Closed curves in $\mathbf{R}^3$: a characterization in terms of curvature and torsion, the Hasimoto map and periodic solutions of the Filament Equation.

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Abstract

If a curve in $\mathbb{R}^3$ is closed, then the curvature and the torsion are periodic functions satisfying some additional constraints. We show that these constraints can be naturally formulated in terms of the spectral problem for a 2×2 matrix differential operator. This operator arose in the theory of the self-focusing Nonlinear Schrödinger Equation.

A simple spectral characterization of Bloch varieties generating periodic solutions of the Filament Equation is obtained. We show that the method of isoperiodic deformations suggested earlier by the authors for constructing periodic solutions of soliton equations can be naturally applied to the Filament Equation.

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1 Introduction

In our article we shall study the periodic problem for the Filament Equation

$$\frac{\partial \vec{\gamma}(s,t)}{\partial t} = k(s,t)\vec{b}(s,t) \tag{1}$$

where $\gamma(s,t)$ is t-dependent family of smooth curves in $\mathbb{R}^3$, s is a natural parameter on these curves, $k(s,t)$ is the curvature, $\vec{b}(s,t)$ is the binormal vector (the necessary definitions from differential geometry are collected in Section 2).

The Filament Equation describes the motion of a very thin isolated vortex filament in an incompressible unbounded fluid. It was derived by Da Rios [4] in the year 1906 and rediscovered in the 60's by R. J. Arms, F. R. Hama, R. J. Betchov (see the historical article [17] for more details).

By a periodic problem we mean constructing solutions of (1) such that for any $t = t_0$ the curve $\gamma(s,t_0)$ is closed:

$$\vec{\gamma}(s+l,t_0) = \vec{\gamma}(s,t_0) \tag{2}$$

Without loss of generality we shall assume the length l of the curve to be equal 2π. A closed curve can be naturally interpreted as a smooth isometric map

$$\gamma : S^1 \to \mathbb{R}^3 \tag{3}$$

In 1972 H. Hasimoto [10] found a change of variables connecting (1) with the the self-focusing Nonlinear Schrödinger Equation (NLS)

$$i\frac{\partial q(s,t)}{\partial t} + \frac{\partial^2 q(s,t)}{\partial s^2} + \frac{1}{2}\left|q(s,t)\right|^2 q(s,t) = 0. \tag{4}$$

This change of variables associates with a curve $\gamma(s)$ a complex function $q(s)$:

$$\mathcal{H} : \gamma(s) \to q(s) = k(s)e^{i\int^s \kappa(\tilde{s})d\tilde{s}} \tag{5}$$

(In [3] A. Doliwa and P. M. Santini have shown that under some natural assumptions any integrable motion of a curve in S^3 results in the NLS hierarchy.)

The periodic problem for NLS is well-studied (see Section 3). Unfortunately results from the periodic NLS theory can not be applied directly to the Filament Equation because the Hasimoto map (5) does not map periodic functions to the periodic ones. It is easy to check, that

1. For a generic closed curve $\vec{\gamma}(s+2\pi) = \vec{\gamma}(s)$ the corresponding potential $q(s)$ is quasi-periodic:

$$q(s+2\pi) = e^{i\phi} q(s). \tag{6}$$

2. For a generic periodic potential $q(s+2\pi) = q(s)$ neither $\vec{\gamma}(s)$ nor the velocity vector $\vec{v}(s)$ are periodic.

The problem of constructing periodic algebro-geometric solutions of the Filament Equation (1) was studied by A. M. Calini in her Ph.D. dissertation [1]. In [1] a number of interesting results were obtained. In particular explicit exact solutions were constructed. Unfortunately, no characterization of periodic solutions of the Filament equation was given in [1]. As pointed out by S. P. Novikov, without such a characterization the periodic problem for (1) can not be considered as completely solved.

Thus we are faced with the following differential-geometrical problem: If a curve in $\mathbb{R}^3$ is closed, then the curvature and the torsion functions are also periodic. But the periodicity of the curvature and the torsion functions does not imply automatically that the corresponding curve is closed. To obtain a closed curve we have to add additional constraints on the curvature and the torsion. We show that these constraints can be naturally written in terms of the spectral problem for a 2×2 matrix differential operator associated with NLS equation (see Sections 3, 4).

Another interesting question associated with the Hasimoto map arose in the Hamiltonian theory of the NLS equation. J. J. Millson and B. Zombro have shown in [14] that the space of smooth isometric maps of the unit circle into $\mathbb{R}^3$ modulo proper Euclidean motions has a natural Kähler structure. The imaginary part of this structure coincides with one of the higher NLS symplectic structures. In order to study this relation it is important to have an explicit description of the spaces connected by the Hasimoto map. We belive that the characterization obtained in Section 4 may be applied to this problem.

The problem of characterizing the spectral data corresponding to the periodic NLS solutions it rather non-trivial. A convenient approach to this problem based on the so-called isoperiodic deformations was suggested by the authors in [9]. In Section 5 we show that this approach can be naturally applied to the Filament Equation.

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2 Curves in 3-dimensional Euclidean space

Let $\vec{\gamma}(s)$ be a smooth, parameterized curve in Euclidean 3-space:

$$\vec{\gamma}(s) = (x^1(s), x^2(s), x^3(s)) \in \mathbb{R}^3. \tag{7}$$

Denote by $\vec{v}(s)$ and $\vec{w}(s)$ the velocity and the acceleration respectively:

$$\vec{v}(s) = \frac{d\vec{\gamma}(s)}{ds} = \left(\frac{dx^1(s)}{ds}, \frac{dx^2(s)}{ds}, \frac{dx^3(s)}{ds} \right) \tag{8}$$

$$\vec{w}(s) = \frac{d^2\vec{\gamma}(s)}{ds^2} = \left(\frac{d^2x^1(s)}{ds^2}, \frac{d^2x^2(s)}{ds^2}, \frac{d^2x^3(s)}{ds^2} \right) \tag{9}$$

We shall assume that s is the natural parameter, i.e. the length along the curve. In other words, the velocity has unit length:

$$|\vec{v}(s)|^2 = \left(\frac{dx(s)}{ds} \right)^2 + \left(\frac{dy(s)}{ds} \right)^2 + \left(\frac{dz(s)}{ds} \right)^2 = 1 \tag{10}$$

Then the acceleration vector is orthogonal to the velocity

$$< \vec{v}(s), \vec{w}(s) >= 0, \tag{11}$$

where $< \, , \, >$ denotes the standard scalar product in $\mathbb{R}^3$

$$< \vec{v}, \vec{w} >= v^1 w^1 + v^2 w^2 + v^3 w^3. \tag{12}$$

The magnitude of the acceleration vector is called the **curvature** of the curve

$$k(s) = |\vec{w}(s)| = \sqrt{< \vec{w}(s), \vec{w}(s) >}. \tag{13}$$

For each value of s such that $k(s) \neq 0$ we have a natural orthogonal reference frame $\left(\vec{v}(s), \vec{n}(s), \vec{b}(s)\right)$, where

$$\vec{n}(s) = \frac{\vec{w}(s)}{|\vec{w}(s)|}, \quad \vec{b}(s) = \vec{v}(s) \times \vec{n}(s). \tag{14}$$

Here $\times$ denotes the vector product in $\mathbb{R}^3$

$$\vec{v} \times \vec{n} = \left(v^2 n^3 - v^3 n^2, v^3 n^1 - v^1 n^3, v^1 n^2 - v^2 n^1\right) \tag{15}$$

The vector $\vec{n}(s)$ is called the principal normal to the curve, the vector $\vec{b}(s)$ is called the binormal.

