Minimization of Conformally Invariant Energies in Homotopy Classes

Frank Duzaar Ernst Kuwert

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1 Introduction

Suppose $X \subset \mathbb{R}^N$ is compact and $\alpha \in \pi_n(X)$ is a given nontrivial homotopy class of maps $u : S^n \to X$. The purpose of this paper is to analyse the behavior of a minimizing sequence $u_k \in \alpha$ for a conformally invariant functional, e.g. the n-energy

$$E(u) = \int_{S^n} |du(\xi)|^n d\xi.$$

The minimizing sequence may not converge strongly due to the phenomenon of *separation of n-spheres* first described in the pioneering paper [14]. We formalize this by saying that a collection $(u^i)_{i\in I}$ of nonconstant maps is a *weak limit set* of a sequence u_k if there exist mutually unbounded (as $k \to \infty$) conformal automorphisms g^i_k of the sphere such that $u_k \circ g^i_k$ converges to u^i weakly for any $i \in I$. If in addition the sum of the energies of the u^i equals the limit of energy along the sequence, then we say the convergence is *in energy*. Concerning the relationship between the topological constraint and this notion of convergence we prove in section 4 a compactness and a continuity result:

Theorem 1. *If u_k is a sequence of homotopically nontrivial maps into X with equibounded n-energy, then after passing to a subsequence there exists a weak limit set $(u^i)_{i\in I}$ such that for any k the homotopy class of u_k belongs to the composition set of the homotopy classes of the u^i.*

Theorem 2. *If a sequence $u_k \in \alpha$ converges in energy to $(u^i)_{i\in I}$, then the collection $(\alpha^i)_{i\in I}$ of corresponding homotopy classes is a decomposition of α.*

The analysis of minimizing sequences follows readily from theorem 1. In fact, the energy is lower semicontinuous with respect to weak limit sets as shown in section 2, and the weak limit set provided by theorem 1 essentially satisfies the topological constraint; therefore a comparison argument (lemma 1 in section 4) yields the convergence in energy. In this sense our reasoning entirely follows the direct method in the Calculus of Variations. On the other hand, if convergence in energy is known for some (not necessarily minimizing) sequence, then theorem 2 shows that the topology is well-behaved.

The subject of this paper falls into the general theory of *concentrated compactness* as developed in[9], and the problem to analyse the fine structure of concentrations – i.e. to capture all the spheres which split off – has attracted great interest. As a common line of attack all authors first prove a crucial ε_0-compactness-alternative to the extent that if no more than ε_0 of the energy accumulates on a ball, then one gets strong subconvergence locally on that ball. Using this property one immediately splits off a first nontrivial limit by rescaling around points where the energy concentrates most, and produces a *remainder sequence* by cutting off that limit; note however that the ε_0-alternative is *not* automatically satisfied by the remainder. But the property of being a Palais-Smale sequence can be preserved in the cut-off and thus for these a proof of the ε_0-alternative is sufficient to finish the analysis. In particular this was achieved for the constant mean curvature equation [1], [2] Yamabe's equation [16] and the Yang-Mills equation [17], Unfortunately, the alternative is not known for Palais-Smale sequences for the n-energy even if $n = 2$.

In other situations, where an inductive argument for the remainder is not obvious, a more delicate analysis of the maps on neck domains seems to be required. This is carried out in [7, Chpt. 4] for saddle point sequences (which have been regularized in a balayage process in advance), in [12], [4] for certain time slices of the parabolic flow and in [11] for sequences of solutions; all these consider the Dirichlet integral in two dimensions. We also mention related work on pseudoholomorphic curves [6], [20]. The study of an *arbitrary* minimizing sequence performed here was also motivated by [5] where the authors employ the framework of Cartesian currents to investigate the loss of compactness from a very different point of view.

For minimizing sequences (and regular X) the ε_0-alternative holds but the remainder sequence may not belong to a fixed homotopy class; this difficulty immediately leads to the question answered by theorem 1. Our proof partially follows related arguments in the free boundary value problem for disk-type minimal surfaces [8]; in particular we use hyperbolic harmonic extension to the ball. Besides simplicity this allows us to obtain results without *any* smoothness or curvature hypotheses on X.

Among the functionals with power growth the n-energy is the only one for which existence holds without being obvious from Sobolev's embedding theorem [18]. An interesting phenomenon has recently been observed in [13] for the 3-energy of Hopf fibrations and n-harmonic maps have also played a role in quasi-conformal map theory. Although the geometric significance of the n-energy is certainly limited due to the large set of conformal structures on the domain, it may still be useful to have examples which are not just solutions to a Dirichlet problem.

In this paper we only consider the case when the domain is a round sphere. However, it poses no difficulties to take as domain a closed Riemannian manifold, at least if the target is sufficiently regular so that only finitely many concentration points can occur. The situation at a concentration point can be transplanted to the sphere where our method applies.

2 Conformally equivariant extension and notation

Let G be the group of orientation-preserving conformal automorphisms of $B^{n+1} = \{x \in \mathbb{R}^{n+1} : |x| < 1\}$ where $n \geq 2$; G also acts conformally on $S^n = \{\xi \in \mathbb{R}^{n+1} : |\xi| = 1\}$ by restriction. The ball is a model of hyperbolic space $\mathbb{H}^{n+1}$ and G is its oriented isometry group. We use this to define the pseudometric

$$d(g, h) := \mathrm{dist}_{\mathbb{H}^{n+1}}(g(0), h(0)) \geq 0\,; \tag{1}$$

for g, h_1, $h_2 \in G$ and T, $S \in SO(n+1)$ we have $d(gh_1, gh_2) = d(h_1, h_2) = d(h_1S, h_2T)$. A sequence $g_k \in G$ is bounded (divergent) if and and only if $d(g_k, \mathrm{id})$ is bounded (divergent). For $x \in B^{n+1} \setminus \{0\}$ we have the orientation-reversing automorphism

$$\begin{aligned} \sigma_x y &:= -x^* + \left(\frac{1}{|x|^2} - 1\right) \frac{y + x^*}{|y + x^*|^2} \quad \text{where } x^* = \tfrac{x}{|x|^2}, \\ \tau_x y &:= y - 2\langle \frac{x}{|x|}\,,\, y\rangle \frac{x}{|x|}\,. \end{aligned}$$

We obtain a smooth map $h \colon B^{n+1} \to G$ with the property $h_x(0) = x$ by letting

$$h_x := \begin{cases} \tau_x \circ \sigma_x & \text{for } x \neq 0, \\ \mathrm{id} & \text{for } x = 0. \end{cases} \tag{2}$$

For $\xi \in S^n$ the curve $h_\xi(s) := h_{(\tanh \frac{s}{2})\xi}$ is a one-parameter subgroup satisfying $d(h_\xi(s), \mathrm{id}) = |s|$; the Jacobian of $h_\xi(s)$ on S^n is given by the formula

$$J_n h_\xi(s)(\eta) = \frac{1}{(\cosh s + \sinh s\langle \xi, \eta\rangle)^n}\,. \tag{3}$$

For $\xi \in S^n$ we let $\xi^\perp := \{y \in \mathbb{R}^{n+1} : \langle y, \xi\rangle = 0\}$, $S^{n-1}(\xi) := S^n \cap \xi^\perp$ and introduce the stereographic projections

$$\begin{cases} St_\xi(\eta) = \dfrac{\eta - \langle \xi, \eta\rangle \xi}{1 - \langle \xi, \eta\rangle} \in \xi^\perp\,, \\ St_\xi^{-1}(y) = \dfrac{2y}{1 + |y|^2} - \dfrac{1 - |y|^2}{1 + |y|^2}\xi\,. \end{cases} \tag{4}$$

We let $D_r(\xi) = \{y \in \xi^\perp : |y| < r\}$ and $B_r(\xi) = St_{-\xi}^{-1}(D_r(\xi))$; note that $B_r(\xi)$ is the geodesic ball around ξ of radius $\vartheta = 2\arctan r$. For $\eta \in S^{n-1}(\xi)$ and $r > 0$ we get

$$(J_n St_{-\xi}^{-1})(r\eta) = \left(\frac{2}{1 + r^2}\right)^n\,. \tag{5}$$

Finally we observe the "law of reciprocal radii"

$$|St_\xi(\eta)||St_{-\xi}(\eta)| = 1 \quad \text{for } \eta \notin \{\xi, -\xi\}. \tag{6}$$

A main tool in this paper is the extension operator

$$(\mathcal{E}u)(x) := \fint_{S^n} u(h_x\omega)\, d\omega\,, \tag{7}$$

where $u \in L^1(S^n, \mathbb{R}^N)$ and "$\fint \ldots$" denotes the average. Of course $\mathcal{E}$ is nothing but the Poisson kernel on $\mathbb{H}^{n+1}$; for us it is important because of conformal invariance, i.e.

$$\mathcal{E}(u \circ g) = (\mathcal{E}u) \circ g \quad \text{for all } g \in G. \tag{8}$$

For $0 \leq s \leq \infty$ we view the extension as a homotopy to the mean value by putting $(\mathcal{E}_s u)(\xi) := (\mathcal{E}u)(\tanh \frac{s}{2}\xi)$. From (3) and the equivariance we obtain

$$(\mathcal{E}_s u)(\xi) = \fint_{S^n} \frac{u(\omega)\, d\omega}{(\cosh s - \sinh s\langle \omega, \xi\rangle)^n}\,; \tag{9}$$

$$d(\mathcal{E}_s u)(\xi) = \fint_{S^n} \frac{du(\omega) \circ (\langle \omega, \xi\rangle \mathrm{id} - \omega \otimes \xi)\ d\omega}{(\cosh s - \sinh s\langle \omega, \xi\rangle)^n}\,; \tag{10}$$

$$(\frac{\partial}{\partial s}(\mathcal{E}_s u))(\xi) = \fint_{S^n} \frac{du(\omega)(\xi - \langle \omega, \xi\rangle \omega)\, d\omega}{(\cosh s - \sinh s\langle \omega, \xi\rangle)^n}\,. \tag{11}$$

The operators $\mathcal{E}_s$ approximate the identity as $s \to \infty$. We shall only need standard cases which are stated without proof; here the operator norm $\|du(\xi)\| = sup\{|du(\xi)\eta| : \eta \in S^{n-1}(\xi)\}$ is used.

Lemma 2.

(12) *For $u \in C^0(S^n, \mathbb{R}^N)$ the image of $\mathcal{E}u$ is contained in the convex hull of the image of u and $\mathcal{E}u$ is continuous on the closed ball $\overline{B^{n+1}}$.*

(13) *If $u \in L^p(S^n, \mathbb{R}^N)$ where $1 \leq p < \infty$, then $\|\mathcal{E}_s u\|_{L^p} \leq \|u\|_{L^p}$ and $\|u - \mathcal{E}_s u\|_{L^p(S^n)} \to 0$ as $s \to \infty$.*

(14) *For $u \in C^1(S^n, \mathbb{R}^N)$ we have the pointwise inequality*

$$\|d(\mathcal{E}_s u)\|(\xi) \leq \mathcal{E}_s(\|du\|)(\xi)$$

and $\|du - d(\mathcal{E}_s u)\|_{C^0} \to 0$ as $s \to \infty$.

(15) *If $u \in H^{1,p}(S^n, \mathbb{R}^N)$ where $1 \leq p < \infty$ then $\|d(\mathcal{E}_s u)\|_{L^p} \leq \|du\|_{L^p}$ and $\|du - d(\mathcal{E}_s u)\|_{L^p} \to 0$ as $s \to \infty$.*

The Poincaré inequlity on S^n is often used in this paper and therefore we include a proof which yields an explicit constant.

Lemma 3 (Poincaré inequality). *For $1 \leq p < \infty$ and $u \in H^{1,p}(S^n, \mathbb{R}^N)$ we have*

$$\left(\fint_{S^n} \left|u(\xi) - \fint_{S^n} u(\omega)\, d\omega\right|^p d\xi\right)^{\frac{1}{p}} \leq \frac{\pi}{2}\left(\fint_{S^n} \|du(\xi)\|^p\, d\xi\right)^{\frac{1}{p}}. \tag{16}$$

Proof. A computation shows

$$\int_0^\infty \fint_{S^n} \frac{\sqrt{1 - \langle \omega, \xi\rangle^2}}{(\cosh s - \sinh s\langle \omega, \xi\rangle)^n}\, d\omega\, ds = \frac{\pi}{2}\,.$$

From Jensen's inequality we conclude

$$\fint_{S^n} \left|u(\xi) - \fint_{S^n} u(\omega)\, d\omega\right|^p d\xi$$

$$
\begin{aligned}
&= \fint_{S^n}\left|\int_0^\infty \fint_{S^n}\frac{du(\omega)(\xi-\langle\omega,\xi\rangle\omega)}{(\cosh s-\sinh s\langle\omega,\xi\rangle)^n}\,d\omega\,ds\right|^p d\xi \\
&\le \left(\frac{\pi}{2}\right)^p \fint_{S^n}\left(\int_0^\infty \fint_{S^n}\|du(\omega)\|\frac{\sqrt{1-\langle\omega,\xi\rangle^2}}{(\cosh s-\sinh s\langle\omega,\xi\rangle)^n}\frac{d\omega\,ds}{\pi/2}\right)^p d\xi \\
&\le \left(\frac{\pi}{2}\right)^p \fint_{S^n}\int_0^\infty \fint_{S^n}\|du(\omega)\|^p\frac{\sqrt{1-\langle\omega,\xi\rangle^2}}{(\cosh s-\sinh s\langle\omega,\xi\rangle)^n}\frac{d\omega\,ds}{\pi/2}d\xi \\
&= \left(\frac{\pi}{2}\right)^p \fint_{S^n}\|du(\omega)\|^p d\omega\,.
\end{aligned}
$$

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3 Weak limits and lower semicontinuity

We consider functionals of the type

$$F(u,\Omega)=\int_{S^n} f(u(\xi),du(\xi))\,d\xi\,,$$

where $\Omega\subset S^n$ is open, $f:\mathbb{R}^N\times(T^*S^n\otimes\mathbb{R}^N)\to\mathbb{R}_0^+$ is continuous and satisfies the following conditions:

(1) There exist constants $0<\lambda\le\Lambda<\infty$ such that $\lambda\|A\|^n\le f(z,A)\le\Lambda\|A\|^n$ for all z, A;

(2) $f(z,A)$ is convex in $A\in T^*_\xi S^n\otimes\mathbb{R}^N$ for any z, ξ;

(3) if $C:T_\xi S^n\to T_\eta S^n$ is linear, preserves orientation and is conformal (i.e. $C^*C=|J_nC|^{2/n}\,\mathrm{id}$), then $f(z,AC)=f(z,A)|J_nC|$ for all z, $A\in T^*_\eta S^n\otimes\mathbb{R}^N$.

