Extensions of Conformal Nets and Superselection Structures

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Abstract

Starting with a conformal Quantum Field Theory on the real line, we show that the dual net is still conformal with respect to a new representation of the Möbius group. We infer from this that every conformal net is normal and conormal, namely the local von Neumann algebra associated with an interval coincides with its double relative commutant inside the local von Neumann algebra associated with any larger interval. The net and the dual net give together rise to an infinite dimensional symmetry group, of which we study a class of positive energy irreducible representations. We mention how superselsection sectors extend to the dual net and we illustrate by examples how, in general, this process generates solitonic sectors. We describe the free theories associated with the lowest weight n representations of $\mathtt{PSL}(2,\mathbb{R})$, showing that they violate 3-regularity for $n > 2$. When $n \geq 2$, we obtain examples of non Möbius-covariant sectors of a 3-regular (non 4-regular) net.

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0 Introduction.

Haag duality is one of the most important properties in Quantum Field Theory for the analysis of the superselection structure. It basically says that the locality principle holds maximally.

Concerning Quantum Field Theory on the usual Minkowski spacetime, duality may be always assumed, in a Wightman theory, because wedge duality automatically holds and one can enlarge the net to a dual net without affecting the superselection structure [3, 26].

Nevertheless there might be good reasons in lower dimensional theories for Haag duality not to be satisfied, [24]. An important case occurs in Conformal QFT: as such a theory naturally lives on a larger spacetime, duality may fail on the original spacetime because contributions at infinity are possibly not detectable there. Moreover in low spacetime dimensions, the superselection structure of the dual net may change due to the occurrence of soliton sectors [31, 26], and a new information being contained in the inclusion of the two nets.

This paper is devoted to an analysis of conformal QFT on the real line, namely one-dimensional components of a two-dimensional chiral conformal QFT.

The first aspect that we discuss concerns the symmetries of the dual net. Starting with a conformal net $\mathcal{A}$ on $\mathbb{R}$, the Bisognano-Wichmann property holds automatically true [5, 14], thus the dual net

$$\mathcal{A}^d(a,b) \doteq \mathcal{A}(-\infty,b) \cap \mathcal{A}(a,\infty) \qquad a < b$$

on $\mathbb{R}$ is local and obviously translation-dilation covariant with respect to the same translation-dilation unitary representation. We shall however prove that $\mathcal{A}^d$ is even conformally covariant with respect to a new unitary representation of $\mathsf{PSL}(2,\mathbb{R})$.

The construction of the new symmetries is achieved by a new characterization of local conformal precosheaves on the circle in terms of what we call a +hsm (half-sided modular) factorization, namely a quadruple $(\mathcal{N}_i, i \in \mathbb{Z}_3; \Omega)$ where the $\mathcal{N}_i$'s are mutually commuting von Neumann algebras with a joint cyclic separating vector Ω such that $\mathcal{N}_i \subset \mathcal{N}_{i+1}$ is a +hsm modular inclusion in the sense of [35] for all $i \in \mathbb{Z}_3$. This characterization makes only use of the modular operators and not of the modular conjugations as in [36].

As second result we shall deduce a structural property for any conformal net $\mathcal{A}$: $\mathcal{A}$ is automatically normal and conormal, namely if $I \subset \tilde{I}$ is an inclusion of proper intervals then

$$\mathcal{A}(I)^{cc} = \mathcal{A}(I), \tag{1}$$

$$\mathcal{A}(\tilde{I}) = \mathcal{A}(I) \vee \mathcal{A}(I)^c \tag{2}$$

where $\mathcal{X}^c = \mathcal{X}' \cap \mathcal{A}(\tilde{I})$ denotes the relative commutant in $\mathcal{A}(\tilde{I})$ and $\mathcal{X}^{cc} = (\mathcal{X}^c)^c$. This property is useful in the analysis of the superselection structure, as discussed below.

Next issue will be to compare the net $\mathcal{A}$ with the dual net $\mathcal{A}^d$. We shall give a detailed study of the inclusion $\mathcal{A} \subset \mathcal{A}^d$ in a particular model, namely when

$\mathcal{A}$ is the net associated with the n-th derivative of the $U(1)$-current algebra, the latter turning out to coincide with $\mathcal{A}^d$. This is a first quantization analysis and to this end we shall give formulas relating two irreducible lowest weight representations of $\mathtt{PSL}(2,\mathbb{R})$ that agree on the upper triangular matrix subgroup $\mathtt{P}_0$. In other words we are studying a class of representations of a certain infinite dimensional Lie group, the amalgamated free product $\mathtt{PSL}(2,\mathbb{R}) *_{\mathtt{P}_0} \mathtt{PSL}(2,\mathbb{R})$, where a classification is simply obtained. For example an explicit formula for the unitary γ whose second quantization implements the canonical endomorphism associated with the inclusion $\mathcal{A}(-1,1) \subset \mathcal{A}^d(-1,1)$ (the product of the ray inversion unitaries of the two nets) will be given as a function of the skew-adjoint generator E of the dilation one-parameter subgroup

$$\gamma = \frac{(E-1)(E-2)\cdots(E-n)}{(E+1)(E+2)\cdots(E+n)}.$$

We shall show that the net $\mathcal{A}$ is not 4-regular if $n \geq 2$ and not 3-regular if $n \geq 3$, where $\mathcal{A}$ is said to be k-regular if $\mathcal{A}$ remains irreducible after removing $k-1$ points from $\mathbb{R}$. The 4-regularity property played a role in the covariance analysis in [16] where the problem of its general validity remained open.

We generalize the construction of the Buchholz, Mack and Todorov sectors for the current algebra ([7]) to the n-th derivative of the current algebra, showing that they form a group isomorphic to $\mathbb{R}^{2n+1}$, and that none of them is covariant w.r.t. the conformal group (if $n \neq 0$). In particular this shows results in [16] to be optimal, at least on the real line.

Finally we shall illustrate by examples how sectors of $\mathcal{A}$ localized in a bounded interval may have extension to $\mathcal{A}^d$ with soliton localization.

1 Structural properties of conformal local precosheaves on S^1.

In the algebraic approach, see [18], chiral conformal field theories are described as conformally covariant local precosheaves $\mathcal{A}$ of von Neumann algebras on proper intervals of the circle S^1. We start by reviewing some aspects of this framework.

An open interval I of S^1 is called *proper* if I and the interior I' of its complement are not empty. The circle will be explicitly described either as the points with modulus one in $\mathbb{C}$ or as the one-point compactification of $\mathbb{R}$, these two description being related by the Cayley transform: $C : S^1 \to \mathbb{R} \cup \{\infty\}$ given by $z \to i(z+1)(z-1)^{-1}$. The group $\mathtt{PSL}(2,\mathbb{R})$ acts on S^1 via its action on $\mathbb{R} \cup \{\infty\}$ as fractional transformations. Intervals are labeled either by the coordinates on $\mathbb{R}$ or by complex coordinates of the endpoints in $S^1 \subset \mathbb{C}$, where in the later case intervals are represented in positive cyclic order.

A precosheaf $\mathcal{A}$ is a covariant functor from the category $\mathcal{J}$ of proper intervals with inclusions as arrows to the category of von Neumann algebras on a Hilbert space $\mathcal{H}$ with inclusions as arrows, i.e., a map

$$I \to \mathcal{A}(I)$$

that satisfies:

A. Isotony. If $I_1 \subset I_2$ are proper intervals, then

$$\mathcal{A}(I_1) \subset \mathcal{A}(I_2).$$

The precosheaf $\mathcal{A}$ will be a (local) *conformal precosheaf* if in addition it satisfies the following properties:

B. Locality. If I_1 and I_2 are disjoint proper intervals, then

$$\mathcal{A}(I_1) \subset \mathcal{A}(I_2)',$$

C. Conformal invariance. There exists a strongly continuous unitary representation U of $\mathtt{PSL}(2,\mathbb{R})$ on $\mathcal{H}$ such that

$$U(g)\mathcal{A}(I)U(g)^* \;=\; \mathcal{A}(gI), \qquad g \in \mathtt{PSL}(2,\mathbb{R}),\; I \in \mathcal{J}.$$

D. Positivity of the energy. The generator of the rotation subgroup $U(R(\cdot))$ (conformal Hamiltonian) is positive. Here $R(\vartheta)$ denotes the rotation of angle ϑ on S^1.

E. Existence of the vacuum. There exists a unit vector $\Omega \in \mathcal{H}$ (vacuum vector) which is $U(\mathtt{PSL}(2,\mathbb{R}))$-invariant and cyclic for $\vee_{I\in\mathcal{J}}\mathcal{A}(I)$.

The Reeh-Schlieder Theorem now states that the vacuum vector Ω is cyclic and separating for any local algebra $\mathcal{A}(I)$.

Let us recall that uniqueness of the vacuum is equivalent both to the irreducibility of the precosheaf or to the factoriality property for local algebras. We shall denote by $\mathtt{Mob}$ the Möbius group, namely the group of conformal transformations in $\mathbb{C}$ that leave the unit circle globally invariant. The group $\mathtt{PSL}(2,\mathbb{R})$ is then identified with the subgroup of orientation preserving transformations and $\mathtt{Mob}$ is generated by $\mathtt{PSL}(2,\mathbb{R})$ and an involution.

Let I_1 be the upper semi-circle parameterized as $(0,+\infty)$; we associate to I_1 the following two one-parameter subgroups of $\mathtt{Mob}$: First the dilations (relative to I_1),

$$\Lambda_{I_1}(t) = \begin{pmatrix} e^{-t/2} & 0 \\ 0 & e^{t/2} \end{pmatrix},$$

leaving I_1 globally stable, second the translations

$$T_{I_1}(s) = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}.$$

mapping I_1 into itself for positive $s \geq 0$. In general, if I is any interval in S^1, there exists a $g \in \mathtt{PSL}(2, \mathbb{R})$ such that $I = gI_1$ and we set

$$\Lambda_I = g\Lambda_{I_1}g^{-1}, \qquad T_I = gT_{I_1}g^{-1}.$$

The definition of the dilations does not depend on g, while the translations of I are defined up to a rescaling of the parameter, that however does not play any role in the following, because we are only interested in the subgroups generated by them.

The subgroup generated by $\Lambda_{I_1}(\cdot)$ and $T_{I_1}(\cdot)$, denoted by $\mathtt{P}_0$, is the subgroup of upper triangular matrices of $\mathtt{PSL}(2, \mathbb{R})$ and plays an important role in the following, especially in the next Section.

We shall associate with any proper interval I a diffeomorphism r_I of S^1, the reflection mapping I onto the causal complement I', i.e. fixing the boundary points of I. In the case of $I_1 = (0, +\infty)$

$$r_{I_1}x = -x$$

and one can extend this definition to a generic I as before. Notice that r_{I_1} is orientation reversing.

By a (anti-)representation U of $\mathtt{Mob}$ we shall mean the obvious generalization of the notion of unitary representation where $U(r_I)$ is anti-unitary.

For a general conformal precosheaf the *Bisognano-Wichmann Property* holds [5, 14]: U extends to a unitary (anti-)representation of $\mathtt{Mob}$ such that, for any $I \in \mathcal{J}$,

$$U(\Lambda_I(2\pi t)) = \Delta_I^{it}, \tag{3}$$

$$U(r_I) = J_I, \tag{4}$$

where Δ_I and J_I are the Tomita-Takesaki modular operator and modular conjugation associated with $(\mathcal{A}(I), \Omega)$. This implies Haag duality:

$$\mathcal{A}(I)' = \mathcal{A}(I'), \quad I \in \mathcal{I}.$$

Let now $(\mathcal{N} \subset \mathcal{M}, \Omega)$ be a triple where $\mathcal{N} \subset \mathcal{M}$ is an inclusion of von Neumann algebras acting on a Hilbert space $\mathcal{H}$ and $\Omega \in \mathcal{H}$ is a cyclic and separating vector for $\mathcal{N}$ and $\mathcal{M}$.

1. $(\mathcal{N} \subset \mathcal{M}, \Omega)$ is said to be *standard* if Ω is cyclic also for the relative commutant $\mathcal{N}^c \doteq \mathcal{M} \cap \mathcal{N}'$ of $\mathcal{N}$ in $\mathcal{M}$, see [10].

2. If $\sigma_t^{\mathcal{M}}$ denotes the modular automorphism associated to $(\mathcal{M}, \Omega)$, then the triple $(\mathcal{N} \subset \mathcal{M}, \Omega)$ is said to be $\pm$ *half-sided modular* ($\pm$ hsm) if $\sigma_{\mp t}^{\mathcal{M}}(\mathcal{N}) \subset \mathcal{N}$ for, respectively, all $t \geq 0$ or all $t \leq 0$.

3. A $\pm$*hsm factorization* of von Neumann algebras is a quadruple $(\mathcal{N}_0, \mathcal{N}_1, \mathcal{N}_2, \Omega)$, where $\{\mathcal{N}_i, i \in \mathbb{Z}_3\}$ is a set of pairwise commuting von Neumann algebras, Ω is a cyclic separating vector for each $\mathcal{N}_i$ and $(\mathcal{N}_i \subset \mathcal{N}_{i+1}', \Omega)$ is a $\pm$hsm inclusion for each $i \in \mathbb{Z}_3$.

