SPECTRAL ANALYSIS OF VOLTERRA OPERATORS AND INVERSE PROBLEMS FOR SYSTEMS OF ORDINARY DIFFERENTIAL EQUATIONS

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§0. Introduction

This paper consists of two parts. The purpose of the first part is a treatment of integral Volterra operators

$$K: \ f \to \int_0^x k(x,t)f(t)dt, \qquad f = (f_1, \dots, f_n), \tag{0.1}$$

with $k(x,t)$ being an $n \times n$ matrix kernel, in the spaces $L_p[0,1] \otimes \mathbb{C}^n$ of vector functions. The following questions are considered: a description of cyclic, invariant and hyperinvariant subspaces, spectral multiplicity and similarity.

But the principle aim of the paper is to investigate inverse problems for some systems of ordinary differential equation (ODE) on a finite interval. We mean the systems

$$B \otimes \frac{dy}{dx} + Q(x)y = \lambda y, \tag{0.2}$$

with B being an $n \times n$ nonsingular diagonal matrix and $Q(x)$ being a summable $n \times n$ matrix potential with zero diagonal. The approach provided is a further developement of the one applied in [M4] to the generalization of the well-known Borg result [Bo] for higher order ODE, that is to the unique recovery of an n-th order ODE from n spectra of the boundary value problems.

The main idea of the approach just mentioned is to reduce the questions of uniqueness in inverse problems to the investigation of the uniqueness of Goursat type problems for a certain partial differential equation for the kernel of a triangular transformation operator.

Now the connection between two topics under investigation becomes clear. In order to sudy the inverse problems we need a trianglular transformation operator for systems of ODE. The existence of such an operator is closely connected with

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the similarity of integral Volterra operators with kernel $k(x,t)$ being the Green function of the Cauchy problem for ODE with zero initial data and the simple Volterra operator $J \otimes B$ being the tensor product of the integration operator and a matrix B. For the sake of completeness of the exposition, the lattices Lat$A$ and Hyplat$A$ of invariant and hyperinvariant subspaces of the operator $A = J^\alpha \otimes B$ (with Re$\alpha > 0$) are investigated.

Now we briefly sketch the content of each section.

The main result of §1 is a criterion for a system of vectors to generate a cyclic subspace for the operator $K \otimes I_n$ in $L_p[0,1] \otimes \mathbb{C}^n$ $(1 \le p < \infty)$, assuming K to be unicellular in $L_p[0,1]$ $(1 \le p < \infty)$.

In §2 we obtain the following relations

$$\{J^\alpha \otimes B\}' = \oplus_{j=1}^r \{J^\alpha \otimes I_{n_j}\}' \tag{0.3}$$

$$Lat(J^\alpha \otimes B) = \oplus_{j=1}^r Lat(J^\alpha \otimes I_{n_j}) \tag{0.4}$$

$$Hyplat(J^\alpha \otimes B) = \oplus_{j=1}^r Hyplat(J^\alpha \otimes I_{n_j}) \tag{0.5}$$

with the conditions

$$\arg \lambda_j \neq \arg \lambda_i, \qquad 1 \le i \neq j \le r \tag{0.6}$$

being imposed on the matrix $B = \mathrm{diag}(\lambda_1 I_{n_1}, \dots, \lambda_r I_{n_r})$. It turns out, that each of the relations (0.3)-(0.5) is equivalent to the condition (0.6). In particular, the relation (0.4) provides a complete description of Lat$(J^\alpha \otimes B)$ under the condition (0.6) if only $n_1 = \ldots = n_r = 1$. When the condition (0.6) is violated, we present a simple criterion (in geometric terms) for a subspace $E \subset L_p[0,1] \otimes \mathbb{C}^n$ to be a cyclic subspace for the operator $A = J^\alpha \otimes B$.

In §3 a description of Hyplat$A$ is obtained for an arbitrary nonsingular matrix B. Simultaneously a description of the commutant $\{A\}'$ is presented.

In §4 some sufficient conditions are obtained for the similarity in $L_p[0,1] \otimes \mathbb{C}^n$ of an integral Volterra operator K of the form (0.1) to $J \otimes B$.

In §5some sufficient conditions for an operator K of the form (0.1) to be similar to the operator $J^\alpha \otimes B$ with Re$\alpha > 0$ in $L_p[0,1] \otimes \mathbb{C}^n$ are announced.

The proofs will be published elsewhere. We observe only that in the previous results on this theme an exponent α was assumed to be positive.

Combining the results of §§1-3 with the just mentioned results one arrives at a description of the lattices Lat$K$ and Hyplat$K$ for an operator K satisfying the similarity conditions.

In §6 the property of a Volterra operator K to be cyclic in $L_p[0,1] \otimes \mathbb{C}^n$ $(n \ge 1)$ or to be unicellular in $L_p[0,1]$ is studied. An answer to the Gohberg and Krein's conjecture on the equivalence of unicellularity and cyclicity for dissipative Volterra operators (see [GK, p.421]) is provided.

Further, in §7 we obtain two kinds of results devoted to inverse problems for systems of the type (0.2). Namely, for some systems, including the Dirac system, we prove the unique recovery of the potential matrix $Q(x)$ from a part of the monodromy matrix. For $n = 2$ this result amounts to the unique recovery of $Q(x)$ from two spectra and may be considered as a further generalization of the Borg's theorem [Bo]. The question of the unique recovery of the potential from mixed gived data is also studied. Namely, Theroems 7.3, 7.5 show that a "twise" less part of the monodromy matrix uniquely determines the system (0.2) provided the potential matrix $Q(x)$ is known over a fraction of the interval $[0,1]$. In Theorem 7.6 we generalize the Hochstadt-Liberman's result [HL] to the case of a matrix Sturm-Liouville operator. Such a type of inverse problems has been investigated in numerous papers. Quite recently Gesztesy and Simon [GS2] posed and solved a new type of inverse spectral problems beeing a further developement of the result from [HL]. Note also, that Theorems 7.3-7.6 are close to the results from [GS2].

Finally, §8 is devoted to the stability in inverse problems, considered in §7. We show that the matrix potential $Q(x)$ continuously depends on the part of the monodromy matrix as well as on another given data. The case of Hamiltonian system is also considered.

It should be pointed out, that the unique recovery of the Hamiltonian system from the monodromy matrix amounts to the real unicellularity of the auxiliary Volterra operator (see [BL]).

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§1. Cyclic subspaces.

1.1. Let R be any commutative ring, in general without unity, let $A=(a_{ij})$ be an arbitrary $n\times n$ matrix with entries $a_{ij}\in R$ and $\widetilde{A}$ be the ajoint matrix. Many of the well-known properties of matrices considered over a field remain valid for matrices with entries from R. In particular, the well-known Binet-Cauchy formula for the minors of the product AB of such two matrices A and B is valid. We note also that the following formula

$$A\cdot\widetilde{A}=\widetilde{A}\cdot A=\det A\cdot I_n=:\Delta_A\cdot I_n \tag{1.1}$$

holds. Here Δ_A is the determinant of the matrix A and I_n is the $n\times n$ identity matrix. It follows that the rows and columns of A are linearly dependent over R iff $\det A$ is a zero divisor in R.

1.2. It is well-known that the space $L_2[0,1]$ with the convolution product

$$(f*g)(x)=\int\limits_0^x f(x-t)g(t)dt=\int\limits_0^x g(x-t)f(t)dt=(g*f)(x) \tag{1.2}$$

is a commutative Banach algebra without unity. The description of zero divisors in $L_2[0,1]$ is given by the Titchmarsh convolution theorem [B2], [GK]:

(1.3)

$$(f*g)(x)=0\ \ \forall x\in[0,1]\Longrightarrow \operatorname{supp}(f)\subset[\alpha,1]\,,\ \operatorname{supp}(g)\subset[\beta,1] \text{ and } \alpha+\beta\geq 1$$

In particular, $f(x)$ is not a zero divisor in $L_p[0,1]$ if and only if

$$(1.4) \qquad \int_0^{\varepsilon} |f(x)|^p dx > 0 \ , \quad \forall \varepsilon > 0$$

This condition will be referred to as the ε-condition.

The determinant of a functional matrix $F(x) = (f_{ij}(x))_{i,j=1}^n$ ($f_{ij} \in L_2[0,1]$) calculated with respect to the convolution product (1.2) will be called the $*$-determinant and will be denoted by $* - \det F(x)$. Similarly the $*$-minors of $F(x)$ are the minors with respect to the convolution product. Finally $* - rankF(x)$ will be the highest order of $*$-minors of $F(x)$ satisfying the ε -condition .

1.3. Let B be an $n \times n$ diagonal matrix with eigenvalues λ_j $(1 \le j \le r)$ of equal arguments:

$$(1.5) \quad B = \oplus_{j=1}^r \lambda_j I_{n_j} \in [\mathbb{C}^n] \, , \ \arg\lambda_j = \arg\lambda_1 \quad \forall j \in \{1, \dots, r\} \, , \, n = n_1 + \dots + n_r.$$

In the following theorem we describe the cyclic subspaces of the operator $A = J^\alpha \otimes B$. We need some definitions to state the theorem.

Recall that a subspace E of some Banach space X is called a cyclic subspace for an operator $T \in [X]$ if $span\{T^n E : n \ge 0\} = X$.
A vector $f (\in X)$ is called cyclic if $span\{T^n f : n \ge 0\} = X$.
The set of all cyclic vectors of an operator T is denoted by $Cyc(T)$.
We set

$$\mu_T := \inf_E \{\dim E : E \text{ is a cyclic subspace of the operator } T \in [X]\}.$$

μ_T is called the multiplicity of the spectrum of an operator $T \in [X]$. Note that μ_T can be ∞.

It is clear that an operator T is cyclic iff $\mu_T = 1$.

Theorem 1.1. *Let $Re\alpha > 0, B$ be a matrix of the form (1.5), where $\lambda_j = \lambda_1 / s_j^\alpha$, $1 = s_1 < s_2 < \dots < s_r$, and*

$$f_{ij} = \{f_{ij1}, \dots, f_{ijn_j}\} \in L_2[0,1] \otimes \mathbb{C}^{n_j} \qquad 1 \le i \le N \, , \qquad 1 \le j \le r$$

Then $\mu_A = n$ and the system $\{f_i\}_{i=1}^N$ of vectors $f_i = \oplus_{j=1}^r f_{ij}$ generates a cyclic subspace in the space $L_2[0,1] \otimes \mathbb{C}^n$ for the operator

$$(1.6) \qquad A = J^\alpha \otimes B = \oplus_{j=1}^r \lambda_j J^\alpha \otimes I_{n_j}$$

if and only if $N \ge n$ and the matrices

$$(1.7) \qquad F_p(x) = \begin{pmatrix} f_{11}(s_1 x) & f_{12}(s_2 x) & \dots & f_{ip}(s_p x) \\ \dots & \dots & \dots & \dots \\ f_{N1}(s_1 x) & f_{N2}(s_2 x) & \dots & f_{Np}(s_p x) \end{pmatrix}$$

are of maximal $* - ranks$ for all $p \in \{1, \dots, r\}$,namely

$$(1.8) \qquad * - rankF_p(x) = n_1 + n_2 + \dots + n_p \qquad \forall p \in \{1, \dots, r\}$$

Proof. *Sufficiency.* Let $N \geq n = n_1 + n_2 + \ldots + n_r$ and the conditions (1.8) be satisfied.

