CLUSTER PROPERTIES OF ONE PARTICLE SCHRÖDINGER OPERATORS. II

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Abstract

We continue the study of cluster properties of spectral and scattering characteristics of Schrödinger operators with potentials given as a sum of two wells, begun in our preceeding article [Rev. Math. Phys. **6** (1994) 833 – 853] and where we determined the leading behaviour of the spectral shift function and the scattering amplitude as the separation of the wells tends to infinity. In this article we determine the explicit form of the subleading contributions, which in particular show strong oscillatory behaviour. Also we apply our methods to the critical and subcritical double well problems.

1 Introduction

We consider Schrödinger operators with double well potentials when the distance between the wells tends to infinity. More precisely the main subject of our study is the asymptotic behaviour as $|d| \to \infty$ of Hamiltonians in $L^2(\mathbb{R}^3)$ of the form

$$H(d) = -\Delta + V_d, \;\; d \in \mathbb{R}^3, \tag{1.1}$$

with

$$V_d = V_1 + V_2(\cdot - d). \tag{1.2}$$

Here Δ is the Laplace operator, and V_1 and V_2 are real valued functions in the Rollnik class $\mathcal{R}$, acting as multiplication operators on $L^2(\mathbb{R}^3)$. Recall that $V \in \mathcal{R}$ iff

$$\int \frac{|V(x)||V(y)|}{|x-y|^2} dx dy < \infty.$$

The present work is a direct continuation of our previous paper [27], where we established the limiting interrelations (as $|d| \to \infty$) between the resolvent, scattering matrix and the spectral shift function of $H(d)$ and those of the single well Hamiltonians

$$H_i = -\Delta + V_i, \;\; i = 1, 2. \tag{1.3}$$

Following the terminology of many particle scattering theory [16, 19, 20] we call the aforementioned relations cluster properties. The main aim of the present article is to exhibit the explicit structure of the subleading terms in the asymptotic expansion of the spectral shift function and the scattering amplitude when $|d| \to \infty$.

Cluster properties of the discrete spectrum of $H(d)$ have been the subject of intensive study above all due to the method of Born and Oppenheimer in the theory of molecules. This property can be described as follows [15]: every eigenvalue of $H(d)$ (1.1) in the limit $|d| \to \infty$ tends to some eigenvalue of H_1 or H_2. If H_1 and H_2 have a common eigenvalue (say E_0) then $H(d)$ has a pair of eigenvalues $E_\pm(d)$, which are asymptotically degenerated, i.e. $E_\pm(d) \to E_0$ for $|d| \to \infty$. The splitting between asymptotically degenerated eigenvalues has been studied in [15, 25, 9].

Cluster properties of the discrete spectrum for long-range potentials which are not in $\mathcal{R}$ (in particular, Coulomb potentials, $V_i(x) = |x|^{-1}$) were discussed in [26, 30, 15, 3, 8, 31]. It is interesting to note that for such potentials the limit $|d| \to \infty$ is essentially equivalent to the semiclassical (large coupling constant) limit, since in the case of the Coulomb potential, say, $H(d)$ is unitarily equivalent (up to a scaling factor $|d|^{-2}$) to the Hamiltonian

$$-\Delta - |d|\Big(\frac{1}{|x|} + \frac{1}{|x-\hat{d}|}\Big), \tag{1.4}$$

with $\hat{d} = d|d|^{-1}$.

The study of the so-called critical double well problem was initiated by Ovchinnikov and Sigal [34] in order to investigate the Efimov effect [12] and by Klaus and Simon [24]. These authors considered the potential V_d (1.2) where V_i have compact support and the Hamiltonians H_i (1.3) have no negative bound states, but have zero energy resonances (see below for the precise definition of this notion). Under these assumptions Klaus and Simon [24] showed that for large $|d|$ the operator $H(d)$ has the only eigenvalue

$$E(d) = -\alpha^2/d^2 + O(|d|^{-3}), \tag{1.5}$$

where α is the (unique) real solution of the equation $e^{-\alpha} = \alpha$. Later Høegh-Krohn and Mebkhout [17] found an infinite sequence of resonances $E_n(d)$, $n = 1, 2, \ldots$, tending to zero as $|d| \to \infty$, such that $E_n(d) = -\gamma_n^2/d^2 + O(|d|^{-3})$. The constants γ_n do not depend on the potentials V_i and are the complex solutions of $e^{-\alpha} = \alpha$. On the other hand Tamura [43] extended the results of [24] to the case of potentials with noncompact supports satisfying $|V_i(x)| \leq C(1+|x|)^{-2-\epsilon}$, $\epsilon > 0$ (however with a lack of uniqueness, i.e. the proof in [43] does not prohibit the existence of other eigenvalues different from $E(d)$ (1.5) and tending to zero as $|d| \to \infty$).

The low-energy scattering properties of the Hamiltonian $H(d)$ (1.1) as $|d| \to \infty$ were studied in [18], displaying the connection between (1.1) and Hamiltonians with point interaction. Let h_0 be the Hamiltonian of the two-fixed-center point interaction, which is formally

$$h_0 = -\Delta + \nu_1 \delta(x) + \nu_2 \delta(x - \hat{d}). \tag{1.6}$$

The rigorous meaning of h_0 and a detailed investigation of its properties can be found in [2]. Let W_r $(r > 0)$be the unitary scaling in $L^2(\mathbb{R}^3)$ given by $(W_r\psi)(x) = r^{3/2}\psi(rx)$. Then $W_{|d|} H(d) W_{|d|}^{-1} = |d|^{-2} h_\epsilon$, where

$$h_\epsilon = -\Delta + \epsilon^{-2}\left(V_1\left(\frac{x}{\epsilon}\right) + V_2\left(\frac{x-\hat{d}}{\epsilon}\right)\right),$$

with $\epsilon = |d|^{-1}$. When $\epsilon \to 0$ for compactly supported $V_i \in \mathcal{R}$, h_ϵ converges to h_0 (1.6) in the norm sense [2]. The limit is, however, very delicate since it depends very crucially on detailed properties of the spectral point zero for one center operators (1.3). One can expect that all zero-energy characteristics of $H(d)$ in the limit $|d| \to \infty$ are well described by the Hamiltonian h_0 (1.6). Indeed, it is proven in [18] for compactly supported V_i's that

$$S_d(E/d^2) = s_\epsilon(E),$$

and

$$s_\epsilon(E) = s_0(E) + O(\epsilon), \tag{1.7}$$

again with $\epsilon = |d|^{-1}$. Here the operators $S_d(E)$, $s_\epsilon(E)$, and $s_0(E)$ are the on-shell scattering matrices at energy E of $H(d)$, h_ϵ, and h_0 respectively.

A result of another type was proved by the present authors in [27]. Let $S_1(E)$, $S_2(E; d)$ and $S_d(E)$ denote the on-shell scattering matrices of energy E for the pairs (H_1, H_0), $(H_2(d) = H_0 + V_2(\cdot - d), H_0)$ and $(H(d), H_0)$ respectively. It was shown that both $S_d(E) - S_1(E)S_2(E;d)$ and $S_d(E) - S_2(E;d)S_1(E)$ tend to zero in Hilbert-Schmidt norm as $|d| \to \infty$ uniformly in E on compact sets in $(0, +\infty)$. For $V_i \in \mathcal{R} \cap L^1(\mathbb{R}^3)$ the result holds also in the trace norm sense. Since $s_0(E)$ does not possess the cluster property, relation (1.7) shows that in general uniform convergence cannot hold on compact sets in $[0, +\infty)$.

We turn to a description of the main results of the present article.

For $d \in \mathbb{R}^3$, let $U(d)$ denote the unitary translation operator

$$(U(d)f)(x) = f(x-d), \ f \in L^2, \tag{1.8}$$

such that

$$U(d)V_2U(d)^{-1} = V_2(\cdot - d). \tag{1.9}$$

Let

$$H_2(d) = U(d)H_2U(d)^{-1} = -\Delta + V_2(\cdot - d). \tag{1.10}$$

We denote by $R(z;d)$, $R_1(z)$, $R_2(z;d)$ and $R_0(z)$ the resolvents of $H(d)$, H_1, $H_2(d)$ and H_0 respectively and with the sign convention $R_0(z) = (H_0 - z)^{-1}$ etc. We note that obviously $R_2(z;d) = U(d)R_2(z)U(d)^{-1}$, where $R_2(z)$ is the resolvent of H_2. It is well known (see [41]) that for V_1 and V_2 in $\mathcal{R} \cap L^1$ the differences $R(z;d) - R_0(z)$, $R_1(z) - R_0(z)$ and $R_2(z;d) - R_0(z)$ are trace class for Im z $\neq 0$ and for all $d \in \mathbb{R}^3$.

Let $\mathcal{E}_i$, $i = 1, 2$ be the set of $z \in \mathbb{C}$ for which the homogeneous equation $\phi = V_i^{1/2}R_0(z)|V_i|^{1/2}\phi$ has a solution in $L^2(\mathbb{R}^3)$, where $V_i^{1/2}(x) = |V_i(x)|^{1/2}\mathrm{sign}V_i(x)$ such that $V_i = V_i^{1/2}|V_i|^{1/2}$. Note that $\mathcal{E}_i^{(-)} = \mathcal{E}_i \cap (-\infty, 0)$ represents the negative discrete spectrum of H_i. The sets $\mathcal{E}_i^{(+)} = \mathcal{E}_i \cap (0, +\infty)$ are bounded closed sets of Lebesgue measure zero. More precisely, they are the unions of two subsets, belonging to the upper and lower lips of the cut $(0, +\infty)$ respectively. Let $\Pi_0 = \mathbb{C} \setminus [0, +\infty)$ and let $\overline{\Pi_0}$ be the closure of Π_0 with the two lips of the cut added. We write $\mathcal{E} = \mathcal{E}_1 \cup \mathcal{E}_2$.

For real valued V_1 and V_2 in $\mathcal{R} \cap L^1$ Krein's spectral shift functions $\xi_1(E)$, $\xi_2(E;d)$, and $\xi(E;d)$ for the pairs (H_1, H_0), $(H_2(d), H_0)$ and $(H(d), H_0)$ respectively exist, satisfy $(1 + |\cdot|^2)^{-1}\xi \in L^1(\mathbb{R})$ and for each of them the following trace relations hold

$$\begin{aligned} \mathrm{tr}\Big(\phi(H_1) - \phi(H_0)\Big) &= \int_{\mathbb{R}} \phi'(E)\xi_1(E)dE, \\ \mathrm{tr}\Big(\phi(H_2(d)) - \phi(H_0)\Big) &= \int_{\mathbb{R}} \phi'(E)\xi_2(E;d)dE, \\ \mathrm{tr}\Big(\phi(H(d)) - \phi(H_0)\Big) &= \int_{\mathbb{R}} \phi'(E)\xi(E;d)dE \end{aligned} \tag{1.11}$$

with ϕ being a function in a suitable class of continuously differentiable functions. For instance we may take ϕ to be in $C_0^\infty(\mathbb{R})$, also $\phi(E) = e^{-tE}$, $t > 0$, $\phi(E) = (E-z)^{-1}$, Im$z \neq 0$ are in this class. Note that $\xi_2(E;d)$ is independent of d such that $\xi_2(E;d) = \xi_2(E)$, where $\xi_2(E)$ is spectral shift function for the pair (H_2, H_0).

The spectral shift functions are not uniquely defined by (1.11). They may be changed by an additive constant. We normalize the spectral shift functions by the conditions that $\xi(E;d)$, $\xi_1(E)$, and $\xi_2(E)$ are identically zero for E below the spectra of $H(d)$, H_1 and H_2 respectively ($H(d)$ is bounded below uniformly in d by the Kato inequality). With this normalization for $E<0$ one has

$$\xi_i(E-0) = -N_i(E),\ \xi(E-0;d) = -N(E;d),$$

where $N_i(E)$ and $N(E;d)$ are the counting functions for the Hamiltonians H_i and $H(d)$ respectively.

By the Birman-Krein theorem [4] the spectral shift function $\xi(E)$ for the pair $(H_0+V,\ H_0)$ is related to the scattering matrix $S(E)$ for fixed energy E:

$$\xi(E) = -\frac{1}{2\pi i}\log\det S(E),\ E>0. \tag{1.12}$$

The normalization of the specral shift function introduced above leads to the relation (see e.g. [32, 7])

$$\lim_{E\to+\infty}\left\{\xi(E) - \frac{\sqrt{E}}{4\pi^2}\int V(x)dx\right\} = 0, \tag{1.13}$$

which fixes the branch of the logarithm in (1.12) uniquely.

Theorem 1 *Let V_1 and V_2 be in $L^1\cap\mathcal{R}$. Let the spectral shift functions be normalized as above. Then*

$$\lim_{|d|\to\infty}\int_{\mathbb{R}}\Big(\xi(E;d)-\xi_1(E)-\xi_2(E)\Big)\phi(E)dE = 0 \tag{1.14}$$

for arbitrary $\phi\in C_0^\infty(\mathbb{R})$.

This result extends results in [27], where we only established that $\xi'(E;d) - \xi_1'(E) - \xi_2'(E)$ tends to zero in the sense of distributions as $|d|\to\infty$.

For potentials satisfying slightly more restrictive conditions we will provide a local information on the convergence of $\xi(E;d)-\xi_1(E)-\xi_2(E)$ in the limit $|d|\to\infty$. We impose the following additional condition on the potentials V_i:

Property A *The potential V satisfies $(1+|\cdot|)^2V\in L^1(\mathbb{R}^3)\cap\mathcal{R}$. There are constants $R_0>0$ and $C>0$ such that $|V(x)|\le C$ for all $|x|\ge R_0$ and $|x||V(x)|\to 0$ for $|x|\to\infty$.*

The assumption $(1+|\cdot|)^2V\in L^1(\mathbb{R}^3)\cap\mathcal{R}$ implies that V is an Agmon potential, and therefore $\sigma_{sc}(H)=\emptyset$ [38]. The fact that V is bounded outside some ball and $|x||V(x)|\to 0$ for $|x|\to\infty$ guarantees that $\sigma_p(H)\cap(0,\infty)=\emptyset$ [38]. Hence for V_i's having Property A the sets $\mathcal{E}_i$ satisfy $\mathcal{E}_i\cap(0,\infty)=\emptyset$.

Let $A_i(\omega,\omega';E)$, $i=1,2$; $\omega,\omega'\in\mathbb{S}^2$, $E\in\mathbb{R}_+$ be the scattering amplitude for the Hamiltonian H_i. We recall that it can be expressed in terms of the integral kernel $S_i(\omega, \omega'; E)$ of the on-shell S-matrix $S_i(E)$ (see e.g. [37]):

$$A_i(\omega, \omega'; E) = -\frac{2\pi i}{\sqrt{E}} \left(S_i(\omega, \omega'; E) - \delta(\omega - \omega') \right).$$

We recall the notation $\hat{d} = d|d|^{-1} \in \mathbb{S}^2$.

Theorem 2 *Let the potentials V_i, $i = 1, 2$ have Property A. Then the function*

$$\xi_{12}(E; d) = \xi(E; d) - \xi_1(E) - \xi_2(E) \tag{1.15}$$

for sufficiently large $|d|$ is jointly continuous in d and $E \in (0, +\infty)$ and has the asymptotic representation

$$\xi_{12}(E; d) = \frac{\cos(2\sqrt{E}|d|)}{|d|^2} \mathrm{Im}\ a(E; \hat{d}) + \frac{\sin(2\sqrt{E}|d|)}{|d|^2} \mathrm{Re}\ a(E; \hat{d}) + o(|d|^{-2}), \tag{1.16}$$

where

$$a(E; \hat{d}) = -\frac{1}{\pi} A_1(\hat{d}, -\hat{d}; E) A_2(-\hat{d}, \hat{d}; E).$$

The error term $o(|d|^{-2})$ is uniform in E on compact sets in $(0, +\infty)$.

Remarks: 1. For C_0^∞-potentials V_1 and V_2 the functions $\xi(E; d)$, $\xi_1(E)$ and $\xi_2(E)$ are infinitely differentiable in E on $(0, +\infty)$ [39]. Moreover, one can show that for any $E \in (0, +\infty)$ $\xi'(E; d) \to \xi_1'(E) + \xi_2'(E)$ as $|d| \to \infty$. The statement of Theorem 2 shows in particular that the second derivative $\xi''(E; d)$ does not converge pointwise as $|d| \to \infty$.

2. Under additional restrictions on the potentials (for instance, the V_i's have compact support) the error term in (1.16) can be replaced by $O(|d|^{-3})$.

3. The formulas (1.15), (1.16) remain valid for the Hamiltonians with point interactions centered at $x = 0$ and $x = d$.

The behaviour of $\xi_{12}(E; d)$ for $E = 0$ as $|d| \to \infty$ is more delicate and strongly depends on the properties of the spectral point zero for H_1 and H_2. We recall that zero is said to be a resonance for H, if the differential equation $Hu = 0$ has a nontrivial solution $u \in L^2_s(\mathbb{R}^3)$ for some s such that $-3/2 \le s < -1/2$ and $u \notin L^2(\mathbb{R}^3)$. Here $L^2_s(\mathbb{R}^3) = \{u \in L^2_{loc}(\mathbb{R}^3) | (1 + x^2)^{s/2} u \in L^2(\mathbb{R}^3)\}$. A detailed discussion of zero energy resonances can be found in [21].

Following Jensen and Kato [21] we call the point $E = 0$ *regular* for H if it is neither an eigenvalue nor a resonance of H. If $E = 0$ is a resonance but not an eigenvalue, it is said to be an *exceptional point of the first kind.* We note that the last case is in no way pathological: even for a square well potential there are isolated values of the coupling constant for which $E = 0$ is an exceptional point of the first kind. Also one can construct examples of square wells for which $E = 0$

is an eigenvalue but not a resonance (see e.g. [38]). In this case $E = 0$ is said to be an *exceptional point of the second kind*. If $E = 0$ is both a resonance and an eigenvalue, it is said to be an *exceptional point of the third kind*. We note that if H has no negative eigenvalues but has a resonance at zero energy, then $E = 0$ cannot be an eigenvalue of H (see [44]).

Let $c_0(V)$ denote the scattering length of the Hamiltonian $H_0 + V$ (see Section 5). Then we have

Theorem 3 *Let the potentials V_1 and V_2 satisfy Property A.*

(i) Let $E = 0$ be either a regular point or an exceptional point of the first kind for both H_1 and H_2. Then $\xi_{12}(0+; d) = 0$ for all sufficiently large $|d|$.

(ii) Let $E = 0$ be a regular point for one of the Hamiltonians H_i, $i = 1, 2$ (say H_1) and an exceptional point of the first kind for H_2. Then $\xi_{12}(0+; d) = -\mathrm{sign} c_0(V_1)/2$ for all sufficiently large $|d|$.

We note that our definition of the scattering length (5.2) agrees with that used by Kato and Jensen [21] and differs by a sign from the definition customary used in physical literature [1, 6].

Now we apply Theorem 3 to an analysis of the discrete spectrum of $H(d)$. The value of the spectral shift function at zero energy gives information on the multiplicity of the non-positive point spectrum of the Hamiltonian. By the generalized Levinson theorem [32, 10, 11, 6] $\xi(0+)$ equals minus the number of nonpositive eigenvalues counting their multiplicities minus $1/2$, if zero is a resonance for $H = H_0 + V$. Thus, we have

Corollary 4 *Let the potentials V_1 and V_2 have Property A. Let $E = 0$ be either an exceptional point of the first kind for both H_1 and H_2, or a regular point for one of the Hamiltonians H_i with positive scattering length and an exceptional point of the first kind for the other one. Moreover, let the negative spectrum of both H_1 and H_2 be empty. Then for all sufficiently large $|d|$ the operator $H(d)$ has a unique (nonpositive) eigenvalue.*

Now we comment on the case when $E = 0$ is an exceptional point of the first kind for (say) H_2 and the regular one for H_1. For the case of compactly supported potentials we can easily prove that $E(d)$ satisfies the equation

$$\sqrt{-E} = c_0(V_1) \frac{e^{-2\sqrt{-E}|d|}}{|d|^2} + O(|d|^{-3}) + O(-E). \tag{1.17}$$

Without explicit pointing out the connection with the scattering length the equation (1.17) was proved by Klaus and Simon [24] for the case when $V_1 \leq 0$ and H_1 is subcritical (i.e. $E = 0$ is a regular point for H_1 and the operator $H_0 + (1+\epsilon)V$ for all $-1 \leq \epsilon < \delta$ has no bound states). It is easy to show (see Appendix A below) that if $V_1 \geq 0$ and is not identically zero then the scattering length $c_0(V_1) < 0$. If

$V_1 \leq 0$ and the discrete spectrum of $H_0 + V_1$ is empty then $c_0(V_1) > 0$. In this case the equation (1.17) has a solution. For stronger attractive potentials V_1 which bind several states the scattering length can be positive or negative negative (for rotationally symmetric potentials see the discussion in [33]). The existence of the solution to (1.17) is controlled by $\mathrm{sign}\, c_0(V_1)$ in agreement with the statement of Theorem 3.

Neither equation (1.17) nor Theorem 3 give an information on the low-energy spectrum of $H(d)$ if $c_0(V_1) = 0$. To avoid a possible reader's question whether this is possible or not, we show in the Appendix A how to construct the potentials V for which $c_0(V) = 0$.

Consider now potentials V_i satisfying the inequality $|V_i(x)| \leq C(1 + |x|)^{-5-\epsilon}$ for some $C > 0$ and $\epsilon > 0$. Combining Corollary 4 with the results of Tamura [43] we can conclude that for the case of critical double well potentials (the negative spectra of both H_1 and H_2 are empty and $E = 0$ is an exceptional point of the first kind for both H_1 and H_2) the Hamiltonian $H(d)$ has a unique negative eigenvalue with asymptotics for $|d| \to \infty$ given by formula (1.5).

We note that our methods also can be applied to study other characteristics of $H(d)$ in the limit $|d| \to \infty$. As an example we consider the scattering amplitude

$$A_d(\omega, \omega'; E) = -\frac{2\pi i}{\sqrt{E}} \left(S_d(\omega, \omega'; E) - \delta(\omega - \omega') \right),$$

where now $S_d(\omega, \omega'; E)$ ($\omega, \omega' \in \mathbb{S}^2$, $E \in \mathbb{R}_+$) is the integral kernel of the on-shell S-matrix $S_d(E)$ for the pair of Hamiltonians $(H(d), H_0)$. Let us note that for $V_i \in \mathcal{R} \cap L^1$ and for all $E \in \mathbb{R}_+ \setminus \mathcal{E}$ the scattering amplitude is a bounded function of its arguments.

Let $S_2(E; d)$ be the on-shell S-matrix for the Hamiltonian $H_2(d)$. By (1.10) we obviously have $S_2(E; d) = U(d) S_2(E) U(d)^{-1}$. Note that since $U(d)$ commutes with H_0, it has a corresponding spectral decomposition inducing translations in the spaces of fixed energy of H_0, also denoted by $U(d)$. Hence for the corresponding scattering amplitude $A_2(\omega, \omega'; E; d)$ we have the relation

$$A_2(\omega, \omega'; E; d) = e^{-i\sqrt{E}\langle \omega - \omega', d\rangle} A_2(\omega, \omega'; E). \tag{1.18}$$

Property B *The potential V has property A and satisfies $(1 + |\cdot|)^4 V \in L^1(\mathbb{R}^3)$.*

We will prove the following analogue of Theorem 2.

Theorem 5 *Let the V_i's have Property B. Then the function*

$$A_{12}(\omega, \omega'; E; d) = A_d(\omega, \omega'; E) - A_1(\omega, \omega'; E) - A_2(\omega, \omega'; E; d)$$

for all $\omega, \omega' \in \mathbb{S}^2$ and for sufficiently large $|d|$ is continuous with respect to d and has the asymptotic representation

$$A_{12}(\omega, \omega'; E; d) =$$

$$\frac{1}{|d|}\Big[A_2(\omega,\hat{d};E;d)A_1(\hat{d},\omega';E)$$
$$+A_1(\omega,-\hat{d};E)A_2(-\hat{d},\omega';E;d)\Big]+o(|d|^{-1}). \tag{1.19}$$

The error term $o(|d|^{-1})$ is uniform in $\omega,\omega'\in\mathbb{S}^2$ and E on compact sets in $\mathbb{R}_+$.

Intuitively this result is clear (see Fig. 1). By construction of A_{12}, to A_{12} both scattering centers have to contribute. Therefore to leading order in $|d|^{-1}$, the particle has to move in direction $\hat{d}$, if it first has hit center 1, in order to hit center 2 (and in direction $-\hat{d}$ if it first hits center 2 and then center 1). Below we will also give (besides a rigorous proof) a formal proof based on this geometric picture and the Born series for the scattering amplitude.