In the work [37], local conformal precosheaves have been characterized in terms of $\pm$hsm standard inclusions of von Neumann algebras and the adjoint action of the modular conjugations. (This work is based on a statement about hsm modular inclusions [34], whose correct proof is contained in [2].) We shall give here below an alternative characterization in terms of a $\pm$hsm factorization, that has the advantage of using only the modular groups and not the modular conjugations.

Lemma 1.1. *Let G be the universal group (algebraically) generated by 3 one-parameter subgroups $\Lambda_i(\cdot)$, $i \in \mathbb{Z}_3$, such that Λ_i and Λ_{i+1} have the same commutation relations of Λ_{I_i} and $\Lambda_{I_{i+1}}$ for each $i \in \mathbb{Z}_3$, where I_0, I_1, I_2 are intervals forming a partition of S^1. Then G is isomorphic to $\overline{\mathtt{PSL}}(2,\mathbb{R})$, the universal covering group of $\mathtt{PSL}(2,\mathbb{R})$, and the Λ_i's are continuous one parameter subgroups naturally corresponding to Λ_{I_i}.*

Proof. Obviously G has a quotient isomorphic to $\overline{\mathtt{PSL}}(2,\mathbb{R})$, and we denote by q the quotient map. As the exponential map is a local diffeomorphism of the Lie algebra of a Lie group and the Lie group itself, there exists a neighbourhood $\mathcal{U}$ of the origin $\mathbb{R}^3$ such that the map $(t_0, t_1, t_2) \to \Lambda_{I_0}(2\pi t_0)\Lambda_{I_1}(2\pi t_1)\Lambda_{I_2}(2\pi t_2)$ is a diffeomorphism of $\mathcal{U}$ with a neighbourhood of the identity of $\overline{\mathtt{PSL}}(2,\mathbb{R})$. Therefore the map $\Phi : (t_0, t_1, t_2) \in \mathcal{U} \to \Lambda_0(2\pi t_0)\Lambda_1(2\pi t_1)\Lambda_2(2\pi t_2) \in G$ is still one-to-one. It is easily checked that the maps $g\Phi : \mathcal{U} \to G$, $g \in G$, form an atlas on G, thus G is a manifold. In fact G is a Lie group since the group operations are smooth, as they are locally smooth. Now G is connected by construction and q is a local diffeomorphism of G with $\overline{\mathtt{PSL}}(2,\mathbb{R})$, hence a covering map, that has to be an isomorphism because of the universality property of $\overline{\mathtt{PSL}}(2,\mathbb{R})$. $\square$

Theorem 1.2. *Let $(\mathcal{N}_0, \mathcal{N}_1, \mathcal{N}_2, \Omega)$ be a +hsm factorization of von Neumann algebras and let I_0, I_1, I_2 be intervals forming a partition of S^1 in counterclockwise order. There exists a unique local conformal precosheaf $\mathcal{A}$ on S^1 such that $\mathcal{A}(I_i) = \mathcal{N}_i$, $i \in \mathbb{Z}_3$, with Ω the vacuum vector. The (unique) positive energy unitary representation U of $\mathtt{PSL}(2,\mathbb{R})$ is determined by the modular prescription $U(\Lambda_{I_i}(2\pi t)) = \Delta_{I_i}^{it}$.*

Notice that every +hsm factorization of von Neumann algebras arises by considering the von Neumann algebras associated to 3 intervals of S^1 as in the above theorem, due to the geometric property of the modular group (3).

Proof. The subgroup of $\mathtt{PSL}(2,\mathbb{R})$ generated by the one-parameter subgroups $\Lambda_{I_i}(2\pi t)$ and $\Lambda_{I_{i+1}}(2\pi s)$, $i \in \mathbb{Z}_3$, is a two-dimensional Lie group $\mathtt{P}_i$ isomorphic to the translation-dilation group $\mathtt{P}_0$. As $(\mathcal{N}_i, \mathcal{N}'_{i+1}, \Omega)$ is a +hsm standard inclusion, by a result first stated in [34] with an erroneous proof and whose correct proof is given in [2], the unitary group generated by $\Delta_{I_i}^{it}$ and $\Delta_{I_{i+1}}^{is}$ is isomorphic to $\mathtt{P}_i$, indeed there exists a unitary representation U_i of $\mathtt{P}_i$ determined by $U_i(\Lambda_{I_i}(2\pi t)) = \Delta_{I_i}^{it}$ and $U_i(\Lambda_{I_{i+1}}(-2\pi s)) = \Delta_{I_{i+1}}^{is}$, therefore by Lemma 1.1, there exists a unitary representation U of $\overline{\mathtt{PSL}}(2,\mathbb{R})$, such that $U|_{\mathtt{P}_i} = U_i$.

Let $t_0 = \frac{1}{2\pi}\ln 2$. Then we have ([34, 2], see the remarks above)

$$\mathrm{Ad}\ \Delta_{I_0}^{it_0}\Delta_{I_1}^{it_0}(\mathcal{N}_0) = \mathcal{N}_1', \tag{5}$$

and similarly

$$\mathrm{Ad}\ \Delta_{I_2}^{it_0}\Delta_{I_0}^{it_0}\Delta_{I_1}^{it_0}\Delta_{I_2}^{it_0}\Delta_{I_0}^{it_0}\Delta_{I_1}^{it_0}(\mathcal{N}_0) = \mathcal{N}_0'. \tag{6}$$

The element $\Lambda_{I_2}(2\pi t_0)\Lambda_{I_0}(2\pi t_0)\Lambda_{I_1}(2\pi t_0)\Lambda_{I_2}(2\pi t_0)\Lambda_{I_0}(2\pi t_0)\Lambda_{I_1}(2\pi t_0)$ is easily seen to be conjugate to the rotation by π in $\mathtt{PSL}(2,\mathbb{R})$, hence equation (6) entails $U(2\pi)$ to implement an automorphism on $\mathcal{N}_0$.

Set $\mathcal{A}(I_0) := \mathcal{N}_0$. If I is an interval of S^1, then $I = gI_0$ for some $g \in \mathtt{PSL}(2,\mathbb{R})$, and we set $\mathcal{A}(I) = U(g)\mathcal{A}(I_0)U(g)^*$.

Since the group G_{I_0} of all $g \in \overline{\mathtt{PSL}}(2,\mathbb{R})$ such that $gI_0 = I_0$ is generated by $\Lambda_{I_0}(t),\ t \in \mathbb{R}$ and by rotations of $2k\pi,\ k \in \mathbb{Z}$, then $U(g)\mathcal{A}(I_0)U(g)^* = \mathcal{A}(I_0)$ for all $g \in G_{I_0}$ and the von Neumann algebra $\mathcal{A}(I)$ is well defined.

The isotony of $\mathcal{A}$ follows if we show that $gI_0 \subset I_0 \implies \mathcal{A}(gI_0) \subset \mathcal{A}(I_0)$. Indeed any such g is a product of an element in G_{I_0} and translations $T_{I_0}(\cdot)$ and $T_{I_0'}(\cdot)$ mapping I_0 into itself, hence the isotony follows by the half-sided modular conditions.

By (5) we have

$$\mathrm{Ad}\ \Delta_{I_1}^{it_0}\Delta_{I_2}^{it_0}\Delta_{I_0}^{it_0}\Delta_{I_1}^{it_0}(\mathcal{N}_0) = \mathcal{N}_2$$

and since the corresponding element in $\overline{\mathtt{PSL}}(2,\mathbb{R})$ maps I_0 onto I_2, we get $\mathcal{N}_2 = \mathcal{A}(I_2)$ and analogously $\mathcal{N}_1 = \mathcal{A}(I_1)$.

The locality of $\mathcal{A}$ now follows by the factorization property.

Finally U is a true representation of $\mathtt{PSL}(2,\mathbb{R})$ by the vacuum conformal spin-statistics theorem [17], and the positivity of the energy follows by the Bisognano-Wichmann property (3), see [36, 37]. $\square$

Although a conformal precosheaf satisfies Haag duality on S^1, duality on $\mathbb{R}$ does not necessarily hold.

Lemma 1.3. *Let $\mathcal{A}$ be a local conformal precosheaf on S^1. The following are equivalent:*

(i) The restriction of $\mathcal{A}$ to $\mathbb{R}$ satisfies Haag duality:

$$\mathcal{A}(I) = \mathcal{A}(\mathbb{R}\backslash I)'$$

(ii) $\mathcal{A}$ is strongly additive*: If $I_1,\ I_2$ are the connected components of the interval I with one internal point removed, then*

$$\mathcal{A}(I) = \mathcal{A}(I_1) \vee \mathcal{A}(I_2)$$

(iii) if I, I_1, I_2 are intervals as above

$$\mathcal{A}(I_1)' \cap \mathcal{A}(I) = \mathcal{A}(I_2)$$

Proof. Note that by Möbius covariance we may suppose that the point removed in (i) and (ii) is the point ∞. Now $(i) \Leftrightarrow (ii)$ because $\mathbb{R}\backslash I$ consists of two contiguous intervals in S^1 whose union has closure equal I', and by Haag duality $\mathcal{A}(I) = \mathcal{A}(I')'$. Similarly $(ii) \Leftrightarrow (iii)$ because, after taking commutants and renaming the intervals, one relation becomes equivalent to the other one. □

Examples of conformal precosheaf on S^1 that are not strongly additive, i.e. not Haag dual on the line, were first given in [19, 8] and [38]. We will look in some detail at an example of [38] in Section 3. Haag duality on S^1 entails duality for half-lines on $\mathbb{R}$ hence essential duality, namely the dual net of the restriction $\mathcal{A}_0$ to $\mathbb{R}$ is local:

$$I \mapsto \mathcal{A}_0^d(I) \doteq \mathcal{A}(\mathbb{R}\backslash I)', \qquad I \subset \mathbb{R}.$$

Due to locality the net $\mathcal{A}_0^d$ is larger than the original one, namely

$$\mathcal{A}_0(I) \subset \mathcal{A}_0^d(I), \qquad I \subset \mathbb{R}.$$

$\mathcal{A}_0^d$ is usually called the dual net, see [3, 8] and its main feature is that it obeys Haag duality on $\mathbb{R}$. The dual net does not in general transform covariantly under the covariance representation of the starting net.

Theorem 1.4. *Let $\mathcal{A}$ be a local net of von Neumann algebras on the intervals of $\mathbb{R}$, Ω a cyclic and separating vector for the von Neumann algebra $\mathcal{A}(I)$ associated with each interval $I \subset \mathbb{R}$ and U a Ω-fixing unitary representation of the translation-dilation group acting covariantly on $\mathcal{A}$. The following are equivalent:*

- (i) *$\mathcal{A}$ extends to a conformal precosheaf on S^1.*
- (ii) *The Bisognano-Wichmann property holds for $\mathcal{A}$, namely*

$$\Delta^{it}_{\mathbb{R}^+} = U(\Lambda_{\mathbb{R}^+}(2\pi t)). \tag{7}$$

Proof. $(i) \Rightarrow (ii)$: See [5, 14].

$(ii) \Rightarrow (i)$: Note first that, by translation covariance, $\Delta^{it}_{(a,\infty)} = U(\Lambda_{(a,\infty)}(2\pi t))$ for all $a \in \mathbb{R}$. Hence $\mathcal{A}(-\infty, a)$ is a von Neumann subalgebra of $\mathcal{A}(a,\infty)'$ that is cyclic on Ω and globally invariant under the modular group of $\mathcal{A}(a,\infty)'$ with respect to Ω, hence, by the Tomita-Takesaki theory, duality for half-lines holds

$$\mathcal{A}(a,\infty)' = \mathcal{A}(-\infty, a).$$

Recall now that if $\mathcal{N} \subset \mathcal{M}$ is an inclusion of von Neumann algebras and Ω is a cyclic and separating vector for both $\mathcal{N}$ and $\mathcal{M}$, then $(\mathcal{N} \subset \mathcal{M}, \Omega)$ is $+$hsm iff $(\mathcal{M}' \subset \mathcal{N}', \Omega)$ is $-$hsm [34, 2].

Then it is immediate to check $(\mathcal{A}(-\infty,-1), \mathcal{A}(-1,1), \mathcal{A}(1,\infty), \Omega)$ to be a $+$hsm factorization of von Neumann algebras, so we get a conformal precosheaf from Theorem 1.2. Due to Bisognano-Wichmann property this is indeed an extension to S_1 of the original net. □

Note as a consequence that a local net on $\mathbb{R}$ as above with property (7) automatically has a PCT symmetry, namely

$$J_{\mathbb{R}^+}\mathcal{A}(I)J_{\mathbb{R}^+} = \mathcal{A}(-I), \quad \forall \,\text{interval}\, I \subset \mathbb{R}.$$

Now, if $\mathcal{A}$ is a local conformal precosheaf on S^1, its restriction $\mathcal{A}_0$ to $\mathbb{R}$ does not depend, up to isomorphism, on the point we cut S^1, because of Möbius covariance. The local precosheaf on S^1 extending $\mathcal{A}_0^d$ is thus well defined up to isomorphism. We call it the dual precosheaf of $\mathcal{A}$ and denote it by $\mathcal{A}^d$.