The natural reference frame $\left(\vec{v}(s), \vec{n}(s), \vec{b}(s)\right)$ satisfies the Serret-Frenet equations (see for example [2])

$$\begin{aligned}
\frac{d\vec{v}(s)}{ds} &= k(s)\vec{n}(s) \\
\frac{d\vec{n}(s)}{ds} &= -k(s)\vec{v}(s) + \kappa(s)\vec{b}(s) \\
\frac{d\vec{b}(s)}{ds} &= -\kappa(s)\vec{n}(s)
\end{aligned} \tag{16}$$

The function $\kappa(s)$ is called the **torsion**.

Further we shall use another reference frame $(\vec{e}_1(s), \vec{e}_2(s), \vec{e}_3(s))$ associated with a smooth curve $\vec{\gamma}(s)$ parameterized by the natural parameter. Let

$$\begin{aligned}
\vec{e}_1(s) &= \vec{v}(s) \\
\vec{e}_2(s) &= \cos(\theta(s))\ \vec{n}(s) - \sin(\theta(s))\ \vec{b}(s) \\
\vec{e}_3(s) &= \sin(\theta(s))\ \vec{n}(s) + \cos(\theta(s))\ \vec{b}(s)
\end{aligned} \tag{17}$$

where

$$\theta(s) = \int^s \kappa(\tilde{s}) d\tilde{s}. \tag{18}$$

This reference frame satisfies

$$\frac{d\vec{e}_1(s)}{ds} = k(s)\cos(\theta(s))\ \vec{e_2} + k(s)\sin(\theta(s))\ \vec{e_3}(s)$$

$$\frac{d\vec{e_2}(s)}{ds} = -k(s)\cos(\theta(s))\ \vec{e_1}(s) \tag{19}$$
$$\frac{d\vec{e_3}(s)}{ds} = -k(s)\sin(\theta(s))\ \vec{e_1}(s)$$

Equations (19) are written in terms of the Lie algebra $so(3)$. Let us rewrite them in terms of $su(2)$.

Let I, J, K be a basis in the space of skew-hermitian matrices

$$I = i\sigma_z = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix},\quad J = -i\sigma_y = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},\quad K = -i\sigma_x = \begin{bmatrix} 0 & -i \\ -i & 0 \end{bmatrix}, \tag{20}$$

where $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ are the Pauli matrices.

These matrices satisfy the multiplication rules for the basic quaternions

$$IJ = -JI = K,\ JK = -KJ = I,\ KI = -IK = J,\ I^2 = J^2 = K^2 = -1, \tag{21}$$

and they form an orthonormal reference frame in the space of 2×2 skew-hermitian matrices with the following scalar product

$$< A, B >= -\frac{1}{2}\mathrm{trace} AB. \tag{22}$$

It is easy to check that the following map:

$$\vec{w} = (w^1, w^2, w^3) \to \hat{W} = w^1 \cdot I + w^2 \cdot J + w^3 \cdot K. \tag{23}$$

is an isometry between the Euclidean space $\mathbb{R}^3$ and the space of 2×2 skew-hermitian matrices with inverse given by

$$w^1 = -\frac{1}{2}\mathrm{trace}\,\hat{W}I,\ w^2 = -\frac{1}{2}\mathrm{trace}\,\hat{W}J,\ w^3 = -\frac{1}{2}\mathrm{trace}\,\hat{W}K.$$

It generates the famous map $SU(2) \to SO(3)$: any unitary matrix $g \in SU(2)$ generates an isometric map of the space of 2×2 hermitian matrices

$$x \to gxg^{-1}. \tag{24}$$

It is well-known (see for example [2] p. 431) that any isometric orientation preserving map of the the space of 2×2 hermitian matrices is generated by (24) and the matrix g is defined uniquely up to multiplication by -1.

Denote the matrices corresponding to the vectors $\vec{e}_1(s)$, $\vec{e}_2(s)$ $\vec{e}_3(s)$ by $\hat{E}_1(s)$, $\hat{E}_2(s)$, $\hat{E}_3(s)$ respectively. They form an orthonormal reference frame. Thus there exists a 2×2 unitary matrix $\Omega(s)$ such that

$$\begin{aligned} \hat{E}_1(s) &= \Omega^{-1}(s) I \Omega(s), \\ \hat{E}_2(s) &= \Omega^{-1}(s) J \Omega(s), \\ \hat{E}_3(s) &= \Omega^{-1}(s) K \Omega(s). \end{aligned} \tag{25}$$

In terms of these 2×2 matrices the system of equations (19) takes the form

$$\begin{aligned} [I, \omega(s)] &= k(s) \cos(\theta(s))\ J + k(s) \sin(\theta(s))\ K \\ [J, \omega(s)] &= -k(s) \cos(\theta(s))\ I \\ [K, \omega(s)] &= -k(s) \sin(\theta(s))\ I \end{aligned} \tag{26}$$

where

$$\omega(s) = \frac{d\Omega(s)}{ds} \Omega^{-1}(s). \tag{27}$$

Taking into account that $\omega(s)$ is skew-hermitian we obtain

$$\omega(s) = -k(s) \cos(\theta(s))\ K + k(s) \sin(\theta(s))\ J = \begin{bmatrix} 0 & \frac{iq(s)}{2} \\ \frac{i\bar{q}(s)}{2} & 0 \end{bmatrix} \tag{28}$$

where

$$q(s) = k(s) e^{i\theta(s)}. \tag{29}$$

Starting from a curve $\vec{\gamma}(s) \in \mathbb{R}^3$ we have thus constructed a potential $q(s)$. Formula (29) coincides with the Hasimoto transformation (5).

Let us discuss the inverse map (the map from the space of complex-valued functions of one real variable to the space of curves in $\mathbb{R}^3$).

Lemma 1 *Let $q(s)$ be a complex-valued smooth function of one real variable s such that $q(s) \neq 0$ for all s. Then there exists an unique (up to a proper isometry of $\mathbb{R}^3$) curve $\gamma(s)$ such that $\mathcal{H}\gamma(s) = e^{i\phi} q(s)$ where $\mathcal{H}$ is the Hasimoto map. (Recall that the image of the Hasimoto map is defined up to an arbitrary constant phase ϕ). The curve $\gamma(s)$ can be constructed by using the following procedure:*

1. *Define a matrix* $\omega(s)$ *by (28).*

2. *Define a* 2×2 *matrix function* $\Omega(s)$ *as a solution of the following linear ordinary differential equation:*
$$\frac{d}{ds}\Omega(s) = \omega(s)\Omega(s) \tag{30}$$
such that $\Omega(0)$ *is an unitary matrix. (From (30) it follows that the matrix* $\Omega(s)$ *is unitary for all* s*).*

3. *Let* $\hat{E}_1(s)$ *be a skew-hermitian matrix defined by the formula (25), and* $\vec{e}_1(s)$ *be the corresponding vector in* $\mathbb{R}^3$*.*

4. *The curve* $\vec{\gamma}(s)$ *and the corresponding matrix-valued function* $\hat{\Gamma}(s)$ *are defined by:*
$$\vec{\gamma}(s) = \vec{\gamma}(0) + \int_0^s \vec{e}_1(\tilde{s})d\tilde{s}, \;\; \hat{\Gamma}(s) = \hat{\Gamma}(0) + \int_0^s \hat{E}_1(\tilde{s})d\tilde{s} \tag{31}$$

The proof of Lemma 1 is standard, so we will not present it here.