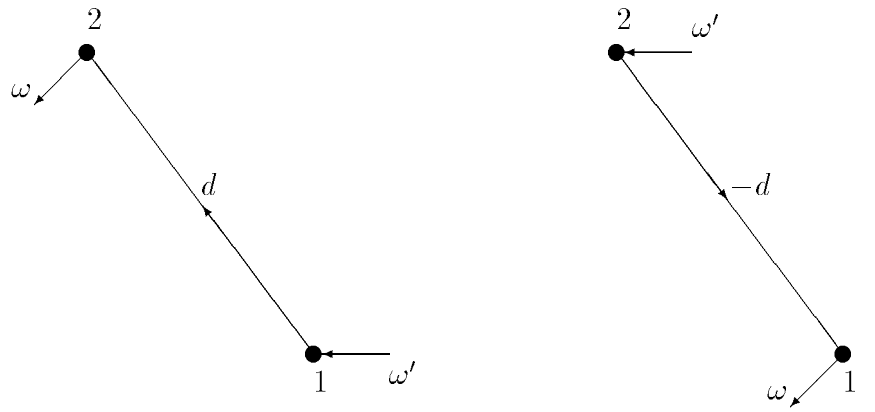

Figure 1: Dominant contributions to A_{12}.

The scattering amplitude A_{12} also exhibits strong oscillations in the limit $|d|\to\infty$, since by means of (1.18) we can rewrite (1.19) in the form

$$A_{12}(\omega,\omega';E,d)=$$
$$\frac{e^{i\sqrt{E}|d|}}{|d|}\Big[e^{-i\sqrt{E}\langle\omega,\hat{d}\rangle}A_2(\omega,\hat{d};E)A_1(\hat{d},\omega';E)$$
$$+e^{i\sqrt{E}\langle\omega',\hat{d}\rangle}A_1(\omega,-\hat{d};E)A_2(-\hat{d},\omega';E)\Big]+o(|d|^{-1}).$$

We remark that the present results can be easily generalized to potentials which are the sum of $n>2$ terms by considering the situation where all the separations of their respective centers tend to infinity.

The paper is organized as follows. In Section 2 we prove several auxiliary lemmas. The Section 3, 4, 5 are devoted to the proofs of Theorems 2, 3 and 5 respectively. All the proofs can be made much easier, when considering the case of compactly supported potentials. The major efforts are made to accomodate the case of potentials having Properties A and B. Throughout the paper we freely do not distinguish between the formal expressions of the form $V^{\alpha}R_0(z)|V|^{1-\alpha}$ defined on $\mathcal{D}(V^{1-\alpha})$ and their closures defined on the whole $L^2(\mathbb{R}^3)$. All formal manipulations with the operators of this form can be easily justified.

Hereafter we will use the notations $\mathcal{J}_p$ $(p \geq 1)$ for the trace ideals of p-summable compact operators. In particular, $\mathcal{J}_1$ denotes the ideal of trace class operators, and $\mathcal{J}_2$ stands for the set of all Hilbert-Schmidt operators. The corresponding norms are denoted by $\|\cdot\|_{\mathcal{J}_p}$.

2 Auxiliary Results

Hereafter we assume that $V_i \in \mathcal{R}$, $i = 1, 2$. We start with the formulation of some technical results from our preceeding paper [27] (Theorem 1.1 and Lemma 2.2), which we recall here for the reader's convenience.

Theorem T *Let $V_1, V_2 \in \mathcal{R}$. Then*

(i) the operator $K_{12}(z; d) = V_1^{1/2}R_0(z)|V_2|^{1/2}(\cdot - d)$ tends to zero in Hilbert-Schmidt norm uniformly in $z \in \overline{\Pi_0}$ as $|d| \to \infty$,

(ii) there is a constant $c_0 > 0$ such that the Hamiltonians H_1, H_2, and $H(d)$ defined in the form sense are bounded below by the constant $-c_0$ for all $d \in \mathbb{R}^3$. Moreover, if $V_1, V_2 \in \mathcal{R} \cap L^1$ the operator

$$R(z; d) - R_1(z) - R_2(z; d) + R_0(z) \tag{2.1}$$

tends to zero as $|d| \to \infty$ in trace norm uniformly in z on compact subsets of $\Pi_0 \setminus \mathcal{E}$.

We note that a result similar to (ii) for H_0-form compact potentials, for which the convergence of (2.1) to zero holds in the operator norm, was proved earlier by Klaus [25, Appendix].

Lemma 2.1 *Let $V_1, V_2 \in \mathcal{R}$. Then for all $z \in \overline{\Pi_0}$ the operator $K_{12}(z; d)$ is continuous in $d \in \mathbb{R}^3$ in Hilbert-Schmidt norm.*

Proof. We recall that the translation operator $U(d)$ is strongly continuous in all L^p-spaces with $1 \leq p < \infty$. By means of simple limiting arguments and by the inequality

$$\|V\|_{\mathcal{R}} \leq 3^{1/2}(2\pi)^{1/3}\|V\|_{L^2}^{2/3}\|V\|_{L^1}^{1/3}, \; V \in L^1(\mathbb{R}^3) \cap L^2(\mathbb{R}^3)$$

(see [41]) one can easily prove that $U(d)$ is also strongly continuous in the Rollnik norm,

$$\|V\|_{\mathcal{R}} = \left(\int \frac{|V(x)||V(y)|}{|x-y|^2} dx dy \right)^{1/2}.$$

We recall that with this norm $\mathcal{R}$ is a complete normed vector space.

For arbitrary d_1 and d_2 we estimate

$$\begin{aligned}
& \|K(z;d_1) - K(z;d_2)\|_{\mathcal{J}_2} \\
\le\ & \|K(z;d_1) - V_{d_1}^{1/2} R_0(z)|V_{d_2}|^{1/2}\|_{\mathcal{J}_2} \\
& + \|V_{d_1}^{1/2} R_0(z)|V_{d_2}|^{1/2} - K(z;d_2)\|_{\mathcal{J}_2} \\
\le\ & \|V_{d_1}\|_{\mathcal{R}} \left\| (|V_{d_1}|^{1/2} - |V_{d_2}|^{1/2})^2 \right\|_{\mathcal{R}} \\
& + \|V_{d_2}\|_{\mathcal{R}} \left\| (V_{d_1}^{1/2} - V_{d_2}^{1/2})^2 \right\|_{\mathcal{R}}.
\end{aligned} \tag{2.2}$$

Now we use the inequality

$$\left(|a+b|^{1/2} - |a|^{1/2}\right)^2 \le |b|, \tag{2.3}$$

which is valid for all $a, b \in \mathbb{R}$ and which is a simple consequence of $|a+b|^{1/2} \le |a|^{1/2} + |b|^{1/2}$. Due to (2.3) we have

$$\begin{aligned}
& \left(|V_{d_1}(x)|^{1/2} - |V_{d_2}(x)|^{1/2}\right)^2 \\
=\ & \left(|V_{d_1}(x) + V_2(x-d_2) - V_2(x-d_1)|^{1/2} - |V_{d_1}(x)|^{1/2}\right)^2 \\
\le\ & |V_2(x-d_2) - V_2(x-d_1)|.
\end{aligned}$$

Therefore the r.h.s. of (2.2) can be bounded by

$$\left(\|V_{d_1}\|_{\mathcal{R}} + \|V_{d_2}\|_{\mathcal{R}}\right) \|V_2(x-d_2) - V_2(x-d_1)\|_{\mathcal{R}}.$$

Since the shift operator $U(d)$ is strongly continuous in Rollnik norm this completes the proof of the lemma. ∎

Now for potentials having Property A we provide a more detailed information on the decay of $K_{12}(z;d)$ as $|d| \to \infty$.

For $z \in \overline{\Pi_0}$ we set

$$\Phi_1(x;z,\hat{d}) = |V_1(x)|^{1/2} \exp\{-i\sqrt{z}\langle x, \hat{d}\rangle\}, \tag{2.4}$$

$$\Phi_2(x;z,\hat{d}) = |V_2(x)|^{1/2} \exp\{-i\sqrt{z}\langle x, \hat{d}\rangle\}. \tag{2.5}$$

The functions $\Phi_i(\cdot;z,\hat{d})$ are in $L^2_{loc}(\mathbb{R}^3)$. However for non-compactly supported potentials they are generally not in $L^2(\mathbb{R}^3)$. This is a main obstacle for extending the results of [24, 25] to the case of non-compactly supported potentials.

For $E \ge 0$ we define

$$\Phi_i^{(\pm)}(x;E,\hat{d}) = |V_i(x)|^{1/2} \exp\{\mp i\sqrt{E}\langle x, \hat{d}\rangle\}, \ i = 1, 2, \tag{2.6}$$

where $\langle x, \hat{d}\rangle$ is the inner product of x and $\hat{d} = d|d|^{-1}$ in $\mathbb{R}^3$. Obviously $\Phi_i^{(\pm)}(\cdot;E,\hat{d}) \in L^2(\mathbb{R}^3)$, $i = 1, 2$. The functions $\Phi^{(\pm)}(\cdot;E,\hat{d})$ defined by (2.6) are the limiting values of (2.4), (2.5) on the upper and lower lips of the cut $[0, +\infty)$ respectively.

Lemma 2.2 *Let the potentials V_i have Property A. Then there is a constant $C > 0$ such that*

$$\|K_{12}(z;d)\|_{\mathcal{J}_2} \leq \frac{C}{|d|}, \quad \|K_{21}(z;d)\|_{\mathcal{J}_2} \leq \frac{C}{|d|}$$

for all sufficiently large $|d|$ and for all $z \in \overline{\Pi_0}$. Moreover for all real $E \geq 0$ the operators

$$|d|K_{12}(E \pm i0;d) - \frac{e^{\pm i\sqrt{E}|d|}}{4\pi} \mathrm{sign} V_1 \Phi_1^{(\pm)}(\cdot;E,\hat{d})(\Phi_2^{(\pm)}(\cdot - d;E,\hat{d}),\cdot), \quad (2.7)$$

$$|d|K_{21}(E \pm i0;d) - \frac{e^{\pm i\sqrt{E}|d|}}{4\pi} \mathrm{sign} V_2(\cdot - d) \Phi_2^{(\mp)}(\cdot - d;E,\hat{d})(\Phi_1^{(\mp)}(\cdot;E,\hat{d}),\cdot) \quad (2.8)$$

tend to zero in Hilbert-Schmidt norm as $|d| \to \infty$. The convergence is uniform in E on compact sets in $(0, +\infty)$.

Remark: For compactly supported potentials $V_i \in \mathcal{R} \cap L^1$ the second claim of Lemma 2.2 can be extended to the complex plane. For instance one can then easily prove that the operator

$$|d|K_{12}(z;d) - \frac{e^{i\sqrt{z}|d|}}{4\pi} \mathrm{sign} V_1 \Phi_1(\cdot;z,\hat{d})(\Phi_2(\cdot - d;z,\hat{d}),\cdot) \to 0 \quad (2.9)$$

in the Hilbert-Schmidt norm uniformly in z on compact sets in $\overline{\Pi_0}$. Moreover, there is a constant $C > 0$ such that

$$\left\| |d|K_{12}(z;d) - \frac{e^{i\sqrt{z}|d|}}{4\pi} \mathrm{sign} V_1 \Phi_1(\cdot;z,\hat{d})(\Phi_2(\cdot - d;z,\hat{d}),\cdot) \right\|_{\mathcal{J}_2} \leq \frac{C}{|d|}$$

uniformly in d for all large $|d|$ and in z on compact sets in $\overline{\Pi_0}$. Formula (2.9) was used by Klaus and Simon [24, 25] to study the critical double well problem.

Our approach heavily uses the formula

$$\int \frac{f(x)g(y)}{|x-y|^2} dx dy = \frac{1}{4\pi} \int \frac{\overline{\widehat{f}(p)}\widehat{g}(p)}{|p|} dp, \quad (2.10)$$

which is valid for all $f, g \in \mathcal{R}$. Here the hat $\widehat{\ }$ denotes the Fourier transform,

$$\widehat{f}(p) = \int e^{-i\langle p,x \rangle} f(x) dx.$$

The proof of the formula (2.10) can be found in [41].

To prove Lemma 2.2 we use the following asymptotics:

Lemma 2.3 *Let $f \in C^2(\mathbb{R}^3)$ and $|\cdot|^{-1}\partial^\alpha f \in L^1(\mathbb{R}^3)$ for all $|\alpha| \leq 2$ (α is multiindex). Then for all $E \geq 0$ and all large $d \in \mathbb{R}^3$ one has*

$$(i) \qquad \int_{\mathbb{R}^3} \frac{f(p)}{|p|} e^{-i\langle d,p\rangle} dp = \frac{4\pi f(0)}{|d|^2} + o(|d|^{-2}), \tag{2.11}$$

$$(ii) \qquad \int_{\mathbb{R}^3} \frac{f(p)e^{-i\langle d,p\rangle}}{p^2 - E \mp i0} dp = \frac{2\pi^2}{|d|} e^{\pm i\sqrt{E}|d|} f(\mp\sqrt{E}\hat{d}) + o(|d|^{-1}), \tag{2.12}$$

where the error term is uniform in E *on compact sets in* $(0, \infty)$.

Remark: Here $(x \mp i0)^{-1}$ is understood in the sense of distributions, such that

$$(x \mp i0)^{-1} = \text{v.p.}\frac{1}{x} \pm i\pi\delta(x).$$

Since the distributions $(x \mp i0)^{-1}$ have order 1 they can be extended to linear continuous functionals on $C^k(\mathbb{R})$ with $k \geq 1$.

Proof. The claim (i) follows from the results of [40]. Asymptotics (2.12) is also well known. Its proof for analytic f is quite elementary. However, since we could not find a proof for the case $f \in C^2(\mathbb{R}^3)$ in the literature, we give it in Appendix B.

Proof of Lemma 2.2. We consider only the operator $K_{12}(z; d)$, since $K_{21}(z; d)$ can be considered similarly. By means of the formula (2.10) one can easily show that

$$\begin{aligned} & |d|^2 \|K_{12}(z; d)\|^2_{\mathcal{J}_2} \\ \leq \quad & \frac{|d|^2}{(4\pi)^2} \int \frac{|V_1(x)||V_2(y-d)|}{|x-y|^2} dxdy = \frac{|d|^2}{(4\pi)^3} \int \frac{\overline{\widehat{|V_1|}(p)}\widehat{|V_2|}(p)e^{-i\langle d,p\rangle}}{|p|} dp. \end{aligned}$$

The Assumption A guarantees that $\widehat{|V_i|}(p) \in C^2(\mathbb{R}^3)$ and $|\cdot|\partial^\alpha \left(\overline{\widehat{|V_1|}(p)}\widehat{|V_2|}(p)\right) \in L^1(\mathbb{R}^3)$. Applying Lemma 2.3 (i) we find that

$$\begin{aligned} & \frac{|d|^2}{(4\pi)^3} \int \frac{\overline{\widehat{|V_1|}(p)}\widehat{|V_2|}(p)e^{-i\langle d,p\rangle}}{|p|} dp \\ = \quad & \frac{1}{(4\pi)^2} \overline{\widehat{|V_1|}(0)}\widehat{|V_2|}(0) + o(1) \\ = \quad & \frac{1}{(4\pi)^2} \|V_1\|_{L^1} \|V_2\|_{L^1} + o(1), \end{aligned} \tag{2.13}$$

which proves the first part of the claim.

Now let $z = E + i0$. To prove the second part of the claim we consider the squared Hilbert-Schmidt norm of the operator (2.7), which is given by

$$\frac{|d|^2}{(4\pi)^2} \int \frac{|V_1(x)||V_2(y-d)|}{|x-y|^2} dxdy + \frac{1}{(4\pi)^2} \int |V_1(x)||V_2(y-d)|dxdy$$

$$-\frac{|d|}{(4\pi)^2}e^{-i\sqrt{E}|d|}\int\frac{|V_1(x)||V_2(y-d)|}{|x-y|}e^{i\sqrt{E}|x-y|}e^{i\sqrt{E}\langle x-y+d,\hat{d}\rangle}dxdy$$
$$-\frac{|d|}{(4\pi)^2}e^{i\sqrt{E}|d|}\int\frac{|V_1(x)||V_2(y-d)|}{|x-y|}e^{-i\sqrt{E}|x-y|}e^{-i\sqrt{E}\langle x-y+d,\hat{d}\rangle}dxdy. \qquad (2.14)$$

We have already shown that the first term coincides asymptotically with the second one.

Now we prove that for all real $f_i \in \mathcal{R}\cap L^1(\mathbb{R}^3)$ and all real $E \geq 0$

$$\frac{1}{4\pi}\int\frac{f_1(x)f_2(y)}{|x-y|}e^{\pm i\sqrt{E}|x-y|}dxdy = \int\frac{\overline{\widehat{f_1}(p)}\widehat{f_2}(p)}{p^2-E\mp i0}dp. \qquad (2.15)$$

First we note that the integral on the l.h.s. of (2.15) is well defined. Indeed,

$$\int\frac{|f_1(x)||f_2(x)|}{4\pi|x-y|}dxdy \leq \left\||f_1|^{1/2}R_0(0)|f_2|^{1/2}\right\|_{\mathcal{J}_2}\||f_1|^{1/2}\|_{L^2}\||f_2|^{1/2}\|_{L^2}$$
$$= \left\||f_1|^{1/2}R_0(0)|f_2|^{1/2}\right\|_{\mathcal{J}_2}\|f_1\|_{L^1}^{1/2}\|f_2\|_{L^1}^{1/2}. \qquad (2.16)$$

Now consider the integral

$$\frac{1}{4\pi}\int\frac{f_1(x)f_2(y)}{|x-y|}e^{i\sqrt{z}|x-y|}dxdy \qquad (2.17)$$

for $z \in \Pi_0$. First let us suppose that $f_i \in L^2\cap L^1$. Then by the convolution formula (see e.g. [41])

$$\int f(x)h(x-y)g(y)dxdy = \frac{1}{(2\pi)^3}\int\overline{\widehat{f}(p)}\widehat{h}(p)\widehat{g}(p)dp,$$

where $f,h \in L^2$, $g\in L^1$ we have that (2.17) equals

$$\int\frac{\overline{\widehat{f_1}(p)}\widehat{f_2}(p)}{p^2-z}dp.$$

Since the integrand in (2.17) is dominated by $|f_1(x)||f_2(x)||x-y|^{-1}$, by the Lebesgue dominated convergence theorem the limit $\mathrm{Im}z \to \pm 0$ exists. Thus (2.15) is proven for $f \in L^2\cap L^1$. A simple limiting procedure and (2.16) proves (2.15) for $f_i \in \mathcal{R}\cap L^1$.

Consider now the third and fourth terms of (2.14). By (2.15) we have

$$\frac{|d|}{(4\pi)^2}e^{\mp i\sqrt{E}|d|}\int\frac{|V_1(x)|\,|V_2(y-d)|}{|x-y|}e^{\pm i\sqrt{E}|x-y|}e^{\pm i\sqrt{E}\langle x,\hat{d}\rangle}e^{\mp i\sqrt{E}\langle y-d,\hat{d}\rangle}dxdy$$
$$= \frac{|d|}{(4\pi)^2}\frac{1}{2\pi^2}e^{\mp i\sqrt{E}|d|}\int\frac{\overline{\widehat{|V_1|}(p\pm\sqrt{E}\hat{d})}\widehat{|V_2|}(p\pm\sqrt{E}\hat{d})e^{-i\langle d,p\rangle}}{p^2-E\mp i0}dp. \qquad (2.18)$$

Applying Lemma 2.3 (ii) we see that the r.h.s. of (2.18) has the asymptotics

$$(4\pi^2)^{-1}\overline{\widehat{|V_1|}(0)}\widehat{|V_2|}(0)+o(1) = (4\pi^2)^{-1}\|V_1\|_{L^1}\|V_2\|_{L^1}+o(1),$$

where the error term is uniform in E on compact sets in $\mathbb{R}_+$. Summing up all contributions we obtain the claim of the lemma. The case $z = E - i0$ can be considered in exactly the same way. ■

To proceed further we need the following technical

Lemma 2.4 *Let the V_i's have Property A. Then for every $\alpha \in (0,1)$ both integrals*

$$\int \frac{|V_1(x)|^\alpha |V_2(x-d)|^{1-\alpha}|V_1(y)|}{|x-y|^2} dxdy$$

and

$$\int \frac{|V_1(x)|^\alpha |V_2(x-d)|^{1-\alpha}|V_2(y-d)|}{|x-y|^2} dxdy$$

are $o(|d|^{-2})$ as $|d| \to \infty$.

Proof. First we note that $V_i \in \mathcal{R}$, $i = 1, 2$ implies $|V_1|^\alpha |V_2|^{1-\alpha} \in \mathcal{R}$ for any $\alpha \in (0,1)$ (see [27]). Moreover by Hölder inequality

$$\begin{aligned}
&\left\|(1+|\cdot|)^2 |V_1|^\alpha |V_2|^{1-\alpha}(\cdot - d)\right\|_{L^1} \\
&\quad \le \left\|(1+|\cdot|)^{2\alpha}|V_1|^\alpha\right\|_{L^{1/\alpha}} \cdot \left\|(1+|\cdot|)^{2(1-\alpha)}|V_2|^{1-\alpha}(\cdot - d)\right\|_{L^{1/(1-\alpha)}} \\
&\quad = \left\|(1+|\cdot|)^2|V_1|\right\|_{L^1}^\alpha \cdot \left\|(1+|\cdot|)^2|V_2|(\cdot - d)\right\|_{L^1}^{1-\alpha}.
\end{aligned}$$

Therefore if the potentials V_i have Property A, then $|V_1|^\alpha |V_2|^{1-\alpha}(\cdot - d)$ has Property A also for all $\alpha \in (0,1)$ and all $d \in \mathbb{R}^3$.

By means of formula (2.10) we obtain

$$\begin{aligned}
&\frac{|d|^2}{4\pi} \int \frac{|V_1(x)|^\alpha |V_2(x-d)|^{1-\alpha}|V_1(y)|}{|x-y|^2} dxdy \\
&= \frac{|d|^2}{(4\pi)^2} \int \frac{\overline{F_1(p;d)}\widehat{|V_1|}(p)}{|p|} dp, \qquad (2.19)
\end{aligned}$$

$$\begin{aligned}
&\frac{|d|^2}{4\pi} \int \frac{|V_1(x)|^\alpha |V_2(x-d)|^{1-\alpha}|V_2(y-d)|}{|x-y|^2} dxdy \\
&= \frac{|d|^2}{4\pi} \int \frac{|V_1(x+d)|^\alpha |V_2(x)|^{1-\alpha}|V_2(y)|}{|x-y|^2} dxdy \\
&= \frac{|d|^2}{(4\pi)^2} \int \frac{\overline{F_2(p;d)}\widehat{|V_2|}(p)}{|p|} dp, \qquad (2.20)
\end{aligned}$$

where

$$F_1(p;d) = \int e^{-i\langle p,x\rangle}|V_1(x)|^\alpha |V_2(x-d)|^{1-\alpha} dx$$

$$= e^{-i\langle p,d\rangle} \int e^{-i\langle p,x\rangle} |V_1(x+d)|^\alpha |V_2(x)|^{1-\alpha} dx, \tag{2.21}$$
$$F_2(p;d) = \int e^{-i\langle p,x\rangle} |V_1(x+d)|^\alpha |V_2(x)|^{1-\alpha} dx$$
$$= e^{i\langle p,d\rangle} \int e^{-i\langle p,x\rangle} |V_1(x)|^\alpha |V_2(x-d)|^{1-\alpha} dx.$$

Consider first the r.h.s. of (2.19). With the help of Lemma 2.3 we get that in the limit $|d| \to \infty$

$$\frac{|d|^2}{(4\pi)^2} \int \frac{\overline{F_1(p;d)} \widehat{|V_1|}(p)}{|p|} dp$$
$$= \frac{1}{4\pi} \widehat{|V_1|}(0) \int |V_1(x+d)|^\alpha |V_2(x)|^{1-\alpha} dx + o(1).$$

In the case $\alpha = 1/2$ by Riemann-Lebesgue lemma we have

$$\int |V_1(x+d)|^{1/2} |V_2(x)|^{1/2} dx = \int e^{-i\langle d,p\rangle} \overline{\widehat{|V_1|^{1/2}}(p)} \widehat{|V_2|^{1/2}}(p) = o(1). \tag{2.22}$$

Consider now the general case $\alpha \in (0,1)$. Without loss of generality we can assume that $\alpha < 1/2$. Then by Hölder inequality

$$\begin{aligned} & \int |V_1(x+d)|^\alpha |V_2(x)|^{1-\alpha} dx \\ = & \int |V_1(x+d)|^\alpha |V_2(x)|^\alpha |V_2(x)|^{1-2\alpha} dx \\ \le & \left(\int |V_1(x+d)|^{1/2} |V_2(x)|^{1/2} dx \right)^{2\alpha} \left(\int |V_2(x)| dx \right)^{1-2\alpha}, \end{aligned}$$

which is $o(1)$ by (2.22). The r.h.s. of (2.20) can be considered similarly. ■

3 Proof of Theorem 1

As already mentioned in the Introduction the spectra of $H(d)$, H_1 and H_2 are bounded below by a common constant [27]. Pick a real positive number c_0 such that $-c_0 < \min\{\inf \sigma(H(d)), \inf \sigma(H_1), \inf \sigma(H_2)\}$. Then for $V_i \in \mathcal{R} \cap L^1$ the differences

$$\begin{aligned} \mathcal{V}_d &= R(-c_0;d) - R_0(-c_0), \\ \mathcal{V}_i &= R_i(-c_0) - R_0(-c_0), \ i = 1,2, \\ \mathcal{V}_2(d) &= R_2(-c_0;d) - R_0(-c_0) \end{aligned} \tag{3.1}$$

with $\mathcal{V}_2(d=0) = \mathcal{V}_2$ are trace class for all $d \in \mathbb{R}^3$.