Condition (3) implies the conformal invariance $F(u\circ g,\Omega)=F(u,g\Omega)$ for all $g\in G$. The main example is the n-energy $E(u)=\int_{S^n}|du(\xi)|^n\,d\xi$, where $|A|^2=\mathrm{tr}(A^*A)$ and $\|A\|\le|A|\le\sqrt{n}\|A\|$; we shall also use $E_\infty(u)=\int_{S^n}\|du(\xi)\|^n\,d\xi$.

If u_k is a minimizing sequence for F in some homotopy class, then so is $u_k\circ h_k$ for any choice of $h_k\in G$; however the two sequences may have nonconstant weak limits which are not related by any $g\in G$. In other words, if one divides out the action of G on $H^{1,n}(S^n,\mathbb{R}^N)$, the weak topology on the quotient will have nonunique limits and not be Hausdorff. We formalize this in the following

Definition. An indexed set $(u^i)_{i\in I}\subset H^{1,n}(S^n,\mathbb{R}^N)$ is called *a weak limit set* of a sequence $(u_k)_{k\in\mathbb{N}}\subset H^{1,n}(S^n,\mathbb{R}^N)$ if the following holds:

(4) for all $i\in I$ the map u^i is nonconstant;

(5) there exist $h^i_k\in G$ such that

  (a) $u_k\circ h^i_k\to u^i$ weakly in $H^{1,n}(S^n,\mathbb{R}^N)$;

  (b) $d(h^i_k,h^j_k)\to\infty$ as $k\to\infty$ for all i, $j\in I$, $i\ne j$.

We observe that if $(u^i)_{i\in I}$ is a weak limit set of $(u_k)_{k\in\mathbb{N}}$ with h^i_k chosen according to (5), then $(v^i=u^i\circ h^i)_{i\in I}$ is a weak limit set of $(v_k=u_k\circ h_k)_{k\in\mathbb{N}}$ for any choice of h^i,

$h_k \in G$. In fact we let $g^i_k := h_k^{-1} h^i_k h^i$ and have $v_k \circ g^i_k = (u_k \circ h^i_k) \circ h^i \to u^i \circ h^i = v^i$ weakly; furthermore $d(g^i_k, g^j_k) = d(h^i_k h^i, h^j_k h^j) \geq d(h^i_k, h^j_k) - d(h^i, \mathrm{id}) - d(h^j, \mathrm{id}) \to \infty$ as $k \to \infty$ for $i \neq j$. This observation is used in the proof of the lower semicontinuity statement below; there we also need the following version of the Courant-Lebesgue lemma.

Lemma 2. *Let $v \in H^{1,n}(B^n, \mathbb{R}^N)$ and $\varrho \in (0,1)$. There exists a set $I \subset (\varrho^2, \varrho)$ of positive measure, such that for any $r \in I$ the map $v_r\colon S^{n-1} \to \mathbb{R}^N$, $v_r(\omega) := v(r\omega)$, satisfies the inequalities*

$$\int_{S^{n-1}} \|dv_r\|^n \, d\omega \;\leq\; \frac{2}{\log 1/\varrho} \int_{B^n_\varrho(0)} \|dv\|^n \, dy\,; \tag{6}$$

$$\int_{S^{n-1}} |v_r|^n \, d\omega \;\leq\; \frac{2}{r^n \log 1/\varrho} \int_{B^n_\varrho(0)} |v|^n \, dy\,. \tag{7}$$

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Theorem 1 (lower semicontinuity). *Let $(u^i)_{i\in I}$ be a weak limit set of a sequence $(u_k)_{k\in\mathbb{N}}$ and let $F(\cdot)$ satisfy (1), (2), and (3). Then I is countable and in fact*

$$\sum_{i\in I} F(U^i) \leq \liminf_{k\to\infty} F(U_k)\,. \tag{8}$$

Proof. It is sufficient to prove (8) in the case $m := \operatorname{card} I < \infty$. For $m = 1$ the statement is immediate; we now assume inductively that (8) holds for any weak limit set of cardinality at most m of an arbitrary sequence. Given the weak limit set $(u^i)_{i=1}^{m+1}$ of the sequence $(u_k)_{k\in\mathbb{N}}$ and automorphisms h^i_k according to (5), we may assume by passing to a subsequence that

$$F(u_k) \to F_* < \infty\,, \tag{9}$$

$$E_\infty(u_k) \leq E < \infty \quad \text{for all } k, \tag{10}$$

$$\int_{S^n} |u_k \circ h^i_k(\omega)| \, d\omega \leq L^i < \infty \quad \text{for all } k,\ 1 \leq i \leq m+1. \tag{11}$$

By the observation above we can arrange that $h^{m+1}_k \equiv \mathrm{id}$. We have $h^i_k = h_{x^i_k} \circ T^i_k$ where $x^i_k = (\tanh \frac{s^i_k}{2}) \xi^i_k \in B^{n+1}$ and $T^i_k \in SO(n+1)$ for $1 \leq i \leq m$. In fact we can assume that $T^i_k \equiv \mathrm{id}$ for all i, k: otherwise we select a subsequence such that $T^i_k \to T^i \in SO(n+1)$ for $1 \leq i \leq m$. We then have $u_k \circ h_{x^i_k} = u_k \circ h^i_k \circ (T^i_k)^{-1} \to u^i \circ (T^i)^{-1}$ weakly, $d(h_{x^i_k}, h_{x^j_k}) = d(h^i_k, h^j_k) \to \infty$ for $i \neq j$, and $F(u^i \circ (T^i)^{-1}) = F(u^i)$ for all i. Thus w.l.o.g. $h^i_k = h_{x^i_k}$, and by selecting a further subsequence

$$0 < s^i_k = d(h^i_k, \mathrm{id}) = d(h^i_k, h^{m+1}_k) \to \infty \quad \text{for } 1 \leq i \leq m, \tag{12}$$

$$\xi^i_k \to \xi^i \in S^n \quad \text{for } 1 \leq i \leq m. \tag{13}$$

Let $l \in \mathbb{N}$ and note that $\langle \omega, \xi^i \rangle \leq \frac{l^4-1}{l^4+1} < 1$ for $\omega \in S^n \setminus B_{1/l^2}(\xi^i)$. Choose $k(l) \geq l$ minimal such that for $1 \leq i \leq m$ and all $k \geq k(l)$ we have $h^i_k(B_{l^2}(\xi^i)) \subset B_{1/l^2}(\xi^i)$, and furthermore $\cosh s^i_k - \sinh s^i_k \langle \omega, \xi^i_k \rangle \geq 1$ for $\omega \in S^n \setminus B_{1/l^2}(\xi^i)$. We then pass to the subsequence $(u_{k(l)})_{l\in\mathbb{N}}$; after renumbering we have

$$h^i_k(B_{k^2}(\xi^i)) \subset B_{1/k^2}(\xi^i) \quad \text{for } k \in \mathbb{N},\ 1 \leq i \leq m; \tag{14}$$

$$\cosh s^i_k - \sinh s^i_k \langle \omega, \xi^i_k \rangle \geq 1 \quad \text{for } \omega \in S^n \setminus B_{1/k^2}(\xi^i). \tag{15}$$

Let $J \subset \{1,\dots,m\}$ be such that $(\xi^j)_{j\in J}$ are pairwise different and $\bigcup_{j\in J}\{\xi^j\} = \bigcup_{i=1}^m \{\xi^i\}$. For $j \in J$ we have local representations

$$v_k^j := u_k \circ St_{-\xi^j}^{-1} \colon (\xi^j)^\perp \to \mathbb{R}^N . \tag{16}$$

We compute from (10) and (11)

$$\begin{aligned} \int_{D_{1/k}(\xi^j)} \|dv_k^j(y)\|^n\, dy &= \int_{B_{1/k}(\xi^j)} \|du_k(\omega)\|^n\, d\omega \le E\,, \\ \int_{D_{1/k}(\xi^j)} |v_k^j(y)|^n\, dy &\le \int_{B_{1/k}(\xi^j)} |u_k(\omega)|^n\, d\omega \le L^{m+1} =: L\,. \end{aligned}$$

From lemma 1 we obtain the existence of a radius $r_k^j \in (\frac{1}{k^2}, \frac{1}{k})$ such that

$$\int_{S^{n-1}(\xi^j)} \|d(v_k^j(r_k^j, \,\bullet\,))\|^n\, d\omega \le \frac{2E}{\log k}\,; \tag{17}$$

$$\int_{S^{n-1}(\xi^j)} |v_k^j(r_k^j,\omega)|^n\, d\omega \le \frac{2L}{(r_k^j)^n \log k}\,. \tag{18}$$

We put $p_k^j := ⨍_{S^{n-1}(\xi^j)} v_k^j(r_k^j,\omega)\, d\omega$, and have

$$|p_k^j|^n \le \frac{2L}{\omega_{n-1}(r_k^j)^n \log k}\,. \tag{19}$$

For $r_k^j \le r \le 2r_k^j$ and $\omega \in S^{n-1}(\xi^j)$ we define

$$w_k^j(r,\omega) := \left(\frac{r}{r_k^j} - 1\right) p_k^j + \left(2 - \frac{r}{r_k^j}\right) v_k^j(r_k^j,\omega)\,.$$

Using the Poincaré inequality (2.16) we can estimate

$$\int_{r_k^j}^{2r_k^j} r^{n-1} \int_{S^{n-1}(\xi^j)} \left|\frac{\partial w_k^j}{\partial r}\right|^n (r,\omega)\, d\omega\, dr \le \frac{2\pi^n E}{n \log k}\,.$$

For the tangential derivatives we have

$$\int_{r_k^j}^{2r_k^j} \frac{1}{r} \int_{S^{n-1}(\xi^j)} \|d(w_k^j(r, \,\bullet\,))\|^n\, d\omega\, dr \le \frac{2E}{\log k}\,.$$

Combining the two estimates we obtain

$$E_\infty(w_k^j, \{r_k^j \le r \le 2r_k^j\}) \le 2^{n-1}\left(\frac{\pi^n}{n} + 1\right)\frac{2E}{\log k} \le \frac{2^{n+1}\pi^n}{n}\frac{E}{\log k}\,. \tag{20}$$

We now define for $j \in J$

$$u_k^j(\omega) := \begin{cases} u_k(\omega) & \text{for } \omega \in B_{r_k^j}(\xi^j), \\ w_k^j(St_{-\xi^j}(\omega)) & \text{for } \omega \in B_{2r_k^j}(\xi^j) \setminus B_{r_k^j}(\xi^j), \\ p_k^j & \text{else.} \end{cases}$$

Claim. For each $j \in J$ the set $(u^i)_{\{i\in I:\xi^i=\xi^j\}}$ is a weak limit set of the sequence $(u_k^j)_{k\in\mathbb{N}}$.

Proof of the claim. We show that if $i \in \{1,\dots,m\}$ satisfies $\xi^i = \xi^j$, then the sequence $(u_k^j \circ h_k^i)_{k\in\mathbb{N}}$ converges weakly to u^i; for this we first prove that the sequence is bounded in $H^{1,n}(S^n,\mathbb{R}^N)$. By (10) and (20) we have

$$E(u_k^j \circ h_k^i) = E(u_k, B_{r_k^j}(\xi^j)) + E(w_k^j, \{r_k^j \le r \le 2r_k^j\}) \le E + \frac{2^{n+1}\pi^n}{n\log k}E\,.$$

Next we observe $u_k^j \circ h_k^i = u_k \circ h_k^i$ on $(h_k^i)^{-1}B_{r_k^j}(\xi^j)$, whence by estimate (11)

$$\int_{(h_k^i)^{-1}B_{r_k^j}(\xi^j)} |u_k^j \circ h_k^i(\omega)|^n\, d\omega \le L^i < \infty \quad \text{for all } k.$$

We further estimate

$$\begin{aligned}
&\int_{(h_k^i)^{-1}(B_{2r_k^j}(\xi^j)\setminus B_{r_k^j}(\xi^j))} |u_k^j \circ h_k^i(\zeta)|^n\, d\zeta \\
&= \int_{B_{2r_k^j}(\xi^j)\setminus B_{r_k^j}(\xi^j)} |u_k^j(\omega)|^n \frac{d\omega}{(\cosh s_k^i - \sinh s_k^i\langle\omega,\xi_k^i\rangle)^n} \\
&\le \int_{B_{2r_k^j}(\xi^j)\setminus B_{r_k^j}(\xi^j)} |u_k^j(\omega)|^n\, d\omega \quad (\text{using } \xi^j=\xi^i,\ r_k^j \ge \tfrac{1}{k^2}, \text{ and } (15)) \\
&\le 2^{2n-1}\int_{r_k^j}^{2r_k^j} r^{n-1}\int_{S^{n-1}(\xi^j)} \left(|p_k^j|^n + |v_k^j(r_k^j,\eta)|^n\right) d\eta dr \\
&\le 2^{2n-1}\frac{4L}{(r_k^j)^n\log k}\frac{2^n-1}{n}(r_k^j)^n \quad (\text{by } (18),\ (19)).
\end{aligned}$$

Therefore

$$\int_{(h_k^i)^{-1}(B_{2r_k^j}(\xi^j)\setminus B_{r_k^j}(\xi^j))} |u_k^j \circ h_k^i(\omega)|^n\, d\omega \le \frac{2^{3n+1}L}{n\log k}\,.$$

Now observe from (14)

$$B_{k^2}(\xi^j) \subset (h_k^i)^{-1}B_{1/k^2}(\xi^j) \subset (h_k^i)^{-1}(B_{r_k^j}(\xi^j)) \subset (h_k^j)^{-1}(B_{2r_k^j}(\xi^j))\,,$$

whence by the law of reciprocal radii (2.6)

$$(h_k^i)^{-1}(S^n \setminus B_{2r_k^j}(\xi^j)) = S^n \setminus ((h_k^i)^{-1}B_{2r_k^j}(\xi^j)) \subset S^n \setminus B_{k^2}(\xi^j) = B_{1/k^2}(-\xi^j)\,.$$

Thus by (19)

$$\begin{aligned}
\int_{(h_k^i)^{-1}(S^n\setminus B_{2r_k^j}(\xi^j))} |u_k^j \circ h_k^i(\omega)|^n\, d\omega &\le |p_k^j|^n\mathcal{H}^n(B_{1/k^2}(-\xi^j)) \\
&\le \frac{2L}{\omega_{n-1}(r_k^j)^n\log k}\frac{2^n\omega_{n-1}}{nk^{2n}} \\
&\le \frac{2^{n+1}L}{n\log k} \quad (\text{since } r_k^j \ge \tfrac{1}{k^2}).
\end{aligned}$$

This proves that $(u_k^j\circ h_k^i)_{k\in\mathbb{N}}$ is bounded in $H^{1,n}(S^n,\mathbb{R}^N)$. On the other hand, since $u_k^j\circ h_k^i = u_k\circ h_k^i$ on $(h_k^i)^{-1}(B_{r_k^j}(\xi^j)) \supset B_{k^2}(\xi^j)$, it follows that the sequence converges weakly to u^i locally in $S^n\setminus\{-\xi^j\}$ and hence on S^n, and the claim is proved.

