Minimizing p–Harmonic Maps at a Free Boundary

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1 Introduction

In this paper we investigate the regularity properties of maps $u: M \to N$ between Riemannian manifolds which minimize locally the p–energy amongst maps satisfying a partially free boundary condition $u(\Sigma) \subset \Gamma$. The **parameter domain** M for our maps is a compact connected Riemannian manifold of dimension $m \geq 2$ and the **free boundary** Σ is a non-empty, relatively open subset of ∂M. As **target manifold** N we have a compact Riemannian manifold of dimension $n \geq 1$ which we assume to be isometrically embedded in $\mathbb{R}^{n+k}$ for some $k \geq 0$. The **supporting manifold** Γ for the free boundary values is a closed submanifold of N of dimension d, $0 \leq d \leq n$. We are then interested in mappings $u: M \to N$ of Sobolev class $W^{1,p}(M, N) := \{u \in W^{1,p}(M, \mathbb{R}^{n+k}) : u(x) \in N \text{ for almost all } x \in M\}$ which minimize locally the p–**energy**

$$E(u) = \int_M |\nabla u|^p \, dvol$$

with respect to the free boundary condition $u(\Sigma) \subset \Gamma$. Here $|\nabla u| = \{\sum_{i=1}^{n+k} |\nabla u^i|^2\}^{1/2}$. A map $u \in W^{1,p}(M, N)$ is termed to be **locally p–energy minimizing** on $M \cup \Sigma$ with respect to the free boundary condition $u(\Sigma) \subset \Gamma$ if there exists an open covering $\mathcal{X}$ of $M \cup \Sigma$ such that $E(u) \leq E(v)$ for every $v \in W^{1,p}(M, N)$ which satisfies $v(\Sigma) \subset \Gamma$ and which coincides with u outside X, for some $X \in \mathcal{X}$. A point $x \in M \cup \Sigma$ is called a **regular point** of u if u coincides with a continuous function on a neighbourhood of x in $M \cup \Sigma$. The set of regular points is denoted by $\operatorname{Reg} u$ and its complement $(M \cup \Sigma) \setminus \operatorname{Reg} u$ is termed the singular set $\operatorname{Sing} u$. By the Sobolev embedding theorem regularity in the case $p > m$ follows trivially. Therefore we restrict ourselves to the case $1 < p \leq m$. Our main result reads as follows.

Theorem. *If $u \in W^{1,p}(M, N)$ is locally p–energy minimizing on $M \cup \Sigma$ with respect to the free boundary condition $u(\Sigma) \subset \Gamma$, then*

$$\mathcal{H}\!-\dim(\Sigma \cap \operatorname{Sing} u) \leq m - [p] - 1 \,,$$

where $[p] := \max\{l \in \mathbb{N} : l \leq p\}$. Moreover, $\Sigma \cap \operatorname{Sing} u$ is discrete in $M \cup \Sigma$ if $m - 1 \leq p < m$.

With regard to interior regularity the corresponding theorem was proved by Schoen and Uhlenbeck [SU] in the quadratic case $p = 2$, and independently by Fuchs [Fu], Hardt and Lin [HL1], and Luckhaus [Lu] in the general case $1 < p \leq m$. Regularity for minimizing maps at a general free boundary was considered by Duzaar and Steffen [DS1], [DS2], and Hardt and Lin [HL2] in the case $p = 2$. Finally in [DG] the first author and Grotowski obtained an optimal partial regularity result when $\partial\Gamma \neq \varnothing$ is allowed and $p = 2$ (i.e. they studied a vectorvalued thin obstacle problem).

2 Notation and general assumptions

First we describe our assumptions on the parameter domain $M \cup \Sigma$. We assume that $M \cup \Sigma$ is a connected Riemannian manifold with boundary $\partial M = \Sigma \neq \varnothing$ and interior M of dimension $m \geq 2$ and differentiability class 2. Introducing local coordinates around $x_0 \in \Sigma$ we specialize the parameter domain M to the unit upper half ball $B^+ := \{x \in \mathbb{R}^m : |x| < 1\,,\ x^m > 0\}$ equipped with a C^1–Riemannian metric which is close to the Euclidean metric, and Σ to its equatorial part $D = \{x \in \mathbb{R}^m : |x| < 1\,,\ x^m = 0\}$. Then, similarly to [DS1, section 1] and [HL1, section 7], we may restrict ourselves to the situation where the metric is in fact Euclidean.

Next, we specify the assumptions on the target manifold N and the supporting manifold for the free boundary values Γ. We assume that N is a compact C^2–submanifold of $\mathbb{R}^{n+k}$, that Γ is a closed submanifold of N, and that Γ as a submanifold of $\mathbb{R}^{n+k}$ is of class C^2. These assumptions imply that N admits a uniform tubular neighbourhood $\mathbf{U}_\sigma(N) := \{q \in \mathbb{R}^{n+k} : \operatorname{dist}(q, N) < \sigma\}$ for some $\sigma > 0$, and that the associate nearest point map $\Pi\colon \mathbf{U}_\sigma(N) \to N$ is well-defined and Lipschitz continuous with Lipschitz constants satisfying

$$\operatorname{Lip}(\Pi|_{\mathbf{U}_{t\sigma}(N)}) \downarrow 1 \quad \text{as } t \downarrow 0.$$

Similarly, the nearest point map onto Γ, which is Lipschitz continuous and well-defined on $\mathbf{U}_\varrho(\Gamma)$ for some $\varrho > 0$, is denoted by R and satisfies

$$\operatorname{Lip}(R|_{\mathbf{U}_{t\varrho}(\Gamma)}) \downarrow 1 \quad \text{as } t \downarrow 0.$$

3 Extension and compactness

Throughout this section we use the notation $[\![\, p \,]\!] := \min\{l \in \mathbb{N} : p \leq l\}$.