Corollary 1.5. *The dual precosheaf of a conformal precosheaf on S^1 is a strongly additive conformal precosheaf on S^1.*

Proof. By construction, the dual net satisfies Haag duality on $\mathbb{R}$, hence strong additivity by Lemma 1.3. □

Remark . Let us compare the precosheaves $\mathcal{A}$ and $\mathcal{A}^d$ on S^1. First we observe that equality holds if and only if the conformal precosheaf $\mathcal{A}$ is strongly additive. As mentioned above locality implies

$$\mathcal{A}(I) \subset \mathcal{A}^d(I) \quad \text{if } -1 \notin I,$$

i.e. if I does not contain the point infinity, while Haag duality on S^1 implies

$$\mathcal{A}^d(I) \subset \mathcal{A}(I) \quad \text{if } -1 \in I.$$

Therefore the observable algebras assciated with bounded intervals of the real line are enlarged while the others, associated with intervals containing the point at infinity, decrease. The observable algebras associated with half-lines, i.e. with intervals having -1 as a boundary point, remain fixed. Due to the Bisognano-Wichmann property, which holds for all conformal precosheaves, altering the algebras implies a change in the representation of the conformal group $\mathtt{PSL}(2,\mathbb{R})$. Moreover, since the algebras associated with half-lines coincide, both representations agree on the isotropy group of the point at infinity, i.e. on the subgroup $\mathtt{P}_0$ of $\mathtt{PSL}(2,\mathbb{R})$ generated by translations and dilations.

An inclusion $\mathcal{N} \subset \mathcal{M}$ of von Neumann algebras is said to be *normal* if $\mathcal{N} = \mathcal{N}^{cc}$, where $\mathcal{X}^c = \mathcal{X}' \cap \mathcal{M}$ denotes the relative commutant of $\mathcal{X}$ in $\mathcal{M}$, and *conormal* if $\mathcal{M}$ is generated by $\mathcal{N}$ and its relative commutant w.r.t. $\mathcal{M}$, i.e., $\mathcal{M} = \mathcal{N} \vee \mathcal{N}^c$ (i.e. $\mathcal{M}' \subset \mathcal{N}'$ is normal).

We shall then say that a local conformal precosheaf $\mathcal{A}$ is (co-)normal if the inclusion $\mathcal{A}(I_1) \subset \mathcal{A}(I_2)$ is (co-)normal for any pair $I_1 \subset I_2$ of proper intervals of S^1. By Haag duality, normality and conormality are equivalent properties of conformal precosheaves.

Theorem 1.6. *Any conformal precosheaf on S^1 is normal and conormal.*

Proof. Let us consider first an inclusion of two proper intervals $I_1 \subset I_2$ with a common boundary point.

If $\mathcal{A}$ is strongly additive, the inclusion of von Neumann algebras $\mathcal{A}(I_1) \subset \mathcal{A}(I_2)$ is conormal as in this case $\mathcal{A}(I_1)' \cap \mathcal{A}(I_2) = \mathcal{A}(I_2 \backslash I_1)$. In the general case, by conformal invariance we may assume that I_1 and I_2 are respectively the intervals of the real line $(1, +\infty)$ and $(0, +\infty)$. By definition then $\mathcal{A}(I_1) = \mathcal{A}^d(I_1)$, $\mathcal{A}(I_2) = \mathcal{A}^d(I_2)$, with $\mathcal{A}^d$ the dual net, hence the inclusion $\mathcal{A}(I_1) \subset \mathcal{A}(I_2)$ is conormal by Corollary 1.5 and the above argument. As $\mathcal{A}(I_2)' \subset \mathcal{A}(I_1)'$ is conormal, $\mathcal{A}(I_1) \subset \mathcal{A}(I_2)$ is also normal.

It remains to show the normality of $\mathcal{A}(I_1) \subset \mathcal{A}(I_2)$ when $I_1 \subset I_2$ are intervals with no common boundary point, e.g. $I_1 = (b, c)$ and $I_2 = (a, d)$, with $a < b < c < d$. Then we set $I_3 = (a, c)$ and $I_4 = (b, d)$, therefore $I_1 = I_3 \cap I_4$ and both I_3 and I_4 are subintervals of I_2 with a common boundary point. Then the double relative commutant of $\mathcal{A}(I_1)$ in $\mathcal{A}(I_2)$ is given by

$$\mathcal{A}(I_1)^{cc} \subset \mathcal{A}(I_3)^{cc} \cap \mathcal{A}(I_4)^{cc} = \mathcal{A}(I_3) \cap \mathcal{A}(I_4) = \mathcal{A}(I_1) \tag{8}$$

where the last equality is a consequence of duality and additivity and implies the first inclusion; the opposite inclusion is elementary $\square$

Corollary 1.7. *Let $(\mathcal{N} \subset \mathcal{M}, \Omega)$ be a +hsm standard inclusion of von Neumann algebras. In this case:*

- *The inclusion $\mathcal{N} \subset \mathcal{M}$ is normal and conormal.*
- *There exists a unique strongly additive local conformal precosheaf $\mathcal{A}$ of von Neumann algebras on S^1 with $\mathcal{M} = \mathcal{A}(0, +\infty)$, $\mathcal{N} = \mathcal{A}(1, +\infty)$, and Ω the vacuum vector.*
- *There exists a bijection between local conformal precosheaves $\mathcal{A}$ of von Neumann algebras on S^1 with $\mathcal{M} = \mathcal{A}(0, +\infty)$, $\mathcal{N} = \mathcal{A}(1, +\infty)$, Ω the vacuum vector, and von Neumann subalgebras $\mathcal{N}_0$ of $\mathcal{N}' \cap \mathcal{M}$ cyclic on Ω such that $(\mathcal{N}_0 \subset \mathcal{M}, \Omega)$ and $(\mathcal{N}_0 \subset \mathcal{N}', \Omega)$ are $-$hsm inclusions.*

Proof. Starting with the last point, notice that $(\mathcal{M}', \mathcal{N}_0, \mathcal{N}, \Omega)$ is a +hsm factorization of von Neumann algebras, and clearly any +hsm factorization arises in this way, therefore the thesis is a consequence of Theorem 1.2.

In the special case $\mathcal{N}_0 = \mathcal{N}' \cap \mathcal{M}$ we then obtain the second statement by Lemma 1.3(ii)$\Longleftrightarrow$ (iii).

The first point is then a consequence of Theorem 1.6. $\square$

Note as a consequence that a hsm modular standard inclusion $(\mathcal{N} \subset \mathcal{M}, \Omega)$ is also *pseudonormal* :

$$\mathcal{N} \vee J\mathcal{N}J = \mathcal{M} \cap J\mathcal{M}J$$

where J is the modular conjugation of $(\mathcal{N}' \cap \mathcal{M}, \Omega)$ and has the continuous interpolation property (see [10]).

The irreducibility of $\mathcal{A}$ in Corollary 1.7 is equivalent in particular to the factoriality of $\mathcal{N}$ and $\mathcal{M}$, see [17]. This is also equivalent to the center of $\mathcal{N}$ and $\mathcal{M}$ to have trivial intersection. Thus we have the following.

Corollary 1.8. *Let $(\mathcal{N} \subset \mathcal{M}, \Omega)$ be +hsm and standard. Then $\mathcal{N}$ and $\mathcal{M}$ have the same center $\mathcal{Z}$ and $(\mathcal{N} \subset \mathcal{M}, \Omega)$ has a direct integral decomposition $\mathcal{N} = \int_{\mathcal{Z}}^{\oplus} \mathcal{N}_\lambda d\mu(\lambda)$, $\mathcal{M} = \int_{\mathcal{Z}}^{\oplus} \mathcal{M}_\lambda d\mu(\lambda)$, $\Omega = \int_{\mathcal{Z}}^{\oplus} \Omega_\lambda d\mu(\lambda)$, where each $(\mathcal{N}_\lambda \subset \mathcal{M}_\lambda, \Omega_\lambda)$ is either a +hsm standard inclusion of III_1 factors or trivial ($\mathcal{N} = \mathcal{M} = \mathbb{C}$).*

Proof. The modular group acts trivially on the center, so that

$$\mathrm{Ad}\,\Delta_{\mathcal{M}}^{it}(\mathcal{N} \cap \mathcal{M}') = \mathcal{N} \cap \mathcal{M}', \quad \forall t \in \mathbb{R}.$$

Since $\mathrm{Ad}\,\Delta_{\mathcal{M}}^{it_0}\mathcal{N} = \mathcal{M}$ for a suitable t_0, we immediately obtain $\mathcal{N} \cap \mathcal{M}' = \mathcal{M} \cap \mathcal{M}'$ and

$$\mathcal{M} \cap \mathcal{M}' = \mathcal{N} \cap \mathcal{M}' \subset \mathcal{N} \cap \mathcal{N}',$$

i.e. $\mathcal{Z}(\mathcal{M}) \subset \mathcal{Z}(\mathcal{N})$. Using the commutants we obtain the equality and the direct integral decomposition as stated. Applying [34], Theorem 12, and [2], we finish the proof.

□

The following Corollary summarizes part of the above discussion, based on results in [4, 34, 2].

Corollary 1.9. *There exists a one-to-one correspondence between:*

- *Isomorphism classes of standard +half-sided modular inclusions $(\mathcal{N} \subset \mathcal{M}, \Omega)$*
- *Isomorphism classes of Borchers triples $(\mathcal{M}, U, \Omega)$, (i.e. $\mathcal{M}$ is a von Neumann algebra with a cyclic separating unit vector Ω and U is a one-parameter Ω-fixing unitary group with positive generator s. t. $U(t)\mathcal{M}U(-t) \subset \mathcal{M}$, $t > 0$) such that Ω is cyclic for $U(t)\mathcal{M}'U(-t) \cap \mathcal{M}$ for some, hence for all, $t > 0$.*
- *Isomorphism classes of translation-dilation covariant, Haag dual nets on $\mathbb{R}$ with the Bisognano-Wichmann property $\Delta_{\mathbb{R}^+}^{it} = U(\Lambda(2\pi t))$.*
- *Isomorphism classes of strongly additive local conformal precosheaves of von Neumann algebras on S^1.*

The notion of isomorphism in the above setting has an obvious meaning. Note however that an isomorphism between local conformal precosheaves can be defined as an isomorphism of precosheaves relating the vacuum states, as in this case it will automatically intertwine the Möbius representations as these are unique, being fixed by the modular prescriptions.

2 Representations of $\mathtt{PSL}(2, \mathbb{R})$ and derivatives of the $U(1)$-current

2.1 A class of representations of $\mathtt{PSL}(2, \mathbb{R}) *_{\mathtt{P}_0} \mathtt{PSL}(2, \mathbb{R})$

In section 1 we have seen that we may associate with any conformal precosheaf on S^1 another conformal precosheaf on S^1 which is its Haag dual net on $\mathbb{R}$.

This amounts to "cut the circle," namely to fix a special point ("∞") and to redefine the local algebras associated to intervals which are relatively compact in $S^1 \setminus \{\infty\}$ in such a way that Haag duality holds on $S^1 \setminus \{\infty\}$. The representations of $\mathtt{PSL}(2,\mathbb{R})$ associated with the two nets coincide when restricted to the group $\mathtt{P}_0$ generated by translations and dilations, therefore give a representation of $\mathtt{PSL}(2,\mathbb{R}) *_{\mathtt{P}_0} \mathtt{PSL}(2,\mathbb{R})$, i.e., the free product of two copies of $\mathtt{PSL}(2,\mathbb{R})$ amalgamated by the subgroup $\mathtt{P}_0$.

Let us denote by i_1, resp. i_2, the embeddings of $\mathtt{PSL}(2,\mathbb{R})$ into the first, resp. the second, component of the free product, and by i the immersion of $\mathtt{P}_0$ in the amalgamated free product. Then we shall consider on $\mathtt{PSL}(2,\mathbb{R}) *_{\mathtt{P}_0} \mathtt{PSL}(2,\mathbb{R})$ the topology generated by the maps i_1 i_2, namely a unitary representation U of $\mathtt{PSL}(2,\mathbb{R}) *_{\mathtt{P}_0} \mathtt{PSL}(2,\mathbb{R})$ is strongly continuous if and only if $U \circ i_1$ and $U \circ i_2$ are strongly continuous.