The resolvent $(I - \lambda\lambda_k J^\alpha)^{-1}$ of the operator $\lambda_k J^\alpha$ is

$$(1.9) \qquad (I - \lambda\lambda_k J^\alpha)^{-1} h = h(x) + \lambda\lambda_k \int_0^x (x-t)^{\alpha-1} E_{1/\alpha}(\lambda\lambda_k (x-t)^\alpha; \alpha) h(t) dt$$

where $h = h(x) \in L_2[0,1]$ and

$$(1.10) \qquad E_{1/\alpha}(\lambda(x-t)^\alpha; \alpha) = \sum_{k=0}^{\infty} \frac{\lambda^k (x-t)^{\alpha k}}{\Gamma(\alpha + \alpha k)}$$

is the Mittag-Leffler function.

Let $g = \oplus_{j=1}^r g_j$, where $g_j = (g_{j1}, \ldots, g_{jn_j}) \in L_2[0,1] \otimes \mathbb{C}^{n_j}$ be a certain vector ortogonal to the subspace $H = span\{A_k f_i : k \geq 0\,, 1 \leq i \leq N\}$. Let us show that $g = 0$.

Actually, it follows from (1.9) and the obvious formula

$$(I - \lambda A)^{-1} = \oplus_{k=1}^r \left((I - \lambda_k \lambda J^\alpha)^{-1} \otimes I_{n_k} \right)$$

that for an arbitrary $i \in \{1, \ldots, N\}$

$$0 = \left((I - \lambda A)^{-1} f_i, g \right) = \sum_{j=1}^r \sum_{k=1}^{n_j} \int_0^1 f_{ijk}(x) \overline{g_{jk}(x)} dx$$

$$+ \sum_{j=1}^r \sum_{k=1}^{n_j} \lambda\lambda_j \int_0^1 \overline{g_{jk}(x)} dx \int_0^x (x-t)^{\alpha-1} E_{1/\alpha}(\lambda\lambda_j (x-t)^\alpha; \alpha) f_{ijk}(t) dt$$

$$= \sum_{j=1}^r \sum_{k=1}^{n_j} \left[\int_0^1 f_{ijk}(x-t) \overline{g_{jk}(x)} dx + \lambda\lambda_j \int_0^1 t^{\alpha-1} E_{1/\alpha}(\lambda\lambda_j t^\alpha; \alpha) dt \int_t^1 f_{ijk}(x-t) g_{jk}(x) dx. \right]$$

Making use of the change of variables $t_j = t/s_j\,, x_j = x/s_j$ $(1 \leq j \leq r)$ and denoting $\mu = \lambda\lambda_1$, we can rewrite last identity in the form

$$(1.11) \quad \sum_{j=1}^r \sum_{k=1}^{n_j} \left[\int_0^1 f_{ijk}\,(x-t)\, \overline{g_{jk}(x)}\, dx \, + \mu \int_0^1 t^{\alpha-1} E_{1/\alpha}\, (\mu\, t^\alpha; \alpha\,)\, \varphi_{ijk}(t)\, dt \right] = 0\, (1 \leq i \leq N)$$

where

$$\varphi_{ijk}(t) = \begin{cases} s_j^{\alpha+1} \int_t^{s_j^{-1}} f_{ijk}((x-t)s_j) g_{jk}(x s_j) dx & \forall t \in [o, s_j^{-1}]; \\ 0 & \forall t \in [s_j^{-1}, 1]. \end{cases}$$

It is easy to see that the system (1.11) is equivalent to the following one

$$\sum_{j=1}^{r}\sum_{k=1}^{n_j}\varphi_{ijk}(t)=0 \quad \forall t\in[0,s_1] \qquad (1\le i\le N). \tag{1.13}$$

We'll show step by step that (1.13) implies the equalities

$$g_{jk}(x)=0 \quad \forall x\in[0,1]\,,\, j\in\{1,\dots,r\}\,,\; k\in\{1,\dots,n_j\}. \tag{1.14}$$

On the first step we consider the system (1.13) for $t\in[s_2^{-1},1]$ and prove that

$$g_{1k}=0 \quad \forall\, x\in[s_2^{-1},1]\,,\;\; k\in\{1,\dots,n_1\}. \tag{1.15}$$

Actually, for $t\in[s_2^{-1},1]$ the system (1.13) takes the form

$$\sum_{k=1}^{n_1}\int_t^1 f_{i1k}\,(x-t)\;g_{1k}\,(x)\;dx\;=0\;,\;\; t\in[s_2^{-1},1)\;(1\;\le i\le\, n_1). \tag{1.16}$$

According to the first of the conditions (1.8) there exists an $n_1\times n_1$-submatrix $\widetilde{F}_1(x)$ of the matrix $F_1(x)$ such that $*-rank\,\widetilde{F}_1(x)\;=\;*-rank\,F_1(x)\;=\;n_1$, i.e. $\Delta_1(x)\;=\;*-\det\widetilde{F}_1(x)$ is not a zero divisor in $L_2\,[0,1]$. Let for the definitness

$$\widetilde{F}_1(x)\;=\;\begin{pmatrix} f_{11}(x)\\ \dots\\ f_{n_11}(x)\end{pmatrix}\;=\;\begin{pmatrix} f_{111}(x) & f_{112}(x) & \dots & f_{11n_1}(x)\\ \dots & \dots & \dots & \dots\\ f_{n_111}(x) & f_{n_112}(x) & \dots & f_{n_11n_1}(x)\end{pmatrix}$$

be such a matrix. Denoting by $A_{i1p}(x)$ the $*-$cofactor of an element $f_{i1p}(x)$ $(1\le p\le n_1)$ of the matrix $\widetilde{F}_1(x)$, we obtain from (1.16) and (1.1)

$$0\;=\;\sum_{i=1}^{n_1}\sum_{k=1}^{n_1}\int_y^1 A_{i1p}(t-y)\,dt\int_t^1 f_{i1k}\;(x-t)\;\overline{g_{1k}}\;(x)\;dx\;\;=$$

$$=\;\sum_{i=1}^{n_1}\sum_{k=1}^{n_1}\int_y^1\overline{g_{1k}}\;(x)\;dx\int_y^x A_{i1p}\,(t-y)\,f_{i1k}\;(x-t)\;dt\;\;=$$

$$=\;\sum_{k=1}^{n_1}\int_y^1\overline{g_{1k}}\;(x)\;dx\int_0^{x-y}\sum_{i=1}^{n_1}A_{i1p}\,(x-y-t)f_{i1k}\,(t)\;dt\;\;=$$

$$=\;\int_y^1\overline{g_{1p}}\,(x)\,\Delta_1\,(x-y)\,dx \qquad \forall\,y\in[s_2^{-1},1]\;\;(1\;\le p\le\;n_1) \tag{1.17}$$

Since $\Delta_1(x)$ satisfies the ε-condition (1.4), it follows from the Titchmarsh convolution theorem and (1.17) that the equalities (1.15) are fulfilled. Further, suppose by induction, that on the (m-1)-th step the relations

$$g_{jk}(x) = 0 \quad \forall\, x \in [s_m^{-1} s_j^{-1}, 1],\ j \in \{1, \dots, m-1\},\ k \in \{1, \dots, n_j\} \tag{1.18}$$

are established. Then for $t \in [s_{m+1}, s_m]$ the system (1.13) may be rewritten in the form

$$\sum_{j=1}^{m} \sum_{k=1}^{n_j} (s_j^{\alpha+1}) \int_{l}^{s_m^{-1}} f_{ijk}\ ((x-t)s_j)\ \overline{g_{jk}}\,(xs_j)\ dx \ = \ 0. \tag{1.19}$$

Let $N_m = n_1 + \dots + n_m$ and $\widetilde{F}_m(x)$ be an $N_m \times N_m$ submatrix of the matrix $F_m(x)$ such that $* - rank\ \widetilde{F}_m(x) \ = \ * - rank\ F_m(x) \ = \ N_m$. Without loss of generality we may assume that $\widetilde{F}_m(x)$ consists of the first N_m rows of the matrix $F_m(x)$. Denoting by $A_{ijp}(x)$ the $*$-cofactor of an element $f_{ipq}(xs_p)$ $(1 \le i \le N_m,\ 1 \le p \le m,\ 1 \le q \le n_p)$ of $\widetilde{F}_m(x)$ and repeating the arguments used above, we obtain from (1.19) and (1.11)

$$0 = \sum_{i=1}^{N_m} \sum_{j=1}^{m} \sum_{k=1}^{n_j} \int_0^{s_m^{-1}} A_{ipq}\,(t-y)\,dt \int_t^{s_m^{-1}} s_j^{\alpha+1}\ f_{ijk}\ ((x-t)s_j)\ \overline{g_{jk}}\ (xs_j)\ dx \ =$$

$$= \sum_{j=1}^{m} \sum_{k=1}^{n_j} \int_y^{s_m^{-1}} \overline{g}_{jk}\ (xs_j)\ dx \int_0^{x-y} \sum_{i=1}^{N_m} s_j^{\alpha+1}\ A_{ipq}\,(x-y-t)\,f_{ijk}\,(ts_j)\,dt \ =$$

$$= \int_y^{s_m^{-1}} \overline{g_{pq}\,(xs_p)}\,\Delta_m(x-y)\,dy \quad \forall\, y \in [s_{m+1}^{-1}, s_m^{-1}]. \tag{1.20}$$

Since $\Delta_m \ = \ * - \det\, \widetilde{F}_m(x)$ is not a zero divisor in $L_2\,[0,1]$, it follows from (1.20) that

$$g_{jk}(x) \ = \ 0 \ \ \forall\, x \in \ [s_{m+1}^{-1} s_j^{-1}, s_m^{-1} s_j^{-1}],\ \ 1 \le j \le m,\ 1 \le k \le n_j. \tag{1.21}$$

The equalities (1.21), (1.18) prove the induction hypothesis.