3.1 Lemma (extension). *For $1 < p < \infty$, $\frac{[\![p]\!]-1}{p} < \beta < 1$, there exist constants $c_1(m, \beta, p)$ and $c_2(m, p)$ such that whenever $K > 0$, $\varepsilon \in \,]0, 1[$, $\lambda \in \,]0, 1]$ and $u, v \in W^{1,p}(S^+, \mathbb{R}^{n+k})$ with $u(S^+) \subset \Gamma$ and $v(S^+) \subset \Gamma$ satisfy*

(i) $$\int_{S^+} |\nabla u|^p + |\nabla v|^p + \frac{|u - v|^p}{\varepsilon^p}\, d\mathcal{H}^{m-1} \leq K^p$$

and

(ii) $$d := c_1 \varepsilon^{1-\beta} \lambda^{([\![p]\!]-m)/p} K < \varrho,$$

then there exists an extension $w \in W^{1,p}([0, \lambda] \times S^+, \mathbb{R}^{n+k})$ such that $w(0, x) = u(x)$, $w(\lambda, x) = v(x)$ for almost all $x \in S^+$, $w([0, \lambda] \times \partial S^+) \subset \Gamma$, and

(3.1) $$\int_{[0,\lambda] \times S^+} |\nabla w|^p\, d\mathcal{H}^m \leq c_2 (1 + \operatorname{Lip} R|_{\mathbf{U}_d(\Gamma)})^p \lambda (1 + (\frac{\varepsilon}{\lambda})^p) K^p$$

and

$$\operatorname{dist}(w(t,x),\operatorname{Im} u\cup\operatorname{Im} v)\le d \quad \textit{for } \mathcal{H}^m\textit{–almost–all } (t,x)\in[0,\lambda]\times S^+. \tag{3.2}$$

Proof. Like [DG] we assume $\lambda = 3^{-\nu}$ and decompose the unit cube $Q := [-1,1]^{m-1}$ in $\mathbb{R}^{m-1}$ into $3^{\nu(m-1)}$ cubes of edge length 2λ. For $\ell = 0,\dots,m-1$ we denote by Q^ℓ the ℓ–skeleton of this decomposition. Q^ℓ is the union of the closed ℓ–cells Q^ℓ_i. We define

$$Z := \{x\in Q : \operatorname{dist}(x,\partial Q)\le\lambda\}$$

and observe that there exists a bi–Lipschitz homeomorphism (with bi–Lipschitz constants not depending on λ) $\varphi : Z\to[0,\lambda]\times S^{m-2}$ such that for $\ell = 1,\dots,m-1$

$$\varphi(Q^\ell\cap Z\setminus\partial Q) = \,]0,\lambda]\times\varphi(Q^{\ell-1}\cap\partial Q). \tag{3.3}$$

The construction from [DG] yields a bi–Lipschitz homeomorphism $\psi : Q\to S^+$ (cf. (2.7) of [DG]) such that for $\ell = 0,\dots,m-1$

$$\int_{\psi(Q^\ell\setminus\partial Q)} |\nabla u|^p + |\nabla v|^p + \frac{|u-v|^p}{\varepsilon^p}\,d\mathcal{H}^\ell \le c_3(m)\lambda^{\ell-m+1}K^p. \tag{3.4}$$

Interpolating linearly on $[0,\lambda]\times Y$ between u and v, i.e.

$$z(t,x) := (1-\frac{t}{\lambda})u(x) + \frac{t}{\lambda}v(x)$$

where $Y := \psi(Q^{[\![p]\!]-1}\setminus\partial Q)$, we obtain $z\in W^{1,p}([0,\lambda]\times\overline{Y},\mathbb{R}^{n+k})$ satisfying

$$\int_{[0,\lambda]\times Y} |\nabla z|^p\,d\mathcal{H}^{[\![p]\!]} \le c_4(m)\lambda^{[\![p]\!]-m+1}(1+(\frac{\varepsilon}{\lambda})^p)K^p \tag{3.5}$$

and

$$|z(t,x)-u(x)| \le c_5(m,\beta,p)\varepsilon^{1-\beta}\lambda^{([\![p]\!]-m)/p}K\,.$$

In particular

$$\operatorname{dist}(z(t,x),\operatorname{Im} u\cup\operatorname{Im} v)\le c_5(m,\beta,p)\varepsilon^{1-\beta}\lambda^{([\![p]\!]-m)/p}K \tag{3.6}$$

for almost every $(t,x)\in[0,\lambda]\times Y$. (3.5) follows from (3.4), and (3.6) follows from [Lu, proof of Lemma 1].

Our aim now is to deform z on a neighbourhood of $[0,\lambda]\times\partial S^+$ such that the new mapping w will obey the free boundary condition $w(t,x)\in\Gamma$ for $x\in\partial S^+$, $t\in[0,\lambda]$, in addition to (3.5) and (3.6).