Since $\operatorname{card}\{i:\xi^i=\xi^j\}\le m$ for all $j\in J$ we can apply the inductive hypothesis to the sequence $(u_k^j)_{k\in\mathbb{N}}$ to obtain

$$\sum_{i\in I\,:\,\xi^i=\xi^j} F(U^i) \le \liminf_{k\to\infty} F(u_k^j) \quad \text{for all } j\in J. \tag{21}$$

Let $\Omega_k := S^n\setminus\bigcup_{j\in J}\overline{B_{2r_k^j}(\xi^j)}$. Then the lower semicontinuity and positivity of F imply

$$F(u^{m+1}) \le \liminf_{k\to\infty} F(u_k,\Omega_k)\,. \tag{22}$$

For sufficiently large k the balls $\{B_{2r_k^j}(\xi^j) : j\in J\}$ are pairwise disjoint; therefore by (20)

$$\begin{aligned} F(u_k) &\ge F(u_k,\Omega_k) + \sum_{j\in J} F(u_k^j) - \sum_{j\in J} F(u_k^j, B_{2r_k^j}(\xi^j)\setminus B_{r_k^j}(\xi^j)) \\ &\ge F(u_k,\Omega_k) + \sum_{j\in J} F(u_k^j) - \frac{m\Lambda 2^{n+1}\pi^n}{n}\frac{E}{\log k}\,. \end{aligned}$$

We thus conclude

$$\begin{aligned} \sum_{i=1}^{m+1} F(U^i) &= F(U^{m+1}) + \sum_{j\in J}\sum_{i\,:\,\xi^i=\xi^j} F(U^i) \\ &\le \liminf_{k\to\infty} F(u_k,\Omega_k) + \sum_{j\in J}\liminf_{k\to\infty} F(u_k^j) \\ &\le \liminf_{k\to\infty} F(U_k)\,. \end{aligned}$$

■

Of course one is interested in weak limit sets having a certain maximality property, the most interesting being that the energies add up; in this case we now prove uniqueness of the weak limit set.

Definition. Let $(u^i)_{i\in I}$ be a weak limit set of $(u_k)_{k\in\mathbb{N}}$. The sequence u_k *converges in F to $(u^i)_{i\in I}$* iff

$$\sum_{i\in I} F(U^i) = \lim_{k\to\infty} F(U_k)\,.$$

Theorem 2 (uniqueness of F-limits). *Suppose that $(u_k)_{k\in\mathbb{N}}$ converges in F to $(u^i)_{i\in I}$. Then for any weak limit set $(v^j)_{j\in J}$ of $(u_k)_{k\in\mathbb{N}}$ there exists an injection $\varphi\colon J\to I$ such that for any $j\in J$ the maps $u^{\varphi(j)}$ and v^j are equal up to conformal reparametrization by an element of G.*

Proof. Let h_k^i, $g_k^j\in G$ be chosen as in (5) for $(u^i)_{i\in I}$ and $(v^j)_{j\in J}$ respectively. For any $j\in J$ and any subsequence $\mathfrak{K}\subset\mathbb{N}$ we consider

$$I_{\mathfrak{K}}(j) := \left\{ i\in I : (d(h_k^i,g_k^j))_{k\in\mathfrak{K}} \text{ has a bounded subsequence}\right\}\,.$$

For $i\in I_{\mathfrak{K}}(j)$ there is a subsequence $\mathfrak{K}'\subset\mathfrak{K}$ such that $(g_k^j)^{-1}h_k^i$ converges to some $f\in G$ as $k\in\mathfrak{K}'$ goes to infinity; from this we get

$$u^i = \lim_{k\to\infty} u_k\circ h_k^i = \lim_{\mathfrak{K}'\ni k\to\infty} (u_k\circ g_k^j)\circ(g_k^j)^{-1}h_k^i = v^j\circ f\,.$$

Thus we have the implication

$$(23) \qquad i \in I_{\mathfrak{K}}(j) \quad \Rightarrow \quad u^i = v^j \ \text{ (modulo } G).$$

We further claim that

$$(24) \qquad I_{\mathfrak{K}}(j) \text{ is a nonempty, finite set.}$$

In fact, if $I_{\mathfrak{K}}(j)$ were empty, then $((u^i)_{i\in I}, v^j)$ is again a weak limit set of $(u_k)_{k\in\mathfrak{K}}$ and theorem 1 implies $F(v^j) = 0$ which contradicts the nonconstancy of v^j. The finiteness of $I_{\mathfrak{K}}(j)$ follows from (23) and theorem 1 again. Next we observe that

$$(25) \qquad (d(h_k^i, g_k^j))_{k\in\mathfrak{K}} \text{ bounded } \Rightarrow \begin{cases} \lim\limits_{\mathfrak{K}\ni k\to\infty} d(h_k^{i'}, g_k^j) = \infty & \text{for } i' \neq i, \\ \lim\limits_{\mathfrak{K}\ni k\to\infty} d(h_k^i, g_k^{j'}) = \infty & \text{for } j' \neq j. \end{cases}$$

From (24) and (25) we obtain

$$(26) \qquad (d(h_k^i, g_k^j))_{k\in\mathfrak{K}} \text{ bounded } \quad \Leftrightarrow \quad I_{\mathfrak{K}}(j) = \{i\}\,.$$

By theorem 1 we can assume $J = \{j \in \mathbb{N} : 1 \leq j < m+1\}$ where $m \in \mathbb{N} \cup \{\infty\}$. Suppose inductively we have constructed an injection $\varphi\colon \{1, \ldots, j\} \to I$ and a subsequence $\mathfrak{K}_j$ such that

$$(27) \qquad \mathbb{N} =: \mathfrak{K}_0 \supset \mathfrak{K}_1 \supset \ldots \supset \mathfrak{K}_j\,, \quad (d(h_k^{\varphi(l)}, g_k^l))_{k\in\mathfrak{K}_j} \text{ is bounded for } 1 \leq l \leq j.$$

By (24) we have $I_{\mathfrak{K}_j}(j+1) = \{i_1, \ldots, i_r\}$ for some $r \in \mathbb{N}$. Now if $r = 1$ then we let $\varphi(j+1) = i_1$ and $\mathfrak{K}_{j+1} = \mathfrak{K}_j$. Otherwise by (26) we can pass to a subsequence $\mathfrak{K}_j'$ such that $d(h_k^{i_1}, g_k^{j+1}) \to \infty$ as $k \in \mathfrak{K}_j'$ goes to infinity; therefore $I_{\mathfrak{K}_j'}(j+1) \subset \{i_2, \ldots, i_r\}$. Repeating this argument finitely many times we find a subsequence $\mathfrak{K}_{j+1} \subset \mathfrak{K}_j$ with $I_{\mathfrak{K}_{j+1}}(j+1) = \{i_\nu\}$ for some fixed $\nu \in \{1, \ldots, r\}$ and put $\varphi(j+1) := i_\nu$. Then (27) follows from (26) and and $\varphi(j+1) \notin \varphi\{1, \ldots, j\}$ by (25). This finishes the inductive construction of φ and proves the theorem. ∎

4 Homotopy classes of finite energy maps

In this section we recall (with slight modifications) the definition of homotopy classes for maps $u \in H^{1,n}(S^n, X)$; we assume that $X \subset \mathbb{R}^N$ is compact and $U_\delta(X) = \{p \in \mathbb{R}^N : \operatorname{dist}(p, X) < \delta\}$ is connected for any $\delta > 0$. The assignment of topological data to Sobolev maps is dicussed in detail in [18], [19]; the case of a critical embedding is also considered in [3]. The main point for us is that the smoothings $\mathcal{E}_s u$ are uniformly close to X for s sufficiently large.

For $u \in C^0(S^n, \mathbb{R}^N)$ we let $d(u, X) = \max\{\operatorname{dist}(u(\xi), X) : \xi \in S^n\}$ and introduce the relation

$$(1) \qquad u \overset{\delta}{\sim} v \ :\Leftrightarrow u,\, v \text{ are freely homotopic in } U_\delta(X).$$

We then consider the sequence space

$$(2) \qquad \Pi_n(X) := \{\mathbf{u} = (u_k)_{k\in\mathbb{N}} : u_k \in C^0(S^n, \mathbb{R}^N), \text{ for any } \delta > 0 \text{ there exists } k_\delta \in \mathbb{N} \text{ such that } u_k \overset{\delta}{\sim} u_l \text{ for } k,\, l \geq k_\delta\}.$$

Dividing by the obvious equivalence relation

$$\mathbf{u} \sim \mathbf{v} \ :\Leftrightarrow \ u_k \overset{\delta}{\sim} v_l \ \text{ for } k,\, l \text{ sufficiently large}$$

we obtain $\hat{\pi}_n(X) := \Pi_n(X)/\sim = \lim_{\delta \to 0} \pi_n(U_\delta(X))$. We have well-defined maps

$$\begin{aligned} i\colon \pi_n(X) &\to \hat{\pi}_n(X)\,, \quad \{u\} \mapsto \{(u_k \equiv u)_{k\in\mathbb{N}}\}\,; \\ i_\delta\colon \hat{\pi}_n(X) &\to \pi_n(U_\delta(X))\,, \quad \{\mathbf{u}\} \mapsto \{u_{k_\delta}\}\,. \end{aligned} \tag{3}$$

For α, $\beta \in \hat{\pi}_n(X)$ we can measure the distance of α, β and the size of α by putting

$$\begin{aligned} \operatorname{dist}(\alpha,\beta) &:= \inf\{\delta > 0 : i_\delta(\alpha) = i_\delta(\beta)\}\,, \\ \|\alpha\| &:= \inf\{\delta > 0 : i_\delta(\alpha) \text{ is trivial}\}\,. \end{aligned} \tag{4}$$

It is not difficult to see that $(\hat{\pi}_n(X), \operatorname{dist}(\bullet,\bullet))$ is a complete metric space and totally diconnected in the sense that any $\gamma \in C^0([0,1], \hat{\pi}_n(X))$ must be constant (see [8]). In a similar fashion we let

$$\|\mathbf{u}\| := \inf\{\delta > 0 : \text{there exists } k_\delta \in \mathbb{N} \text{ such that } u_k \text{ is contractible in } U_\delta(X) \text{ for all } k \geq k_\delta\} \tag{5}$$

for any sequence $\mathbf{u} = (u_k)_{k\in\mathbb{N}} \subset C^0(S^n, \mathbb{R}^N)$ satisfying $d(u_k, X) \to 0$; if $\|\mathbf{u}\| > 0$ then $\mathbf{u}$ will be called *nontrivial*. Replacing $\pi_n(X)$ by $\hat{\pi}_n(X)$ allows us to proceed without regularity hypothesis on X. The two notions coincide under a mild regularity assumption:

Lemma 2. *Suppose X admits a continuous retraction $R_{\delta_0}\colon U_{\delta_0}(X) \to X$, i.e. $R_{\delta_0}|_X = \mathrm{id}_X$. Then $i\colon \pi_n(X) \to \hat{\pi}_n(X)$ is an isomorphism and* $\operatorname{dist}(\alpha,\beta) > 0$ *(*$\|\alpha\| > 0$, $\|\mathbf{u}\| > 0$*) implies* $\operatorname{dist}(\alpha,\beta) \geq \delta_0$ *(*$\|\alpha\| \geq \delta_0$, $\|\mathbf{u}\| \geq \delta_0$*).*

The simple proof is omitted. In a way similar to the above we define

$$\mathcal{H}(X) := \{\mathbf{u} = (u_k)_{k\in\mathbb{N}} : u_k \in C^0(S^n, \mathbb{R}^N),\ d(u_k, X) \to 0,\ \mathbf{u} \text{ is a Cauchy sequence in } H^{1,n}(S^n, \mathbb{R}^N)\}, \tag{6}$$

$$\mathbf{u} \approx \mathbf{v} \quad \Leftrightarrow \quad \|u_k - v_k\|_{H^{1,n}} \to 0 \ \text{ as } k \to \infty, \tag{7}$$

and $H^{1,n}(S^n, X) := \mathcal{H}(X)/\approx$.

We shall denote by $\lambda_n > 0$ the biggest constant satisfying the Poincaré inequality

$$\lambda_n \fint_{S^n} \left| u(\xi) - \fint_{S^n} u(\omega)\, d\omega \right|^n d\xi \leq \int_{S^n} \|du(\xi)\|^n \, d\xi\,, \tag{8}$$

whenever $u \in H^{1,n}(S^n, \mathbb{R}^N)$ for some $N \geq 1$. Recall from lemma 3, section 2 that $(\frac{2}{\pi})^n \leq \frac{\lambda_n}{\omega_n} \leq 1$ for all n. We shall need the following "localized" version of the Poincaré inequality.

Lemma 3. *There is a constant c_0 depending only on n such that for $u \in H^{1,n}(S^n, \mathbb{R}^N)$, $\xi \in S^n$, $s > 0$ and $0 < \varrho \leq \frac{1}{2}$ we have*

$$\begin{aligned} &\int_{S^n} \left| u(h_\xi(s)\omega) - \fint_{S^n} u(h_\xi(s)\omega')\, d\omega' \right|^n d\omega \\ &\quad\leq\ c_0 \left[\int_{B_\varrho(\xi)} \|du(\omega)\|^n \, d\omega + \frac{e^{-ns}}{\varrho^{3n}} \int_{S^n} \|du(\omega)\|^n \, d\omega \right]. \end{aligned} \tag{9}$$

Proof. Let $v := u \circ St_{-\xi}^{-1}$ and $p := \fint_{D_\varrho(\xi)\setminus D_{\varrho/2}(\xi)} v(x)\,dx$. Choosing

$$\eta_\varrho(r) := \begin{cases} 1 & \text{for } r \le \frac{\varrho}{2}, \\ 0 & \text{for } r \ge \varrho, \\ 2(1-\frac{r}{\varrho}) & \text{for } \frac{\varrho}{2} \le r \le \varrho, \end{cases}$$

where $r = |x|$, we put $w(x) := \eta_\varrho(r)v(x) + (1-\eta_\varrho(r))p$ and estimate, using Poincaré's inequality on the annulus $\{\varrho/2 < r < \varrho\}$ and the conformal invariance of the energy,

$$E_\infty(w, \{\varrho/2 < r < \varrho\}) \le c(n) \int_{B_\varrho(\xi)\setminus B_{\varrho/2}(\xi)} \|du(\omega)\|^n \, d\omega \,. \tag{10}$$

We define on S^n the map

$$\tilde{u}(\omega) := \begin{cases} u(\omega) & \text{if } \omega \in B_{\varrho/2}(\xi), \\ (w \circ St_{-\xi})(\omega) & \text{if } \omega \in B_\varrho(\xi) \setminus B_{\varrho/2}(\xi), \\ p & \text{else.} \end{cases}$$