By Krein's theorem [28] (see also [5]) for self-adjoint operators A_2 and A_1 such that $A_2 - A_1$ is trace class the spectral shift function $\xi(\lambda; A_2, A_1)$ exists and is given by

$$\xi(\lambda; A_2, A_1) = \pi^{-1} \lim_{\epsilon \to +0} \arg\det\left(I + (A_2 - A_1)(A_1 - \lambda - i\epsilon)^{-1}\right). \tag{3.2}$$

For all $z \in \mathbb{C}$ with $\mathrm{Im} z \neq 0$ it has the property

$$\int_{\mathbb{R}} \frac{\xi(\lambda; A_2, A_1)}{\lambda - z} d\lambda = \log\det\Big(I + (A_2 - A_1)(A_1 - z)^{-1}\Big)$$

The branch of the logarithm is uniquely fixed by the condition

$$\log\det\Big(I + (A_2 - A_1)(A_1 - z)^{-1}\Big) \to 0,$$

when $\mathrm{Im} z \to \infty$. We have

$$\begin{aligned} \int_{\mathbb{R}} \xi(\lambda; A_2, A_1) d\lambda &= \mathrm{tr}(A_2 - A_1), \\ \int_{\mathbb{R}} |\xi(\lambda; A_2, A_1)| d\lambda &= \|A_2 - A_1\|_{\mathcal{J}_1}. \end{aligned} \tag{3.3}$$

In particular this guarantees the existence of the spectral shift functions

$$\begin{aligned} \tilde{\xi}(\lambda; d) &= \xi(\lambda; R(-c_0; d), R_0(-c_0)), \\ \tilde{\xi}_1(\lambda) &= \xi(\lambda; R_1(-c_0), R_0(-c_0)), \\ \tilde{\xi}_2(\lambda; d) &= \xi(\lambda; R_2(-c_0; d), R_0(-c_0)). \end{aligned}$$

Note that $\tilde{\xi}_2(\lambda; d) = \tilde{\xi}_2(\lambda) = \xi(\lambda; R_2(-c_0), R_0(-c_0))$ by the unitarity of the translations (1.8).

By the invariance principle (see e.g. [5, 22]) we can define the spectral shift functions $\xi(E; d)$, $\xi_1(E)$ and $\xi_2(E; d)$ for the pairs $(H(d),\ H_0)$, $(H_1,\ H_0)$, and $(H_2(d),\ H_0)$ by

$$\begin{aligned} \xi(E; d) &= -\tilde{\xi}((E + c_0)^{-1}; d), \\ \xi_1(E) &= -\tilde{\xi}_1((E + c_0)^{-1}), \\ \xi_2(E; d) &= -\tilde{\xi}_2((E + c_0)^{-1}; d). \end{aligned} \tag{3.4}$$

Obviously, $\xi(E; d) = \xi_1(E) = \xi_2(E; d) = 0$ for any $E \leq -c_0$ and for all $d \in \mathbb{R}^3$.

Remark: Note that from (3.3) it follows that if $A_2 \geq A_1$ then $\xi(\lambda; A_2, A_1) \geq 0$ for all $\lambda \in \mathbb{R}$. Therefore by the monotonicity of the resolvent and by the invariance principle for arbitrary real W_1 and W_2 in $\mathcal{R} \cap L^1$ satisfying $W_1 \geq W_2$ pointwise we have $\xi(E; H_0 + W_1, H_0) \geq \xi(E; H_0 + W_2, H_0)$ for all $E \in \mathbb{R}$. This yields an elementary proof of Kato's monotonicity theorem [23] (see also [13]).

Lemma 3.1 *The limit relation*

$$\lim_{|d|\to\infty}\int_{\mathbb{R}}\frac{\tilde{\xi}(\lambda;d)}{\lambda-z}d\lambda=\int_{\mathbb{R}}\frac{\tilde{\xi}_1(\lambda)+\tilde{\xi}_2(\lambda)}{\lambda-z}d\lambda$$

holds for every $z\in\mathbb{C}$ *with* $\mathrm{Im}z\neq 0$.

Proof. Using the relation between the spectral shift function and the perturbational determinant [5] we have

$$\int_{\mathbb{R}}\frac{\tilde{\xi}(\lambda;d)}{\lambda-z}d\lambda=\log\det(I+\mathcal{V}_d r_0(z)),$$

$$\int_{\mathbb{R}}\frac{\tilde{\xi}_i(\lambda)}{\lambda-z}d\lambda=\log\det(I+\mathcal{V}_i r_0(z)),\ \ i=1,2, \tag{3.5}$$

where $\mathcal{V}_d$ and $\mathcal{V}_i$, $i=1,2$ are defined by (3.1), and

$$r_0(z)=[R_0(-c_0)-z]^{-1}.$$

In the representations (3.5) the branch of the logarithm is fixed uniquely by the conditions

$$\log\det(I+\mathcal{V}_d r_0(z))\to 0,$$

$$\log\det(I+\mathcal{V}_i r_0(z))\to 0,\ \ i=1,2, \tag{3.6}$$

when $\mathrm{Im}z\to\infty$. Since $\xi_2(\lambda)=\xi_2(\lambda;d)$ it suffices to prove that

$$\lim_{|d|\to\infty}\log\det\Big\{(I+\mathcal{V}_d r_0(z))(I+\mathcal{V}_2(d)r_0(z))^{-1}(I+\mathcal{V}_1 r_0(z))^{-1}\Big\}=0. \tag{3.7}$$

for all nonreal z. If we now show that

$$\det\Big\{(I+\mathcal{V}_d r_0(z))(I+\mathcal{V}_2(d)r_0(z))^{-1}(I+\mathcal{V}_1 r_0(z))^{-1}\Big\}\to 1 \tag{3.8}$$

as $|d|\to\infty$ and that the convergence is uniform in $\mathrm{Im}z\in[\epsilon,+\infty)$ (or $\mathrm{Im}z\in(-\infty,-\epsilon]$) for all $\epsilon>0$, then the conditions (3.6) guarantee (3.7).

We note that the operators $I+\mathcal{V}_2(d)r_0(z)$ and $I+\mathcal{V}_1 r_0(z)$ are invertible for nonreal z. The operator norms of $(I+\mathcal{V}_1 r_0(z))^{-1}$ and $(I+\mathcal{V}_2(d)r_0(z))^{-1}$ can be bounded uniformly in $\mathrm{Im}z\in[\epsilon,+\infty)$ ($\mathrm{Im}z\in(-\infty,-\epsilon]$) for all $\epsilon>0$. Therefore relation (3.8) will be proved once we show that

$$\Big\|I+\mathcal{V}_d r_0(z)-(I+\mathcal{V}_1 r_0(z))(I+\mathcal{V}_2(d)r_0(z))\Big\|_{\mathcal{J}_1}\to 0$$

when $|d|\to\infty$ uniformly in $z\in\overline{\Pi_0}$. By Theorem T we have that $\mathcal{V}_d-\mathcal{V}_1-\mathcal{V}_2(d)\to 0$ in trace norm. Thus, it suffices to prove that

$$\lim_{|d|\to\infty}\|\mathcal{V}_1 r_0(z)\mathcal{V}_2(d)\|_{\mathcal{J}_1}=0. \tag{3.9}$$

To establish (3.9) we note that by the second resolvent identity and due to the obvious relation

$$zr_0(z)R_0(-c_0) = -R_0(z^{-1}-c_0),$$

we have

$$\begin{aligned}
\mathcal{V}_1 r_0(z)\mathcal{V}_2(d) &= [R_1(-c_0)-R_0(-c_0)]r_0(z)[R_2(-c_0;d)-R_0(-c_0)] \\
&= R_1(-c_0)V_1R_0(-c_0)r_0(z)R_0(-c_0)V_2(\cdot-d)R_2(-c_0;d) \\
&= R_1(-c_0)V_1R_0(-c_0)V_2(\cdot-d)R_2(-c_0;d) \\
&\quad +zR_1(-c_0)V_1r_0(z)R_0(-c_0)V_2(\cdot-d)R_2(-c_0;d) \\
&= R_1(-c_0)V_1R_0(-c_0)V_2(\cdot-d)R_2(-c_0;d) \\
&\quad -R_1(-c_0)V_1R_0(z^{-1}-c_0)V_2(\cdot-d)R_2(-c_0;d). \qquad (3.10)
\end{aligned}$$

Consider the first term on the r.h.s. of (3.10). We decompose

$$\begin{aligned}
&R_1(-c_0)V_1R_0(-c_0)V_2(\cdot-d)R_2(-c_0;d) \\
&= R_1(-c_0)|V_1|^{1/2}V_1^{1/2}R_0(-c_0)|V_2|^{1/2}(\cdot-d)V_2^{1/2}(\cdot-d)R_2(-c_0;d).
\end{aligned}$$

Note that the Hilbert-Schmidt norms of $R_1(-c_0)|V_1|^{1/2}$ and $V_2^{1/2}(\cdot-d)R_2(-c_0;d)$ are uniformly bounded in d. Moreover, (i) of Theorem T yields

$$\| V_1^{1/2}R_0(-c_0)|V_2|^{1/2}(\cdot-d)\|_{\mathcal{J}_2} \to 0$$

when $|d|\to\infty$. The second term on the r.h.s. of (3.10) can be discussed in a similar way. In this case Theorem T says that the last term tends to zero in Hilbert-Schmidt norm as $|d|\to\infty$ uniformly in $z\in\overline{\Pi_0}$, thus proving (3.9), and by the previous remark also Lemma 3.1. ■

Now we use the continuity of the Stieltjes transform (see e.g. [35, Appendix A]):

Lemma 3.2 *Let the sequence of measurable functions $\mu_n(t)$ on $\mathbb{R}$ ($n\in\mathbb{N}$) satisfy the following properties:*

$$\begin{aligned}
&\int_{\mathbb{R}} \frac{|\mu_n(t)|}{1+|t|}dt < \infty, \\
&\lim_{c\to\infty}\sup_{n\geq 1}\int_{|t|\geq c}\frac{|\mu_n(t)|}{|t|}dt = 0. \qquad (3.11)
\end{aligned}$$

If the sequence $\int\frac{\mu_n(t)dt}{t-z}$ converges to a function $f(z)$ for every nonreal z, then there exists a measurable function $\mu(t)$ such that $\mu_n(t)\to\mu(t)$ in the sense of distributions and f is the Stieltjes transform of μ, i.e.

$$f(z) = \int\frac{\mu(t)dt}{t-z}.$$

We show that the condition (3.11) is satisfied for $\tilde{\xi}(\lambda; d)$. Obviously,

$$\int_{|t|\geq c} \frac{|\tilde{\xi}(t;d)|}{|t|} dt \leq \frac{1}{c} \int_{|t|\geq c} |\tilde{\xi}(t;d)| dt \leq \frac{1}{c} \|\mathcal{V}_d\|_{\mathcal{J}_1}.$$

The norm $\|\mathcal{V}_d\|_{\mathcal{J}_1}$ is uniformly bounded in d since

$$\begin{aligned}
& \|R(-c_0; d) - R_0(-c_0)\|_{\mathcal{J}_1} \\
\leq\ & \|R(-c_0; d) - R_1(-c_0) - R_2(-c_0; d) + R(-c_0)\|_{\mathcal{J}_1} \\
& + \|R_1(-c_0) - R_0(-c_0)\|_{\mathcal{J}_1} + \|R_2(-c_0) - R_0(-c_0)\|_{\mathcal{J}_1} \\
\leq\ & \sup_{d\in\mathbb{R}^3} \left\{ \|R(-c_0; d) - R_1(-c_0) - R_2(-c_0; d) + R(-c_0)\|_{\mathcal{J}_1} \right\} \\
& + \|R_1(-c_0) - R_0(-c_0)\|_{\mathcal{J}_1} + \|R_2(-c_0) - R_0(-c_0)\|_{\mathcal{J}_1}.
\end{aligned}$$

By Theorem T the norm

$$\|R(-c_0; d) - R_1(-c_0) - R_2(-c_0; d) + R_0(-c_0)\|_{\mathcal{J}_1}$$

can be bounded by a constant which is independent of d.

Now, it follows from Lemmas 3.1 and 3.2 that

$$\lim_{|d|\to\infty} \int_{\mathbb{R}} \Big(\tilde{\xi}(\lambda; d) - \tilde{\xi}_1(\lambda) - \tilde{\xi}_2(\lambda) \Big) \psi(\lambda) d\lambda = 0$$

for all $\psi \in C_0^\infty(\mathbb{R})$. Using (3.4) we get

$$\lim_{|d|\to\infty} \int_{\mathbb{R}} \Big(\xi(E; d) - \xi_1(E) - \xi_2(E) \Big) \psi((E + c_0)^{-1}) \frac{dE}{(E + c_0)^2} = 0$$

for arbitrary $\psi \in C_0^\infty(\mathbb{R})$. Let us consider functions $\psi \in C_0^\infty(\mathbb{R})$ with $\mathrm{supp}\psi \subset (0, +\infty)$. Then obviously

$$\phi(E) = \frac{\psi((E + c_0)^{-1})}{(E + c_0)^2} \tag{3.12}$$

is infinitely differentiable and has compact support lying in $(-c_0, +\infty)$. Conversely, an arbitrary C_0^∞-function ϕ with $\mathrm{supp}\phi \subset (-c_0, +\infty)$ can be represented in the form (3.12) with $\psi \in C_0^\infty(\mathbb{R})$. Since c_0 can be taken arbitrary large this proves Theorem 1.

4 Proof of Theorem 2

In the proof of Theorem 2 we will use the representation of the spectral shift function for the pair of Hamiltonians $H = H_0 + V$ and H_0 in terms of regularized Fredholm determinants. We recall that for an arbitrary Hilbert-Schmidt operator

A the regularized Fredholm determinant $\det_2(I+A)$ is defined as the product $\prod_j(1+\lambda_j(A))e^{-\lambda_j(A)}$, where the $\lambda_j(A)$ are the eigenvalues of A.

Lemma 4.1 *For $V \in \mathcal{R} \cap L^1$ the spectral shift for the pair ($H = H_0 + V$, H_0) function can be represented in the form*

$$\xi(E;H,H_0) = \frac{1}{2\pi i}\log\left[\frac{\det_2(I+V^{1/2}R_0(E+i0)|V|^{1/2})}{\det_2(I+V^{1/2}R_0(E-i0)|V|^{1/2})}\exp\left\{\theta(E)\frac{i\sqrt{E}}{2\pi}\int_{\mathbb{R}} V\,dx\right\}\right], \tag{4.1}$$

where $\theta(E)$ is the Heaviside unit step function, $\theta(t)=1$, $t>0$, and $\theta(t)=0$ otherwise. The branch of the logarithm is chosen so that $\lim_{|\mathrm{Im} z|\to\infty}\log\det_2(I+V^{1/2}R_0(z)|V|^{1/2})=0$.

This formula has previously appeared in [32, 14, 7]. Since its proof for $V \in \mathcal{R}\cap L^1$ still seems to be unpublished, for the reader's convenience it will be given in Appendix C.

Remark: We note that the operators $V^{1/2}R_0(E\pm i0)|V|^{1/2}$ are continuous in the Hilbert-Schmidt norm in $E \in \mathbb{R}_+ \setminus \mathcal{E}$ [37]. Since the determinant $\det_2 A$ is continuous with respect to A (see e.g. [42]), the functions $\det_2(I+V^{1/2}R_0(E\pm i0)|V|^{1/2})$ are both continuous in $E\in\mathbb{R}_+\setminus\mathcal{E}$. Since the operators $I+V^{1/2}R_0(E\pm i0)|V|^{1/2}$ are invertible for all $E\in\mathbb{R}_+\setminus\mathcal{E}$ and due to (4.1) the spectral shift function is continuous in $E\in\mathbb{R}_+\setminus\mathcal{E}$. For the potentials having Property A the intersection $\mathcal{E}\cap(0,+\infty)=\emptyset$. Therefore in this case $\xi(E;H,H_0)$ is continuous on $(0,+\infty)$.

Let

$$\begin{aligned}
K_i(z) &= V_i^{1/2}R_0(z)|V_i|^{1/2},\ i=1,2,\\
K_2(z;d) &= V_2^{1/2}(\cdot-d)R_0(z)|V_2|^{1/2}(\cdot-d),\\
K_{12}(z;d) &= V_1^{1/2}R_0(z)|V_2|^{1/2}(\cdot-d),\\
K_{21}(z;d) &= V_2^{1/2}(\cdot-d)R_0(z)|V_1|^{1/2},\\
K(z;d) &= V_d^{1/2}R_0(z)|V_d|^{1/2}.
\end{aligned}$$

We recall that for arbitrary Hilbert-Schmidt operators A,B,C the following identity holds (see e.g. [42])

$$\begin{aligned}
&\det_2((I+A)(I+B)(I+C))\\
&= \det_2(I+A)\det_2(I+B)\det_2(I+C)\\
&\cdot\exp\Big\{-\mathrm{tr}[AB+AC+BC+ABC]\Big\}.
\end{aligned} \tag{4.2}$$

We can write

$$I+K(z;d) = (I+K_1(z))(I+K_2(z;d))(I+L(z;d)), \tag{4.3}$$

with

$$L(z;d) = (I + K_2(z;d))^{-1}(I + K_1(z))^{-1}$$
$$\cdot(K(z;d) - K_1(z) - K_2(z;d) - K_1(z)K_2(z;d)). \tag{4.4}$$

Applying the formula (4.2) to (4.3) we get

$$\det{}_2(I + K(z;d)) = \det{}_2(I + K_1(z))$$
$$\cdot\det{}_2(I + K_2(z;d))\det{}_2(I + L(z;d))e^{-t(z;d)},$$

where

$$\begin{aligned} t(z;d) \;\; = \;\; & \mathrm{tr}K_1(z)K_2(z;d) + \mathrm{tr}K_1(z)L(z;d) \\ & + \mathrm{tr}K_2(z;d)L(z;d) + \mathrm{tr}K_1(z)K_2(z;d)L(z;d). \end{aligned} \tag{4.5}$$

Now by means of Lemma 4.1 we get the representation

$$\xi(E;d) = \xi_1(E) + \xi_2(E) + \xi_{12}(E;d),$$

where

$$\xi_{12}(E;d) = \frac{1}{2\pi i}\log\left[\frac{\det_2(I + L(E+i0;d))}{\det_2(I + L(E-i0;d))}\exp\{-t(E+i0;d) + t(E-i0;d)\}\right]. \tag{4.6}$$

Due to (1.13) the branch of the logarithm can be fixed uniquely by the condition

$$\lim_{E\to+\infty}\xi_{12}(E;d) = 0.$$

By Theorem T $L(z;d)$ tends to zero in Hilbert-Schmidt norm uniformly in z on compact sets in $\overline{\Pi_0}$ when $|d| \to \infty$. Therefore the operator $I + L(E \pm i0;d)$ is invertible for all sufficiently large $|d|$. From Lemma 2.1 it follows that the operator $K_2(z;d)$ is $\mathcal{J}_2$-continuous in d (special case $V_1 = 0$). Thus the operator $L(z;d)$ and the function $t(z;d)$ are continuous in d. By the continuity of the determinant $\det_2$, $\xi_{12}(E;d)$ is continuous in d for all sufficiently large $|d|$ and fixed E. Combining these arguments with the remark after Lemma 4.1 we can easily prove the joint continuity of $\xi_{12}(E;d)$ in E and d.

As an easy consequence of Lemma 2.4 we get

Lemma 4.2 *Let the potentials V_i have Property A. Then for all $z \in \overline{\Pi_0}$*

$$\|K_1(z)K_2(z;d)\|_{\mathcal{J}_1} = o(|d|^{-2}).$$

as $|d| \to \infty$

Proof. It is easy to see that

$$
\begin{aligned}
&\|K_1(z)K_2(z;d)\|_{\mathcal{J}_1}\\
&\le \left\|V_1^{1/2}R_0(z)|V_1|^{1/4}|V_2|^{1/4}(\cdot-d)\right\|_{\mathcal{J}_2}\\
&\quad\cdot\left\||V_1|^{1/4}|V_2|^{1/4}(\cdot-d)R_0(z)|V_2|^{1/2}(\cdot-d)\right\|_{\mathcal{J}_2}\\
&\le \frac{1}{(4\pi)^2}\Big(\int\frac{|V_1(x)||V_1(y)|^{1/2}|V_2(y-d)|^{1/2}}{|x-y|^2}dxdy\Big)^{1/2}\\
&\cdot\frac{1}{(4\pi)^2}\Big(\int\frac{|V_1(x)|^{1/2}|V_2(x-d)|^{1/2}|V_2(y-d)|}{|x-y|^2}dxdy\Big)^{1/2},
\end{aligned}
$$

which is $o(|d|^{-2})$ due to Lemma 2.4. ■

We define the operator $\widetilde{K}(z;d)$ as

$$\widetilde{K}(z;d)=K(z;d)-K_1(z)-K_2(z;d)-K_{12}(z;d)-K_{21}(z;d).$$

Lemma 4.3 *Let the potentials V_i have Property A. Then $|d|\|\widetilde{K}(z;d)\|_{\mathcal{J}_2}$ tends to zero as $|d|\to\infty$ uniformly in z on compact subsets of $\overline{\Pi}_0$.*

Remark: For compactly supported potentials the operator $\widetilde{K}(z;d)$ is identically zero for sufficiently large $|d|$.

Proof. Obviously,

$$
\begin{aligned}
\|\widetilde{K}(z;d)\|_{\mathcal{J}_2}\le&\left\|\left(V_d^{1/2}-V_1^{1/2}-V_2^{1/2}(\cdot-d)\right)R_0(z)|V_d|^{1/2}\right\|_{\mathcal{J}_2}\\
+&\left\|V_1^{1/2}R_0(z)\Big(|V_d|^{1/2}-|V_1|^{1/2}-|V_2|^{1/2}(\cdot-d)\Big)\right\|_{\mathcal{J}_2}\\
+&\left\|V_2^{1/2}(\cdot-d)R_0(z)\Big(|V_d|^{1/2}-|V_1|^{1/2}-|V_2|^{1/2}(\cdot-d)\Big)\right\|_{\mathcal{J}_2}.
\end{aligned}\tag{4.7}
$$

Due to Lemma 2.3 of [27]

$$\Big(V_d^{1/2}(x)-V_1^{1/2}(x)-V_2^{1/2}(x-d)\Big)^2\le 4|V_1(x)|^{1/2}|V_2(x-d)|^{1/2}.\tag{4.8}$$

Also $V_i\in\mathcal{R}\cap L^1$ implies $|V_1|^{1/2}|V_2|^{1/2}(\cdot-d)\in\mathcal{R}\cap L^1$.

Let us consider the first term on the r.h.s. of (4.7). The two other terms can be discussed in a completely similar way. It is easy to see that

$$
\begin{aligned}
&|d|^2\left\|\left(V_d^{1/2}-V_1^{1/2}-V_2^{1/2}(\cdot-d)\right)R_0(z)|V_d|^{1/2}\right\|_{\mathcal{J}_2}^2\\
\le\;&\frac{|d|^2}{4\pi^2}\int\frac{|V_1(x)|^{1/2}|V_2(x-d)|^{1/2}|V_d(y)|}{|x-y|^2}dxdy
\end{aligned}
$$

$$\leq \frac{|d|^2}{4\pi^2}\int \frac{|V_1(x)|^{1/2}|V_2(x-d)|^{1/2}|V_1(y)|}{|x-y|^2}dxdy$$
$$+\frac{|d|^2}{4\pi^2}\int \frac{|V_1(x)|^{1/2}|V_2(x-d)|^{1/2}|V_2(y-d)|}{|x-y|^2}dxdy.$$

Due to Lemma 2.4 both these integrals are $o(1)$, which completes the proof of the lemma. ■

We can represent the operator $L(z;d)$ (4.4) in the form

$$\begin{aligned} L(z;d) = (I+K_2(z;d))^{-1}(I+K_1(z))^{-1} \\ \cdot [K_{12}(z;d)+K_{21}(z;d)] + \widetilde{L}(z;d), \end{aligned} \tag{4.9}$$

where

$$\begin{aligned} \widetilde{L}(z;d) = (I+K_2(z;d))^{-1}(I+K_1(z))^{-1} \\ \cdot \left[\widetilde{K}(z;d) - K_1(z)K_2(z;d)\right], \end{aligned}$$

Due to Lemma 4.2 and Lemma 4.3 for all $z \in \mathbb{C}\setminus\mathcal{E}$ and all large $|d|$ the operator $\widetilde{L}(z;d)$ can be estimated as

$$\|\widetilde{L}(z;d)\|_{\mathcal{J}_2} = o(|d|^{-1}).$$

We estimate now the square of the operator $L(z;d)$.

Lemma 4.4 *For all $z \in \mathbb{C}\setminus\mathcal{E}$ and sufficiently large $|d|$,*

$$\begin{aligned} L^2(z;d) &= (I+K_1(z))^{-1}K_{12}(z;d)(I+K_2(z))^{-1}K_{21}(z;d) \\ &+ (I+K_2(z;d))^{-1}K_{21}(z;d)(I+K_1(z))^{-1}K_{12}(z;d) + o(|d|^{-2}), \end{aligned}$$

where the error term is understood in the sense of trace norm und uniform in z on compact sets in $\mathbb{C}\setminus\mathcal{E}$.