Necessity. Let the subspace $E \ = \ span\{f_i \ : \ 1 \le i \le N\}$ be cyclic for the operator A. At first we suppose that $N \ge n$ and show by contradiction that the conditions (1.8) hold. Let one of the conditions (1.8) be violated. Assume, for example, that $* - rank F_r(x)$ is less then n and denote by $F_r \begin{pmatrix} i_1 & i_2 & \dots & i_n \\ 1 & 2 & \dots & n \end{pmatrix}$ the $n \times n$ $*$-minor of the $N \times n$ matrix $F_r(x)$ (here $i_1 < i_2 < \dots < i_n$ are the numbers of rows). Since each such a minor is a zero divisor in $L_2[0,1]$, there exists $l \in (0,1]$, such that

$$(1.22)\quad F_r\begin{pmatrix} i_1 & i_2 & \dots & i_n \\ 1 & 2 & \dots & n \end{pmatrix}(x) = 0 \ \ \forall\, x \in [0,l]\,,\ 1 \le i_1 \le i_2 \le \ldots \le i_n \le N.$$

Further, for any n vectors $g_i \in H = span\{A^m f_k : m \ge 0\,,\ 1 \le k \le N\}$ and for an arbitrary $\varepsilon > 0$ there exist $n \cdot N$ polynomials $P^{\varepsilon}_{ik}(t)$ of one variable such that

$$\left\| \sum_{k=1}^{N} P^{\varepsilon}_{ik}(A)\, f_k - g_i \right\|_{L_2^n[0,1]} < \varepsilon \quad , \quad P^{\varepsilon}_{ik}(t) = \sum_{m=0}^{d_{ik}} a_{ikm}\, t^m \ \ (1 \le i \le n).$$

Setting
(1.23)

$$g^{\varepsilon}_{ij} = \sum_{k=1}^{N} P^{\varepsilon}_{ik}\left(s_j^{-\alpha} J^{\alpha} \otimes I_{n_j}\right) f_{kj}, \qquad \widetilde{P}^{\varepsilon}_{ik}(t) = \sum_{m=0}^{d_{ik}} a_{ikm}(t^{m\alpha-1}/\Gamma(m\alpha))\,,$$

we obtain

$$(1.24)\quad \begin{aligned} g^{\varepsilon}_{ij}\,(xs_j) &= \sum_{k=1}^{N}\sum_{m=0}^{d_{ik}} a_{ikm}\, s_j^{-\alpha m} \int_0^{xs_j} \frac{(xs_j - t)^{m\alpha-1}}{\Gamma(\alpha m)}\, f_{kj}(t)\, dt \\ &= \sum_{k=1}^{N}\sum_{m=0}^{d_{ik}} a_{ikm}\, s_j^{-\alpha m} \int_0^{x} \frac{(x - t)^{m\alpha-1} s_j^{m\alpha-1}}{\Gamma(\alpha m)}\, f_{kj}\ (ts_j)\, s_j dt \\ &= \sum_{k=1}^{N} \widetilde{P}^{\varepsilon}_{ik}(t) * f_{kj}\,(ts_j) \end{aligned}$$

Let $g^0_i = g_j = \oplus_{j=1}^{r} g_{ij}$,

$$(1.25)\qquad G^{\varepsilon}_r(x) = G_r(g^{\varepsilon}_1, \dots, g^{\varepsilon}_n) = \begin{pmatrix} g^{\varepsilon}_{11}(xs_1) & g^{\varepsilon}_{12}(xs_2) & \dots & g^{\varepsilon}_{1r}(xs_r) \\ \dots & \dots & \dots & \dots \\ g^{\varepsilon}_{n1}(xs_1) & g^{\varepsilon}_{n2}(xs_2) & \dots & g^{\varepsilon}_{nr}(xs_r) \end{pmatrix}$$

be the $n\times n$-matrix-function, consisting of the vectors $g^{\varepsilon}_i = \oplus_{j=1}^{r} g^{\varepsilon}_{ij} = \sum_{1\le k\le N} P_{ik}(A) f_m$, and $P^{\varepsilon}(x) = \left(\widetilde{P}^{\varepsilon}_{ik}(x)\right)$ be the $n \times N$- matrix-function with entries $\widetilde{P}^{\varepsilon}_{ik}(x)$. It follows from (1.24), (1.25) and (1.7) that

$$G^{\varepsilon}_r(x) = P^{\varepsilon}(x) * F_r(x)$$

and therefore in view of the Binet-Cauchy formula

$$* - \det G^{\varepsilon}_r(x) = \sum_{i_1<\ldots<i_n} P^{\varepsilon}\begin{pmatrix} 1 & 2 & \dots & n \\ i_1 & i_2 & \dots & i_n \end{pmatrix} \cdot F_r\begin{pmatrix} i_1 & i_2 & \dots & i_n \\ 1 & 2 & \dots & n \end{pmatrix}$$

where $P^\varepsilon\begin{pmatrix}1 & 2 & \dots & n\\ i_1 & i_2 & \dots & i_n\end{pmatrix}$ are the $n\times n - *$-minors of the matrix $P^\varepsilon(x)$. Consequently, by virtue of (1.22) we have

$$(1.26)\qquad * - \det G_r^\varepsilon(x) = 0 \quad \forall x\in[0,l] \quad \forall\varepsilon>0$$

This property is preserved when ε tends to zero:

$$(1.27)\qquad * - \det G_r(g_1,\dots,g_n)(x) = 0 \quad \forall x\in[0,l].$$

Thus, the invariant subspace H is not all of $L_2[0,1]\otimes\mathbb{C}^n$, because, for example, for the vectors $g_i = \{\delta_{ij}\}_{j=1}^n(\delta_{ij}$ is Kronecker delta) we have $G_r(g_1,\dots,g_n) = \underbrace{1*1*\dots*1}_{n} = \frac{x^{n-1}}{(n-1)!}$.

It remains to note that, in the case $N<n$, we can adjoin zero vectors $f_{N+1} = \dots = f_n = 0$ to the system $\{f_i\}_1^N$ and, repeating previous arguments, obtain the equality (1.27) for an arbitrary system $\{g_i\}_1^n$ of vectors $g_i\in L_2[0,1]\otimes\mathbb{C}^n$. □

Corollary 1.1. *The system $\{f_i\}_{i=1}^N$ of vectors*

$$(1.28)\qquad f_i(x) = \{f_{i1}(x),\dots,f_{in}(x)\}\in L_2[0,1]\otimes\mathbb{C}^n \quad (1\le i\le N)$$

generates a cyclic subspace in $L_2[0,1]\otimes\mathbb{C}^n$ for the operator $A = J^\alpha\otimes I_n = \underbrace{J^\alpha\oplus\dots\oplus J^\alpha}_{n}$ if and only if the matrix function

$$(1.29)\qquad F_n(x) = \begin{pmatrix} f_{11}(x) & f_{12}(x) & \dots & f_{1n}(x)\\ \dots & \dots & \dots & \dots\\ f_{N1}(x) & f_{N2}(x) & \dots & f_{Nn}(x)\end{pmatrix}$$

is of maximal $*-rank$, that is, $*-rank F_n(x) = n$.

In particular, if $N=n$ the subspace $E = span\{f_i(x) : 0\le i\le n\}$ will be a cyclic subspace for $A = J^\alpha\otimes I_n$ iff $* - \det F_n(x)$ satisfies the ε-condition (1.4).

1.4. The following result is the main result of the section. It may be proven in just the same manner as Theorem 1.1.

Theorem 1.2. *Let*

$$K:\ f\to\int_0^x k(x-t)f(t)dt, \qquad k(x)\in L_1[0,1]$$

be a unicellular operator in $L_p[0,1]$ for some $p\in[1,\infty)$ and let $A = K\otimes I_n$. Then $\mu_A = n$ and a system $\{f_i\}_{i=1}^N$ of vectors $f_i = \oplus_{j=1}^r f_{ij}$ generates a cyclic subspace in $L_p[0,1]\otimes\mathbb{C}^n$ for the operator A iff $N\ge n$ and the conditions (1.8) for the matrices (1.7) are satisfied.

Remark 1.1. In the scalar case Theorem 1.1 (as well as Corollary 1.1) gives a description of the set $\mathrm{Cyc}(J^\alpha)$ of the operator J^α acting in $L_p[0,1]$:

$$f\in\mathrm{Cyc}(J^\alpha)\Leftrightarrow f \text{ satisfies the } \varepsilon \text{ - condition (1.4).}$$

This fact is well-known and is equivalent to the unicellularity of J^α (see [B2], [GK], [N1], [FR2]). There are two essentially different proofs of this fact concerning J in the literature: one is based on the Titchmarsh's convolution theorem (see, for example [GK]), the other, suitable only for the space $L_2[0,1]$ uses either the Livsic's theory of characteristic functions or the Nagy-Foias functional model [N1], [NF], [S1] (see also [FR2] and papers cited therein).

In the present paper we develop the first approach, though Corollary 1.4 for the space $L_2[0,1]$ can be also proved just like it was done by Sarason in the proof of unicellularity of J([S1]). But the proof we give is preferable for the following reasons: a) it is suitable for J^α with $\alpha \neq 1$, for the operator (1.6) with distinct λ_j and even for more general convolution operators; b) it is suitable for Banach function spaces like $L_p[0,1](1 \le p < \infty)$, $C[0,1], W_p^k[0,1]$.

§2 The lattices of invariant subspaces.

2.1. Recall that an operator $T \in [X]$ is called unicellular if its lattice of invariant subspaces $\mathrm{Lat}T$ is linearly ordered. It is well-known that the operator J^α acting in $L_2[0,1]$ is unicellular and

$$\mathrm{Lat}J^\alpha = \{L_2[a,1] : 0 \le a \le 1\}, \tag{2.1}$$

i.e. Lat J^α is anti-izomorphic to the segment $[0,1]$. In (2.1) we identify $L_2[a,1]$ with the subspace E_a of $L_2[0,1]$, such that $E_a = \{f \in L_2[0,1] : f(x) = 0 \ \ \forall x \in [0,a]\}$.

In this section some cases of disintegration of the lattice of invariant subspaces of an operator $A = J^\alpha \otimes B$ are investigated.

This result reduces the description of Lat$A$ to the cases described in Theorem 1.1 and (2.1).

