{so}(4)_++i\mathfrak{so}(4)_-$ is isomorphic to the Lie algebra $\mathfrak{so}(4,2)^{\mathrm{neut}}$ of $SO(4,2)^{\mathrm{neut}}$. The isomorphism is given by

$$\begin{aligned} E_{13} &\longmapsto E_{13}\\ E_{24} &\longmapsto E_{24}\\ iE_{12} &\longmapsto -A_{12}\\ iE_{14} &\longmapsto -A_{14}\\ iE_{23} &\longmapsto A_{23}\\ iE_{34} &\longmapsto -A_{34}. \end{aligned}$$

Then we continue as above up to the fact that now $J_0^-\in\mathfrak{so}(4)_-$ and $iJ_0^-\in i\mathfrak{so}(4)_-\subset\mathfrak{so}(4)'=\mathfrak{so}(4,2)^{\mathrm{neut}}$ equals J_1^-. $\square$

Proposition 6.3 *1.* $J^-(\mathbb{R}^4) = U(2)/U(1) \times U(1)$

2. $J^{-+}(\mathbb{R}^{4,2}) = U(2,1)/U(1) \times U(1)$

3. $S^-(\mathbb{R}^{4,2}_{\mathrm{neut}}) = (G_1^+)^+/(G_1^+)^+ \cap G_1^-$*, where* $(G_1^+)^+$ *is isomorphic to* $GL^+(2,\mathbb{R})$ *and* $(G_1^+)^+ \cap G_1^-$ *is isomorphic to* $SO^+(2,1) \times SO^+(2,1) \cup SO^-(2,1) \times SO^-(2,1)$ *.*

4. $S^-(\mathbb{R}^{4,2}) = U(2,1)/SO(2,1) \times U(1)$ *.*

Proof. We have to prove the transitivity of the group actions. As for the $SU(2)$-action on $J^-(\mathbb{R}^4)$ this follows because of the compactness of $SU(2)$ by comparing the dimensions of the corresponding Lie algebras.

First we will prove that $U(2,1)$ acts transitively by conjugation on $J^{-+}(\mathbb{R}^{4,2})$. Let J be an arbitrary element of $J^{-+}(\mathbb{R}^{4,2})$ different from J_0^-. Because of the properties (1) and (2) in (17) J has to be of the form

$$J = \begin{pmatrix} 0 & -a & b & c \\ a & 0 & -c & b \\ b & -c & 0 & a \\ c & b & -a & 0 \end{pmatrix} \quad \text{or} \quad J = \begin{pmatrix} 0 & -a & b & c \\ a & 0 & c & -b \\ b & c & 0 & -a \\ c & -b & a & 0 \end{pmatrix},$$

where $a^2 - b^2 - c^2 = 1$. Because of property (3) of (17) the second case is impossible. Furthermore, since J is in $J^{-+}(\mathbb{R}^{4,2})$ the pair

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ a \\ b \\ c \end{pmatrix}$$

has to be positively time oriented, hence, $a > 0$. Now we obtain from $a^2 - b^2 - c^2 = 1$ the condition $a \geq 1$. Since $J \neq J_0^-$ we conclude that $a > 1$. Consequently, the vectors

$$X = \frac{1}{\sqrt{2a-2}} \begin{pmatrix} -b \\ c \\ 0 \\ a-1 \end{pmatrix} \qquad JX = J_0^+ X = \frac{1}{\sqrt{2a-2}} \begin{pmatrix} -c \\ -b \\ -a+1 \\ 0 \end{pmatrix}$$

$$Y = \frac{1}{\sqrt{2a+2}} \begin{pmatrix} -b \\ c \\ 0 \\ a+1 \end{pmatrix} \qquad -JY = J_0^+ Y = \frac{1}{\sqrt{2a+2}} \begin{pmatrix} -c \\ -b \\ -a-1 \\ 0 \end{pmatrix}$$

constitute a positively space and time oriented pseudo-orthonormal basis of $\mathbb{R}^{4,2}$. We consider the transformation of the standard basis into this new basis. The corresponding transformation matrix is in $U(2,1)$ and J is transformed into J_0^-. The isotropy group of J_0^- equals $U(1) \times U(1)$.

Now we will prove assertion 3. We denote by G_1^+ the group

$$G_1^+ \;:=\; \{A \in SO(4,2)^{\mathrm{neut}} \mid [A, J_1^+] = 0\} \tag{20}$$

Its Lie algebra equals

$$\mathfrak{g}_1^+ = \mathrm{span}\{A_{14} + A_{23}\,,\, E_{13} + E_{24}\,,\, A_{12}\,,\, A_{34}\}\,.$$

Conjugation by

$$B = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix}$$

yields

$$B\,\mathfrak{g}_1^+\,B^{-1} = \{\begin{pmatrix} M & 0 \\ 0 & -M^t \end{pmatrix} \mid M \in \mathbb{R}(2)\} \cong \mathfrak{gl}(2,\mathbb{R}).$$

Hence, G_1^+ is isomorphic to $GL(2)$ and the connected component $(G_1^+)^+ = G_1^+ \cap SO^+(4,2)^{\mathrm{neut}}$ of the unity of G_1^+ is isomorphic to $GL^+(2)$.

$(G_1^+)^+$ acts transitively on $S^-(\mathbb{R}^{4,2}_{\mathrm{neut}})$. Let J be an arbitrary element of $S^-(\mathbb{R}^{4,2}_{\mathrm{neut}})$. Then we have because of properties (1) and (2) of (17)

$$J = \begin{pmatrix} 0 & a & -b & c \\ a & 0 & c & -b \\ b & c & 0 & -a \\ c & b & -a & 0 \end{pmatrix} \qquad \text{or} \qquad J = \begin{pmatrix} 0 & a & -b & c \\ a & 0 & -c & b \\ b & -c & 0 & a \\ c & -b & a & 0 \end{pmatrix},$$

where $a^2 - b^2 + c^2 = 1$. Because of property (3) of (17) the second case is impossible. We consider the vectors

$$X_1 = \frac{1}{\sqrt{|2+2a|}}\begin{pmatrix} b \\ -c \\ 0 \\ a+1 \end{pmatrix} \qquad X_2 = -JX_1 = J_1^+X_1 = \frac{1}{\sqrt{|2+2a|}}\begin{pmatrix} -c \\ b \\ a+1 \\ 0 \end{pmatrix}$$
$$X_3 = \frac{1}{\sqrt{|2-2a|}}\begin{pmatrix} b \\ -c \\ 0 \\ a-1 \end{pmatrix} \qquad X_4 = JX_3 = J_1^+X_3 = \frac{1}{\sqrt{|2-2a|}}\begin{pmatrix} -c \\ b \\ a-1 \\ 0 \end{pmatrix}.$$

We have

$$\begin{aligned} ||X_1||^2 &= -\,\mathrm{sgn}(2+2a) = -||X_2||^2 \\ ||X_3||^2 &= -\,\mathrm{sgn}(2-2a) = -||X_4||^2 \quad . \end{aligned}$$

Consider first the case $a < -1$. Then either

$$X_1\,,\, X_2\,,\, X_4\,,\, X_3$$

or

$$X_1\,,\, X_2\,,\, -X_4\,,\, -X_3$$

constitute a $(G_1^+)^+$-basis, i.e. a basis such that the matrix of the transformation from the standard basis into this one is in $(G_1^+)^+$. With respect to this basis J is given by the matrix J_1^-.
In case $-1 < a < 1$ either

$$X_2\,,\, X_1\,,\, X_4\,,\, X_3$$

or

$$X_2\,,\, X_1\,,\, -X_4\,,\, -X_3$$

is a $(G_1^+)^+$-basis and with respect to this basis J again is given by the matrix J_1^-.
In case $a = 1$ the vectors $e_3, e_4, -e_1, -e_2$ constitute a $(G_1^+)^+$-basis such that J is given by the matrix J_1^- with respect to this basis.
Finally, we consider the case $a > 1$. Then either

$$X_2\,,\, X_1\,,\, X_3\,,\, X_4$$

or

$$X_2\,,\, X_1\,,\, -X_3\,,\, -X_4$$

constitute a $(G_1^+)^+$-basis and also with respect to this basis J is given by the matrix J_1^-.

The isotropy group of J_1^- equals

$$\begin{aligned}
(G_1^+)^+ \cap G_1^- &= \{ \begin{pmatrix} a_1 & a_2 & 0 & 0 \\ a_2 & a_1 & 0 & 0 \\ 0 & 0 & a_3 & a_4 \\ 0 & 0 & a_4 & a_3 \end{pmatrix} \in SO(4,2)^{\mathrm{neut}} \mid a_1, ..., a_4 \in \mathbb{R}\,,\, a_1 \cdot a_3 > 0\} \\
&= (SO^+(2,1) \times SO^+(2,1)) \cup (SO^-(2,1) \times SO^-(2,1))
\end{aligned}$$

It remains statement 4. We prove that $U(2,1)$ acts transitively on $S^-(\mathbb{R}^{4,2})$. Let J be an arbitrary element of $S^-(\mathbb{R}^{4,2}_{\mathrm{neut}})$. We proceed in the same way as above. We have because of properties (1) and (2) of (17)

$$J = \begin{pmatrix} 0 & -a & b & c \\ a & 0 & -c & b \\ b & -c & 0 & a \\ c & b & -a & 0 \end{pmatrix} \qquad \text{or} \qquad J = \begin{pmatrix} 0 & -a & b & c \\ a & 0 & c & -b \\ b & c & 0 & -a \\ c & -b & a & 0 \end{pmatrix},$$

where $-a^2 + b^2 + c^2 = 1$. Because of property (3) of (17) the second case is impossible. Now we consider the vectors

$$X_1 = \frac{1}{2\sqrt{|a-c|}} \begin{pmatrix} 1+c-a \\ b \\ 1+a-c \\ b \end{pmatrix} \qquad X_2 = J_0^- X_1 = \frac{1}{2\sqrt{|a-c|}} \begin{pmatrix} -b \\ 1+c-a \\ -b \\ 1+a-c \end{pmatrix}$$

$$X_3 = JX_1 = \frac{1}{2\sqrt{|a-c|}} \begin{pmatrix} b \\ 1+a-c \\ b \\ 1-a+c \end{pmatrix} \qquad X_4 = JJ_0^- X_1 = \frac{1}{2\sqrt{|a-c|}} \begin{pmatrix} -1-a+c \\ b \\ -1+a-c \\ b \end{pmatrix}$$

which constitute a $U(2,1)$-basis in case $a < c$. If $a > c$

$$X_3\,,\, X_4\,,\, X_1\,,\, X_2$$

is a $U(2,1)$-basis. With respect to both of these bases J is given by the matrix $J_* = A_{13} + A_{24}$. If $a = c$ then $b = \pm 1$. In case $b = -1$

$$\mathrm{diag}(-1,-1,1,1) \cdot J \cdot \mathrm{diag}(-1,-1,1,1) = J_*.$$

The isotropy group of J_* with respect to the $U(2,1)$-action equals

$$\begin{aligned}
&\{A \in U(2,1) \mid A = \begin{pmatrix} \alpha & \beta \\ \beta & \alpha \end{pmatrix}\,,\, \alpha\,,\, \beta \in \mathbb{C}\} \\
&= \{\begin{pmatrix} r_1\alpha & r_2\alpha \\ r_2\alpha & r_1\alpha \end{pmatrix} \mid r_1, r_2 \in \mathbb{R}\,,\, r_1^2 - r_2^2 = 1\,,\, \alpha \in \mathbb{C}\,,\, |\alpha| = 1\} \\
&= SO(2,1) \times U(1)\ .
\end{aligned}$$

□

Here the spaces $J^-(\mathbb{R}^{4,2})\,,\, S^-(\mathbb{R}^{4,2}_{\mathrm{neut}})\,,\, S^-(\mathbb{R}^{4,2})$ again arise by dualisation from $J^-(\mathbb{R}^4)$. The involutive automorphism $t_1 : \mathfrak{u}(2) \to \mathfrak{u}(2)$ defined by

$$\begin{aligned}
\mathfrak{u}(2)_+ &= \mathrm{span}\{iD_{11}\,,\, iD_{22}\} \\
\mathfrak{u}(2)_- &= \mathrm{span}\{E_{12}\,,\, iA12\}
\end{aligned} \tag{21}$$

yields $J^{-+}(\mathbb{R}^{4,2})$. Analogously,

$$\begin{aligned}
t_2 :\quad \mathfrak{u}(2)_+ &= \mathrm{span}\{E_{12}\} \\
\mathfrak{u}(2)_- &= \mathrm{span}\{iD_{11}\,,\, iD_{22}\,,\, iA12\}
\end{aligned} \tag{22}$$

and

$$\begin{aligned}
t_3 :\quad \mathfrak{u}(2)_+ &= \mathrm{span}\{Z_1 = \frac{1}{2}\begin{pmatrix} i & i \\ i & i \end{pmatrix}\,,\, Z_2 = \frac{1}{2}\begin{pmatrix} i & -i \\ -i & i \end{pmatrix}\} \\
\mathfrak{u}(2)_- &= \mathrm{span}\{Z_3 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\,,\, Z_4 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}\}
\end{aligned} \tag{23}$$

yield $S^-(\mathbb{R}^{4,2}_{\mathrm{neut}})$ and $S^-(\mathbb{R}^{4,2})$, respectively. As for t_3 we remark, that by conjugation by $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ the basis $\{Z_1, Z_2, Z_3, Z_4\}$ of $\mathfrak{u}(2)' = \mathfrak{u}(2)_+ \oplus i\mathfrak{u}(2)_-$ becomes the standard basis $\{iD_{11}\,,\, iD_{22}\,,\, iE_{12}\,,\, A_{12}\}$ of $\mathfrak{u}(2,1)$. The basis $\{Z_1 + Z_2, iZ_4\}$ of $(\mathfrak{u}(1) \oplus \mathfrak{u}(1))'$ becomes $\{\begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}\,,\, \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}\}$.

Let now $(M^4, g, \mathcal{O})$ be an oriented pseudo-Riemannian manifold of even index. We define the twistor space $Z^-(M^4)$ by

$$Z^-(M^4) = \bigcup_{m \in M^4} J^-(T_m M^4).$$

If the index equals 2 and M^4 is space and time oriented, then we also consider

$$Z^{-+}(M^4) = \bigcup_{m \in M^4} J^{-+}(T_m M^4).$$

The reflector space $S^-(M^4)$ of a 4-dimensional oriented pseudo-Riemannian manifold of index 2 is defined as (see [JR])

$$Z_-(M^4) = \bigcup_{m \in M^4} S^-(T_m M^4).$$

We will now describe the twistor spaces of the spheres S^4, $S^{4,4}/\mathbb{Z}_2$, $S^{4,2}$ and the complex projective spaces $\mathbb{C}P^2$, $\mathbb{C}P^{2,2}$ and $\mathbb{C}P^{2,1}$ as well as the reflector spaces of $S^{4,2}$ and $\mathbb{C}P^{2,1}$. In all these examples we have the following situation.

