Lagrangian Embeddings and Holomorphic Discs

Yakov Eliashberg * Klaus Mohnke†

Abstract

For a given embedded Lagrangian in the complement of a complex line we construct a holomorphic disc in the complement having boundary on that Lagrangian.

1 Introduction

We consider Lagrangians in $\mathbb{C}^2$, i.e. immersed (real) surfaces $L \subset \mathbb{C}^2$ such that the restriction $\omega|_L = 0$ for the symplectic 2-form $\omega = d\theta = dx \wedge dy + du \wedge dv$, where $\theta = xdy + udv$ is the Liouville form and $z = x + \mathrm{i}y, w = u + \mathrm{i}v$ are the (complex) coordinates on $\mathbb{C}^2$. Note that then $\theta|_L$ is a closed 1-form. In [1] Gromov constructed non constant holomorphic discs with boundary on any embedded, closed Lagrangian in $\mathbb{C}^2$. He concluded that such a Lagrnagian is never exact, i.e. $[\theta|_L] \in \mathbf{H^1}(L;\mathbb{Z})$ defines a non vanishing class in de Rham cohomology: the symplectic volume of the disc is positive by the compatibility of complex and symplectic structure and so is the integral of θ over the boundary of the disc by Stokes' theorem.

There are further topological obstructions against Lagrange embeddings in $\mathbb{C}^2$. Note that any Lagrangian is also a totally real surface: the direct sum of the tangent bundle with the image of the tangent bundle under the (alomost) complex structure J_0, given by multiplication with i, is the whole tangent bundle of the ambient space, which is trivial. It follows that the direct sum of the tangent space of an immersed totally real surface is trivial and thus the only possible immersed totally real surfaces are tori and Klein bottles. It is conjectured that there is no embedded Lagrangian Klein bottle in $\mathbb{C}^2$. Now for any immersed totally real surface $i : L \hookrightarrow \mathbb{C}^2$ consider the Gaußmap

*Stanford University, supported by NSF ..., e-mail: eliash@math.stanford.edu

†Humboldt-Universität zu Berlin, SFB 288 der DFG, e-mail: mohnke@mathematik.hu-berlin.de

$$G : L \to \mathcal{R} \subset \mathbf{Gr}(2; \mathbb{R}^4)$$

taking values in the open, dense submanifold $\mathcal{R}$ of planes totally real w.r.t. the complex structure given by $\mathbb{R}^4 \cong \mathbb{C}^2$. Now $\pi_1(\mathcal{R}) \cong \mathbb{Z}$ by

$$\exp(2\pi i t) \mapsto \mathbb{R} \oplus \exp(\pi i t)\mathbb{R} \subset \mathbb{C}^2.$$

With that a class $\mu \in \mathbf{Hom}(\mathbf{H_1}(L; \mathbb{Z}), \mathbb{Z}) \cong \mathbf{H^1}(L; \mathbb{Z})$ is welldefined by $\mu = G_*$, the so called *Maslov class* (see [2]). The *Maslov index* is given by the non negative integer $m(L, i)$ defined by $\mu(\mathbf{H_1}(L; \mathbb{Z})) = m(L, i)\mathbb{Z}$. Observe that if L is oriented then μ is an even class and $m(L, i)$ is even. It is not hard to see that $m = 2$ for the standard torus $\mathbf{S^1} \times \mathbf{S^1} \subset \mathbb{C}^2$. Note that this easy definition as well as the extension to the class of totally real surfaces only makes sense in (real) dimension 4. Now for any even m there exists an embedded totally real torus and an immersed Lagrangian torus of that Maslov index, the latter simply given by covering the standard torus. Polterovich [4] and Viterbo [5] proved with different methods that there is no Lagrangian torus with Maslov index other than 2. Polterovich derived that result by constructing a (pseudo)holomorphic disc with boundary on the Lagrangian for a generic almost complex structure on $\mathbb{C}^2$: this forces the Maslov index of the boundary to lie between 1 and 3 and thus to be equal 2. This observatio confirmed the conjecture that all embedded Lagrangian tori in $\mathbb{C}^2$ are Lagrange isotopic to the standard torus. For this as well as the conjecture on the Klein bottle some progress has been reported by Hofer and Luttinger [3].