Using the bi-Lipschitz homeomorphisms φ and ψ, we will work on $[0,\lambda]\times S^{m-2}$ instead of a neighbourhood of ∂S^+ in S^+. We define

$$\tilde u : [0,\lambda]\times S^{m-2}\to\mathbb{R}^{n+k},\quad \tilde u := u\circ\psi\circ\varphi^{-1},$$
$$\tilde v : [0,\lambda]\times S^{m-2}\to\mathbb{R}^{n+k},\quad \tilde v := v\circ\psi\circ\varphi^{-1}.$$

From (3.3) and the definition of Y we infer $\psi^{-1}(Y) = Q^{[\![p]\!]-1}\setminus\partial Q$, and therefore, using (3.3),

$$\varphi(Z\cap\psi^{-1}(\overline{Y})) = [0,\lambda]\times\varphi(Q^{[\![p]\!]-2}\cap\partial Q) =: [0,\lambda]\times X.$$

We also define

$$\tilde{z} : [0,\lambda]^2 \times X \to \mathbb{R}^{n+k}, \quad \tilde{z} := z \circ (\mathrm{id} \times (\psi \circ \varphi^{-1})).$$

Then (3.4)–(3.6) directly imply

$$\int_{[0,\lambda]\times X} |\nabla \tilde{u}|^p + |\nabla \tilde{v}|^p + \frac{|\tilde{u}-\tilde{v}|^p}{\varepsilon^p}\, d\mathcal{H}^{[\![p]\!]-1} \le c_6(m)\lambda^{[\![p]\!]-m} K^p, \tag{3.7}$$

$$\int_{[0,\lambda]^2\times X} |\nabla \tilde{z}|^p \, d\mathcal{H}^{[\![p]\!]} \le c_7(m)\lambda^{[\![p]\!]-m+1}(1 + (\frac{\varepsilon}{\lambda})^p)K^p, \tag{3.8}$$

and

$$\operatorname{dist}(\tilde{z}(t,r,x), \operatorname{Im} u \cup \operatorname{Im} v) \le c_5(m,\beta,p)\varepsilon^{1-\beta}\lambda^{([\![p]\!]-m)/p} K \tag{3.9}$$

almost everywhere on $[0,\lambda]^2 \times X$.

For $s > 0$ we now define

$$\begin{aligned} a(t) &:= \frac{\lambda}{2} - |t - \frac{\lambda}{2}| \quad \text{for } 0 \le t \le \lambda, \\ A_s &:= \{(t, sa(t)) : 0 \le t \le \lambda\}, \\ D_s &:= \{(t,r) : 0 \le t \le \lambda,\, 0 \le r \le sa(t)\}. \end{aligned}$$

By the coarea formula we have

$$\int_{D_\sigma \times X} |\nabla \tilde{z}|^p \, d\mathcal{H}^{[\![p]\!]} = \int_0^\sigma \int_{A_s\times X} \frac{a(t)}{\sqrt{1+s^2}} |\nabla \tilde{z}|^p \, d\mathcal{H}^{[\![p]\!]-1}\, ds.$$

Therefore for each $\sigma \in]0,2]$ there exists $s \in [\frac{\sigma}{2}, \sigma]$ such that

$$\begin{aligned} \int_{A_s\times X} \frac{a(t)}{\sqrt{1+s^2}} |\nabla \tilde{z}|^p \, d\mathcal{H}^{[\![p]\!]-1} &\le \frac{2}{\sigma} \int_{D_\sigma \times X} |\nabla \tilde{z}|^p \, d\mathcal{H}^{[\![p]\!]} \\ &\le c_8(m)\sigma^{-1}\lambda^{[\![p]\!]-m+1}(1 + (\frac{\varepsilon}{\lambda})^p)K^p, \end{aligned} \tag{3.10}$$

the last inequality following from (3.8).

Let $\delta := \min\{\lambda,\, \varepsilon^{(1-\beta)/\mu}\}$ and $r \in]0,\delta]$. Then, for $x \in X$ the Sobolev inequality and (3.7) imply

$$\begin{aligned} |\tilde{u}(r,x) - \tilde{u}(0,x)| &\le c_9(m,p) r^\mu \Big(\int_{[0,\lambda]\times X} |\nabla \tilde{u}|^p \, d\mathcal{H}^{[\![p]\!]-1}\Big)^{1/p} \\ &\le c_{10}(m,p) r^\mu \lambda^{([\![p]\!]-m)/p} K, \end{aligned} \tag{3.11}$$

where $\mu := 1 - \frac{[\![p]\!]-1}{p}$. Note that $r^\mu \le \varepsilon^{1-\beta}$ and $\delta \le \lambda$. Then from (3.11) we infer

$$|\tilde{u}(r,x) - \tilde{u}(0,x)| \le c_{10}(m,p)\varepsilon^{1-\beta}\lambda^{([\![p]\!]-m)/p} K. \tag{3.12}$$

Recalling (3.9) and the definition of $\tilde{u}$ we obtain for $(t,r,x) \in [0,\lambda]^2 \times X$

$$|\tilde{z}(t,r,x) - \tilde{u}(r,x)| \le c_5(m,\beta,p)\varepsilon^{1-\beta}\lambda^{([\![p]\!]-m)/p} K. \tag{3.13}$$

Combining (3.12), (3.13), and assumption (ii) we infer for any $(t,r,x) \in [0,\lambda] \times [0,\sigma] \times X$

$$\operatorname{dist}(\tilde{z}(t,r,x), \Gamma) \le c_1(m,\beta,p)\varepsilon^{1-\beta}\lambda^{([\![p]\!]-m)/p} K = d < \varrho, \tag{3.14}$$

which, of course, yields that $R(\tilde{z}(t,r,x))$ is well–defined for all specified arguments (t,r,x).