Remark 1 *If the potential* $q(s)$*is known, so is the curvature and the torsion. Indeed one has*
$$k(s) = |q(s)|, \;\; \kappa(s) = \frac{d}{ds} arg\, q(s), \tag{32}$$
and the curve $\gamma(s)$ *is defined uniquely up to an isometry (see for example [2]). However the reconstruction procedure described above is essential for the approach used in this article.*

From Lemma 1 it follows directly:

Lemma 2 *The curve* $\vec{\gamma}(s)$ *constructed in Lemma 1 is periodic with period* $l = 2\pi$ *if and only if the following two conditions are fulfilled:*

1. *The matrix* $\hat{E}_1(s)$ *defined by the formula (25) is periodic with period* 2π
$$\hat{E}_1(s+2\pi) \equiv \hat{E}_1(s). \tag{33}$$

2. *The integral of* $\hat{E}_1(s)$ *over one period vanishes, i.e.*
$$\int_0^{2\pi} \hat{E}_1(s)ds = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}. \tag{34}$$

3 Periodic theory of Nonlinear Schrödinger Equation

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3.1 The zero-curvature representation

The self-focusing Nonlinear Schrödinger Equation (NLS)

$$i\frac{\partial q(s,t)}{\partial t} + \frac{\partial^2 q(s,t)}{\partial s^2} + \frac{1}{2}\left|q(s,t)\right|^2 q(s,t) = 0. \tag{35}$$

is one of the most important soliton equations. In 1971 V. E. Zakharov and A. B. Shabat [20] proved that NLS can be integrated by the inverse scattering method. This method is based on the so-called zero-curvature representation for NLS (a good introduction to the NLS theory can be found in the book [7] by L. D. Faddeev and L. A. Takhtajian):

Consider the following pair of linear problems:

$$\frac{\partial F(x,t,\lambda)}{\partial x} = U(x,t,\lambda)F(x,t,\lambda), \tag{36}$$

$$\frac{\partial F(x,t,\lambda)}{\partial t} = V(x,t,\lambda)F(x,t,\lambda) \tag{37}$$

where $F(x,t,\lambda)$ is a vector-valued function

$$F(x,t,\lambda) = \begin{pmatrix} f_1(x,t,\lambda) \\ f_2(x,t,\lambda) \end{pmatrix}, \tag{38}$$

$U(x,t,\lambda)$, $V(x,t,\lambda)$ are the following 2×2 matrices, which depend polynomially on spectral parameter λ:

$$U(x,t,\lambda) = \begin{bmatrix} -\frac{1}{2}\,i\,\lambda & \frac{1}{2}\,i\,q(\,x,t\,) \\ \frac{1}{2}\,i\,\bar{q}(\,x,t\,) & \frac{1}{2}\,i\,\lambda \end{bmatrix} \tag{39}$$

$$V(x,t,\lambda) = \begin{bmatrix} \frac{1}{4}\,i\,q(\,x,t\,)\,\bar{q}(\,x,t\,) - \frac{1}{2}\,i\,\lambda^2\,, & -\frac{1}{2}\left(\frac{\partial}{\partial x}\,q(\,x,t\,)\right) + \frac{1}{2}\,i\,q(\,x,t\,)\,\lambda \\ \frac{1}{2}\left(\frac{\partial}{\partial x}\,\bar{q}(\,x,t\,)\right) + \frac{1}{2}\,i\,\bar{q}(\,x,t\,)\,\lambda\,, & -\frac{1}{4}\,i\,q(\,x,t\,)\,\bar{q}(\,x,t\,) + \frac{1}{2}\,i\,\lambda^2 \end{bmatrix} \tag{40}$$

The system (36), (37) is compatible if and only if

$$\frac{\partial U(x,t,\lambda)}{\partial t} - \frac{\partial V(x,t,\lambda)}{\partial x} + [U(x,t,\lambda), V(x,t,\lambda)] = 0. \tag{41}$$

(Here [,] is the standard matrix commutator $[A, B] = AB - BA$.) A simple direct calculation shows, that (41) is equivalent to (4).

The representation (41) is called the **zero-curvature** representation (see e.g. [7] for additional information).

Remark 2 *The word "zero-curvature" means the following. The matrices $U(x,t,\lambda)$, $V(x,t,\lambda)$ can be interpreted as local connection coefficients in the trivial bundle with base $\mathbb{R}^2$ and fiber $\mathbb{C}^2$. Here (x,t) is a point of the base, $F(x,t,\lambda)$ takes values in the fiber and λ is a parameter. Then (41) means exactly that the curvature of this connection vanishes for all λ.*

3.2 The auxiliary linear problem and gauge transformations

The linear problem (36) is called the **auxiliary linear problem** for the NLS equation. It plays a crucial role in the inverse scattering method. We have a spectral problem for a first-order ordinary differential operator in the variable x, with a spectral parameter λ. This operator also depend on an additional parameter t. We shall study this spectral problem for a fixed $t = t_0$. Henceforth we will drop the t-dependence from the notations and write $U(x,\lambda)$, $F(x,\lambda)$, $q(x)$ instead of $U(x,t_0,\lambda)$, $F(x,t_0,\lambda)$, $q(x,t_0)$ respectively. Also we will rewrite (36) in the following form

$$L(\lambda)F(x,\lambda) = 0, \quad L(\lambda) = \frac{d}{dx} - \begin{bmatrix} -\frac{1}{2}\,i\,\lambda & \frac{1}{2}\,i\,q(x) \\ \frac{1}{2}\,i\,\overline{q(x)} & \frac{1}{2}\,i\,\lambda \end{bmatrix} \tag{42}$$

We are looking for a characterization of potentials corresponding to periodic curves of length $l = 2\pi$. Thus we shall study the direct spectral transform for the problem (42) in the class of smooth complex-valued potentials $q(x)$ such that

$$q(s + 2\pi) = e^{i\phi} q(s), \quad \phi \in \mathbb{R}. \tag{43}$$

(It is well-known that ϕ is an integral of motion for the filament equations).