Splitting the left-hand side of (9) and using (8) (applied to $\tilde{u} \circ h_\xi(s)$), (10) and (2.3) we obtain

$$\begin{aligned} &\fint_{S^n} \left| u(h_\xi(s)\omega) - \fint_{S^n} u(h_\xi(s)\omega')\,d\omega' \right|^n d\omega \\ \le\ & 3^{n-1}\left\{ \fint_{S^n} \left| \tilde{u}(h_\xi(s)\omega) - \fint_{S^n} \tilde{u}(h_\xi(s)\omega')\,d\omega' \right|^n d\omega + 2\fint_{S^n} |u(h_\xi(s)\omega) - \tilde{u}(h_\xi(s)\omega)|^n \, d\omega \right\} \\ \le\ & 3^{n-1}\left\{ \frac{1}{\lambda_n}\int_{S^n} \|d\tilde{u}(\omega)\|^n \, d\omega + 2\fint_{S^n\setminus B_{\varrho/2}(\xi)} \frac{|u(\omega)-\tilde{u}(\omega)|^n \, d\omega}{(\cosh s - \sinh s \langle \omega, \xi\rangle)^n} \right\} \\ \le\ & c(n)\left\{ \int_{B_\varrho(\xi)} \|du(\omega)\|^n \, d\omega + \int_{S^n\setminus B_{\varrho/2}(\xi)} \frac{|u(\omega)-\tilde{u}(\omega)|^n \, d\omega}{(\cosh s - \sinh s\langle\omega,\xi\rangle)^n} \right\}. \end{aligned}$$

For $\omega \in S^n \setminus B_{\varrho/2}(\xi)$ we have $\langle \omega, \xi \rangle \le \frac{1-(\varrho/2)^2}{1+(\varrho/2)^2}$ and hence $\cosh s - \sinh s \langle \omega, \xi\rangle \ge \frac{4}{17}(e^{-s} + \varrho^2 e^s)$. Therefore we can bound the second term in the brackets by (recalling (2.5))

$$\begin{aligned} 2^{\text{nd}}\text{-term} &\le \frac{5^n e^{-ns}}{(\varrho^2 + e^{-2s})^n} \int_{S^n \setminus B_{\varrho/2}(\xi)} |u(\omega) - \tilde{u}(\omega)|^n \, d\omega \\ &= \frac{5^n e^{-ns}}{(\varrho^2 + e^{-2s})^n} \int_{D_{2/\varrho}(-\xi)} |(u-\tilde{u})(St_\xi^{-1}(y))|^n \left(\frac{2}{1+|y|^2}\right)^n dy \\ &\le \frac{10^n e^{-ns}}{(\varrho^2 + e^{-2s})^n} \int_{D_{2/\varrho}(-\xi)} |(u-\tilde{u})(St_\xi^{-1}(y))|^n \, dy \\ &\le \frac{c(n) e^{-ns}}{\varrho^n(\varrho^2 + e^{-2s})^n} \int_{S^n\setminus B_{\varrho/2}(\xi)} \|d(u-\tilde{u})(\omega)\|^n \, d\omega \,, \end{aligned}$$

where the last step follows from Poincaré's inequality on the Euclidean ball $D_{2/\varrho}(-\xi)$. We finally infer from inequality (10), with a different constant $c(n)$,

$$2^{\text{nd}}\text{-term} \le \frac{c(n)e^{-ns}}{\varrho^n(\varrho^2 + e^{-2s})^n} \int_{S^n} \|du(\omega)\|^n \, d\omega \,,$$

and the claim follows. $\blacksquare$

Corollary 1 (Distance estimate). *There is a constant $c_1 > 0$ depending only on n such that for any $u \in W^{1,n}(S^n, X) := \{u \in H^{1,n}(S^n, \mathbb{R}^N) : u(\xi) \in X$ for a.e. $\xi\}$ and all $\xi \in S^n$, $s > 0$ we have*

$$\text{(11)}\quad \operatorname{dist}((\mathcal{E}_s u)(\xi), X) \le c_1 \left\{ \left(\int_{B_{e^{-s/6}}(\xi)} \|du(\omega)\|^n \, d\omega \right)^{\frac{1}{n}} + e^{-s/2} \left(\int_{S^n} \|du(\omega)\|^n \, d\omega \right)^{\frac{1}{n}} \right\}.$$

In particular $H^{1,n}(S^n, X) \cong W^{1,n}(S^n, X)$ is a complete metric space which is closed under weak convergence.

Proof. Since $u(h_\xi(s)\omega) \in X$ for a.e. $\omega \in S^n$ we have

$$[\operatorname{dist}((\mathcal{E}_s u)(\xi), X)]^n \le \fint_{S^n} \left| u(h_\xi(s)\omega) - \fint_{S^n} u(h_\xi(s)\omega')\, d\omega' \right|^n d\omega\,.$$

For $s \ge 6$ we choose $\varrho = e^{-s/6}$ in (9) and obtain the inequality with $c_1 = \sqrt[n]{c_0}$, while for $s \le 6$ application of the global Poincaré inequality (8) implies (11) with $c_1 = \frac{e^3}{\sqrt[n]{\lambda_n}}$. For the second statement we note that the canonical isometric embedding $J\colon H^{1,n}(S^n, X) \to W^{1,n}(S^n, X)$ given by $J(\{\mathbf{u}\}) := \lim_{k\to\infty} u_k$ is surjective since by (11) for $u \in W^{1,n}(S^n, X)$

$$\text{(12)}\qquad d(\mathcal{E}_s u, X) \to 0 \quad \text{as } s \to \infty$$

and $\mathcal{E}_s u \to u$ in $H^{1,n}(S^n, \mathbb{R}^N)$ by lemma 2 of section 2. $\blacksquare$

Theorem 3 (Assignment of homotopy classes). *We have $\mathcal{H}(X) \subset \Pi_n(X)$ and this inclusion induces a map $\{\,\cdot\,\}\colon H^{1,n}(S^n, X) \to \hat{\pi}_n(X)$ defining a homotopy class. The map $\{\,\cdot\,\}$ is continuous as a map between metric spaces and bounded; more precisely*

$$\text{(13)}\qquad \|\{u\}\| \le \sqrt[n]{\frac{E_\infty(u)}{\lambda_n}} \quad \textit{for } u \in H^{1,n}(S^n, X)$$

where λ_n is as in (8). $\{\,\cdot\,\}$ is constant on path components of $H^{1,n}(S^n, X)$.

Proof. Let $\mathbf{u} = (u_k)_{k\in\mathbb{N}} \in \mathcal{H}(X)$ and let $\delta > 0$ be given. Since the sequence $\|du_k\|^n \in L^1(S^n)$ is equiintegrable and $d(u_k, X) \to 0$ we can use (11) applied to $\overline{U_{\delta_n}(X)}$ instead of X to obtain $s_0 = s_0(\delta) > 0$ and $k_0 = k_0(\delta)$ such that $d(\mathcal{E}_s u_k, X) \le \frac{\delta}{3}$ for all $s \ge s_0(\delta)$, $k \ge k_0(\delta)$. Furthermore

$$|(\mathcal{E}_{s_0} u_k)(\xi) - (\mathcal{E}_{s_0} u_l)(\xi)| \le e^{s_0} \left(\fint_{S^n} |u_k(\omega) - u_l(\omega)|^n \, d\omega \right)^{\frac{1}{n}}.$$

Enlarging $k_0(\delta)$ if necessary such that $\left(\fint_{S^n} |u_k - u_l|^n \, d\omega \right)^{\frac{1}{n}} \le e^{-s_0} \frac{\delta}{3}$ we infer that $\|\mathcal{E}_{s_0} u_k - \mathcal{E}_{s_0} u_l\|_{C^0} \le \frac{\delta}{3}$ for $k, l \ge k_0(\delta)$. We thus obtain a homtopy from u_k to u_l by first deforming u_k to $\mathcal{E}_{s_0} u_k$ in $U_{\delta/3}(X)$ by $\mathcal{E}_s u_k$, $\infty \ge s \ge s_0$ (and u_l to $\mathcal{E}_{s_0} u_l$ likewise) and finally connecting $\mathcal{E}_{s_0} u_k$ to $\mathcal{E}_{s_0} u_l$ via the affine homotopy in $U_\delta(X)$; this proves the inclusion $\mathcal{H}(X) \subset \Pi_n(X)$. The fact that the inclusion descends and that the induced map is continuous is proved in [8, sec. 1], see also [3].

Let $\varepsilon > 0$ and let $\mathbf{u} = (u_k)_{k\in\mathbb{N}} \subset C^0(S^n, \mathbb{R}^N)$ satisfy $\limsup_{k\to\infty} E_\infty(u_k) \le \varepsilon$ and $d(u_k, X) \to 0$. We claim that

$$\text{(14)}\qquad \|\mathbf{u}\| \le \sqrt[n]{\frac{\varepsilon}{\lambda_n}}\,.$$

Namely, consider the extensions $\mathcal{E}u_k \in C^0(\overline{B^{n+1}}, \mathbb{R}^N)$ and assume w.l.o.g. that $d(\mathcal{E}u_k, X) =: \delta_k \geq 0$ is attained at $x = 0$; then the Poincaré inequality (8) yields

$$\limsup_{k\to\infty} \delta_k^n \leq \limsup_{k\to\infty} ⨍_{S^n} \left| u_k(\xi) - ⨍_{S^n} u_k(\omega)\, d\omega \right|^n d\xi \leq \frac{\varepsilon}{\lambda_n}.$$

Thus if $\delta > \sqrt[n]{\varepsilon/\lambda_n}$ then u_k is contractible in $U_\delta(X)$ for k sufficiently large. This proves (14) and implies (13) immediately. The final statement is a consequence of the continuity of $\{\,\cdot\,\}$ and lemma 1. ∎

Corollary 2. *If $\mathbf{u} = (u_k)_{k\in\mathbb{N}} \subset C^0(S^n, \mathbb{R}^N)$ satisfies $d(u_k, X) \to 0$ and $E_\infty(u_k) \to 0$ as $k \to \infty$, then $\mathbf{u}$ is a trivial sequence.*

Corollary 3. *If $u \in H^{1,n}(S^n, X) \cong W^{1,n}(S^n, X)$ and if $s_k \to \infty$, then the sequence $(\mathcal{E}_{s_k}u)_{k\in\mathbb{N}}$ is admissible for the definition of $\{u\} \in \hat{\pi}_n(X)$.*

For $\alpha \in \hat{\pi}_n(X)$ and a conformally invariant functional $F(\,\cdot\,)$ as in (3.1)–(3.3) we define

$$F_*(\alpha) := \inf\{\liminf_{k\to\infty} F(u_k) : (u_k)_{k\in\mathbb{N}} \in \alpha\} \in [0, \infty], \tag{15}$$

and also introduce the numbers

$$\varepsilon_*(X) := \inf\{\varepsilon > 0 : \text{there is a nontrivial sequence } (u_k)_{k\in\mathbb{N}} \text{ with } \limsup_{k\to\infty} F(u_k) \leq \varepsilon\}, \tag{16}$$

$$\varepsilon_0(X) := \inf\{F_*(\alpha) : \|\alpha\| > 0\}. \tag{17}$$

Note that $0 \leq \varepsilon_*(X) \leq \varepsilon_0(X) \leq \infty$.

Corollary 4. *We have with $\lambda_n > 0$ as in (8)*

$$F_*(\alpha) \geq \lambda\lambda_n \|\alpha\|^n. \tag{18}$$

If X admits a continuous retraction $R_{\delta_0}\colon U_{\delta_0}(X) \to X$ then

$$\varepsilon_*(X) \geq \lambda\lambda_n\delta_0^n > 0. \tag{19}$$

Definition. Let $F(\,\cdot\,)$ be a functional as in (3.1)–(3.3). A minimizing sequence for $F_*(\alpha)$ is a sequence $(u_k)_{k\in\mathbb{N}} \in \alpha$ with $F(u_k) \to F_*(\alpha)$ as $k \to \infty$.

A simple diagonal argument shows that there is always a smooth minimizing sequence for α.

While the map $\{\,\cdot\,\}$ is continuous with respect to strong convergence in $H^{1,n}(S^n, X)$ and bounded in the sense of (13) it is clearly not continuous with respect to weak convergence; in the next section we will also show that it is not compact. For this we shall now discuss the topological notion of decomposition.

Let $U \subset \mathbb{R}^N$ be a connected open set and let $p \in U$. A class $\alpha \in \pi_n(U, p)$ is represented either by a map $u\colon (S^n, -e_{n+1}) \to (U, p)$ or by a map $v\colon (B^n, \partial B^n) \to (U, p)$; here u and $\{u \circ \Phi\}$ define the same class for $\Phi(x) = St^{-1}_{-e_{n+1}}\left(\frac{x}{1-|x|}\right)$. The action of $\pi_1(U, p)$ on $\pi_n(U, p)$ is described as follows: if $v \in \alpha \in \pi_n(U, p)$, $c \in \gamma \in \pi_1(U, p)$, then $\gamma_*(\alpha)$ is the class of the map

$$(c * v)(x) := \begin{cases} u(2x) & \text{for } |x| \leq \frac{1}{2}, \\ c(3 - 4|x|) & \text{for } \frac{1}{2} \leq |x| \leq 1. \end{cases}$$

One has a surjective projection $[\,\cdot\,]\colon \pi_n(U,p) \to \pi_n(U)$ given by *forgetting the basepoint* and $[\alpha] = [\beta]$ if and only if $\beta = \gamma_* \alpha$ for some $\gamma \in \pi_1(U,p)$.

For $\alpha, \alpha^1, \ldots, \alpha^m \in \pi_n(U)$ we say that the indexed set $(\alpha^i)_{i=1}^m$ is a *decomposition of* α, $(\alpha^i)_{i=1}^m \in \mathcal{D}(\alpha)$, or equivalently that α belongs to the composition set $\mathcal{C}((\alpha^i)_{i=1}^m)$, iff there exist $\tilde{\alpha}^i \in \pi_n(U,p)$ for $1 \le i \le m$ such that $[\tilde{\alpha}^i] = \alpha^i$ and $[\tilde{\alpha}^1 + \ldots + \tilde{\alpha}^m] = \alpha$. This notion does not depend on the choice of the basepoint. Alternatively we may view this concept in terms of extensions.

Lemma 4. *Let α, $(\alpha^i)_{i=1}^m \subset \pi_n(U)$ and let $D = B^{n+1} \setminus \bigcup_{i=1}^m K^i$, where the K^i, $1 \le i \le m$, are closed round balls contained in B^{n+1} with $K^i \cap K^j = \emptyset$ for $i \neq j$. Then the following statements are equivalent:*

(20) $(\alpha^i)_{i=1}^m \in \mathcal{D}(\alpha)$;

(21) *there exists a map $H \in C^0(\overline{D}, U)$ with $H|_{\partial B^{n+1}} \in \alpha$ and $H|_{\partial K^i} \in \alpha^i$ for $1 \le i \le m$ (here ∂K^i is identified with S^n by an orientation-preserving homeomorphism);*

(22) *if $h \in C^0(\partial D, U)$ satisfies $h|_{\partial B^{n+1}} \in \alpha$, $h|_{\partial K^i} \in \alpha^i$ for $1 \le i \le m$, then there exists $H \in C^0(\overline{D}, U)$ with $H|_{\partial D} = h$.*

We shall now extend the notion of (de-) composition to $\hat{\pi}_n(X)$. Let $\boldsymbol{\alpha} := (\alpha^i)_{i=1}^m \subset \hat{\pi}_n(X)$ where $1 \le m \le \infty$; for $m = \infty$ we require additionally that $m(\boldsymbol{\alpha}, \delta) := \operatorname{card} I(\boldsymbol{\alpha}, \delta)$ is finite for any $\delta > 0$, where $I(\boldsymbol{\alpha}, \delta) := \{i : \|\alpha^i\| \ge \delta\}$.