Remark 6.1 *Let be given Lie algebras $\mathfrak{g} \supset \mathfrak{h} \supset \mathfrak{k}$ and subspaces $\mathfrak{m}, \mathfrak{n} \subset \mathfrak{g}$ such that*

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}, \;\; \mathfrak{h} = \mathfrak{k} \oplus \mathfrak{n}, \;\; [\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}, \;\; [\mathfrak{k}, \mathfrak{n}] \subset \mathfrak{n}.$$

Furthermore, let on $\mathfrak{m}$ be given an $\mathrm{ad}(\mathfrak{h})$-invariant inner product $\langle \cdot\,, \cdot \rangle_{\mathfrak{m}}$ and on $\mathfrak{n}$ an $\mathrm{ad}(\mathfrak{k})$-invariant inner product $\langle \cdot\,, \cdot \rangle_{\mathfrak{n}}$. Now let $T : \mathfrak{g} \to \mathfrak{g}$ be an involutive automorphism such that

$$T(\mathfrak{k}) = \mathfrak{k} \; , \;\; T(\mathfrak{h}) = \mathfrak{h} \; , \;\; T(\mathfrak{m}) = \mathfrak{m} \; , \;\; T(\mathfrak{n}) = \mathfrak{n}$$

and $T|_{\mathfrak{m}} : \mathfrak{m} \to \mathfrak{m}$ and $T|_{\mathfrak{n}} : \mathfrak{n} \to \mathfrak{n}$ are isometries. If $G \supset H \supset K$ are Lie groups such that G/H is a reductive space associated to $(\mathfrak{g}, \mathfrak{h}, \mathfrak{m}, \langle \cdot\,, \cdot \rangle_{\mathfrak{m}})$ and H/K is associated to $(\mathfrak{h}, \mathfrak{k}, \mathfrak{n}, \langle \cdot\,, \cdot \rangle_{\mathfrak{n}})$ and if $G' \supset H' \supset K'$ are Lie groups such that G'/H', H'/K' are T-dual to G/H and H/K, respectively, then obviously also $(\mathfrak{g}, \mathfrak{k}, \mathfrak{m} \oplus \mathfrak{n}, \langle \cdot\,, \cdot \rangle_{\mathfrak{m}} \oplus \langle \cdot\,, \cdot \rangle_{\mathfrak{n}})$ and $(\mathfrak{g}', \mathfrak{k}', \mathfrak{m}' \oplus \mathfrak{n}', \langle \cdot\,, \cdot \rangle'_{\mathfrak{m}} \oplus \langle \cdot\,, \cdot \rangle'_{\mathfrak{n}})$ are T-dual and $G \times_H H/K = G/K$ and $G' \times_{H'} H'/K' = G'/K'$ are T-dual.

Now we begin with the description of $Z^-(S^4)$. According to the notation introduced above we consider

$$\begin{aligned}
\mathfrak{g} &= \mathfrak{so}(5) = \mathrm{span}\{E_{12}, \ldots, E_{45}\} \\
\mathfrak{h} &= \mathfrak{so}(4) = \mathrm{span}\{E_{12}, \ldots, E_{34}\} \\
\mathfrak{k} &= \mathfrak{g}_0^- = \mathrm{span}\{E_{14} + E_{23}\, , \, E_{13} - E_{24}\, , \, E_{12}\, , \, E_{34}\} \\
\mathfrak{m} &= \mathrm{span}\{E_{15}, \ldots, E_{45}\} \\
\mathfrak{n} &= \mathrm{span}\{E_{14} - E_{23}\, , \, E_{13} + E_{24}\}.
\end{aligned}$$

We define the inner product B on $\mathfrak{so}(5)$ by $B(Z_1, Z_2) = -\frac{1}{2}\,\mathrm{tr}(Z_1 Z_2)$ for $Z_1, Z_2 \in \mathfrak{so}(5)$ and then $\langle \cdot\,, \cdot \rangle_{\mathfrak{m}}$ and $\langle \cdot\,, \cdot \rangle_{\mathfrak{n}}$ by

$$\langle \cdot\,, \cdot \rangle_{\mathfrak{m}} = B|_{\mathfrak{m}} \; , \quad \langle \cdot\,, \cdot \rangle_{\mathfrak{m}} = tB|_{\mathfrak{n}} \; .$$

Then the sphere S^4 is the homogeneous space $SO(5)/SO(4)$ associated to $(\mathfrak{g}, \mathfrak{h}, \mathfrak{m}, \langle \cdot\,, \cdot \rangle_{\mathfrak{m}})$ and $J_-(\mathbb{R}^4)$ is the homogeneous space $SO(4)/G_0^-$ associated to $(\mathfrak{h}, \mathfrak{k}, \mathfrak{n}, \langle \cdot\,, \cdot \rangle_{\mathfrak{n}})$ and we have

$$\begin{aligned}
Z^-(S^4) &= SO(5) \times_{SO(4)} J^-(\mathfrak{m}) \\
&= SO(5) \times_{SO(4)} J^-(\mathbb{R}^4),
\end{aligned}$$

where $SO(4)$ acts on $J^-(\mathbb{R}^4)$ by conjugation. According to $J_-(\mathbb{R}^4) = SO(4)/G_0^-$ we conclude that $Z^-(S^4)$ is the homogeneous space

$$Z^-(S^4) = SO(5) \times_{SO(4)} (SO(4)/G_0^-) = SO(5)/G_0^-$$

associated to $(\mathfrak{g}, \mathfrak{k}, \mathfrak{m} \oplus \mathfrak{n}, \langle \cdot\,,\cdot \rangle_{\mathfrak{m}} \oplus \langle \cdot\,,\cdot \rangle_{\mathfrak{n}})$. The vectors

$$\begin{gathered} X_1 = E_{15}\,,\ X_2 = E_{25}\,,\ X_3 = E_{35}\,,\ X_4 = E_{45}\,, \\ X_5 = \tfrac{1}{\sqrt{2t}}(E_{13} + E_{24})\,,\ X_6 = \tfrac{1}{\sqrt{2t}}(E_{14} - E_{23}) \end{gathered}$$

constitute an orthonormal basis of $\mathfrak{m} \oplus \mathfrak{n}$. With respect to this basis the map

$$\Lambda = \Lambda_{\mathfrak{m}\oplus\mathfrak{n}} : \mathfrak{m} \oplus \mathfrak{n} \longrightarrow \mathfrak{sm} \oplus \mathfrak{n})$$

which describes the Levi-Civita connection is given by

$$\begin{aligned} \Lambda(X_1) &= \frac{\sqrt{t}}{2\sqrt{2}}(E_{35} + E_{46}) \\ \Lambda(X_2) &= \frac{\sqrt{t}}{2\sqrt{2}}(-E_{36} + E_{45}) \\ \Lambda(X_3) &= \frac{\sqrt{t}}{2\sqrt{2}}(-E_{15} + E_{26}) \\ \Lambda(X_4) &= \frac{\sqrt{t}}{2\sqrt{2}}(-E_{16} - E_{25}) \\ \Lambda(X_5) &= \frac{t-2}{2\sqrt{2t}}(E_{13} - E_{24}) \\ \Lambda(X_6) &= \frac{t-2}{2\sqrt{2t}}(E_{14} + E_{23}). \end{aligned}$$

Calculating the curvature tensor of the metric g_t on $Z^-(S^4)$ induced by $\langle \cdot\,,\cdot \rangle = \langle \cdot\,,\cdot \rangle_{\mathfrak{m}} \oplus \langle \cdot\,,\cdot \rangle_{\mathfrak{n}}$ one can see, that g_t is an Einstein metric for $t = 1$ and $t = 2$.

We define linear $\mathrm{Ad}(G_0^-)$-invariant maps $I_1, I_2 : \mathfrak{m} \oplus \mathfrak{n} \to \mathfrak{m} \oplus \mathfrak{n}$ by

$$\begin{aligned} &I_{1,2}(X_1) = X_2 \\ &I_{1,2}(X_3) = -X_4 \\ &I_1(X_5) = -X_6 \quad I_2(X_5) = X_6 \\ &(I_1)^2 = (I_2)^2 = -\mathrm{Id}\,. \end{aligned}$$

In case $t = 2$ the almost complex structure on $Z^-(S^4)$ induced by I_1 is Kählerian, since $I_1 \circ \Lambda(X) = \Lambda(X) \circ I_1$ holds for all $X \in \mathfrak{m} \oplus \mathfrak{n}$. I_2 induces a non-integrable almost complex structure.

We now summarize the above observations.

Proposition 6.4 *The twistor space $Z^-(S^4)$ is the homogeneous space $(SO(5)/G_0^-, g_t)$ associated to $(\mathfrak{so}(5), \mathfrak{g}_0^-, \mathfrak{m} \oplus \mathfrak{n}, \langle \cdot\,,\cdot \rangle)$. In case $t = 1$ or $t = 2$ the metric g_t satisfies the Einstein condition. Furthermore, in case $t = 2$ the space $(Z^-(S^4), g_t)$ admits an $SO(5)$-invariant Kähler structure.*

Now we consider the involutive automorphism $T_0 : \mathfrak{so}(5) \to \mathfrak{so}(5)$ defined by the decomposition

$$\begin{aligned}\mathfrak{so}(5)_+ &= \operatorname{span}\{E_{12}\,,\, E_{13}\,,\, E_{14}\,,\, E_{23}\,,\, E_{24}\,,\, E_{34}\} \\ &= \mathfrak{so}(4) \\ \mathfrak{so}(5)_- &= \operatorname{span}\{E_{15}\,,\, E_{25}\,,\, E_{35}\,,\, E_{45}\}.\end{aligned}$$

In particular, we have $T_0|_{\mathfrak{so}(4)} = \mathrm{Id}$.

Then $S^{4,4}/\mathbb{Z}_2 = SO^+(5,4)/SO(4)$ equipped with the standard metric is T_0-dual to $S^4 = SO(5)/SO(4)$. Furthermore, $J^-(\mathbb{R}^{4,4}) = SO(4)/U(2)$ is T_0-dual to $J^-(\mathbb{R}^4)$. The twistor space $Z^-(S^{4,4}/\mathbb{Z}_2)$ equals

$$\begin{aligned}Z^-(S^{4,4}/\mathbb{Z}_2) &= SO^+(4,1) \times_{SO(4)} J^-(\mathfrak{m}') \\ &= SO^+(5,4) \times_{SO(4)} (SO(4)/U(2)) \\ &= SO^+(5,4)/U(2).\end{aligned}$$

Since $U(2)$ is connected the inner product $\langle \cdot\,,\cdot \rangle'$ on $\mathfrak{m}' \oplus \mathfrak{n}'$ which is T_0-dual to $\langle \cdot\,,\cdot \rangle$ induces an $SO^+(5,4)$-invariant metric g_t' on $Z^-(S^{4,4}/\mathbb{Z}_2)$. Hence, we proved

Proposition 6.5 *$(Z^-(S^{4,4}/\mathbb{Z}_2), g_t')$ is T_0-dual to $(Z^-(S^4), g_t)$. In particular, in case $t = 1$ or $t = 2$ the metric g_t' is an Einstein metric. For $t = 2$ the Kähler structure on $(Z^-(S^4), g_t)$ becomes a parallel complex structure on $(Z^-(S^{4,4}/\mathbb{Z}_2), g_t')$.*

Now we consider a further involutive automorphism. We define $T_1 : \mathfrak{so}(5) \to \mathfrak{so}(5)$ by

$$\begin{aligned}\mathfrak{so}(5)_+ &= \operatorname{span}\{E_{12}\,,\, E_{34}\,,\, E_{35}\,,\, E_{45}\} \\ \mathfrak{so}(5)_- &= \operatorname{span}\{E_{13}\,,\, E_{14}\,,\, E_{15}\,,\, E_{23}\,,\, E_{24}\,,\, E_{25}\}.\end{aligned}$$

In particular, $T_1|_{\mathfrak{so}(4)}$ equals the automorphism t_1 defined in (18).

Then $S^{4,2} = SO^+(5,2)/SO^+(4,2)$ equipped with the standard metric is T_1-dual to $S^4 = SO(5)/SO(4)$. Furthermore, $J^{-+}(\mathbb{R}^{4,2}) = SO^+(4,2)/(G_0^-)'$ is T_1-dual to $J^-(\mathbb{R}^4)$. The twistor space $Z^{-+}(S^{4,2})$ equals

$$\begin{aligned}Z^{-+}(S^{4,2}) &= SO^+(5,2) \times_{SO^+(4,2)} J^{-+}(\mathfrak{m}') \\ &= SO^+(5,2) \times_{SO^+(4,2)} (SO^+(4,2)/(G_0^-)') \\ &= SO^+(5,2)/(G_0^-)'.\end{aligned}$$

Since $(G_0^-)'$ is connected the inner product $\langle \cdot\,,\cdot \rangle'$ on $\mathfrak{m}' \oplus \mathfrak{n}'$ which is T_1-dual to $\langle \cdot\,,\cdot \rangle$ induces an $SO^+(5,2)$-invariant metric g_t' on $Z^{-+}(S^{4,2})$. We obtain

Proposition 6.6 *$(Z^{-+}(S^{4,2}), g_t')$ is T_1-dual to $(Z^-(S^4), g_t)$. In particular, in case $t = 1$ or $t = 2$ the metric g_t' is an Einstein metric. For $t = 2$ the Kähler structure on $(Z^-(S^4), g_t)$ becomes a parallel complex structure on $(Z^{-+}(S^{4,2}), g_t')$.*

Finally, let $T_2 : \mathfrak{so}(5) \to \mathfrak{so}(5)$ be defined by

$$\begin{aligned}\mathfrak{so}(5)_+ &= \operatorname{span}\{E_{13}\,,\, E_{15}\,,\, E_{24}\,,\, E_{35}\}\\ \mathfrak{so}(5)_- &= \operatorname{span}\{E_{12}\,,\, E_{14}\,,\, E_{23}\,,\, E_{25}\,,\, E_{34}\,,\, E_{45}\}.\end{aligned}$$

In particular, $T_2|_{\mathfrak{so}(4)}$ equals the automorphism t_2 defined in (19). We denote by $SO^+(5,2)^{\mathrm{neut}}$ the group $SO^+(\mathbb{R}^5, g = \operatorname{diag}(1,-1,1,-1,1))$. $S^{4,2} = SO^+(5,2)^{\mathrm{neut}}/SO^+(4,2)^{\mathrm{neut}}$ equipped with the standard metric is T_2-dual to $S^4 = SO(5)/SO(4)$. The reflector space $S^-(\mathbb{R}^{4,2}_{\mathrm{neut}}) = SO^+(4,2)^{\mathrm{neut}}/(G_1^-)^+$ is T_2-dual to $J^-(\mathbb{R}^4)$. The reflector space $Z_-(S^{4,2})$ equals

$$\begin{aligned}Z_-(S^{4,2}) &= SO^+(5,2)^{\mathrm{neut}} \times_{SO^+(4,2)^{\mathrm{neut}}} S^-(\mathfrak{m}')\\ &= SO^+(5,2)^{\mathrm{neut}} \times_{SO^+(4,2)^{\mathrm{neut}}} (SO^+(4,2)^{\mathrm{neut}}/(G_1^-)^+)\\ &= SO^+(5,2)^{\mathrm{neut}}/(G_1^-)^+.\end{aligned}$$

Since $(G_1^-)^+$ is connected the inner product $\langle\cdot\,,\cdot\rangle'$ on $\mathfrak{m}' \oplus \mathfrak{n}'$ which is T_2-dual to $\langle\cdot\,,\cdot\rangle$ induces an $SO^+(5,2)^{\mathrm{neut}}$-invariant metric g_t' on $Z_-(S^{4,2})$. Hence, we have

Proposition 6.7 *$(Z_-(S^{4,2}), g_t')$ is T_2-dual to $(Z^-(S^4), g_t)$. In particular, in case $t = 1$ or $t = 2$ the metric g_t' is an Einstein metric. For $t = 2$ the Kähler structure on $(Z^-(S^4), g_t)$ becomes a parallel bilagrangian structure on $(Z_-(S^{4,2}), g_t')$.*