Proof. Due to (4.9) and Lemma 4.3 for all $z \in \mathbb{C}\setminus\mathcal{E}$ and all sufficiently large $|d|$ we have

$$\begin{aligned} L^2(z;d) &= (I+K_2(z;d))^{-1}(I+K_1(z))^{-1}K_{12}(z;d) \\ &\quad \cdot(I+K_2(z;d))^{-1}(I+K_1(z))^{-1}K_{12}(z;d) \\ &+ (I+K_2(z;d))^{-1}(I+K_1(z))^{-1}K_{12}(z;d) \\ &\quad \cdot(I+K_2(z;d))^{-1}(I+K_1(z))^{-1}K_{21}(z;d) \\ &+ (I+K_2(z;d))^{-1}(I+K_1(z))^{-1}K_{21}(z;d) \\ &\quad \cdot(I+K_2(z;d))^{-1}(I+K_1(z))^{-1}K_{12}(z;d) \\ &+ (I+K_2(z;d))^{-1}(I+K_1(z))^{-1}K_{21}(z;d) \\ &\quad \cdot(I+K_2(z;d))^{-1}(I+K_1(z))^{-1}K_{21}(z;d) + o(|d|^{-2}), \end{aligned} \tag{4.10}$$

where the error term $o(|d|^{-2})$ is understood in the sense of trace norm.

Consider the first term in the r.h.s. of (4.10) (the third term can be considered in exactly the same way). Let us estimate the operator

$$|d|^2 K_{12}(z;d)(I+K_2(z;d))^{-1}(I+K_1(z))^{-1}K_{12}(z;d).$$

For the case of compactly supported potentials this operator is identically zero for all large $|d|$. Obviously, for all z in compact sets in $\mathbb{C}\setminus\mathcal{E}$ we have

$$\begin{aligned}
& |d|^2\|K_{12}(z;d)(I+K_2(z;d))^{-1}(I+K_1(z))^{-1}K_{12}(z;d)\|_{\mathcal{J}_1} \\
= \ & |d|^2\|(I+K_2(z;d))^{-1}(I+K_1(z))^{-1}K_{12}^2(z;d)\|_{\mathcal{J}_1} \\
\le \ & C|d|^2\|K_{12}^2(z;d)\|_{\mathcal{J}_1} \\
\le \ & C\|V_1^{1/2}R_0(z)|V_1|^{1/4}|V_2|^{1/4}(\cdot-d)\|_{\mathcal{J}_2} \\
& \|V_1^{1/4}|V_2|^{1/4}(\cdot-d)R_0(z)|V_2|^{1/2}(\cdot-d)\|_{\mathcal{J}_2}.
\end{aligned} \tag{4.11}$$

for some $C>0$. The r.h.s. of (4.11) can be bounded by

$$\begin{aligned}
& \frac{C}{(4\pi)^2}\Big(\int\frac{|V_1(x)||V_1(y)|^{1/2}|V_2(y-d)|^{1/2}}{|x-y|^2}dxdy\Big)^{1/2} \\
& \cdot\Big(\int\frac{|V_1(x)|^{1/2}|V_2(x-d)|^{1/2}|V_2(y-d)|}{|x-y|^2}dxdy\Big)^{1/2},
\end{aligned}$$

which is $o(|d|^{-2})$ by Lemma 2.4.

We turn to the discussion of the second term in (4.10) (the fourth term can be considered in exactly the same way). We show that

$$\begin{aligned}
& |d|^2\Big\{(I+K_2(z;d))^{-1}(I+K_1(z))^{-1}K_{12}(z;d) \\
& \cdot(I+K_2(z;d))^{-1}(I+K_1(z))^{-1}K_{21}(z;d) \\
& -(I+K_1(z))^{-1}K_{12}(z;d)(I+K_2(z;d))^{-1}K_{21}(z;d)\Big\}
\end{aligned} \tag{4.12}$$

tends to zero in trace norm as $|d|\to\infty$. Again for the case of compactly supported potentials the expression (4.12) is identically zero for all large $|d|$. First we express the difference (4.12) in the following form

$$\begin{aligned}
& |d|^2(I+K_2(z;d))^{-1}(I+K_1(z))^{-1}K_{12}(z;d) \\
& \cdot(I+K_2(z;d))^{-1}\Big\{(I+K_1(z))^{-1}-I\Big\}K_{21}(z;d) \\
+ \ & |d|^2\Big\{(I+K_2(z;d))^{-1}-I\Big\}(I+K_1(z))^{-1}K_{12}(z;d) \\
& \cdot(I+K_2(z;d))^{-1}K_{21}(z;d).
\end{aligned} \tag{4.13}$$

We show that

$$|d|^2\Big\|\Big\{(I+K_1(z))^{-1}-I\Big\}K_{21}(z;d)\Big\|_{\mathcal{J}_1}\to 0$$

as $|d| \to \infty$. Clearly,

$$(I + K_1(z))^{-1} - I = -(I + K_1(z))^{-1} K_1(z).$$

Thus, it suffices to prove that

$$|d|^2 \|K_1(z) K_{21}(z; d)\|_{\mathcal{J}_1} \to 0$$

as $|d| \to \infty$. To this end we use the estimate

$$\begin{aligned} &\|K_1(z) K_{21}(z; d)\|_{\mathcal{J}_1} \\ \le \; & \|V_1^{1/2} R_0(z) |V_1|^{1/4} |V_2|^{1/4} (\cdot - d)\|_{\mathcal{J}_2} \\ & \cdot \|V_1^{1/4} V_2^{1/4} (\cdot - d) R_0(z) |V_1|^{1/2}\|_{\mathcal{J}_2} \\ \le \; & \frac{1}{(4\pi)^2} \int \frac{|V_1(x)||V_1(y)|^{1/2} |V_2(y-d)|^{1/2}}{|x-y|^2} dx dy. \end{aligned}$$

This integral is $o(|d|^{-2})$ due to Lemma 2.4.

Similarly we can show that

$$|d|^2 \left\| K_{21}(z; d) \left\{ (I + K_2(z; d))^{-1} - I \right\} \right\|_{\mathcal{J}_1} \to 0 \qquad (4.14)$$

as $|d| \to \infty$.

Consider the second term in (4.13). Obviously,

$$\begin{aligned} & |d|^2 \left\| \left\{ (I + K_2(z; d))^{-1} - I \right\} (I + K_1(z))^{-1} \right. \\ & \left. \cdot K_{12}(z; d) (I + K_2(z; d))^{-1} K_{21}(z; d) \right\|_{\mathcal{J}_1} \\ = \; & |d|^2 \left\| K_{21}(z; d) \left\{ (I + K_2(z; d))^{-1} - I \right\} \right. \\ & \left. \cdot (I + K_1(z))^{-1} K_{12}(z; d) (I + K_2(z; d))^{-1} \right\|_{\mathcal{J}_1}. \end{aligned}$$

Due to (4.14) this tends to zero as $|d| \to \infty$. ■

Consider now the function $t(z; d)$ given by (4.5).

Lemma 4.5 *For all $z \in \mathbb{C} \setminus \mathcal{E}$ and sufficiently large $|d|$,*

$$t(z; d) = o(|d|^{-2})$$

uniformly in z on compact sets in $\mathbb{C} \setminus \mathcal{E}$.

Remark: For V_i's with compact supports $t(z; d) = 0$ for all sufficiently large $|d|$.

Proof. From (4.5) by Lemma 4.2 it follows that

$$t(z; d) = \mathrm{tr} K_1(z) L(z; d) + \mathrm{tr} K_2(z; d) L(z; d) + o(|d|^{-2}).$$

Consider $\mathrm{tr} K_1(z)L(z;d)$. By (4.4) and again by Lemma 4.2 we have

$$
\begin{aligned}
&|\mathrm{tr} K_1(z)L(z;d)| = |\mathrm{tr} L(z;d)K_1(z)| \\
&= \left|\mathrm{tr}\Big[(I+K_2(z;d))^{-1}(I+K_1(z))^{-1}\left\{K(z;d)-K_1(z)\right\}K_1(z)\Big]\right| + o(|d|^{-2}) \\
&\le C\|K_2(z;d)K_1(z)\|_{\mathcal{J}_1} + C\|K_{12}(z;d)K_1(z)\|_{\mathcal{J}_1} \\
&+C\|K_{21}(z;d)K_1(z)\|_{\mathcal{J}_1} + C\|\widetilde{K}(z;d)K_1(z)\|_{\mathcal{J}_1} + o(|d|^{-2}).
\end{aligned}
$$

In the course of the proof of Lemma 4.4 we have already shown that

$$\|K_1(z)K_{21}(z;d)\|_{\mathcal{J}_1} = o(|d|^{-2})$$

when $d \to 0$. In exactly the same way we can show that $\|K_{12}(z;d)K_1(z)\|_{\mathcal{J}_1}$ is $o(|d|^{-2})$ also. The norm $\|K_2(z;d)K_1(z)\|_{\mathcal{J}_1}$ is $o(|d|^{-2})$ by Lemma 4.2. We consider now $\|\widetilde{K}(z;d)K_1(z)\|_{\mathcal{J}_1}$. As in the course of the proof of Lemma 4.3 using (4.8) we can estimate

$$
\begin{aligned}
&\|K_1(z)\widetilde{K}(z;d)\|_{\mathcal{J}_1} \\
&\le \left\|K_1(z)\Big(V_d^{1/2}-V_1^{1/2}-V_2^{1/2}(\cdot-d)\Big)R_0(z)|V_d|^{1/2}\right\|_{\mathcal{J}_1} \\
&\quad+\left\|V_1^{1/2}R_0(z)\Big(|V_d|^{1/2}-|V_1|^{1/2}-|V_2|^{1/2}(\cdot-d)\Big)K_1(z)\right\|_{\mathcal{J}_1} \\
&\quad+\left\|V_2^{1/2}R_0(z)\Big(|V_d|^{1/2}-|V_1|^{1/2}-|V_2|^{1/2}(\cdot-d)\Big)K_1(z)\right\|_{\mathcal{J}_1} \\
&\le 2\left\|V_1^{1/2}R_0(z)|V_1|^{3/8}|V_2|^{1/8}(\cdot-d)\right\|_{\mathcal{J}_2} \\
&\qquad\cdot\left\||V_1|^{3/8}|V_2|^{1/8}(\cdot-d)R_0(z)|V_d|^{1/2}\right\|_{\mathcal{J}_2} \\
&\quad+2\left\|V_1^{1/2}R_0(z)|V_1|^{3/8}|V_2|^{1/8}(\cdot-d)\right\|_{\mathcal{J}_2} \\
&\qquad\cdot\left\|V_1^{3/8}|V_2|^{1/8}(\cdot-d)R_0(z)|V_1|^{1/2}\right\|_{\mathcal{J}_2} \\
&\quad+2\left\|V_2^{1/2}(\cdot-d)R_0(z)|V_1|^{3/8}|V_2|^{1/8}(\cdot-d)\right\|_{\mathcal{J}_2} \\
&\qquad\cdot\left\|V_1^{3/8}|V_2|^{1/8}(\cdot-d)R_0(z)|V_1|^{1/2}\right\|_{\mathcal{J}_2}.
\end{aligned}
$$

Due to Lemma 2.4 this is $o(|d|^{-2})$. In exactly the same way one can estimate the remaining term $\mathrm{tr} K_2(z;d)L(z;d)$, thus completing the proof of the lemma. ■

To proceed further with the proof of Theorem 2 we use the obvious estimate

$$\left|\log\det{}_2(I+A)+\frac{1}{2}\mathrm{tr}A^2\right| \le \|A^3\|_{\mathcal{J}_1}, \tag{4.15}$$

where A is an arbitrary Hilbert-Schmidt operator with operator norm satisfying $\|A\| < 1/2$. (This estimate easily follows from the definition of the modified

Fredholm determinant). We apply (4.15) to the operator $L(E\pm i0;d)$. Let us note that

$$\|L^3(E\pm i0;d)\|_{\mathcal{J}_1}\le \|L(E\pm i0;d)\|\ \|L^2(E\pm i0;d)\|_{\mathcal{J}_1}\le \|L(E\pm i0;d)\|^3_{\mathcal{J}_2}.$$

¿From (4.9), Lemmas 2.2 – 2.4 and 4.3 it follows that

$$\|L(E\pm i0;d)\|^3_{\mathcal{J}_2}\le \frac{C}{|d|^3}$$

for some $C>0$. Therefore, it follows from (4.6) and Lemma 4.5 that

$$\xi_{12}(E;d)=-\frac{1}{4\pi i}\mathrm{tr}\left[L^2(E+i0;d)-L^2(E-i0;d)\right]+o(|d|^{-2}). \tag{4.16}$$

Applying Lemmas 4.4 and 2.2 we get that

$$\begin{aligned}&|d|^2\mathrm{tr}L^2(E\pm i0;d)-2\frac{e^{\pm 2i\sqrt{E}|d|}}{(4\pi)^2}\\ &\cdot\left(\Phi_1^{(\mp)}(\cdot;E,\hat d),(I+K_1(E\pm i0))^{-1}\mathrm{sign}V_1\Phi_1^{(\pm)}(\cdot;E,\hat d)\right)\\ &\cdot\left(\Phi_2^{(\pm)}(\cdot-d;E,\hat d),(I+K_2(E\pm i0;d))^{-1}\mathrm{sign}V_2(\cdot-d)\Phi_2^{(\mp)}(\cdot-d;E,\hat d)\right)\end{aligned}$$

tends to zero when $|d|\to\infty$. By translation invariance

$$\begin{aligned}&\left(\Phi_2^{(\pm)}(\cdot-d;E,\hat d),(I+K_2(E\pm i0;d))^{-1}\mathrm{sign}V_2(\cdot-d)\Phi_2^{(\mp)}(\cdot-d;E,\hat d)\right)\\ &\qquad=\left(\Phi_2^{(\pm)}(\cdot;E,\hat d),(I+K_2(E\pm i0))^{-1}\mathrm{sign}V_2\Phi_2^{(\mp)}(\cdot;E,\hat d)\right).\end{aligned}$$

Therefore to leading order in $|d|^{-1}$

$$\begin{aligned}&\mathrm{tr}\left[L^2(E+i0;d)-L^2(E-i0;d)\right]\\ &=\frac{2}{(4\pi)^2}\frac{e^{2i\sqrt{E}|d|}}{|d|^2}\left(\Phi_1^{(-)}(\cdot;E,\hat d),(I+K_1(E+i0))^{-1}\mathrm{sign}V_1\Phi_1^{(+)}(\cdot;E,\hat d)\right)\\ &\qquad\cdot\left(\Phi_2^{(+)}(\cdot;E,\hat d),(I+K_2(E+i0))^{-1}\mathrm{sign}V_2\Phi_2^{(-)}(\cdot;E,\hat d)\right)\\ &-\frac{2}{(4\pi)^2}\frac{e^{-2i\sqrt{E}|d|}}{|d|^2}\left(\Phi_1^{(+)}(\cdot;E,\hat d),(I+K_1(E-i0))^{-1}\mathrm{sign}V_1\Phi_1^{(-)}(\cdot;E,\hat d)\right)\\ &\qquad\cdot\left(\Phi_2^{(-)}(\cdot;E,\hat d),(I+K_2(E-i0))^{-1}\mathrm{sign}V_2\Phi_2^{(+)}(\cdot;E,\hat d)\right)+o(|d|^{-2}).\end{aligned}$$

With the help of the well known representation for the scattering amplitude (see e.g. [37, 1])

$$\begin{aligned}&A_i(\omega,\omega';E)\\ &\quad=-\frac{1}{4\pi}\left(e^{i\sqrt{E}\langle\omega,\cdot\rangle}|V_i|^{1/2},(I+K_i(E+i0))^{-1}V_i^{1/2}e^{i\sqrt{E}\langle\omega',\cdot\rangle}\right),\end{aligned} \tag{4.17}$$

the symmetry relation

$$\begin{aligned}
&\left(|V_i|^{1/2}e^{-i\sqrt{E}\langle\hat{d},\cdot\rangle}, (I+K_i(E-i0))^{-1}V_i^{1/2}e^{i\sqrt{E}\langle\hat{d},\cdot\rangle}\right)\\
&=\overline{\left(|V_i|^{1/2}e^{i\sqrt{E}\langle\hat{d},\cdot\rangle}, (I+K_i(E+i0))^{-1}V_i^{1/2}e^{-i\sqrt{E}\langle\hat{d},\cdot\rangle}\right)}
\end{aligned}$$

and (2.6) one gets

$$\begin{aligned}
&\operatorname{tr}\left[L^2(E+i0;d)-L^2(E-i0;d)\right]\\
&= 2\frac{e^{2i\sqrt{E}|d|}}{|d|^2}A_1(\hat{d},-\hat{d};E)A_2(-\hat{d},\hat{d};E)\\
&\quad -2\frac{e^{-2i\sqrt{E}|d|}}{|d|^2}\overline{A_1(\hat{d},-\hat{d};E)A_2(\hat{d},-\hat{d};E)}+o(|d|^{-2}),
\end{aligned}$$

which together with (4.16) gives (1.16).

5 Proof of Theorem 3

Let $E=0$ be a regular point for both H_1 and H_2. Hence the operators $I+K_i(0)$, $i=1,2$ are invertible. Therefore all the arguments used to prove Theorem 2 can be repeated verbatim in the case $E=0$, thus yielding

$$\xi_{12}(0+;d)=\frac{1}{|d|^2}\mathrm{Im}\ a(0;\hat{d})+o(|d|^{-2}), \tag{5.1}$$

where

$$a(0;\hat{d})=-\frac{1}{\pi}A_1(\hat{d},-\hat{d};0)A_2(-\hat{d},\hat{d};0).$$

We recall that

$$c_0(V_i):=A_i(\omega,\omega';0)=-\frac{1}{4\pi}\left(|V_i|^{1/2},(I+K_i(0))^{-1}V_i^{1/2}\right), \tag{5.2}$$

is the scattering length, which is obviously real. Therefore $\mathrm{Im}\ a(0;\hat{d})=0$. From (5.1) and the fact that $\xi_{12}(0+;d)$ is integer or half-integer, it follows that for all sufficiently large $|d|$

$$\xi_{12}(+0;d)=0.$$

Now we consider the case where $E=0$ is an exceptional point of the first kind at least for one of the operators H_1 and H_2. It is well known (see e.g. [1]) that $E=0$ is an exceptional point of the first kind for H_i iff the equation

$$K_i(0)\varphi_i=-\varphi_i, \tag{5.3}$$

has a unique solution $\varphi_i\in L^2(\mathbb{R}^3)$ and $(|V_i|^{1/2},\varphi_i)\neq 0$. The corresponding eigenprojector we denote by $\mathcal{P}_i$, i.e.

$$\mathcal{P}_i=\frac{\varphi_i(\tilde{\varphi}_i,\cdot)}{(\tilde{\varphi}_i,\varphi_i)},$$

where $\tilde{\varphi}_i = (\mathrm{sign} V_i)\varphi_i$. Obviously $\tilde{\varphi}_i$ satisfies the equation $K_i(0)^*\tilde{\varphi}_i = -\tilde{\varphi}_i$. It is easy to see that $(\tilde{\varphi}_i, \varphi_i) \neq 0$ (see [2, pp. 21-22]). The function $\psi_i = R_0(0)|V_i|^{1/2}\varphi_i \notin L^2$ is called a zero energy resonance wave function and satisfies $H_i\psi_i = 0$ in the sense of distributions.

For potentials having Property A we can expand the operators $K_i(z)$ in the powers of $\sqrt{z}$:

$$K_i(z) = K_i(0) + \frac{i\sqrt{z}}{4\pi} V_i^{1/2}(|V_i|^{1/2}, \cdot) + zN_i + o(z), \tag{5.4}$$

where N_i is a Hilbert-Schmidt operator with integral kernel

$$N_i(x, y) = -\frac{1}{8\pi} V_i^{1/2}(x)|x - y||V_i|^{1/2}(y).$$

.

The low-energy expansion obtained in [1] yields the operator relation

$$(I + K_i(z))^{-1} = \frac{4\pi}{i\sqrt{z}} \mathcal{Q}_i + M_i(z), \tag{5.5}$$

with

$$\mathcal{Q}_i = \frac{\varphi_i(\tilde{\varphi}_i, \cdot)}{|(|V_i|^{1/2}, \varphi_i)|^2}, \tag{5.6}$$

being a rank 1 operator and $M_i(z)$ is a bounded operator in a neighborhood of $z = 0$. For $z \to 0$ $M_i(z)$ has a representation

$$\begin{aligned} M_i(z) = T_i &- \frac{\varphi_i(\tilde{\varphi}_i, V_i^{1/2})(T_i^*|V_i|^{1/2}, \cdot)}{|(|V_i|^{1/2}, \varphi_i)|^2} - \frac{T_i V_i^{1/2}(|V_i|^{1/2}, \varphi_i)(\tilde{\varphi}_i, \cdot)}{|(|V_i|^{1/2}, \varphi_i)|^2} \\ &+ \frac{\varphi_i(|V_i|^{1/2}, T_i V_i^{1/2})(\tilde{\varphi}_i, \cdot)}{|(|V_i|^{1/2}, \varphi_i)|^2} + (4\pi)^2 \frac{\varphi_i(\tilde{\varphi}_i, N_i\varphi_i)(\tilde{\varphi}_i, \cdot)}{|(|V_i|^{1/2}, \varphi_i)|^4} + o(1), \end{aligned} \tag{5.7}$$

where

$$T_i = n - \lim_{\epsilon \to 0+} (I + V_i^{1/2} R_0(0)|V_i|^{1/2} + \epsilon)^{-1}(I - \mathcal{P}_i).$$

The error term in (5.7) is understood in the sense of the operator norm.

Below we will use the formula

$$\begin{aligned} (I + K_i(z))M_i(z) &= I - \frac{4\pi}{i\sqrt{z}}(I + K_i(z))\mathcal{Q}_i \\ &= I - V_i^{1/2}(|V_i|^{1/2}, \cdot)\mathcal{Q}_i + O(\sqrt{z}). \end{aligned} \tag{5.8}$$

To prove this we multiply (5.5) by $K_i(z)$ thus obtaining

$$K_i(z)(I + K_i(z))^{-1} = \frac{4\pi}{i\sqrt{z}} K_i(z)\mathcal{Q}_i + K_i(z)M_i(z). \tag{5.9}$$

On the other hand

$$K_i(z)(I+K_i(z))^{-1} = I-(I+K_i(z))^{-1} = -\frac{4\pi}{i\sqrt{z}}\mathcal{Q}_i + I - M_i(z). \qquad (5.10)$$

Comparing (5.9) and (5.10) proves the first part of (5.8). Now expanding $K_i(z)$ around $z=0$ in $\sqrt{z}$ and using $K_i(0)\mathcal{Q}_i = -\mathcal{Q}_i$ we obtain

$$(I+K_i(z))\mathcal{Q}_i = \frac{i\sqrt{z}}{4\pi}V_i^{1/2}(|V_i|^{1/2},\cdot)\mathcal{Q}_i + O(z),$$

thus proving (5.8).