In this paper we consider Lagrangians in the complement of a real line in $\mathbb{C}^2$. In Chapter 2 we show that specifying the data used by Gromov and Polterovich for the construction of the disc we can control the process to obtain a disc in the complement of the line. This is not exciting unless the Lagrangian and the complex line is nontrivial. Then we obtain some new obstructions against Lagrangian embeddings. This is described in Chapter 3. Finally, in Chapter 4 we discuss an obstruction for certain totally real tori in the unit sphere $\mathbf{S^3} \subset \mathbb{C}^2$ to be *pre-Lagrangian*, i.e. the image of some Lagrangian in $\mathbb{C}^2 \setminus \{0\}$ under normalization.

2 On the existence of holomorphic discs with boundary on a Lagrangian

We call a submanifold $\Sigma \subset \mathbb{C}^2$ of *finite geometry* if

1. the injectivity radii of the normal exponential map and the exponential map of the induced Riemannian metric are bounded away from zero and

2. there are constants $\epsilon, c > 0$ such that $\mathrm{d}_\Sigma(x,y) < c|x-y|$ for $x, y \in \Sigma$ with $|x-y| < \epsilon$.

We reformulate a statement of Polterovich [4]:

Proposition 2.1 *Let $\Sigma \subset V := \mathbb{C}^2 \setminus H$ be any embedded Lagrangian in the complement of the complex line H of finite geometry. Suppose there is a complex linear function which is bounded on Σ and unbounded on H. Then there is a callibrating and taming almost complex structure J on $\mathbb{C}^2$ which coincides with the standard structure outside an arbitrary small fixed neighbourhood of Σ such that there exist a pseudoholomorphic disc in V with boundary on Σ which has Maslov-index between* 1 *and* 3.

Remark 2.2 *The result specifies the cycle on Σ found independently by Polterovich [4] and Viterbo [5] with different methods carefully studying the arguments of the first author.*

Proof. W.l.o.g. suppose $H = \mathbb{C}\times\{0\}$. Perturb Σ such that there exists an antiholomorphic involution τ of a sufficiently small neighbourhood $\mathcal{U} \subset V$ of Σ such that the set of fixed points is $\mathrm{Fix}(\tau) = \Sigma$.

Denote by $\mathcal{J}$ the set of almost complex structures on V compatible with and callibrating ω, with finite Floer norm, coinciding with the standard complex structure $J_0 := \mathrm{i}\cdot$ outside a fixed small neighbourhood of Σ such that τ is still antiholomorphic on $\mathcal{U}$.

Fix $r \geq 2, s \in (0,1)$. Given $\alpha \in \mathbf{H_2}(V, \Sigma; \mathbf{Z}) \setminus \{0\}$ we define

$$\begin{aligned}\mathcal{F}_\alpha \quad := \quad & \{u : (\mathbf{\Delta}, \partial\mathbf{\Delta}) \longrightarrow (V, \Sigma)| \\ & u \in \mathbf{C^{r,s}}; [u] = \alpha; f^{-1}(f(z)) = \{z\}, Df(z) \neq 0 \text{ for a } z \in \partial\mathbf{\Delta}\}.\end{aligned}$$

Set

$$M_\alpha := \{u \in \mathcal{F}_\alpha \times \mathcal{J} | \overline{\partial}_J u = 0\}.$$

Proposition 2.3 *M_α is a Banach submanifold of $\mathcal{F}_\alpha \times \mathcal{J}$. The natural projection $P : M_\alpha \longrightarrow \mathcal{J}$ is a Fredholm map of index $\mu(\partial\alpha) + 2$ where $\partial : \mathbf{H_2}(V, \Sigma; \mathbf{Z}) \longrightarrow \mathbf{H_1}(\Sigma; \mathbf{Z})$ is the boundary homomorphism of the long exact sequence.*$\Box$

Put $M_\alpha(J) := P^{-1}(J)$. The 3–dimensional group of Möbius transformations acts freely on the (non constant) maps in M_α.

Corollary 2.4 *Let $J \in \mathcal{J}$ be a regular value of P. If $M_\alpha(J) \neq \emptyset$ then $\mu(\partial\alpha) \geq 1$.* $\square$

Let $\mathcal{G}_J$ be the space of conjugated complex homomorphisms from $T\mathbf{\Delta}$ to TV bounded in the Floer norm such that $g_2 = (\mathrm{pr}_2)_* g = 0$ in a small fixed neighbourhood of $\mathbb{C} \times \{0\}$. The following is the crucial step in the proof of the proposition:

Proposition 2.5 *There exists a neighbourhood $\mathcal{J}_0 \subset \mathcal{J}$ of J_0 such that for each $J \in \mathcal{J}_0$ there is an open set $\mathcal{G}'_J \in \mathcal{G}_J$ with $\overline{\partial}_J^{-1}(\mathcal{G}'_J) = \emptyset$.*