Next we discuss $Z^-(\mathbb{C}P^2)$. According to our notation we consider

$$\begin{aligned}\mathfrak{g} &= \mathfrak{su}(3) = \operatorname{span}\{i(D_{11}-D_{22})\,,\, i(D_{11}-D_{33})\,,\, E_{12}\,,\, iA_{12}\\ &\qquad\qquad E_{13}\,,\, iA_{13}\,,\, E_{23}\,,\, iA_{23}\}\\ \mathfrak{h} &= \mathfrak{u}(2) = \operatorname{span}\{i(D_{11}-D_{22})\,,\, i(D_{11}-D_{33})\,,\, E_{23}\,,\, iA_{23}\}\\ \mathfrak{k} &= \mathfrak{u}(1)\oplus\mathfrak{u}(1) = \operatorname{span}\{i(D_{11}-D_{22})\,,\, i(D_{11}-D_{33})\}\\ \mathfrak{m} &= \operatorname{span}\{E_{12}\,,\, iA_{12}\,,\, E_{13}\,,\, iA_{13}\}\\ \mathfrak{n} &= \operatorname{span}\{E_{23}\,,\, iA_{23}\}.\end{aligned}$$

We define the inner product B on $\mathfrak{su}(3)$ by $B(Z_1, Z_2) = -\frac{1}{2}\operatorname{Re}\operatorname{tr}(Z_1 Z_2)$ for $Z_1, Z_2 \in \mathfrak{su}(3)$ and then $\langle\cdot\,,\cdot\rangle_{\mathfrak{m}}$ and $\langle\cdot\,,\cdot\rangle_{\mathfrak{n}}$ by

$$\langle\cdot\,,\cdot\rangle_{\mathfrak{m}} = B|_{\mathfrak{m}}\,, \quad \langle\cdot\,,\cdot\rangle_{\mathfrak{n}} = tB|_{\mathfrak{n}} \quad (t > 0).$$

The complex projective space $\mathbb{C}P^2$ is the homogeneous space $SU(3)/U(2)$ associated to $(\mathfrak{g}, \mathfrak{h}, \mathfrak{m}, \langle\cdot\,,\cdot\rangle_{\mathfrak{m}})$ where the embedding is given by

$$U(2) \ni A \longmapsto \begin{pmatrix} \frac{1}{\det A} & 0 \\ 0 & A \end{pmatrix} \in SU(3)\,.$$

The isotropy representation of $U(2)$ satisfies

$$\begin{array}{rcl}\mathrm{Ad}: \qquad U(2) & \longrightarrow & SO(\mathfrak{m})\\ B = \begin{pmatrix} \frac{1}{\det A} & 0 \\ 0 & A \end{pmatrix} & \longmapsto & (\det A)\cdot A \in U(2) \hookrightarrow SO(4)\end{array}$$

where $U(2) \hookrightarrow SO(4)$ is the standard embedding.

$J^-(\mathbb{R}^4)$ is the homogeneous space $U(2)/(U(1)\times U(1))$. Hence, the twistor space of $\mathbb{C}P^2$ equals

$$\begin{aligned} Z^-(\mathbb{C}P^2) &= SU(3)\times_{U(2)} J^-(\mathbb{R}^4) \\ &= SU(3)\times_{U(2)} (U(2)/(U(1)\times U(1))) \\ &= SU(3)/(U(1)\times U(1))\,. \end{aligned}$$

The vectors

$$X_1 = E_{12}\,,\ X_2 = iA_{12}\,,\ X_3 = E_{13}\,,\ X_4 = iA_{13}\,,\ X_5 = \tfrac{1}{\sqrt{t}}E_{23}\,,\ X_6 = \tfrac{i}{\sqrt{t}}A_{23}$$

constitute an orthonormal basis of $\mathfrak{m}\oplus\mathfrak{n}$. With respect to this basis the map

$$\Lambda = \Lambda_{\mathfrak{m}\oplus\mathfrak{n}} : \mathfrak{m}\oplus\mathfrak{n} \longrightarrow \mathfrak{so}(\mathfrak{m}\oplus\mathfrak{n})$$

which describes the Levi-Civita connection is given by

$$\begin{aligned} \Lambda(X_1) &= \frac{\sqrt{t}}{2}(E_{35}+E_{46}) \\ \Lambda(X_2) &= \frac{\sqrt{t}}{2}(-E_{36}+E_{45}) \\ \Lambda(X_3) &= \frac{\sqrt{t}}{2}(-E_{15}+E_{26}) \\ \Lambda(X_4) &= \frac{\sqrt{t}}{2}(-E_{16}-E_{25}) \\ \Lambda(X_5) &= \frac{2-t}{2\sqrt{t}}(E_{13}+E_{24}) \\ \Lambda(X_6) &= \frac{2-t}{2\sqrt{t}}(E_{14}-E_{23}). \end{aligned}$$

Calculating the curvature tensor of the metric g_t on $Z^-(\mathbb{C}P^2)$ induced by $\langle\cdot\,,\cdot\rangle = \langle\cdot\,,\cdot\rangle_{\mathfrak{m}}\oplus\langle\cdot\,,\cdot\rangle_{\mathfrak{n}}$ one can see, that g_t is an Einstein metric for $t=1$ and $t=2$.

We define linear $\mathrm{Ad}(U(1)\times U(1))$-invariant maps $I_1, I_2 : \mathfrak{m}\oplus\mathfrak{n}\to\mathfrak{m}\oplus\mathfrak{n}$ by

$$\begin{aligned} &I_{1,2}(X_1) = X_2 \\ &I_{1,2}(X_3) = -X_4 \\ &I_1(X_5) = -X_6 \quad I_2(X_5) = X_6 \\ &(I_1)^2 = (I_2)^2 = -\,\mathrm{Id}\,. \end{aligned}$$

In case $t=2$ the almost complex structure on $Z^-(\mathbb{C}P^2)$ induced by I_1 is Kählerian, since $I_1\circ\Lambda(X) = \Lambda(X)\circ I_1$ holds for all $X\in\mathfrak{m}\oplus\mathfrak{n}$. I_2 induces a non-integrable almost complex structure.

We now summarize the above observations.

Proposition 6.8 *The twistor space $Z^-(\mathbb{C}P^2)$ is the homogeneous space $(SU(3)/(U(1)\times U(1)), g_t)$ associated to $(\mathfrak{su}(3), \mathfrak{u}(1)\oplus\mathfrak{u}(1), \mathfrak{m}\oplus\mathfrak{n}, \langle\cdot\,,\cdot\rangle)$. In case $t=1$ or $t=2$ the metric g_t satisfies the Einstein condition. Furthermore, in case $t=2$ the space $(Z^-(\mathbb{C}P^2), g_t)$ admits an $U(1)\times U(1)$-invariant Kähler structure.*

First we consider the involutive automorphism $T_0 : \mathfrak{su}(3) \to \mathfrak{su}(3)$ defined by the decomposition

$$\begin{aligned}
\mathfrak{su}(3)_+ &= \operatorname{span}\{i(D_{11}-D_{22})\,,\, i(D_{11}-D_{33})\,,\, E_{23}\,,\, iA_{23}\} \\
&= \mathfrak{u}(2) \\
\mathfrak{su}(3)_- &= \operatorname{span}\{E_{12}\,,\, iA_{12}\,,\, E_{13}\,,\, iA_{13}\} \\
&= \mathfrak{m}.
\end{aligned}$$

In particular, we have $T_0|_{\mathfrak{u}(2)} = \mathrm{Id}$.

Then $\mathbb{C}P^{2,2} = SU(3,1)/U(2)$ equipped with the standard metric is T_0-dual to $\mathbb{C}P^2 = SU(3)/U(2)$. Furthermore, $J^-(\mathbb{R}^{4,4}) = U(2)/(U(1)\times U(1))$ is T_0-dual to $J^-(\mathbb{R}^4) = J^-(\mathbb{R}^{4,4})$. The twistor space $Z^-(\mathbb{C}P^{2,2})$ equals

$$\begin{aligned}
Z^-(\mathbb{C}P^{2,2}) &= SU(3,1)\times_{U(2)} J^-(\mathfrak{m}') \\
&= SU(3,1)\times_{U(2)} (U(2)/(U(1)\times U(1))) \\
&= SU(3,1)/(U(1)\times U(1)).
\end{aligned}$$

Since $U(1)\times U(1)$ is connected the inner product $\langle\cdot\,,\cdot\rangle'$ on $\mathfrak{m}'\oplus\mathfrak{n}'$ which is T_0-dual to $\langle\cdot\,,\cdot\rangle$ induces an $SU(3,1)$-invariant metric g_t' on $Z^-(\mathbb{C}P^{2,2})$. We conclude

Proposition 6.9 *$(Z^-(\mathbb{C}P^{2,2}), g_t')$ is T_0-dual to $(Z^-(\mathbb{C}P^2), g_t)$. In particular, in case $t=1$ or $t=2$ the metric g_t' is an Einstein metric. For $t=2$ the Kähler structure on $(Z^-(\mathbb{C}P^2), g_t)$ becomes a parallel complex structure on $(Z^-(\mathbb{C}P^{2,2}), g_t')$.*

Next we consider the involutive automorphism $T_1 : \mathfrak{su}(3) \to \mathfrak{su}(3)$ defined by the decomposition

$$\begin{aligned}
\mathfrak{su}(3)_+ &= \operatorname{span}\{i(D_{11}-D_{22})\,,\, i(D_{11}-D_{33})\,,\, E_{12}\,,\, iA_{12}\} \\
\mathfrak{su}(3)_- &= \operatorname{span}\{E_{13}\,,\, iA_{13}\,,\, E_{23}\,,\, iA_{23}\}.
\end{aligned}$$

In particular, we have $T_1|_{\mathfrak{u}(2)} = t_1$ in (21).

Then $\mathbb{C}P^{2,1} = SU(3,2)/U(1,1)$ equipped with the standard metric is T_1-dual to $\mathbb{C}P^2 = SU(3)/U(2)$. Furthermore, $J^{-+}(\mathbb{R}^{4,2}) = U(2,1)/(U(1)\times U(1))$ is T_1-dual to $J^-(\mathbb{R}^4) = U(2)/(U(1)\times U(1))$. The twistor space $Z^{-+}(\mathbb{C}P^{2,1})$ equals

$$\begin{aligned}
Z^{-+}(\mathbb{C}P^{2,1}) &= SU(3,2)\times_{U(2,1)} J^{-+}(\mathfrak{m}') \\
&= SU(3,2)\times_{U(2,1)} (U(2,1)/(U(1)\times U(1))) \\
&= SU(3,2)/(U(1)\times U(1)).
\end{aligned}$$

Since $U(1)\times U(1)$ is connected the inner product $\langle\cdot\,,\cdot\rangle'$ on $\mathfrak{m}'\oplus\mathfrak{n}'$ which is T_1-dual to $\langle\cdot\,,\cdot\rangle$ induces an $SU(3,2)$-invariant metric g_t' on $Z^-(\mathbb{C}P^{2,1})$. We conclude

Proposition 6.10 *$(Z^-(\mathbb{C}P^{2,1}), g'_t)$ is T_1-dual to $(Z^-(\mathbb{C}P^2), g_t)$. In particular, in case $t=1$ or $t=2$ the metric g'_t is an Einstein metric. For $t=2$ the Kähler structure on $(Z^-(\mathbb{C}P^2), g_t)$ becomes a parallel complex structure on $(Z^-(\mathbb{C}P^{2,1}), g'_t)$.*

Finally we consider the involutive automorphism $T_2 : \mathfrak{su}(3) \to \mathfrak{su}(3)$ defined by the decomposition

$$
\begin{aligned}
\mathfrak{su}(3)_+ \;&=\; \mathrm{span}\{Z_1 = \frac{1}{2}\begin{pmatrix} -2i & 0 & 0 \\ 0 & i & i \\ 0 & i & i \end{pmatrix}, Z_2 = \frac{1}{2}\begin{pmatrix} -2i & 0 & 0 \\ 0 & i & -i \\ 0 & -i & i \end{pmatrix} \\
&\qquad Z_3 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -1 & -1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, Z_4 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & i & i \\ i & 0 & 0 \\ i & 0 & 0 \end{pmatrix}\} \\
\mathfrak{su}(3)_- \;&=\; \mathrm{span}\{Z_5 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, Z_6 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & i & -i \\ i & 0 & 0 \\ -i & 0 & 0 \end{pmatrix} \\
&\qquad Z_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, Z_8 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & -i \end{pmatrix}\}.
\end{aligned}
$$

In particular, we have $T_2|_{\mathfrak{u}(2)} = t_2$ in (22). Conjugation by

$$
A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \quad (= A^{-1})
$$

transforms the basis $Z_1, \ldots, Z_4, iZ_5, \ldots, iZ_8$ of $\mathfrak{su}(3)_+ \oplus \mathfrak{su}(3)_-$ into the basis $i(-D_{11} + D_{22})$, $i(-D_{11}+D_{33})$, E_{12}, iA_{12}, iE_{13}, $-A_{13}$, E_{23}, $-A_{23}$ of $\mathfrak{su}(3,2)$. $\mathfrak{h}'$ is transformed into $\mathfrak{u}(2,1)$. $\mathfrak{k}' = \mathrm{span}\{Z_1{+}Z_2\,,\, iZ_8\}$ becomes $\mathrm{span}\{i(-2D_{11}{+}D_{22}{+}D_{33})\,,\, -A_{23}\}$ which is the Lie algebra of the subgroup

$$
\begin{aligned}
SO(2,1) \times U(1) \;&=\; \{\begin{pmatrix} r_1\alpha & r_2\alpha \\ r_2\alpha & r_1\alpha \end{pmatrix} \mid r_1, r_2 \in \mathbb{R}\,,\, r_1^2 - r_2^2 = 1\,,\, \alpha \in \mathbb{C}\,,\, |\alpha| = 1\} \\
&\subset\; U(2,1) \;\subset\; SU(3,2)
\end{aligned}
$$

We define

$$
\begin{aligned}
G' &= A\, SU(3,2)\, A \\
H' &= A\, U(2,1)\, A \\
K' &= A\, (SO(2,1) \times U(1))\, A\,.
\end{aligned}
$$

The T_2-dual space G'/H' of $\mathbb{C}P^2 = SU(3)/U(2)$ is isometric to

$$
A\, G'\, A \,/\, A\, H'\, A = SU(3,2)/U(2,1) = \mathbb{C}P^{2,1}
$$

with the standard metric. Furthermore, the T_2-dual space H'/K' of $J^-(\mathbb{R}^4) = U(2)/(U(1)\times U(1))$ is isometric to

$$A\,H'\,A\,/\,A\,K'\,A = U(2,1)/(SO(2,1)\times U(1)) = S^-(R^{4,2})$$

Finally, $G'/K' = G'\times_{H'}(H'/K')$ can be identified with

$$\begin{aligned} SU(3,2)\times_{U(2,1)}(U(2,1)/(SO(2,1)\times U(1))) &= SU(3,2)\times_{U(2,1)} S^-(R^{4,2}) \\ &= Z_-(\mathbb{C}P^{2,1}). \end{aligned}$$

$\mathfrak{n}'$ is not only $\mathrm{ad}(\mathfrak{k}')$-invariant but also $\mathrm{Ad}(K')$-invariant. Indeed, it follows from $[\mathfrak{k}',\mathfrak{n}']\subset\mathfrak{n}'$ that $\mathrm{Ad}(K_0')\mathfrak{n}'\subset\mathfrak{n}'$ holds for the connected component K_0' of the unity of K'. On the other hand $\mathrm{Ad}(D_{11}-D_{22}-D_{33})(\mathfrak{n}')\subset\mathfrak{n}'$ holds and K' is generated by K_0' and $D_{11}-D_{22}-D_{33}$. In the same way one proves that the inner product $\langle\cdot\,,\cdot\rangle'_{\mathfrak{n}}$ on $\mathfrak{n}'$ which is T_2-dual to $\langle\cdot\,,\cdot\rangle_{\mathfrak{n}}$ is $\mathrm{Ad}(K')$-invariant. Obviously, $\mathfrak{m}'$ and $\langle\cdot\,,\cdot\rangle'_{\mathfrak{m}}$ are $\mathrm{Ad}(K')$-invariant since they are in fact $\mathrm{Ad}(H')$-invariant. The inner product $\langle\cdot\,,\cdot\rangle'$ on $\mathfrak{m}'\oplus\mathfrak{n}'$ which is T_2-dual to $\langle\cdot\,,\cdot\rangle$ induces a G'-invariant metric g_t' on $G'/K' = Z_-(\mathbb{C}P^{2,1})$.