Definition. We say that $\boldsymbol{\alpha} = (\alpha^i)_{i=1}^m$ is a decomposition of $\alpha \in \hat{\pi}_n(X)$ if and only if $(i_\delta(\alpha^i))_{i \in I(\boldsymbol{\alpha},\delta)} \in \mathcal{D}(i_\delta(\alpha))$ for any $\delta > 0$. We further introduce the sequence space

$$\Pi_n(\boldsymbol{\alpha}) := \{\mathbf{u} = (u_k)_{k\in\mathbb{N}} : u_k \in C^0(S^n, \mathbb{R}^N),\ d(u_k, X) \to 0,\ \text{for any } \delta > 0 \text{ there exists } k(\mathbf{u}, \delta) \in \mathbb{N} \text{ such that } \{u_k\} \in \mathcal{C}((i_\delta(\alpha^i))_{i\in I(\boldsymbol{\alpha},\delta)}) \text{ for all } k \ge k(\mathbf{u}, \delta)\}. \tag{23}$$

We shall adopt the notation $\alpha \in \mathcal{C}(\boldsymbol{\alpha})$, $\boldsymbol{\alpha} \in \mathcal{D}(\alpha)$ for α, $\boldsymbol{\alpha} = (\alpha^i)_{i=1}^m \subset \hat{\pi}_n(X)$.

Remark. By lemma 4 we have $\mathcal{C}((\alpha^i)_{i=1}^m) = \mathcal{C}((\alpha^{\sigma(i)})_{i=1}^m)$ for any reordering σ of $\{i \in \mathbb{N} : 1 \le i < m+1\}$.

We finally mention that if $[\![\alpha]\!] \in H_n(U, \mathbb{Z})$ denotes the homology class of $\alpha \in \pi_n(U)$ given by $[\![\alpha]\!] := u_*[\![S^n]\!]$ where $u \in \alpha$ and $[\![S^n]\!]$ is the fundamental cycle, then $(\alpha^i)_{i=1}^m \in \mathcal{D}(\alpha)$ implies (but is in general stronger than) $[\![\alpha^1]\!] + \ldots + [\![\alpha^m]\!] = [\![\alpha]\!]$ by lemma 4.

5 Topological compactness results

In this section we take up the following questions: suppose that $(u_k)_{k\in\mathbb{N}} \subset H^{1,n}(S^n, X)$ is a sequence with equibounded energies $E_\infty(u_k) \le E < \infty$. How much control can we obtain for the homotopy classes $\{u_k\} \in \hat{\pi}_n(X)$, assuming that we are allowed to pass to a subsequence? If $(u_k)_{k\in\mathbb{N}}$ is a nontrivial sequence, is it possible to reflect this nontriviality by construction of a suitable weak limit set? As a consequence of our analysis we will obtain an optimal compactness result for minimizing sequences in a given homotopy class. Throughout this section we assume again

$$U_\delta(X) \text{ is connected for any } \delta > 0. \tag{1}$$

Note that (1) implies that there is a *unique* trivial element $\{0\} \in \hat{\pi}_n(X)$; any trivial sequence $(u_k)_{k\in\mathbb{N}}$ represents $\{0\}$ and $\|\alpha\| = \boldsymbol{\delta}(\alpha, \{0\})$ for any $\alpha \in \hat{\pi}_n(X)$.

Lemma 1. *Let F be a functional as in (3.1)–(3.3), and suppose $(\alpha^i)_{i=1}^m \subset \hat{\pi}_n(X)$ ($1 \le m \le \infty$) is a decomposition of $\alpha \in \hat{\pi}_n(X)$. Then*

$$F_*(\alpha) \le \sum_{i=1}^m F_*(\alpha^i)\,. \tag{2}$$

Proof. We may assume $0 < F_*(\alpha^i) < \infty$ for all i. For $k \in \mathbb{N}$ there exist by assumption $v_k^i \in C^0((S^n, -e_{n+1}), (U_{1/k}(X), p))$ representing an element $\tilde{\alpha}_k^i \in \pi_n(U_{1/k}(X), p)$ with $[\tilde{\alpha}_k^i] = i_{1/k}(\alpha^i)$ such that

$$\Bigg[\sum_{i\in I(\boldsymbol{\alpha},1/k)} \tilde{\alpha}_k^i\Bigg] = i_{1/k}(\alpha)\,; \tag{3}$$

here $p \in X$ is an arbitrary fixed point. Let $\varepsilon > 0$ be given and choose for each i, $1 \le i < m+1$, a smooth minimizing sequence $\mathbf{u}^i = (u_k^i)_{k\in\mathbb{N}}$ for $F_*(\alpha^i)$ satisfying the conditions

$$k(\mathbf{u}^i, \tfrac{1}{2\nu}) \le \nu \text{ for all } \nu \in \mathbb{N}, \text{ and} \tag{4}$$

$$F(u_k^i) \le F_*(\alpha^i) + \frac{\varepsilon}{m(\boldsymbol{\alpha}, \frac{1}{k})} \quad \text{for all } k \in \mathbb{N}. \tag{5}$$

By the choice of u_k^i and v_k^i there exists a homotopy $H_k^i \in C^0(S^n \times [0,1], U_{1/k}(X))$ with $H_k^i(\,\cdot\,, 0) = u_k^i$ and $H_k^i(\,\cdot\,, 1) = v_k^i$; we obtain the path $\tilde{c}_k^i(s) := H_k^i(-e_{n+1}, s)$, $0 \le s \le 1$, with initial point $\tilde{c}_k^i(0) = u_k^i(-e_{n+1}) =: p_k^i$ and endpoint $\tilde{c}_k^i(1) = v_k^i(-e_{n+1}) = p$. Let $c_k^i\colon [0,1] \to U_{1/k}(X)$ be a polygon inscribed to $\tilde{c}_k^i$ of length $L(c_k^i) =: \ell_k^i < \infty$, which is homotopic to $\tilde{c}_k^i$ in $U_{1/k}(X)$ with endpoints fixed and which is parametrized proportionally to its arclength.

Let $(u_k^i)_+ := u_k^i \circ St_{e_{n+1}}^{-1}\colon \mathbb{R}^n \to U_{1/k}(X)$. By smoothness of u_k^i there exists $r_k \in (0, \frac{1}{k})$ such that if we define for $r_k \le r \le 2r_k$, $\xi \in S^{n-1}$,

$$(w_k^i)_+(r\xi) := \left(\frac{r}{r_k} - 1\right)(u_k^i)_+(2r_k\xi) + \left(2 - \frac{r}{r_k}\right) p_k^i\,, \tag{6}$$

then for all $i \in I(\boldsymbol{\alpha}, \frac{1}{k})$ the following assertions hold:

$$|(u_k^i)_+(x) - p_k^i| < \frac{1}{2k} \text{ for } x \in D_{2r_k}(-e_{n+1}), \text{ and} \tag{7}$$

$$\int_{\{r_k<|x|<2r_k\}} \|d(w_k^i)_+(x)\|^n\, dx \le \frac{\varepsilon}{m(\boldsymbol{\alpha}, \frac{1}{k})}\,, \text{ and} \tag{8}$$

$$\frac{\omega_{n-1}(\ell_k^i)^n}{(\log 1/r_k)^{n-1}} \le \frac{\varepsilon}{m(\boldsymbol{\alpha}, \frac{1}{k})}\,. \tag{9}$$

We now define the map

$$w_k^i(\omega) := \begin{cases} u_k^i(\omega) & \text{for } \omega \in S^n \setminus B_{2r_k}(-e_{n+1}), \\ (w_k^i)_+(St_{e_{n+1}}(\omega)) & \text{for } \omega \in B_{2r_k}(-e_{n+1}) \setminus B_{r_k}(-e_{n+1}), \\ c_k^i\left(\dfrac{\log r/r_k}{\log r_k}\right) & \text{for } \omega = St_{e_{n+1}}^{-1}(r\xi),\ r_k^2 \le r \le r_k, \\ p & \text{for } \omega \in B_{r_k^2}(-e_{n+1}). \end{cases} \tag{10}$$

We compute, using conformal invariance and the choice of c_k^i:

$$\int_{B_{r_k}(-e_{n+1})} \|dw_k^i(\omega)\|^n \, d\omega = \frac{\omega_{n-1}(\ell_k^i)^n}{(\log 1/r_k)^{n-1}}\,.$$

It follows from (5), the properties of F, (8), (9) and (10) that

$$F(w_k^i) \leq F_*(\alpha^i) + \frac{2\Lambda+1}{m(\boldsymbol{\alpha},\frac{1}{k})}\,\varepsilon\,. \tag{11}$$

Now choose pairwise different points $(\omega_k^i)_{i\in I(\boldsymbol{\alpha},\frac{1}{k})} \subset S^n$, pairwise disjoint balls $B_{\varrho_k}(\omega_k^i)$ (with suitable $\varrho_k > 0$) and elements $h_k^i \in G$ such that $S^n \setminus B_{r_k^2}(-e_{n+1}) \Subset h_k^i(B_{\varrho_k}(\omega_k^i))$. We then define the map

$$w_k(\omega) := \begin{cases} w_k^i(h_k^i(\omega)) & \text{for } \omega \in B_{\varrho_k}(\omega_k^i), \\ p & \text{else.} \end{cases} \tag{12}$$

We claim that the sequence $(w_k)_{k\in\mathbb{N}}$ belongs to α. For this it is sufficient to show that $w_k^i\colon (S^n, -e_{n+1}) \to (U_{1/k}(X), p)$ belongs to $\tilde{\alpha}_k^i$, because then by construction of w_k we have $w_k \in \left[\sum_{i\in I(\boldsymbol{\alpha},1/k)} \tilde{\alpha}_k^i\right] = i_{1/k}(\alpha)$ by (3). We thus have to show that w_k^i is homotopic to v_k^i with fixed basepoint in $U_{1/k}(X)$. Working in stereographic coordinates around $-e_{n+1}$ we first consider the homotopy for $r_k \leq r \leq 2r_k$

$$h(t, r\xi) := (1-t)(w_k^i)_+(r\xi) + t(u_k^i)_+(2(r-r_k)\xi) \tag{13}$$

for $0 \leq t \leq 1$. Note that $h(t, r_k\xi) \equiv p_k^i$ and $h(t, 2r_k\xi) \equiv (u_k^i)_+(2r_k\xi)$. Furthermore by the definitions (6), (13) and (7) we infer that $|h(t,r\xi) - p_k^i| < \frac{1}{2k}$; by (4) (i.e. $d(u_k^i, X) \leq \frac{1}{2k}$) and $p_k^i = u_k^i(-e_{n+1})$ we conclude $d(h, X) < \frac{1}{k}$. Next we observe that we can replace c_k^i in definition (10) by $\tilde{c}_k^i$ without changing the class in $\pi_n(U_{1/k}(X), p)$; we denote the resulting map by $\tilde{u}_k^i$. Now let $\Psi_k \in C^0(S^n, S^n)$ be the map which is the identity on $S^n \setminus B_{2r_k}(-e_{n+1})$ and which is given in stereographic coordinates on $B_{2r_k}(-e_{n+1})$ by

$$(St_{e_{n+1}} \circ \Psi_k \circ St_{e_{n+1}}^{-1})(r\xi) = \begin{cases} 2(r-r_k)\xi & \text{for } r_k \leq r \leq 2r_k, \\ 0 & \text{for } 0 \leq r \leq r_k. \end{cases} \tag{14}$$

Let $0 \leq s \leq 1$ and consider the homotopy

$$G_k^i(\omega, s) := \begin{cases} H_k^i(\Psi_k(\omega), s) & \text{for } \omega \in S^n \setminus B_{r_k}(-e_{n+1}), \\ \tilde{c}_k^i\left((1-s)\dfrac{\log r/r_k}{\log r_k} + s\right) & \text{for } \omega = St_{e_{n+1}}^{-1}(r\xi),\ r_k^2 \leq r \leq r_k, \\ p & \text{for } \omega \in B_{r_k^2}(-e_{n+1}). \end{cases} \tag{15}$$

Using the properties of H_k^i, $\tilde{c}_k^i$, (13) and the definition of $\tilde{u}_k^i$ we obtain $G_k^i \in C^0(S^n \times [0,1], U_{1/k}(X))$, $G_k^i(-e_{n+1}, \,\cdot\,) \equiv p$, $G_k^i(\,\cdot\,, 0) = \tilde{u}_k^i$ and

$$G_k^i(\omega, 1) = \begin{cases} v_k^i(\Psi_k(\omega)) & \text{for } \omega \in S^n \setminus B_{r_k}(-e_{n+1}), \\ p & \text{for } \omega \in B_{r_k}(-e_{n+1}). \end{cases} \tag{16}$$

Finally it is obvious that $G_k^i(\,\cdot\,,1)$ is homotopic to v_k^i with preserved basepoint and therefore the claim is proved; this means that the sequence $(w_k)_{k\in\mathbb{N}}$ is admissible for comparison in the definition of $F_*(\alpha)$. By (11) and (12) we obtain in the limit $k\to\infty$

$$F_*(\alpha) \le \liminf_{k\to\infty} F(w_k) \le \sum_{i=1}^m F_*(\alpha^i) + (2\Lambda+1)\varepsilon\,.$$

As $\varepsilon > 0$ is arbitrary the lemma is proved. ■

We now come to the question about the compactness of the map $\{\,\cdot\,\}\colon H^{1,n}(S^n,X)\to \hat{\pi}_n(X)$. Let us assume here that X is a compact, path-wise connected, continuous retract of $U_{\delta_0}(X)$ for some $\delta_0 > 0$. Since $\hat{\pi}_n(X) \cong \pi_n(X)$ is then a discrete metric space, compactness of $\{\,\cdot\,\}$ would mean that any sequence $(u_k)_{k\in\mathbb{N}} \subset H^{1,n}(S^n,X)$ with $E(u_k)\le E$ for all k and some $E<\infty$ has a subsequence $(u_{k(l)})_{l\in\mathbb{N}}$ such that $\{u_{k(l)}\}\equiv\alpha$ for all l and some *fixed* $\alpha\in\pi_n(X)$. However, this is not true in general.