First let us consider the case when $E=0$ is a regular point for H_1 and an exceptional one of the first kind for H_2. In this case we represent the operator $L(z;d)$ (4.4) in the form

$$L(z;d) = \frac{4\pi}{i\sqrt{z}}L^{(0)}(z;d) + L^{(1)}(z;d),$$

where

$$\begin{aligned} L^{(0)}(z;d) &= \mathcal{Q}_2(d)(I+K_1(z))^{-1} \\ &\quad \cdot(K_{12}(z;d)+K_{21}(z;d)+\widetilde{K}(z;d)-K_1(z)K_2(z;d)), \\ L^{(1)}(z;d) &= M_2(z;d)(I+K_1(z))^{-1} \\ &\quad \cdot(K_{12}(z;d)+K_{21}(z;d)+\widetilde{K}(z;d)-K_1(z)K_2(z;d)). \qquad (5.11)\end{aligned}$$

Here $\mathcal{Q}_2(d) = U(d)\mathcal{Q}_2U(d)^{-1}$ and $M_2(z;d) = U(d)M_2(z)U(d)^{-1}$. We note that the operator $L^{(0)}(z;d)$ has rank 1. Due to Theorem T for all sufficiently large $|d|$ and all small z one has $\|L^{(1)}(z;d)\| < 1$. Therefore $I+L^{(1)}(z;d)$ is then invertible. Hence,

$$\begin{aligned} &\det{}_2(I+L(z;d)) \\ &= \det{}_2(I+L^{(1)}(z;d))\ \det{}_2\left[I+\frac{4\pi}{i\sqrt{z}}L^{(0)}(z;d)(I+L^{(1)}(z;d))^{-1}\right] \\ &\quad \cdot\exp\left\{-\frac{4\pi}{i\sqrt{z}}\mathrm{tr}L^{(0)}(z;d)(I+L^{(1)}(z;d))^{-1}L^{(1)}(z;d)\right\} \\ &= \det{}_2(I+L^{(1)}(z;d))\ \det\left[I+\frac{4\pi}{i\sqrt{z}}L^{(0)}(z;d)(I+L^{(1)}(z;d))^{-1}\right] \\ &\quad \cdot\exp\left\{-\frac{4\pi}{i\sqrt{z}}\mathrm{tr}L^{(0)}(z;d)\right\}.\end{aligned}$$

Since for any rank 1 operator A, $\det(I+A) = 1+\mathrm{tr}A$, using (5.6) we can easily calculate

$$\det\left[I+\frac{4\pi}{i\sqrt{z}}L^{(0)}(z;d)(I+L^{(1)}(z;d))^{-1}\right]$$

$$
\begin{aligned}
&= 1+\frac{4\pi}{i\sqrt{z}}\mathrm{tr}L^{(0)}(z;d)(I+L^{(1)}(z;d))^{-1}\\
&= 1+\frac{4\pi}{i\sqrt{z}}\mathrm{tr}\Big[\mathcal{Q}_2(d)(I+K_1(z))^{-1}\\
&\quad\cdot\Big(K_{12}(z;d)+K_{21}(z;d)+\widetilde{K}(z;d)-K_1(z)K_2(z;d)\Big)\,(I+L^{(1)}(z;d))^{-1}\Big]\\
&= 1+\frac{4\pi}{i\sqrt{z}}\Big(\tilde{\varphi}_2(\cdot-d),(I+K_1(z))^{-1}\\
&\quad\cdot\Big[K_{12}(z;d)+K_{21}(z;d)+\widetilde{K}(z;d)-K_1(z)K_2(z;d)\Big]\\
&\qquad\cdot(I+L^{(1)}(z;d))^{-1}\varphi_2(\cdot-d)\Big)|(|V_2|^{1/2},\varphi_2)|^{-2}.
\end{aligned}
$$

Further we calculate $t(z;d)$ defined in (4.5),

$$
t(z;d)=\frac{4\pi}{i\sqrt{z}}t^{(0)}(z;d)+t^{(1)}(z;d),
$$

where

$$
\begin{aligned}
&t^{(0)}(z;d)=\mathrm{tr}L^{(0)}(z;d)\,(K_1(z)+K_2(z;d)+K_1(z)K_2(z;d))\\
&=\Big(\tilde{\varphi}_2(\cdot-d),(I+K_1(z))^{-1}\Big[K_{12}(z;d)+K_{21}(z;d)+\widetilde{K}(z;d)-K_1(z)K_2(z;d)\Big]\\
&\qquad\cdot\Big[K_1(z)+K_2(z;d)+K_1(z)K_2(z;d)\Big]\varphi_2(\cdot-d)\Big)|(|V_2|^{1/2},\varphi_2)|^{-2},\\
&t^{(1)}(z;d)=\mathrm{tr}L^{(1)}(z;d)\Big[K_1(z)+K_2(z;d)+K_1(z)K_2(z;d)\Big]+\mathrm{tr}K_1(z)K_2(z;d).
\end{aligned}
$$

Now we study the limit $E\to+0$ of the expression

$$
\pm\frac{4\pi}{i\sqrt{E}}\mathrm{tr}L^{(0)}(E\pm i0;d)\pm\frac{4\pi}{i\sqrt{E}}t^{(0)}(E\pm i0;d). \tag{5.12}
$$

Simple calculations shows that (5.12) equals

$$
\begin{aligned}
\pm\ &\frac{4\pi}{i\sqrt{E}}\Big(\tilde{\varphi}_2(\cdot-d),(I+K_1(E\pm i0))^{-1}[K_{12}(E\pm i0;d)+K_{21}(E\pm i0;d)\\
&+\widetilde{K}(E\pm i0;d)-K_1(E\pm i0)K_2(E\pm i0;d)]\varphi_2(\cdot-d)\Big)|(|V_2|^{1/2},\varphi_2)|^{-2}\\
\pm\ &\frac{4\pi}{i\sqrt{E}}\Big(\tilde{\varphi}_2(\cdot-d),(I+K_1(E\pm i0))^{-1}[K_{12}(E\pm i0;d)+K_{21}(E\pm i0;d)\\
&+\widetilde{K}(E\pm i0;d)-K_1(E\pm i0)K_2(E\pm i0;d]\\
&\quad\cdot[K_1(E\pm i0)+K_2(E\pm i0;d)\\
&+K_1(E\pm i0)K_2(E\pm i0;d)]\varphi_2(\cdot-d)\Big)|(|V_2|^{1/2},\varphi_2)|^{-2}.
\end{aligned}\tag{5.13}
$$

Now we prove

Lemma 5.1 *Let the V_i's have Property A. Then for $E \to +0$*

$$\frac{1}{\sqrt{E}}\Big[\Big(K_1(E \pm i0) + K_2(E \pm i0; d) + K_1(E \pm i0)K_2(E \pm i0; d)\Big)\varphi_2(\cdot - d)$$
$$+\varphi_2(\cdot - d)\Big] = \pm\frac{i}{4\pi}\Big[(I + K_1(0))V_2^{1/2}(\cdot - d)(|V_2|^{1/2}, \varphi_2)\Big] + O(\sqrt{E}) \quad (5.14)$$

in L^2-norm uniformly in $d \in \mathbb{R}^3$.

Proof. We expand the operators $K_1(E \pm i0)$ and $K_2(E \pm i0; d)$ in Taylor series in $\sqrt{E}$ at $E = 0$:

$$K_1(E \pm i0) = K_1(0) \pm \frac{i\sqrt{E}}{4\pi}V_1^{1/2}(|V_1|^{1/2}, \cdot) + O(E), \quad (5.15)$$
$$K_2(E \pm i0; d) = K_2(0; d) \pm \frac{i\sqrt{E}}{4\pi}V_2^{1/2}(\cdot - d)(|V_2|^{1/2}(\cdot - d), \cdot) + O(E), \quad (5.16)$$

where the error terms $O(E)$ are understood in the sense of Hilbert-Schmidt norm. The equation (5.14) follows immediately from (5.15) and (5.16). ■

Remark: The low energy expansion for the operator $K_{12}(z; d)$

$$K_{12}(z; d) = K_{12}(0; d) + \frac{i\sqrt{z}}{4\pi}V_1^{1/2}(|V_2|^{1/2}(\cdot - d), \cdot) + zN_{12}(d) + o(z)$$

is not uniform in d. Here $N_{12}(d)$ is the Hilbert-Schmidt operator with integral kernel

$$N_{12}(x, y; d) = -\frac{1}{8\pi}V_1^{1/2}(x)|x - y||V_2|^{1/2}(y - d).$$

It is easy to show that $\|N_{12}\|_{\mathcal{J}_2}$ increases linearly with d. For this reason the limit $E \to +0$ of $\xi_{12}(E; d)$ is not uniform in d.

Now by Lemma 5.1 the r.h.s. of (5.13) equals $C(d) + O(\sqrt{E})$ (independent of the sign in (5.13)), where $C(d)$ is a d-dependent constant. Therefore

$$\lim_{E \to +0} \frac{4\pi}{i\sqrt{E}}\Big\{\mathrm{tr}L^{(0)}(E + i0; d) - \mathrm{tr}L^{(0)}(E - i0; d)$$
$$+t^{(0)}(E + i0; d) - t^{(0)}(E - i0; d)\Big\} = 0$$

for all fixed $d \in \mathbb{R}^3$.

Since the operator $I + L^{(1)}(0; d)$ is invertible for all sufficiently large $|d|$, due to (4.6) we have

$$\xi_{12}(+0; d) = \frac{1}{2\pi i}\lim_{E \to +0} \log\Big\{\Big[1 + \frac{4\pi}{i\sqrt{E}}\Big(\tilde{\varphi}_2(\cdot - d),$$
$$(I + K_1(0))^{-1}[K_{12}(0; d) + K_{21}(0; d) + \widetilde{K}(0; d) - K_1(0)K_2(0; d)]$$

$$
\begin{aligned}
&\cdot (I + L^{(1)}(0;d))^{-1}\varphi_2(\cdot - d)\Big) |(|V_2|^{1/2}, \varphi_2)|^{-2}\Big] \\
&\cdot \Big[1 - \frac{4\pi}{i\sqrt{E}}\Big(\tilde\varphi_2(\cdot - d), (I + K_1(0))^{-1} \\
&\quad \cdot [K_{12}(0;d) + K_{21}(0;d) + \widetilde{K}(0;d) - K_1(0)K_2(0;d)] \\
&\cdot (I + L^{(1)}(0;d))^{-1}\varphi_2(\cdot - d)\Big) |(|V_2|^{1/2}, \varphi_2)|^{-2}\Big]\Big\}. \qquad (5.17)
\end{aligned}
$$

To proceed further with our calculations of $\xi_{12}(+0;d)$ we prove the following

Lemma 5.2 *For all sufficiently large $|d|$*

$$
\begin{aligned}
&\Big(\tilde\varphi_2(\cdot - d), (I + K_1(0))^{-1}[K_{12}(0;d) + K_{21}(0;d) \\
&\qquad + \widetilde{K}(0;d) - K_1(0)K_2(0;d)](I + L^{(1)}(0;d))^{-1}\varphi_2(\cdot - d)\Big) \\
&= \frac{|(|V_2|^{1/2}, \varphi_2)|^2}{4\pi|d|^2} c_0(V_1) + o(|d|^{-2}),
\end{aligned}
$$

where $c_0(V_1)$ is the scattering length (5.2).

Proof. First we show that

$$\|(I + K_1(0)^*)^{-1}\tilde\varphi_2(\cdot - d) - \tilde\varphi_2(\cdot - d)\|_{L^2} = o(|d|^{-2})$$

for sufficiently large $|d|$. To this end we use

$$K_2(0;d)^*\tilde\varphi_2(\cdot - d) = -\tilde\varphi_2(\cdot - d)$$

and write

$$
\begin{aligned}
&(I + K_1(0)^*)^{-1}\tilde\varphi_2(\cdot - d) - \tilde\varphi_2(\cdot - d) \\
&= -(I + K_1(0)^*)^{-1}K_1(0)^*\tilde\varphi_2(\cdot - d) \\
&= (I + K_1(0)^*)^{-1}K_1(0)^*K_2(0;d)^*\tilde\varphi_2(\cdot - d).
\end{aligned}
$$

Its L^2-norm can be bounded by

$$C\|K_1(0)K_2(0;d)\|_{\mathcal{J}_1}\|\varphi_2\|_{L^2},$$

which is $o(|d|^{-2})$ by Lemma 4.2.

Therefore it suffices to consider

$$
\begin{aligned}
&\Big(\tilde\varphi_2(\cdot - d), \Big[K_{12}(0;d) + K_{21}(0;d) + \widetilde{K}(0;d) \\
&\quad - K_1(0)K_2(0;d)\Big](I + L^{(1)}(0;d))^{-1}\varphi_2(\cdot - d)\Big) \\
&= \Big(\tilde\varphi_2(\cdot - d), K_{12}(0;d)(I + L^{(1)}(0;d))^{-1}\varphi_2(\cdot - d)\Big) \\
&\quad + \Big(\tilde\varphi_2(\cdot - d), \widetilde{K}(0;d)(I + L^{(1)}(0;d))^{-1}\varphi_2(\cdot - d)\Big) \\
&\quad - \Big(\tilde\varphi_2(\cdot - d), K_1(0)K_2(0;d)(I + L^{(1)}(0;d))^{-1}\varphi_2(\cdot - d)\Big) \\
&\quad + \Big(\tilde\varphi_2(\cdot - d), K_{21}(0;d)(I + L^{(1)}(0;d))^{-1}\varphi_2(\cdot - d)\Big). \qquad (5.18)
\end{aligned}
$$

Here the first summand is $o(|d|^{-2})$ since

$$\begin{aligned} & K_{12}(0;d)^*\tilde{\varphi}_2(\cdot-d) = -K_{12}(0;d)^*K_2(0;d)^*\tilde{\varphi}_2(\cdot-d) \\ = \; & -|V_2|^{1/2}(\cdot-d)R_0(0)V_1^{1/4}V_2^{1/4}(\cdot-d) \\ & \cdot|V_1|^{1/4}|V_2|^{1/4}(\cdot-d)R_0(0)|V_2|^{1/2}(\cdot-d)\tilde{\varphi}_2(\cdot-d). \end{aligned}$$

Its L^2-norm can be bounded by

$$\begin{aligned} & \frac{1}{(4\pi)^2}\left(\int\frac{|V_2(x-d)||V_1(y)|^{1/2}|V_2(y-d)|^{1/2}}{|x-y|^2}dxdy\right)^{1/2} \\ \cdot & \left(\int\frac{|V_1(x)|^{1/2}|V_2(x-d)|^{1/2}|V_2(y-d)|}{|x-y|^2}dxdy\right)^{1/2}\|\varphi_2\|_{L^2}, \end{aligned}$$

which indeed is $o(|d|^{-2})$ by Lemma 2.4.

Consider the second term on the r.h.s. of (5.18). Since $L^{(1)}(0;d)\to 0$ in Hilbert-Schmidt norm as $|d|\to\infty$, it suffices to consider $(\tilde{\varphi}_2(\cdot-d),\widetilde{K}(0;d)\varphi_2(\cdot-d))$ only. Mimicking the idea of the proof of Lemma 4.3 we can write

$$\begin{aligned} & \Big(\tilde{\varphi}_2(\cdot-d),\widetilde{K}(0;d)\varphi_2(\cdot-d)\Big) \\ = \; & -\Big(\tilde{\varphi}_2(\cdot-d),K_2(0;d)(V_d^{1/2}-V_1^{1/2}-V_2^{1/2}(\cdot-d))R_0(0)|V_d|^{1/2}\varphi_2(\cdot-d)\Big) \\ & +\Big(\tilde{\varphi}_2(\cdot-d),K_2(0;d)V_1^{1/2}R_0(0)(|V_d|^{1/2}-|V_1|^{1/2}-|V_2|^{1/2}(\cdot-d)) \\ & \qquad \cdot K_2(0;d)\varphi_2(\cdot-d)\Big) \\ & -\Big(\tilde{\varphi}_2(\cdot-d),V_2^{1/2}(\cdot-d)R_0(0)(|V_d|^{1/2}-|V_1|^{1/2}-|V_2|^{1/2}(\cdot-d)) \\ & \qquad \cdot K_2(0;d)\varphi_2(\cdot-d)\Big). \end{aligned} \tag{5.19}$$

With the help of inequality (4.8) the first term on the r.h.s. of (5.19) can be bounded by

$$\begin{aligned} & \frac{2}{4\pi^2}\left(\int\frac{|V_2(x-d)||V_2(y-d)|^{3/4}|V_1(y)|^{1/4}}{|x-y|^2}dxdy\right)^{1/2} \\ \cdot & \left(\int\frac{|V_1(x)|^{1/4}|V_2(x-d)|^{3/4}|V_d(y)|}{|x-y|^2}dxdy\right)^{1/2}\|\varphi_2\|_{L^2}^2. \end{aligned}$$

By Lemma 2.4 this is $o(|d|^{-2})$. This completes the proof of that the first term on the r.h.s. of (5.19) is $o(|d|^{-2})$. The two other terms can be treated analogously. Hence the second term on the r.h.s. of (5.18) is $o(|d|^{-2})$.

The third term on the r.h.s. of (5.18) is $o(|d|^{-2})$ due to Lemma 4.2.

Finally consider the fourth term on the r.h.s. of (5.18). We replace $(I+L^{(1)}(0;d))^{-1}$ by the first two terms of its Neumann series expansion. It follows from (5.11) and Lemma 2.2 that

$$(I+L^{(1)}(0;d))^{-1} = I-L^{(1)}(0;d)+O(|d|^{-2}).$$

Hence

$$\Big(\tilde\varphi_2(\cdot - d), K_{21}(0;d)(I+L^{(1)}(0;d))^{-1}\varphi_2(\cdot - d)\Big)$$
$$= \Big(\tilde\varphi_2(\cdot - d), K_{21}(0;d)\varphi_2(\cdot - d)\Big)$$
$$-\Big(\tilde\varphi_2(\cdot - d), K_{21}(0;d)L^{(1)}(0;d)\varphi_2(\cdot - d)\Big) + O(|d|^{-3}).$$

Obviously,

$$\Big|\Big(\tilde\varphi_2(\cdot - d), K_{21}(0;d)\varphi_2(\cdot - d)\Big)\Big|$$
$$= \Big|\Big(\tilde\varphi_2(\cdot - d), K_{21}(0;d)K_2(0;d)\varphi_2(\cdot - d)\Big)\Big|$$
$$\leq \frac{1}{(4\pi)^2}\left(\frac{|V_2(x-d)||V_2(y-d)|^{1/2}|V_1(y)|^{1/2}}{|x-y|^2}dxdy\right)^{1/2}$$
$$\cdot\left(\frac{|V_1(x)|^{1/2}|V_2(x-d)|^{1/2}|V_2(y-d)|}{|x-y|^2}dxdy\right)^{1/2}\quad \|\varphi\|^2_{L^2},$$

which is $o(|d|^{-2})$ due to Lemma 2.4. Now applying Lemma 2.2 we get

$$-\Big(\tilde\varphi_2(\cdot - d), K_{21}(0;d)L^{(1)}(0;d)\varphi_2(\cdot - d)\Big)$$
$$= -\frac{1}{4\pi|d|}(\tilde\varphi_2, V_2^{1/2})\Big(|V_1|^{1/2}, L^{(1)}(0;d)\varphi_2(\cdot - d)\Big) + o(|d|^{-2}). \tag{5.20}$$

Using the representation (5.11) we find that $(|V_1|^{1/2}, (I+K_1(0))^{-1}K_{12}(0;d)\varphi_2(\cdot - d))$ describes the leading term of the asymptotics of $(|V_1|^{1/2}, L^{(1)}(0;d)\varphi_2(\cdot - d))$ as $|d|\to\infty$ (here we omit the corresponding calculations). Applying Lemma 2.2 once more we obtain that (5.20) is given by

$$-\frac{1}{(4\pi)^2|d|^2}(\tilde\varphi_2, V_2^{1/2})(|V_1|^{1/2}, (I+K_1(0))^{-1}V_1^{1/2})(|V_2|^{1/2}, \varphi_2) + o(|d|^{-2})$$
$$= \frac{|(|V_2|^{1/2}, \varphi_2)|^2}{4\pi|d|^2}c_0(V_1) + o(|d|^{-2}),$$

which proves the lemma. ■

Now we apply Lemma 5.2 to (5.17). Then for sufficiently large $|d|$ we have

$$\xi_{12}(+0;d) = \frac{1}{2\pi i}\lim_{E\to+0}\log\left\{\left[1+\frac{1}{i\sqrt{E}}\left(\frac{c_0(V_1)}{|d|^2} + o(|d|^{-2})\right)\right]\right.$$
$$\left.\left[1-\frac{1}{i\sqrt{E}}\left(\frac{c_0(V_1)}{|d|^2} + o(|d|^{-2})\right)\right]^{-1}\right\}.$$

Noting that for $E>0$ and $c_0(V_1)\neq 0$

$$\arg\left(1+\frac{c_0(V_1)}{i\sqrt{E}}\right)\ \operatorname{sign} c_0(V_1) < 0,$$

we calculate the limit $E \to +0$ thus obtaining that for all sufficiently large $|d|$

$$\xi_{12}(+0; d) = \frac{1}{2\pi i} \log \exp\{-i\pi \mathrm{sign}\ c_0(V_1)\} = -\frac{1}{2} \mathrm{sign} c_0(V_1).$$

Thus the claim (ii) of Theorem 3 is proved.

Next let us consider the case when $E = 0$ is an exceptional point of the first kind for both H_1 and H_2. In this case we decompose the operator $L(z; d)$ into its singular and regular parts,

$$\begin{aligned} L(z; d) &= L^{(0)}(z; d) + L^{(1)}(z; d), \\ L^{(0)}(z; d) &= -\frac{(4\pi)^2}{z} \mathcal{Q}_2(d) \mathcal{Q}_1 \\ &\quad \cdot \Big[K_{12}(z; d) + K_{21}(z; d) + \widetilde{K}(z; d) - K_1(z) K_2(z; d)\Big] \\ &\quad + \frac{4\pi}{i\sqrt{z}} \mathcal{Q}_2(d) M_1(z) \\ &\quad \cdot \Big[K_{12}(z; d) + K_{21}(z; d) + \widetilde{K}(z; d) - K_1(z) K_2(z; d)\Big] \\ &\quad + \frac{4\pi}{i\sqrt{z}} M_2(z; d) \mathcal{Q}_1 \\ &\quad \cdot \Big[K_{12}(z; d) + K_{21}(z; d) + \widetilde{K}(z; d) - K_1(z) K_2(z; d)\Big], \qquad (5.21) \\ L^{(1)}(z; d) &= M_2(z; d) M_1(z) \\ &\quad \cdot \Big[K_{12}(z; d) + K_{21}(z; d) + \widetilde{K}(z; d) - K_1(z) K_2(z; d)\Big]. \end{aligned}$$

We note that the operator $L^{(0)}(z; d)$ has rank 2. Due to Theorem T for all sufficiently large $|d|$ and all small z one has $\|L^{(1)}(z; d)\| < 1$. Therefore the operator $I + L^{(1)}(z; d)$ is invertible. Hence,

$$\begin{aligned} \det{}_2(I + L(z; d)) &= \det{}_2(I + L^{(1)}(z; d)) \\ &\quad \cdot \det{}_2\Big[I + L^{(0)}(z; d)(I + L^{(1)}(z; d))^{-1}\Big] \\ &\quad \cdot \exp\Big\{ - \mathrm{tr} L^{(0)}(z; d)(I + L^{(1)}(z; d))^{-1} L^{(1)}(z; d)\Big\} \\ &= \det{}_2(I + L^{(1)}(z; d)) \\ &\quad \cdot \det\Big[I + L^{(0)}(z; d)(I + L^{(1)}(z; d))^{-1}\Big] \\ &\quad \cdot \exp\Big\{ - \mathrm{tr} L^{(0)}(z; d)\Big\}. \qquad (5.22) \end{aligned}$$

It is easy to show that

$$\|M_2(z; d)\varphi_1 - \varphi_1\|_{L^2} \to 0,$$
$$(\varphi_2(\cdot - d), \varphi_1) \to 0$$

as $|d| \to \infty$. We note that $M_2(z;d)\varphi_1$ and $\varphi_2(\cdot - d)$ for sufficiently large $|d|$ depend on d continuously (in L^2-norm). Therefore for sufficiently large $|d|$ the set $\{M_2(z;d)\varphi_1, \varphi_2(\cdot - d)\}$ forms a basis (in general non-orthogonal) in $\mathrm{Ran}(L^{(0)}(z;d))$. Let P be the projector onto $\mathrm{Ran}(L^{(0)}(z;d))$, such that

$$P = M_2(z;d)\varphi_1(\psi_1(z;d),\cdot) + \varphi_2(\cdot - d)(\psi_2(z;d),\cdot),$$

where

$$\begin{aligned}
\psi_1(z;d) &= c_{11}(d)M_2(z;d)\varphi_1 + c_{12}(z;d)\varphi_2(\cdot - d),\\
\psi_2(z;d) &= c_{21}(z;d)M_2(z;d)\varphi_1 + c_{22}(z;d)\varphi_2(\cdot - d)
\end{aligned}$$

is the basis in $\mathrm{Ran}(L^{(0)}(z;d))$ dual with respect to $\{M_2(z;d)\varphi_1, \varphi_2(\cdot - d)\}$. The coefficients c_{ij} are given by

$$\begin{aligned}
c_{11}(d) &= (\varphi_2,\varphi_2)D^{-1},\\
c_{12}(z;d) &= -(\varphi_2(\cdot - d), M_2(z;d)\varphi_1)D^{-1},\\
c_{21}(z;d) &= -(M_2(z;d)\varphi_1, \varphi_2(\cdot - d))D^{-1},\\
c_{22}(z;d) &= (M_2(z;d)\varphi_1, M_2(z;d)\varphi_1)D^{-1},\\
D &= (\varphi_2,\varphi_2)(M_2(z;d)\varphi_1, M_2(z;d)\varphi_1)\\
&\quad -(\varphi_2(\cdot - d), M_2(z;d)\varphi_1)(M_2(z;d)\varphi_1, \varphi_2(\cdot - d)).
\end{aligned}$$

We note that D is nothing but Gram's determinant of the vectors $\varphi_2(\cdot - d)$ and $M_2(z;d)\varphi_1$. Since for sufficiently large $|d|$ these vectors are linear independent, we have that $D \neq 0$.

Due to the identity $\det(I + AB) = \det(I + BA)$ one has

$$\begin{aligned}
&\det\left[I + L^{(0)}(z;d)(I + L^{(1)}(z;d))^{-1}\right]\\
=\ &\det\left[I + PL^{(0)}(z;d)(I + L^{(1)}(z;d))^{-1}\right]\\
=\ &\det\left[I + PL^{(0)}(z;d)(I + L^{(1)}(z;d))^{-1}P\right]\\
=\ &\det\begin{pmatrix} 1 + a_{11} & a_{12}\\ a_{21} & 1 + a_{22}\end{pmatrix},
\end{aligned}$$

where

$$\begin{aligned}
a_{i1} &= \Big(\psi_i(z;d), L^{(0)}(z;d)(I + L^{(1)}(z;d))^{-1}M_2(z;d)\varphi_1\Big),\\
a_{i2} &= \Big(\psi_i(z;d), L^{(0)}(z;d)(I + L^{(1)}(z;d))^{-1}\varphi_2(\cdot - d)\Big)
\end{aligned}$$

for $i = 1, 2$.