Proposition 6.11 *$(Z_-(\mathbb{C}P^{2,1}), g_t')$ is T_2-dual to $(Z^-(\mathbb{C}P^2), g_t)$. In particular, g_t' is an Einstein metric for $t=1$ and $t=2$. For $t=2$ the Kähler structure on $(Z^-(\mathbb{C}P^2), g_t)$ becomes a parallel bilagrangian structure on $Z_-(\mathbb{C}P^{2,1})$.*

7 pseudo-Riemannian homogeneous spaces with Killing spinors

7.1 Spinor representations, Killing spinors

Let $g_{n,k}$ be that inner product with index k of $\mathbb{R}^n$, that is given with respect to the standard basis $e_1,\dots,e_n$ by the matrix

$$\mathrm{diag}(\underbrace{-1,\dots,-1}_{k\text{ times}},\ \underbrace{1,\dots,1}_{(n-k)\text{ times}})$$

Let $\mathcal{C}_{n,k}$ denote the complexified Clifford algebra $\mathrm{Cliff}^{\mathbb{C}}(\mathbb{R}^n,-g_{n,k})$ of $-g_{n,k}$. It is generated by $e_1,\dots,e_n$ with relations $e_ie_j+e_je_i = -2g_{n,k}(e_i,e_j)$. If n is even, i.e. $n=2m$, then $\mathcal{C}_{n,k}$ is isomorphic to the algebra $\mathbb{C}(2^m)$ of complex $2^m\times 2^m$ matrices. There exists exactly one irreducible representation of $\mathcal{C}_{n,k}$. If we restrict this one to $Spin(2m,k)\subset\mathcal{C}_{2m,k}$ we obtain a representation $Spin(2m,k)\hookrightarrow GL(\Delta_{2m,k})$ of dimension $\dim_{\mathbb{C}}\Delta_{2m}=2^m$. This representation decomposes into the sum of two inequivalent irreducible 2^{m-1}-dimensional representations. If n is odd, i.e. $n=2m+1$, then $\mathcal{C}_{n,k}$ is isomorphic to $\mathbb{C}(2^m)\oplus\mathbb{C}(2^m)$. In this case there exist two irreducible representations of $\mathcal{C}_{n,k}$. If one restricts these to $Spin(2m+1,k)$ one obtains two equivalent irreducible representations $Spin(2m+1,k)\hookrightarrow GL(\Delta_{2m+1,k})$ and $\dim_{\mathbb{C}}\Delta_{2m+1,k}=\dim_{\mathbb{C}}\hat{\Delta}_{2m+1,k}=2^m$.

The representations $\Delta_{n,k}$ and $\hat{\Delta}_{n,k}$ are called spinor representations. If we have only in mind the vector spaces, $\Delta_{n,k}$ and $\hat{\Delta}_{n,k}$ are called spinor modules. $\mathcal{C}_{n,k}$ acts on $\Delta_{n,k}$ and defines because of $\mathbb{R}^n \subset \Delta_{n,k}$ a multiplication

$$\begin{array}{rcl} \mathbb{R}^n \times \Delta_{n,k} & \longrightarrow & \Delta_{n,k} \\ (X, \psi) & \longmapsto & X \cdot \psi \end{array}$$

called Clifford multiplication. Analogously this is true for $\hat{\Delta}_{n,k}$. In this article the following isomorphisms $\mathcal{C}_{2m,k} \cong \mathbb{C}(2^m)$ and $\mathcal{C}_{2m+1,k} \cong \mathbb{C}(2^m) \oplus \mathbb{C}(2^m)$ are used. Let E, T, g_1, g_2 denote the matrices

$$E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad T = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \quad g_1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \quad g_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

Furthermore, let τ be

$$\tau(j) = \begin{cases} i & \text{if } j \leq k \\ 1 & \text{if } j > k \end{cases}$$

Then we define

$$\Phi_{2m,k} : \mathcal{C}_{2m,k} \xrightarrow{\sim} \mathbb{C}(2^m)$$

by

$$\begin{aligned} \Phi_{2m,k}(e_{2j-1}) &= \tau(2j-1) \cdot E \otimes \ldots \otimes E \otimes g_1 \otimes \underbrace{T \otimes \ldots \otimes T}_{(j-1)\times} \\ \Phi_{2m,k}(e_{2j}) &= \tau(2j) \cdot E \otimes \ldots \otimes E \otimes g_2 \otimes \underbrace{T \otimes \ldots \otimes T}_{(j-1)\times} \end{aligned} \tag{24}$$

$$(j = 1, \ldots, m)$$

The restriction of $\Phi_{2m,k}$ to $Spin(2m,k)$ yields the spinor representation $\Delta_{2m,k}$ which we use here. And we define

$$\Phi_{2m+1,k} : \mathcal{C}_{2m,k} \xrightarrow{\sim} \mathbb{C}(2^m) \oplus \mathbb{C}(2^m)$$

by

$$\begin{aligned} \Phi_{2m+1,k}(e_j) &= (\Phi_{2m,k}(e_j), \Phi_{2m,k}(e_j)) \quad (1 \leq j \leq 2m) \\ \Phi_{2m+1,k}(e_{2m+1}) &= \tau(2m+1)(iT \otimes \ldots \otimes T, -iT \otimes \ldots \otimes T) \end{aligned} \tag{25}$$

The restriction of $\Phi_{2m+1,k}$ to $Spin(2m+1,k)$ and the projection onto the first component yields the spinor representation $\Delta_{2m,k}$, which we use here. We identify the Lie algebra $\mathfrak{spin}(n,k)$ of $Spin(n,k)$ with $\text{span}\{e_i e_j | 1 \leq i < j \leq n\} \subset \mathcal{C}_{n,k}$. Let

$$\lambda : Spin(n,k) \longrightarrow SO(n,k)$$

be the 2-fold covering. Let again D_{ij} denote the $n \times n$-matrix, whose (i,j)-entry is 1 and all of whose other entries are 0. We set $E_{ij} = -D_{ij} + D_{ji}$ and $A_{ij} = D_{ij} + D_{ji}$. We obtain for the differential $\lambda_* : \mathfrak{spin}(n,k) \to \mathfrak{so}(n,k)$ of λ

$$\lambda_*(e_i e_j) = \begin{cases} -2E_{ij} & \text{if} \quad 1 \le i < j \le k \\ -2A_{ij} & \text{if} \quad 1 \le i \le k \quad \text{and} \quad k+1 \le j \le n \\ \;\;\, 2E_{ij} & \text{if} \quad k+1 \le i < j \le n. \end{cases}$$

Let $(V, \langle \cdot\,, \cdot \rangle)$ be an n-dimensional pseudo-Euclidean vector space. Then V is isometric to $\mathbb{R}^{n,k} = (\mathbb{R}^n, g_{n,k})$ for $1 \le k \le n$. We write $SO(V)$ for the corresponding special orthogonal group and given a pseudo-orthogonal basis in V we identify $SO(V)$ with $SO(n,k)$. We write $Spin(V), \Delta_V, \ldots$ for the corresponding spin structure, its representation etc.

Let now $(G/H, g)$ be a reductive homogeneous space associated to $(\mathfrak{g}, \mathfrak{h}, \mathfrak{m}, \langle \cdot\,, \cdot \rangle)$. Suppose that $(G/H, g)$ is space and time oriented. Then the image of the adjoint representation of H is contained in the connected component $SO^+(\mathfrak{m})$ of the unity in $SO(\mathfrak{m})$. We assume that $\mathrm{Ad} : H \to SO^+(\mathfrak{m})$ admits a lift

$$\begin{tikzcd} & Spin^+(\mathfrak{m}) \arrow[d, "\lambda"] \\ H \arrow[ur, "\widetilde{\mathrm{Ad}}"] \arrow[r, "\mathrm{Ad}"'] & SO^+(\mathfrak{m}) \end{tikzcd}$$

into the connected component $Spin^+(\mathfrak{m})$ of the unity in $Spin(\mathfrak{m})$. This lift defines a $Spin^+$-structure

$$Q = G \times_{\widetilde{\mathrm{Ad}}} Spin^+(\mathfrak{m})$$

on M. The associated spinor bundle S is defined to be

$$S = Q \times_{Spin^+(\mathfrak{m})} \Delta_{\mathfrak{m}}.$$

$\mathcal{Z}$ induces a connection $\widetilde{\mathcal{Z}}$ on $Q \subset G \times_{\widetilde{\mathrm{Ad}}} GL(\Delta_{\mathfrak{m}})$. G acts on $G \times_{\mathrm{Ad}} GL(\Delta_{\mathfrak{m}})$ by left translations. $\widetilde{\mathcal{Z}}$ is invariant under this action and corresponds to the linear map

$$\widetilde{\Lambda_{\mathfrak{m}}} = \lambda_*^{-1} \circ \Lambda_{\mathfrak{m}} : \mathfrak{m} \longrightarrow \mathfrak{spin}(\Delta_{\mathfrak{m}}).$$

Definition 7.1 *A section ψ of the spinor bundle S of a pseudo-Riemannian spin manifold is called Killing spinor, if there exists a real number $\lambda \neq 0$ such that for any vector field $X \in \mathfrak{X}(M)$ the differential equation*

$$\nabla_X \psi = \lambda X \cdot \psi$$

is satisfied. λ is called Killing number of ψ. $K(M,g)_\lambda$ denotes the vector space of Killing spinors of (M,g) with Killing number λ.

If (M, g) is a Riemannian spin manifold of dimension n then the existence of a non-trivial Killing spinor ψ with Killing number λ implies that g is an Einstein metric with scalar curvature $R = 4n(n-1)\lambda^2$. For a Riemannian metric g with positive scalar curvature R we denote by $K(M^n, g)_+$ or shortly K_+ the space of Killing spinors on (M, g) with Killing number $\frac{1}{2}\sqrt{\frac{R}{n(n-1)}}$ and by $K_-(M^n, g)$ or shortly K_- the space of Killing spinors on (M^n, g) with Killing number $-\frac{1}{2}\sqrt{\frac{R}{n(n-1)}}$.

Remark 7.1 *1. Let $M = G/H$ be a compact reductive space with G-invariant metric g such that M admits Killing spinors. Then $(G/H, g)$ is an Einstein space with positive scalar curvature. Therefore, the maximal semi-simple subgroup G_1 of G acts transitively on M, ([Bs], Th. 7.53). Hence, $M = G_1/(H \cap G_1)$. Clearly, g is G_1-invariant. Consequently, in the compact case we may assume without loss of generality that G is semi-simple.*

2. *If we understand a spinor field $\varphi \in \Gamma(S)$ as an $\widetilde{\mathrm{Ad}}(H)$- equivariant map $\widetilde{\varphi} : G \to \Delta_{\mathfrak{m}}$, then we have by Equation 1*

$$\widetilde{\nabla_{l_{g^*}X}\varphi}(g) = X(\widetilde{\varphi})(g) + \widetilde{\Lambda_{\mathfrak{m}}}(X)\widetilde{\varphi}(g),$$

where X is a vector in $\mathfrak{m} = T_{x_0}M$ and simultaneously the corresponding left-invariant vector field on G. On the other hand we have

$$(l_{g^*}X) \cdot \varphi(g) = [g, X] \cdot [g, \widetilde{\varphi}(g)] = [g, X \cdot \widetilde{\varphi}(g)]$$

i.e.

$$\widetilde{(l_{g^*}X) \cdot \varphi(g)} = X \cdot (\widetilde{\varphi}(g))$$

Consequently, φ is a Killing spinor if and only if

$$X(\widetilde{\varphi}) + \widetilde{\Lambda_{\mathfrak{m}}}(X)\widetilde{\varphi} = \lambda X \cdot \widetilde{\varphi} \tag{26}$$

is satisfied for all $X \in \mathfrak{m}$.

3. *G acts on $K(M, g)_\lambda$:*
G acts on $S = G \times_{\widetilde{Ad}} \Delta_{\mathfrak{m}}$ by left translations

$$\begin{array}{rcl} G \times S & \longrightarrow & S \\ (g_0, [g, u]) & \longmapsto & [g_0 g, u]; \qquad u \in \Delta_{\mathfrak{m}}; \quad g_0, g \in G \end{array}$$

Let now $\psi \in \Gamma(S)$ be a Killing spinor and $\widetilde{\psi} : G \to \Delta_m$ the corresponding H-equivariant function, i.e. $\psi(gH) = [g, \widetilde{\psi}(g)]$. Then $g_0 \in G$ acts by

$$g_0 \cdot \psi(gH) = [g_0 g, \widetilde{\psi}(g)] = [g_0 g, \widetilde{g_0 \psi}(g_0 g)],$$

i.e. $\widetilde{g_0\psi}(g_0 g) = \widetilde{\psi}(g)$ or, equivalently, $\widetilde{g_0\psi}(g) = \widetilde{\psi}(g_0^{-1}g)$. Now let be $X \in \mathfrak{m}$. Since ψ is a Killing spinor we have

$$X(\widetilde{\psi}) + \widetilde{\Lambda_{\mathfrak{m}}}(X)\widetilde{\psi} = \lambda X\widetilde{\psi}. \tag{27}$$

Since on the other hand

$$
\begin{aligned}
X(\widetilde{g_0\psi})(g) &= \frac{d}{dt}\widetilde{g_0\psi}(g\cdot \exp tX)|_{t=0} \\
&= \frac{d}{dt}\widetilde{\psi}(g_0^{-1}g\ \exp tX)|_{t=0} \\
&= X\widetilde{\psi}(g_0^{-1}g)
\end{aligned}
$$

holds, (27) results

$$
\begin{aligned}
X(\widetilde{g_0\psi})(g) &= X(\widetilde{\psi})(g_0^{-1}g) \\
&= \lambda\cdot X\cdot\widetilde{\psi}(g_0^{-1}g) - \widetilde{\Lambda_{\mathfrak{m}}}(X)\widetilde{\psi}(g_0^{-1}g) \\
&= \lambda\cdot X\cdot\widetilde{g_0\psi}(g) - \widetilde{\Lambda_{\mathfrak{m}}}(X)\widetilde{g_0\psi}(g).
\end{aligned}
$$

Hence, $g_0\psi$ is a Killing spinor.