Proposition 2.1. *Let $\alpha > 0$, B be a diagonal matrix:*

$$B = \oplus_{j=1}^r B_j \ \ , \qquad B_j = diag(\lambda_{j1}, \dots, \lambda_{jn_j}) \quad (1 \le j \le r) \tag{2.2}$$

where $\lambda_{jk} = \lambda_{j1}/s_{jk}^\alpha$, $1 = s_{j1} \le s_{j2} \le \dots \le s_{jn_j}$ and arg $\lambda_{j1} \neq arg\lambda_{k1}(\mod 2\pi)$ for $j \neq k$. Let also $A = J^\alpha \otimes B \in [L_2[0,1] \otimes \mathbb{C}^n]$, where $n = n_1 + \dots + n_r$. Then

$$\mu_A = \max_{1 \le j \le r} n_j \qquad \text{and} \qquad LatA = \oplus_{j=1}^r Lat(J^\alpha \otimes B_j) \tag{2.3}$$

Proof. Let π_j be the orthogonal projection from $L_2[0,1] \otimes \mathbb{C}^n$ onto $L_2[0,1] \otimes \mathbb{C}^{n_j}$ in the decomposition $L_2^n[0,1] = \oplus_{j=1}^r (L_2[0,1] \otimes \mathbb{C}_{n_j})$. For an arbitrary nontrivial invariant subspace $H(\subset L_2[0,1])$ of the operator A we introduce the subspaces

$$H_j = span\{(J^\alpha \otimes B_j)^k f_j : k \ge 0 \,, \ \ f_j \in \pi_j H\} \ , \quad 1 \le j \le r. \tag{2.4}$$

It is clear that $H \subseteq \oplus_{j=1}^r H_j$. Establish the opposite inclusion.

For arbitrary vectors $f = \oplus_{j=1}^r f_j \in H$ and $g = \oplus_{j=1}^r g_j \in H^\perp$ where

$$f_j = \{f_{j1}, \dots, f_{jn_j}\}, \ \ g_j = \{g_{j1}, \dots, g_{jn_j}\} \in L_2[0,1] \otimes \mathbb{C}^{n_j}$$

let us define the functions

$$
\varphi_{jk}(t) = \begin{cases} s_{jk}^{\alpha+1} \int\limits_{t}^{s_{jk}^{-1}} f_{jk}((x-t)s_{jk}) g_{jk}(x s_{jk}) dx & \forall t \in [o, s_{jk}^{-1}]; \\ 0 & \forall t \in [s_{jk}^{-1}, 1]. \end{cases} \tag{2.5}
$$

and

$$
\varphi_j(t) = \sum_{k=1}^{n_j} \varphi_{jk}(t) \qquad (1 \le j \le r).
$$

Since $R_\lambda(A)f \in H \quad \forall f \in H$ and

$$
R_\lambda(A) = (I - \lambda A)^{-1} = \oplus_{j=1}^{n} (I - \lambda J^\alpha \otimes B_j)^{-1} \tag{2.6}
$$

it follows from (1.9),(2.5) and (2.6) that

$$
\begin{aligned}
0 &= \sum_{j=1}^{r} \sum_{k=1}^{n_j} \int_0^1 g_{jk}(x)dx \int_0^x (x-t)^{\alpha-1} E_{1/\alpha}(\lambda\lambda_{jk}((x-t)^\alpha;\alpha) f_{jk}(t)dt = \\
&= \sum_{j=1}^{r} \sum_{k=1}^{n_j} \int_0^1 t^{\alpha-1} E_{1/\alpha}(\lambda\lambda_j s_{jk}^{-\alpha} t^\alpha;\alpha)dt \int_t^1 f_{jk}(x-t) g_{jk}(x)dx = \\
&= \sum_{j=1}^{r} \int_0^1 t^{\alpha-1} E_{1/\alpha}(\lambda\lambda_j t^\alpha;\alpha)\varphi_j(t)dt =: \sum_{j=1}^{n} \Phi_j(\lambda).
\end{aligned} \tag{2.7}
$$

Here $\lambda_j := \lambda_{j1}$ for $(1 \le j \le r)$. We will obtain the inclusion $H \supset \oplus_{j=1}^{r} H_j$ proving the implication

$$
\sum_{j=1}^{r} \Phi_j(\lambda) = 0 \Longrightarrow \Phi_j(\lambda) = 0 \quad \forall j \in \{1, \dots, r\} \tag{2.8}
$$

Assuming the contrary we establish that there exists a ray such that on this ray one of the summands in (2.6) is of larger growth than the others. To indicate this ray we use the asymptotic behavior of the function $E_{1/\alpha}(z,\alpha)$ considering the cases $\alpha \ge 2$ and $0 < \alpha < 2$ separately.

i) In the case $2 \le \alpha < \infty$ the following asymptotic formula

$$
E_{1/\alpha}(z,\alpha) = \sum_k \left(z^{1/\alpha}\exp(i2\pi k\alpha^{-1})\right)^{1-\alpha} \cdot \exp\left(z^{1/\alpha} e^{i\frac{2\pi k}{\alpha}}\right) + O(|z|^{-2}) \tag{2.9}
$$

holds as $|z| \longrightarrow \infty$. (see[Dj]). Here the summation is over those $k \in \mathbb{Z}$ for which $|\arg z + 2\pi n| \le \pi\alpha/2$.

It is easily seen that the right hand side of (2.8) does not depend on the choice of a branch of the multifunction $z^{1/\alpha}$.

Denoting by $[a_j, b_j]$ the minimal segment containing the support of the function $\varphi_j(t)$ we suppose for the definitness that

$$b_1|\lambda_1|^{1/\alpha} = \max\{b_j|\lambda_j|^{1/\alpha} \; : \; 1 \le j \le r\} \tag{2.10}$$

We will show that the ray $\Lambda = \{\lambda \in \mathbb{C} : \arg\lambda = -\arg\lambda_1\}$ is the desired one. Substituting $z = (xt)^\alpha \lambda_j \lambda_1^{-1}$ $(x, t \in \mathbb{R}_+)$ into (2.8) and letting

$$\beta_1 = 0\,, \quad \beta_j = \arg\lambda_j\lambda_1^{-1}(\neq 0)\,, \quad 2 \le j \le r, \tag{2.11}$$

we note that in view of (2.10)

$$\varepsilon := 1 - \max_{j,k}\left\{\cos\frac{\beta_j + 2\pi k}{\alpha} \; : \; |\beta_j + 2\pi k| \le \pi\alpha/2\,,\, 2 \le j \le r\right\} > 0. \tag{2.12}$$

Now it follows from (2.11) and (2.12) that
(2.13)

$$\int_0^1 t^{\alpha-1}E_{1/\alpha}((xt)^\alpha\lambda_j\lambda_1^{-1};\alpha)\varphi_j(t)dt = \int_{a_j}^{b_j} t^{\alpha-1}E_{1/\alpha}((xt)^\alpha\lambda_j\lambda_1^{-1};\alpha)\varphi_j(t)dt =$$

$$= O\left(x^{1-\alpha}\sum_k \exp\left(xb_j\left|\frac{\lambda_j}{\lambda_1}\right|^{1/\alpha}\cos\left(\frac{\beta_j + 2\pi k}{\alpha}\right)\right)\right) = O(\exp(b_1(1-\varepsilon)x)\,(2 \le j \le r)$$

where the summation is taken over those $k \in \mathbb{Z}$ for which $|\arg\beta_j + 2\pi k| \le \pi\alpha/2$.

At the same time from (2.11) we see that asymptotically as $x \longrightarrow \infty$

$$\begin{aligned} \int_{a_1}^{b_1} t^{\alpha-1}E_{1/\alpha}((xt)^\alpha;\alpha)\varphi_1(t)dt &= \sum_{|k|\le\alpha/4} x^{1-\alpha}\int_{a_1}^{b_1}\exp\left(xt\cos\frac{2\pi k}{\alpha}\right)\varphi(t)dt+ \\ &+o(1) = x^{1-\alpha}\int_{a_1}^{b_1} e^{xt}\varphi_1(t)dt + O\left(e^{b_1cx}\right) \end{aligned} \tag{2.14}$$

where

$$c := \max\{\cos 2\pi k/\alpha \; : \; k \in \mathbb{Z}\setminus 0\,,\; |k| \le \alpha/4\} < 1. \tag{2.15}$$

Since both functions $\varphi_1(t - a_1)$ and $\varphi_2(b_1 - t)$ satisfy the ε-condition (1.4), the indicator diagram of the function

$$F(z) = \int_{a_1}^{b_1} e^{zt}\varphi_1(t)dt \tag{2.16}$$

is the segment $[a_1, b_1] \subseteq [0,1]$ (see [L]). Therefore the indicator function $h_F(\theta) = \overline{\lim}_{r\to\infty} r^{-1}\ln|F(re^{i\theta})|$ is of the form

$$h_F(\theta) = 2^{-1}(b_1 - a_1)|\cos\theta| + 2^{-1}(a_1 + b_1)\cos\theta$$

and consequently

$$\overline{\lim}_{r\to+\infty}\frac{\ln|F(r)|}{r}=h_F(0)=b_1 \tag{2.17}$$

On the other hand it follows from (2.6) and (2.13)-(2.15) that

$$\overline{\lim}_{r\to+\infty}\frac{\ln|F(r)|}{r}\le b_1(1-\varepsilon). \tag{2.18}$$

Comparing the relations (2.17) and (2.18) we obtain $b_1=0$ and bearing in mind (2.9) we see that $b_j=0\quad\forall j\in\{1,\dots,r\}$. This means that $\varphi_j(t)=0\quad\forall j\in\{1,\dots,n\}$ and hence

$$\left((I-\lambda J^\alpha\otimes B_j)^{-1}f_j,g_j\right)=0\qquad\forall f\in\pi_jH$$

that is $g_j\in H_j^\perp$. Thus $H=\oplus_{j=1}^r H_j$.

ii) Let now $0<\alpha<2$. Then for arbitrary γ, $\pi\alpha/2<\gamma<\min\{\pi,\pi\alpha\}$ we have the following asymptotic formulas as $|z|\to\infty$ [Dj]:

$$\begin{aligned} E_{1/\alpha}(z;\alpha)&=\frac{1}{\alpha}z^{(1-\alpha)\alpha^{-1}}\exp\left(z^{1/\alpha}\right)+O\left(\frac{1}{|z|^2}\right), \quad |\arg z|\le\gamma\\ E_{1/\alpha}(z;\alpha)&=O\left(\frac{1}{|z|^2}\right),\qquad \gamma\le|\arg z|\le\pi\end{aligned} \tag{2.19}$$

Taking into account (2.19) one can repeat previous arguments using the estimates

$$t^{\alpha-1}E_{1/\alpha}(\lambda_1^{-1}\lambda_j(xt)^\alpha;\alpha)=O\left(\frac{1}{x^2}\right)$$

instead of (2.1) when $|\beta_j|\in[\gamma,\pi]$. One can complete the proof as for the case i). $\square$