Usually the quasi-periodic spectral theory is much more complicated than the periodic one. Fortunately in the present case we can handle the situation (this fact was pointed out in [7]). The reason is that the linear problem (42) is invariant under the following gauge transformations:

$$F(x,\lambda) = \left(\begin{array}{c} f_1(x,\lambda) \\ f_2(x,\lambda) \end{array} \right) \to \tilde{F}(x,\tilde{\lambda}) = \left(\begin{array}{c} e^{\frac{i}{2}\alpha x} f_1(x,\lambda) \\ e^{-\frac{i}{2}\alpha x} f_2(x,\lambda) \end{array} \right), \tag{44}$$

$$\lambda \to \tilde{\lambda} = \lambda - \alpha, \;\; q(x) \to \tilde{q}(x) = e^{i\alpha x} q(x) \tag{45}$$

$$\tilde{L}(\tilde{\lambda})\tilde{F}(x,\tilde{\lambda}) = 0, \quad \tilde{L}(\tilde{\lambda}) = \frac{d}{dx} - \left[\begin{array}{cc} -\frac{1}{2}\, i\, \tilde{\lambda} & \frac{1}{2}\, i\, \tilde{q}(x) \\ \frac{1}{2}\, i\, \overline{\tilde{q}(x)} & \frac{1}{2}\, i\, \tilde{\lambda} \end{array} \right] \tag{46}$$

parameterized by $\alpha \in \mathbb{R}$.

With the choice

$$\alpha = -\frac{\phi}{2\pi} \tag{47}$$

we have transformed (42) to a spectral problem with a purely periodic potential $\tilde{q}(x+2\pi) = \tilde{q}(x)$.

The gauge transformation (44), (45) respects the formula (25). More precisely let $\Omega(x)$ be a 2×2 matrix solution of the equation

$$L(0)\Omega(x) = 0. \tag{48}$$

Then

$$\tilde{\Omega}(x) = e^{-\frac{i\phi}{4\pi} x\, \sigma_z}\Omega(x), \quad \sigma_z = \left[\begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right] \tag{49}$$

satisfy

$$\tilde{L}\left(\frac{\phi}{2\pi}\right)\tilde{\Omega}(x) = 0. \tag{50}$$

and

$$\hat{E}_1(x) = \Omega^{-1}(x) I \Omega(x) = \tilde{\Omega}^{-1}(x) I \tilde{\Omega}(x). \tag{51}$$

The gauge transformation (44), (45) shifts the spectral parameter λ. Thus the $\lambda = 0$ eigenfunctions of the problem (42) with an arbitrary ϕ are equivalent to eigenfunctions of the purely periodic problem with an arbitrary real λ.

Taking into account the gauge transformation properties of (42) we obtain the following modification of the reconstruction procedure described in Lemma 1.

Lemma 3 *Let $q(x)$ be a complex-valued smooth function of one real variable x such that $q(x) \neq 0$ for all x. Let $\Lambda_0 \in \mathbb{R}$ be a real point in the spectral plane. Then there exists a curve $\gamma(x)$ (unique up to an isometry of $\mathbb{R}^3$) such that $\mathcal{H}\gamma(x) = e^{i\phi - i\Lambda_0 x} q(x)$ where $\mathcal{H}$ is the Hasimoto map. The curve $\gamma(x)$ can be obtained by the following procedure:*

1. *Let $\Omega(x)$ be an arbitrary 2×2 matrix solution of the following equation:*
$$L(\Lambda_0)\Omega(x) = 0 \tag{52}$$
such that $\Omega(0)$ is a unitary matrix.

2. *Let $\hat{E}_1(x)$ be the skew-hermitian matrix defined by*
$$\hat{E}_1(x) = \Omega^{-1}(x) I \Omega(x). \tag{53}$$

3. *The function $\hat{\Gamma}(x)$ is defined by*
$$\hat{\Gamma}(x) = \hat{\Gamma}(0) + \int_0^x \hat{E}_1(\tilde{x}) d\tilde{x}. \tag{54}$$

3.3 Periodic spectral problem and Bloch variety

For the remainder of this section we shall assume that $q(x)$ is a smooth complex-valued periodic function with period 2π.

If we impose no boundary conditions, then the equation (42) has a two-dimensional space of solutions for any complex λ. A point $\lambda \in \mathbb{C}$ belongs to the spectrum of (42) if and only if this space contains at least one function bounded on the whole x-line.

The structure of the spectrum of the problem (42) may be rather complicated (this structure was studied by Y. Li and D. W.McLaughlin in [12]). Fortunately we do not have to know this structure in detail to construct explicit solutions.

Let us fix a basis of solutions $\vec{\varphi}^{(1)}(x,\lambda)$, $\vec{\varphi}^{(2)}(x,\lambda)$:

$$L(\lambda)\vec{\varphi}^{(1)}(x,\lambda) = L(\lambda)\vec{\varphi}^{(2)}(x,\lambda) = 0, \;\; \vec{\varphi}^{(1)}(0,\lambda) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \; \vec{\varphi}^{(2)}(0,\lambda) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{55}$$

Denote by $\Phi(x,\lambda)$ the 2×2 fundamental solution of (42):

$$\Phi(x,\lambda) = \left[\vec{\varphi}^{(1)}(0,\lambda)\vec{\varphi}^{(2)}(0,\lambda)\right]. \tag{56}$$

The operator $L(\lambda)$ commutes with the shift operator

$$f(x) \to f(x+2\pi). \tag{57}$$

Thus we have

$$\Phi(x+2\pi,\lambda) = \Phi(x,\lambda)T(\lambda), \text{ where } T(\lambda) = \Phi(2\pi,\lambda) \tag{58}$$

The matrix $T(\lambda)$ is called the monodromy matrix. For generic λ it can be diagonalized. The common eigenfunctions of $L(\lambda)$ and of the shift operator

$$\vec{\psi}^{(1)}(x,\lambda) = \begin{pmatrix} \psi_1^{(1)}(x,\lambda) \\ \psi_2^{(1)}(x,\lambda) \end{pmatrix}, \qquad \vec{\psi}^{(2)}(x,\lambda) = \begin{pmatrix} \psi_1^{(2)}(x,\lambda) \\ \psi_2^{(2)}(x,\lambda) \end{pmatrix}$$

$$\begin{aligned} L(\lambda)\vec{\psi}^{(1)}(x,\lambda) = 0, \;\; \vec{\psi}^{(1)}(x+2\pi,\lambda) = w^{(1)}(\lambda)\vec{\psi}^{(1)}(x,\lambda), \\ L(\lambda)\vec{\psi}^{(2)}(x,\lambda) = 0, \;\; \vec{\psi}^{(2)}(x+2\pi,\lambda) = w^{(2)}(\lambda)\vec{\psi}^{(2)}(x,\lambda) \end{aligned} \tag{59}$$

are called the **Bloch functions**. The functions $p^{(1)}(\lambda)$, $p^{(2)}(\lambda)$ are called quasimomenta. The matrix $U(x,\lambda)$ is traceless, thus

$$w^{(1)}(\lambda)w^{(2)}(\lambda) = 1.$$

A point $\lambda \in \mathbb{C}$ belongs to the spectrum of (42) if and only if

$$\left|w^{(1)}(\lambda)\right| = 1.$$

It is convenient to fix a normalization of the Bloch functions by the condition

$$\psi_1^{(1)}(0,\lambda) + \psi_2^{(1)}(0,\lambda) = \psi_1^{(2)}(0,\lambda) + \psi_2^{(2)}(0,\lambda) = 1. \tag{60}$$

It is easy to check that

1. The matrix $T(\lambda)$ is holomorphic in λ in the whole λ-plane.
2. $\det T(\lambda) \equiv 1$.
3. For all $\lambda \in \mathbb{R}$ $T(\lambda)$ is unitary, i.e. $T^{-1}(\lambda) = T^*(\lambda)$ where $*$ denotes hermitian conjugation.