Often the dimension of $K(M,g)_\lambda$ is "small" and the representation of G on $K(M,g)_\lambda$ must therefore be trivial. In this case for any Killing spinor ψ and all $g\in G$ $g\cdot\psi=\psi$ holds. Hence, ψ corresponds to a <u>constant</u> map $\widetilde{\psi}: G\to\Delta_{\mathfrak{m}}$.

Proposition 7.1 *Let $(M^n=G/H,g), n\geq 5$, be a compact Riemannian homogeneous spin manifold of non-constant sectional curvature. Let $\psi\in\Gamma(S)$ be a Killing spinor on M^n. Suppose that G is semi-simple and fix a decomposition $\mathfrak{g}=\mathfrak{m}\oplus\mathfrak{h}$ of the Lie algebra $\mathfrak{g}$ of G into the Lie algebra $\mathfrak{h}$ and a subspace $\mathfrak{m}\subset\mathfrak{g}$ such that $[\mathfrak{h},\mathfrak{m}]\subset\mathfrak{h}$. If G does not contain a factor of type A_1 or if $n\not\equiv 3$ mod 8 then the H-equivariant function $\widetilde{\psi}: G\to\Delta_{\mathfrak{m}}$ corresponding to $\psi\in\Gamma(S)$ is constant.*

Proof. We consider the universal covering $(\widetilde{G}/\widetilde{H}_0=\widetilde{M},\widetilde{g})$ of $(G/H,g)$. Spinors on G/H lift to spinors on $\widetilde{G}/\widetilde{H}_0$. In particular, Killing spinors on M^n lift to Killing spinors on $\widetilde{M}$. The action of G on $K(M^n,g)_\pm$ lifts to an action of $\widetilde{G}$ on (a subspace) of Killing spinors on $\widetilde{M}$. We have to prove that under the condition of the proposition these representations are trivial. We use now the description of the possible dimensions $\dim K_\pm$ of spaces of Killing spinors due to Ch. Bär [Br]. Consider first the case that $(\widetilde{M},\widetilde{g})$ does not admit a 3-Sasakian structure. Then $\dim K(\widetilde{M},\widetilde{g})\leq 2$. Hence, if $\widetilde{G}$ does not contain a $SU(2)$-factor, there does not exist a non-trivial representation of $\widetilde{G}$ on subspaces of $K(\widetilde{M},\widetilde{g})_\pm$. If $\dim K(\widetilde{M},\widetilde{g})_+=2$ or $\dim K(\widetilde{M},\widetilde{g})_-=2$, then $n\equiv 3,7$ (8). In case $n\equiv 7$ (8) the spinor representation Δ_n has a real structure which commutes with the Clifford multiplication. On $\Gamma(S)$ it is parallel and commutes with the action of G. Hence, the representation of $\widetilde{G}$ on (a subspace of) $K(\widetilde{M},\widetilde{g})_\pm$ is real and therefore trivial.

Now suppose that $(\widetilde{M},\widetilde{g})$ is a 3-Sasakian manifold of dimension $n=4m-1$. Then one of the spaces $K(\widetilde{M},\widetilde{g})_\pm$ has dimension 0 and the other one $d=m+1$. We now use classification results of C. P. Boyer, K. Galicki, and B. M. Mann. Since $(\widetilde{M},\widetilde{g})$ is not a space of constant sectional curvature it has to be 3-Sasakian homogeneous and therefore it is one of the following spaces (see [BGM]).

case 1
$\widetilde{M} = SO(k)/(SO(k) \times Sp(1)) \quad k > 5$
In this case we have $\dim \widetilde{M} = 4(k-3) - 1$ and $d = k-2$. Since there do not exist non-trivial representations of $SO(k)$ of dimension smaller than k the assertion follows.

case 2
$\widetilde{M} = SU(k)/S(U(k-2) \times U(1)) \qquad k > 2$
Here we have $\dim \widetilde{M} = 4(k-1) - 1$ and $d = k$. The spinor representation $\Delta_{4(k-1)-1}$ admits a real structure in case k is odd and a quaternionic structure in case k is even. Both structures commute with the Clifford multiplication. On $\Gamma(S)$ they are parallel and commute with the action of $SU(k)$. Hence, the representation of $SU(k)$ on the k-dimensional space of Killing spinors is either real or quaternionic. However, the smallest non-trivial representation of $SU(k)$ is the k-dimensional standard representation which is neither real nor quaternionic if $k > 2$.

case 2 exceptional spaces
As above we compare the dimension d of the space of Killing spinors with the dimension d_1 of the smallest complex irreducibel representation of $\widetilde{G}$.

$$
\begin{array}{llll}
\widetilde{M} = G_2/Sp(1) & \dim(\widetilde{M}) = 11 & d = 4 & d_1 = 7 \\[1em]
\widetilde{M} = F_4/Sp(3) & \dim(\widetilde{M}) = 31 & d = 9 & d_1 = 26 \\[1em]
\widetilde{M} = E_6/SU(6) & \dim(\widetilde{M}) = 43 & d = 12 & d_1 = 27 \\[1em]
\widetilde{M} = E_7/Spin(12) & \dim(\widetilde{M}) = 67 & d = 18 & d_1 = 56 \\[1em]
\widetilde{M} = E_8/E_7 & \dim(\widetilde{M}) = 115 & d = 30 & d_1 = 248
\end{array}
$$

□

Let $(G/H, g)$ be a compact Riemannian reductive space associated to $(\mathfrak{g}, \mathfrak{h}, \mathfrak{m}, \langle \cdot\,, \cdot \rangle)$ with semi-simple G and $T : \mathfrak{g} \to \mathfrak{g}$ an involutive automorphism such that $T(\mathfrak{h}) = \mathfrak{h}$, $T(\mathfrak{m}) = \mathfrak{m}$ and $T|_{\mathfrak{m}}$ is an isometry. Furthermore, let $(G'/H', g')$ be a reductive space associated to $(\mathfrak{g}', \mathfrak{h}', \mathfrak{m}', \langle \cdot\,, \cdot \rangle')$ and assume that $(\mathfrak{g}', \mathfrak{h}', \mathfrak{m}', \langle \cdot\,, \cdot \rangle')$ is T-dual to $(\mathfrak{g}, \mathfrak{h}, \mathfrak{m}, \langle \cdot\,, \cdot \rangle)$.

Definition 7.2 *Under the above assumptions $(G'/H', g'), \langle \cdot\,, \cdot \rangle')$ is called strongly T-dual to $(G/H, g)$, if*

1. *G' and $H' \subset G'$ are connected Lie groups and if*
2. *the connected subgroup H_1' of H' with Lie algebra $\mathfrak{h}' \cap \mathfrak{g}_+ \subset \mathfrak{h}'$ is naturally isomorphic to the connected subgroup H_1 of H with Lie algebra $\mathfrak{h} \cap \mathfrak{g}_+$, i.e. if there*

is an isomorphism of H_1' and H_1 such that the diagram

$$\begin{array}{ccccccc} G' & \supset & H_1' & \xrightarrow{\ \sim\ } & H_1 & \subset & G \\ \uparrow{\scriptstyle \exp} & & \uparrow{\scriptstyle \exp} & & \uparrow{\scriptstyle \exp} & & \uparrow{\scriptstyle \exp} \\ \mathfrak{g}' & \supset & \mathfrak{h}' \cap \mathfrak{g}_+ & \xrightarrow{\ \sim\ } & \mathfrak{h} \cap \mathfrak{g}_+ & \subset & \mathfrak{g} \end{array} \tag{28}$$

commutes.

Remark 7.2 *H_1' is maximal compact in H'.*

Proposition 7.2 *Let as above $(G/H, g)$ and $(G'/H', g')$ be strongly T-dual reductive spaces whose dimension is now at least 4 and assume that $(G/H, g)$ admits a homogeneous spin structure. Then there exists a homogeneous spin structure on $(G'/H', g')$.*

Proof. We use the same notation as above. The statement is equivalent to the existence of a lift

$$\begin{array}{ccc} & & Spin^+(\mathfrak{m}') \\ & \overset{\widetilde{Ad'}}{\nearrow} & \downarrow{\scriptstyle \lambda} \\ H_1' \subset H' & \xrightarrow{Ad'} & SO^+(\mathfrak{m}') \end{array} \tag{29}$$

of the adjoint representation $\mathrm{Ad}' : H' \to SO^+(\mathfrak{m}')$ to a group homomorphism $\widetilde{\mathrm{Ad}}' : H' \to Spin^+(\mathfrak{m}')$. We have to check

$$\mathrm{Ad}'_{\#}\pi_1(H', e) \subset \lambda_{\#}\pi_1(Spin^+(m'), 1)$$

To do this it suffices to prove

$$\mathrm{Ad}'_{\#}\pi_1(H_1', e) \subset \lambda_{\#}\pi_1(Spin^+(m'), 1)$$

since H_1' is a maximal compact subgroup of H'. According to (9) we have

$$\mathrm{ad}'|_{\mathfrak{h}'\cap\mathfrak{g}_+} = \mathrm{ad}|_{\mathfrak{h}\cap\mathfrak{g}_+} : \mathfrak{h}' \cap \mathfrak{g}_+ = \mathfrak{h} \cap \mathfrak{g}_+ \longrightarrow \mathfrak{so}(\mathfrak{m})_+ = \mathfrak{so}(\mathfrak{m}')_+$$

We choose an orthonormal basis $b_1, \ldots, b_k$ of $\mathfrak{m} \cap \mathfrak{g}_-$ and an orthonormal basis $b_{k+1}, \ldots, b_n$ of $\mathfrak{m} \cap \mathfrak{g}_+$. Then, in particular, $b_1, \ldots, b_n$ is an orthonormal basis of $\mathfrak{m}$ and $ib_1, \ldots, ib_k, b_{k+1}, \ldots, b_n$ a pseudo-orthonormal basis of $\mathfrak{m}'$. Using these bases we identify $\mathfrak{so}(\mathfrak{m})_+ = \mathfrak{so}(\mathfrak{m}')_+$ with $\mathfrak{so}(k) \times \mathfrak{so}(n-k)$. Because of (28) we obtain

$$\mathrm{Ad}|_{H_1} = \mathrm{Ad}'|_{H_1'} : H_1 \cong H_1' \to SO(k) \times SO(n-k).$$

Let now $\gamma : I = [0.1] \to H_1'$ be a curve with $\gamma(0) = \gamma(1) = e \in H_1'$ and $[\gamma(t)]$ its equivalence class in $\pi_1(H_1', e)$. Then we have

$$\mathrm{Ad}'_{\#}[\gamma(t)] = [\mathrm{Ad}'\gamma(t)] = \left[\left(\begin{array}{c|c} \gamma_1(t) & 0 \\ \hline 0 & \gamma_2(t) \end{array}\right)\right] \in \pi_1(SO^+(n-k), \mathrm{Id})$$

where γ_1, γ_2 are pathes $\gamma_1 : I \to SO(k), \gamma_2 : I \to SO(n-k)$. In $\pi_1(SO^+(n-k), \mathrm{Id}) = \pi_1(SO(k) \times SO(n-k), \mathrm{Id})$ we have

$$\left[\left(\begin{array}{c|c} \gamma_1(t) & 0 \\ \hline 0 & \gamma_2(t) \end{array}\right)\right] = \begin{cases} \underline{\text{case 1} \qquad k=1} \\ \left[\left(\begin{array}{c|cc} 1 & 0 & 0 \\ \hline 0 & B(t) & 0 \\ 0 & 0 & E_{n-3} \end{array}\right)\right] \\ \text{where } B : I \to SO(2) \\ \\ \underline{\text{case 2} \qquad 1 < k < n-1} \\ \left[\left(\begin{array}{cc|cc} A(t) & 0 & 0 & 0 \\ 0 & E_{k-2} & 0 & 0 \\ \hline 0 & 0 & B(t) & 0 \\ 0 & 0 & 0 & E_{n-k-2} \end{array}\right)\right] \\ \text{where } A, B : I \to SO(2) \\ \\ \underline{\text{case 3} \qquad k = n-1} \\ \left[\left(\begin{array}{cc|c} A(t) & 0 & 0 \\ 0 & E_{k-2} & 0 \\ \hline 0 & 0 & 1 \end{array}\right)\right] \\ \text{where } A : I \to SO(2) \\ \\ \underline{\text{case 4} \qquad k = n} \\ \left[\left(\begin{array}{cc} A(t) & 0 \\ 0 & E_{n-2} \end{array}\right)\right] \\ \text{where } A : I \to SO(2) \end{cases}$$

Here E_j denotes the unity matrix of rank j. Taking into consideration that with identification $H_1 = H_1'$

$$\mathrm{Ad}'(\gamma(t)) = \mathrm{Ad}(\gamma(t)) \subset SO(k) \times SO(n-k) \subset SO(n)$$

holds and that by assumption (Ad admits a lift!) $[\mathrm{Ad}[\gamma(t))]$ is trivial in $\pi_1(SO(n), Id)$

we obtain

$$\underline{\text{in case 1:}} \qquad \left[\left(\begin{array}{c|c} B(t) & 0 \\ \hline 0 & E_{n-3} \end{array}\right)\right] = 0 \in \pi_1(SO(n-1))$$

$$\underline{\text{in case 2:}} \qquad [A(t)] \equiv [B(t)] \mod 2\mathbb{Z} \quad \text{in } \pi_1(SO(2)) = \mathbb{Z}$$

$$\underline{\text{in cases 3 and 4:}} \qquad \left[\begin{pmatrix} A(t) & 0 \\ 0 & E_{k-2} \end{pmatrix}\right] = 0 \in \pi_1(SO(k))$$

We conclude in all cases

$$\mathrm{Ad}'_{\#}[\gamma(t)] \in \lambda_{\#}\pi_1(Spin^+(\mathfrak{m}'), 1)$$

since we know

$$\lambda_{\#}\pi_1(Spin^+(n,k),1) = \begin{cases} (0 \oplus 0) \cup (1 \oplus 1) & \text{if} \quad 3 \leq k \leq [\frac{n}{2}] \\ (2\mathbb{Z} \oplus 0) \cup ((2\mathbb{Z}+1) \oplus 1) & \text{if} \quad k = 2, n \geq 5 \\ (2\mathbb{Z} \oplus 2\mathbb{Z}) \cup ((2\mathbb{Z}+1) \oplus (2\mathbb{Z}+1)) & \text{if} \quad k = 4, n = 4 \end{cases}$$

□

Theorem 7.1 *Assume that $(G/H, g)$ admits a homogeneous spin structure. Let*

$$\widetilde{\psi} : G \longrightarrow \Delta_{\mathfrak{m}}$$
$$\widetilde{\psi}(g) \equiv u \in \Delta_{\mathfrak{m}}$$

be a constant H-equivariant map from G into the spinor modul $\Delta_{\mathfrak{m}}$ of $\mathfrak{m}$ such that the induced section of the spinor bundle S of G/H is a Killing spinor with Killing number λ. Then there exists on each strongly T-dual homogeneous space $(G'/H', g')$ of $(G/H, g)$ a $Spin^+$-structure and

$$\widetilde{\psi'} : G' \longrightarrow \Delta_{\mathfrak{m}'} = \Delta_{\mathfrak{m}}$$
$$\widetilde{\psi}(g) \equiv u \in \Delta'_{\mathfrak{m}}$$

defines a Killing spinor on $(G'/H', g')$.