Remark 2.1. a) If $\alpha=p(\in\mathbb{Z}_+)$ is an integer a prooof immediately follows from a simple fact about the sum of entire functions. Inded, in this case setting $\lambda=\mu^p$ one can easily obtain from (1.10) the following explicit formula

$$t^{p-1}E_{1/p}\left((\mu t)^p;p\right)=\frac{1}{\mu^{p-1}}\sum_{k=0}^{\infty}\frac{(\mu t)^{kp+p-1}}{(kp+p-1)!}=\frac{1}{p\mu^{p-1}}\sum_{k=1}^{p}\varepsilon_k e^{\varepsilon_k\mu t} \tag{2.20}$$

instead of the asymptotic formulas (2.8),(2.19). Here in (2.20) $\varepsilon_k=\exp(i2\pi k/p)$ are distinct roots of unity. It follows from (2.20) and the well-known Polya theorem[L] that each entire function

$$\Phi_j(\mu)=\int_0^1 t^{p-1}E_{1/p}(\lambda_j(\mu t)^p;p)\varphi(t)dt=\frac{1}{p\mu^{p-1}}\sum_{k=1}^{p}\varepsilon_k\int_{a_j}^{b_j}e^{\varepsilon_k\mu t}\varphi_j(t)dt$$

is of exponential type b_j and its indicator diagram is the polygon with vertices

$$b_j|\lambda_j|\exp(-i2\pi k/p)\ , \quad 1 \le k \le p$$

To obtain the implication

$$\sum_{1\le j\le n} \Phi_j(\mu) \equiv 0 \Longrightarrow \Phi_j(\mu) = 0 \quad \forall j \in \{1,\dots,r\}$$

it remains to note that, as it follows from the Polya's Theorem , the indicator diagram of a sum of entire functions of exponential type contains the convex hull of those extreme points of each of the indicator diagrams that do not lie in any of the other diagrams [L].

b) We also mention the papers [CW] and [N2], devoted to the splitting of $\mathrm{Lat}(T_1 \oplus T_2)$. In [CW] the splitting of $\mathrm{Lat}(T_1 \oplus T_2)$ is investigated for Hilbert space contractions of class C_0, or, what is the same, for dissipative operators T_1 and T_2. It should be pointed out that the operators $\lambda_1 J$ and $\lambda_2 J$ are simultaneously dissipative if and only if $\arg\lambda_1 = \arg\lambda_2$.

In [N2] the splitting of the lattice $\mathrm{Lat}(T_1 \oplus T_2)$ is studied under the spectral independence condition: $\sigma(T_1) \cap \sigma(T_2) = \emptyset$ (see, for example, Theorem 1.75 and Corollaries 1.69, 1.71, 1.77 in [N2]).

2.2. Now we are ready to formulate the main result of the section.

Theorem 2.1. *Let $\alpha_1 > \alpha_2 > \ldots > \alpha_n > 0$, B_{kj} be $n_{kj} \times n_{kj}$ diagonal matrices with eigenvalues λ_{kji} of equal arguments :*

$$B_{kj} = diag(\lambda_{kj1},\dots,\lambda_{kjn_j}),\ \arg\lambda_{kj1} = \arg\lambda_{kji}\ (1 \le i \le n_{kj})$$

and $\arg\lambda_{kj_1 1} \ne \arg\lambda_{kj_2 1}$ if $j_1 \ne j_2$. Let also

$$B_k = \oplus_{j=1}^{n_k} B_{kj}\ ,\ \ A_k = J^{\alpha_k} \otimes B_k\ ,\ \ A = \oplus_{k=1}^{n} A_k. \tag{2.21}$$

Then

$$Lat A = \oplus_{k=1}^{n} \oplus_{j=1}^{n_k} Lat(J^{\alpha_k} \otimes B_{kj}). \tag{2.22}$$

Proof. Let

$$N_k = \sum_{1\le j\le n_k} n_{kj}\ ,\ \ N = \sum_{1\le k\le n} N_k,$$

and let π_k be the orthogonal projection from $L_2[0,1]\otimes\mathbb{C}^N$ onto $L_2[0,1]\otimes\mathbb{C}^{N_k}$ in the decomposition $L_2^N = \oplus_{k=1}^n L_2^{N_k}$. In just the same way as in the proof of Proposition 2.1 we introduce the subspaces

$$H_k = span\{(J^{\alpha_k}\otimes B_k)^p f_k : p \ge 0,\ \ f_k \in \pi_k H\},\ \ 1 \le k \le n$$

for an artbitrary invariant subspace $H(\subset L_2^N[0,1])$ of the operator A.
The equality $H = \oplus_{k=1}^n H_k$ amounts to the inclusion $H \supset \oplus_{k=1}^n H_k$ because the contrary inclusion is obvious.

Let $f = \oplus_{k=1}^{n} f_k \in H,\ g = \oplus_{k=1}^{n} g_k \in H^{\perp}$, where $f_k \in H_k,\ g_k \in L_2^{N_k}[0,1]$. When made use of, the formula

$$(I - \lambda A)^{-1} = \oplus_{k=1}^{n} (I - \lambda (J^{\alpha_k} \otimes B_k))^{-1} \tag{2.23}$$

yields

$$\sum_{k=1}^{n} ((I - \lambda (J^{\alpha_k} \otimes B_k)) f_k, g_k) = 0. \tag{2.24}$$

Since each non-zero summand on the left-hand-side of the equality (2.24) is an entire function $\Phi_k(\lambda)$ of growth α_k, then the sum $\sum_k \Phi_k(\lambda)$ has the same order and type as the summand of larger growth. Therefore the equality (2.24) is equivalent to the system

$$\Phi_k(\lambda) := \big((I - \lambda (J^{\alpha_k} \otimes B_k))^{-1} f_k, g_k\big) = 0, \quad 1 \le k \le n. \tag{2.25}$$

Thus $g = \oplus_{k=1}^{n} g_k \in H^{\perp} \Leftrightarrow g_k \in H_k^{\perp} \quad \forall k \in \{1, \dots, n\}$ and hence $H = \oplus_{k=1}^{n} H_k$. To complete the proof it remains to apply Proposition 2.1 to each of the operators $J^{\alpha_k} \otimes B_k$ acting in the space $L_2[0,1] \otimes \mathbb{C}^{N_k} \quad (1 \le k \le n)$. $\square$

Let π_{kj} be the orthogonal projection from $L_2^N = L_2[0,1] \otimes \mathbb{C}^N$ onto $L_2^{n_{kj}} = L_2[0,1] \otimes \mathbb{C}^{n_{kj}}$ in the decomposition

$$L_2^N = \oplus_{(k,j) \in P} L_2^{n_{kj}}\ , \ P := \{(k,j):\ 1 \le k \le n,\ 1 \le j \le n_k\},\ N = \mathrm{card} P. \tag{2.26}$$

The following corollary immediately follows from Theorems 1.1 and 2.1.

Corollary 2.1. *Under the conditions of Theorem 1.1 the following are true :*

1) The multiplicity μ_A satisfies

$$\mu_A = \max\{n_{kj} : 1 \le k \le n,\ 1 \le j \le n_k\}. \tag{2.27}$$

2) A subspace E in $L_2[0,1] \otimes \mathbb{C}^N$ is a cyclic subspace for the operator A of the form (2.21) if and only if for each pair $(k,j) \in P$ the subspace $E_{kj} = \pi_{kj} E$ is a cyclic subspace for the operator $A_{kj} = J^{\alpha_k} \otimes B_{kj}$.

3) A system $\{f_i\}_{i=1}^{M}$ of vectors $f_i = \oplus_{k=1}^{n} \oplus_{j=1}^{n_k} f_{ikj}$, where $f_{ikj} \in L_2[0,1] \otimes \mathbb{C}^{n_{kj}}$, generates a cyclic subspace for the operator A iff $M \ge \mu_A$ and for each pair $(k,j) \in P$ the system $\{f_{ikj}\}_{i=1}^{M}$ satisfies the conditions of Theorem 1.1.

Corollary 2.2. *Let $A = \oplus_{k=1}^{r} J^{\alpha_k} \otimes B_k,\ B_k = diag(\lambda_{k1}, \dots, \lambda_{kn_k}),\ n = n_1 + \dots + n_r$ and $\arg \lambda_{ki} \ne \arg \lambda_{kj}$ for $i \ne j$. Then the following are true:*

1) The operator A is cyclic in $L_2[0,1] \otimes \mathbb{C}^n$.

2) The lattice $LatA$ is anti-isomorphic to the cube $\square_n = \{x \in \mathbb{R}^n :\ 0 \le x_i \le 1\}$:

$$LatA = \{\oplus_{i=1}^{n} L_2[a_i, 1] :\ 0 \le a_i \le 1\} = \oplus_{i=1}^{n} LatJ^{\alpha_i} \tag{2.28}$$

3) A vector $f = \oplus_{k=1}^{n} \oplus_{j=1}^{n_k} f_{kj}$, where $f_{kj} \in L_2[0,1]$, is cyclic ($f \in C^A$) iff for each $(k,j) \in P$ the vector $f_{kj}(x)$ satisfies the ε-condition (1.4).

Remark 2.2. Setting $\alpha := \alpha_1 = \dots = \alpha_r$ Corollary 2.2 yields that the $\mathrm{Lat}(J^{\alpha} \otimes B_1)$ is of the form (2.28) with $n = n_1$. For the operator $J \otimes B_1$ (i.e. for $\alpha = 1$) a quite different proof of this fact, based on the Lavrentyev's Approximation Theorem , was obtained in [OS1], [OS2].

2.3. To state the next corollary we recall the following definition.

Definition 2.1. Let S be an involution [GK] in a Hilbert space H.

a) An operator $T(\in [H])$ is called S-real if $AS = SA$;

b) An S-real operator T is said to be S-unicellular if $\mathrm{Lat}_S T := \mathrm{Lat}T \cap \mathrm{Lat}S$ is totally ordered by st inclusion.