We shall classify the class of strongly continuous unitary positive energy irreducible representations of $\mathtt{PSL}(2,\mathbb{R}) *_{\mathtt{P}_0} \mathtt{PSL}(2,\mathbb{R})$ whose restrictions $U \circ i_k$ are still irreducible or, equivalently, such that $U((H_1 - H_2)T)$ is a scalar, where T is the generator of the translations belonging to $\mathfrak{p}_0$, the Lie algebra of $\mathtt{P}_0$ and $H_k = i_k(H)$ are the generators of the rotation subgroup. This amounts to classify the unitary positive energy representation with $U((H_1 - H_2)T)$ central, as these decompose into a direct integral of irreducible representations in the previous class. As we shall see, this is the general case in a free theory.

Theorem 2.1. *Let U be an irreducible unitary representation of $\mathtt{PSL}(2,\mathbb{R}) *_{\mathtt{P}_0} \mathtt{PSL}(2,\mathbb{R})$ with positive energy, namely $-iU(T)$ is positive. Then $U \circ i_k$ is irreducible for some $k = 1, 2$ if and only if both the $U \circ i_k$ are irreducible, and if and only if $U((i_1(H) - i_2(H))i(T)) \in \mathbb{C}$, where H resp. T generate rotations resp. translations in $\mathtt{PSL}(2,\mathbb{R})$. Moreover, such representations are classified by pairs of natural numbers (n_1, n_2), where n_k is the lowest weight of the representation $U \circ i_k$ of $\mathtt{PSL}(2,\mathbb{R})$.*

As is known the matrices

$$E = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad T = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad S = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}.$$

form a basis for the Lie algebra $\mathfrak{sl}(2,\mathbb{R})$ and verify the commutation relations

$$[E, T] = T, \quad [E, S] = -S, \quad [T, S] = -2E \ . \tag{9}$$

Let us remind us that the conformal Hamiltonian is $H = \frac{i}{2}(T + S)$ and that the lowest weight of the representation is its lowest eigenvalue. The Casimir operator

$$\lambda = E(E-1) - TS \tag{10}$$

is a central element of the universal enveloping Lie algebra, thus its value in an irreducible unitary representation is a scalar. If U is a unitary irreducible non-trivial lowest weight representation of $\mathtt{PSL}(2,\mathbb{R})$, then the selfadjoint generator $-iU(T)$ of $U(e^{tT})$ is positive and non-singular, therefore $U(e^{tE})$ and

$e^{it\log(-iU(T))}$ give a representation of the Weyl commutation relations, namely U restricts to a unitary representation of $\mathtt{P}_0$, that has to be irreducible because any bounded operator commuting with E and T also commutes with S due to the formula (10). The von Neumann uniqueness theorem then implies that the restriction of U to $\mathtt{P}_0$ is unitarily equivalent to the Schrödinger representation, therefore $E \mapsto d/dx$, $T \mapsto -ie^x$ on $L^2(\mathbb{R})$.

We now describe all lowest weight representations of $\mathtt{PSL}(2,\mathbb{R})$ (or its universal covering group $\overline{\mathtt{PSL}}(2,\mathbb{R})$) as extensions of the representation of $\mathtt{P}_0$.

Let us fix now the unitary irreducible representation of $\mathtt{PSL}(2,\mathbb{R})$ with lowest weight 1 and denote by E_0, T_0 and S_0 the image in this representation of the above Lie algebra generators E, T and S.

Proposition 2.2.
- *Each non-trivial irreducible unitary representation U of $\overline{\mathtt{PSL}}(2,\mathbb{R})$ with lowest weight ≥ 1 is unitarily equivalent to the representation obtained by exponentiation of the operators $T_\lambda = T_0$, $E_\lambda = E_0$, $S_\lambda = S_0 - \lambda T_0^{-1}$, for some $\lambda > 0$*[1]*.*
- *All $\lambda > 0$ appear and $\lambda = \alpha(\alpha-1)$ if U has lowest weight α*
- *λ may be written as $\lambda = \frac{m}{n}(\frac{m}{n}-1)$, $m,n \in \mathbb{N}$, if and only if U is a representation of the n-th covering of $\mathtt{PSL}(2,\mathbb{R})$*

Proof. The first two statements follow from the above discussion since the value of the Casimir operator in the unitary representation with lowest weight α is equal to $\lambda = \alpha(\alpha-1)$, see [20], and one gets the formula for S_λ by multiplying both sides of (10) by T^{-1}.

To check the last point, first observe that when $\lambda \geq 0$, $\lambda = \alpha(\alpha-1)$, $\alpha \geq 1$, we get an orthonormal set of eigenvectors for the (self-adjoint) conformal Hamiltonian

$$H_\lambda = \frac{1}{2}\left(e^x - \frac{d}{dx}\left(e^{-x}\frac{d}{dx}\right) + \lambda e^{-x}\right).$$

In fact, set $\phi_\alpha = e^{\alpha x}e^{-e^x}$ and define the following operators $a^\alpha_\pm = 2E_\lambda \pm i(T_\lambda + S_\lambda)$. We also set for simplicity of notation $H \doteq H_{\alpha(\alpha-1)}$. Since $Ha^\alpha_\pm = a^\alpha_\pm(H \pm 1)$, $H\phi_\alpha = \alpha\phi_\alpha$ and $a^\alpha_-\phi_\alpha = 0$ then $\phi^n_\alpha \doteq (a^\alpha_+)^n\phi_\alpha$ is an orthogonal set of eigenvectors of H with eigenvalues $\alpha + n$. An application of the Stone-Weierstrass theorem shows that it is actually a basis, and the generated vector space is a Gårding domain for H_λ, T, E. The rest of the statement follows easily. □

Proof of Theorem 2.1: If $U((i_1(H) - i_2(H))i(T))$ is a scalar, $U \circ i_2(e^{tH})$ belongs to $(U \circ i_1(\mathtt{PSL}(2,\mathbb{R}) *_{\mathtt{P}_0} \mathtt{PSL}(2,\mathbb{R})))''$ and $U \circ i_1(e^{tH})$ belongs to $(U \circ i_2(\mathtt{PSL}(2,\mathbb{R}) *_{\mathtt{P}_0} \mathtt{PSL}(2,\mathbb{R})))''$, therefore, since U is irreducible, $U \circ i_k$ is irreducible too, $k = 1,2$. On the other hand, if say $U \circ i_1$ is irreducible, we may identify it with one of the representations described in Proposition 2.2 for some $\alpha \in \mathbb{R}$.

[1] If A and B are linear operator with closable sum, the closure of their sum is denoted simply by $A + B$.

Then, since U is irreducible and $U \circ i_1|_{\mathbf{P}_0} = U \circ i_2|_{\mathbf{P}_0}$, $U \circ i_2$ too has to be of the form described in Proposition 2.2, hence $U((i_1(H) - i_2(H))i(T))$ is a scalar. The rest of the statement is now obvious. □

Corollary 2.3. *Let $I \to \mathcal{A}(I)$ be a second quantization conformal precosheaf on S^1 as described in the following subsection, $I \to \mathcal{A}^d(I)$ be its dual net and U be the above representation of $\mathtt{PSL}(2,\mathbb{R}) *_{\mathbf{P}_0} \mathtt{PSL}(2,\mathbb{R})$, then the irreducible components of U belongs to the family described in Theorem 2.1.*

Proof. Since the dual net may be described in terms of the local algebras $\mathcal{A}^d(a,b) \doteq \mathcal{A}(-\infty,b) \cap \mathcal{A}(a,\infty)$ and the map which associates a local algebra with a local subspace is an isomorphism of complemented lattices (cf. [1]), the representation U is indeed a second quantization. On the one-particle space, the construction of the dual net may be done on any irreducible component, and the result follows. □

Corollary 2.4. *Every irreducible lowest weight representation of $\mathtt{PSL}(2,\mathbb{R})$ extends to a (anti-)representation of $\mathtt{Mob}$ in a unique (up to a phase) way.*

Proof. Let E_λ, T_λ and S_λ be the generators of the representation of lowest weight α as above. Since $\mathtt{Mob}$ is generated by $\mathtt{PSL}(2,\mathbb{R})$ and e.g. the matrix $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$, which correspond to the change of sign on $\mathbb{R}$, we need an antiunitary C which satisfies $CE_\lambda C = E_\lambda$, $CT_\lambda C = -T_\lambda$ and $CS_\lambda C = -S_\lambda$. (Because E_λ, T_λ and S_λ are generators, C is then uniquely defined up to a phase.) Since in the Schrödinger representation the complex conjugation C satisfies the mentioned commutation relations with T_0, S_0 and E_0, it trivially has the prescribed commutation relations with T_λ and E_λ, and the last relation follows by the formula $S_\lambda = S_0 - \lambda T_0^{-1}$. □

2.2 A modular construction of free conformal fields on S^1

For a certain class of pairs (M,G), where M ifs a homogeneous space for the symmetry group G, modular theory may be used to construct a net of local algebras on M starting from a suitable (anti-) representation of the symmetry group G [6]. For related works, pointing also to other directions, the interested reader should consult [27, 28, 29]. We sketch here the case of the action of the Möbius group on S^1.

We recall that a real subspace $\mathcal{K}$ of a complex Hilbert space $\mathcal{H}$ is called *standard* if $\mathcal{K} \cap i\mathcal{K} = \{0\}$ and $\mathcal{K} + i\mathcal{K}$ is dense in $\mathcal{H}$, and Tomita operators j, δ are canonically associated with any standard space (cf. [25]). One may easily show that the subspaces $\mathcal{K}'$, $i\mathcal{K}$ and $i\mathcal{K}'$ are standard subspaces if $\mathcal{K}$ is such, where the symplectic complement $\mathcal{K}'$ is defined by $\mathcal{K}' = \{h \in \mathcal{H} \ ; \ \mathrm{Im}(h,g) = 0 \ \forall g \in \mathcal{K}\}$.

As shown in [6], with any positive energy representation of $\mathtt{Mob}$ on a Hilbert space $\mathcal{H}$ we may uniquely associate a family $I \to \mathcal{K}(I)$ of standard subspaces attached to proper intervals I in S^1 satisfying the following properties:

1) $I_1 \subset I_2 \Rightarrow \mathcal{K}(I_1) \subset \mathcal{K}(I_2)$ (isotony)
2) $\mathcal{K}(I)' = \mathcal{K}(I')$ (duality)
3) $U(g)\mathcal{K}(I) = \mathcal{K}(gI)$ (conformal covariance)
4) $\delta_I^{it} = U(\Lambda_I(t)),\ j_I = U(r_I)$ (Bisognano-Wichmann property)

that is to say, $I \to \mathcal{K}(I)$ is a local conformal precosheaf of standard subspaces of $\mathcal{H}$ on the proper intervals of S^1. The subspaces $\mathcal{K}(I)$ are defined as

$$\mathcal{K}(I) \doteq \{h \in \mathcal{H} \mid j_I \delta_I^{\frac{1}{2}} h = h\}.$$

We notice that the precosheaf is irreducible, i.e. $\vee\mathcal{K}(I)$ is dense in $\mathcal{H}$, if and only if U does not contain the trivial representation. Applying the second quantization functor, we then get a local conformal precosheaf of von Neumann algebras acting on the Fock space $e^{\mathcal{H}}$.

Now we observe that we may extend the lowest weight representations (with integral $\alpha = n$) described in Proposition 2.2, to an (anti-)representation of $\mathtt{Mob}$ (cf. Corollary 2.4) in a coherent way, e.g. choosing the complex conjugation C for any α as in the proof of Corollary 2.4, and we get a family of conformal precosheaves $I \to \mathcal{K}_n(I)$ of standard spaces on S^1. The groups e^{tE} and e^{tT} have a unique fixed point, namely $\{\infty\}$, in S^1 and we may therefore identify $S^1 \setminus \{\infty\}$ with $\mathbb{R}$. Then, since the modular groups of the half-lines do not depend on n by construction, we get $\mathcal{K}_n(I) = \mathcal{K}_1(I)$ when I is a half-line, namely $\{\infty\}$ is one of its edges.

Theorem 2.5. *With the above notations, let $I \to \mathcal{K}(I)$ be a conformal precosheaf of standard subspaces of $\mathcal{H}$ on S^1 such that $\mathcal{K}(I) = \mathcal{K}_1(I)$ for any half-line I. Then, there exists $n \in \mathbb{N}$ such that $\mathcal{K}(I) = \mathcal{K}_n(I)$ for any interval I.*

Proof. By the Bisognano-Wichmann Theorem for conformal precosheaves on S^1 (cf. [5, 14]) we get that $U|_{\mathtt{P}_0}$ coincides with the restriction to $\mathtt{P}_0$ of the $n = 1$ lowest weight representation of $\overline{\mathtt{PSL}}(2, \mathbb{R})$. Therefore, since U is a positive energy representation, it should be of the form described in Proposition 2.2. □

Suppose now we start with the unique irreducible positive energy unitary representation U of the translation-dilation group, with non-trivial restriction to the translation subgroup, on a Hilbert space $\mathcal{H}$. According to [6], we may then consider the associated precosheaf of standard subspaces $I \to \mathcal{K}(I)$ on the half-lines $I \subset \mathbb{R}$. The following Corollary summarizes some properties discussed in Section 1 and some results of the previous subsection.