Proof. By Proposition 7.2 $(G'/H', g')$ admits a $Spin^+$-structure. Let S' be the associated spinor bundle

$$S' = G' \times_{\widetilde{\mathrm{Ad}}'} \Delta_{\mathfrak{m}'}.$$

We denote by $\widetilde{\mathrm{ad}}$ and $\widetilde{\mathrm{ad}}'$ the maps

$$\begin{aligned} \widetilde{\mathrm{ad}} &= \lambda_*^{-1} \circ \mathrm{ad} : \mathfrak{h} \to \mathfrak{spin}(\mathfrak{m}) \\ \widetilde{\mathrm{ad}}' &= \lambda_*^{-1} \circ \mathrm{ad}' : \mathfrak{h}' \to \mathfrak{spin}(\mathfrak{m}') \end{aligned}$$

Let now $\psi \in \Gamma(S)$ be a Killing spinor on G/H and $\widetilde{\psi} : G \longrightarrow \Delta_n$ the corresponding H-equivariant map. Suppose that $\widetilde{\psi}$ is constant, i.e.

$$\begin{array}{rcl} \widetilde{\psi} : G & \longrightarrow & \Delta_{\mathfrak{m}} \\ \widetilde{\psi} & \equiv & u \in \Delta_{\mathfrak{m}} \qquad \text{for all} \quad g \in G \end{array}$$

Then $u = \widetilde{\psi}(g \cdot h) = \widetilde{\mathrm{Ad}}(h^{-1}) \cdot \psi(g) = \widetilde{\mathrm{Ad}}(h^{-1})u$ holds for all $h \in H, g \in G$. This is equivalent to

$$\widetilde{\mathrm{Ad}}(h)u = u \qquad \text{for all} \quad h \in H.$$

Hence,

$$\widetilde{\mathrm{ad}}(X)u = 0 \qquad \text{for all} \quad X \in \mathfrak{h}.$$

Let $\widetilde{\psi}'$ be the map

$$\begin{array}{c} \widetilde{\psi}' : G' \longrightarrow \Delta_{\mathfrak{m}'} \\ \widetilde{\psi}'(g') = u. \end{array}$$

We will see that $\widetilde{\psi}'$ defines again a Killing spinor. To begin with we verify that $\widetilde{\psi}'$ is H'-equivariant. The map $\widetilde{\psi}'$ is H'-equivariant if and only if

$$\widetilde{\mathrm{Ad}}'(h')(u) = u$$

is true for all $h' \in H'$, and since H' is connected if and only if

$$\widetilde{\mathrm{ad}}'(Z)(u) = 0$$

holds for all $Z \in \mathfrak{h}'$. We choose as above an orthonormal basis $b_1, \ldots, b_k$ of $\mathfrak{m} \cap \mathfrak{g}_-$ and an orthonormal basis $b_{k+1}, \ldots, b_n$ of $\mathfrak{m} \cap \mathfrak{g}_+$. Then we identify $(\mathfrak{m}, \langle \cdot , \cdot \rangle)$ with the Euclidean space $\mathbb{R}^n$ and $(\mathfrak{m}', \langle \cdot , \cdot \rangle')$ with $\mathbb{R}^{n,k}$ using the basis $b_1, \ldots, b_n$ and $ib_1, \ldots, ib_k, b_{k+1}, \ldots, b_n$, respectively. Let now X be in $\mathfrak{h} \cap \mathfrak{g}_-$. Then we have

$$\mathrm{ad}(X) = \left(\begin{array}{c|c} 0 & -M(X) \\ \hline M(X) & 0 \end{array} \right) = \sum_{i=1}^{k} \sum_{j=k+1}^{n} m_{ij}(X) E_{ij} \in \mathfrak{so}(n)$$

and therefore,

$$\mathrm{ad}'(iX) = \left(\begin{array}{c|c} 0 & -M(X) \\ \hline -M(X) & 0 \end{array} \right) = \sum_{i=1}^{k} \sum_{j=k+1}^{n} -m_{ij}(X) A_{ij} \in \mathfrak{so}(n,k)$$

For $Y \in \mathfrak{h} \cap \mathfrak{g}_+ = \mathfrak{h}' \cap \mathfrak{g}_+$ we have

$$\begin{aligned} \mathrm{ad}(Y) = \mathrm{ad}'(Y) &= \left(\begin{array}{c|c} Q(Y) & 0 \\ \hline 0 & P(Y) \end{array} \right) \\ &= \sum_{i,j=1}^{k} q_{ij}(Y) E_{ij} + \sum_{i,j=k+1}^{n} p_{ij}(Y) E_{ij} \in \mathfrak{so}(k) \oplus \mathfrak{so}(n-k)\,. \end{aligned}$$

This yields for $X \in \mathfrak{h} \cap \mathfrak{g}_-$ and $Y \in \mathfrak{h} \cap \mathfrak{g}_+ = \mathfrak{h}' \cap \mathfrak{g}_+$

$$
\begin{aligned}
\widetilde{\mathrm{ad}}(X) &= \frac{1}{2}\sum_{i=1}^{k}\sum_{j=k+1}^{n} m_{ij}(X)e_ie_j \in \mathfrak{spin}(n) \\
\widetilde{\mathrm{ad}}'(iX) &= \frac{1}{2}\sum_{i=1}^{k}\sum_{j=k+1}^{n} m_{ij}(X)e_ie_j \in \mathfrak{spin}(n,k) \\
\widetilde{\mathrm{ad}}(Y) &= \frac{1}{2}\left\{\sum_{i,j=1}^{k} q_{ij}(Y)e_ie_j + \sum_{i,j=k+1}^{n} p_{ij}(Y)e_ie_j\right\} \in \mathfrak{spin}(n) \\
\widetilde{\mathrm{ad}}'(Y) &= \frac{1}{2}\left\{\sum_{i,j=1}^{k} (-q_{ij}(Y))e_ie_j + \sum_{i,j=k+1}^{n} p_{ij}(Y)e_ie_j\right\} \in \mathfrak{spin}(n,k)
\end{aligned}
$$

Hence,

$$(\Phi_{n,k} \circ \widetilde{\mathrm{ad}}')(iX) = i(\Phi_{n,0} \circ \widetilde{\mathrm{ad}})(X) \in GL(2^{[\frac{n}{2}]}, \mathbb{C})$$

and

$$(\Phi_{n,k} \circ \widetilde{\mathrm{ad}}')(Y) = i(\Phi_{n,0} \circ \widetilde{\mathrm{ad}})(Y) \in GL(2^{[\frac{n}{2}]}, \mathbb{C}).$$

In particular,

$$
\begin{aligned}
u \in \ker(\Phi_n \circ \widetilde{\mathrm{ad}}(X)) &\iff u \in \ker(\Phi_{n,k} \circ \widetilde{\mathrm{ad}}'(iX)) \\
u \in \ker(\Phi_n \circ \widetilde{\mathrm{ad}}(Y)) &\iff u \in \ker(\Phi_{n,k} \circ \widetilde{\mathrm{ad}}'(Y))
\end{aligned}
$$

holds. Consequently, we obtain $\widetilde{\mathrm{ad}}'(Z)u = 0$ for all $Z \in \mathfrak{h}'$ and conclude that $\widetilde{\psi}'$ defines a spinor $\psi' \in \Gamma(S')$. Next we prove that ψ is a Killing spinor. Let be $1 \leq \alpha \leq k$ and $k < \beta \leq n$. From Proposition 4.1 we obtain

$$\Lambda(b_\alpha) = \left[\left(\begin{array}{c|c} 0 & -L^\alpha \\ \hline L^\alpha & 0 \end{array}\right)\right] = \sum_{i=1}^{k}\sum_{j=k+1}^{n} l_{ij}^\alpha E_{ij} \in \mathfrak{so}(n)$$

$$\Lambda'(ib_\alpha) = \left[\left(\begin{array}{c|c} 0 & -L^\alpha \\ \hline -L^\alpha & 0 \end{array}\right)\right] = -\sum_{i=1}^{k}\sum_{j=k+1}^{n} l_{ij}^\alpha A_{ij} \in \mathfrak{so}(n)$$

$$\Lambda(b_\beta) = \left[\left(\begin{array}{c|c} R^\beta & 0 \\ \hline 0 & S^\beta \end{array}\right)\right] = \sum_{i,j=1}^{k} r_{ij}^\beta E_{ij} + \sum_{i,j=k+1}^{n} s_{ij}^\beta E_{ij} \in \mathfrak{so}(n)$$

$$\Lambda'(b_\beta) = \left[\left(\begin{array}{c|c} R^\beta & 0 \\ \hline 0 & S^\beta \end{array}\right)\right] = \sum_{i,j=1}^{k} r_{ij}^\beta E_{ij} + \sum_{i,j=k+1}^{n} s_{ij}^\beta E_{ij} \in \mathfrak{so}(n,k)\,.$$

and, therefore, $\widetilde{\Lambda} = \lambda_*^{-1} \circ \Lambda$ and $\widetilde{\Lambda'} = \lambda_*^{-1} \circ \Lambda'$ satisfy

$$\begin{aligned}
\widetilde{\Lambda}(b_\alpha) &= \frac{1}{2}\sum_{i=1}^{k}\sum_{j=k+1}^{n} l_{ij}^{\alpha} e_i e_j \in \mathfrak{spin}(n) \\
\widetilde{\Lambda'}(ib_\alpha) &= \frac{1}{2}\sum_{i=1}^{k}\sum_{j=k+1}^{n} l_{ij}^{\alpha} e_i e_j \in \mathfrak{spin}(n,k) \\
\widetilde{\Lambda}(b_\beta) &= \frac{1}{2}\left\{\sum_{i,j=1}^{k} r_{ij}^{\beta} e_i e_j + \sum_{i,j=k+1}^{n} s_{ij}^{\beta} e_i e_j\right\} \in \mathfrak{spin}(n) \\
\widetilde{\Lambda'}(b_\beta) &= \frac{1}{2}\left\{-\sum_{i,j=1}^{k} r_{ij}^{\beta} e_i e_j + \sum_{i,j=k+1}^{n} s_{ij}^{\beta} e_i e_j\right\} \in \mathfrak{spin}(n,k)
\end{aligned}$$

Using $\Phi_{n,0}$ and $\Phi_{n,k}$ we think of Λ and $\widetilde{\Lambda}$ as maps into $GL(2^{[\frac{n}{2}]}, \mathbb{C})$ and obtain

$$\begin{aligned}
i\Phi_{n,0}(\widetilde{\Lambda}(b_\alpha)) &= \Phi_{n,k}(\widetilde{\Lambda'}(ib_\alpha)) \\
\Phi_n(\widetilde{\Lambda}(b_\beta)) &= \Phi_{n,k}(\widetilde{\Lambda'}(b_\beta))\,. \qquad (30)
\end{aligned}$$

Since the Killing spinor $\psi \in \Gamma(S)$ corresponds to a constant map $\widetilde{\psi} : G \to \Delta_n, \widetilde{\psi}(g) \equiv u$, Equation (26) results

$$\Lambda(b_i)u = \lambda b_i u,$$

i.e. $u \in \ker \Phi_n(\widetilde{\Lambda}(b_i) - \lambda e_i)$ for $i = 1, \dots, n$. According to (30) and (24), (25) we have

$$\begin{aligned}
\Phi_{n,k}(\widetilde{\Lambda'}(ib_\alpha) - \lambda e_\alpha) &= i\Phi_{n,0}(\widetilde{\Lambda}(ib_\alpha) - \lambda e_\alpha) \qquad \alpha = 1, \dots, k \\
\Phi_{n,k}(\widetilde{\Lambda'}(b_\beta) - \lambda e_\beta) &= \Phi_{n,0}(\widetilde{\Lambda}(b_\beta) - \lambda e_\beta) \qquad \beta = k+1, \dots, n.
\end{aligned}$$

In particular, u is contained in the kernels of $\Phi_{n,k}(\widetilde{\Lambda}(ib_\alpha) - \lambda e_\alpha)$ and $\Phi_{n,k}(\widetilde{\Lambda}(b_\beta) - \lambda e_\beta)$. Hence,

$$\widetilde{\Lambda'}(ib_\alpha)u = \lambda b_\alpha \cdot u \qquad \text{and} \qquad \widetilde{\Lambda'}(b_\beta)u = \lambda b_\beta \cdot u.$$

Consequently, $\psi' \in \Gamma(S'), \psi'(g'H') = [g'H', u]$ is a Killing spinor.

7.2 Examples

In this section we collect a series of examples, most of them of dimension 5, 6 and 7. We begin with remarks concerning specialities of these dimensions.

n=5

If (M^5, g) is a compact Riemannian manifold of positive scalar curvature then $\dim K_+ > 1$ and $\dim K_- > 1$ holds only if g has constant sectional curvature. If (M^5, g) is simply connected then $\dim K_+ = \dim K_-$ holds (see [K1],[BFGK]).

n=6

In dimension 6 we replace $\Phi_{6,k}$ defined in Section 7.1 by

$$\begin{array}{rcl} \Phi_{6,k}:\mathcal{C}_{6.k} & \longrightarrow & \mathbb{C}(8) \\ e_1 & \longmapsto & \tau(1)(E_{18}+E_{27}-E_{36}-E_{45}) \\ e_2 & \longmapsto & \tau(2)(-E_{17}+E_{28}+E_{35}-E_{46}) \\ e_3 & \longmapsto & \tau(3)(-E_{16}+E_{25}-E_{38}+E_{47}) \\ e_4 & \longmapsto & \tau(4)(-E_{15}-E_{26}-E_{37}-E_{48}) \\ e_5 & \longmapsto & \tau(5)(-E_{13}-E_{24}+E_{57}+E_{68}) \\ e_6 & \longmapsto & \tau(6)(E_{14}-E_{23}-E_{58}+E_{67}). \end{array}$$

In particular, the induced spinor representation of $Spin(6)$ is real, i.e. $\Delta_{6,0}=\mathbb{R}^6$ If (M^6,g) is a compact Riemannian spin manifold of positive scalar curvature then $\dim K_+=\dim K_-$ holds. If (M^6,g) is connected and not isometric to the standard sphere S^6, then we have $\dim K_+=\dim K_-\leq 1$

n=7

We replace $\Phi_{7,k}$ defined in Section 7.1 by

$$\begin{array}{rcl} \Phi_{7,k}:\mathcal{C}_{7,k} & \longrightarrow & \mathbb{C}(8)\oplus\mathbb{C}(8) \\ e_1 & \longmapsto & (\tau(1)(E_{18}+E_{27}-E_{36}-E_{45}),\tau(1)(E_{18}+E_{27}-E_{36}-E_{45})) \\ e_2 & \longmapsto & (\tau(2)(-E_{17}+E_{28}+E_{35}-E_{46}),\tau(2)(-E_{17}+E_{28}+E_{35}-E_{46})) \\ e_3 & \longmapsto & (\tau(3)(-E_{16}+E_{25}-E_{38}+E_{47}),\tau(3)(-E_{16}+E_{25}-E_{38}+E_{47})) \\ e_4 & \longmapsto & (\tau(4)(-E_{15}-E_{26}-E_{37}-E_{48}),\tau(4)(-E_{15}-E_{26}-E_{37}-E_{48})) \\ e_5 & \longmapsto & (\tau(5)(-E_{13}-E_{24}+E_{57}+E_{68}),\tau(5)(-E_{13}-E_{24}+E_{57}+E_{68})) \\ e_6 & \longmapsto & (\tau(6)(E_{14}-E_{23}-E_{58}+E_{67}),\tau(6)(E_{14}-E_{23}-E_{58}+E_{67})) \\ e_7 & \longmapsto & (\tau(7)(E_{12}-E_{34}-E_{56}+E_{78}),\tau(7)(-E_{12}+E_{34}+E_{56}-E_{78}))\,. \end{array}$$

In particular spinor representation of $Spin(7)$ is real, i.e. $\Delta_{7,0}=\mathbb{R}^8$

Let (M^7,g) be a compact, connected Riemannian manifold of positive scalar curvature. If $\dim_{\mathbb{R}} K_+>3$ or $\dim_{\mathbb{R}} K_->3$, then (M^7,g) has constant sectional curvature. If $\dim_{\mathbb{R}} K_+>0$ and $\dim_{\mathbb{R}} K_->0$, then M^7 is isometric to the standard sphere S^7.