Now elementary calculations give

$$\det\left[I + L^{(0)}(z;d)(I + L^{(1)}(z;d))^{-1}\right] = \frac{\alpha_1(z;d)}{z} + \frac{\alpha_{1/2}(z;d)}{iz^{1/2}} + 1, \tag{5.23}$$

where the functions $\alpha_i(z;d)$ $(i = 1/2, 1)$ are regular at $z = 0$ and are given by

$$\begin{aligned}
\alpha_1(z;d) = &\ \frac{(4\pi)^2}{|(|V_1|^{1/2},\varphi_1)|^2|(|V_2|^{1/2},\varphi_2)|^2} \\
&\cdot\Big\{\Big(\tilde{\varphi}_1, [K_{12}+K_{21}+\widetilde{K}-K_1K_2](I+L^{(1)})^{-1}\varphi_2(\cdot-d)\Big) \\
&\Big(\tilde{\varphi}_2(\cdot-d), M_1(z)[K_{12}+K_{21}+\widetilde{K}-K_1K_2](I+L^{(1)})^{-1}M_2(z;d)\varphi_1\Big) \\
- &\ \Big(\tilde{\varphi}_1, [K_{12}+K_{21}+\widetilde{K}-K_1K_2](I+L^{(1)})^{-1}M_2(z;d)\varphi_1\Big) \\
&\cdot\Big(\tilde{\varphi}_2(\cdot-d), M_1(z)[K_{12}+K_{21}+\widetilde{K}-K_1K_2](I+L^{(1)})^{-1}\varphi_2(\cdot-d)\Big) \\
- &\ (\tilde{\varphi}_2(\cdot-d),\varphi_1)\Big(\tilde{\varphi}_1, [K_{12}+K_{21}+\widetilde{K}-K_1K_2](I+L^{(1)})^{-1}\varphi_2(\cdot-d)\Big)\Big\}
\end{aligned}$$

and

$$\begin{aligned}
&\alpha_{1/2}(z;d) \\
&= \frac{4\pi}{|(|V_1|^{1/2},\varphi_1)|^2}\Big(\tilde{\varphi}_1, [K_{12}+K_{21}+\widetilde{K}-K_1K_2](I+L^{(1)})^{-1}M_2(z;d)\varphi_1\Big) \\
&+ \frac{4\pi}{|(|V_2|^{1/2},\varphi_2)|^2}\Big(\tilde{\varphi}_2(\cdot-d), [K_{12}+K_{21}+\widetilde{K}-K_1K_2](I+L^{(1)})^{-1}\varphi_2(\cdot-d)\Big).
\end{aligned}$$

Here we have used the abbreviations K_{12}, K_{21}, $\widetilde{K}$, K_1, K_2 and $L^{(1)}$ for $K_{12}(z;d)$, $K_{21}(z;d)$, $\widetilde{K}(z;d)$, $K_1(z)$, $K_2(z;d)$ and $L^{(1)}(z;d)$ respectively.

Lemma 5.3 *For sufficiently large* $|d|$

$$\alpha_1(0;d) = |d|^{-2} + o(|d|^{-2}).$$

Proof. The calculations of the expression

$$\Big(\tilde{\varphi}_1, [K_{12}+K_{21}+\widetilde{K}-K_1K_2](I+L^{(1)})^{-1}\varphi_2(\cdot-d)\Big) \tag{5.24}$$

are very similar to those used in the course of the proof of Lemma 5.2. First one can easily show that the dominant contribution to (5.24) is given by $(\tilde{\varphi}_1, K_{12}(0;d)\varphi_2(\cdot - d))$. Its asymptotic can be calculated by means of Lemma 2.2 finally giving

$$\frac{1}{4\pi|d|}(\varphi_1, |V_1|^{1/2})(|V_2|^{1/2},\varphi_2) + o(|d|^{-1}).$$

In almost the same way we can calculate the expression

$$\Big(\tilde{\varphi}_2(\cdot-d), M_1(z)[K_{12}+K_{21}+\widetilde{K}-K_1K_2](I+L^{(1)})^{-1}M_2(z;d)\varphi_1\Big),$$

thus obtaining

$$\frac{1}{4\pi|d|}(\varphi_2, |V_2|^{1/2})(|V_1|^{1/2},\varphi_1) + o(|d|^{-1}).$$

Also the expressions

$$\Big(\tilde{\varphi}_1, [K_{12} + K_{21} + \widetilde{K} - K_1K_2](I + L^{(1)})^{-1}M_2(z; d)\varphi_1\Big)$$

and

$$\Big(\tilde{\varphi}_2(\cdot - d), M_1(z)[K_{12} + K_{21} + \widetilde{K} - K_1K_2](I + L^{(1)})^{-1}\varphi_2(\cdot - d)\Big)$$

are $O(|d|^{-2})$. Finally we estimate $(\tilde{\varphi}_2(\cdot - d), \varphi_1)$. To this end we write

$$\begin{aligned}(\tilde{\varphi}_2(\cdot - d), \varphi_1) &= (K_2(0; d)^*\tilde{\varphi}_2(\cdot - d), K_1(0)\varphi_1) \\ &= (\tilde{\varphi}_2(\cdot - d), K_2(0; d)K_1(0)\varphi_1),\end{aligned}$$

which is $o(|d|^{-2})$ by Lemma 4.2. ∎

Hence we have

$$\lim_{E\to+0} \log \frac{\det[I + L^{(0)}(E + i0; d)(I + L^{(1)}(E + i0; d))^{-1}]}{\det[I + L^{(0)}(E - i0; d)(I + L^{(1)}(E - i0; d))^{-1}]} = 0$$

for all sufficiently large $|d|$.

Now we turn to the estimate of the function $\xi_{12}(E; d)$ as $E \to +0$. By (4.6) and (5.22) we have

$$\begin{aligned}&\xi_{12}(E; d) \\ = \;& \frac{1}{2\pi i}\log\Big[\frac{\det_2(I + L^{(1)}(E + i0; d))}{\det_2(I + L^{(1)}(E + i0; d))}\Big] \\ & \frac{1}{2\pi i}\log\Big[\frac{\det(I + L^{(0)}(E + i0; d)(I + L^{(1)}(E + i0; d))^{-1})}{\det(I + L^{(0)}(E - i0; d)(I + L^{(1)}(E - i0; d))^{-1})}\Big] \\ & \frac{1}{2\pi i}\Big[- t(E + i0; d) + t(E + i0; d) \\ & \quad -\mathrm{tr}L^{(0)}(E + i0; d) + \mathrm{tr}L^{(0)}(E - i0; d)\Big].\end{aligned} \tag{5.25}$$

Since the operator $I + L^{(1)}(0; d)$ is invertible for all large $|d|$, the first term on the r.h.s. of (5.25) equals zero in the limit $E \to +0$ for all sufficiently large $|d|$. By our previous discussion the second term is also zero for $E \to +0$ and for all sufficiently large $|d|$. Consider the third term of (5.25). Obviously

$$\begin{aligned}& t(E \pm i0; d) + \mathrm{tr}L^{(0)}(E \pm i0; d) \\ = \;& \mathrm{tr}K_1(E \pm i0)K_2(E \pm i0; d) \\ & +\mathrm{tr}K_1(E \pm i0)L^{(1)}(E \pm i0; d) \\ & +\mathrm{tr}K_2(E \pm i0; d)L^{(1)}(E \pm i0; d) \\ & +\mathrm{tr}K_1(E \pm i0)K_2(E \pm i0; d)L^{(1)}(E \pm i0; d) \\ & +\mathrm{tr}\Big[K_1(E \pm i0) + K_2(E \pm i0; d) + \\ & K_1(E \pm i0)K_2(E \pm i0; d) + I\Big]L^{(0)}(E \pm i0; d).\end{aligned} \tag{5.26}$$

Here the first four terms are regular at $E = 0$. Consider the fifth term.

Lemma 5.4 *For every fixed* $d \in \mathbb{R}^3$

$$\lim_{E \to +0} \mathrm{tr}\Big[K_1(E \pm i0) + K_2(E \pm i0; d) + K_1(E \pm i0)K_2(E \pm i0; d) + I\Big] L^{(0)}(E \pm i0; d) = C(d),$$

where $C(d)$ *is a* d*-dependent constant.*

Proof. By means of (5.21) we can write

$$\begin{aligned}
&\mathrm{tr}\Big\{[K_1(E \pm i0) + K_2(E \pm i0; d) \\
&\qquad + K_1(E \pm i0)K_2(E \pm i0; d) + I\Big] L^{(0)}(E \pm i0; d)\Big\} \\
= \; & -\frac{(4\pi)^2}{E}\mathrm{tr}\Big\{[K_1(E \pm i0) + K_2(E \pm i0; d) \\
&\qquad + K_1(E \pm i0)K_2(E \pm i0; d) + I\Big] \mathcal{Q}_2(d)\mathcal{Q}_1 \\
&\cdot [K_{12}(E \pm i0; d) + K_{21}(E \pm i0; d) \\
&\qquad + \widetilde{K}(E \pm i0; d) - K_1(E \pm i0)K_2(E \pm i0; d)]\Big\} \\
\pm \; & \frac{4\pi}{i\sqrt{E}}\mathrm{tr}\Big\{[K_1(E \pm i0) + K_2(E \pm i0; d) \\
&\qquad + K_1(E \pm i0)K_2(E \pm i0; d) + I\Big] \mathcal{Q}_2(d)M_1(E \pm i0) \\
&\cdot [K_{12}(E \pm i0; d) + K_{21}(E \pm i0; d) \\
&\qquad + \widetilde{K}(E \pm i0; d) - K_1(E \pm i0)K_2(E \pm i0; d)]\Big\} \\
\pm \; & \frac{4\pi}{i\sqrt{E}}\mathrm{tr}\Big\{[K_1(E \pm i0) + K_2(E \pm i0; d) \\
&\qquad + K_1(E \pm i0)K_2(E \pm i0; d) + I\Big] M_2(E \pm i0; d)\mathcal{Q}_1 \\
&\cdot [K_{12}(E \pm i0; d) + K_{21}(E \pm i0; d) \\
&\qquad + \widetilde{K}(E \pm i0; d) - K_1(E \pm i0)K_2(E \pm i0; d)]\Big\}. \qquad (5.27)
\end{aligned}$$

By Lemma 5.1 the second term on the r.h.s. of (5.27) is

$$\Big(\tilde{\varphi}_2(\cdot - d), M_1(0)[K_{12}(0; d) + K_{21}(0; d) + \widetilde{K}(0; d) - K_1(0)K_2(0; d)] (I + K_1(0))V_2^{1/2}(\cdot - d)\Big)(\varphi_2, |V_2|^{1/2})^{-1} + O(\sqrt{E}).$$

Consider the last term on the r.h.s. of (5.27). Using the identity (5.8) we can write

$$\Big[K_1(E \pm i0) + K_2(E \pm i0; d) + K_1(E \pm i0)K_2(E \pm i0; d) + I\Big]$$

$$
\begin{aligned}
&\quad \cdot M_2(E \pm i0; d)\mathcal{Q}_1 \\
&= (I + K_1(E \pm i0))(I + K_2(E \pm i0; d))M_2(E \pm i0; d)\mathcal{Q}_1 \\
&= (I + K_1(E \pm i0))\mathcal{Q}_1 \\
&\quad \mp \frac{4\pi}{i\sqrt{E}}(I + K_1(E \pm i0))(I + K_2(E \pm i0; d))\mathcal{Q}_2(d)\mathcal{Q}_1
\end{aligned}
$$

Now we use the identity

$$
(I + K_i(z))\mathcal{Q}_i = \frac{i\sqrt{z}}{4\pi} V_i^{1/2}(|V_i|^{1/2}, \cdot)\mathcal{Q}_i + zN_i\mathcal{Q}_i + o(z),
$$

which is a direct consequence of (5.4) and of $K_i(0)\mathcal{Q}_i = -\mathcal{Q}_i$. Thus we obtain

$$
\begin{aligned}
&\Big[K_1(E \pm i0) + K_2(E \pm i0; d) + K_1(E \pm i0)K_2(E \pm i0; d) + I\Big] \\
&\quad \cdot M_2(E \pm i0; d)\mathcal{Q}_1 \\
&= -(I + K_1(0))V_2^{1/2}(\cdot - d)(|V_2|^{1/2}(\cdot - d), \cdot)\mathcal{Q}_2(d)\mathcal{Q}_1 \\
&\quad \pm \frac{i\sqrt{E}}{4\pi} V_1^{1/2}(|V_1|^{1/2}, \cdot)\mathcal{Q}_1 \\
&\quad \mp \frac{i\sqrt{E}}{4\pi} V_1^{1/2}(|V_1|^{1/2}, V_2^{1/2}(\cdot - d))(|V_2|^{1/2}(\cdot - d), \cdot)\mathcal{Q}_2(d)\mathcal{Q}_1 \\
&\quad \pm 4\pi i\sqrt{E}(I + K_1(0))N_2(d)\mathcal{Q}_2(d)\mathcal{Q}_1 + o(\sqrt{E}),
\end{aligned}
$$

where $N_2(d) = U(d)N_2U(d)^{-1}$. Therefore the last term on the r.h.s. of (5.27) equals

$$
\begin{aligned}
&\mp \frac{4\pi}{i\sqrt{E}}\Big(\tilde{\varphi}_1, [K_{12}(0; d) + K_{21}(0; d) + \widetilde{K}(0; d) - K_1(0)K_2(0; d)] \\
&\cdot (I + K_1(0))V_2^{1/2}(\cdot - d)\Big)(\tilde{\varphi}_2(\cdot - d), \varphi_1)(\varphi_2, |V_2|^{1/2})^{-1}|(\varphi_1, |V_1|^{1/2})|^{-2} \\
&\qquad + C_3(d) + o(1),
\end{aligned}
$$

where $C_3(d)$ is a d-dependent constant.

Now we consider the first term on the r.h.s. of (5.27). By Lemma 5.1 this term equals

$$
\begin{aligned}
&\pm \frac{4\pi}{i\sqrt{E}}\Big(\tilde{\varphi}_1, [K_{12}(0; d) + K_{21}(0; d) + \widetilde{K}(0; d) + K_1(0)K_2(0; d)] \\
&\cdot (I + K_1(0))V_2^{1/2}(\cdot - d)\Big)(\tilde{\varphi}_2(\cdot - d), \varphi_1)(\varphi_2, |V_2|^{1/2})^{-1}|(\varphi_1, |V_1|^{1/2})|^{-2} \\
&\qquad + C_1(d) + o(1),
\end{aligned}
$$

Summing up all the contibutions completes the proof of the lemma. ∎

Thus from (5.26) and Lemma 5.4 it follows that

$$
\lim_{E \to +0} \Big[t(E + i0; d) + \mathrm{tr}L^{(0)}(E + i0; d) - t(E - i0; d) - \mathrm{tr}L^{(0)}(E - i0; d)\Big] = 0
$$

for all $d \in \mathbb{R}^3$. This completes the proof of Theorem 3.

6 Proof of Theorem 5

Before we present the rigorous proof of Theorem 5 we give a simple derivation based on the Born series for the scattering amplitudes (see e.g. [37]). Here we restrict ourselves to the leading terms in the Born series since the consideration of higher order terms is essentially the same.

The first contribution in the Born series of the scattering amplitude $A_d(\omega, \omega'; E)$ involving both potentials V_1 and $V_2(\cdot - d)$ is given by the Born approximation

$$
\begin{aligned}
&A_{12}^{Born}(\omega, \omega'; E; d) = \\
&\quad \frac{2\pi^2}{(2\pi)^6} e^{i\sqrt{E}\langle d, \omega'\rangle} \int \frac{\widehat{V}_1(\sqrt{E}\omega - q)\widehat{V}_2(q - \sqrt{E}\omega')}{q^2 - E - i0} e^{-i\langle d, q\rangle} dq \\
&\quad + \frac{2\pi^2}{(2\pi)^6} e^{-i\sqrt{E}\langle d, \omega\rangle} \int \frac{\widehat{V}_2(\sqrt{E}\omega - q)\widehat{V}_1(q - \sqrt{E}\omega')}{q^2 - E - i0} e^{i\langle d, q\rangle} dq.
\end{aligned}
$$

Applying now Lemma 2.3 we find that

$$
\begin{aligned}
&A_{12}^{Born}(\omega, \omega'; E; d) = \\
&\quad \frac{(2\pi^2)^2}{(2\pi)^6} |d|^{-1} \widehat{V}_1(\sqrt{E}(\omega + \hat{d}))\widehat{V}_2(\sqrt{E}(-\omega' - \hat{d})) e^{i\sqrt{E}\langle d, \hat{d} + \omega'\rangle} \\
&\quad + \frac{(2\pi^2)^2}{(2\pi)^6} |d|^{-1} \widehat{V}_2(\sqrt{E}(\omega - \hat{d}))\widehat{V}_1(\sqrt{E}(-\omega' + \hat{d})) e^{i\sqrt{E}\langle d, \hat{d} - \omega\rangle} + o(|d|^{-1}).
\end{aligned}
$$

Noting that the first term in the Born series for the amplitudes A_1 and A_2 is given by the Born approximation

$$
\begin{aligned}
A_1^{Born}(\omega, \omega'; E) &= -\frac{2\pi^2}{(2\pi)^3} \widehat{V}_1(\sqrt{E}(\omega - \omega')), \\
A_2^{Born}(\omega, \omega'; E; d) &= -\frac{2\pi^2}{(2\pi)^3} \widehat{V}_2(\sqrt{E}(\omega - \omega')) e^{-i\sqrt{E}\langle d, \omega - \omega'\rangle},
\end{aligned}
$$

we arrive at the claim of Theorem 5 in the Born approximation. Theorem 5 then "follows" by replacing A_i^{Born} by A_i.

We now turn to the rigorous proof and start with

Lemma 6.1 *Let the V_i's have the Property A. Then*

$$
\begin{aligned}
|d| \Big[&(I + K(z; d))^{-1} - (I + K_1(z))^{-1}(I + K_2(z; d))^{-1} \\
&+ (I + K_1(z))^{-1} K_{12}(z; d)(I + K_2(z; d))^{-1} \\
&+ (I + K_2(z; d))^{-1} K_{21}(z; d)(I + K_1(z))^{-1} \Big] \to 0
\end{aligned}
$$

as $|d| \to \infty$ in Hilbert-Schmidt norm uniformly in z on compact sets in $\overline{\Pi_0} \setminus \mathcal{E}$.

Proof. We start with a remark on the invertibility of the operator $I + K(z; d)$. Let us denote by $\mathcal{E}(d)$ the set of all $z \in \overline{\Pi_0}$ for which $I + K(z; d)$ has no bounded inverse. We proved in Lemma 2.7 in [27] that for an arbitrary compact set $\mathcal{I} \subset \overline{\Pi_0} \setminus \mathcal{E}$ there is $d(\mathcal{I}) > 0$ such that for all $|d| > d(\mathcal{I})$ the set $\mathcal{E}(d)$ has an empty intersection with $\mathcal{I}$, $\mathcal{E}(d) \cap \mathcal{I} = \emptyset$.

Therefore we fix a compact set $\mathcal{I} \subset \overline{\Pi_0} \setminus \mathcal{E}$ and suppose that $|d| > d(\mathcal{I})$.

First we show that

$$|d| \Big\| (I + K_1(z) + K_2(z;d))^{-1} - (I + K_1(z))^{-1}(I + K_2(z;d))^{-1} \Big\|_{\mathcal{J}_2} \to 0 \qquad (6.1)$$

as $|d| \to \infty$. This follows immediately from the equality

$$\begin{aligned} &(I + K_1(z) + K_2(z;d))^{-1} - (I + K_1(z))^{-1}(I + K_2(z;d))^{-1} \\ &= (I + K_1(z))^{-1}(I + K_2(z;d))^{-1} K_2(z;d) K_1(z)(I + K_1(z))^{-1} \\ &\cdot [I - (I + K_2(z;d))^{-1} K_2(z;d) K_1(z)(I + K_1(z))^{-1}]^{-1}(I + K_2(z;d))^{-1} \end{aligned}$$

and Lemma 4.2. Consider the difference

$$\begin{aligned} &(I + K(z;d))^{-1} - (I + K_1(z))^{-1}(I + K_2(z;d))^{-1} \\ &= (I + K(z;d))^{-1} - (I + K_1(z) + K_2(z;d))^{-1} \\ &\quad + (I + K_1(z) + K_2(z;d))^{-1} - (I + K_1(z))^{-1}(I + K_2(z;d))^{-1}. \end{aligned}$$

Due to (6.1) the second term is $o(|d|^{-1})$ in the sense of Hilbert-Schmidt norm. Due to the second resolvent identity the first term can be written as follows

$$\begin{aligned} &(I + K(z;d))^{-1} - (I + K_1(z) + K_2(z;d))^{-1} \\ &= -(I + K_1(z) + K_2(z;d))^{-1}[K(z;d) - K_1(z) - K_2(z;d)](I + K(z;d))^{-1} \\ &= -(I + K_1(z) + K_2(z;d))^{-1} \\ &\qquad \cdot [K_{12}(z;d) + K_{21}(z;d) + \widetilde{K}(z;d)](I + K(z;d))^{-1} \\ &= -(I + K_1(z) + K_2(z;d))^{-1}[K_{12}(z;d) + K_{21}(z;d)](I + K_1(z) + K_2(z;d))^{-1} \\ &\quad + (I + K_1(z) + K_2(z;d))^{-1}[K_{12}(z;d) + K_{21}(z;d)](I + K_1(z) + K_2(z;d))^{-1} \\ &\qquad \cdot [K_{12}(z;d) + K_{21}(z;d) + \widetilde{K}(z;d)](I + K(z;d))^{-1} \\ &\quad - (I + K_1(z) + K_2(z;d))^{-1}\widetilde{K}(z;d)(I + K(z;d))^{-1}. \end{aligned} \qquad (6.2)$$

Due to Lemmas 2.2 and 4.3 the second and third terms on the r.h.s. of (6.2) are $o(|d|^{-1})$ in the sense of Hilbert-Schmidt norm. The relation (6.1) shows that the first term on the r.h.s. of (6.2) up to corrections of order $o(|d|^{-1})$ is given by

$$\begin{aligned} -(I + K_1(z))^{-1}(I + K_2(z;d))^{-1}[K_{12}(z;d) + K_{21}(z;d)] \\ \cdot (I + K_1(z))^{-1}(I + K_2(z;d))^{-1}. \end{aligned}$$

Hence

$$\begin{aligned} |d| \Big\| (I + K(z;d))^{-1} - (I + K_1(z))^{-1}(I + K_2(z;d))^{-1} \\ + (I + K_1(z))^{-1}(I + K_2(z;d))^{-1}[K_{12}(z;d) + K_{21}(z;d)] \\ \cdot (I + K_1(z))^{-1}(I + K_2(z;d))^{-1} \Big\|_{\mathcal{J}_2} \to 0. \end{aligned}$$

To complete the proof of the lemma it remains to apply once more the arguments used in the proof of Lemma 4.4. ■

Due to Lemma 6.1 and the representation (4.17) for the scattering amplitude to prove Theorem 5 it suffices to show that

$$|d|\Big[\Big(e^{i\sqrt{E}\langle\omega,\cdot\rangle}|V_d|^{1/2},(I+K_1(E+i0))^{-1}(I+K_2(E+i0;d))^{-1}V_d^{1/2}e^{i\sqrt{E}\langle\omega',\cdot\rangle}\Big) \\ +4\pi A_1(\omega,\omega';E)+4\pi A_2(\omega,\omega';E;d)\Big]\to 0, \quad (6.3)$$

$$4\pi A_1(\omega,-\hat{d};E)A_2(-\hat{d},\omega';E;d) \\ -|d|\Big(e^{i\sqrt{E}\langle\omega,\cdot\rangle}|V_d|^{1/2},(I+K_1(E+i0))^{-1}K_{12}(E+i0;d) \\ \cdot(I+K_2(E+i0;d))^{-1}V_d^{1/2}e^{i\sqrt{E}\langle\omega',\cdot\rangle}\Big)\to 0, \quad (6.4)$$

$$4\pi A_2(\omega,\hat{d};E;d)A_1(\hat{d},\omega';E) \\ -|d|\Big(e^{i\sqrt{E}\langle\omega,\cdot\rangle}|V_d|^{1/2},(I+K_2(E+i0;d))^{-1}K_{21}(E+i0;d) \\ \cdot(I+K_1(E+i0))^{-1}V_d^{1/2}e^{i\sqrt{E}\langle\omega',\cdot\rangle}\Big)\to 0 \quad (6.5)$$

as $|d|\to\infty$ uniformly in $\omega,\omega'\in\mathbb{S}^2$ and in E on compact sets in $\mathbb{R}_+$.