Corollary 2.6. *Let $I \to \mathcal{K}(I)$ the above described precosheaf on the half-lines of $\mathbb{R}$. Then there exists a bijective correspondence between*

- *Extensions of $\mathcal{K}$ to a local conformal precosheaf on the intervals of S^1.*

- *Real standard subspaces of* $\mathcal{K}(-\infty,1)'\cap\mathcal{K}(0,\infty)$ *-halfsided invariant w. r. t. the subgroup of dilations centered in* 0 *and +halfsided invariant w.r.t. the subgroup of dilations centered in* 1.
- *The real linear spaces* $\mathcal{K}_n(0,1)$, $n\in\mathbb{N}$.

2.3 Multiplicative perturbations: a formula for the canonical endomorphism

We now give an alternative way to pass from the representation of lowest weight 1 to the representation with lowest weight $\alpha\geq 1$. In this subsection we denote by E,T,S the Lie algebra generators in the lowest weight 1 representation, and with E,T,S_α the corresponding generators in the lowest weight α case. Instead of defining the generator S_α as $S-\lambda T^{-1}$, $\lambda=\alpha(\alpha-1)$, we will define the unitary R_α corresponding to the ray inversion or, equivalently, the unitary

$$\gamma=\gamma_\alpha=R_\alpha R=J_\alpha J \tag{11}$$

where J, resp. J_α is the modular conjugation of $\mathcal{K}(-1,1)$, resp. $\mathcal{K}_\alpha(-1,1)$, as $J=CR$ and $J_\alpha=CR_\alpha$ with the same anti-unitary conjugation commuting with them. In the examples below the second quantization of γ will implement the canonical endomorphism of the inclusion of algebras $\mathcal{A}_\alpha(-1,1)\subset\mathcal{A}(-1,1)$ given by $(\alpha-1)$-derivative of the current algebra (in case α is an integer).

We now make some formal motivation calculations, that may however be given a rigorous meaning. First note that γ commutes with E, because both J and J_α commute with E, hence γ must be a bounded Borel function of E

$$\gamma=f_\alpha(E) \tag{12}$$

because the bounded Borel functions of E form a maximal abelian von Neumann algebra.

In order to determine $f=f_\alpha$, note that the formulas

$$RTR=S \tag{13}$$

$$R_\alpha TR_\alpha=S_\alpha=S-\lambda T^{-1} \tag{14}$$

implies $\gamma^*T\gamma=T-\lambda RT^{-1}R$, hence

$$\gamma^*T\gamma T^{-1}=1-\lambda RT^{-1}RT^{-1}=1-\lambda(TRTR)^{-1}. \tag{15}$$

On the other hand

$$TRTR=TS=E(E-1) \tag{16}$$

by (10), and since $TET^{-1}=E-1$, thus

$$Tf(E)T^{-1}=f(E-1), \tag{17}$$

formula (15) implies f to satisfy the functional equation

$$\frac{f(z-1)}{f(z)} = 1 - \frac{\lambda}{z(z-1)} \tag{18}$$

and $|f(z)| = 1$ for all $z \in i\mathbb{R}$.

Proposition 2.7. *If $\alpha = n$ is an integer, then*

$$\gamma = \frac{(E-1)(E-2)\cdots(E-n+1)}{(E+1)(E+2)\cdots(E+n-1)}. \tag{19}$$

In the general case, $\gamma = f_\alpha(E)$ with

$$f_\alpha(z) = \frac{\Gamma(z+1)\Gamma(z)}{\Gamma(z+\alpha)\Gamma(z-\alpha+1)} \tag{20}$$

where Γ is the Euler Gamma-function.

Proof. Let γ_α be given by the formula (19). In order to check that γ_α gives (up to a phase) the unitary (11) it is enough to check that

$$\gamma_\alpha E \gamma_\alpha^* = R_\alpha E R_\alpha = E \tag{21}$$

$$\gamma_\alpha S \gamma_\alpha^* = R_\alpha T R_\alpha = S_\alpha \tag{22}$$

because the representation generated by E and S is irreducible, see (10) and the remarks below it.

The first equation is obvious because γ_α is a function of E. To verify the second equation we notice that from $S = RTR$ we get S positive and non-singular and (10) shows that this also holds for $E(E+1) = TS$. Since $SES^{-1} = E + 1$ the functional equation for f_α implies

$$f_\alpha(E)Sf_\alpha(E)^* = f_\alpha(E)f_\alpha(E+1)^*S = (1 - \frac{\lambda}{E(E+1)})S = S - \lambda T^{-1} = S_\alpha.$$

$\square$

2.4 Lowest weight representations of $\mathrm{PSL}(2,\mathbb{R})$ and derivatives of the $U(1)$-current

On the space $\mathcal{C}^\infty(S^1,\mathbb{R})$ of real valued smooth functions on the circle S^1, we consider the seminorm

$$||\phi||^2 = \sum_{k=1}^{\infty} k|\hat{\phi}_k|^2$$

and the operator $\mathcal{I} : \widehat{\mathcal{I}\phi}_k = -i\mathrm{sign}(k)\hat{\phi}_k$, where the $\hat{\phi}_k$'s denote the Fourier coefficients of ϕ.

Since $\mathcal{I}^2 = -1$ and $\mathcal{I}$ is an isometry w.r.t. $\|\cdot\|$, $(\mathcal{C}^\infty(S^1,\mathbb{R}), \mathcal{I}, \|\cdot\|)$ becomes a complex vector space with a positive bilinear form, defined by polarization. Thus, taking the quotient by constant functions and completing, we get a complex Hilbert space $\mathcal{H}$.

We note that the symplectic form ω may be written as

$$\omega(f,g) = \mathrm{Im}(f,g) = \frac{-i}{2}\sum_{k\in\mathbb{Z}} k\hat{f}_{-k}\hat{g}_k = \frac{1}{2}\int_{S^1} g df.$$

One might recognize this form as coming from the commutation relations for $U(1)-$currents. The natural action of $\mathtt{PSL}(2,\mathbb{R})$ on S^1 gives rise to a unitary representation on $\mathcal{H}$:

$$U(g)\phi(t) = \phi(g^{-1}t)$$

Then, observing that $\mathcal{I}\cos kt = \sin kt$ for $k \geq 1$, it is easy to see that $\cos kt$ is an eigenvector of the rotation subgroup $U(\theta)$:

$$U(\theta)\cos kt = \cos k(t-\theta) = (\cos k\theta + \sin k\theta\mathcal{I})\cos kt = e^{ik\theta}\cos kt, \qquad k \geq 1$$

and that all the eigenvectors have this form. Therefore the representation has lowest weight 1.

We need another description of the Hilbert space $\mathcal{H}$ which is more suitable to be generalized. First we choose another coordinate on S^1, namely $x = \tan(t/2)$, $x \in \mathbb{R}$, and therefore identify $\mathcal{C}^\infty(S^1,\mathbb{R})$ with $\mathcal{C}^\infty(\dot{\mathbb{R}})$, $\dot{\mathbb{R}}$ being the one-point compactification of $\mathbb{R}$. Since the symplectic form is the integral of a differential form it does not depend on the coordinate:

$$\omega(f,g) = \frac{1}{2}\int_{\mathbb{R}} g(x)df(x)$$

A computation shows that the anti-unitary $\mathcal{I}$ applied to a function f coincides up to an additive constant with the convolution of f with the distribution $1/(x+i0)$ on $\mathbb{R}$, therefore, since the symplectic form is trivial on the constants, the (real) scalar product may be written as

$$\begin{aligned}\langle f,g\rangle = \omega(f,\mathcal{I}g) &= \frac{1}{2}\int\left(\frac{1}{x+i0} * g(x)\right) f'(x)dx \\ &= \frac{1}{4\pi}\int f(x)g(y)\frac{1}{|x-y+i0|^2}dxdy = const\int_0^\infty p\hat{f}(-p)\hat{g}(p)dp\end{aligned} \tag{23}$$

and $\mathcal{H}$ may be identified with the completion of $\mathcal{C}^\infty(\dot{\mathbb{R}})$ w.r.t. this norm.

Note that since $\mathcal{I}f = -if$ if $\mathrm{supp}\hat{f} \subset [0,+\infty)$, $\mathcal{H}$ is also the completion of $C^\infty(\mathbb{R},\mathbb{C})$ modulo $\{f | \hat{f}|_{(-\infty,0]} = 0\}$ with scalar product $(f,g) = \int_0^\infty p\overline{\hat{f}(p)}\hat{g}(p)dp$.

Let us now consider the space $X^n \doteq \mathcal{C}^\infty(\dot{\mathbb{R}}) + \mathbb{R}^{2(n-1)}[x]$, $n \geq 1$, where $\mathbb{R}^p[x]$ denotes the space of real polynomials of degree p, and the bilinear form on it given by

$$\langle f,g\rangle_n = \frac{1}{4\pi}\int f(x)g(y)\frac{1}{|x-y+i0|^{2n}}dxdy$$

It turns out that $\langle\ \cdot\ ,\ \cdot\ \rangle_n$ is a well defined positive semi-definite bilinear form on X^n which degenerates exactly on $\mathbb{R}^{2(n-1)}[x]$. On this space one may define also a symplectic form by

$$\omega_n(f,g) = \frac{1}{2}\int f(x)g(y)\delta_0^{(2n-1)}(x-y)dxdy$$

This form might be read as the restriction of ω_1 to the n-th derivatives. Therefore we can recognize this symplectic form as coming from the commutation relations for the n-th derivatives of $U(1)-$currents. This form again degenerates exactly on $\mathbb{R}^{2(n-1)}[x]$, and the operator $\mathcal{I}$ defined before connects the positive form with the symplectic form for any n in such a way that $(\cdot,\cdot)_n \doteq \langle\cdot,\cdot\rangle_n + i\omega_n(\cdot,\cdot)$ becomes a complex bilinear form on $(X^n,\mathcal{I})$. We shall denote by $\mathcal{H}^n$ the complex Hilbert space obtained by completing the quotient $X^n/\mathbb{R}^{2(n-1)}[x]$.

With any matrix $g = \begin{pmatrix} a & b \\ c & d\end{pmatrix}$ in $\mathtt{SL}(2,\mathbb{R})$ we may associate the rational transformation $x \to gx = \frac{ax+b}{cx+d}$ and then, for any $n \geq 1$, the operators $U^n(g)$ on X^n:

$$U^n(g)f(x) = (cx-a)^{2(n-1)}f(g^{-1}x).$$

It turns out that $g \to U^n(g)$ is a representation of $\mathtt{PSL}(2,\mathbb{R})$, $n \geq 1$, and that the positive form is preserved (cf. [38]) as well as the symplectic form and the operator $\mathcal{I}$, therefore U^n extends to a unitary representation of $\mathtt{PSL}(2,\mathbb{R})$ on $\mathcal{H}^n$.

We remark that while X^n and $\mathbb{R}^{2(n-1)}[x]$ are globally preserved by U^n, the space $\mathcal{C}^\infty(\dot{\mathbb{R}})$ is not, and that explains why the space X^n had to be introduced.

By definition the space $\mathcal{H}^1$ coincides with the space $\mathcal{H}$ and the representation U^1 with the representation U, which we proved to be lowest weight 1. We observe that, for functions in $\mathcal{C}^\infty(\dot{\mathbb{R}})$, one gets

$$\langle f,g\rangle_n = \frac{1}{2}\int_{\mathbb{R}} |p|^{2n-1}\hat{f}(-p)\hat{g}(p)$$
$$\omega(f,g)_n = \frac{1}{2}\int_{\mathbb{R}} p^{2n-1}\hat{f}(-p)\hat{g}(p)$$

hence

$$(f,g)_n = (D^{n-1}f, D^{n-1}g)_1,$$

i.e. D^{n-1} is a unitary between $\mathcal{H}^n$ and $\mathcal{H}^1 \equiv \mathcal{H}$, where D is the derivative operator. The following holds:

Theorem 2.8. *The representation U^n has lowest weight n.*

Proof. Making use of the results of Proposition 2.7, we have to show that

$$R_nR = \prod_{k=1}^{n-1}\left(\frac{E-k}{E+k}\right), \qquad n \geq 1$$

where $R_n = D^{n-1}U^n(r)(D^{n-1})^*$ with r the ray inversion, $R = R_1$. This amounts to prove

$$D^{n-1}U^n(r) = \prod_{k=1}^{n-1} \left(\frac{E-k}{E+k}\right) U(r)D^{n-1}. \tag{24}$$

Now we take equation (24) as an inductive hypothesis. Then, equation(24) for $n+1$ can be rewritten, using the inductive hypothesis and the relation $U^{n+1}(r) = x^2U^n(r)$, as

$$D^n(x^2U^n(r)) = \left(\frac{E-n}{E+n}\right) D^{n-1}U^n(r)D. \tag{25}$$

Finally we observe that $U^n(r)D = x^2DU^n - 2(n-1)xU^n$, hence equation (25) is equivalent to

$$(E+n)D^n(x^2\cdot) = (E-n)D^{n-1}(x^2D\cdot -2(n-1)x\cdot). \tag{26}$$

Since $E = -xD$, equation (26) follows by a straightforward computation. □

Proposition 2.9. *The unitary representations of* $\mathtt{PSL}(2,\mathbb{R})$ *on* $\mathcal{H}$ *given by*

$$D^{n-1}U^n(D^{n-1})^*, \qquad n \geq 1$$

coincide when restricted to the subgroup of translations and dilations on $\mathbb{R}$.