We now let $\sigma := \frac{2\delta}{\lambda}$ (such that $\frac{\lambda}{2}\sigma = \delta$) and choose an $s \in [\frac{\sigma}{2}, \sigma]$ according to (3.10). In view of the inclusion $D_s \subset [0,\lambda] \times [0, s\frac{\lambda}{2}] \subset [0,\lambda] \times [0,\delta]$ we can define $\tilde{w} \in W^{1,p}([0,\lambda]^2 \times X, \mathbb{R}^{n-k})$ by

$$\tilde{w}(t,r,x) := \begin{cases} R(\tilde{z}(t,sa(t),x)) & \text{on } [0,\lambda] \times \{0\} \times X, \\ \tilde{z}(t,r,x) & \text{on } ([0,\lambda]^2 \setminus D_s \cup A_s) \times X, \\ \frac{r}{sa(t)} R(\tilde{z}(t,sa(t),x)) + (1 - \frac{r}{sa(t)})\tilde{z}(t,sa(t),x) & \text{on } D_s \times X. \end{cases}$$

On $D_s \times X$ we compute $\frac{\partial}{\partial t}\tilde{w}$, $\frac{\partial}{\partial r}\tilde{w}$, and $\nabla_x \tilde{w}$ and get, using $0 \le r \le sa(t)$ and $|a'| \equiv 1$,

$$\begin{aligned} |\frac{\partial}{\partial t}\tilde{w}(t,r,x)| &\le \frac{1}{a(t)} |R(\tilde{z}(t,sa(t),x)) - \tilde{z}(t,sa(t),x)| \\ &\qquad + |\frac{\partial}{\partial t} R(\tilde{z}(t,sa(t),x))| + |\frac{\partial}{\partial t}\tilde{z}(t,sa(t),x)|, \\ |\frac{\partial}{\partial r}\tilde{w}(t,r,x)| &= \frac{1}{sa(t)} |R(\tilde{z}(t,sa(t),x)) - \tilde{z}(t,sa(t),x)|, \\ |\nabla_x \tilde{w}(t,r,x)| &\le |\nabla_x R(\tilde{z}(t,sa(t),x))| + |\nabla_x \tilde{z}(t,sa(t),x)|. \end{aligned}$$

These inequalities together imply

$$\begin{aligned} |\nabla \tilde{w}(t,r,x)|^p &\le c_{11}(p)\Big\{\frac{1}{(sa(t))^p} \operatorname{dist}(\tilde{z}(t,sa(t),x),\Gamma)^p \\ &\qquad + (1 + \operatorname{Lip} R|_{\mathbf{U}_d(\Gamma)})^p |\nabla_{(t,x)}\tilde{z}(t,sa(t),x)|^p\Big\}. \end{aligned} \tag{3.15}$$

To estimate the first summand in the right hand side of (3.15) we observe that

$$\begin{aligned} \operatorname{dist}(\tilde{z}(t,sa(t),x),\Gamma) &\le |\tilde{z}(t,sa(t),x) - \tilde{u}(0,x)| \\ &\le \tfrac{t}{\lambda}|\tilde{u}(sa(t),x) - \tilde{v}(sa(t),x)| + \int_0^{sa(t)} |\frac{\partial}{\partial r}\tilde{u}(r,x)|\,dr \\ &\le \tfrac{t}{\lambda}|\tilde{u}(sa(t),x) - \tilde{v}(sa(t),x)| + (sa(t))^{1-1/p}\Big(\int_0^{sa(t)} |\frac{\partial}{\partial r}\tilde{u}(r,x)|^p dr\Big)^{\frac{1}{p}}. \end{aligned}$$

The same estimate with $\tilde{u}(0,x)$ replaced by $\tilde{v}(0,x)$ shows

$$\operatorname{dist}(\tilde{z}(t,sa(t),x),\Gamma) \le \tfrac{\lambda-t}{\lambda}|\tilde{u}(sa(t),x) - \tilde{v}(sa(t),x)| + (sa(t))^{1-1/p}\Big(\int_0^{sa(t)} |\frac{\partial}{\partial r}\tilde{v}(r,x)|^p dr\Big)^{\frac{1}{p}}.$$

Both inequalities together with the definition of $a(t)$ imply for $t \in [0,\lambda]$, $x \in X$,

$$\begin{aligned} \operatorname{dist}(\tilde{z}(t,sa(t),x),\Gamma)^p &\le c_{12}(p)\Big\{\frac{a(t)^p}{\lambda^p} |\tilde{u}(sa(t),x) - \tilde{v}(sa(t),x)|^p \\ &\qquad + (sa(t))^{p-1} \int_0^{sa(t)} |\frac{\partial}{\partial r}\tilde{u}(r,x)|^p + |\frac{\partial}{\partial r}\tilde{v}(r,x)|^p\,dr\Big\}. \end{aligned} \tag{3.16}$$