For a generic $\lambda \in \mathbb{C}$ $T(\lambda)$ has two eigenvalues $w^{(1)}(\lambda)$, $w^{(2)}(\lambda)$ and a pair of corresponding Bloch functions. In fact there is a holomorphic function $w(\mu)$ on a hyperelliptic Riemann surface Y and $w^{(1)}(\lambda) = w(\mu_1)$, $w^{(2)}(\lambda) = w(\mu_2)$ where μ_1 and μ_2 are the pre-images of the point λ under the projection $Y \to \mathbb{C}$ (a Riemann surface is called hyperelliptic if is a two-sheeted ramified covering of the Riemann sphere). For generic potentials the surface Y is connected, but there exist exceptional potentials such that $Y = \mathbb{C} \cup \mathbb{C}$.

Any hyperelliptic Riemann surface has a natural holomorphic involution given transposing the sheets. Let us denote this involution by σ:

$$\sigma\lambda = \lambda, \quad \sigma w(\mu) = w^{-1}(\mu). \tag{61}$$

Let

$$\vec{\psi}(x,\mu) = \begin{pmatrix} \psi_1(x,\mu) \\ \psi_2(x,\mu) \end{pmatrix} \tag{62}$$

be a Bloch solution of $L(\lambda(\mu))\vec{\psi}(x,\mu) = 0$, $\vec{\psi}(x+2\pi,\mu) = w(\mu)\vec{\psi}(x,\mu)$. Then

$$\vec{\psi}^+(x,\mu) = \begin{pmatrix} -\bar{\psi}_2(x,\mu) \\ \bar{\psi}_1(x,\mu) \end{pmatrix} \tag{63}$$

is a Bloch solution of $L(\bar{\lambda}(\mu))\vec{\psi}^+(x,\mu) = 0$, $\vec{\psi}^+(x+2\pi,\mu) = \bar{w}(\mu)\vec{\psi}^+(x,\mu)$.

Thus on Y there is in addition an antiholomorphic involution $\sigma\tau$ (we use this notation for historical reasons):

$$\sigma\tau\lambda = \bar{\lambda}, \quad \sigma\tau w(\mu) = \bar{w}(\mu). \tag{64}$$

The Bloch solution of (42) $\vec{\psi}(x,\mu)$ with the normalization (60) is meromorphic in μ on Y with poles which do not depend on x.

The function

$$p(\mu) = \frac{1}{2\pi i} \ln w(\mu) \tag{65}$$

is called the **quasimomentum function**. It's differential

$$dp(\mu) = \left[\frac{d}{d\mu}p(\mu)\right] d\mu \tag{66}$$

is called the **quasimomentum differential**. Of course $p(\mu)$ is defined up to adding an arbitrary integer. For a generic potential $q(x)$ the function $p(\mu)$ is essentially multivalued on Y, i.e. it's increment is non-zero along some cycles in Y. The quasimomentum differential is a well-defined holomorphic differential on the finite part of Y.

The Riemann surface Y is called the **Bloch variety**. It plays a crucial role in the inverse problem for periodic potentials. The structure of Y was studied in details by one of the authors (M.S.) in [18]. Let us recall some basic facts.

A point $\lambda \in \mathbb{C}$ is called regular if $w^{(1)}(\lambda) \neq w^{(2)}(\lambda)$ and irregular if $w^{(1)}(\lambda) = w^{(2)}(\lambda)$. We shall distinguish 3 types of irregular points.

1. Branch points.

2. Non-removable double points.

3. Removable double points.

An irregular point λ_0 is called a branch point if going around this point we come from one sheet of Y to the other one (i.e. the monodromy around this point is non-trivial). If the monodromy around an irregular point λ_0 is trivial, then λ_0 is called a double point. In a neighbourhood of a double point we have a pair of locally holomorphic functions $w^{(1)}(\lambda)$, $w^{(2)}(\lambda)$, and the Bloch functions $\vec{\psi}^{(1)}(x,\lambda)$, $\vec{\psi}^{(2)}(x,\lambda)$ are locally meromorphic. A double point λ_0 is called non-removable if $\vec{\psi}^{(1)}(x,\lambda_0) = \vec{\psi}^{(2)}(x,\lambda_0)$, and removable otherwise.

Lemma 4 *Let λ be a real point of the spectral plane, i.e. $\bar{\lambda} = \lambda$. Then:*

1. *$trT(\lambda)$ is a real function of λ and $|trT(\lambda)| \leq 2$.*

2. *λ lies in the spectrum of (42).*

3. *The point λ is regular if and only if $|trT(\lambda)| < 2$.*

4. If $trT(\lambda) = \pm 2$ then the matrix $T(\lambda)$ is diagonal

$$T(\lambda) = \pm \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{67}$$

and λ is a removable double point.

The fact that all irregular points on the real line are removable double points was proved in [18]. All other statements of Lemma 4 follow directly from the unitarity of $T(\lambda)$.

For generic $q(x)$ Y has infinitely many branch points (and the genus of Y is infinite). But the asymptotic structure of Y near infinity is rather simple. Let ε be a real positive constant. Then there exists a constant R depending on $q(x)$ and ε such that:

1. If $\lambda \in \mathbb{C}$, $|\lambda| > R$ and $|\lambda - k| > \varepsilon$ for any integer $k \in \mathbb{Z}$, then the point λ is regular.
2. Let $k \in \mathbb{Z}$ be an integer such, that $|k| > R$. Then the ε-neighbourhood of the point k in the complex plane contains either a pair of complex-conjugate branch points or one removable real double point.

A potential $q(x)$ is called **finite-gap** or **algebro-geometric** if Y has only a finite number of branch points (the number of non-removable double point is always finite). A finite-gap potential has an explicit representation in terms of the Riemann θ-functions. Using the methods of [18] and [9] it is rather easy to prove that any smooth potential can be approximated by the finite-gap potentials.

Finite-gap potentials in the context of the soliton theory first appeared in the soliton theory in the article [15] by S. P. Novikov (see the book [19] for additional information). The finite-gap theory of first-order matrix differential operators, including the θ-function formulas was developed by B. A. Dubrovin (see [5]). The algebro-geometrical solutions of the NLS equation were studied by E. Previato in [16].

An analog of the finite-gap theory for generic periodic potentials can also be developed. For the first-order matrix operators including the NLS case it was done by one of the authors (M.S.) in [18].

Let us recall some useful formulas from [18].