Proof. We define

$$\begin{aligned} \mathcal{J}_0 &:= \{J \in \mathcal{J} | \quad |J - J_0| < c_1\} \\ \mathcal{G}'_J &:= \{g \in \mathcal{G}_J | \Re g_1 > A, \|g\|_{2,\infty} < c_2 A\}, \end{aligned}$$

with $A, c_1, c_2 > 0$ constants apropriately chosen below. Let $\overline{\partial}_J f = g$ for a $g \in \mathcal{G}'_J$ for an $f \in \mathcal{F}_0$. Then

$$g_1(z, f(z)) = \overline{\partial} f_1 + R(z)$$

where the rest term $|R| < c_1 |\nabla f|$. Σ is Lagrange and $[f] = 0$. Thus $\int_{\mathbf{\Delta}} f^*\omega = 0$. But the latter is given by the $\mathbf{L^2}$-pairing

$$\langle \overline{\partial}_J f, \partial_J \overline{f} \rangle = \int_{\mathbf{\Delta}} \overline{\partial}_J f \wedge \partial_J \overline{f}.$$

Hence we have $\|g(z, f(z))\|_{2,J} = \|\nabla f\|_{2,J}$ and therefore

$$\|\nabla f\|_2 \leq c_3 \|g\|_{2,\infty} < c_3 c_2 A.$$

To put things together we write

$$\left| \Re \int_{\mathbf{\Delta}} g_1(z, f(z)) dx dy \right| \leq \left| \Re \int_{\mathbf{\Delta}} \frac{\partial f_1}{\partial \overline{z}} dx dy \right| + \left| \Re \int_{\mathbf{\Delta}} R dx dy \right|.$$

From the boundedness of $h(\Sigma)$ we see

$$\left|\Re\int_{\mathbf{\Delta}}\frac{\partial f_1}{\partial\overline{z}}dxdy\right|\leq c_4\max_{\partial\mathbf{\Delta}}|f_1|\leq c_4\max_{\Sigma}|z_1|\leq c_5.$$

The bound on R is given by the above computations:

$$\left|\Re\int_{\mathbf{\Delta}}Rdxdy\right|\leq c_6\|R\|_2\leq c_6c_1\|\nabla f\|_2\leq c_1c_2c_3c_6A.$$

Together with our definition of $\mathcal{G}'_J$ we obtain $2\pi A<c_5+c_1c_2c_3c_6A$.

Thus for $c_1c_2<(c_3c_6)^{-1}$ and $A>\frac{c_5}{2\pi-1}$ we obtain a contradiction and the desired property for the subset $\mathcal{G}'_J$.$\square$

Now let $J\in\mathcal{J}_0$ be a regular value of P. Consider $\Sigma_1:=\partial\mathbf{\Delta}\times\Sigma\subset\mathbb{C}\times V=:V_1$ and define an almost complex structure on V_1 via $J_g(\xi,\eta)=(\mathrm{i}\xi,g\xi+J\eta)$ for $\xi\in T\mathbb{C}$, $\eta\in TV$ where J is the complex structure on $\mathbb{C}^2$ and $g\in\mathcal{G}$. Define

$$\Phi_\beta:=\{\varphi\in\mathbf{C}^{r,s}((\mathbf{\Delta},\partial\mathbf{\Delta}),(V_1,\Sigma_1))|\quad\varphi(z)=(z,\hat\varphi(z)),[\hat\varphi]=\beta\in\mathbf{H_2}(V,\Sigma;\mathbf{Z})\}.$$

Let $N_\beta:=\{(\varphi,g)\in\Phi_\beta\times\mathcal{G}|\overline{\partial}_{J_g}=0\}$. Now, $\overline{\partial}_{J_g}\varphi=0$ iff $\overline{\partial}_J\hat\varphi=g$. Again we have

Proposition 2.6 *N_β is a Banach submanifold in $\Phi_\beta\times\mathcal{G}$. The projection $P_1:N_\beta\longrightarrow\mathcal{G}$ is Fredholm with index $\mu(\partial(\beta))+2$.* $\square$

Let $\gamma:[0,1]\longrightarrow\mathcal{G}_J$ be a path connecting the regular values $\gamma(0),\gamma(1)$ of P_1 such that P_1 is transversal to γ for all $\beta\in\mathbf{H_2}(V,\Sigma;\mathbf{Z})$. Assume that $\gamma(1)\in\mathcal{G}$ and $\gamma(0)=0$. Now if $Y_\beta:=P_1^{-1}(\gamma)\cap N_\beta$ is nonempty then $\dim Y_\beta=\mu(\partial(\beta))+3\geq0$ and thus $\mu(\partial(\beta))\geq-3$.