Integrating (3.16) over $D_s \times X$ we obtain, using (3.7),

$$\begin{aligned} &\int_{D_s \times X} (sa(t))^{-p} \operatorname{dist}(\tilde{z}(t,sa(t),x),\Gamma)^p \, d\mathcal{H}^{[\![p]\!]}(t,r,x) \\ &\le c_{12}(p)\Big\{s^{-p}\lambda^{1-p} \int_{[0,\lambda]\times X} |\tilde{u} - \tilde{v}|^p \, d\mathcal{H}^{[\![p]\!]-1} + \lambda \int_{[0,\lambda]\times X} |\nabla \tilde{u}|^p + |\nabla \tilde{v}|^p \, d\mathcal{H}^{[\![p]\!]-1}\Big\} \\ &\le c_{12}(p) c_6(m) \lambda^{[\![p]\!]-m+1}\Big(1 + \frac{\varepsilon^p}{s^p\lambda^p}\Big)K^p \end{aligned}$$

$$\le c_{13}(m,p)\lambda^{[\![p]\!]-m+1}\Big(1 + \max\{\varepsilon^{p(1-\frac{1-\beta}{\mu})}, (\frac{\varepsilon}{\lambda})^p\}\Big)K^p, \tag{3.17}$$

the last estimate following from $s \geq \frac{\sigma}{2} = \frac{\delta}{\lambda} = \min\{1, \lambda^{-1}\varepsilon^{\frac{1-\beta}{\mu}}\}$. Since $\beta > \frac{[\![p]\!]-1}{p}$ (by assumption) we have $\varepsilon^{p(1-\frac{1-\beta}{\mu})} < 1$, and from (3.17) we derive

$$\text{(3.18)}\quad \int_{D_s\times X} \frac{\operatorname{dist}(\tilde{z}(t,sa(t),x),\Gamma)^p}{sa(t)^p}\, d\mathcal{H}^{[\![p]\!]}(t,r,x) \leq 2c_{13}(m,p)\lambda^{[\![p]\!]-m+1}K^p(1+(\frac{\varepsilon}{\lambda})^p).$$

Now, we estimate the integral of the second summand of the right hand side of (3.15). Using (3.10) we find

$$\begin{aligned}
&\int_{D_s\times X} |\nabla_{(t,x)}\tilde{z}(t,sa(t),x)|^p\, d\mathcal{H}^{[\![p]\!]} \\
&\leq 2^{p/2}s\int_X\int_0^\lambda \frac{a(t)}{\sqrt{1+s^2}}|\nabla\tilde{z}|^p(t,sa(t),x)\sqrt{1+s^2}\,dt\, d\mathcal{H}^{[\![p]\!]-2}x \\
&= 2^{p/2}s\int_{A_s\times X}\frac{a(t)}{\sqrt{1+s^2}}|\nabla\tilde{z}|^p\, d\mathcal{H}^{[\![p]\!]-1} \\
&\leq c_{14}(m,p)\frac{s}{\sigma}\lambda^{[\![p]\!]-m+1}(1+(\frac{\varepsilon}{\lambda})^p)K^p \\
\text{(3.19)}\qquad &\leq c_{14}(m,p)\lambda^{[\![p]\!]-m+1}(1+(\frac{\varepsilon}{\lambda})^p)K^p.
\end{aligned}$$

Combining (3.15), (3.18), and (3.19) we finally arrive at

$$\text{(3.20)}\qquad \int_{D_s\times X}|\nabla\tilde{w}|^p\, d\mathcal{H}^{[\![p]\!]} \leq c_{15}(m,p)(1+\operatorname{Lip} R_{|\mathbf{U}_d(\Gamma)})^p\lambda^{[\![p]\!]-m+1}(1+(\frac{\varepsilon}{\lambda})^p)K^p.$$

Now, we define $w \in W^{1,p}([0,\lambda]\times\overline{Y},\mathbb{R}^{n+k})$ compatible with the free boundary condition by

$$w := \begin{cases} \tilde{w}\circ(\operatorname{id}\times(\varphi\circ\psi^{-1})) & \text{on } [0,\lambda]\times\overline{Y}\cap\psi(Z), \\ z & \text{on } [0,\lambda]\times Y\setminus\psi(Z). \end{cases}$$

The estimates (3.5), (3.8), and (3.20) yield

$$\text{(3.21)}\qquad \int_{[0,\lambda]\times Y}|\nabla w|^p\, d\mathcal{H}^{[\![p]\!]} \leq c_{16}(m,p)\lambda^{[\![p]\!]-m+1}(1+(\frac{\varepsilon}{\lambda})^p)K^p,$$

and from (3.6), (3.9) and (3.14) we obtain

$$\text{(3.22)}\qquad \operatorname{dist}(w(t,x),\operatorname{Im}u\cup\operatorname{Im}v) \leq c_{17}(m,\beta,p)\varepsilon^{1-\beta}\lambda^{([\![p]\!]-m)/p}K$$

for almost all $(t,x)\in[0,\lambda]\times\overline{Y}$. The inductive procedure of homogeneous extension from [Lu] together with the modifications described in [DG] can now be used to construct the map $w \in W^{1,p}([0,\lambda]\times S^+,\mathbb{R}^{n+k})$ satisfying the assertions of the lemma. ■

We apply the extension lemma to prove the following compactness theorem for p–energy minimizing maps at a free boundary.