Lemma 5 *1. Let $\lambda \in \mathbb{C}$ be a regular or a removable double point and let μ_1 and $\mu_2 = \sigma\mu_1$ be the pre-images of λ. Then*

$$dp(\mu_1) = -\frac{\int\limits_0^{2\pi} [\psi_1(x,\mu_1)\psi_2(x,\mu_2) + \psi_2(x,\mu_1)\psi_1(x,\mu_2)]\, dx}{4\pi\, [\psi_1(0,\mu_1)\psi_2(0,\mu_2) - \psi_2(0,\mu_1)\psi_1(0,\mu_2)]} d\lambda(\mu_1) \quad (68)$$

2. Let $\lambda \in \mathbb{C}$ be a removable double point, μ_1 be one of the pre-images of λ. Then

$$\int_0^{2\pi} \psi_1(x,\mu_1)\psi_2(x,\mu_1) dx = 0. \quad (69)$$

Proof of Lemma 5. Let μ, ν be an arbitrary pair of points in Y and $\lambda(\mu)$, $\lambda(\nu)$ their projections to the λ-plane. A direct calculation shows that

$$\frac{d}{dx} [\psi_1(x,\mu)\psi_2(x,\nu) - \psi_2(x,\mu)\psi_1(x,\nu)] =$$

$$= -\frac{i}{2}(\lambda(\mu) - \lambda(\nu)) [\psi_1(x,\mu)\psi_2(x,\nu) + \psi_2(x,\mu)\psi_1(x,\nu)] . \quad (70)$$

Integrating (70) over a period we get

$$(w(\mu)w(\nu) - 1) [\psi_1(0,\mu)\psi_2(0,\nu) - \psi_2(0,\mu)\psi_1(0,\nu)] =$$

$$-\frac{i}{2}(\lambda(\mu) - \lambda(\nu)) \int_0^{2\pi} [\psi_1(x,\mu)\psi_2(x,\nu) + \psi_2(x,\mu)\psi_1(x,\nu)] \quad (71)$$

To prove part 1 let us assume that $\nu = \mu_2$, $\mu = \mu_1 + \delta$, $\delta \to 0$. From the fact λ is a regular or a removable double point it follows that the Wronskian in the denominator of (68) is non-zero. Then (68) follows directly from (71).

To prove part 2 let us assume that $\mu = \mu_1$, $\nu = \mu_1 + \delta$ where $\delta \to 0$. The left-hand side of (71) has at least a second-order zero as $\delta \to 0$, $(\lambda(\mu) - \lambda(\nu))$ has a first-order zero, thus the integral at the right-hand side of (71) vanishes as $\delta \to 0$. This completes the proof.

4 Riemann surfaces corresponding to periodic curves

We now state the main result of our article.

Theorem 1 *Let $q(x)$ be a complex-valued smooth periodic function of one variable x, $q(x) \neq 0$ for all x, $q(x+2\pi) = q(x)$, $\Lambda_0 \in \mathbb{R}$ an arbitrary real number, $\hat{\Gamma}(x)$ the corresponding curve constructed in Lemma 3. Then*

1. *The matrix $\hat{E}_1(x)$ is periodic with period 2π, i.e. $\hat{E}_1(x+2\pi) = \hat{E}_1(x)$, if and only if Λ_0 is a double point of the Bloch variety Y (let us recall that any real double point is automatically removable).*

2. *The function $\hat{\Gamma}(x)$ is periodic with period 2π $\hat{\Gamma}(x+2\pi) = \hat{\Gamma}(x)$, if and only if Λ_0 is a double point of Y and $dp(\mu_1) = 0$, $dp(\mu_2) = 0$ where μ_1, μ_2 are the pre-images of Λ_0 under the projection $Y \to \mathbb{C}$.*

Proof of Theorem 1. Let μ_0 be one of the pre-images of Λ_0, let

$$\vec{\psi}(x,\mu_0) = \begin{pmatrix} \psi_1(x,\mu_0) \\ \psi_2(x,\mu_0) \end{pmatrix} \tag{72}$$

be a Bloch solution of (42) with a normalization such that

$$\psi_1(0,\mu_0)\bar{\psi}_1(0,\mu_0) + \psi_2(0,\mu_0)\bar{\psi}_2(0,\mu_0) = 1. \tag{73}$$

Λ_0 is real thus

$$\vec{\psi}(x,\sigma\mu_0) = \begin{pmatrix} -\bar{\psi}_2(x,\mu_0) \\ \bar{\psi}_1(x,\mu_0) \end{pmatrix} \tag{74}$$

and the matrix

$$\Omega(x) = \begin{bmatrix} \psi_1(x,\mu_0) & -\bar{\psi}_2(x,\mu_0) \\ \psi_2(x,\mu_0) & \bar{\psi}_1(x,\mu_0) \end{bmatrix} \tag{75}$$

is unitary, satisfies (52) and one has

$$\Omega(x+2\pi) = \Omega(x) \begin{bmatrix} w(\mu_0) & 0 \\ 0 & w^{-1}(\mu_0) \end{bmatrix}. \tag{76}$$

Thus for the function $\hat{E}_1(x)$ defined by (53) we have

$$\hat{E}_1(x+2\pi) = \begin{bmatrix} w^{-1}(\mu_0) & 0 \\ 0 & w(\mu_0) \end{bmatrix} \hat{E}_1(x) \begin{bmatrix} w(\mu_0) & 0 \\ 0 & w^{-1}(\mu_0) \end{bmatrix}. \tag{77}$$

From (77) it follows that $\hat{E}_1(x+2\pi) \equiv \hat{E}_1(x)$ if and only if one of the following two conditions is fulfilled:

1. $\hat{E}_1(x)$ is diagonal for all x.

2. $w(\mu_0) = \pm 1$.

If the matrix $\hat{E}_1(x)$ is diagonal for all x then $q(x) \equiv 0$, but $q(x)$ is everywhere non-zero by assumption. Thus $w(\mu_0) = \pm 1$ and Λ_0 is an irregular point. By Lemma 4 Λ_0 is therefore a removable double point.

A simple calculation shows that

$$\int_0^{2\pi} \hat{E}_1(x)dx = \begin{bmatrix} a & -\bar{b} \\ b & -a \end{bmatrix} \tag{78}$$

where

$$a = i\int_0^{2\pi} \psi_1(x,\mu_0)\bar{\psi}_1(x,\mu_0) - \psi_2(x,\mu_0)\bar{\psi}_2(x,\mu_0)dx, \tag{79}$$

$$b = -2i\int_0^{2\pi} \psi_1(x,\mu_0)\psi_2(x,\mu_0)dx. \tag{80}$$

By Lemma 5

$$b = 0, \quad a = -4\pi \left.\frac{\partial p(\mu)}{\partial \lambda(\mu)}\right|_{\mu=\mu_0}. \tag{81}$$

Thus $a = 0$ and $\hat{\Gamma}(x + 2\pi) = \hat{\Gamma}(x)$ if and only if $dp(\mu_0) = 0$. Theorem 1 is proved.

5 Deformations of Bloch varieties and periodic solutions of the Filament Equation

It is well-known that if the Bloch variety is algebraic, then the solutions of the soliton equation can be written explicitly in terms of the Riemann θ-functions. Such solutions are called algebro-geometric or finite-gap. But if we start from a generic algebraic Riemann surface, then the corresponding θ-functional solutions are quasi-periodic in space. Riemann surfaces generating purely periodic solutions form a rather complicated transcendental subvariety in the moduli space of all Riemann surfaces (let us call it the "Periodic subvariety").