Proposition 2.7 *There exist classes $\beta,\alpha_1,\dots\alpha_n\in\mathbf{H_2}(V,\Sigma;\mathbf{Z})$ such that $\alpha_i\neq0$, $M_{\alpha_i}\neq\emptyset$, $Y_\beta\neq\emptyset$, $\beta+\sum_{i=1}^n\alpha_i=0$ and $\beta\cdot[H]=\alpha_i\cdot[H]=0$ for the class of the hyperplane in the compactification $\mathbb{C}P^2$ of $\mathbb{C}^2$.*

Proof. Fix a regular value $\sigma\in\Sigma$ of $A:Y_0\longrightarrow\Sigma$, $(\varphi,g)\mapsto\hat\varphi(1)$. $X:=A^{-1}$ is a nonempty curve. If it is compact then $\sharp P_1^{-1}(g_1)\cap X\equiv1\mathrm{mod}2$ which contradicts Proposition 2.5. Thus X is noncompact. Now all maps $\hat\varphi\in\mathcal{F}_0$ with $\hat\varphi(1)=\sigma$ and $\overline{\partial}_J\hat\varphi\in\gamma$ are bounded. So Gromov's compactness theorem gives a J_g–cusp curve with boundary in Σ_1 which is the union of a curve $\varphi \in Y_\beta$ and some nonconstant curves $u_1, \dots u_k$ which lie in the fibres $\{z_i\} \times V$. Here $g \in \gamma$ is some point in the interior of γ. The cusp curve is a limit of some sequence in Y_0. Thus $\hat{u_i} \in M_{\alpha_i}(J)$, $\overline{\partial}_J \hat{\varphi} = g$ and $\beta + \sum_{i=1}^{k} \alpha_i = 0$. In order to prove the last statement it suffices to show that for all elements φ_i of that sequence $\hat{\varphi}_i(\mathbf{\Delta}) \cap H = \emptyset$. Note that this is true for $A^{-1}(g_0)$. Assume $\varphi_i \in P_1^{-1}(\gamma(t_i)) \cap N_0$ for an increasing sequence $\{t_i\}_i$ with $\lim_{i \to \infty} = t_0 < 1$ such that $P_1^{-1}([0, t']) \cap N_0$ *is* compact for all $t' < t_0$. It suffices to show that $\{t \in [0, t_0) | \quad \varphi \in P_1^{-1}(\gamma(t)) \cap N_0 \Rightarrow \hat{\varphi}(\mathbf{\Delta}) \cap H = \emptyset\}$ is a closed subset of $[0, t_0)$. Pick a point $(z, 0) \in \hat{\varphi}(\mathbf{\Delta}) \cap H$. We claim that intersection index is positive. Note that $\mathrm{pr}_2 \hat{\varphi}$ is a holomorphic function near the intersection. The intersection index is given by the degree of the map at zero. It is positive if that map does not vanish identically for some neighbourhood of the intersection point. But in that case $\hat{\varphi}$ would map into H which is impossible. □

Proof of Proposition 2.1. The proposition follows immediately. Moreover, from $\mu(\partial\beta) \geq -3$ we have that $1 \geq \mu(\partial\alpha_i) \leq 3$. □

3 Obstructions to Lagrangian Embeddings

The refinement of Polterovich' result gives some more obstructions to Lagrangian embeddings of finite geometry. Let $H \subset \mathbb{C}^2 \setminus \Sigma$ be a complex line in the complement of the submanifold Σ. Then $[H]_\Sigma \in \mathbf{H^1}(\Sigma; \mathbb{Z})$ denotes the element given by linking.

Proposition 3.1 *Suppose $\Sigma \cong \mathbb{R} \times \mathbf{S^1}$ is a Lagrange embedding of finite geometry in $\mathbb{C}^2$ such that there exists a complex line H in its complement with $[H]_\Sigma \neq 0$. Then each complex linear function h which is non constant on H is unbounded on Σ.*

Proof. Assume there is a bounded complex linear function which is constant on H then the hypotheses of Proposition 2.1 are satisfied and in the complement of H there is a pseudo-holomorphic discs with boundary on Σ. However, this contradicts $[H]_\Sigma \neq 0$. □

Proposition 3.2 *Let Σ be a closed, compact Lagrange embedding such that there exists a complex line H in its complement with $[H]_\Sigma \neq 0$. Then $\Sigma \cong \mathbf{T^2}$ is the 2-torus.*