Proof. We have to prove that $D^{n-1}U^n(g) = U(g)D^{n-1}$ when g is a translation or a dilation. For translations, $U^n(t)f(x) = f(x-t)$, and the equality is obvious; for dilations, $g = \begin{pmatrix} e^{\lambda/2} & 0 \\ 0 & e^{-\lambda/2}\end{pmatrix}$, $U^n(\lambda)f(x) = e^{\lambda n}f(e^{-\lambda}x)$, hence $D^{n-1}U^n(\lambda)f(x) = f^{(n)}(e^{-\lambda}x) = U(\lambda)D^nf(x)$. □

The family of representations $D^{n-1}U^n(D^{n-1})^*$ on the Hilbert space $\mathcal{H}$ constitute a concrete realization of the family of (integral) lowest weight representations described in Proposition 2.2, therefore we may construct a family of local conformal precosheaves of standard subspaces of $\mathcal{H}$ as explained in Subsection 2.2. In the next subsection we shall give another description of these precosheaves, showing that they coincide with the ones described in [38].

2.5 Relations among local spaces

Let us fix an $n \geq 1$ and, for any proper interval I of $\dot{\mathbb{R}}$, let us set

$$X^n(I) = \{f \in X^n : f|_{I'} \equiv 0\}.$$

It is easy to check that these spaces satisfy the properties

1. $I_1 \subset I_2 \Longrightarrow X^n(I_1) \subset X^n(I_2)$ (isotony),
2. $I_1 \cap I_2 = \emptyset \Longrightarrow X^n(I_1) \subset X^n(I_2)'$ (locality),
3. $U^n(g)X^n(I) = X^n(gI), \forall g \in \mathtt{PSL}(2,\mathbb{R})$ (covariance),

and that the immersion $i_n^I : X^n(I) \to \mathcal{H}^n$ is injective. Therefore the spaces $\mathcal{K}^n(I) \doteq (i_n^I X^n(I))^-$, where the closure is taken w.r.t. $\|\cdot\|_n$, form a local conformal precosheaf of subspaces of $\mathcal{H}^n$, and the following property obviously holds:

$$4. \ \bigvee_{I \subset \dot{\mathbb{R}}} \mathcal{K}^n(I) = \mathcal{H}^n \quad \text{(irreducibility)}.$$

Therefore, by the first quantization version of results mentioned in Section 1, these spaces are standard, the Bisognano-Wichmann property and duality on the circle hold.

Now we identify $\mathcal{H}^n$ with $\mathcal{H}$ via the unitary D^{n-1}, and set $\mathcal{K}_n(I) \doteq D^{n-1}\mathcal{K}^n(I)$. Then, if I is compact in $\mathbb{R}$ and $f \in \mathcal{K}_n(I)$, f may be integrated $n-1$ times, giving a function which has still support in I, therefore

$$\mathcal{K}_n(I) = \{[f] \in \mathcal{H} : f|_{I'} = 0, \ \int t^j f = 0, \ j = 0, \ldots, n-2\}, \quad I \subset\subset \mathbb{R} \qquad \text{(a)}$$

where $[f]$ denotes the equivalence class of f modulo polynomials.

If I is a half line in $\mathbb{R}$, $\mathcal{K}_n(I)$ is an invariant subspace of the dilation subgroup, which is the modular group of $\mathcal{K}(I)$. Using Takesaki's result, see [32], this implies that

$$\mathcal{K}_n(I) = \mathcal{K}(I), \qquad I \text{ a half-line}, \qquad \text{(b)}$$

(For an alternative proof of this fact, see [38].)

Then, by duality and the formula for the compact case, we obtain

$$\mathcal{K}_n(I) = \{[f] \in \mathcal{H} : f|_{I'} = p_{f,I} \in \mathbb{R}^{n-1}[x]\} \quad I' \subset\subset \mathbb{R}. \qquad \text{(c)}$$

Finally we observe that, since Bisognano-Wichmann property holds, these precosheaves coincide with those abstractly constructed in Subsection 2.2.

Now, we fix a bounded interval in $\mathbb{R}$, e.g. $(-1,1)$, and consider the family $\mathcal{K}_n \doteq \mathcal{K}_n((-1,1))$. The concrete characterization of $\mathcal{K}_n$ given in the preceding subsection shows that $\mathcal{K}_m \subseteq \mathcal{K}_n$ if $m \geq n$. Now, we may show

Theorem 2.10. *The following dimensional relations hold:*

$$\begin{aligned} &codim(\mathcal{K}_m \subset \mathcal{K}_n) = m - n, \qquad m \geq n, \\ &dim(\mathcal{K}_m' \cap \mathcal{K}_n) = \max((m-n-1), 0). \end{aligned}$$

Before proving Theorem 2.10, we discuss some of its consequences.

Definition 1. A precosheaf $\mathcal{K}$ is said $n-$*regular* if, for any partition of S^1 into n intervals $I_1, \ldots, I_n$, the linear space $\bigvee_{j=1}^n \mathcal{K}(I_j)$ is dense in $\mathcal{H}$.

We recall that irreducible conformal precosheaves are 2-regular, because duality holds and local algebras are factors.

Corollary 2.11. *The conformal precosheaf $\mathcal{K}_1$ is n-regular for any n. The conformal precosheaf $\mathcal{K}_2$ is 3-regular but it is not 4-regular. The conformal precosheaves $\mathcal{K}_n$, $n \geq 3$, are not 3-regular. Moreover, strong additivity and duality on the line hold for the precosheaf $I \to \mathcal{K}_1(I)$ only, therefore it is the dual precosheaf of $I \to \mathcal{K}_n(I)$ for any n.*

Proof. First we recall that a precosheaf is strongly additive if and only if it coincides with its dual precosheaf. Then, the precosheaf $\mathcal{K} \equiv \mathcal{K}_1$ is strongly additive because its dual net should be of the form $\mathcal{K}_n$ (cf. Corollary 2.6) and should satisfy $\mathcal{K}^d(-1,1) \supseteq \mathcal{K}(-1,1)$. As a consequence, $\mathcal{K}$ is n-regular for any n.

Then, since the spaces for the half-lines do not depend on n, the dual net of $\mathcal{K}_n$ does not depend on n either, hence coincides with $\mathcal{K}$.

Since $\mathtt{PSL}(2,\mathbb{R})$ acts transitively on the triples of distinct points, we may study 3-regularity for the special triple $(-1,1,\infty)$ in $\mathbb{R} \cup \{\infty\}$. Then,

$$\begin{aligned}
&(\mathcal{K}_n(\infty,-1) \vee \mathcal{K}_n(-1,1) \vee \mathcal{K}_n(1,\infty))' \\
&\qquad = (\mathcal{K}_1(\infty,-1) \vee \mathcal{K}_n(-1,1) \vee \mathcal{K}_1(1,\infty))' \\
&\qquad = (\mathcal{K}_1(-1,1)' \vee \mathcal{K}_n(-1,1))' = \mathcal{K}_n(-1,1)' \wedge \mathcal{K}_1(-1,1)
\end{aligned}$$

where we used strong additivity and duality for $\mathcal{K}_1$. By Theorem 2.10, 3-regularity holds if and only if $n = 1, 2$.

Violation of 4-regularity for $\mathcal{K}_2$ may be proved by exhibiting a function which is localized in the complement of any of the intervals $(\infty,-1)$, $(-1,0)$, $(0,1)$, $(1,\infty)$, i.e. belongs to $\mathcal{K}_2(-1,0)' \cap \mathcal{K}_2(0,1)' \cap \mathcal{K}_1(-1,1)$:

$$\phi(x) = \begin{cases} 1+x & \text{if} \quad -1 \geq x \geq 0, \\ 1-x & \text{if} \quad 0 \geq x \geq 1, \\ 0 & \text{if} \quad |x| \geq 1 \end{cases}$$

In the same way we may construct a function which violates 3-regularity for $\mathcal{K}_3$, namely

$$\phi(x) = \begin{cases} x^2-1 & \text{if} \quad |x| < 1, \\ 0 & \text{if} \quad |x| \geq 1 \end{cases}$$

Clearly, $\phi \in \mathcal{K}_3(\infty,-1)' \cap \mathcal{K}_3(-1,1)' \cap \mathcal{K}_3(1,\infty)' = \mathcal{K}_3' \cap \mathcal{K}_1$. ☐

Lemma 2.12. *codim*$(\mathcal{K}_{m+1} \subset \mathcal{K}_m) = 1$.

Proof. Since $\mathcal{K}_{m+1} = \{\phi \in \mathcal{K}_m \ : \ \int x^{m-1}\phi(x)dx = 0\}$, and we may find a function $\psi_{m-2} \in \mathcal{C}_0^\infty(\mathbb{R}) \ : \ \psi'_{m-1}(x) = x^{m-1}$, $x \in (-1,1)$, we get $\mathcal{K}_{m+1} = \{\phi \in \mathcal{K}_m \ : \ \omega(\psi_{m-1},\phi) = 0\}$. Because the functional $\phi \longrightarrow \omega(\psi_m,\phi)$ is continuous and non zero on $\mathcal{K}_m$, the thesis follows. ☐

Proof of Theorem 2.10: The first statement of the Theorem easily follows from Lemma 2.12. Now, let us consider the relative commutants $\mathcal{K}'_{m+p} \cap \mathcal{K}_m$. We observe that, by Poincaré inequality, the norm on $\mathcal{K}_1$ is equivalent to the

Sobolev norm for the space $H^{1/2}$, i.e., we may identify $\mathcal{K}_1 \simeq H^{1/2}(-1,1)$ as real Hilbert spaces. We also recall that the Dirac measure δ does not belong to $H^{-1/2}$, but belongs to $H^{-1/2-\epsilon}$ for each $\epsilon > 0$ (see, e.g., [33]). Then

$$\begin{aligned}\mathcal{K}'_{m+p} \cap \mathcal{K}_m &= \{\phi \in \mathcal{K}_m \ : \ \langle \phi', \psi \rangle = 0 \ \forall \psi \in \mathcal{K}_{m+p}\} \\ &= \{\phi \in H^{1/2}(-1,1) \ : \ \int t^j \phi = 0, \ j = 0, \ldots, m-2, \\ &\qquad\qquad \langle \phi^{(m+p)}, \psi \rangle = 0, \ \forall \psi \in H^{m+p-1/2}(-1,1)\}.\end{aligned}$$

Then $f \doteq \phi^{(m+p)} \in H^{1/2-m-p}\{-1,1\}$, i.e., f should be a combination of Dirac's δ measures with supports in $\{-1,1\}$ and their derivatives. Since $f \in H^{1/2-m-p}$, it has the form $f = \sum_{j=0}^{m+p-2}(c_j \delta^{(j)}_{(-1)} + d_j \delta^{(j)}_{(1)})$. The condition $\phi \in \mathcal{K}_m$ may be written as

$$\langle f, t^q \rangle = 0, \qquad q = 0, \ldots, 2m+p-2. \tag{27}$$

The dimension of $\mathcal{K}'_{m+p} \cap \mathcal{K}_m$ will be the difference between the dimension of the space $\Delta \doteq \{\sum_{j=0}^{m+p-2}(c_j \delta^{(j)}_{(-1)} + d_j \delta^{(j)}_{(1)}) \ : \ c_j, d_j \in \mathbb{R}\}$, which is $2(m+p-1)$, and the number of independent conditions in equation (27). We may also write the conditions in equation (27) as

$$\langle f, P \rangle = 0 \qquad \text{where} \ \ P \ \text{is a polynomial of degree} \ 2m+p-2.$$

They are independent if the only polynomial P of degree $\leq 2m+p-2$ satisfying $\langle f, P \rangle = 0$ for any $f \in \Delta$ is the null polynomial. Indeed, such polynomial should have zeroes with multiplicities greater than $m+p-1$ for the points -1 and 1, therefore, either $p = 0$, and then there exists exactly one non trivial such polynomial, or $p > 0$, and the null polynomial is the unique solution. In conclusion, if $p > 0$, the conditions in equation (27) are independent, and the dimension of $\mathcal{K}'_{m+p} \cap \mathcal{K}_m$ is

$$2(m+p-1) - (2m+p-1) = p-1.$$

If $p = 0$ the independent conditions in equation (27) are $2m-2$, and the dimension is $2(m-1) - (2m-2) = 0$, which corresponds to the general fact (see, [17]) that local algebras of irreducible conformal theories are factors. □

3 Examples of superselection sectors for the first derivative of the $U(1)$-current

In this section we shall discuss examples of superselection sectors of the first-derivative theory. All these sectors are abelian, i.e. are equivalence classes of automorphisms, and we will see that they are non covariant under the conformal group. In particular, recalling that the first-derivative precosheaf is 3-regular but not 4-regular, this shows that the assumption of 4-regularity in [16] cannot be avoided in general in order to obtain the automatic covariance of superselection sectors.