We will only consider the involution S of complex conjugation in $L_2[0,1]$ and the involution $S_n := S \otimes I_n$ in $L_2[0,1] \otimes \mathbb{C}^n$ given by

$$S_n = S \otimes I_n : (f_1, \dots, f_n) \to (\overline{f}_1, \dots, \overline{f}_n). \tag{2.29}$$

Corollary 2.3. *Let S_2 be the involution in $L_2[0,1] \otimes \mathbb{C}^2$ defind by (2.29), $B = \begin{pmatrix} b & d \\ -d & b \end{pmatrix}$. Then the operator $A = J^\alpha \otimes B$ is S_2-unicellular in $L_2 \otimes \mathbb{C}^2$ and*

$$Lat_S A = \{L_2[a,1] \oplus L_2[a,1] : a \in [0,1]\}.$$

Proof. Since

$$U^{-1}BU = d := \begin{pmatrix} b+id & 0 \\ 0 & b-id \end{pmatrix}, \qquad U = \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}$$

it is clear that

$$E \in \mathrm{Lat}(J^\alpha \otimes B) \Longleftrightarrow (I \otimes U^{-1})E \in \mathrm{Lat}(J^\alpha \otimes d).$$

In accordance with Proposition 2.1 there exist $a_1, a_2 \in [0,1]$ such that

$$E = \{\{f_1, f_2\} : f_1(x) - if_2(x) = 0 \quad \forall x \in [0, a_1], \quad f_1(x) + if_2(x) = 0 \forall x \in [0, a_2]\}.$$

It is easily seen that $a_1 = a_2$ if in addition $E \in \mathrm{Lat}S$. Indeed, if for example $a_1 < a_2$, one can find a pair of functions $\{f_1^0, f_2^0\} \in E$ such that

$$\mathrm{mes}\{x \in [a_1, a_2] : f_1(x)^0 + if_2(x) \neq 0\} > 0. \tag{2.30}$$

But, since $E \in \mathrm{Lat}S$

$$f_1^0(x) = -if_2^0(x), \quad \overline{f_1^0(x)} = -i\overline{f_2^0(x)} \qquad \forall x \in [0, a_2],$$

that is $f_1^0 = f_2^0 = 0 \quad \forall x \in [0, a_2]$, which contradicts (2.30). $\square$

Remark 2.3. Quite different proofs of the S_2-unicellularity of the operator $J \otimes B$ have been obtained earlier in [B1] for $B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and in [OS2] for $B = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix}$.

§3. THE LATTICES OF HYPERINVARIANT SUBSPACES.

3.1. We recall that an invariant subspace E of an operator $A \in [H]$ is called hyperinvariant for A if E is invariant for any bounded operator that commutes with A, that is it is invariant for all the operators from the commutant $\{A\}'$. The lattice of all the hyperinvariant subspaces of an operator A will be denoted by Hyplat A. To describe them we need the descriptions of commutants $\{A\}'$.

One can obtain the following proposition in just the same way as Proposition 2.1.

Proposition 3.1. *Let* $X = L_p[0,1]$ $p \in [1,\infty)$, $X^n = \underbrace{X \oplus \ldots \oplus X}_{n}$, *$B$ be a diagonal matrix with simple nonzero eigenvalues of different arguments:*

$$B = diag(\lambda_1, \ldots, \lambda_n)\ , \quad \arg\lambda_i \neq \arg\lambda_j \quad \forall i \neq j. \tag{3.1}$$

Then the commutant $\{A\}'$ *of the operator* $A = J^\alpha \otimes B (\in [X^n])$ *is*

$$\{A\}' = \{K : K = diag(K_1, \ldots, K_n), K_j \in [X]\} \tag{3.2}$$

where all the K_j *are of the form:*

$$K_j f = \frac{d}{dx}\int_0^x k_j(x-t)f(t)dt \qquad k_j(x) \in L_q[0,1] \qquad (q^{-1}+p^{-1}=1). \tag{3.3}$$

Proof. Let $K = (K_{ij})_{i,j=1}^n$ be the block-matrix partition of the operator $K$with respect to the direct sum decomposition $X^n = \underbrace{X \oplus \ldots \oplus X}_{n}$. Then the equality $KJ^\alpha = J^\alpha K$ is equivalent to the following system

$$\lambda_j K_{ij} J^\alpha = \lambda_i J^\alpha K_{ij} \quad (1 \leq i,\ j \leq n). \tag{3.4}$$

It was shown in [M1] that there are no bounded operators intertwining the operators J^α and cJ^α for $c \in \mathbb{C}$ and $\arg c \neq 0$, that is $K_{ij} = 0$ for $i \neq j$.

A brief direct proof can be easily derived from (1.9), (2.8) and (2.19) as well as the implication (2.7). In fact, it follows from (3.4) that

$$K_{ij}(I - \lambda_j J^\alpha)^{-1} = (I - \lambda_i J^\alpha)^{-1} K_{ij} \quad (i \neq j) \tag{3.5}$$

Therefore for arbitrary vectors $f, g \in L_2[0,1]$ we have

$$\int_0^1 t^{\alpha-1} E_{1/\alpha}(\lambda\lambda_1 t^\alpha; \alpha)\varphi_1(t)dt + \int_0^1 t^{\alpha-1} E_{1/\alpha}(\lambda\lambda_2 t^\alpha; \alpha)\varphi_2(t)dt = 0 \tag{3.6}$$

where

$$\varphi_1(t) = \int_t^1 f(x-t)g_1(x)dx\ , \quad \varphi_2(t) = -\int_t^1 f_1(x-t)g(x)dx$$

and $g_1 = K_{ij}^* g$, $f_1 = K_{ij} f$. Denote by $[a_j, b_j]$ $(j = 1,2)$ the minimal segment containig the support of the function $\varphi_j(t)$ and suppose that $b_1|\lambda_1|^{1/\alpha} \geq b_2|\lambda_2|^{1/\alpha} > 0$. The asymptotic formulas (2.8) and (2.9) yield that on the ray $\lambda_1^{-1}\mathbb{R}_+ = \{\lambda_1^{-1}x : x \geq 0\}$ the first summand in (3.6) is of larger growth than the second one. It means that $b_1 = b_2 = 0$ that is $\varphi_1(t) = \varphi_2(t) = 0$ $\forall t \in [0,1]$ and due to the Titchmarsh's Theorem $K_{ij} = 0$ for $i \neq j$.

It suffices to prove that each bounded operator $K \in [H]$ commuting with J^α has the form (3.3). This fact is well-known but we provide here a brief and simple proof.

Since $P := JK$ maps $L_p[0,1]$ into $C[0,1]$, it is an integral operator (see [Ka]), with a kernel $P(x,t) \in L_\infty \hat{\otimes} L_q$:

$$Pf = \int_0^1 P(x,t)f(t)dt \ , \qquad M = \operatorname*{vraisup}_{0\leq x\leq 1} \int_0^1 |P(x,t)|^q dt < \infty.$$

Let $R(x,t)$ be the kernel of the operator $R := JPJ$. Then

$$R(x,t) = \int_0^x d\xi \int_t^1 P(\xi,\eta)d\eta \tag{3.7}$$

and moreover $R(x,t)$ is an absolutely continuos function with respect to x and t under fixed $t \in [0,1]$ and $x \in [0,1]$ respectively. Since $RJ^\alpha = J^\alpha R$, we have $RJ^{\alpha m} = J^{\alpha m}R$ for all $m \in \mathbb{Z}_+$ and the Muntz-Szasz's Theorem [PW] yields $RJ = JR$, that is

$$\int_t^1 R(x,s)ds = \int_0^x R(s,t)ds \tag{3.8}$$

Applying the operator $D_x D_t$ to the equation (3.8) we obtain $(D_x + D_t)R(x,t) = 0$. This means that $R(x,t) = R(x-t)$, i.e. $R(x,t)$ depends only on the difference of the variables. We can now rewrite (3.8) in the form

$$0 = \int_{x-1}^{x-t} R(s)ds - \int_{-t}^{x-t} R(s)ds = \int_{x-1}^{-t} R(s)ds.$$

It follows that $R(s) = 0 \ \ \forall s \in [-1,0]$. Taking (3.7) into account we have

$$P(x,t) = D_x D_t R(x,t) = -R''(x-t) = P(x-t) \qquad \textit{and} \qquad P(x) = 0 \ \forall x \in [-1,0].$$

Thus $JKf = P(x) * f(x)$ and therefore K is of the form (3.3) and $M = \|P(x)\|_{L_q[0,1]}$. $\square$

Observe that for $p = 2$ the bondedness criterion for a convolution operator in $L_2[0,1]$ is the same as in $L_2(0,\infty)$. This fact together with the investigation of the spectrum of such an operator will be published elsewhere.

It allows us to present for $p = 2$ the following refinement of the previous result.

Proposition 3.1$'$. *Let under the conditions of Proposition 3.1 $p = 2$. Then $K \in \{A\}'$ iff it is of the form (3.2), (3.3) where the kernels $k_j(x) \in L_2[0,1]$ and satisfy the condition*

$$\lambda \hat{k}_j(\lambda) \in L_\infty(\mathbb{R}), \quad (1 \le j \le n)$$

with $\hat{k}_j(\lambda)$ being the Fourier transform of $k_j(x)$.

Remark 3.1. i) In the case $p = 2$ one concludes that $P = JK$ admits an integral representation (3.6) because P is a Hilbert-Smidt operator. Another description of the commutant $\{J\}'$ in $L_2[0,1]$ was obtained by Sarason [S2].

ii) Dixmier [D] has found that in the space $L_1[0,1]$ the commutant $\{J\}'$ consists precisely of the convolution operators induced by measures on $[0,1)$. This fact can be established also from the representation (3.3).

One can consider an operator K_j of the form (3.3) as a convolution operator $: f \to f * k_j'$ with k_j' being a distribution. However, not all the operators in the commutant $\{J\}'$ of J on $L_p[0,1]$ for $p > 1$ are generated by measures on $[0,1)$. For example $k(x) = x^i$ generates a bounded operator of the form (3.3), although x^i is not a function of bounded variation on $[0,1]$.

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Remark 3.2. It is obvious that $\mathrm{Lat}(A_1 \oplus A_2) \supset \mathrm{Lat}A_1 \oplus \mathrm{Lat}A_2$. The opposite inclusion, which is equivalent to the splitting of $\mathrm{Lat}(A_1 \oplus A_2)$, is closely connected with the existence of an operator $T \in [H_1, H_2]$ intertwining the operators $A_i \in [H_i]$ $(i = 1,2)$. In fact, if there exists a bounded operator $(O \ne)T \in [H_1, H_2]$ such that $TA_1 = A_2T$ then

$$\mathrm{Lat}(A_1 \oplus A_2) \ne (\mathrm{Lat}A_1 \oplus \mathrm{Lat}A_2)$$

Indeed the subspace $E = \mathrm{gr}T = \{(h, Th) : h \in H_1\} \subset H_1 \oplus H_2$ is invariant for $A_1 \oplus A_2$ because of

$$\begin{pmatrix} h \\ Th \end{pmatrix} \in E \Rightarrow (A_1 \oplus A_2) \begin{pmatrix} h \\ Th \end{pmatrix} = \begin{pmatrix} A_1 h \\ A_2 Th \end{pmatrix} = \begin{pmatrix} A_1 h \\ TA_1 h \end{pmatrix} \in E,$$

but $E \notin \mathrm{Lat}A_1 \oplus \mathrm{Lat}A_2$.

Therefore the first part of Proposition 3.1, i.e. the diagonal form of an operator $K \in \{A\}'$, immediately follows from Proposition 2.1. $\Box$

The following two results are simple corollaries of the previous one.