Example. Let $\mathbb{Z}$ act on $\mathbb{R}^3 = \mathbb{R}^2\times\mathbb{R}$ by $\tau_k(w,z) = (w, z+4k)$. Consider $\tilde{X} = \{(w,z) : |w| \le 2\} \setminus \bigcup_{k\in\mathbb{Z}} \tau_k(B_1(0))$ and denote by $\pi\colon(\tilde{X},\tilde{p}) \to (X,p)$ the projection onto the quotient; we may take $\tilde{p} = -e_3$. Choosing an arbitrary path $\tilde{c}_k\colon[-1,1]\to\tilde{X}$ satisfying $\tilde{c}_k(-1) = \tilde{p}$ and $\tilde{c}_k(1) = \tau_k(\tilde{p})$ we define the maps

$$\tilde{v}_k(x) = \begin{cases} \tau_k(St^{-1}_{-e_3}(2x)) & \text{for } |x| \le \frac{1}{2}, \\ \tilde{c}_k(3-4|x|) & \text{for } \frac{1}{2} \le |x| \le 1. \end{cases}$$

By Hurewicz's theorem the homotopy classes $\tilde{\alpha}_k$ of $\tilde{v}_k$ generate $\pi_2(\tilde{X},\tilde{p})$ and $\pi_2(X,p) = \bigoplus_{k\in\mathbb{Z}} \mathbb{Z}\cdot\alpha_k$ where $\alpha_k = \pi_*(\tilde{\alpha}_k)$. It is also easily seen that for $\gamma_k = \{\pi\circ\tilde{c}_k\} \in \pi_1(X,p)$ there holds $(\gamma_k)_*(\alpha_l) = \alpha_{k+l}$. We conclude that $[\alpha_0+\alpha_k]\in\pi_2(X)$ $k\in\mathbb{N}_0$, are pairwise different free homotopy classes. On the other hand, since $[\alpha_k] = [\alpha_0]$, the set $([\alpha_0],[\alpha_0])$ is a decomposition of $[\alpha_0+\alpha_k]$ whence $E_*([\alpha_0+\alpha_k]) \le 2E_*([\alpha_0]) \le \text{const}$.

In order to answer the questions posed at the beginning of this section, we now study sequences $u_k \in C^0(S^n,\mathbb{R}^N)$ satisfying the following conditions:

$$d(u_k,X) \le \frac{1}{k} \ \text{ for all } k\in\mathbb{N}, \text{ and} \tag{17}$$

$$E_\infty(u_k) \le E < \infty \ \text{ for all } k\in\mathbb{N}. \tag{18}$$

We define

$$A((\mathcal{E}u_k)_{k\in\mathbb{N}}) := \{p\in\mathbb{R}^N\colon \text{ there exists a subsequence } \mathfrak{K}\subset\mathbb{N} \text{ and } x_k\in B^{n+1} \text{ for } k\in\mathfrak{K} \text{ such that } (\mathcal{E}u_k)(x_k)\to p \text{ as } \mathfrak{K}\ni k\to\infty\}. \tag{19}$$

$A((\mathcal{E}u_k)_{k\in\mathbb{N}})$ is the closed set of accumulation points of $(\mathcal{E}u_k)(B^{n+1})$. Using conformal invariance of the energy and the global Poincaré inequality (4.8) as well as the definition of $\mathcal{E}$ we get

$$A((\mathcal{E}u_k)_{k\in\mathbb{N}}) \subset \overline{U_{\delta(E)}(X)} \ \text{ where } \delta(E) = \sqrt[n]{\frac{E}{\lambda_n}}. \tag{20}$$

We are aiming to construct by induction the following data:

(21) A number $m\in\mathbb{N}_0\cup\{\infty\}$;

(22) a subsequence $\mathfrak{K}_m = (k_m(l))$; in the following it is assumed that k, l are coupled by $k = k_m(l)$;

(23) elements $h_k^i := h_{x_k^i} \in G$ for $1 \le i < m+1$ and $k \in \mathfrak{K}_m$, where $x_k^i \in B^{n+1} =: B$;

(24) hyperbolic radii $\tau_k^i \in (0, \infty)$ for $1 \le i < m+1$, $k \in \mathfrak{K}$.

Suppose that data as in (17), (18), (21)–(24) are given. Then we may define the following associated data:

(25) hyperbolic balls $B_k^i := B_{\tau_k^i}(x_k^i)$ for $1 \le i < m+1$, $k \in \mathfrak{K}_m$;

(26) $\Omega_k^i := B \setminus \bigcup_{j=1}^{i} \overline{B}_k^j$; we put $\Omega_k^0 = B$ for all k;

(27) $t_k^{i,j} := d(h_k^i, h_k^j) = \mathrm{dist}_{\mathbb{H}^{n+1}}(x_k^i, x_k^j)$ for $1 \le i, j < m+1$;

(28) $\omega_k^{i,j} \in S^n$ such that $x_k^j = h_k^i((\tanh \frac{t_k^{i,j}}{2})\omega_k^{i,j})$ for $1 \le i, j < m+1$.

Lemma 2. *Suppose $(u_k)_{k \in \mathbb{N}}$ is a sequence satisfying (17) and (18). Then there exist data (21)–(24) with the following properties:*

(29) *if $m < \infty$ then $A((\mathcal{E}u_k|_{\Omega_k^m})_{k \in \mathfrak{K}_m}) \subset X$;*

(30) *for $1 \le i < m+1$ there are $\delta^i > 0$ with $\delta(E) \ge \delta^1 \ge \delta^2 \ge \dots$ such that $A((\mathcal{E}u_k|_{\Omega_k^i})_{k \in \mathfrak{K}_m}) \subset \overline{U_{\delta^i}(X)}$;*

(31) *for $1 \le i < m+1$ the sequence $(u_k \circ h_k^i)_{k \in \mathfrak{K}_m}$ converges weakly to $u^i \in H^{1,n}(S^n, X)$;*

(32) *$\frac{i}{i+1}\delta^i \le d(\mathcal{E}u^i, X) \le \delta^i$ for $1 \le i < m+1$; in particular $E_\infty(u^i) \ge \lambda_n \left(\frac{i}{i+1}\delta^i\right)^n > 0$ and u^i is not constant;*

(33) *$\tau_k^i \ge l$ for $k = k_m(l)$, $1 \le i < m+1$;*

(34) *$\|u^i - \mathcal{E}_{\tau_k^i}(u_k \circ h_k^i)\|_{H^{1,n}(S^n, \mathbb{R}^N)} \le \frac{1}{l}$ for $1 \le i < m+1$;*

(35) *$d(\mathcal{E}u_k|_{B_{2\tau_k^i}(x_k^i) \setminus B_{\tau_k^i}(x_k^i)}, X) \le \frac{1}{l}$ for $1 \le i < m+1$;*

(36) *$B_k^i \Subset B$, $\overline{B}_k^i \cap \overline{B}_k^j = \emptyset$ for $1 \le i, j < m+1$, $i \ne j$;*

(37) *$t_k^{i,j} \ge 2l$ for $1 \le i, j < m+1$, $i \ne j$;*

(38) *$\omega_k^{i,j} \to \omega^{i,j} \in S^n$ for $1 \le i, j < m+1$, $i \ne j$;*

(39) *let $u_l^i := (\mathcal{E}u_k)|_{\partial B_k^i}$ for $k = k_m(l)$, $1 \le i < m+1$, and let $\alpha^i := \{u^i\} \in \hat{\pi}_n(X)$; then $\mathbf{u}^i \in \alpha^i$ with $u_l^i \sim u_{l'}^i$ in $U_{1/\nu}(X)$ for l, $l' \ge \nu$; in particular $d(\mathcal{E}u_k|_{\partial\Omega_k^i}, X) \le \frac{1}{l}$ for $1 \le i < m+1$;*

(40) *we have $(\mathcal{E}u_k)(\Omega_k^i) \subset \left[\overline{U_{2\delta^i}(X)} \cup (\mathbb{R}^N \setminus B_i^N(0))\right]$ for $1 \le i < m+1$ and all $k \in \mathfrak{K}_m$;*

(41) *for any functional $F(\,\cdot\,)$ as in (3.1)–(3.3) we have $\sum_{i=1}^{m} F(u^i) \le \liminf_{k \to \infty} F(u_k)$;*

(42) $$\sum_{i=1}^{m} \left(\frac{i}{i+1}\delta^i\right)^n \le \frac{E}{\lambda_n} < \infty.$$

Proof. We shall inductively construct subsequences $\mathfrak{K}_0 \supset \mathfrak{K}_1 \supset \mathfrak{K}_2 \supset \dots$ and data (21)–(24), such that (30)–(42) hold at each step. For $m = 0$ we put $\mathfrak{K}_0 = \mathbb{N}$, i.e. $k_0(l) = l$;

then there is nothing to prove. Now suppose that for some $m \in \mathbb{N}_0$ data as in (21)–(24) satisfying (30)–(42) are already constructed.

Case 1. $A((\mathcal{E}u_k|_{\Omega_k^m})_{k\in\mathfrak{K}_m})\backslash X{=}\emptyset$. Then the lemma is proved for the given number m.

Case 2. $A((\mathcal{E}u_k|_{\Omega_k^m})_{k\in\mathfrak{K}_m}) \setminus X \neq \emptyset$. We start with the following important

Observation: if $\mathfrak{K} \subset \mathfrak{K}_m$ is any subsequence and if h_k^i, τ_k^i are the subdata corresponding to $\mathfrak{K}$, then (30)–(42) hold with the same δ^i for the subsequence $\mathfrak{K}$.

Now put

$$\delta^{m+1} := \sup\left\{d(q, X) : q \in A((\mathcal{E}u_k|_{\Omega_k^m})_{k\in\mathfrak{K}_m})\right\}. \tag{43}$$

By assumption we have $\delta^{m+1} > 0$; also from (20) we see that $\delta^1 \leq \delta(E)$. If $m \geq 1$ then by inductive assumption $\mathfrak{K}_m$ is a subsequence of $\mathfrak{K}_{m-1}$ and $\Omega_k^m \subset \Omega_k^{m-1}$ for $k \in \mathfrak{K}_m$; from (43) we then get $\delta^{m+1} \leq \delta^m$ which implies $(30)_{m+1}$. Choose $p \in A((\mathcal{E}u_k|_{\Omega_k^m})_{k\in\mathfrak{K}_m})$ with $\operatorname{dist}(p, X) \geq \frac{m+1}{m+2}\delta^{m+1}$. There exists a subsequence $\mathfrak{K}'_m \subset \mathfrak{K}_m$ and $x_k^{m+1} \in \Omega_k^m$ such that

$$(\mathcal{E}u_k)(x_k^{m+1}) \to p \ \text{ for } \mathfrak{K}'_m \ni k \to \infty. \tag{44}$$

Using (35) we have, after discarding finitely many elements of $\mathfrak{K}'_m$,

$$x_k^{m+1} \in B \setminus \bigcup_{i=1}^{m} B_{2\tau_k^i}(x_k^i). \tag{45}$$

Choose a subsequence $\mathfrak{K}''_m$ such that $(u_k \circ h_k^{m+1})_{k\in\mathfrak{K}''_m}$ converges weakly to a map $u^{m+1} \in H^{1,n}(S^n, X)$; here $h_k^{m+1} := h_{x_k^{m+1}}$ as in (23). Note that this is possible since

$$⨍_{S^n}(u_k \circ h_k^{m+1})(\omega)\, d\omega = (\mathcal{E}u_k)(h_k^{m+1}(0)) = (\mathcal{E}u_k)(x_k^{m+1}) \to p\,.$$

It follows that $(\mathcal{E}u^{m+1})(0) = p$ and in particular by (20)

$$\frac{m+1}{m+2}\delta^{m+1} \leq \operatorname{dist}(p, X) \leq d(\mathcal{E}u^{m+1}, X) \leq \delta(E)\,.$$

From (45) we have $\operatorname{dist}_{\mathbb{H}^{n+1}}(x_k^{m+1}, B_k^i) \geq \tau_k^i$ for $1 \leq i \leq m$ and hence by $(33)_m$

$$\operatorname{dist}_{\mathbb{H}^{n+1}}(0, (h_k^{m+1})^{-1}B_k^i) \geq l \to \infty \ \text{ for } \mathfrak{K}''_m \ni k \to \infty \text{ and } 1 \leq i \leq m. \tag{46}$$

Therefore for any $x \in B$ there exists a sequence $y_k \in (h_k^{m+1})^{-1}\Omega_k^m$, $k \in \mathfrak{K}''_m$, satisfying

$$(\mathcal{E}u^{m+1})(x) = \lim_{\mathfrak{K}''\ni k\to\infty} \mathcal{E}(u_k \circ h_k^{m+1})(y_k)\,;$$

this and the definition (43) imply $d(\mathcal{E}u^{m+1}, X) \leq \delta^{m+1}$ which shows $(31)_{m+1}$ and $(32)_{m+1}$. By corollaries 1 and 3 of section 4 and lemma 2 of section 2 we can find a sequence $\tau_l \geq l$ ($l \in \mathbb{N}$) with the following properties:

$$d(\mathcal{E}u^{m+1}|_{B\setminus B_{\tau_l}(0)}, X) \leq \frac{1}{2l}\,; \tag{47}$$

(48) $\quad (\mathcal{E}_{\tau_l}u^{m+1})_{l\in\mathbb{N}}$ represents $\alpha^{m+1} := \{u^{m+1}\} \in \hat{\pi}_n(X)$ with $\mathcal{E}_{\tau_l}u^{m+1} \sim \mathcal{E}_{\tau_{l'}}u^{m+1}$ in $U_{1/2\nu}(X)$ for $l,\, l' \geq \nu$;

$$\|u^{m+1} - \mathcal{E}_{\tau_l}(u^{m+1})\|_{H^{1,n}} \leq \frac{1}{2l}\,. \tag{49}$$

As the sequence $(\mathcal{E}(u_k \circ h_k^{m+1}))_{k\in\mathfrak{K}''_m}$ converges in $C^\infty_{\mathrm{loc}}(B,\mathbb{R}^N)$, we can pass to a subsequence $\mathfrak{K}'''_m \subset \mathfrak{K}''_m$ such that

$$\|\mathcal{E}u^{m+1} - \mathcal{E}(u_k \circ h_k^{m+1})\|_{C^0(B_{2\tau_l}(0),\mathbb{R}^N)} < \frac{1}{2l}\,; \tag{50}$$

$$\|\mathcal{E}_{\tau_l}u^{m+1} - \mathcal{E}_{\tau_l}(u_k \circ h_k^{m+1})\|_{H^{1,n}} \le \frac{1}{2l}\,. \tag{51}$$

By (46) we can choose $\mathfrak{K}'''_m$ such that we additionally have

$$(h_k^{m+1})^{-1}B_k^i \cap B_{2\tau_l}(0) = \emptyset \text{ for } 1 \le i < m+1,\ k = k'''_m(l). \tag{52}$$

We now put $\tau_k^{m+1} := \tau_l$ for $k = k'''_m(l)$. Then by choice of τ_l we have $\tau_k^{m+1} \ge l$, i.e. $(33)_{m+1}$. Inequalities (49) and (51) yield $(34)_{m+1}$, i.e.

$$\|u^{m+1} - \mathcal{E}_{\tau_k^{m+1}}(u_k \circ h_k^{m+1})\|_{H^{1,n}} \le \frac{1}{l}\,.$$

Furthermore, from (47), (50) and equivariance of $\mathcal{E}$ we obtain

$$d(\mathcal{E}u_k|_{B_{2\tau_k^{m+1}}(x_k^{m+1})\setminus B_{\tau_k^{m+1}}(x_k^{m+1})}, X) \le \frac{1}{l}\,;$$

hence $(35)_{m+1}$ holds. Moreover for $1 \le i \le m$ we have by (52)

$$\mathrm{dist}_{\mathbb{H}^{n+1}}(B_k^{m+1}, B_k^i) = \mathrm{dist}_{\mathbb{H}^{n+1}}(B_{\tau_k^{m+1}}(0), (h_k^{m+1})^{-1}B_k^i) \ge \tau_k^{m+1} \ge l \to \infty\,,$$

and $(36)_{m+1}$ follows.