3.2 Theorem (compactness). *Suppose Γ_i, N_i, and $u_i \in W^{1,p}(B^+,N_i)$ with $u_i(D)\subset\Gamma_i$, $i\in\mathbb{N}\cup\{\infty\}$, satisfy:*

(i) N_i admits a Lipschitz neighbourhood retraction $\Pi_i\colon \mathbf{U}_{\sigma_i}(N_i)\to N_i$ which satisfies $\lim_{\sigma\to 0}\operatorname{Lip}(\Pi_i|_{\mathbf{U}_\sigma(N_i)}) = 1$;

(ii) Γ_i *admits a Lipschitz neighbourhood retraction* $R_i\colon \mathbf{U}_{\varrho_i}(\Gamma_i) \to \Gamma_i$ *for which*

$$\liminf_{i\to\infty} \varrho_i > 0 \quad \textit{and} \quad \liminf_{i\to\infty} \mathrm{Lip}\,(R_i|_{\mathbf{U}_t(\Gamma_i)}) < \infty\,;$$

(iii) *each* $v \in W^{1,p}(B^+, \mathbb{R}^{n+k})$ *with* $v(B^+) \subset N_\infty$ *and* $v(D) \subset \Gamma_\infty$ *is the* $W^{1,p}$*–limit of maps* $v_i \in W^{1,p}(B^+, \mathbb{R}^{n+k})$ *with* $v_i(B^+) \subset N_i$ *and* $v_i(D) \subset \Gamma_i$*; and*

(iv) *the* u_i *are* p*–energy minimizing maps from* B^+ *into* N_i *with respect to the free boundary condition* $u_i(D) \subset \Gamma_i$ *for* $i \in \mathbb{N}$ *and converge weakly in* $W^{1,p}(B^+, \mathbb{R}^{n+k})$ *to* u_∞*.*

Then u_∞ *is* p*–energy minimizing in* $W^{1,p}(B^+, N_\infty)$ *subject to the free boundary condition* $u_\infty(D) \subset \Gamma_\infty$*, and* $u_i \to u_\infty$ *strongly in* $W^{1,p}(B^+_\sigma, \mathbb{R}^{n+k})$ *for any* $0 < \sigma < 1$*.*

The assertions of the compactness theorem follow from our extension lemma using a direct adaptation of the arguments from [Lu, p. 357 ff] to the free boundary situation considered here. One merely has to replace balls B_σ and spheres S_σ by half-balls B^+_σ and hemi-spheres S^+_σ before applying the extension lemma.

3.3 Corollary. *Suppose* $(u_i)_{i\in\mathbb{N}} \subset W^{1,p}(B^+, N)$ *is a sequence of* p*–energy minimizing maps subject to the free boundary condition* $u_i(D) \subset \Gamma$*, and*

$$\sup_{i\in\mathbb{N}} \int_{B^+} |\nabla u_i|^p \, dx < \infty\,.$$

Then there exists a subsequence (u_i) *and a map* $u \in W^{1,p}(B+, N)$ *which is* p*–energy minimizing w.r.t. the free boundary condition* $u(D) \subset \Gamma$ *such that* u_i *converges strongly in* $W^{1,p}(B^+_\sigma, \mathbb{R}^{n+k})$ *to* u *for any* $0 < \sigma < 1$*.*

4 An ε–regularity theorem

Our line of reasoning follows exactly to that of [DG, Section 3] which, of course, is based on [Lu, Proposition 1] (see also [St, Theorem 10.3]). One merely has to replace the scaled 2-energy $\sigma^{2-m}\int_{B^+_\sigma(a)} |\nabla u|^2\,dx$ by $\sigma^{p-m}\int_{B^+_\sigma(a)} |\nabla u|^p\,dx$ for half-balls $B^+_\sigma(a) := \{x \in \mathbb{R}^m : |x-a| < \sigma\,,\ x^m > 0\} \subset B^+$ centered at $a \in D$ by the scaled p–energy $\sigma^{p-m}\int_{B^+_\sigma(a)} |\nabla u|^p dx$, and, instead of [DG, Lemma 3.1], to use:

4.1 Lemma. *Let* $\xi \in W^{1,p}(B^+, \mathbb{R}^n)$ *be* p*–energy minimizing with respect to the free boundary condition* $\xi(D) \subset \mathbb{R}^d \times \{0\} \subset \mathbb{R}^n$*. Then, for any* $0 < \sigma < 1$ *we have*

$$\int_{B^+_\sigma} |\nabla \xi|^p \, dx \le c\sigma^n \int_{B^+} |\nabla \xi|^p \, dx\,,$$

where c *is a constant depending on* m*,* n *and* p *only.*

Proof. For $j = 1, \ldots, d$ we define $\tilde{\xi}^j$ to be the extension of ξ^j to B by even reflection (across D). Moreover, for $j = d+1, \ldots, n$ we let $\tilde{\xi}^j$ to be the extension of ξ^j to B by odd reflection. Then it is easy to check that $\tilde{\xi} := (\tilde{\xi}^1, \ldots, \tilde{\xi}^d)$ is weakly p–harmonic on B, i.e. for any $\varphi \in C_0^1(B, \mathbb{R}^n)$ we have

$$\int_B |\nabla\tilde{\xi}|^{p-2} \nabla\tilde{\xi} \cdot \nabla\varphi \, dx = 0 \, .$$

Hence, from [U, Theorem 3.2] we infer

$$\int_{B_\sigma} |\nabla\tilde{\xi}|^p \, dx \le c(m,n,p)\sigma^m \int_B |\nabla\tilde{\xi}|^p \, dx$$

which clearly yields the corresponding estimate for ξ on B_σ^+. ■

As in the case $p = 2$ [DG, Theorem 3.4] (see also [DS1, Theorem 5.2], [HL2, Theorem 3.4]) we can now state an ε–regularity theorem for minimizing p–harmonic maps at a free boundary.