Remark 7.3 *The spinor representations of $Spin(6,4)$ and $Spin(7,4)$ are also real, but we will not take use of this fact.*

Example 1 $\boxed{G = SO(4) \qquad H = SO(2)}$

$G/H = V_{4,2}$ is the Stiefel manifold of oriented 2-frames in $\mathbb{R}^4$. The dimension of G/H equals 5. The embedding $G \hookrightarrow H$ is given by

$$\begin{array}{ccc} SO(2) & \hookrightarrow & SO(4) \\ A & \longmapsto & \left(\begin{array}{cc} E_2 & 0 \\ 0 & A \end{array}\right), \qquad \text{where} \qquad E_2 = \left(\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array}\right). \end{array}$$

The Lie algebras of G and H equal

$$\begin{aligned} \mathfrak{so}(4) &= \operatorname{span}\{E_{12}, E_{13}, E_{14}, E_{23}, E_{24}, E_{34}\} \\ \mathfrak{so}(2) &= \operatorname{span}\{E_{34}\} \end{aligned}$$

The vectors

$$X_1 = E_{13}\,,\ X_2 = E_{23}\,,\ X_3 = E_{14}\,,\ X_4 = E_{24}\,,\ X_5 = \frac{\sqrt{3}}{2}E_{12}$$

constitute a basis of a suitable complement $\mathfrak{m}$ of $\mathfrak{so}(2) \subset \mathfrak{so}(4)$. We choose the inner product of $\mathfrak{m}$ such that $X_1, \ldots, X_n$ is an orthonormal basis. $SO(4)/SO(2)$ admits a homogeneous spin structure. The constant functions

$$\widetilde{\psi_i} : SO(4) \longrightarrow \Delta_{5,0}, \widetilde{\psi_i} \equiv u_i \qquad i = 1, 4$$

define sections in the corresponding spinor bundle. ψ_1 is a Killing spinor with Killing number $\lambda_+ = \frac{1}{2\sqrt{3}}$. ψ_4 is a Killing spinor with Killing number $\lambda_- = -\frac{1}{2\sqrt{3}}$.

The following involutive automorphisms $T_i : \mathfrak{g} \to \mathfrak{g}$ or, equivalently, the following decompositions $\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_-$ are possible.

Let T_1 be defined by the decomposition

$$\begin{aligned} \mathfrak{g}_+ &= \operatorname{span}\{E_{12}, E_{34}\} \\ \mathfrak{g}_- &= \operatorname{span}\{E_{13}, E_{14}, E_{23}, E_{24}\} \end{aligned}$$

In this case we obtain the reductive space $G'/H' = SO^+(4,2)/SO(2)$ with a homogeneous metric of index $k = 4$.

Let T_2 be defined by

$$\begin{aligned} \mathfrak{g}_+ &= \operatorname{span}\{E_{13}, E_{24}\} \\ \mathfrak{g}_- &= \operatorname{span}\{E_{12}, E_{14}, E_{23}, E_{24}\} \end{aligned}$$

This yields $G'/H' = SO^+(4,2)/SO^+(2,1)$ with a homogeneous metric of index $k = 3$.

Let T_3 be defined by

$$\begin{aligned} \mathfrak{g}_+ &= \operatorname{span}\{E_{12}, E_{13}, E_{23}\} \\ \mathfrak{g}_- &= \operatorname{span}\{E_{14}, E_{24}, E_{34}\}. \end{aligned}$$

We obtain $G'/H' = SO^+(4,3)/SO^+(2,1)$ with a homogeneous metric of index $k = 2$.

Furthermore let T_4 be defined by

$$\begin{aligned}\mathfrak{g}_+ &= \text{span}\{E_{23}, E_{24}, E_{34}\}\\ \mathfrak{g}_- &= \text{span}\{E_{12}, E_{13}, E_{14}\}.\end{aligned}$$

This yields $G'/H' = SO^+(4,3)/SO(2)$ with a metric of index $k = 3$.

Remark 7.4 *The manifolds $V_1 = SO^+(4,2)/SO^+(2,1)$ and $V_2 = SO^+(4,3)/SO(2)$ are not homeomorphic. The exact homotopy sequences of the fibrations yield*

$$\begin{aligned}\pi_1(V_1) = \mathbb{Z} \oplus \mathbb{Z} \quad &, \quad \pi_1(V_2) = 0\\ \pi_2(V_1) = 0 \quad &, \quad \pi_2(V_2) = \mathbb{Z}.\end{aligned}$$

In the same way we treat the following examples

Example 2 $\boxed{G = G_2 \qquad H = SU(3)}$

$G/H = S^6$ sphere
$\dim G/H = 6$
$G_2 \subset Spin(7)$ is the isotropy subgroup of $u_1 \in \Delta_{7,0}$.
$SU(3) \subset G_2 \subset Spin(7)$, where $SU(3)$ is the isotropy subgroup of the ordered pair $(u_1, u_2) \in \Delta_{7,0} \times \Delta_{7,0}$.

$$\begin{aligned}\mathfrak{g}_2 &= \text{span}\{E_{13} - E_{34}, E_{12} - E_{56}, E_{13} + E_{24}, E_{13} - E_{67}, E_{14} - E_{24},\\ &\qquad E_{14} - E_{57}, E_{16} - E_{25}, E_{16} + E_{37}, E_{15} + E_{26}, E_{15} - E_{47},\\ &\qquad E_{17} - E_{36}, E_{17} - E_{45}, E_{27} - E_{35}, E_{27} + E_{46}\}\\ \mathfrak{su}(3) &= \text{span}\{E_{12} - E_{34}, E_{12} - E_{56}, E_{13} + E_{24}, E_{14} - E_{23}, E_{15} + E_{26},\\ &\qquad E_{16} - E_{25}, E_{35} + E_{46}, E_{36} - E_{45}\}\end{aligned}$$

Orthonormal basis of $\mathfrak{m}$:

$$\begin{aligned}X_1 &= -E_{13} + E_{24} + 2E_{67}, & X_2 &= E_{14} + E_{23} - 2E_{57},\\ X_3 &= E_{15} - E_{26} + 2E_{47}, & X_4 &= E_{16} + E_{25} + 2E_{37},\end{aligned}$$

$G_2/SU(3)$ admits a homogeneous spin structure. The constant functions

$$\widetilde{\psi_i} : G_2 \to \Delta_{6,0}, \qquad \psi_i \equiv u_i, \qquad i = 7, 8$$

define spinors ψ_7, ψ_8 on G/H. ψ_7 is a Killing spinor with Killing number $\lambda_- = -1$, ψ_8 is a Killing spinor with Killing number $\lambda_+ = 1$.

Remark 7.5 *On the standard sphere S^6 we have $dim_{\mathbb{R}} K_+ = dim_{\mathbb{R}} K_- = 8$. The representations of G_2 on K_+ and K_- decompose into an irreducible representation of dimension 7 and a 1-dimensional trivial representation.*

The following involutive automorphism is possible.

$$
\begin{aligned}
T: \quad \mathfrak{g}_+ &= \mathrm{span}\{E_{12}-E_{34}, E_{12}-E_{56}, E_{13}+E_{24}, E_{13}-E_{67}, E_{14}-E_{24}, E_{14}-E_{57}\} \\
\mathfrak{g}_- &= \mathrm{span}\{E_{16}-E_{25}, E_{16}-E_{37}, E_{15}+E_{26}, E_{15}+E_{47}, E_{17}-E_{36}, E_{17}-E_{45}, \\
&\qquad E_{27}-E_{35}, E_{27}+E_{46}\}
\end{aligned}
$$

$G'/H' = G^*_{2(2)}/SU(3,2) = S^{6,4} =$ round sphere in $\mathbb{R}^{7,4}$
$k=4$

Example 3 $\boxed{G = SO(5) \qquad H = U(2)}$

$G/H = \mathbb{C}P^3$
$\dim G/H = 6$
The embedding $U(2) \hookrightarrow SO(5)$ is the standard one:

$$
U(2) \hookrightarrow SO(4) \hookrightarrow \left(\begin{array}{c|c} SO(4) & 0 \\ \hline 0 & 1 \end{array} \right) \hookrightarrow SO(5)
$$

$$
\begin{aligned}
\mathfrak{so}(5) &= \mathrm{span}\{E_{12}\,, \ldots\,, E_{45}\} \\
\mathfrak{u}(2) &= \mathrm{span}\{E_{12}\,, E_{34}\,, E_{13}+E_{24}\,, E_{14}-E_{23}\}
\end{aligned}
$$

orthonormal basis of $\mathfrak{m}$:

$$
\begin{gathered}
X_1 = E_{15}\,,\ X_2 = E_{25}\,,\ X_3 = E_{35}\,,\ X_4 = E_{45} \\
X_5 = \tfrac{1}{\sqrt{2}}(-E_{13}+E_{24})\,,\ X_6 = \tfrac{1}{\sqrt{2}}(-E_{14}-E_{23})
\end{gathered}
$$

$SO(5)/U(2)$ admits a homogeneous spin structure.
The constant functions

$$
\widetilde{\psi_i} : SO(5) \to \Delta_{6,0}, \qquad \widetilde{\psi_i} \equiv u_i, \qquad i = 3,4
$$

define spinors ψ_3, ψ_4 on G/H.
ψ_3 is a Killing spinor with Killing number $\lambda_- = -\frac{1}{2\sqrt{2}}$, ψ_8 is a Killing spinor with Killing number $\lambda_+ = \frac{1}{2\sqrt{2}}$.
Let A be the diagonal matrix $A = \mathrm{diag}(-1,1,1,1,-1)$. Then we have

$$
SO(5) = A\, SO(5)\, A^{-1}\,, \qquad U(2) = A\, G_0^-\, A^{-1}
$$

where G_0^- was defined in Section 6. Now we consider the involutive automorphisms $T_i : \mathfrak{so}(5) \to \mathfrak{so}(5)\,,\ i = 1,2,3$ defined in Section 6. Let T be one of these involutions. It is given by a decomposition $\mathfrak{so}(5) = \mathfrak{so}(5)_+ \oplus \mathfrak{so}(5)_-$. Furthermore, let G'/H' be a homogeneous space which is T-dual to $SO(5)/G_0^-$. Define

$$
\widetilde{\mathfrak{so}(5)_+} = A\, \mathfrak{so}(5)_+\, A^{-1}\,, \qquad \widetilde{\mathfrak{so}(5)_-} = A\, \mathfrak{so}(5)_-\, A^{-1}\,.
$$

Then the decomposition $\mathfrak{so}(5) = \widetilde{\mathfrak{so}(5)}_+ \oplus \widetilde{\mathfrak{so}(5)}_-$ gives a further involutive automorphism $\widetilde{T} : \mathfrak{so}(5) \to \mathfrak{so}(5)$. Now let $\widetilde{G'}$ and $\widetilde{H'}$ be the groups

$$\widetilde{G'} = A\,G'\,A^{-1}\,, \qquad \widetilde{H'} = A\,H'\,A^{-1}\,.$$

Then $\widetilde{G'}/\widetilde{H'}$ is $\widetilde{T}$-dual to $SO(5)/U(2)$. On the other hand $\widetilde{G'}/\widetilde{H'}$ can be identified with G'/H'.

Example 4 $\boxed{G = U(3) \qquad H = U(1) \times U(1) \times U(1)}$

G/H is the flag manifold $F(1,2)$.
$\dim G/H = 6$
orthonormal basis of $\mathfrak{g}$:

$$X_1 = E_{12}\,,\ X_2 = iA_{12}\,,\ X_3 = E_{13}\,,\ X_4 = iA_{13}\,,\ X_5 = E_{23}\,,\ X_6 = iA_{23}$$

$U(3)/(U(1) \times U(1) \times U(1))$ admits a homogeneous spin structure. The constant functions

$$\widetilde{\psi_i} : U(3) \to \Delta_{6,0}, \qquad \widetilde{\psi_i} \equiv u_i, \qquad i = 5,6$$

define spinors on G/H.
ψ_5 is a Killing spinor with Killing number $\lambda_- = -\frac{1}{2}$, ψ_6 a Killing spinor with Killing number $\lambda_+ = \frac{1}{2}$.
For suitable involutive automorphisms see Section 6.

Example 5 $\boxed{G = Spin(4) = S^3 \times S^3}$

$\dim G = 6$
orthonormal basis of $\mathfrak{g}$:

$$X_1 = \sqrt{3}E_{13}, X_2 = E_{24}, X_3 = \sqrt{3}E_{23}, X_4 = E_{14}, X_5 = \sqrt{3}E_{12}, X_6 = E_{34}$$

$\widetilde{\psi_7} : G \to \Delta_{6,0}\,,\ \widetilde{\psi_7} \equiv u_7$ defines a Killing spinor $\widetilde{\psi_7}$ with Killing number $\lambda_+ = \frac{1}{2\sqrt{3}}$.
$\widetilde{\psi_8} : G \to \Delta_{6,0}\,,\ \widetilde{\psi_8} \equiv u_8$ defines a Killing spinor $\widetilde{\psi_8}$ with Killing number $\lambda_- = -\frac{1}{2\sqrt{3}}$.
The following involutive automorphisms $T_i : \mathfrak{g} \to \mathfrak{g}$ are possible.

$$T_1 : \quad \mathfrak{g}_+ = \mathrm{span}\{E_{12}, E_{13}E_{23}\} \quad \mathfrak{g}_- = \mathrm{span}\{E_{14}, E_{24}, E_{34}\}$$

$G' = Spin^+(4,3) \quad k = 3$

$$T_2 : \quad \mathfrak{g}_+ = \mathrm{span}\{E_{12}, E_{34}\} \quad \mathfrak{g}_- = \mathrm{span}\{E_{13}, E_{14}E_{23}, E_{24}\}$$

$G' = Spin^+(4,2) \quad k = 4$

Example 6 $\boxed{G = Sp(2) \qquad H = Sp(1)}$

$G/H = S^7$ sphere
dim $G/H = 7$
The embedding $Sp(1) \hookrightarrow Sp(2)$ is given by

$$Sp(1) \ni A \longmapsto \begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix} \in Sp(2).$$

$$\begin{aligned} \mathfrak{sp}(2) &= \mathrm{span}\{iD_{11}\,,\, jD_{11}\,,\, kD_{11}\,,\, iD_{22}\,,\, iD_{22}\,,\, kD_{22}\,,\, E_{12}\,,\, iA_{12}\,,\, jA_{12}\,,\, kA_{12}\} \\ \mathfrak{sp}(1) &= \mathrm{span}\{iD_{11}\,,\, jD_{11}\,,\, kD_{22}\} \end{aligned}$$

orthonormal basis of $\mathfrak{m}$:

$$X_1 = -\tfrac{1}{\sqrt{2}}E_{12}\,,\ X_2 = -\tfrac{i}{\sqrt{2}}A_{12}\,,\ X_3 = -\tfrac{j}{\sqrt{2}}A_{12}\,,\ X_4 = -\tfrac{k}{\sqrt{2}}A_{12}\,,$$
$$X_5 = -\tfrac{j}{\sqrt{2t}}D_{11}\,,\ X_6 = -\tfrac{k}{\sqrt{2t}}D_{11}\,,\ X_7 = -\tfrac{i}{\sqrt{2t}}D_{11}$$