First we prove (6.3). We represent the l.h.s. of (6.3)

$$|d|\Big[\Big(e^{i\sqrt{E}\langle\omega,\cdot\rangle}|V_d|^{1/2},(I+K_1(E+i0))^{-1}(I+K_2(E+i0;d))^{-1}V_d^{1/2}e^{i\sqrt{E}\langle\omega',\cdot\rangle}\Big) \\ +4\pi A_1(\omega,\omega';E)+4\pi A_2(\omega,\omega';E;d)\Big] \\ =|d|\Big[\Big(e^{i\sqrt{E}\langle\omega,\cdot\rangle}|V_d|^{1/2},(I+K_1(E+i0))^{-1}(I+K_2(E+i0;d))^{-1}V_d^{1/2}e^{i\sqrt{E}\langle\omega',\cdot\rangle}\Big) \\ -\Big(e^{i\sqrt{E}\langle\omega,\cdot\rangle}|V_1|^{1/2},(I+K_1(E+i0))^{-1}V_1^{1/2}e^{i\sqrt{E}\langle\omega',\cdot\rangle}\Big) \\ -\Big(e^{i\sqrt{E}\langle\omega,\cdot\rangle}|V_2|^{1/2}(\cdot-d),(I+K_2(E+i0;d))^{-1}V_2^{1/2}(\cdot-d)e^{i\sqrt{E}\langle\omega',\cdot\rangle}\Big)\Big] \quad (6.6)$$

as the sum of the following terms

$$|d|\Big(e^{i\sqrt{E}\langle\omega,\cdot\rangle}(|V_d|^{1/2}-|V_1|^{1/2}-|V_2|^{1/2}(\cdot-d)), \\ (I+K_1(E+i0))^{-1}(I+K_2(E+i0;d))^{-1}V_d^{1/2}e^{i\sqrt{E}\langle\omega',\cdot\rangle}\Big) \\ +\quad |d|\Big(e^{i\sqrt{E}\langle\omega,\cdot\rangle}|V_1|^{1/2},(I+K_1(E+i0))^{-1} \\ \cdot\Big[(I+K_2(E+i0;d))^{-1}V_d^{1/2}-V_1^{1/2}\Big]e^{i\sqrt{E}\langle\omega',\cdot\rangle}\Big) \\ +\quad |d|\Big(e^{i\sqrt{E}\langle\omega,\cdot\rangle}|V_2|^{1/2}(\cdot-d),\Big[(I+K_1(E+i0))^{-1}(I+K_2(E+i0;d))^{-1} \\ \cdot V_d^{1/2}-(I+K_2(E+i0;d))^{-1}V_2^{1/2}(\cdot-d)\Big]e^{i\sqrt{E}\langle\omega',\cdot\rangle}\Big). \quad (6.7)$$

Since $(I+K_1(E+i0))^{-1}$ and $(I+K_2(E+i0;d))^{-1}$ are uniformly norm bounded in E in any compact set in $\mathbb{R}_+$ (say, by a constant C) and due to the Schwarz inequality the first term of (6.7) can be bounded by

$$C^2 \left\| |V_d|^{1/2} - |V_1|^{1/2} - |V_2|^{1/2}(\cdot - d) \right\|_{L^2} \|V_d\|_{L^1}$$
$$\leq C^2 \left\| |V_d|^{1/2} - |V_1|^{1/2} - |V_2|^{1/2}(\cdot - d) \right\|_{L^2} (\|V_1\|_{L^1} + \|V_2\|_{L^1}).$$

Due to the inequality (4.8) we have

$$\left\| |V_d|^{1/2} - |V_1|^{1/2} - |V_2|^{1/2}(\cdot - d) \right\|_{L^2} \leq 2 \left\| |V_1|^{1/2} |V_2|^{1/2}(\cdot - d) \right\|_{L^1}^{1/2}$$
$$= 2 \left(\int \overline{\widehat{|V_1|^{1/2}}(p)} \widehat{|V_2|^{1/2}}(p) e^{-ipd} dp \right)^{1/2}. \qquad (6.8)$$

For the potentials V_i having the Property B one has that $\widehat{|V_i|^{1/2}} \in L^2(\mathbb{R}^3) \cap C^2(\mathbb{R}^3)$.

Now we use the following asymptotics

Lemma 6.2 *Let $f \in L^1(\mathbb{R}^3) \cap C^2(\mathbb{R}^3)$. Then*

$$\int f(p) e^{-ipd} dp = o(|d|^{-2})$$

as $|d| \to \infty$.

This asymptotics can be directly derived from the asymptotic formula (2.11).

Now due to Lemma 6.2 the first term of (6.7) is $o(1)$.

Consider the second term of (6.7). First we show that

$$|d| \Big\| (I + K_2(E + i0; d))^{-1} V_d^{1/2}$$
$$-(I + K_2(E + i0; d))^{-1} V_2^{1/2}(\cdot - d) - V_1^{1/2} \Big\|_{L^2(\mathbb{R}^3)} \to 0 \qquad (6.9)$$

as $|d| \to \infty$. For this aim we represent the L^2-function on the l.h.s. of (6.9) in the form

$$(I + K_2(E + i0; d))^{-1} V_d^{1/2} - (I + K_2(E + i0; d))^{-1} V_2^{1/2}(\cdot - d) - V_1^{1/2}$$
$$= (I + K_2(E + i0; d))^{-1} (V_d^{1/2} - V_1^{1/2} - V_2^{1/2}(\cdot - d))$$
$$-(I + K_2(E + i0; d))^{-1} K_2(E + i0; d) V_1^{1/2}. \qquad (6.10)$$

The first term on the r.h.s. of (6.10) is $o(|d|^{-1})$ in the sense of $L^2(\mathbb{R}^3)$-norm due to (6.8) and Lemma 6.2. The second term on the r.h.s. of (6.10) can be represented in the form

$$(I + K_2(E + i0; d))^{-1} \cdot V_2^{1/2}(\cdot - d)) R_0(E + i0) |V_1|^{1/4} |V_2|^{1/4}(\cdot - d))$$
$$\cdot V_1^{1/4} |V_2|^{1/4}(\cdot - d)).$$

Therefore its L^2-norm is bounded by

$$C \left\| V_2^{1/2}(\cdot - d)) R_0(E+i0) V_1^{1/4} |V_2|^{1/4}(\cdot - d)) \right\|_{\mathcal{J}_2} \left\| V_1^{1/2} |V_2|^{1/2}(\cdot - d)) \right\|_{L^1}^{1/2}$$
$$\leq \frac{C}{4\pi} \Big(\int \frac{|V_2(x-d)||V_1(y)|^{1/2}|V_2(y-d)|^{1/2}}{|x-y|^2} dx dy \Big)^{1/2}$$
$$\cdot \left\| V_1^{1/2} |V_2|^{1/2}(\cdot - d) \right\|_{L^1}^{1/2},$$

which is $o(|d|^{-2})$ due to Lemma 2.4, (6.8) and Lemma 6.2. Thus (6.9) is proven.

Now to prove that the second term of (6.7) is $o(1)$, it suffices to show that

$$|d| \Big(e^{i\sqrt{E}\langle\omega,\cdot\rangle} |V_1|^{1/2}, (I + K_1(E+i0))^{-1}$$
$$\cdot (I + K_2(E+i0;d))^{-1} V_2^{1/2}(\cdot - d) e^{i\sqrt{E}\langle\omega',\cdot\rangle} \Big) \to 0$$

as $|d| \to \infty$ uniformly in $\omega, \omega' \in \mathbb{S}^2$. The proof of this fact is similar to that of Lemma 2.9 in [27]. First we show that

$$|d| \Big[\Big(e^{i\sqrt{E}\langle\omega,\cdot\rangle} |V_1|^{1/2}, (I + K_1(E+i0))^{-1}$$
$$\cdot (I + K_2(E+i0;d))^{-1} V_2^{1/2}(\cdot - d) e^{i\sqrt{E}\langle\omega',\cdot\rangle} \Big)$$
$$- \Big(e^{i\sqrt{E}\langle\omega,\cdot\rangle} |V_1|^{1/2} (I + K_2(E+i0;d))^{-1}$$
$$\cdot (I + K_1(E+i0))^{-1} V_2^{1/2}(\cdot - d) e^{i\sqrt{E}\langle\omega',\cdot\rangle} \Big) \Big] \to 0$$

as $|d| \to \infty$. This follows immediately from the identity

$$\begin{aligned}
&(I + K_1(E+i0))^{-1}(I + K_2(E+i0;d))^{-1} \\
&\qquad -(I + K_2(E+i0;d))^{-1}(I + K_1(E+i0))^{-1} \\
&= (I + K_1(E+i0))^{-1}(I + K_2(E+i0;d))^{-1} \\
&\qquad \cdot [K_1(E+i0)K_2(E+i0;d) - K_2(E+i0;d)K_1(E+i0)] \\
&\qquad \cdot (I + K_2(E+i0;d))^{-1}(I + K_1(E+i0))^{-1}
\end{aligned}$$

and Lemma 4.2. Now we show that

$$|d| \Big[\Big(e^{i\sqrt{E}\langle\omega,\cdot\rangle} |V_1|^{1/2}, (I + K_2(E+i0;d))^{-1}(I + K_1(E+i0))^{-1}$$
$$\cdot V_2^{1/2}(\cdot - d) e^{i\sqrt{E}\langle\omega',\cdot\rangle} \Big) - \Big(e^{i\sqrt{E}\langle\omega,\cdot\rangle} |V_1|^{1/2}, V_2^{1/2}(\cdot - d) e^{i\sqrt{E}\langle\omega',\cdot\rangle} \Big) \Big] \to 0 \qquad (6.11)$$

as $|d| \to \infty$. To this end we can represent the l.h.s. of (6.11) in the form

$$-|d| \Big(e^{i\sqrt{E}\langle\omega,\cdot\rangle} |V_1|^{1/2}, K_2(E+i0;d)(I + K_2(E+i0;d))^{-1}$$
$$\cdot (I + K_1(E+i0))^{-1} V_2^{1/2}(\cdot - d) e^{i\sqrt{E}\langle\omega',\cdot\rangle} \Big)$$
$$-|d| \Big(e^{i\sqrt{E}\langle\omega,\cdot\rangle} |V_1|^{1/2}, (I + K_1(E+i0))^{-1} K_1(E+i0) V_2^{1/2}(\cdot - d) e^{i\sqrt{E}\langle\omega',\cdot\rangle} \Big).$$

Here the second term tends to zero since

$$|d|\|K_1(E+i0)V_2^{1/2}(\cdot-d)\|_{L^2} \le |d|\|V_1^{1/2}R_0(E+i0)|V_1|^{1/4}|V_2|^{1/4}(\cdot-d)\|_{\mathcal{J}_2} \cdot \||V_1|^{1/2}|V_2|^{1/2}(\cdot-d)\|_{L^1}^{1/2}.$$

The first term can be rewritten in the form

$$-|d|\Big(e^{i\sqrt{E}\langle\omega,\cdot\rangle}|V_1|^{1/4}V_2^{1/4}(\cdot-d), |V_1|^{1/4}|V_2|^{1/4}(\cdot-d)R_0(E+i0)|V_2|^{1/2}(\cdot-d) \cdot(I+K_2(E+i0;d))^{-1}(I+K_1(E+i0))^{-1}V_2^{1/2}(\cdot-d)e^{i\sqrt{E}\langle\omega',\cdot\rangle}\Big).$$

Therefore its absolute value can be bounded by

$$\begin{aligned}&|d|C^2\|V_1^{1/2}V_2^{1/2}(\cdot-d)\|_{L^1}^{1/2}\|V_2\|_{L^1}^{1/2} \cdot\Big\|\big||V_1|^{1/4}V_2^{1/4}(\cdot-d)R_0(E+i0)|V_2|^{1/2}(\cdot-d)\big|\Big\|_{\mathcal{J}_2}\\ &\le \frac{|d|C^2}{4\pi}\Big(\int\frac{|V_1(x)|^{1/2}|V_2(x-d)|^{1/2}|V_2(y-d)|}{|x-y|^2}dxdy\Big)^{1/2} \cdot\|V_1^{1/2}V_2^{1/2}(\cdot-d)\|_{L^1}^{1/2}\|V_2\|_{L^1}^{1/2},\end{aligned}$$

which is again $o(1)$ as $|d|\to\infty$.

Now we note that

$$|d|\Big(e^{i\sqrt{E}\langle\omega,\cdot\rangle}|V_1|^{1/2}, V_2^{1/2}(\cdot-d)e^{i\sqrt{E}\langle\omega',\cdot\rangle}\Big) \le |d|\left\|V_1^{1/2}V_2^{1/2}(\cdot-d)\right\|_{L^1},$$

which is again $o(1)$ due to Lemma 6.2. This estimate and (6.11) completes the proof of that the second term in (6.7) tends to zero as $|d|\to\infty$. The third term in (6.7) can be considered in exactly the same way. Thus (6.3) is proven.

Now we prove (6.4). (6.5) can be treated in exactly the same way. Due to Lemma 2.2

$$\begin{aligned}&|d|\Big(e^{i\sqrt{E}\langle\omega,\cdot\rangle}|V_d|^{1/2}, (I+K_1(E+i0))^{-1}K_{12}(E+i0;d) \cdot(I+K_2(E+i0;d))^{-1}V_d^{1/2}e^{i\sqrt{E}\langle\omega',\cdot\rangle}\Big)\\ &-\frac{1}{4\pi}\Big(e^{i\sqrt{E}\langle\omega,\cdot\rangle}|V_d|^{1/2}, (I+K_1(E+i0))^{-1}V_1^{1/2}e^{-i\sqrt{E}\langle\hat{d},\cdot\rangle}\Big)\\ &\Big(e^{-i\sqrt{E}\langle\hat{d},\cdot\rangle}|V_2|^{1/2}(\cdot-d), (I+K_2(E+i0;d))^{-1}V_d^{1/2}e^{i\sqrt{E}\langle\omega',\cdot\rangle}\Big)\end{aligned}$$

tends to zero when $|d|\to\infty$. Therefore to prove (6.4) it suffices to show that both

$$\begin{aligned}&\Big(e^{i\sqrt{E}\langle\omega,\cdot\rangle}|V_d|^{1/2}, (I+K_1(E+i0))^{-1}V_1^{1/2}e^{-i\sqrt{E}\langle\hat{d},\cdot\rangle}\Big) +4\pi A_1(\omega,-\hat{d};E)\\ &=\Big(e^{i\sqrt{E}\langle\omega,\cdot\rangle}|V_d|^{1/2}, (I+K_1(E+i0))^{-1}V_1^{1/2}e^{-i\sqrt{E}\langle\hat{d},\cdot\rangle}\Big)\\ &-\Big(e^{i\sqrt{E}\langle\omega,\cdot\rangle}|V_1|^{1/2}, (I+K_1(E+i0))^{-1}V_1^{1/2}e^{-i\sqrt{E}\langle\hat{d},\cdot\rangle}\Big) \qquad (6.12)\end{aligned}$$

and

$$\begin{aligned}
&\Big(e^{-i\sqrt{E}\langle\hat{d},\cdot\rangle}|V_2|^{1/2}(\cdot-d),(I+K_2(E+i0;d))^{-1}V_d^{1/2}e^{i\sqrt{E}\langle\omega',\cdot\rangle}\Big)\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad +4\pi A_2(\omega,\hat{d};E;d)\\
&=\Big(e^{-i\sqrt{E}\langle\hat{d},\cdot\rangle}|V_2|^{1/2}(\cdot-d),(I+K_2(E+i0;d))^{-1}V_d^{1/2}e^{i\sqrt{E}\langle\omega',\cdot\rangle}\Big)\\
&-\Big(e^{-i\sqrt{E}\langle\hat{d},\cdot\rangle}|V_2|^{1/2}(\cdot-d),(I+K_2(E+i0;d))^{-1}V_2^{1/2}(\cdot-d)e^{i\sqrt{E}\langle\omega',\cdot\rangle}\Big)
\end{aligned}$$

tend to zero. We give the proof of (6.12) only. First we represent (6.12) in the form

$$\begin{aligned}
&\Big(e^{i\sqrt{E}\langle\omega,\cdot\rangle}(|V_d|^{1/2}-|V_1|^{1/2}-|V_2|^{1/2}(\cdot-d)),(I+K_1(E+i0))^{-1}V_1^{1/2}e^{i\sqrt{E}\langle\hat{d},\cdot\rangle}\Big)\\
&\qquad\qquad +\Big(e^{i\sqrt{E}\langle\omega,\cdot\rangle}|V_2|^{1/2}(\cdot-d)),(I+K_1(E+i0))^{-1}V_1^{1/2}e^{i\sqrt{E}\langle\hat{d},\cdot\rangle}\Big).
\end{aligned}$$

The first term is $o(1)$ due to inequality (6.8) and Lemma 6.2. To consider the second term we represent it in the form

$$\begin{aligned}
&-\Big(e^{i\sqrt{E}\langle\omega,\cdot\rangle}|V_2|^{1/2}(\cdot-d)),K_1(E+i0)(I+K_1(E\pm i0))^{-1}V_1^{1/2}e^{i\sqrt{E}\langle\hat{d},\cdot\rangle}\Big)\\
&+\Big(e^{i\sqrt{E}\langle\omega,\cdot\rangle}|V_2|^{1/2}(\cdot-d)),V_1^{1/2}e^{i\sqrt{E}\langle\hat{d},\cdot\rangle}\Big).
\end{aligned}$$

We have already shown that the second term is $o(1)$. The first term can be written as

$$\begin{aligned}
&-\Big(e^{i\sqrt{E}\langle\omega,\cdot\rangle}V_1^{1/4}|V_2|^{1/4}(\cdot-d),|V_1|^{1/4}|V_2|^{1/4}(\cdot-d)R_0(E+i0)|V_1|^{1/2}\\
&\qquad\qquad\qquad\qquad\qquad \cdot(I+K_1(E+i0))^{-1}V_1^{1/2}e^{i\sqrt{E}\langle\hat{d},\cdot\rangle}\Big).
\end{aligned}$$

The absolute value of this expression can be bounded by

$$\begin{aligned}
&C\|V_1\|_{L^1}^{1/2}\Big\|V_1^{1/2}|V_2|^{1/2}(\cdot-d)\Big\|_{L^1}^{1/2}\\
&\cdot\Big\|V_1^{1/4}|V_2|^{1/4}(\cdot-d)R_0(E+i0)|V_1|^{1/2}\Big\|_{\mathcal{J}_2}\\
\le\;&\frac{C}{4\pi}\|V_1\|_{L^1}^{1/2}\Big\|V_1^{1/2}|V_2|^{1/2}(\cdot-d)\Big\|_{L^1}^{1/2}\\
&\cdot\Big(\int\frac{|V_1(x)|^{1/2}|V_2(x-d)|^{1/2}|V_1(y)|}{|x-y|^2}dxdy\Big)^{1/2},
\end{aligned}$$

which is $o(|d|^{-2})$ again due to Lemma 2.4, (6.8) and Lemma 6.2. This completes the proof of Theorem 5.

Appendix A

Here we prove the following statements:

Theorem A.1 *Consider the Hamiltonians $H_\pm = H_0 \pm V$ with $V \in \mathcal{R} \cap L^1(\mathbb{R}^3)$ and V is nonnegative. Moreover suppose that $E = 0$ is a regular point for H_-, which in addition has no negative bound states. Then the scattering lengths*

$$c_0(\pm V) = \mp \frac{1}{4\pi} \left(V^{1/2}, (I \pm V^{1/2} R_0(0) V^{1/2})^{-1} V^{1/2} \right)$$

satisfy $\mp c_0(\pm V) > 0$.

Proof. The operator $V^{1/2} R_0(0) V^{1/2}$ is self-adjoint compact and non-negative. Therefore $(I + V^{1/2} R_0(0) V^{1/2})^{-1} > 0$. By the Birman-Schwinger Principle the total multiplicity of the discrete spectrum of H_- equals the number of eigenvalues λ_k of $-V^{1/2} R_0(0) V^{1/2}$ such that $\lambda_k \leq -1$. Since by assumption the discrete spectrum is empty, we have that $\lambda_k > -1$ for all k, and therefore $I - V^{1/2} R_0(0) V^{1/2} > 0$. Hence $(I - V^{1/2} R_0(0) V^{1/2})^{-1}$ exists and is a positive operator. ■

Theorem A.2 *Let V satisfy the following conditions*
(i) $V \in \mathcal{R} \cap L^1$,
(ii) $\int V dx = 0$,
(iii) V is not identically zero,
(iv) $\|V\|_{\mathcal{R}} < 2\pi$.
Let $W \geq 0$ satisfy (i), (iii), and (iv). Then for any sufficiently small $\lambda > 0$ there is $a \in (0,1)$ such that $c_0(\lambda(V + aW)) = 0$.

Proof. Denote $U_a = V + aW$. The assumptions of the theorem guarantee that the Neumann expansion

$$(I + \lambda U_a^{1/2} R_0(0) |U_a|^{1/2})^{-1} = I - \lambda U_a^{1/2} R_0(0) |U_a|^{1/2} + O(\lambda^2)$$

converges for all $\lambda \in [0,1]$ and $a \in [0,1]$. Therefore

$$\begin{aligned} c_0(\lambda U_a) &= -\frac{\lambda}{4\pi} \int U_a dx + \frac{\lambda^2}{4\pi} \left(|U_a|^{1/2}, U_a^{1/2} R_0(0) |U_a|^{1/2} \cdot U_a^{1/2} \right) + O(\lambda^3) \\ &= -\frac{\lambda a}{4\pi} \int W dx + \frac{\lambda^2}{4\pi} \left(|U_a|^{1/2}, U_a^{1/2} R_0(0) |U_a|^{1/2} \cdot U_a^{1/2} \right) + O(\lambda^3). \quad \text{(A.1)} \end{aligned}$$

We note that

$$\left(|U_a|^{1/2}, U_a^{1/2} R_0(0) |U_a|^{1/2} \cdot U_a^{1/2} \right) > 0.$$

Indeed,

$$\left(|U_a|^{1/2}, U_a^{1/2} R_0(0) |U_a|^{1/2} \cdot U_a^{1/2} \right) = \int \frac{U_a(x) U_a(y)}{4\pi |x - y|} dx dy.$$

By (2.15) we have that

$$\int \frac{U_a(x) U_a(y)}{4\pi |x - y|} dx dy = \int \frac{|\widehat{U}_a(p)|^2}{p^2} dp > 0.$$

Now consider the equation $c_0(\lambda U_a) = 0$, which by (A.1) can be written in the form

$$f(a,\lambda) := -\frac{a}{4\pi}\int W dx + \frac{\lambda}{4\pi}\left(|U_a|^{1/2}, U_a^{1/2} R_0(0)|U_a|^{1/2}\cdot U_a^{1/2}\right) + O(\lambda^2) = 0.$$

Now fix $\lambda_0 > 0$ so small that

$$0 < \frac{\lambda}{4\pi}\left(|U_a|^{1/2}, U_a^{1/2} R_0(0)|U_a|^{1/2}\cdot U_a^{1/2}\right) + O(\lambda^2) < \frac{1}{4\pi}\int W dx$$

for all $a \in [0,1]$ and all $0 < \lambda < \lambda_0$. For every λ with $0 < \lambda < \lambda_0$ we have $f(0,\lambda) > 0$ and $f(1,\lambda) < 0$. Since $f(a,\lambda)$ is continuous in $a \in [0,1]$, there is $a = a(\lambda)$ such that $f(a,\lambda) = 0$. ■

Appendix B

This appendix is devoted to a proof of relation (2.12).

Let us denote

$$I(d) = \int_{\mathbb{R}^3} \frac{f(p)e^{-i\langle d,p\rangle}}{p^2 - E \mp i0} dp. \tag{B.1}$$

We define the function $\eta_\delta(q) \in C_0^\infty(\mathbb{R}_+)$ such that $\eta_\delta(q) \equiv 1$ for $|q - \sqrt{E}| < \delta$ and $\eta_\delta(q) \equiv 0$ for $|q - \sqrt{E}| > 2\delta$. We represent the integral (B.1) as follows

$$\begin{aligned} I(d) &= \int \frac{f(p)\eta_\delta(|p|)e^{-i\langle d,p\rangle}}{p^2 - E \mp i0} dp \\ &+ \int \frac{f(p)(1-\eta_\delta(|p|))e^{-i\langle d,p\rangle}}{p^2 - E \mp i0} dp. \end{aligned} \tag{B.2}$$

The integrand in the second term is in $C^2 \cap L^1$. Therefore due to (2.11) the second integral is $o(|d|^{-2})$.

Consider the first integral on the r.h.s. of (B.2) for which we preserve the notation $I(d)$. Also we write $f_\delta(p)$ for $f(p)\eta_\delta(|p|)$.

Integration by parts for an arbitrary differentiable function $g(\omega)$ on the unit sphere $\mathbb{S}^2$ gives

$$\begin{aligned} &\int_{\mathbb{S}^2} g(\omega)\exp\{-i|p|\langle d,\omega\rangle\} d\omega \\ &= -\frac{2\pi}{i|p||d|}\left\{ g(\hat{d})e^{-i|p||d|} - g(-\hat{d})e^{i|p||d|}\right\} \\ &\quad + \frac{1}{i|p||d|}\int_{\mathbb{S}^2} \exp\{-i|p|\langle d,\omega\rangle\}\langle \hat{d},\nabla\rangle g(\omega) d\omega. \end{aligned} \tag{B.3}$$

By $\langle \hat{d},\nabla\rangle$ we mean the differential operator $\partial/\partial\theta$, if (θ,ϕ) are the polar coordinates of $\omega \in \mathbb{S}^2$ with $\hat{d}$ as polar axis.