Proof. In the complement of H we obtain a pseudoholomorphic disc with boundary on Σ and nonvanishing Maslov index. Thus we obtain a element in the kernel of $[H]_\Sigma$ which is not torsion. Thus $b_1(\Sigma) \geq 2$ and the statement follows. □

4 Pre–Lagrangian Tori in $\mathbf{S^3}$

A curve γ in the unit sphere $\mathbf{S^3} \subset \mathbb{C}^2$ is called Legendrian if it is tangent to the complex tangencies ξ at $\mathbf{S^3}$: $T\gamma \subset \xi|_\gamma$. An embedded totally real torus Σ in the unit sphere $\mathbf{S^3}$ is called *pre-Lagrangian* if there is a function f on Σ such that $f\lambda|_\Sigma$ is closed, where

$$\lambda = \Im(\overline{z}\mathrm{d}z + \overline{w}\mathrm{d}w)$$

in coordinates $(z, w) \in \mathbb{C}^2$. Tubular neighbourhoods of Legendrian knots are all contactomorphic to each other. In particular they are contactomorphic to a tubular neighbourhood

$$N_\epsilon = \{(z, w) \in S^3 | \quad y^2 + v^2 < \epsilon\}$$

of $\gamma_0 := \{(z, w) \in \mathbf{S^3} | y = v = 0\} \subset \mathbf{S^3}$ with

$$\xi = \mathrm{Ker}\lambda,$$

where the contact structure is given by $\xi = \mathrm{Ker}\lambda$ with $\lambda = \Im\{\overline{z}dz + \overline{w}dw\}$. Thus it suffices to show the following proposition for the case of that standard Legendre knot:

Proposition 4.1 *To any Legendre knot there exists a small tubular neighbourhood such that no totally real boundary of a smaller tubular neighbourhood is pre–Lagrangian.*

Remark 4.2 *This property was discovered in a discussion of an example by J. Duval of a totally real torus without Reeb–components which does not admit any holomorphic disc with boundary on it. It is of the above type. Thus the statement contradicts the hypotheses of Duval that his example already shows, that fillability is not a qualitative property of totally real tori in S^3 preserved under isotopies among them.*

Proof. As mentioned before we may restrict ourself to the standard Legendrian γ_0. Obviously all tubular neighbourhoods N_ϵ are in the complement of $H = \{(z, w) | z + \mathrm{i}w = 0\}$ for $\epsilon \leq 1$. Thus a pre–Lagrangian would define a Lagrangian in $\mathbb{C}^2 \setminus H$. They are isotopic through totally real embeddings. Thus there is a cycle on the pre–Lagrangian which bounds a disc in N_1 and has nontrivial Maslov index. Hence if there were such a totally real torus its Maslov index along the meridian would be ± 2.

Now we claim that for such tori the meridian has actually trivial Maslov index. For simlicitiy choose coordinates (θ, z) on N_1 such that the contact structure is given by Kerμ with $\mu = \cos(\theta)dx + \sin(\theta)dy$ and $\gamma_0 = \{x = y = 0\}$. Suppose θ_0 is a regular value of the projection restricted to the surface. Note that there is an obvious trivialization of $\xi|_{\{}\theta^{-1}(\theta_0)\}$ given by $\{\frac{\partial}{\partial\theta}, v = -\sin(\theta_0)\frac{\partial}{\partial x} + \cos(\theta_0)\frac{\partial}{\partial y}\}$. Pick any circle δ in $\theta^{-1}(\theta_0)$. By the above its Maslov index measures the numbers of rotations of $\sphericalangle\,(v, \dot{\delta})$ which is zero for any smooth curve δ and fixed vector v. One of these circles is the meridian which proves the vanishing of the index. $\square$

References

[1] Michail Gromov. Pseudo-holomorphic curves in symplectic manifolds. *Invent. math.*, 82:307–347, 1985.

[2] Helmut Hofer. Pseudoholomorphic curves in symplectizations with applications to the Weinstein conjecture in dimension three. *Invent. math.*, 114:515–563, 1993.

[3] Helmut Hofer and Karl Luttinger. 1996. talk given at the conference ... , Italy.

[4] L.V. Polterovich. The Maslov class of the Lagrange surfaces and Gromov's pseudo-holomorphic curves. *Trans. of the AMS*, 325:241–248, 1991.

[5] C. Viterbo. A new obstruction to embedding Lagrangian tori. *Invent. Math.*, 100:301–320, 1990.