As we shall see, all these sectors will be obtained by generalizing methods of the Buchholz-Mack-Todorov approach to sectors (see [7]).

On the one hand the conformal net on $\mathbb{R}$ associated with the current algebra contains as a subnet the one associated with the first (n-th) derivative of the current algebra (cf. Corollary 2.11 and the Remark after Corollary 1.5), therefore BMT sectors may be restricted to the conformal net on $\mathbb{R}$ associated with the first (n-th) derivative of the current algebra. The sectors described here will be extensions (of such restrictions) to the conformal precosheaf on S^1 associated with the first (n-th) derivative of the current algebra.

On the other hand they may be seen as sectors on a suitable global algebra in a way which is formally identical to the BMT procedure.

Now let $\mathcal{A}$ be a local conformal precosheaf on $\mathbb{R}$ and $\mathcal{A}^d$ its Bisognano-Wichmann dual net. Consider the unitary $\Gamma = JJ_d$, where J and J_d are the modular conjugations of $\mathcal{A}(-1,1)$ and $\mathcal{A}^d(-1,1)$ with respect to the vacuum vector Ω, in other words Γ is the product of the two ray inversion unitaries of the nets $\mathcal{A}$ and $\mathcal{A}^d$. The unitary Γ implements the *canonical endomorphism* γ of $\mathcal{A}^d(-1,1)$ into $\mathcal{A}(-1,1)$. If $a>0$ we have

$$\mathrm{Ad}\Gamma(\mathcal{A}^d(-a,a)) = J\mathcal{A}^d(-\frac{1}{a},\frac{1}{a})'J \subset J\mathcal{A}(-\frac{1}{a},\frac{1}{a})'J = \mathcal{A}(-a,a) \qquad (28)$$

so γ extends to an endomorphism of the quasi-local C*-algebra $\cup_{a>0}\mathcal{A}^d(-a,a)^-$ into the quasi-local C*-algebra $\cup_{a>0}\mathcal{A}(-a,a)^-$. Note also that $\mathrm{Ad}\Gamma$ is an automorphism of the von Neumann algebras $\mathcal{A}^d(0,+\infty)=\mathcal{A}(0,+\infty)$ and $\mathcal{A}^d(-\infty,0)$ $=\mathcal{A}(-\infty,0)$.

If now ρ is a morphism of $\mathcal{A}$ localized in a bounded interval $I\subset\mathbb{R}$ we define the "extension" of ρ to $\mathcal{A}^d$ by

$$\tilde{\rho} = \mathrm{Ad}\Gamma^*\rho\gamma\ .$$

Note that if I is contained in $(0,\infty)$ (resp. $(-\infty,0)$) then $\tilde{\rho}$ is localized in $(0,\infty)$ (resp. $(-\infty,0)$). However $\tilde{\rho}$ may be localized in the full half-line, namely $\tilde{\rho}$ has a *soliton* localization. (In this lower dimensional theory one might also interpret these sectors as coming from order variables, [30], Chapter 3.8.) In subsection 3.3 we show examples of this phenomenon.

Let us remark that if $\tilde{\rho}$ is localized in a bounded interval, then $\tilde{\rho}$ is automatically covariant with respect to the new Möbius group representation of $\mathcal{A}^d$, because $\mathcal{A}^d$ is strongly additive [16].

3.1 Buchholz, Mack and Todorov approach to sectors of the current algebra

We defined the one-particle space for the current algebra as the completion of the space $X = X^1 = \dot{\mathcal{C}}^\infty(\mathbb{R})$ modulo constant functions w.r.t. the norm given in (23). We may then define $\mathcal{A}(S^1)$ as the * - algebra generated by $W(h), h \in X$ with the relations $W(h)W(h)^* = 1$ (unitarity) and $W(h)W(k) = exp(i/2\omega(h,k))W(h+k)$ (CCR).

BMT automorphisms of $\mathcal{A}(S^1)$ are then given in terms of differential forms ϕ on S^1. Setting

$$\alpha_\phi(W(h)) \doteq e^{i\int h\phi}W(h)$$

it is easy to see that α extends to an automorphism of $\mathcal{A}(S^1)$. By CCR, it follows that α_ϕ is inner if and only if the form ϕ is exact, i.e. there exists a function $f \in X$ s.t. $\phi = df$, and that two automorphisms α_ϕ, α_ψ are equivalent if and only if $\int \phi = \int \psi$, i.e. if the two forms give the same cohomology class in $H^1(S^1)$. The constant $Q(\alpha_\phi) := \int \phi$ will be called the charge of α_ϕ.

For any open interval I in S^1 we set $\mathcal{A}(I)$ to be the subalgebra of $\mathcal{A}(S^1)$ generated by Weyl unitaries $W(h)$ such that the support of h is contained in I. Clearly the algebras associated with disjoint intervals commute and $\beta_g\mathcal{A}(I) \doteq \mathcal{A}(gI)$ where $\beta_g(W(f)) \doteq W(U(g)f)$.

We observe that BMT automorphisms are *locally internal*, i.e. for any interval I and any form ϕ there exists a function f with support in some larger interval $\hat{I}$ such that $df|_I \equiv \phi|_I$, therefore $\alpha_\phi|_{\mathcal{A}(I)} \equiv \mathrm{ad}W(f)|_{\mathcal{A}(I)}$.

Also, the superselection sectors corresponding to a given charge are conformally covariant w.r.t. the adjoint action of the conformal group on X, i.e. the automorphisms α_ϕ and $\beta_g \cdot \alpha_\phi \cdot \beta_{g^{-1}}$ are in the same class for any conformal transformation g. Indeed, since the class of inner automorphisms is globally stable under the action of the conformal group and the charge is additive, namely $Q(\alpha_\phi \circ \alpha_\psi) = Q(\alpha_\phi) + Q(\alpha_\psi)$, the action of $\mathtt{PSL}(2,\mathbb{R})$ on BMT automorphisms gives a linear action on BMT charges, i.e. a one dimensional linear representation of $\mathtt{PSL}(2,\mathbb{R})$. Any such representation being trivial, BMT sectors are covariant.

Now we give a local description for these sectors. We observe that the second quantization algebra associated with the standard space $\mathcal{K}(I)$ coincides with $\pi(\mathcal{A}(I))'' = \mathcal{R}(I)$, where π is the vacuum representation of $\mathcal{A}(S^1)$ on the Fock space e^H. Moreover, the map $\pi|_{\mathcal{A}(I)}$ is faithful, and the restriction of α_ϕ to $\mathcal{A}(I)$, being implemented in $\mathcal{A}(\hat{I})$, uniquely extends to a normal automorphism of $\mathcal{R}(I)$. As a consequence, α_ϕ gives rise to a representation $I \to \alpha^I_\phi$ of the precosheaf $\mathcal{A}$ in the sense of [17], where

$$\alpha^I_\phi(\pi(W(h))) = e^{i\int h_I\phi}W(h)$$

and, recalling that $h \in H$ is localized in I if it is equal to a constant c_I in I', we have set $h_I \doteq h - c_I$.

We described BMT locally normal representations via automorphisms of $\mathcal{A}(S^1)$. Conversely, the global algebra $\mathcal{A}(S^1)$ plays the role of the universal algebra w.r.t. the family of the locally normal BMT representations, in the sense that the classes of such representations modulo unitary equivalence appear as classes of global automorphisms of $\mathcal{A}(S^1)$ up to inners.

3.2 Restriction of localized sectors

As we have already seen, local algebras associated with compact intervals on the line for the first-derivative net may be described as

$$\mathcal{R}_2(I) \doteq \{W(h) \in \mathcal{R}(I) : \int h(x)dx = 0\}''.$$

Of course these algebras form a net of local algebras on the real line which is covariant with respect to the action of translations and dilations, but Haag duality does not hold on $\mathbb{R}$. The quasi-local algebra $\mathcal{A}_2(\mathbb{R})$ generated by the algebras of compact intervals is a subalgebra of the quasi-local algebra $\mathcal{A}(\mathbb{R})$ of the current algebra on the line, therefore any BMT automorphism of $\mathcal{A}$ localized in some compact interval I gives a representation of $\mathcal{A}_2(\mathbb{R})$ which is equivalent to the vacuum representation if and only if it has zero charge, but, due to the failure of Haag duality, the intertwining unitary is not necessarily localized in I. Such a unitary exhibits instead a solitonic localization, i.e. it necessarily belongs to the von Neumann algebra of any half line containing the localization region. The restrictions of BMT sectors are then translation and dilation covariant.

On the contrary, if we consider classes of automorphisms of $\mathcal{A}_2(\mathbb{R})$ modulo inners, a new charge appears, i.e. two automorphisms α_ϕ and α_ψ are equivalent if and only if both $\int \phi = \int \psi$ and $\int t\phi = \int t\psi$ are equal. As a consequence, such sectors are no longer translation covariant.

3.3 Conformal solitonic sectors

In the first-derivative theory, the automorphism α_ϕ is localized in a compact interval I of $\mathbb{R}$ when ϕ is constant outside I, therefore solitonic sectors on $\mathcal{A}(\mathbb{R})$ may become localized when restricted to $\mathcal{A}_2(\mathbb{R})$. This shows that, conversely, sectors on $\mathcal{A}_2$ may become solitonic when extended to $\mathcal{A}$ as described at the beginning of this section.

Here we shall consider ϕ as a function on $\mathbb{R}$ rather than as a differential form, identifying $\phi(t)$ with $\phi(t)dt$. If ϕ is constant outside I, α_ϕ is equivalent to the vacuum (as a representation) when both $\phi(+\infty) = \phi(-\infty)$ $(= 0)$ and $\int_{-\infty}^{+\infty} \phi = 0$. As a consequence, superselection sectors are described by two charges:

$$Q_0 = \phi(+\infty) - \phi(-\infty) = \int \phi'(t)dt \qquad Q_1 = \int t\phi'(t)dt.$$

These sectors are clearly transportable, but a simple computation shows that they are covariant under translations and dilations if and only if $Q_0 = 0$, i.e. only if they are restrictions of BMT sectors. Restrictions to $\mathcal{A}_2(\mathbb{R})$ of solitonic sectors on $\mathcal{A}(\mathbb{R})$ give then an example of localized non covariant sectors on the line. As we shall see, these sectors may be extended to transportable sectors on the circle.

3.4 Generalized BMT approch to sectors: Local description

We recall that the conformal precosheaf $\mathcal{R}_2$ of the first-derivative theory may be described as second quantization algebras on the same Fock space as the current algebra, $\mathcal{R}_2(I) = \{W(h) : h \in \mathcal{K}_2(I)\}''$.

We have seen that

$$\mathcal{K}_2(I) = \{[f] \in \mathcal{H} : f|_{I'} \equiv p_{f,I},\ p_{f,I} \in \mathbb{R}^0[x], \int (f(x) - p_{f,I})dx = 0\} \quad I \subset\subset \mathbb{R}$$
$$\mathcal{K}_2(I) = \{[f] \in \mathcal{H} : f|_{I'} \equiv p_{f,I}\ p_{f,I} \in \mathbb{R}^0[x]\} \quad I \text{ half line}$$
$$\mathcal{K}_2(I) = \{[f] \in \mathcal{H} : f|_{I'} \equiv p_{f,I},\ p_{f,I} \in \mathbb{R}^1[x]\} \quad I' \subset\subset \mathbb{R}.$$

In order to extend a BMT automorphism α_ϕ to the first-derivative theory on the circle we have to choose a real number λ and then set

$$\alpha^I_{\phi,\lambda}(W(f)) = e^{i\langle\phi,f\rangle_{I,\lambda}} W(f)$$

where f belongs to $\mathcal{K}_2(I)$, with

$$\langle\phi, f\rangle_{I,\lambda} = \int \phi(x)(f(x) - p_{f,I}(\lambda))dx.$$

Taking ϕ, ψ such that $Q = \int \phi = \int \psi$ we may compute $\alpha_{\phi,\lambda} \cdot \alpha^{-1}_{\psi,\mu}$:

$$\begin{aligned}(\alpha_{\phi,\lambda} \cdot \alpha^{-1}_{\psi,\mu})^I(W(f)) &= e^{i(\langle\phi,f\rangle_{I,\lambda} - \langle\psi,f\rangle_{I,\mu})} W(f) \\ &= e^{-iQ(\lambda-\mu)\langle\delta'_x, f\rangle} \text{ad } W(h)(W(f)),\end{aligned} \tag{29}$$

with $h' = \phi - \psi$, where x is any point in I'. Therefore we have proved that two BMT automorphisms with the same (non zero) charge extend to equivalent automorphisms if and only if $\lambda = \mu$. Since a simple calculation shows that the translated automorphism $\beta_{T(t)}\alpha_{\phi,\lambda}\beta_{T(-t)}$ is equal to $\alpha_{\phi(\,\cdot\,+t),\lambda-t}$, we conclude that non trivial BMT sectors give rise to a one parameter family of non covariant sectors on the circle.