First we consider the case $E > 0$. Let us fix δ such that $E > 4\delta^2$. With the help of the identity (B.3) we represent $I(d)$ as the sum of two terms

$$\begin{aligned} I_1(d) &= -\frac{2\pi}{i|d|}\int_{\sqrt{E}-2\delta}^{\sqrt{E}+2\delta} \frac{f_\delta(q\hat{d})e^{-iq|d|}qdq}{q^2-E\mp i0} \\ &\quad +\frac{2\pi}{i|d|}\int_{\sqrt{E}-2\delta}^{\sqrt{E}+2\delta} \frac{f_\delta(-q\hat{d})e^{iq|d|}qdq}{q^2-E\mp i0}, \end{aligned}$$

and

$$I_2(d) = \frac{1}{i|d|}\int_{\sqrt{E}-2\delta}^{\sqrt{E}+2\delta} \frac{qdq}{q^2-E\mp i0}\int_{\mathbb{S}^2} d\omega e^{-iq\langle d,\omega\rangle} f_{1,\delta}(q\omega), \tag{B.4}$$

where $f_{1,\delta}(q\omega) = \langle \hat{d}, \nabla\rangle f_\delta(q\omega) \in C^1(\mathbb{R}^3)$ by the assumption. Since $(q^2 - E \mp i0)^{-1}$ is the distribution of order 1, the integral (B.4) is well defined.

Due to the distributional identities ($q \in \mathbb{R}$)

$$\frac{1}{q^2-E\mp i0} = \text{v.p.}\frac{1}{q^2-E} \pm i\pi\delta(q^2-E), \tag{B.5}$$
$$\delta(q^2-E) = \frac{1}{2|q|}\left(\delta(q-\sqrt{E})+\delta(q+\sqrt{E})\right),$$

one has

$$I_1(d) = I_1^{(1)}(d) + I_1^{(2)}(d),$$
$$\begin{aligned} I_1^{(1)}(d) &= -\frac{2\pi}{i|d|}\ \text{v.p.}\int_{\sqrt{E}-2\delta}^{\sqrt{E}+2\delta} \frac{f_\delta(q\hat{d})e^{-iq|d|}qdq}{q^2-E} \\ &\quad +\frac{2\pi}{i|d|}\ \text{v.p.}\int_{\sqrt{E}-2\delta}^{\sqrt{E}+2\delta} \frac{f_\delta(-q\hat{d})e^{iq|d|}qdq}{q^2-E}, \\ I_1^{(2)}(d) &= \mp\frac{\pi^2}{|d|}\left(f(\sqrt{E}\hat{d})e^{-i\sqrt{E}|d|} - f(-\sqrt{E}\hat{d})e^{i\sqrt{E}|d|}\right). \end{aligned}$$

Now we rewrite $I_1^{(1)}(d)$ in the following way

$$\begin{aligned} I_1^{(1)}(d) &= -\frac{2\pi}{i|d|}\int_{\sqrt{E}-2\delta}^{\sqrt{E}+2\delta} \frac{[f_\delta(q\hat{d})-\eta_\delta(q)f(\sqrt{E}\hat{d})]e^{-iq|d|}qdq}{q^2-E} \\ &\quad +\frac{2\pi}{i|d|}\int_{\sqrt{E}-2\delta}^{\sqrt{E}+2\delta} \frac{[f_\delta(-q\hat{d})-\eta_\delta(q)f(-\sqrt{E}\hat{d})]e^{iq|d|}qdq}{q^2-E} \\ &\quad -\frac{2\pi}{i|d|}f(\sqrt{E}\hat{d})\ \text{v.p.}\int_{\sqrt{E}-2\delta}^{\sqrt{E}+2\delta} \frac{\eta_\delta(q)e^{-iq|d|}qdq}{q^2-E} \\ &\quad +\frac{2\pi}{i|d|}f(-\sqrt{E}\hat{d})\ \text{v.p.}\int_{\sqrt{E}-2\delta}^{\sqrt{E}+2\delta} \frac{\eta_\delta(q)e^{iq|d|}qdq}{q^2-E}. \end{aligned} \tag{B.6}$$

Due to the fact that the functions

$$\frac{f_\delta(\pm q\hat{d})-\eta_\delta(q)f(\pm\sqrt{E}\hat{d})}{q^2-E}$$

have compact support and are bounded, by Riemann-Lebesgue lemma the first two terms on the r.h.s. of (B.6) are $o(|d|^{-1})$.

The integrands in the third and fourth terms on the r.h.s. of (B.6) are analytic for complex q in a small vicinity of $\sqrt{E}$. Therefore due to the well known property of integrals in the sense of principal value we have

$$\begin{aligned} &\text{v.p.} \int_{\sqrt{E}-2\delta}^{\sqrt{E}+2\delta} \frac{\eta_\delta(q) e^{\mp iq|d|} q dq}{q^2 - E} \\ &= \int_{\gamma_\epsilon^{(\mp)}} \frac{\eta_\delta(q) e^{\mp iq|d|} q dq}{q^2 - E} \mp \frac{i\pi}{2} e^{\mp i|d|\sqrt{E}}, \end{aligned}$$

where the contour $\gamma_\epsilon^{(\pm)}$, $(0 < \epsilon < 2\delta)$ consists of the intervals $[\sqrt{E} - 2\delta, \sqrt{E} - \epsilon]$, $[\sqrt{E} + \epsilon, \sqrt{E} + 2\delta]$ and the half circle $|q - \sqrt{E}| = \epsilon$, $\mp \mathrm{Im} q \geq 0$ respectively.

Since $\eta_\delta(q)$ and all its derivatives are zero at the ends of the contour $\gamma_\epsilon^{(\pm)}$ and since $|e^{\pm i|d|q}| \leq 1$ on $\gamma_\epsilon^{(\pm)}$ respectively, by means of an integration by parts we can show that

$$\int_{\gamma_\epsilon^{(\mp)}} \frac{\eta_\delta(q) e^{\mp iq|d|} q dq}{q^2 - E} = O(|d|^{-\infty}).$$

Thus we arrive at the asymptotic formula

$$I_1(d) = \frac{2\pi^2}{|d|} f(\mp\sqrt{E}\hat{d}) e^{\pm i\sqrt{E}|d|} + o(|d|^{-1}),$$

where the error term $o(|d|^{-1})$ is uniform in E on compact sets in $(0, +\infty)$.

Now we turn to the discussion of the integral $I_2(d)$ given by (B.4). To complete the proof of (2.12) it suffices to show that

$$\int_{\sqrt{E}-2\delta}^{\sqrt{E}+2\delta} \frac{q dq}{q^2 - E \mp i0} \int_{\mathbb{S}^2} e^{-iq\langle\omega, d\rangle} f_{1,\delta}(q\omega) d\omega \to 0 \tag{B.7}$$

uniformly in E on compact sets in $(0, \infty)$ as $|d| \to \infty$. Let us note that

$$f_{1,\delta}(p) = f_1(p)\eta_\delta(|p|), \;\; f_1(p) = \langle\hat{d}, \nabla\rangle f(p).$$

Therefore the integral in (B.7) can be represented as the sum of two terms

$$\int_{\sqrt{E}-2\delta}^{\sqrt{E}+2\delta} \frac{q\eta_\delta(q) dq}{q^2 - E \mp i0} \int_{\mathbb{S}^2} e^{-iq\langle\omega, d\rangle} \left[f_1(q\omega) - f_1(\sqrt{E}\omega)\right] d\omega \tag{B.8}$$

$$+ \int_{\sqrt{E}-2\delta}^{\sqrt{E}+2\delta} \frac{q\eta_\delta(q) dq}{q^2 - E \mp i0} \int_{\mathbb{S}^2} e^{-iq\langle\omega, d\rangle} f_1(\sqrt{E}\omega) d\omega. \tag{B.9}$$

Since f_1 is differentiable, the integrand in (B.8) is regular, and hence by Riemann-Lebesgue lemma the integral tends to zero as $|d| \to \infty$. By (B.5) the integral (B.9) can be represented in the form

$$\begin{aligned} &\text{v.p.} \int_{\sqrt{E}-2\delta}^{\sqrt{E}+2\delta} \frac{q\eta_\delta(q) dq}{q^2 - E} \int_{\mathbb{S}^2} e^{-iq\langle\omega, d\rangle} f_1(\sqrt{E}\omega) d\omega \\ &\pm \frac{i\pi}{2} \int_{\mathbb{S}^2} e^{-iq\langle\omega, d\rangle} f_1(\sqrt{E}\omega) d\omega, \end{aligned}$$

where the second term tend to zero again by Riemann-Lebesgue lemma. Consider the first term,

$$\text{v.p.} \int_{\sqrt{E}-2\delta}^{\sqrt{E}+2\delta} \frac{q\eta_\delta(q)}{q^2-E} F(q,d)dq, \tag{B.10}$$

where

$$F(q,d) = \int_{\mathbb{S}^2} e^{-iq\langle\omega,d\rangle} f_1(\sqrt{E}\omega)d\omega.$$

By (B.3) we have

$$F(q,d) = F_1(q,d) + F_2(q,d),$$
$$\begin{aligned} F_1(q,d) &= -\frac{2\pi}{iq|d|}\left\{f_1(\sqrt{E}\hat{d})e^{-iq|d|} - f_1(-\sqrt{E}\hat{d})e^{iq|d|}\right\}, \\ F_2(q,d) &= \frac{1}{iq|d|}\int_{\mathbb{S}^2} e^{-iq\langle\omega,d\rangle} f_2(\sqrt{E}\omega)d\omega, \end{aligned}$$

where $f_2(\sqrt{E}\omega) = \langle d, \nabla\rangle f_1(\sqrt{E}\omega)$. The arguments already used above allow one to calculate the contribution from $F_1(q,d)$ thus giving

$$\begin{aligned} &\text{v.p.} \int_{\sqrt{E}-2\delta}^{\sqrt{E}+2\delta} \frac{q\eta_\delta(q)}{q^2-E} F_1(q,d)dq \\ &= \frac{\pi^2}{\sqrt{E}|d|}\left(f_1(\sqrt{E}\hat{d})e^{-i|d|\sqrt{E}} + f_1(-\sqrt{E}\hat{d})e^{i|d|\sqrt{E}}\right) + O(|d|^{-\infty}). \end{aligned}$$

Consider now the contribution from $F_2(q,d)$,

$$\frac{1}{i|d|}\text{v.p.} \int_{\sqrt{E}-2\delta}^{\sqrt{E}+2\delta} \frac{\eta_\delta(q)dq}{q^2-E} \int_{\mathbb{S}^2} e^{-iq\langle\omega,d\rangle} f_2(\sqrt{E}\omega)d\omega. \tag{B.11}$$

By the obvious estimate

$$\left|\text{v.p.} \int_{-a}^{a} \frac{\varphi(x)}{x}dx\right| = \left|\int_{-a}^{a} \frac{\varphi(x)-\varphi(0)}{x}dx\right| \leq 2a \max_{x\in[-a,a]} |\varphi'(x)|,$$

we have that the absolute value of (B.11) can be bounded by

$$4\delta \max_{q\in[\sqrt{E}-2\delta,\sqrt{E}+2\delta]} \left\{\frac{1}{|d|}\frac{\partial}{\partial q}\left[\frac{\eta_\delta(q)}{q+E}\int_{\mathbb{S}^2} e^{-iq\langle\omega,d\rangle} f_2(\sqrt{E}\omega)d\omega\right]\right\}.$$

Note that

$$\begin{aligned} &\frac{1}{|d|}\frac{\partial}{\partial q}\int_{\mathbb{S}^2} e^{-iq\langle\omega,d\rangle} f_2(\sqrt{E}\omega)d\omega \\ &= -i\int_{\mathbb{S}^2} e^{-iq\langle\omega,d\rangle}\langle\omega,\hat{d}\rangle f_2(\sqrt{E}\omega)d\omega, \end{aligned}$$

which is $o(1)$ as $|d| \to \infty$ by Riemann-Lebesgue lemma. This completes the proof of (B.7).

The proof of (2.12) in the case $E = 0$ follows from the results of [40].

Appendix C

Here we give the proof of Lemma 4.1. According to (3.2) the spectral shift function for the pair of operators $H = H_0 + V$ and H_0 can be written as

$$\begin{aligned} &\xi(E) = -\xi((E+c_0)^{-1}; R(-c_0), R_0(-c_0)) \\ = -\frac{1}{2\pi i}\lim_{\epsilon\to+0}\log &\frac{\det[I+(R(-c_0)-R_0(-c_0))(R_0(-c_0)-(E+c_0)^{-1}-i\epsilon)^{-1}]}{\det[I+(R(-c_0)-R_0(-c_0))(R_0(-c_0)-(E+c_0)^{-1}+i\epsilon)^{-1}]} \\ &= \frac{1}{2\pi i}\lim_{\epsilon\to+0}\log\frac{\det[I+(R(-c_0)-R_0(-c_0))(R_0(-c_0)-\zeta)^{-1}]}{\det[I+(R(-c_0)-R_0(-c_0))(R_0(-c_0)-\overline{\zeta})^{-1}]}, \end{aligned} \quad (C.1)$$

where $\zeta = (E + i\epsilon + c_0)^{-1}$.

Due to the resolvent equation we have

$$R(-c_0) - R_0(-c_0) = -R_0(-c_0)|V|^{1/2}(I + V^{1/2}R_0(-c_0)|V|^{1/2})^{-1}V^{1/2}R_0(-c_0).$$

Obviuosly $R_0(-c_0)|V|^{1/2}$ and $V^{1/2}R_0(-c_0)$ are Hilbert-Schmidt operator. We recall that for an arbitrary trace class operator A the modified Fredholm determinant is given by

$$\det{}_2(I + A) = \det(I + A)e^{-\mathrm{tr}A}.$$

Therefore

$$\begin{aligned} &\det[I+(R(-c_0)-R(-c_0))(R_0(-c_0)-\zeta)^{-1}] \\ &= \det{}_2[I-(I+V^{1/2}R_0(-c_0)|V|^{1/2})^{-1} \\ &\cdot V^{1/2}R_0(-c_0)(R_0(-c_0)-\zeta)^{-1}R_0(-c_0)|V|^{1/2}] \\ &\cdot \exp\{-\mathrm{tr}[(I+V^{1/2}R_0(-c_0)|V|^{1/2})^{-1} \\ &\cdot V^{1/2}R_0(-c_0)(R_0(-c_0)-\zeta)^{-1}R_0(-c_0)|V|^{1/2}]\}. \end{aligned} \quad (C.2)$$

Now we note that

$$\begin{aligned} R_0(-c_0)(R_0(-c_0)-\zeta)^{-1}R_0(-c_0) &= R_0(-c_0) + \zeta(R_0(-c_0)-\zeta)^{-1}R_0(-c_0) \\ &= R_0(-c_0) - R_0(\zeta^{-1}-c_0) = R_0(-c_0) - R_0(E+i\epsilon). \end{aligned}$$

Therefore the r.h.s. of (C.2) equals

$$\begin{aligned} &\det{}_2\Big\{I-(I+V^{1/2}R_0(-c_0)|V|^{1/2})^{-1}V^{1/2}R_0(-c_0)|V|^{1/2} \\ &+(I+V^{1/2}R_0(-c_0)|V|^{1/2})^{-1}V^{1/2}R_0(E+i\epsilon)|V|^{1/2}\Big\} \\ &\cdot\exp\{-\mathrm{tr}(I+V^{1/2}R_0(-c_0)|V|^{1/2})^{-1} \\ &\cdot(V^{1/2}R_0(-c_0)|V|^{1/2}-V^{1/2}R_0(E+i\epsilon)|V|^{1/2})\}. \end{aligned}$$

Since

$$(I + V^{1/2}R_0(-c_0)|V|^{1/2})^{-1}V^{1/2}R_0(-c_0)|V|^{1/2} = I - (I + V^{1/2}R_0(-c_0)|V|^{1/2})^{-1}, \quad (C.3)$$

we get

$$\begin{aligned}
& \det\left[I + (R_0(-c_0) - R_0(-c_0))(R_0(-c_0) - \zeta)^{-1}\right] \\
= \; & \det{}_2 \left\{(I + V^{1/2}R_0(-c_0)|V|^{1/2})^{-1}(I + V^{1/2}R_0(E+i\epsilon)|V|^{1/2})\right\} \\
& \cdot \exp\{-\operatorname{tr}(I + V^{1/2}R_0(-c_0)|V|^{1/2})^{-1} \\
& \cdot (V^{1/2}R_0(-c_0)|V|^{1/2} - V^{1/2}R_0(E+i\epsilon)|V|^{1/2})\}. \qquad \text{(C.4)}
\end{aligned}$$

It is easy to see that for arbitrary Hilbert-Schmidt operators A and B such that $I + A$ is invertible

$$\det{}_2[(I+A)^{-1}(I+B)] = \frac{\det_2(I+B)}{\det_2(I+A)} e^{\operatorname{tr}(I+A)^{-1}A(B-A)}. \qquad \text{(C.5)}$$

Therefore using (C.3) and (C.5) the expression (C.4) can be represented as

$$\begin{aligned}
& \frac{\det_2(I + V^{1/2}R_0(E+i\epsilon)|V|^{1/2})}{\det_2(I + V^{1/2}R_0(-c_0)|V|^{1/2})} \\
& \cdot \exp\{\operatorname{tr}(V^{1/2}R_0(E+i\epsilon)|V|^{1/2} - V^{1/2}R_0(-c_0)|V|^{1/2})\}.
\end{aligned}$$

Finally we get

$$\begin{aligned}
& \frac{\det[I + (R(-c_0) - R_0(-c_0)(R_0(-c_0) - \zeta)^{-1})]}{\det[I + (R(-c_0) - R_0(-c_0)(R_0(-c_0) - \overline{\zeta})^{-1})]} \\
= \; & \frac{\det_2(I + V^{1/2}R_0(E+i\epsilon)|V|^{1/2})}{\det_2(I + V^{1/2}R_0(E-i\epsilon)|V|^{1/2})} \\
& \cdot \exp\{\operatorname{tr}(V^{1/2}R_0(E+i\epsilon)|V|^{1/2} - V^{1/2}R_0(E-i\epsilon)|V|^{1/2})\}. \qquad \text{(C.6)}
\end{aligned}$$

Taking the limit $\epsilon \to 0$ in (C.6) and using the relation

$$\lim_{\epsilon\to 0} \operatorname{tr}(V^{1/2}R_0(E+i\epsilon)|V|^{1/2} - V^{1/2}R_0(E-i\epsilon)|V|^{1/2}) = \begin{cases} \frac{i\sqrt{E}}{2\pi}\int_{\mathbb{R}^3} V\,dx, & E \geq 0 \\ 0, & E < 0 \end{cases}$$

leads to (4.1).

References

[1] S. Albeverio, F. Gesztesy, R. Høegh-Krohn, "The low energy expansion in nonrelativistic scattering theory", *Ann. Inst. Henri Poincaré, Physique théorique* **37** (1982) 1-28.

[2] S. Albeverio, F. Gesztesy, R. Høegh-Krohn and H. Holden, *Solvable Models in Quantum Mechanics*, Springer, Berlin, 1988.

[3] P. Aventini and R. Seiler, "On the electronic spectrum of the diatomic molecular ion", *Commun. Math. Phys.* **41** (1975) 119-134.

[4] M.Sh. Birman and M.G. Krein, "On the theory of wave operators and scattering operators", *Sov. Math. – Dokady* **3** (1962) 740-744.

[5] M.Sh. Birman and D.R. Yafaev, "The spectral shift function. Work by M.G. Krein and its further development", *St. Petersburg Math. J.* **4** (1993) 833-870.

[6] D. Bollé and S.F.J. Wilk, "Low-energy sum rules for two-particle scattering", *J. Math. Phys.* **24** (1983) 1555-1563.

[7] Y. Colin de Verdière, "Une formule de traces pour l'opérateur de Schrödinger dans $\mathbb{R}^3$", *Ann. scient. Éc. Norm. Sup.* **14** (1981) 27-39.

[8] J.M. Combes and R. Seiler, "Regularity and asymptotic properties of the discrete spectrum of the electronic Hamiltonians", *Int. J. Quantum Chem.* **14** (1978) 213-229.

[9] E.B. Davies, "The twisting trick for double well Hamiltonians", *Commun. Math. Phys.* **85** (1982) 471-479.

[10] T. Dreyfus, "The determinant of the scattering matrix and its relation to the number of eigenvalues", *J. Math. Anal. Appl.* **64** (1978) 114-134.

[11] T. Dreyfus, "The number of states bound by non-central potentials", *Helv. Phys. Acta* **51** (1978) 321-329.

[12] V. Efimov, "Energy levels arising from resonant two-body forces in a three-body system", *Phys. Lett. B* **33** (1970) 563-564.

[13] R. Geisler, V. Kostrykin and R. Schrader, "Concavity properties of Krein's spectral shift function", *Rev. Math. Phys.* **7** (1995) 161-181.

[14] L. Guillopé, "Asymptotique de la phase de diffusion pour l'opérateur de Schrödinger avec potentiel", *C.R. Acad. Sc. Paris, Série I* **293** (1991) 601-603.

[15] E.M. Harrell, "Double wells", *Commun. Math. Phys.* **75** (1980) 239-261.

[16] K. Hepp, "Spatial cluster decomposition properties of the S-matrix", *Helv. Phys. Acta* **37** (1964) 659-662.

[17] R. Høegh-Krohn and M. Mebkhout, "The $\frac{1}{r}$ expansion for the critical multiple well problem", *Commun. Math. Phys.* **91** (1983) 65-73.

[18] H. Holden, R. Høegh-Krohn and M. Mebkhout, "The short-range expansion for multiple well scattering theory", *J. Math. Phys.* **26** (1985) 145-151.

[19] W. Hunziker, "Cluster properties of multiparticle systems", *J. Math. Phys.* **6** (1965) 6-10.

[20] W. Hunziker, "Mathematical theory of multi-particle quantum systems," In *Quantum Theory and Statistical Physics",* ed. A.O. Barut and W.E. Brittin, Lectures in Theoretical Physics, Gordon and Breach, New York, 1968, pp.1-48.

[21] A. Jensen and T. Kato, "Spectral properties of Schrödiger operators and time-decay of the wave functions", *Duke Math. J.* **46** (1979) 583-611.

[22] T. Kato, *Perturbation Theory for Linear Operators,* Springer, Berlin, 1966.

[23] T. Kato, "Monotonicity theorem in scattering theory", *Hadronic J.* **1** (1978) 134-154.

[24] M. Klaus and B. Simon, "Binding of Schrödinger particles through conspirancy of potential wells", *Ann. Inst. H. Poincaré, Phys. Theor.* **30** (1979) 83-87.

[25] M. Klaus, "Some remarks on double – wells in one and three dimensions", *Ann. Inst. H. Poincaré, Phys. Theor.* **34** (1981) 405-417.

[26] I.V. Komarov and S. Yu. Slavyanov, "The two Coulomb problem at large center separation", *J. Phys. B: Atom. Molec. Phys.* **1** (1968) 1066-1072.

[27] V. Kostrykin and R. Schrader, "Cluster properties of one particle Schrödinger operators", *Rev. Math. Phys.* **6** (1994) 833-853.

[28] M.G. Krein, "On a trace formula in perturbation theory", *Matem. Sbornik* **33** (1953) 597-626 (in Russian).

[29] M.G. Krein, "Perturbation determinants and a formula for the traces of unitary and self-adjoint operators", *Sov. Math. – Doklady* **3** (1962) 707-710.

[30] E.H. Lieb and B. Simon, "Monotonicity of the electronic contribution to the Born-Oppenheimer energy", *J. Phys. B: Atom. Molec. Phys.* **11** (1978) L537-L542.

[31] J.D. Morgan III and B. Simon, "Behavior of molecular potential energy curves for large nuclear separations", *Int. J. Quantum Chem.* **17** (1980) 1143-1166.

[32] R.G. Newton, "Noncentral potentials: The generalized Levinson theorem and the structure of the spectrum", *J. Math. Phys.* **18** (1977) 1348-1357.

[33] R.G. Newton, *Scattering Theory of Waves and Particles,* Springer, New York, 1982.

[34] Yu. N. Ovchinnikov and I.M. Sigal, "Number of bound states of three-body systems and Efimov's effect", *Ann. Phys.* **123** (1979) 274-295.

[35] L. Pastur and A. Figotin, *Spectra of Random and Almost-Periodic Operators,* Springer, Berlin, 1992.

[36] M. Reed and B. Simon, *Methods of Modern Mathematical Physics, II: Fourier Analysis, Self-Adjointness,* Academic Press, New York, 1975.

[37] M. Reed and B. Simon, *Methods of Modern Mathematical Physics, III: Scattering Theory,* Academic Press, New York, 1979.

[38] M. Reed and B. Simon, *Methods of Modern Mathematical Physics, IV: Analysis of Operators,* Academic Press, New York, 1978.

[39] R. Schrader, "High energy behaviour for non-relativistic scattering by stationary external metrics and Yang-Mills potentials", *Z. Phys. C: Particles and Fields* **4** (1980) 27-36.

[40] P.N. Shivakumar and R. Wong, "Asymptotic expansion of multiple Fourier transforms", *SIAM J. Math. Anal.* **10** (1979) 1095-1104.

[41] B. Simon, *Quantum Mechanics for Hamiltonians Defined as Quadratic Forms,* Princeton University Press, Princeton, 1971.

[42] B. Simon, *Trace Ideals and Their Applications,* Cambridge University Press, Cambridge, 1979.

[43] H. Tamura, "Existence of bound states for double well potentials and the Efimov effect", in H. Fujita, T. Ikebe, and S.T. Kuroda (Eds.) *Functional-Analytic Methods for Partial Differential Equations* , pp. 173-186. Lecture Notes in Mathematics, Vol. 1450, Springer, Berlin, 1990.

[44] H. Tamura, "The Efimov effect of three-body Schrödinger operators: Asymptotics for the number of negative eigenvalues", *Nagoya Math. J.* **130** (1993) 55-83.