4.2 Theorem. *Given N and Γ satisfying the assumptions given in section 2, and $\alpha \in]0,1[$, there exist constants C and ε_0 depending on α, m, n, p, N and Γ only such that every map $u \in W^{1,p}(B^+, N)$ which is p–energy minimizing w.r.t. the free boundary condition $u(D) \subset \Gamma$ and which has small scaled p–energy*

$$\varepsilon^p := \sigma^{p-m} \int_{B_\sigma^+(x_0)} |\nabla u|^p \, dx \le \varepsilon_0^p$$

for some half-ball $B_\sigma^+(x_0) \subset B^+$, $x_0 \in D$, $|x_0| \le \frac{1}{2}$, satisfies

$$r^{p-m} \int_{B_r^+(x_1)} |\nabla u|^p \, dx \le C\varepsilon^p \left(\frac{r}{\sigma}\right)^{p\alpha}$$

for all $x_1 \in B_{\sigma/2}^+(x_0) \cap D$, $0 < r \le \frac{\sigma}{2}$. ■

Combining Theorem 4.2 with the corresponding interior result [Fu], [HL1], [Lu] we see that p–energy minimizing maps at a free boundary are Hölder continuous with exponent α for all $0 < \alpha < 1$ on $B_{1/4}^+$ provided $\int_{B^+} |\nabla u|^p dx \le \varepsilon_1^p$ where $\varepsilon_1 > 0$ is a constant depending on α, n, m, p, N and Γ only. (Note that we also have to use the monotonicity formula Lemma 5.1.)

5 Partial regularity

We begin this section by deriving a **monotonicity formula** for maps $u \in W^{1,p}(B^+, N)$ which minimize the p–energy subject to the free boundary condition $u(D) \subset \Gamma$ (cf. [P], [SU], [Fu], [HL1], [Lu] for the interior case, and [DS1], [HL2], [DG] for the case $p = 2$ at the free boundary). Given $\varphi \in C_0^1(B, \mathbb{R}^m)$ satisfying $\varphi^m|_D = 0$ we define X_t to be the solution of $\dot{X}_t = \varphi \circ X_t$ with initial condition $X_0 = \mathrm{id}$. Then $u_t := u \circ X_t$ is an admissible variation for u, and the minimizing property of u yields

$$0 = \frac{d}{dt} E(u_t)\Big|_0 = \int_{B^+} \sum_{i,j=1}^m \left(|\nabla u|^p \delta_{ij} - p \nabla_i u \cdot \nabla_j u |\nabla u|^{p-2} \right) \nabla_i \varphi^j \, dx \, .$$

Then, exactly as in [DG, Theorem 4.1] we prove

5.1 Theorem. *Let $a \in B^+ \cup D$ and $0 < r < \sigma < 1 - |a|$ be given. Then, any map $u \in W^{1,p}(B^+, N)$ minimizing the p–energy w.r.t. the free boundary condition $u(D) \subset \Gamma$ satisfies:*

$$
(5.1) \quad \sigma^{p-m}\left[\int_{B_\sigma^+(a)} |\nabla u|^p dx + \int_{B_\sigma^+(a^*)} |\nabla u|^p dx\right] - r^{p-m}\left[\int_{B_r^+(a)} |\nabla u|^p dx + \int_{B_r^+(a^*)} |\nabla u|^p dx\right]
$$

$$
= p\left[\int_{B_\sigma^+(a)\setminus B_r^+(a)} R^{p-m}|\nabla u|^{p-2}\left|\frac{\partial u}{\partial R}\right|^2 dx + \int_{B_\sigma^+(a^*)\setminus B_r^+(a^*)} (R^*)^{p-m}|\nabla u|^{p-2}\left|\frac{\partial u}{\partial R^*}\right|^2 dx\right],
$$

where $a^* = (a^1, \ldots, a^{m-1}, -a^m)$, $R = |x - a|$, and $R^* = |x - a^*|$. In particular,

$$
r^{p-m}\left[\int_{B_r^+(a)} |\nabla u|^p \, dx + \int_{B_r^+(a^*)} |\nabla u|^p \, dx\right]
$$

is monotone non-decreasing on $]0, 1 - |a|[$. ■

As a first consequence of the monotonicity formula we observe that the **density function**

$$
\Theta_u(a) := \lim_{r\to 0} r^{p-m}\left[\int_{B_r^+(a)} |\nabla u|^p \, dx + \int_{B_r^+(a^*)} |\nabla u|^p \, dx\right]
$$

is well defined for $a \in B^+ \cup D$. Note that

$$
\Theta_u(a) = \begin{cases} \lim\limits_{r\to 0} r^{p-m} \displaystyle\int_{B_r(a)} |\nabla u|^p \, dx & \text{for } a \in B^+, \\ 2\lim\limits_{r\to 0} r^{p-m} \displaystyle\int_{B_r^+(a)} |\nabla u|^p \, dx & \text{for } a \in D. \end{cases}
$$

5.2 Theorem. *Suppose that $u_j \in W^{1,p}(B^+, N)$, $j \in \mathbb{N}$, is p–energy minimizing subject to the free boundary condition $u_j(D) \subset \Gamma$. Suppose also that $u_j \to u$ strongly in $L^p(B^+, \mathbb{R}^{n+k})$ and that*

$$
\sup_{j\in\mathbb{N}} \int_{B^+} |\nabla u_j|^p \, dx < \infty \, .
$$

Then for $a \in B^+ \cup D$, $a_j \to a$ implies

$$
\Theta_u(a) \geq \limsup_{j\to\infty} \Theta_{u_j}(a_j) \, ,
$$

i.e. the density function is jointly upper semicontinuous.