$Sp(2)/Sp(1)$ admits a homogeneous spin structure. The constant functions

$$\widetilde{\psi_i} : Sp(2) \to \Delta_{7,0}, \qquad \widetilde{\psi_i} \equiv u_i, \qquad i = 1,2,3,4$$

define spinors on G/H.
case 1 $\boxed{t = 1}$ (round 7-sphere)
ψ_1, ψ_2, ψ_3 are Killing spinors with Killing number $\lambda_+ = \frac{1}{2\sqrt{2}}$
case 2 $\boxed{t = \frac{1}{5}}$ (squashed 7-sphere)
ψ_4 is a Killing spinor with Killing number $\lambda_- = -\frac{3}{2\sqrt{2}\sqrt{5}}$
The following involutive automorphism is possible.

$$\begin{aligned} T:\quad \mathfrak{g}_+ &= \mathrm{span}\{iD_{11}\,,\, jD_{11}\,,\, kD_{11}\,,\, iD_{22}\,,\, jD_{22}\,,\, kD_{22}\} \\ \mathfrak{g}_- &= \mathrm{span}\{E_{12}\,,\, iA_{12}\,,\, jA_{12}\,,\, kA_{12}\} \end{aligned}$$

$G'/H' = Sp(2,1)/Sp(1)$, $k = 4$

Example 7 $\boxed{G = SU(3) \qquad H = S^1}$

$G/H = N(1,1)$ sphere
$\dim G/H = 7$
The embedding $S^1 \hookrightarrow SU(3)$ is given by

$$S^1 \ni e^{i\varphi} \longmapsto \begin{pmatrix} e^{i\varphi} & 0 & 0 \\ 0 & e^{i\varphi} & 0 \\ 0 & 0 & e^{-2i\varphi} \end{pmatrix} \in SU(3).$$

$$
\begin{aligned}
\mathfrak{g} &= \mathrm{span}\{i(D_{11}-D_{33})\,,\, i(D_{22}-D_{33})\,,\, E_{12}\,,\, iA_{12}\,,\, E_{13}\,,\, iA_{13}\,,\, E_{23}\,,\, iA_{23}\} \\
\mathfrak{h} &= \mathrm{span}\{i(D_{11}+D_{22}-2D_{33})\}
\end{aligned}
$$

orthonormal basis of $\mathfrak{m}$:

$$
X_1 = -\tfrac{1}{\sqrt{2}}E_{13}\,,\ X_2 = -\tfrac{i}{\sqrt{2}}A_{13}\,,\ X_3 = -\tfrac{1}{\sqrt{3}}E_{23}\,,\ X_4 = \tfrac{i}{\sqrt{2}}A_{23}\,,
$$
$$
X_5 = -\tfrac{1}{2\sqrt{t}}E_{12}\,,\ X_6 = -\tfrac{i}{2\sqrt{t}}A_{12}\,,\ X_7 = \tfrac{i}{2\sqrt{t}}(D_{11}-D_{22}) \qquad (t>0)
$$

$SU(3)/S^1$ admits a homogeneous spin structure.
The constant functions

$$
\widetilde{\psi_i} : SU(3) \to \Delta_{7,0}, \qquad \widetilde{\psi_i} \equiv u_i, \qquad i = 5,6,7,8
$$

define spinors on G/H.
case 1 $\boxed{t=1}$
ψ_5, ψ_6, ψ_8 are Killing spinors with Killing number $\lambda_+ = \frac{1}{4}$.
case 2 $\boxed{t=\frac{1}{5}}$
ψ_7 is a Killing spinor with Killing number $\lambda_- = -\frac{3}{2\sqrt{5}}$. The following involutive automorphisms $T_i : \mathfrak{g} \to \mathfrak{g}$ are possible.

$$
\begin{aligned}
T_1 : \quad \mathfrak{g}_+ &= \mathrm{span}\{i(D_{11}-D_{33})\,,\, i(D_{22}-D_{33})\,,\, E_{12}\,,\, iA_{12}\} \\
\mathfrak{g}_- &= \mathrm{span}\{E_{13}\,,\, iA_{13}\,,\, E_{23}\,,\, iA_{23}\}
\end{aligned}
$$

$G'/H' = SU(3,2)/S^1$

$$
\begin{aligned}
T_2 : \quad \mathfrak{g}_+ &= \mathrm{span}\{i(D_{11}-D_{33})\,,\, i(D_{22}-D_{33})\,,\, E_{23}\,,\, iA_{23}\} \\
\mathfrak{g}_- &= \mathrm{span}\{E_{12}\,,\, iA_{12}E_{13}\,,\, iA_{13}\}
\end{aligned}
$$

$G'/H' = SU(3,2)/S^1$, $k = 4$

$$
\begin{aligned}
T_3 : \quad \mathfrak{g}_+ &= \mathrm{span}\{E_{12}\,,\, E_{13}E_{23}\} \\
\mathfrak{g}_- &= \mathrm{span}\{i(D_{11}-D_{22})\,,\, i(D_{22}-D_{33})\,,\, iA_{12}\,,\, iA_{13}\,,\, iA_{23}\}
\end{aligned}
$$

$G'/H' = SL(3,\mathbb{R})/\mathbb{R}$, where $\mathbb{R} \hookrightarrow SL(3,\mathbb{R})$ is given by

$$
t \longmapsto \mathrm{diag}(e^{-2t}\,,\, e^t\,,\, e^t)
$$

Remark 7.6 *The homogeneous spaces arising from T_1 and T_2 for $t=1$ are not isometric. In the first case we dont have isotropic spinors, in the second case we have.*

Further examples of 7-dimensional compact homogeneous spaces with Killing spinors one can find in [DNP],[K1],[FKMS].

Example 8 $\boxed{G = SO(n) \qquad H = SO(n-2)}$

$G/H = V_{n,2}$ Stiefel manifold of oriented orthonormal 2-frames in $\mathbb{R}^n$
$\dim G/H = 2n-3$

$$\begin{aligned}\mathfrak{g} &= \operatorname{span}\{E_{\alpha\beta} \mid 1 \leq \alpha < \beta \leq n\} \\ \mathfrak{h} &= \operatorname{span}\{E_{\alpha\beta} \mid 1 \leq \alpha < \beta \leq n-2\}\end{aligned}$$

orthonormal basis of $\mathfrak{m}$:

$$X_1 = E_{1\,n-1}\,,\ X_2 = E_{1\,n}\,,\ X_3 = E_{2\,n-1}\,,\ X_4 = E_{2\,n}\,,\ \ldots\,,\ X_{2n-5} = E_{n-2\,n-1}\,,$$
$$X_{2n-4} = E_{n-2\,n}\,,\ X_{2n-3} = \tfrac{1}{s}E_{n-1\,n} \qquad (s > 0)$$

The adjoint representation of $\mathfrak{h}$ on $\mathfrak{m}$ satisfies

$$\operatorname{ad}(E_{\alpha\beta})|_m = E_{2\alpha-1\,2\beta-1} + E_{2\alpha\,2\beta} \qquad (1 \leq \alpha < \beta \leq n-2)$$

G/H admits a homogeneous spin structure.
For the lift of ad we obtain

$$\widetilde{\operatorname{ad}}(E_{\alpha\beta}) = \frac{1}{2}(e_{2\alpha-1}e_{2\beta-1} + e_{2\alpha}e_{2\beta})$$

On the other hand we have

$$\begin{aligned}\Phi_{2n-3,0}(e_{2\alpha-1}e_{2\beta-1}) &= E \otimes \ldots \otimes E \otimes \overset{\overset{\beta}{\downarrow}}{g_1} \otimes T \ldots \otimes T \otimes \overset{\overset{\alpha}{\downarrow}}{T g_1} \otimes E \otimes \ldots \otimes \overset{\overset{1}{\downarrow}}{E} \\ &= E \otimes \ldots \otimes E \otimes g_1 \otimes T \ldots \otimes T \otimes ig_2 \otimes E \otimes \ldots \otimes E \\ \Phi_{2n-3,0}(e_{2\alpha}e_{2\beta}) &= E \otimes \ldots \otimes E \otimes \overset{\overset{\beta}{\downarrow}}{g_2} \otimes T \ldots \otimes T \otimes \overset{\overset{\alpha}{\downarrow}}{T g_2} \otimes E \otimes \ldots \otimes \overset{\overset{1}{\downarrow}}{E} \\ &= E \otimes \ldots \otimes E \otimes g_2 \otimes T \ldots \otimes T \otimes (-ig_1) \otimes E \otimes \ldots \otimes E.\end{aligned}$$

The vectors $u_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \ldots \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $u_{2^{n-2}} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \ldots \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ are contained in the intersection

$$\bigcap_{1 \leq \alpha < \beta \leq n-2} \ker \Phi_{2n-3,0} \circ \widetilde{\operatorname{ad}}(E_{\alpha\beta})$$

of the kernels of all $\widetilde{\operatorname{ad}}(E_{\alpha\beta})$. Consequently, the constant functions

$$\begin{aligned}\widetilde{\psi_1} &: G \to \Delta_{2n-3,0}, \qquad \widetilde{\psi_1} \equiv u_1 \\ \widetilde{\psi_{-1}} &: G \to \Delta_{2n-3,0}, \qquad \widetilde{\psi_{-1}} \equiv u_{2^{n-2}}\end{aligned}$$

define spinors on G/H. Because of

$$\begin{aligned}[X_{2\alpha-1}, X_{2\alpha}] &= E_{n-1\,n} = sX_{2n-3} \\ [X_{2\alpha-1}, X_{2n-3}] &= -\frac{1}{s}X_{2\alpha} \\ [X_{2\alpha}, X_{2\alpha-1}] &= -E_{n-1\,n} = sX_{2n-3} \\ [X_{2\alpha}, X_{2n-3}] &= \frac{1}{s}X_{2\alpha-1}\end{aligned}$$

we get

$$
\begin{aligned}
U(X_{2\alpha-1}, X_{2n-3}) &= (-\frac{s}{2} + \frac{1}{2s})X_{2\alpha} \\
U(X_{2\alpha}, X_{2n-3}) &= (\frac{s}{2} - \frac{1}{2s})X_{2\alpha-1}
\end{aligned}
$$

and, thus

$$
\begin{aligned}
\Lambda(X_{2\alpha-1}) &= \frac{s}{2}E_{2\alpha\,2n-3} \\
\Lambda(X_{2\alpha}) &= -\frac{s}{2}E_{2\alpha-1\,2n-3} \\
\Lambda(X_{2n-3}) &= \sum_{i=1}^{n-2}(\frac{1}{s} - \frac{s}{2})E_{2\alpha-1\,2\alpha}
\end{aligned}
$$

$$
\begin{aligned}
\widetilde{\Lambda}(X_{2\alpha-1}) &= \frac{s}{4}e_{2\alpha}e_{2n-3} \\
\widetilde{\Lambda}(X_{2\alpha}) &= -\frac{s}{4}e_{2\alpha-1}e_{2n-3} \\
\widetilde{\Lambda}(X_{2n-3}) &= \sum_{i=1}^{n-2}\frac{1}{2}(\frac{1}{s} - \frac{s}{2})e_{2\alpha-1}e_{2\alpha}
\end{aligned}
$$

We conclude

$$
\begin{aligned}
\widetilde{\Lambda}(X_{2\alpha-1}) - \lambda X_{2\alpha-1} &\hat{=} \frac{s}{4}i(T\otimes\ldots\otimes T\otimes \overset{\overset{\alpha}{\downarrow}}{g_2} T\otimes E\otimes\ldots\otimes \overset{\overset{1}{\downarrow}}{E}) \\
&\quad -\lambda(E\otimes\ldots\otimes E\otimes g_1\otimes T\otimes\ldots\otimes T) \\
&= -\frac{s}{4}(T\otimes\ldots\otimes T\otimes g_1\otimes E\otimes\ldots\otimes E) \\
&\quad -\lambda(E\otimes\ldots\otimes E\otimes g_1\otimes T\otimes\ldots\otimes T) \\
\widetilde{\Lambda}(X_{2\alpha}) - \lambda X_{2\alpha} &\hat{=} -\frac{s}{4}i(T\otimes\ldots\otimes T\otimes g_1\otimes E\otimes\ldots\otimes E) \\
&\quad -\lambda(E\otimes\ldots\otimes E\otimes g_2\otimes T\otimes\ldots\otimes T) \\
&= -\frac{s}{4}(T\otimes\ldots\otimes T\otimes g_2\otimes E\otimes\ldots\otimes E) \\
&\quad -\lambda(E\otimes\ldots\otimes E\otimes g_2\otimes T\otimes\ldots\otimes T) \\
\widetilde{\Lambda}(X_{2n-3}) - \lambda X_{2n-3} &\hat{=} \sum_{\alpha=1}^{n-2}\frac{1}{2}(\frac{1}{s} - \frac{s}{2})(-i)(E\otimes\ldots\otimes E\otimes \overset{\overset{\alpha}{\downarrow}}{T}\otimes E\otimes\ldots\otimes E) \\
&\quad -\lambda i T\otimes\ldots\otimes T
\end{aligned}
$$

We conclude that in case n is even ψ_1 is a Killing spinor with Killing number $\lambda_+ = -\frac{1}{2\sqrt{2}}\sqrt{\frac{n-2}{n-1}}$ and ψ_{-1} is a Killing spinor with Killing number $\lambda_- = -\frac{1}{2\sqrt{2}}\sqrt{\frac{n-2}{n-1}}$ and in case n is odd both ψ_1 and ψ_{-1} are Killing spinors with Killing number $\lambda_- = -\frac{1}{2\sqrt{2}}\sqrt{\frac{n-2}{n-1}}$.

The following involutive automorphisms $T_s : \mathfrak{g} \to \mathfrak{g}$ and $t_s : \mathfrak{g} \to \mathfrak{g}$, $(0 \leq s \leq n-2)$ are possible.

$$\begin{array}{rrcl} T_s : & \mathfrak{g}_+ & = & \mathrm{span}\{\ E_{\alpha\beta} \mid \alpha < \beta \leq s \text{ or } s < \alpha < \beta\} \\ & \mathfrak{g}_- & = & \mathrm{span}\{\ E_{\alpha\beta} \mid \alpha \leq s < \beta\} \end{array}$$

$G'/H' = SO^+(n,s)/SO^+(n-2,s)\,, \quad k = 2s$

$$\begin{array}{rrclll} t_s : & \mathfrak{g}_+ & = & \mathrm{span}\{\ E_{\alpha\beta} \mid & & \alpha < \beta \leq s \\ & & & & \text{or} & s < \alpha < \beta \leq n-2 \\ & & & & \text{or} & \beta = n-1\,,\ \alpha \leq s \\ & & & & \text{or} & \beta = n\,,\ s < \alpha < n-1 \quad \} \\ & \mathfrak{g}_- & = & \mathrm{span}\{\ E_{\alpha\beta} \mid & & \alpha \leq s < \beta \leq n-2 \\ & & & & \text{or} & \beta = n-1\,,\ \alpha > s \\ & & & & \text{or} & \beta = n\,,\ \alpha \leq s \\ & & & & \text{or} & \beta = n\,,\ \alpha = n-1 \quad \} \end{array}$$

$G'/H' = SO^+(n,s+1)/SO^+(n-2,s)\,, \quad k = s+1$

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