A characterization of this subvariety for the periodic Korteveg-de Vries equation in terms of conformal maps was obtained by V. A. Marchenko and

I. V. Ostrovski in [13]. Another approach to construct periodic solutions of soliton equations is based on the so-called period preserving deformations of Riemann surfaces and it was suggested by the authors in [9]. Let us recall in brief the results of [9] concerning the self-focusing NLS equation. To explain our ideas, for the sake of transparency we shall consider generic algebro-geometric potentials only. The case of general periodic potentials can also be studied by the same method, but it requires some more details from the soliton theory.

Let $q(x)$ be a generic non-singular algebro-geometric NLS potential with period 2π: $q(x+2\pi)=q(x)$. Then the associated Bloch variety Y has the following properties:

1. Y has a finite even number of branch point $\lambda_1,\ldots,\lambda_{2g+2}$, $\mathrm{Im}\,\lambda_k\neq 0$ for all k, the enumeration can be chosen such that $\lambda_{2k+2}=\bar{\lambda}_{2k+1}$. g is called the genus of Y.

2. Y has two distinct points over $\lambda=\infty$. Denote these points by ∞_+ and ∞_-. A local parameter in a neighborhoods of these points can be chosen as $\nu=1/\lambda$.

3. Y has no non-removable double points.

4. The quasimomentum differential has the following representation:

$$dp(\mu)=-\tfrac{1}{2}\frac{q(\lambda)}{\sqrt{(\lambda-\lambda_1)\ldots(\lambda-\lambda_{2g+2})}}d\lambda,$$
$$\text{where } \lambda=\lambda(\mu),\ \ q(\lambda)=\lambda^{g+1}+q_g\lambda^g+\ldots+q_1\lambda+q_0, \tag{82}$$

   where the real constants q_0, q_1, $\ldots$, q_g are uniquely defined by:

$$\mathrm{Im}\oint_c dp(\mu)=0,\quad \mathrm{res}\Big|_{\mu=\pm\infty}dp(\mu)=0, \tag{83}$$

   where c being an arbitrary closed cycle in Y.

5. Let c be an arbitrary closed cycle in Y. Then

$$\oint_c dp(\mu)\in\mathbb{Z}. \tag{84}$$

6. Let c be a path in Y connecting the points ∞_+ and ∞_-, let $p(\mu)$ be a primitive of $dp(\mu)$ defined on c with the following normalization:

$$p(\mu) = -\frac{\lambda(\mu)}{2} + O\left(\frac{1}{\lambda}\right) \text{ as } \mu \to \infty_+. \tag{85}$$

Then

$$p(\mu) = \frac{\lambda(\mu)}{2} + k + O\left(\frac{1}{\lambda}\right) \text{ as } \mu \to \infty_-, \text{where } k \in \mathbb{Z}. \tag{86}$$

Let Y be an arbitrary hyperelliptic Riemann surface with the properties 1)-3) listed above. Then we can construct a family of NLS solutions corresponding to Y (see for example [16]). The real dimension of this family is $g+1$. Some of these solutions may have singularities, but it is not important in the present context. The quasiperiods of these solutions depend only on Y and are the same for the whole family. It is well-know that Riemann surfaces generating purely periodic in x-space solutions can be characterized in terms of the quasimomentum differential (see for example [9]):

Lemma 6 *Let Y be a hyperelliptic Riemann surface with the properties 1)-3) listed above. Let $q(x)$ be one of the potentials corresponding to Y. Then $q(x)$ is periodic with a period 2π $q(x+2\pi) = q(x)$ if and only if a meromorphic differential $dp(\mu)$ uniquely defined by the formulas (82), (83) has the properties (84)-(86).*

In the finite-gap theory the differential $dp(\mu)$ is called the quasimomentum differential for both the periodic and the quasiperiodic solutions.

Assume now that we have a family of Riemann surfaces $Y(\xi)$ such that the following equations, introduced by I.M. Krichever [11], N.M. Ercolani, M.G. Forest, D.W. McLaughlin, A. Sinha [6], and one of the authors (M.S.) [18] are fulfilled:

$$\left.\frac{\partial p(\mu,\xi)}{\partial \xi}\right|_{\mu=const} = \frac{\omega(\mu,\xi)}{d\lambda(\mu)} \tag{87}$$

where any fixed ξ $\omega(\mu,\xi) = \tilde{\omega}(\mu,\xi)d\lambda(\mu)$ is a meromorphic differential on $Y(\xi)$ having no poles outside the points ∞_+, ∞_- and at most first order-poles at the infinite points. Let us recall that any such meromorphic differential can be written in the following form

$$\begin{gathered}\omega(\mu,\xi)=-\tfrac{1}{2}\tfrac{o(\lambda,\xi)}{\sqrt{(\lambda-\lambda_1(\xi))\dots(\lambda-\lambda_{2g+2}(\xi))}}d\lambda,\\ \text{where } \lambda=\lambda(\mu),\ o(\lambda,\xi)=o_g(\xi)\lambda^g+\ldots+o_1(\xi)\lambda+o_0(\xi).\end{gathered}\tag{88}$$

(We shall assume that $o_k(\xi)$ are smooth functions of ξ). Assume that $Y(0)$ generates periodic potentials, and for all ξ the surface $Y(\xi)$ admits the antiholomorphic involution $\sigma\tau$. Then $Y(\xi)$ generates periodic potentials for all ξ.

The proof of this fact is based on Lemma 6 and the following observation: The function on the right-hand side of (87) is single-valued on Y and decays as $\mu\to\infty_\pm$. Thus the deformation (87) affects neither the periods of $dp(\mu)$ nor the asymptotics of $p(\mu)$ at infinity.

It is well-known that equation (87) is equivalent to the following system of ordinary differential equations on the branch points (see [8])

$$\frac{\partial\lambda_k(\xi)}{\partial\xi}=-\frac{o(\lambda,\xi)}{q(\lambda,\xi)}.\tag{89}$$

The right-hand side of (89) involves hyperelliptic functions of the branch points because we have to calculate integrals (83) over periods of Y to determine the coefficients of $q(\lambda,\xi)$. In [9] it was shown that the system (89) can be naturally embedded into a slightly bigger system of ordinary differential equations with a rational right-hand side.

Denote by $\alpha_k(\xi)$, $k=1,\ldots,g+1$ the zeroes of the polynomial $q(\lambda,\xi)$ (we shall call them the zeroes of the quasimomentum differential). In fact $dp(\mu,\xi)$ has zeroes at both pre-images of each $\alpha_k(\xi)$. Since we consider only generic surfaces, we may assume that all $\alpha_k(\xi)$ are pairwise distinct.