Proposition 3.2. *Under the conditions of Proposition 3.1 the lattice HyplatA is anti-isomorphic to the unit cube$\Box_n = \{x \in \mathbb{R}^n : 0 \le x_i \le 1\}$:*

$$(3.9) \qquad HyplatA = LatA = \{\oplus_{i=1}^n L_2[a,1] : \ 0 \le a \le 1\} = \oplus_{j=1}^n Lat(\lambda_j J^\alpha)$$

Proposition 3.3. *Let $A = \oplus_{k=1}^r J^{\alpha_k} \otimes B_k, \alpha_1 > \ldots > \alpha_r, B_k = diag(\lambda_{k1}, \ldots, \lambda_{kn_k})$ be a nonsingular diagonal matrix, $n = n_1 + \ldots + n_r$ and $\arg \lambda_{ki} \ne \arg \lambda_{kj}$ for $i \ne j$. Then*

1) The commutant $\{A\}'$ of the operator A is of the form (3.2), (3.3)

2) HyplatA is of the form (3.9), that is it is anti-isomorphic to the unit cube $\square_n = \{x \in \mathbb{R}^n : 0 \le x_i \le 1\}$.

3.2. To state the next results we need some notations. Corresponding to any $a \in (0,1)$ we define the unitary operator $U_a : L_2[0,1] \to L_2[0,a^{-1}]$ given by

$$U_a : L_2[0,1] \ni f(x) \to \widetilde{f}(x) = \sqrt{a} f(ax) \in L_2[0,a^{-1}] \tag{3.10}$$

As usual for $a < l$ we will identify $L_2[a,l]$ and $L_2[0,a]$ with the subspaces E_a and $E_a^{\perp}$ of $L_2[0,l]$:

$$\begin{aligned} E_a &= \{f(x) \in L_2[0,l] : f(x) = 0 \quad \forall x \in [0,a]\} \\ E_a^{\perp} &= \{f(x) \in L_2[0,l] : f(x) = 0 \quad \forall x \in [a,l]\} \end{aligned} \tag{3.11}$$

respectively.

Let P_a be the orthogonal projection

$$P_a : L_2[0,a^{-1}] \to E_1^{\perp} \cong L_2[0,1] \tag{3.12}$$

of $L_2[0,a^{-1}]$ onto $E_1^{\perp}$ in the orthogonal decomposition $L_2[0,a^{-1}] = E_1 \oplus E_1^{\perp}$.

According to the decomposition $L_2[0,1] = E_a \oplus E_a^{\perp}$ we introduce the embedding operator $i_a : L_2[0,a] \to L_2[0,1]$ given by

$$i_a : f(x) \to g(x) = \begin{cases} 0, & x \in [0, 1-a] \\ f(x-1+a), & x \in [1-a, 1] \end{cases} \tag{3.13}$$

Proposition 3.4. *Let $H = L_2[0,1], B = diag(\lambda_1, \dots, \lambda_n)$ be a nonsingular diagonal matrix with eigenvalues of equal arguments:*

$$\lambda_j = \lambda_1 s_j^{\alpha}, \qquad 1 = s_1 \le s_2 \le \ldots \le s_n$$

Let also:

a) $P_{ij} = P_{a_{ij}}, U_{ij} = U_{a_{ij}}$ and $a_{ij} = s_i^{-1} s_j < 1$ if $i > j$;

b) $i_{ji} = i_{a_{ji}}$ if $i < j$;

c) $U_{jj} = i_{jj} = I$ (I is identical operator in $L_2[0,1]$).

Then the commutant $\{A\}'$ of the operator $A = J^{\alpha} \otimes B$ is

$$\{A\}' = \{K : K = (K_{ij})_{i,j=1}^n, \quad K_{ij} \in [H]\}$$

where $K_{jj} \in \{J^{\alpha}\}'$, that is $K_{jj} (1 \le j \le n)$ are of the form (3.3);

$$K_{ij} = C_{ij} P_{ij} U_{ij}, \quad C_{ij} \in \{J^{\alpha}\}' \qquad \textit{for } i > j; \tag{3.14}$$

$$K_{ij} = i_{ji} U_{ij}^{-1} C_{ij}, \quad C_{ij} \in \{J^{\alpha}\}' \qquad \textit{for } i < j. \tag{3.15}$$

Proof. In view of (3.4) the equality $KA = AK$ is equivalent to the system

$$a_{ij}^{\alpha} J^{\alpha} K_{ij} = K_{ij} J^{\alpha}, \quad (a_{ij} = s_i^{-1} s_j). \tag{3.16}$$

We consider two cases $i > j$ and $i < j$ separately.

i) Let $i > j$. Then $a_{ij} = s_i^{-1}s_j < 1$. To begin with we verify that K_{ij} of the form (3.14) satisfy (3.16). Setting for an arbitrary $f(x) \in L_2[0,1]$

$$\varphi(x) = \begin{cases} \sqrt{a_{ij}}((a_{ij}x)^{\alpha-1} * f)(a_{ij}x), & x \in [0,1] \\ 0, & x \in [1, a_{ij}^{-1}] \end{cases}$$

we have

$$\Gamma(\alpha)a_{ij}^{\alpha}J^{\alpha}P_{ij}U_{ij}f = a_{ij}^{\alpha}\int_0^x (x-t)^{\alpha-1}\sqrt{a_{ij}}f(a_{ij}t)dt =$$

$$\sqrt{a_{ij}}\int_0^{a_{ij}x}(a_{ij}x-t)^{\alpha-1}f(t)dt = (P_{ij}\varphi)x = \Gamma(\alpha)P_{ij}U_{ij}J^{\alpha}f.$$

Thus for $i > j$ the operator

(3.17) $$N_{ij} = P_{ij}U_{ij} \qquad (i > j)$$

satisfies the commutation relation (3.16). Since $C_{ij} \in \{J^{\alpha}\}'$ the same is true for $K_{ij} = C_{ij}N_{ij}$.

Conversely, let K_{ij} satisfy (3.16). To prove (3.14) we set $f_1 = K_{ij}f, g_1 = K_{ij}^{*}g, \phi_1(t) = \int_t^1 f(x-t)g_1(x)x$ and

(3.18) $$\phi_2(t) = \begin{cases} \frac{1}{a_{ij}^{\alpha+1}}\int_t^{a_{ij}} f_1\left(\frac{x-t}{a_{ij}}\right)g\left(\frac{x}{a_{ij}}\right)dx, & t \in [0, a_{ij}]; \\ 0, & t \in [a_{ij}, 1]. \end{cases}$$

Bearing in mind that $\lambda_j = a_{ij}^{\alpha}\lambda_j$ we obtain from (3.5) and (3.6)

$$\int_0^1 t^{\alpha}E_{1/\alpha}(\lambda t^{\alpha};\alpha)[\phi_1(t) - \phi_2(t)]dt = 0.$$

This means that $\phi_1(t) = \phi_2(t)$ $\forall t \in [0,1]$ and in view of (3.18)

$$\phi_1(t) = \int_t^1 f(x-t)g_1(x)dx = 0 \qquad \forall t \in [a_{ij}, 1].$$

It follows that

(3.19) $$g_1(x) = (K_{ij}^{*}g)(x) = 0 \qquad \forall x \in [a_{ij}, 1]$$

and consequently $E_{a_{ij}}(\cong L_2[a_{ji},1]) \subseteq \ker K_{ij}$. According to (3.17) $\Re(N_{ij}^*) = E_{a_{ij}}$ and in view of Douglas lemma [N1] the including $\Re(K_{ij}^*) \subseteq E_{a_{ij}} = \Re(N_{ij}^*)$ is equivalent to the representation

$$K_{ij} = C_{ij}N_{ij} \qquad (i > j). \tag{3.20}$$

Here C_{ij} is a bounded operator on $L_2[0,1]$ which is uniquelly determined by the additional condition $\ker C_{ij} \supset \ker N_{ij}$.

Substituting the operator K_{ij} from (3.20) into (3.16) and using (3.17) we obtain

$$(J^\alpha C_{ij} - C_{ij}J^\alpha)N_{ij} = 0 \qquad (i > j). \tag{3.21}$$

Since $\Re(N_{ij}) = L_2[0,1]$, one concludes from (3.21) that $C_{ij} \in \{J^\alpha\}'$ for $i > j$ and according to Proposition 3.1 there exist $C_{ij}(x) \in L_2[0,1]$ such that

$$C_{ij}f = \frac{d}{dx}\int_0^x C_{ij}(x-t)f(t)dt \tag{3.22}$$

ii) Let now $i < j$. Then $a_{ji} = a_{ij}^{-1} = s_j^{-1}s_i < 1$ and

$$a_{ji}^\alpha K_{ij}J^\alpha = J^\alpha K_{ij}, \qquad i < j. \tag{3.23}$$

We prove that for each pair (i,j) with $i < j$ the opeartor

$$M_{ij} = i_{a_{ji}}U_{ij}^{-1} \qquad (i < j) \tag{3.24}$$

satisfies the commutation relation (3.22). Indeed, setting $(a < 1)$

$$\varphi(x) = \begin{cases} 0, & x \in [0, 1-a] \\ a\int\limits_0^{(a+x-1)/a} (x-1+a-at)^{\alpha-1}f(t)dt, & x \in [1-a, 1] \end{cases}$$

we have

$$\Gamma(\alpha)a^\alpha i_a U_a^{-1}J^\alpha f = a i_a \left(\int_0^{xa^{-1}} (xa^{-1}-t)^{\alpha-1}f(t)dt\right) = \varphi(x). \tag{3.25}$$

On the other hand

$$\Gamma(\alpha)J^\alpha i_a U_a^{-1}f = \Gamma(\alpha)J^\alpha i_a f\left(\frac{x}{a}\right) = \varphi(x). \tag{3.26}$$

Comparing (3.25) and (3.26) one obtains $a_{ji}^\alpha M_{ij}J^\alpha = J^\alpha M_{ij}$, that is M_{ij} of the form (3.24) is the solution of the equation (3.23). Because of $C_{ij} \in \{J^\alpha\}'$ the same is true for $K_{ij} = M_{ij}C_{ij}$.