For $1 \le i \le m$ we use $(33)_i$ and $(33)_{m+1}$ to conclude

$$t_k^{m+1,i} = \mathrm{dist}_{\mathbb{H}^{n+1}}(x_k^{m+1}, x_k^i) = \mathrm{dist}_{\mathbb{H}^{n+1}}(0, (h_k^{m+1})^{-1}x_k^i) \ge \tau_k^i + 2\tau_k^{m+1} \ge 3l\,;$$

in particular $(37)_{m+1}$ holds. The points $\omega_k^{i,m+1}$, $\omega_k^{m+1,i} \in S^n$ are therefore uniquely defined by (28) and we pass to a subsequence $\mathfrak{K}^{\mathrm{IV}}_m$ such that $(38)_{m+1}$ is satisfied.

In view of (47) and (50) the affine homotopy between $\mathcal{E}_{\tau_k^{m+1}}u^{m+1}$ and $\mathcal{E}_{\tau_k^{m+1}}(u_k \circ h_k^{m+1})$ is contained in $U_{1/l}(X)$ for $k \in k^{\mathrm{IV}}_m(l)$. But then from (48) we see that the sequence $(\mathcal{E}_{\tau_k^{m+1}}(u_k \circ h_k^{m+1}))_{k=k^{\mathrm{IV}}_m(l)}$ belongs to $\alpha^{m+1} \in \hat{\pi}_n(X)$ and that for $l,\ l' \ge \nu$ two members are freely homotopic in $U_{1/\nu}(X)$; this shows $(39)_{m+1}$.

By definition of $\delta^{m+1} \in (0, \delta(E)]$ we observe $A((\mathcal{E}u_k|_{\Omega_k^{m+1}})_{k\in\mathfrak{K}^{\mathrm{IV}}_m}) \subset \overline{U_{\delta^{m+1}}(X)}$. There are at most finitely many $k \in \mathfrak{K}^{\mathrm{IV}}_m$ with

$$\mathcal{E}u_k(\Omega_k^{m+1}) \cap \left(\overline{B}^N_{m+1}(0) \setminus U_{2\delta^{m+1}}(X)\right) \ne \emptyset\,.$$

Discarding finitely many $k \in \mathfrak{K}^{\mathrm{IV}}_m$ if necessary we obtain $(40)_{m+1}$ for $\mathfrak{K}^{\mathrm{IV}}_m$.

Now put $\mathfrak{K}_{m+1} = \mathfrak{K}^{\mathrm{IV}}_m$ so that the data (21)–(24) satisfy (30)–(40) with $m+1$ instead of m. By (27), (31), (32) and (37) the set $(u^i)_{i=1}^{m+1}$ is a weak limit set of the sequence $(u_k)_{k\in\mathfrak{K}_{m+1}}$ and theorem 1 of section 3 yields $(41)_{m+1}$. Finally (18), $(32)_{m+1}$ and $(41)_{m+1}$ (in the special case $F(\,\cdot\,) = E_\infty(\,\cdot\,)$) imply $(42)_{m+1}$; this finishes the inductive step.

Now if the induction stops for some $m \in \mathbb{N}_0$, then (29) is also true for that m and the lemma is proved. Otherwise we obtain $\mathfrak{K}_0 \supset \mathfrak{K}_1 \supset \mathfrak{K}_2 \supset \ldots$, such that (30)–(42) are true at each step and such that the data $(21)_{m+1}$–$(24)_{m+1}$ are constructed from $(21)_m$–$(24)_m$ by choice of the subsequence $\mathfrak{K}_{m+1} \subset \mathfrak{K}_m$. We then let $\mathfrak{K}_\infty := (k_l(l))_{l\in\mathbb{N}}$ be the diagonal sequence and put $m = +\infty$. By the observation at the beginning of the proof (30)–(40) are satisfied, and (41), (42) follow by letting $m \to \infty$ in the corresponding statements for finite m. ∎

Theorem 4 (topological compactness). *Let $X \subset \mathbb{R}^N$ be compact such that $U_\delta(X)$ is connected for any $\delta > 0$. Suppose $(u_k)_{k\in\mathbb{N}} \subset C^0(S^n, \mathbb{R}^N)$ is a nontrivial sequence (for X) with $d(u_k, X) \to 0$ as $k \to \infty$ and $E_\infty(u) \le E < \infty$ for all k. Then there is a subsequence $(u_k)_{k\in\mathfrak{K}}$ with the following properties:*

(53) *the sequence $(u_k)_{k\in\mathfrak{K}}$ has a weak limit set $(u^i)_{i=1}^m$ where $1 \le m \le \infty$;*

(54) *if we put $\alpha^i := \{u^i\} \in \hat{\pi}_n(X)$, then $(u_k)_{k\in\mathfrak{K}}$ belongs $\Pi_n(\boldsymbol{\alpha})$;*

(55) *if $\varepsilon_0(X) > 0$, then* $\operatorname{card}\{i : \|\alpha^i\| > 0\} \le \dfrac{E}{\varepsilon_0(X)} < \infty$.

Proof. We let $\mathfrak{K}$ be the subsequence determined in lemma 2 and let $(u^i)_{i=1}^m$ be the weak limit set given by (31). By the convex hull property of $\mathcal{E}$ there is an $R < \infty$ such that $\mathcal{E}u_k(B) \subset B_R^N(0)$ for all $k \in \mathfrak{K}$; in particular the assumption implies $A((\mathcal{E}u_k)_{k\in\mathbb{N}}) \setminus X$ is nonempty so that $m \ge 1$ by the construction in the proof of lemma 2. Let $\delta > 0$ be given. If $m < \infty$ then by (29) – again using the convex hull property of $\mathcal{E}$ – there exists $l(\delta) \in \mathbb{N}$ with

$$(\mathcal{E}u_k)(\Omega_k^m) \subset U_\delta(X) \ \text{ for } k = k_m(l),\ l \ge l(\delta). \tag{56}$$

By (32) we have $d(\mathcal{E}u^i, X) \le \delta^i$ and thus in particular $\|\alpha^i\| \le \delta^i$; then (42) implies (recall $I(\boldsymbol{\alpha}, \delta) = \{i : \|\alpha^i\| \ge \delta\}$)

$$m(\boldsymbol{\alpha}, \delta) = \operatorname{card} I(\boldsymbol{\alpha}, \delta) < \infty\,. \tag{57}$$

Moreover there exists $m(\delta) \in \mathbb{N}$ with $m(\delta) \ge R + 1$ and $2\delta^i < \delta$ for $i \ge m(\delta)$. Thus if $m = \infty$ we may use (40) to obtain

$$(\mathcal{E}u_k)(\Omega_k^{m(\delta)}) \subset U_\delta(X) \ \text{ for all } k \in \mathfrak{K}. \tag{58}$$

Let $\Omega_k := \Omega_k^m$ if $m < \infty$ and $\Omega_k := \Omega_k^{m(\delta)}$ if $m = \infty$; also put $M := m$ if $m < \infty$ and $M := m(\delta)$ if $m = \infty$. We have $i \le M$ for $i \in I(\boldsymbol{\alpha}, \delta)$ since $\|\alpha^i\| \le \delta^i < \delta$ for $i > M$. Let $1 \le i \le M$ and suppose $i \notin I(\boldsymbol{\alpha}, \delta)$; further let $l \ge \max\{l(\delta), [\frac{1}{\delta}] + 1\}$. By (39) the sequence $(u_l^i := \mathcal{E}u_k|_{\partial B_k^i})_{k=k_m(l)}$ belongs to $\alpha^i \in \hat{\pi}_n(X)$ with $u_l^i \sim u_{l'}^i$ in $U_{1/\nu}(X)$ for l, $l' \ge \nu$, and from the choice of l above we obtain

$$0 = i_\delta(\alpha^i) = \{u_l^i\} \in \pi_n(U_\delta(X))\,.$$

Thus for those i and l as above there exists an extension $E_k^i \in C^0(\overline{B}_k^i, U_\delta(X))$ of $\mathcal{E}u_k|_{\partial B_k^i}$. We replace $\mathcal{E}u_k$ on B_k^i by E_k^i to obtain $E_k \in C^0(\overline{B}, \mathbb{R}^N)$; here we use (36). In fact from (56) ((58) respectively) we see that

$$E_k \in C^0\Big(\overline{B} \setminus \bigcup_{i\in I(\boldsymbol{\alpha},\delta)} B_k^i, U_\delta(X)\Big)\,. \tag{59}$$

Now for $i \in I(\boldsymbol{\alpha},\delta)$ we have $E_k|_{\partial B_k^i} = u_l^i \in i_\delta(\alpha^i)$, by (39) and the choice of $l \geq \max\{l(\delta), [\frac{1}{\delta}]+1\}$. But lemma 4 of section 4 then yields for $k = k_m(l)$ and l as above

$$(i_\delta(\alpha^i))_{i\in I(\boldsymbol{\alpha},\delta)} \in \mathcal{D}(\{u_k\}) \text{ in } \pi_n(U_\delta(X)),$$

and this proves $(u_k)_{k\in\mathfrak{K}} \in \Pi_n(\boldsymbol{\alpha})$. Statement (55) follows from (41) (using $F(\,\cdot\,) = E_\infty(\,\cdot\,)$) and the definition of $\varepsilon_0(X)$. ∎

Corollary 1. *Under the assumptions of theorem 4 we have* $\varepsilon_*(X) = \varepsilon_0(X)$.

Proof. Since $\varepsilon_*(X) \leq \varepsilon_0(X)$ we may assume that $\varepsilon_*(X) < \infty$ and $\varepsilon_0(X) > 0$. For $\varepsilon \in (\varepsilon_*(X), 2\varepsilon_0(X))$ there exists by definition a nontrivial sequence $(u_k)_{k\in\mathbb{N}}$ with $\limsup_{k\to\infty} E_\infty(u_k) \leq \varepsilon$. This means that there is a $\delta > 0$ such that – after passing to a subsequence – none of the u_k is contractible in $U_\delta(X)$. Applying theorem 4 we see that we must have $\|\alpha^i\| > 0$ for exactly one i, $1 \leq i < m+1$, by (55). Therefore $\varepsilon_0(X) \leq E_*(\alpha^i) \leq \limsup_{k\to\infty} E_\infty(u_k) \leq \varepsilon$. Now let $\varepsilon \to \varepsilon_*(X)$. ∎

Theorem 5 (topological continuity with respect to convergence in energy). *Let $X \subset \mathbb{R}^N$ be compact such that $U_\delta(X)$ is connected for any $\delta > 0$. Let $(v_k)_{k\in\mathbb{N}} \subset C^0(S^n,\mathbb{R}^N)$ satisfy $d(v_k, X) \to 0$ as $k \to \infty$ and suppose that $(v_k)_{k\in\mathbb{N}}$ converges in energy to $(v^j)_{j\in J}$. Then $(v_k)_{k\in\mathbb{N}}$ belongs to $\Pi_n(\boldsymbol{\alpha})$ where $\alpha^j := \{v^j\} \in \hat{\pi}_n(X)$. In particular if $(v_k)_{k\in\mathbb{N}}$ belongs to $\alpha \in \hat{\pi}_n(X)$ then $(\{v^j\})_{j\in J}$ is a decomposition of α.*

Proof. Let $\mathfrak{K}$ be again the subsequence from lemma 2 and let $(u^i)_{i=1}^m$ be the weak limit set determined there; here $m = 0$ is possible. By theorem 2 of section 3 there exists an injection $\varphi\colon \{i : 1 \leq i < m+1\} \to J$ with $v^{\varphi(i)} = u^i$ modulo G.

We claim that $(\mathcal{E}v^j)(B) \subset X$ for $j \notin \operatorname{im}(\varphi)$. Namely, recalling the proof of theorem 2, we may assume after passing to a subsequence that we have $g_k \in G$ with $d(g_k, h_k^i) \to \infty$ for all $i < m+1$, and $v_k \circ g_k \to v^j$ weakly in $H^{1,n}(S^n,\mathbb{R}^N)$. For given $x \in B$ we let $y_k^i := (h_k^i)^{-1} g_k(x)$ and observe $\operatorname{dist}_{\mathbb{H}^{n+1}}(y_k^i, 0) \to \infty$. Furthermore

$$(\mathcal{E}v^j)(x) = \lim_{k\to\infty} \mathcal{E}(v_k \circ g_k)(x) = \lim_{k\to\infty} \mathcal{E}(v_k \circ h_k^i)(y_k^i)\,. \tag{60}$$

Let us first suppose that $g_k(x) \in B_k^i$ for some i, $1 \leq i < m+1$, and for k belonging to some subsequence $\mathfrak{K}' \subset \mathfrak{K}$. Then we use corollary 1 of section 4 and (50) (with $m+1$ replaced by i) to obtain $\operatorname{dist}(\mathcal{E}(v_k \circ h_k^i)(y_k^i), X) \to 0$ as $\mathfrak{K}' \ni k \to \infty$; this proves the claim in this case. Now otherwise for any $m_0 \in \mathbb{N}$ we have $g_k(x) \in \Omega_k^{m_0}$ for all k sufficiently large. In the case $m < \infty$ then we use $m_0 := m$ and (29) to conclude $\operatorname{dist}((\mathcal{E}v^j)(x), X) \leq \limsup_{k\to\infty} d(\mathcal{E}v_k|_{\Omega_k^m}, X) = 0$. If $m = +\infty$, then by (30) and the same argument we get $\operatorname{dist}((\mathcal{E}v^j)(x), X) \leq \delta^{m_0}$ for any $m_0 \in \mathbb{N}$, which proves – by letting $m_0 \to \infty$ and using (42) – the claim in the remaining case.

It now follows that for any subsequence $\mathfrak{K} \subset \mathbb{N}$ there exists a further subsequence $\mathfrak{K}' \subset \mathfrak{K}$ such that $(v_k)_{k\in\mathfrak{K}'}$ belongs to $\Pi_n(\boldsymbol{\alpha})$ where α^j is as in the statement of the theorem; this implies in fact that the whole sequence belongs to $\Pi_n(\boldsymbol{\alpha})$ and the theorem is proved. ∎

6 Minimization in homotopy classes

The following result describes the loss of compactness in minimizing sequences.