Proof. In view of Corollary 3.3 we have $u_j \to u$ strongly in $W^{1,p}(B_\sigma^+, \mathbb{R}^{n+k})$ for any $0 < \sigma < 1$. For $a \in D$ consider $r, \varepsilon > 0$ such that $r + \varepsilon < 1 - |a|$. From (5.1) we infer

$$
\Theta_{u_j}(a_j) \leq r^{p-m}\left[\int_{B_r^+(a_j)} |\nabla u_j|^p \, dx + \int_{B_r^+(a_j^*)} |\nabla u_j|^p \, dx\right] \leq 2r^{p-m} \int_{B_{r+\varepsilon}^+(a)} |\nabla u_j|^p \, dx
$$

for any j such that $|a - a_j| < \varepsilon$. Now, the strong $W^{1,p}$–convergence of u_j on $B_{r+\varepsilon}^+(a)$ yields

$$
\limsup_{j\to\infty} \Theta_{u_j}(a_j) \leq 2r^{p-m} \int_{B_{r+\varepsilon}^+(a)} |\nabla u|^p \, dx \, .
$$

Letting first ε and then r tend to zero, the assertion follows. The case $a \in B^+$ follows similarly. ∎

Next we discuss **tangent maps at the free boundary**. For $a \in D$, $0 < r \le r_0 < 1 - |a|$ we define the rescaled map

$$u_{a,r}(x) := u(a + rx) \quad \text{for } x \in B^+_{r/r_0}.$$

Then, the monotonicity formula and Corollary 3.3 provide as in [Si, Section 3] (see also [DG, Section 4]) that a sequence $(u_{a,r_i})_{i\in\mathbb{N}}$, $r_i \to 0$, converges strongly in $W^{1,p}(B^+_\sigma, \mathbb{R}^{n+k})$ for any $\sigma \in]0,\infty[$ to a map $\varphi\colon \mathbb{R}^m_+ \to N$ which is p–energy minimizing w.r.t. the free boundary condition $\varphi(\mathbb{R}^{m-1} \times \{0\}) \subset \Gamma$. Any such map is called a (free boundary) **tangent map to u at a**.

We now follow the arguments of [Si, Section 3] almost verbatim. First we deduce the following important properties of tangent maps at the free boundary:

(5.2) $\qquad \Theta_\varphi(0) \equiv 2r^{p-m} \int_{B_r^+} |\nabla\varphi|^p \, dx = \Theta_u(a)$ for all $r > 0$;

(5.3) $\qquad \varphi$ is homogeneous of degree zero;

(5.4) $\qquad a \in \operatorname{Reg} u \Rightarrow u$ has a constant tangent map at a; and

(5.5) $\qquad$ for $a \in B^+ \cup D$ we have: $\Theta_u(a) = 0 \Leftrightarrow a \in \operatorname{Reg} u$.

5.3 Theorem. *$\mathcal{H}^{m-p}(\operatorname{Sing}(u)) = 0$, in particular $\operatorname{Sing}(u) = \varnothing$ for $p = m$.*

Next we consider homogeneous degree zero maps $\varphi\colon \mathbb{R}^m_+ \to N$ which minimize locally the p–energy subject to the free boundary condition $\varphi(\mathbb{R}^{m-1} \times \{0\}) \subset \Gamma$. Then:

(5.6) $\qquad \Theta_\varphi(\cdot)$ achieves its maximum at 0;

(5.7) $\qquad \varphi \circ \tau_a = \varphi$ for any $a \in \mathcal{S}(\varphi) := \{b \in \mathbb{R}^{m-1} \times \{0\} : \Theta_\varphi(b) = \Theta_\varphi(a)\}$;

(5.8) $\qquad \mathcal{S}(\varphi)$ is a linear subspace of $\mathbb{R}^{m-1} \times \{0\}$;

(5.9) $\qquad$ if $\dim \mathcal{S}(\varphi) = m - [p] + 1$ then $\varphi \equiv$ const; and

(5.10) $\qquad \mathcal{S}(\varphi) \subset \operatorname{Sing}(\varphi)$ if φ is non-constant.

Here $\tau_a(x) := x + a$ for $x \in \mathbb{R}^m$. Finally, we return to the situation of $u \in W^{1,p}(B^+, N)$ being p–energy minimizing w.r.t. $u(D) \subset \Gamma$. For $j = 0, \dots, m - [p]$ we define:

$$\mathcal{S}_j := \{a \in \operatorname{Sing} u \cap D : \dim \mathcal{S}(\varphi) \le j \text{ for every tangent map } \varphi \text{ to } u \text{ at } a\}.$$

5.4 Theorem.

(i) $\mathcal{S}_0 \subset \mathcal{S}_1 \subset \dots \mathcal{S}_{m-[p]-1} \subset \mathcal{S}_{m-[p]} = \operatorname{Sing} \cap D$;

(ii) for each $\lambda > 0$, $\mathcal{S}_0 \cap \{b : \Theta_u(b) = \lambda\}$ *is discrete; and*

(iii) for $j = 0, \dots, m - [p] - 1$ *we have* $\mathcal{H}\text{-}\dim \mathcal{S}_j \le j$.

As an immediate consequence of Theorem 5.4, (iii) and the corresponding interior regularity result of [Fu], [HL1], [Lu] we obtain

5.5 Theorem. *Let $u \in W^{1,p}(B_+, N)$ be p–energy minimizing with respect to the free boundary condition $u(D) \subset \Gamma$. Then*

$$\mathcal{H}-\dim(\operatorname{Sing} u) \leq m - [p] - 1 \quad \textit{if } 1 < p < m-1.$$

In the case $m - 1 \leq p < m$, $\operatorname{Sing} u \cap (D \cup B^+)$ consists of isolated points. ■

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