Lemma 7 *Let $\omega_k(\mu,\xi)$, $k=1,\ldots,g+1$ be the following basis of differentials on $Y(\xi)$*

$$\omega_k(\mu,\xi)=\frac{1}{\lambda-\alpha_k(\xi)}dp(\mu,\xi),\ \ \lambda=\lambda(\mu).\tag{90}$$

Let

$$\omega(\mu,\xi)=\sum_{k=1}^{g+1}c_k(\xi)\omega_k(\mu,\xi)\tag{91}$$

where $c_k(\xi)$ are arbitrary complex constants. Then the flow (87), generated by the differential $\omega(\mu,\xi)$ has the following representation:

$$\begin{gathered}\frac{\partial\lambda_j(\xi)}{\partial\xi} = -\sum_{k=1}^{g+1}\frac{c_k(\xi)}{\lambda_j(\xi)-\alpha_i(\xi)}\\ \frac{\partial\alpha_j(\xi)}{\partial\xi} = \sum_{\substack{k=1,\dots,g+1\\ k\neq j}}\frac{c_k(\xi)+c_j(\xi)}{\alpha_k(\xi)-\alpha_j(\xi)} - \frac{1}{2}\sum_{j=1}^{2g+2}\frac{c_k(\xi)}{\lambda_j(\xi)-\alpha_k(\xi)}\end{gathered} \tag{92}$$

Lemma 7 was proved in [9].

For generic complex $c_k(\xi)$ the flow (92) does not respect the symmetry $\lambda_{2j+2}(\xi) = \bar\lambda_{2j+1}(\xi)$. To construct NLS solution we have to add additional restrictions on $c_k(\xi)$.

If $Y(\xi)$ has the symmetry $\lambda_{2j+2}(\xi) = \bar\lambda_{2j+1}(\xi)$, then the polynomial $q(\lambda)$ has real coefficients, and there are two types of zeroes $\alpha_k(\xi)$:

1. Real zeroes $\alpha_k(\xi) = \bar\alpha_k(\xi)$.
2. Pairs of complex conjugate zeroes $\alpha_l(\xi) = \bar\alpha_k(\xi)$.

Lemma 8 *1. Let the functions $c_k(\xi)$ be chosen such that for all ξ the following conditions are fulfilled*

(a) If $\alpha_k(\xi)$ is a real zero of the quasimomentum differential, then $c_k(\xi) = \bar c_k(\xi)$.

(b) If $\alpha_k(\xi)$, $\alpha_l(\xi)$ is a pair of complex conjugate zeroes of the quasimomentum differential, then $c_l(\xi) = \bar c_k(\xi)$.

Then the flow (92) respects the symmetry $\lambda_{2j+2}(\xi) = \bar\lambda_{2j+1}(\xi)$.

2. Consider the subvariety of all hyperelliptic Riemann surfaces generating periodic solutions of the self-focusing NLS in the moduli space. Let Y be a generic point of this subvariety. Then the flows described above act locally transitive at this subvariety.

The first statement of Lemma 8 follows directly from the formulas (92). To prove the last statement it is sufficient to calculate the number of conditions in a generic point (see [9] for details).

The algorithm for constructing periodic solutions of the self-focusing NLS suggested in [9] is the following: Assume that we know at least one Riemann surface with $2g+2$ branch points generating such solutions. Such a surface can be constructed in a neighbourhood of a constant solution by methods of perturbation theory. Then integrating the system of ordinary differential equations (92) we can construct all NLS solutions in a neighbourhood of the original one. (In fact, any NLS solution can be constructed by this method, but to reach some of them we have to pass through singular points of the system (92).)

Let us show that the method of [9] can be naturally applied to construct periodic solutions in the x-space of the Filament equation. (Let us recall that if the curve $\vec{\gamma}(x,t)$ is periodic in x for a fixed $t=t_0$, then it is periodic in x for all t and the x-period does not depend on t.)

Theorem 2 *Let Y be a hyperelliptic Riemann surface such that:*

1. *Y is a Bloch variety corresponding to a smooth periodic potential $q(x)$, $q(x+2\pi)=q(x)$.*
2. *Y has $2g+2$ branch points and no non-removable double points.*
3. *There exists a point $\Lambda_0 \in \mathbb{R}$ such that*
   - *(a) Λ_0 is a removable double point, i.e. the values of the quasimomentum function $p(\mu)$ at the pre-images of Λ_0 are integer or half-integer.*
   - *(b) Λ_0 coincides with one of the zeroes of the quasimomentum differential. Since the zeroes of the quasimomentum differential α_1, ..., α_{g+1} have no natural order, without loss of generality one may assume therefore $\Lambda_0=\alpha_{g+1}$.*

(Let us recall that by Theorem 1 the pair (Y,Λ_0) generates periodic solutions in x-space of the Filament equation).

Let $c_k(\xi)$, $k=1,\ldots,g+1$ be arbitrary smooth complex-valued functions of a real variable ξ defined in a neighbourhood of the point $\xi=0$ such, that:

1. *For all ξ the $c_k(\xi)$ satisfy the reality conditions of Lemma 8.*
2. *$c_{g+1}(\xi)\equiv 0$.*

Let $\lambda_k(\xi)$, $k=1,\ldots,2g+2$, $\alpha_k(\xi)$, $k=1,\ldots,g+1$ be a solution of the system (92), with the following initial conditions:

1. *$\lambda_k(0)$ are the branch points of Y.*

2. *$\alpha_k(0) = \alpha_k$.*

(This solution is non-singular at least for sufficiently small ξ).

Let $Y(\xi)$ be a family of hyperelliptic Riemann surface with branch points $\lambda_k(\xi)$, $\Lambda_0(\xi) = \alpha_{g+1}(\xi)$.

Then for all sufficiently small real ξ

1. *The pair $(Y(\xi), \Lambda_0(\xi))$ generates periodic solutions in x-space of the Filament equation.*

2. *The non-singular solutions form an open subset in the $g+1$ dimensional variety of all solutions of the Filament Equation corresponding to $Y(\xi)$.*

Proof of Theorem 2. By Lemmas 7 and 8 the Riemann surfaces $Y(\xi)$ generate periodic solutions in x-space of the self-focusing NLS with period 2π. For all ξ $\Lambda_0(\xi)$ is a zero of the quasimomentum differential, which is real. By Theorem 1 it is sufficient to prove that $\Lambda_0(\xi)$ is a double point of $Y(\xi)$. Let $\mu_0(\xi)$ be one of the pre-images of $\Lambda_0(\xi)$. $\Lambda_0(\xi)$ is a double point of $Y(\xi)$ if and only if $p(\mu_0(\xi), \xi) = n$ or $p(\mu_0(\xi), \xi) = n + \frac{1}{2}$ where $n \in \mathbb{Z}$.

From (87) it follows, that

$$\frac{d}{d\xi} p(\mu_0(\xi), \xi) = \frac{\omega(\mu_0(\xi), \xi)}{d\lambda(\mu)} + \frac{d}{d\mu} p(\mu, \xi) \Big|_{\mu=\mu_0(\xi)} \frac{d\mu_0(\xi)}{d\xi} \tag{93}$$

where the differential $\omega(\mu, \xi)$ is defined by the formula (91). Since $\Lambda_0(\xi)$ coincides with one of the zeroes of the quasimomentum differential, the second term on the left-hand side of (93) is equal to zero. Since $c_{g+1}(\xi) \equiv 0$ and thus $\omega(\mu_0(\xi), \xi) = 0$ and therefore

$$\frac{d}{d\xi} p(\mu_0(\xi), \xi) = 0, \;\; p(\mu_0(\xi)) = \text{const}. \tag{94}$$

This completes the proof of the first part. A proof of the second statement can be extracted from [18].

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