Theorem 6. *Let $X \subset \mathbb{R}^N$ be compact and $U_\delta(X)$ be connected for any $\delta > 0$. Suppose $(u_k)_{k\in\mathbb{N}}$ is a given miminizing sequence for F, α where $F(\,\bullet\,)$ is as in (3.1)–(3.3) and $0 < F_*(\alpha) < \infty$. Then there exists a subsequence $(u_k)_{k\in\mathfrak{K}}$ and a weak limit set $(u^i)_{i\in I}$ with the following properties:*

(1) *the sequence $(u_k)_{k\in\mathfrak{K}}$ converges in F to $(u^i)_{i\in I}$;*

(2) *the set $(\alpha^i := \{u^i\})_{i\in I} \subset \hat{\pi}_n(X)$ is a decomposition of α;*

(3) *$\|\alpha^i\| > 0$ and $F(u^i) = F_*(\alpha^i)$ for all $i \in I$, i.e. u^i minimizes F in its own homotopy class α^i;*

(4) *if $\varepsilon_0(X) := \inf\{F_*(\alpha) : \|\alpha\| > 0\} > 0$, then $\operatorname{card} I \le \dfrac{F_*(\alpha)}{\varepsilon_0(X)} < \infty$.*

Proof. We let $\mathfrak{K}$, $(i^i)_{i\in I}$ be as in theorem 4; then statement (2) follows from (5.54). We apply theorem 1 and lemma 1 of section 5 to obtain the inequalities

$$\begin{aligned}\sum_{i\in I} F(u^i) &\le \liminf_{k\to\infty} F(u_k) = \lim_{k\to\infty} F(u_k) = F_*(\alpha)\\ &\le \sum_{i\in I:\|\alpha^i\|>0} F_*(\alpha^i) \le \sum_{i\in I:\|\alpha^i\|>0} F(u^i)\,.\end{aligned}$$

The remaining statements follow. ∎

We say that $(\alpha^i)_{i\in I} \subset \hat{\pi}_n(X)$ is a *proper* decomposition of $\alpha \in \hat{\pi}_n(X)$ iff $\operatorname{card} I \ge 2$ and $\|\alpha^i\| > 0$ for all $i \in I$.

Definition. $\alpha \in \hat{\pi}_n(X)$ satisfies *Douglas's sufficient condition for F* if and only if for any proper decomposition $(\alpha^i)_{i=1}^m$ of α we have the strict inequality

$$F_*(\alpha) < \sum_{i=1}^m F_*(\alpha^i)\,. \tag{5}$$

Corollary 1. *Suppose $\alpha \in \hat{\pi}_n(X)$ satisfies Douglas's sufficient condition for F, where X is as in theorem 6. Then the set of minimizers $\mathcal{F}_*(\alpha) := \{u \in H^{1,n}(S^n, X) : \{u\} = \alpha\,,\ F(u) = F_*(\alpha)\}$ is nonempty and sequentially F-compact in the following sense: for any sequence $(u_k)_{k\in\mathbb{N}} \subset \mathcal{F}_*(\alpha)$ there exist $(h_k)_{k\in\mathbb{N}} \subset G$, a subsequence $\mathfrak{K} \subset \mathbb{N}$ and $u \in \mathcal{F}_*(\alpha)$ such that $u_k \circ h_k \to u$ weakly in $H^{1,n}(S^n, \mathbb{R}^N)$ as $\mathfrak{K} \ni k \to \infty$.*

Proof. By assumption we must have $\operatorname{card} I = 1$ in theorem 6 which implies that $\mathcal{F}_*(\alpha)$ is nonempty. Choose $R > 0$ such that $X \subset B_R(0)$ and consider the set $K(\alpha) := B_R(0) \setminus U_{\frac{\|\alpha\|}{2}}(X)$; then any $u \in \mathcal{F}_*(\alpha)$ satisfies (after conformal reparametrization) $(\mathcal{E}u)(0) \in K(\alpha)$. For a given sequence $(u_k)_{k\in\mathbb{N}} \subset \mathcal{F}_*(\alpha)$ we may therefore assume w.l.o.g. that $(\mathcal{E}u)(0) \in K(\alpha)$ for all $k \in \mathbb{N}$. Passing to a subsequence if necessary we see that $(u_k)_{k\in\mathbb{N}}$ converges weakly to a map $u \in H^{1,n}(S^n, X)$ with $(\mathcal{E}u)(0) \in K(\alpha)$; in particular u is not constant. Moreover, we may find a sequence $s_k \to \infty$ with the following properties:

$$(\mathcal{E}_{s_k} u_k)_{k\in\mathbb{N}} \in \alpha\,; \tag{6}$$

$$\|u_k - \mathcal{E}_{s_k} u_k\|_{H^{1,n}} \le \frac{1}{k} \to 0\,; \tag{7}$$

$$F(\mathcal{E}_{s_k} u_k) \to F_*(\alpha)\,. \tag{8}$$

It follows from (7) that $\mathcal{E}_{s_k} u_k$ also converges weakly to u. On the other hand, by theorem 6 and Douglas condition, we find – after again passing to a subsequence – elements $h_k \in G$ such that $(\mathcal{E}_{s_k} u_k) \circ h_k \to v \in H^{1,n}(S^n, X)$ weakly and $F(v) = F_*(\alpha)$; this means that $(\mathcal{E}_{s_k} u_k) \circ h_k$ converges in F to v. But then we obtain $v = u$ modulo G from theorem 2, and thus we have $F(u) = F(v) = \lim_{k\to\infty} F(\mathcal{E}_{s_k} u_k) = F_*(\alpha) \equiv F(u_k)$; this proves the claim. ■

Corollary 2. *Let $X \subset \mathbb{R}^N$ be a compact, pathwise connected neighborhood retract of $U_{\delta_0}(X)$. If $\alpha \in \pi_n(X)$ satisfies $F_*(\alpha) < \infty$, then there exists a decomposition $(\alpha^i)_{i\in I}$ with $\operatorname{card} I \leq \frac{F_*(\alpha)}{\lambda\lambda_n \delta_0^n} < \infty$ and $\sum_{i\in I} F_*(\alpha^i) = F_*(\alpha)$, such that each of the α^i is representable by an F-minimizing map.*

Proof. This follows directly from theorem 6, the definition (4.8) of λ_n, the definition of the coercivity constant λ, lemma 2 and corollary 4 in section 4. ■

Remark. The statement of corollary 2 can be rephrased as follows: Let $\Gamma_* = \{\tilde{\alpha} \in \pi_n(X,p) : F_*([\tilde{\alpha}]) < \infty\}$ and observe from lemma 1, section 4 that Γ_* is a subgroup of $\pi_n(X,p)$; clearly Γ_* is also closed with respect to the action of $\pi_1(X,p)$. It follows that Γ_* is generated by elements for which $F_*([\tilde{\alpha}])$ is attained; cf. [14], [10], [7]. Note that if X is a Lipschitz neighborhood retract then $\Gamma_* = \pi_n(X,p)$.

Corollary 3. *Let $X \subset \mathbb{R}^N$ be as in corollary 2, and let $\Gamma \subset \Gamma_*$ be a subgroup which is closed under the action of $\pi_1(X,p)$. Then there exists a free homotopy class $\alpha \in [\Gamma_*] \setminus [\Gamma]$ with $F_*(\alpha) = \inf\{F_*(\beta) : \beta \in [\Gamma_*] \setminus [\Gamma]\} =: F_*(\Gamma_* \setminus \Gamma)$, and for which $F_*(\alpha)$ is attained.*

Proof. We first note that the inclusion $X \subset U_{\delta_0}(X)$ and the retraction $R_{\delta_0}\colon U_{\delta_0}(X) \to X$ induce group isomorphisms i_{δ_0}, $r_{\delta_0} = i_{\delta_0}^{-1}$ between $\pi_n(X,p)$ and $\pi_n(U_{\delta_0}(X),p)$ which commute with the projections into the corresponding sets of free homotopy classes; cf. lemma 2 in section 4. Choose $\beta_k \in [\Gamma_*] \setminus [\Gamma]$ and $u_k \in i_{\delta_0}(\beta_k)$ such that $d(u_k, X) \leq \frac{1}{k}$ and $F(u_k) \leq F_*(\Gamma_* \setminus \Gamma) + \frac{1}{k}$. By theorem 4 of section 5 we may assume – after passing to a subsequence – that the sequence $(u_k)_{k\in\mathbb{N}}$ has a weak limit set $(u^i)_{i=1}^m$ with the following properties:

(9) for $\alpha^i := \{u^i\} \in \pi_n(X)$ we have $\|\alpha^i\| > 0$;

(10) $$m \leq \frac{F_*(\Gamma_* \setminus \Gamma)}{\lambda\lambda_n \delta_0^n};$$

(11) the sequence $(u_k)_{k\in\mathbb{N}}$ belongs to $\Pi_n(\boldsymbol{\alpha})$.

Suppose by contradiction that $\alpha^i \in [\Gamma]$ for all i, $1 \leq i \leq m$. By (11) for k sufficiently large we find $\tilde{\alpha}^i \in \pi_n(X,p)$ (*depending on* k), such that $[\tilde{\alpha}^i] = \alpha^i$ and $u_k \in i_{\delta_0}([\tilde{\alpha}^1 + \ldots + \tilde{\alpha}^m])$. Since Γ is closed under the action of $\pi_1(X,p)$ we have $\tilde{\alpha}^i \in \Gamma$ and therefore $u_k \in i_{\delta_0}([\Gamma])$; but this contradicts $u_k \in i_{\delta_0}(\beta_k) \in i_{\delta_0}([\Gamma_*] \setminus [\Gamma])$.

By renumbering we may thus assume $\alpha^1 \in [\Gamma_*] \setminus [\Gamma]$; but then from theorem 1 of section 3 we get

$$\sum_{i=1}^m F(u^i) \leq \lim_{k\to\infty} F(u_k) = F_*(\Gamma_* \setminus \Gamma) \leq F_*(\alpha^1) \leq F(u^1),$$

and the claim follows. ■

Remarks. Using corollary 3 we can inductively determine a sequence $(\tilde{\alpha}_j)_{j=1}^J \subset \Gamma_*$ such that $F_*([\tilde{\alpha}_j])$ is attained for all j and $F_*([\tilde{\alpha}_{j+1}]) = F_*(\Gamma_* \setminus \Gamma(\tilde{\alpha}_1, \ldots, \tilde{\alpha}_j))$ for $0 \leq j < J$;

here we denote by $\Gamma(\tilde{\alpha}_1, \ldots, \tilde{\alpha}_j)$ the smallest $\pi_1(X,p)$-invariant subgroup of Γ_* containing $\tilde{\alpha}_1, \ldots, \tilde{\alpha}_j$ and $\Gamma(\emptyset) := \{0\}$. Note that if the dimension of Γ_* – viewed as a module over the ring $\mathbb{Z}[\pi_1(X,p)]$ consisting of all formal sums $\sum_{l=1}^r \nu_l \gamma_l$ where $\nu_l \in \mathbb{Z}$ and $\gamma_l \in \pi_1(X,p)$ – is finite, then the process has to stop for some $J < \infty$. The statement of corollary 3 was first observed (for $n = 2$, the Dirichlet integral and C^3-submanifolds $X \subset \mathbb{R}^N$) in [10, thm. 5]; see also [7, cor. 4.4.3]. Note that these authors make crucial use of a-priori estimates in $C^{1,\mu}$; therefore their results do not apply if X is (for example) only a Lipschitz neighborhood retract, or if the functional is not differentiable.

References

[1] F. Bethuel and O. Rey, *Multiple solutions of the Plateau problem for nonconstant mean curvature.* Math. J. **73** (1994), 593-646.

[2] H. Brézis and J.M. Coron, *Convergence of solutions of H-systems or how to blow bubbles*, Comm. Pure Appl. Math. **37** (1984), 149-187.

[3] H. Brézis and L. Nirenberg, *Degree theory of BMO: Compact manifolds without boundaries*. Sel. Math. New Ser **1** (1995), 197-263.

[4] W. Ding and G. Tian, *Energy identity for a class of approximate harmonic maps from surfaces*, Comm. Anal. Geom. **4** (1995), 543-554.

[5] M. Giaquinta, G. Modica and J. Souček, *Variational problems for the conformally invariant integral $\int |du|^n$*. In: C. Bandle, J. Bemalmanns, M. Chipot, M. Grüter and J. Saint Jean Paulin (Eds.), Progress in partial differential equations: calculus of variations, applications. Longman London 1992.

[6] M. Gromov, *Pseudo-holomorphic curves in symplectic manifolds*, Invent. Math. **82** (1985), 307-347.

[7] J. Jost, *Two dimensional geometric variational problems.* Wiley Interscience, Chichester 1991.

[8] E. Kuwert, *Area-minimizing immersions of the disk with boundary in a given homotopy class*, Habilitationsschrift Universität Bonn, 1995.

[9] P.L. Lions, *The concentration compactness principle in the calculus of variations, the limit case*, Rev. Mat. Iberoamericana **1** (1985), 45-121 and 145-201.

[10] W.H. Meeks and S.T. Yau, *Topology of three-manifolds and the embedding problems in minimal surface theory*, Ann. Math. **112** (1980), 441–484.

[11] T. Parker, *Bubble tree convergence for harmonic maps*, to appear in J. Diff. Geom.

[12] J. Qing, *On singularities of the heat flow for harmonic maps from surfaces into spheres*, Comm. Anal. Geom. **3** (1995), 297-315.

[13] T. Riviére, *Minimizing fibrations and p-harmonic maps in homotopy classes from S^3 into S^2*, preprint 1996.

[14] J. Sacks and K. Uhlenbeck, *The existence of minimal immersions of 2-spheres*, Ann. of Math. **113** (1981), 1-24.

[15] M. Struwe, *On the evolution of harmonic maps of Riemannian surfaces.* Comm. Math. Helv. **60** (1985) 558-581.

[16] M. Struwe, *A global compactness result for elliptic boundary value problems involving limiting nonlinearities*, Math. Z. **187** (1984), 511-517.

[17] C. Taubes, *Path-connected Yang-Mills moduli spaces*, J. Diff. Geom. **25** (1986).

[18] B. White, *Infima of energy functionals in homotopy classes*, J. Diff. Geom. **17** (1984) 127-142.

[19] B. White, *Homotopy classes in Sobolev spaces and the existence of energy minimizing maps.* Acta. Math. **160** (1988), 1–17.

[20] R. Ye, *Gromov's compactness theorem for pseudo-holomorphic curves.* Trans. Am. Math. Soc. **342** (1994), 671-694.

Institut für Mathematik, Humboldt Universität zu Berlin
(Sitz: Ziegelstr. 13a), Unter den Linden 6, D-10099 Berlin, Germany
and
Mathematisches Institut der Universität Bonn
Wegelerstr. 10, D-53115 Bonn, Germany