Moreover, when $\lambda \neq \mu$, the automorphism $\alpha_{\phi,\lambda} \cdot \alpha^{-1}_{\psi,\mu}$ is equivalent to a new automorphism θ_c, $c = -Q(\lambda - \mu)$:

$$\theta^I_c(W(f)) = e^{ic\langle\delta'_x, f\rangle} W(f), \quad x \in I'. \tag{30}$$

Since $p_{f,I}$ is constant whenever $I \subset \mathbb{R}$ and hence $\langle\delta'_x, f\rangle$ vanishes, we conclude that θ_c is localized in only one point, the point at infinity. We may easily show that θ_c is invariant under translations and that dilations act on the charge c.

By conjugating θ_c with a conformal transformation we will get a family of automorphisms localized in different points of the real line. In particular, requiring the automorphisms to be localized in zero, we get a family ζ_c:

$$\zeta_c(W(f)) \doteq e^{ic\int_0^x (f(y) - p_{f,I}(y))dy} W(f) \qquad x \in I'. \tag{31}$$

Indeed it is easy to see that, if $f \in \mathcal{K}_2(I)$, then $\int_0^x (f(y) - p_{f,I}(y))dy$ does not depend on $x \in I'$, and is equal to zero when $0 \notin I$. Now let 0 be in I, and suppose for simplicity that $I \subset \mathbb{R}$.

Then formula (31) may be obtained by formula (30) using $R(\pi)$, the rotation by π, which in the real line coordinates is $t \to -1/t$. As in Proposition 2.9, such rotation is implemented by $DU^{(2)}(R(\pi))D^*$ on the Hilbert space $\mathcal{H}$, and, since f has compact support, we may set $x = \infty$ in formula (31). Hence the correspondence between (31) and (30) follows by the equality

$$2\int_0^\infty (f(y) - p_{f,I})(y) = \langle \delta_0, 2x \int_0^{-\frac{1}{x}} (f - p_{f,I})(y)dy + f\left(-\frac{1}{x}\right)\rangle$$

When f is localized in $\mathbb{R}$ and we chose $x = \infty$ as before, the automorphisms ζ_c furnish extensions to the circle of the restriction to $\mathcal{A}_2(\mathbb{R})$ of solitonic sectors on $\mathcal{A}(\mathbb{R})$. It is not difficult to see that dilations act on these automorphisms dilating the charge, therefore these sectors are non covariant too.

In the following subsection we shall see that the Weyl algebra on the symplectic space (X^2, ω_2) is a global algebra for all these sectors, i.e. the given automorphisms of the precosheaf modulo unitaries on the Fock space are described by automorphisms of this global algebra modulo inners. In doing that we shall see that the described sectors form a group isomorphic to $\mathbb{R}^3$.

3.5 Generalized BMT approach to sectors: Global description

Now we describe some natural automorphisms of the Weyl algebra $\mathcal{A}^2(\dot{\mathbb{R}})$ on (X^2, ω_2). If ϕ is a measure on $\dot{\mathbb{R}}$ such that $\int (1 + t^2)d|\phi|(t) < \infty$, we set

$$\alpha^2_\phi(W(h)) = e^{i\int h d\phi} W(h) \qquad h \in X^2,$$

and α^2_ϕ extends to an automorphism of $\mathcal{A}^2(\dot{\mathbb{R}})$. This automorphism is inner if and only if it is of the form $\mathrm{ad}W(h)$ where $h''' = \phi$, i.e. if and only if the first three moments (charges) of ϕ vanish:

$$Q_k(\phi) \doteq \int t^k d\phi(t) = 0, \qquad k = 0, 1, 2.$$

As a consequence, two such automorphisms are equivalent if all their charges coincide. We now consider the corresponding sectors, i.e. classes of automorphisms modulo inners.

Proposition 3.1. *The only covariant sector on $\mathcal{A}^2(\mathbb{R})$ in the above class is the identity sector.*

Proof. First we see the behavior of the automorphisms under translations:

$$Q_0(\beta_{T(t)}\alpha^2_\phi\beta_{T(-t)}) = Q_0(\alpha^2_\phi)$$
$$Q_1(\beta_{T(t)}\alpha^2_\phi\beta_{T(-t)}) = \int (x-t)d\phi(x) = Q_1(\alpha^2_\phi) - tQ_0(\alpha^2_\phi)$$
$$Q_2(\beta_{T(t)}\alpha^2_\phi\beta_{T(-t)}) = \int (x-t)^2 d\phi(x) = Q_2(\alpha^2_\phi) - 2tQ_1(\alpha^2_\phi) + t^2 Q_0(\alpha^2_\phi).$$

Then we compute the charges of automorphisms transformed with the ray inversion $r : x \to -1/x$:

$$Q_0(\beta_r\alpha^2_\phi\beta_r) = \int x^2 d\phi(x) = Q_2(\alpha^2_\phi)$$
$$Q_1(\beta_r\alpha^2_\phi\beta_r) = \int x^2(-1/x)d\phi(x) = -Q_1(\alpha^2_\phi)$$
$$Q_2(\beta_r\alpha^2_\phi\beta_r) = \int x^2(-1/x)^2 d\phi(x) = Q_0(\alpha^2_\phi).$$

From the first equations we derive that a translation covariant sector has $Q_0 = Q_1 = 0$, while covariance under ray inversion amounts to $Q_0 = Q_2$ and $Q_1 = 0$, from which the thesis easily follows. □

Remark . If we generalize the preceding construction to the case of n derivatives, thus obtaining sectors parameterized by $2n+1$ charges, the preceding proof generalizes as well, then showing that the identity sector is the only covariant sector in that case also.

Remark . As BMT sectors, also the sectors described above are additive in the sense that the vector charge (Q_0, Q_1, Q_2) of the composition of two sectors is just the sum of the two charges. Then the action of $\mathtt{PSL}(2,\mathbb{R})$ on these sectors gives a linear representation of this group on $\mathbb{R}^3$. The absence of covariant sectors means that the action is free and therefore has no one-dimensional representations, i.e. it is irreducible.

The local subalgebras for $\mathcal{A}^2(\mathbb{R})$ are the sub-algebras generated by the Weyl unitaries whose test functions are zero outside I. Then the representation of $\mathcal{A}^2(\mathbb{R})$ on the Fock space $e^{\mathcal{H}^2}$ is faithful when restricted to local algebras, i.e. $\mathcal{A}^2(I)$ may be seen as a weakly dense subalgebra of the second quantization algebra of the space $\mathcal{K}^2(I)$.

By a classical Sobolev embedding argument, the functions in $\mathcal{K}^2(I)$ are continuous, therefore the automorphisms $\alpha^2_\phi|_{\mathcal{A}^2(I)}$ uniquely extend to normal automorphisms of $\mathcal{R}^2(I)$, so that α^2_ϕ gives rise to an automorphism of the precosheaf $I \to \mathcal{R}^2(I)$.

Proposition 3.2. *All sectors described above may be localized in two points.*

Proof. First we observe that some of them may be localized even in one point, in fact the multiples of the δ_0 function give sectors with $Q_1 = Q_2 = 0$, while for the measures $cx^{-2}\delta_\infty(x)$ we have $Q_0 = 0$, $Q_1 = 0$, $Q_2 = c$, therefore we may restrict to the case $(Q_0, Q_1) \neq (0,0)$. Then we have to show that for any triple Q_i, $i = 0,1,2$, $(Q_0, Q_1) \neq (0,0)$, we may find a measure $\phi = \lambda\delta_a + \mu\delta_b$ with the given momenta for some $\lambda, \mu, a, b \in \mathbb{R}$, or equivalently solve the system

$$\begin{cases} Q_0 & = \lambda + \mu \\ Q_1 & = \lambda a + \mu b \\ Q_2 & = \lambda a^2 + \mu b^2 \end{cases},$$

whose solutions are obtained choosing a b for which $Q_1 - Q_0 b \neq 0$ and $Q_0 b^2 - 2Q_1 b + Q_2 \neq 0$ and then setting $a = (Q_2 - Q_1 b)(Q_1 - Q_0 b)^{-1}$, $\lambda = (Q_1 - Q_0 b)^2 (Q_0 b^2 - 2Q_1 b + Q_2)^{-1}$, $\mu = Q_0 - \lambda$. □

The preceding proposition constitutes indeed another proof that these sectors are non covariant, since the following theorem holds:

Proposition 3.3. *Let $I \to \mathcal{A}(I)$ a local conformal precosheaf on S^1. Then a covariant sector ρ with finite index which may be localized in two points is trivial.*

Proof. We may suppose that the two points are $\{0, \infty\}$. We first observe that ρ is indeed an automorphism because, if j is the antiunitary modular conjugation for $\mathcal{R}(0,\infty)$, $j\rho j$ is still localized in the same two points and then any intertwiner between $\rho j\rho j$ and the identity (which exists e.g. by [17]) is localized in these two points and is therefore a number by two-regularity.

The same argument shows that ρ commutes with the dilations, because the cocycle in the covariance equation for the dilation group is then trivial.

Now the state $\omega_0 \cdot \rho^{-1}$ is dilation invariant, and therefore, by a cluster argument, coincides with the vacuum state, which ends the proof. □

In this last part of the subsection we show the relation between the local and the global picture of the superselection sectors of the first-derivative theory or, more precisely, we show that all the sectors described in subsection 3.4 are (normal extensions of) the sectors of $\mathcal{A}^2(\dot{\mathbb{R}})$ described here.

Proposition 3.4. *The sectors $[\alpha_{\phi,\lambda}]$, $[\theta_c]$ $[\zeta_c]$ are of the form $[\alpha^2_\mu]$, μ measure.*

Proof. Given a non trivial sector $[\alpha_{\phi,\lambda}]$, we may choose a representative s.t. supp $\phi \subset I_0$, where I_0 is a given compact interval in $\mathbb{R}$. In order $[\alpha_{\phi,\lambda}]$ to be localized in I_0, we should have $\langle \phi, f\rangle_{I_0,\lambda} = 0$ for any f localized in I_0', i.e., since on I_0' f coincides with p_{f,I_0}, we get $\int \phi(x)(x-\lambda)dx = 0$, i.e. $\lambda = \lambda_0 = \frac{\int x\phi(x)}{\int \phi(x)}$ (the denominator does not vanish since $[\alpha_{\phi,\lambda}]$ is non trivial). Then, according to formula (29), one has

$$[\alpha_{\phi,\lambda}] = [\theta_{-Q(\lambda-\lambda_0)}] \cdot [\alpha_{\phi,\lambda}]$$

hence, since the class $\{\alpha_\mu,\ \mu \text{ measure}\}$ is closed under composition, it is enough to prove the statement for $[\alpha_{\phi,\lambda_0}]$, $[\theta_c]$ and $[\zeta_c]$.

As far as α_{ϕ,λ_0} is concerned, we have $\int \phi(x)p_{f,I}(\lambda_0)dx = \int \phi(x)p_{f,I}(x)dx$, therefore $\langle \phi, f\rangle_{I,\lambda_0}$ coincides with $\int \phi(x)(f(x) - p_{f,I}(x))dx$ and therefore, integrating by parts, with $-\int(\int(f - p_{f,I}))d\phi(x)$. Observing that $\int(f - p_{f,I})$ is exactly the representative in X^2 of $D^{-1}[f]$ which vanishes outside I we conclude that the automorphism α_{ϕ,λ_0} comes from the automorphism $\alpha^2_{-d\dot\phi}$ on $\mathcal{A}^2(\mathbb{R})$, and the relation among the charges is $Q_0(\alpha^2_{-d\phi}) = 0$, $Q_1(\alpha^2_{-d\phi}) = Q(\alpha_{\phi,\lambda})$, $Q_2(\alpha^2_{-d\phi}) = 2\lambda_0 Q(\alpha_{\phi,\lambda})$.

In the same way we may show that the "solitonic" automorphisms ζ_c in equation (31) come from the automorphisms α^2_μ on $\mathcal{A}^2(\mathbb{R})$ with $\mu = -c\delta_0$. Conjugating ζ_c with the ray inversion we see that the automorphisms θ_c in equation (30) localized at infinity come from the automorphisms α^2_μ on $\mathcal{A}^2(\dot{\mathbb{R}})$ with $\mu = cx^{-2}\delta_\infty$. □

Remark . In [9] it is shown that in any diffeomorphism covariant theory on S^1, in physics terms a theory with a stress-energy tensor, superselection sectors are covariant. It is well known that the usual way to associate a stress-energy tensor to the derivative of the $U(1)$-current formally leads to a conformal charge $c = \infty$. Our result then shows that the $U(1)$-current derivative theories indeed do not have a stress energy tensor.

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