Futher, let K_{ij} be an arbitrary bounded solution of the relation (3.23). We observe that

$$(K_{ij}f)(x) = 0 \qquad \forall x \in [0, 1-a]$$

and consequently $\Re(K_{ij}) \supseteq L_2[19a, 1] = E_{1-a}$. This fact can be deduced from (3.5) and (3.6) in the same way as the relation (3.19), but it immmediately follows from (3.19) because of

$$a_{ij}(J^\alpha)^* K_{ij}^* = K_{ij}^*(J^\alpha)^*.$$

Taking arrangement of Douglas' Lemma we can deduce from the inclusion $\Re(K_{ij}) \supseteq E_{1-a} = \Re(M_{ij})$ that $K_{ij} = M_{ij}C_{ij}$ with some $C_{ij} \in [H]$. One can conclude now using (3.23) that $M_{ij}(C_{ij}J^\alpha - J^\alpha C_{ij}) = 0$. Since $\ker M_{ij} = \{0\}$ the last identity is equivalent to the following one : $C_{ij}J^\alpha - J^\alpha C_{ij} = 0$, that is $C_{ij} \in \{J^\alpha\}'$ and consequently C_{ij} is of the form (3.22). $\square$

Remark 3.3. It follows from (3.14) and (3.15) that there is no quasiaffinity T (i.e. an operator T such that $\ker T = \{0\}$ and $\overline{\Re(T)} = H$) intertwinig the operators sJ^α and J^α ($s \in \mathbb{R} \setminus \{1\}$). Really, if $sJ^\alpha T = TJ^\alpha$ and $s < 1$ then $\ker T \supseteq E_s (\cong L_2[s, 1])$, otherwise if $sJ^\alpha T = TJ^\alpha$ and $s > 1$ then $\Re(T) \subset E_{1-s} (\cong L_2[1-s, 1])$. In particular the operators J^α and sJ^α are not quasisimilar in the sence of B. Nagy and C. Foias [SF]. This assertion was obtained earlier in [M4].

Proposition 3.5. *Let $A = J^\alpha \otimes B$ and B satisfy the conditions of Proposition 3.4. Then*

$$\text{(3.27)} \qquad Hyplat A = \{\oplus_{j=1}^n E_{a_j} : (a_1, \ldots, a_n) \in P(s_1, \ldots, s_n)\}$$

where

(3.28)

$$P(s_1, \ldots, s_n) = \{(a_1, \ldots, a_n) \in \square_n : s_j a_{j+1} \le s_{j+1} a_j \le s_{j+1} - s_j + s_j a_{j+1}, 1 \le j \le n-1\},$$

that is the lattice $Hyplat A$ is anti-isomorphic to the oblique parallelepiped $P(s_1, \ldots, s_n)$. The dimension $dim P(s_1, \ldots, s_n)$ of this parallelepiped is equal to the number of different s_j $(1 \le j \le n)$.

Proof. Let $\lambda_j \in \mathbb{C}$ and $\arg\lambda_j \ne \arg\lambda_i$ for $i \ne j$. Since $\oplus_{j=1}^n \lambda_j J^\alpha \in \{A\}'$ it follows from (3.9) that

$$\operatorname{Hyplat} A \subset \operatorname{Lat}\left(\oplus_{j=1}^n \lambda_j J^\alpha\right) = \{\oplus_{j=1}^n E_{a_j} : 0 \le a_j \le 1\}.$$

The following implications can be easily derived from (3.14) and (3.15):

$$f_j(x) \in E_{a_j} \Rightarrow K_{ij}f_j \in E_{a_j s_i s_j^{-1}} \qquad \forall i > j \ge 1$$

$$f_j(x) \in E_{a_j} \Rightarrow K_{ij}f_j \in E_{1-s_j^{-1}s_i(1-a_j)}, \qquad i+1 \le j \le n$$

To complete the proof if suffices to observe that the system of inequalities

$$s_j a_k \le s_k a_j \le s_k - s_j + s_j a_k \ (1 \le j \le n-1,\ j+1 \le k \le n)$$

is equivalent to the reduced one described in (3.28). $\square$

Corollary 3.1. *Let $A = J^\alpha \otimes I$. Then*

$$Hyplat(J^\alpha \otimes I) = \{\oplus_{j=1}^n E_{a_j} : a_1 = a_2 = \ldots = a_n \in [0,1]\} \tag{3.29}$$

3.3. Now we are ready to state the main results of this section. Combining Propositions 3.1 and 3.4 we obtain the first of them.

Theorem 3.1. *Let $B = \oplus_{p=1}^r B_p$, $B_p = diag(\lambda_{p1}, \lambda_{p2}, \ldots, \lambda_{pn_p})$ be a nonsingular diagonal matrix with eigenvalues $\lambda_{pk} (1 \le k \le n_p)$ of equal arguments:*

$$\lambda_{pk} = \lambda_{p1} s_{pk}^\alpha, \qquad 1 = s_{p1} \le s_{p2} \le \ldots \le s_{pn_p}$$

and $\arg \lambda_{p1} \ne \arg \lambda_{q1}$ for $p \ne q$. Then

$$\{J^\alpha \otimes B\}' = \oplus_{p=1}^r \{J^\alpha \otimes B_p\}', \tag{3.30}$$

that is $X \in \{J^\alpha \otimes B\}'$ if and only if

$$X = diag(X_{11}, X_{22}, \ldots, X_{rr}). \tag{3.31}$$

where $X_{pp} = (K_{pij})_{i,j=1}^{n_p} \in \{J^\alpha \otimes B_p\}'$ is an $n_p \times n_p$ operator-matrix $(1 \le p \le r)$ with entries K_{pij} described in Proposition 3.4.

Combining Propositions 3.2, 3.5 and Theorem 3.1 we obtain the concluding result of this section, namely the description of hyperinvariant subspaces of an operator $J^\alpha \otimes B$ with arbitrary nonsingular diagonable matrix B.

Theorem 3.2. *Let B satisfy the conditions of Theorem 3.1 and $A = J^\alpha \otimes B$. Then*

$$\begin{aligned} Hyplat(J^\alpha \otimes B) &= \oplus_{p=1}^r Hyplat(J^\alpha \otimes B_p) = \\ &\oplus_{p=1}^r \{\oplus_{j=1}^{n_p} E_{a_{pj}} : (a_{p1}, \ldots, a_{pn_p}) \in P(s_{p1}, \ldots, s_{pn_p})\} \end{aligned} \tag{3.32}$$

where $P(s_{p1}, \ldots, s_{pn_p})$ is the parallelepiped, defined by the relation (3.28).

§4 Similarity of Volterra Operators to the Operator $J \otimes B$.

4.1. Let X be a Banach space. Recall that two operators $A, B \in [X]$ are called similar if there exists a bounded invertible operator $T \in [X]$ such that $T^{-1} \in [X]$ and $B = T^{-1}AT$.

In this section some sufficient conditions for the operator

$$K : f \longrightarrow \int_0^x k(x,t) f(t) dt \ , \quad f(t) = (f_1(t), \ldots, f_n(t) \in L_2^n[0,1]) \tag{4.1}$$

to be similar to the operator $J \otimes B$ are investigated. Here $k(x,t)$ is the $n \times n$ matrix kernel of the operator K.

Theorem 4.1. *Let the $n \times n$-matrix kernel $k(x,t)$ of an operator K satisfy the following conditions:*

1) $B := k(x,x)$ is a constant nonsingular matrix with real spectrum and simple elementary divisors.

2) k(x,t) is absolutely continuous with respect to x for almost all $t \in [0,1]$ and $k_1(x,t) = D_x k(x,t) \in L_\infty(\Omega) \otimes \mathbb{C}^{n^2}$.

3) k(x,t) is absolutely continuous with respect to t for almost all $x \in [0,1]$ and $D_t k_1(x,t) = D_t D_x k(x,t) \in L_\infty(\Omega) \otimes \mathbb{C}^{n^2}$, where $\Omega = \{0 \le t \le x \le 1\}$.

Then the operator K is similar to the operator $J \otimes B$ in $L_p[0,1] \otimes \mathbb{C}^n$ $(1 \le p \le \infty)$.

Proof. i) We find an expression for the inverse operator K^{-1}. Differentiating the equality $Kf = g$ and taking the conditions 1) and 2) into account we obtain

$$(I \otimes B)[f(x) + \int_0^x B^{-1} k_1(x,t) f(t) dt] = \frac{dg}{dx}\ ,\ g(0) = 0 \tag{4.2}$$

where $f(x) = (f_1(x), \dots, f_n(x))$, $g(x) = (g_1(x), \dots, g_n(x))$, $k_1(x,t) = D_x k(x,t)$. Since $k_1(x,t) \in L_\infty(\Omega) \otimes \mathbb{C}^{n^2}$, the operator

$$I + K_1 :\ f(t) \longrightarrow f(x) + \int_0^x B^{-1} k_1(x,t) f(t) dt,\ f(t) \in L_p^n[0,1] \tag{4.3}$$

is invertible. Moreover

$$(I + K_1)^{-1} = I + P : f \to f(x) + \int_0^x P(x,t) f(t) dt. \tag{4.4}$$

It is easily seen that the kernels $P(x,t)$ and $k_1(x,t)$ are related by the formula

$$P(x,t) + k_1(x,t) + \int_t^x P(x,s) k_1(s,t) ds = 0. \tag{4.5}$$

It follows from (4.5) and the condition 3) of Theorem , that for almost all $x \in [0,1]$ the matrix kernel $P(x,t)$ is absolutely continuous with respect to t. Taking advantage of this fact we now apply the operator $I + P$ to (4.2) and, integrating by parts, we arrive at

$$\begin{aligned} f(x) = K^{-1} g = (I \otimes B^{-1}) \frac{dg}{dx} + \int_0^x P(x,t) B^{-1} = \\ = B^{-1} \frac{dg}{dx} + P(x) g(x) - \int_0^x P_1(x,t) g(t) dt \end{aligned} \tag{4.6}$$

where

$$P(x) = P(x,x)B^{-1}, \quad P_1(x,t) = D_t P(x,t)B^{-1}. \tag{4.7}$$

ii) According to the condition 1) the matrix B is diagonalizable, that is $S^{-1}BS = B_1$ with some invertible transformation $S \in [\mathbb{C}^n]$ where B_1 is a diagonal matrix. Therefore, replacing, if necessary, the operator K^{-1} by the operator $(I \otimes S^{-1})K^{-1}(I \otimes S)$, we can assume that the matrix B^{-1} in (4.6) is diagonal.

Let $\lambda_1, \dots, \lambda_r$ be all distinct eigenvalues of the matrix B with multiplicities $n_1, \dots, n_r$. Thus $B = \oplus_{j=1}^r \lambda_j I_{n_j}$ and $P(x)$ admits the representation $P(x) = (P_{ij})_{i,j=1}^r$ with respect to the decomposition $\mathbb{C}^n = \oplus_{j=1}^r \ker(B - \lambda_j)$.

Let $W(x)$ be the fundamental $n \times n$ matrix solution of the following Cauchy problem

$$W'(x) = -BP_1(x)W(x), \quad W(0) = I_n \tag{4.8}$$

with $n \times n$ matrix $P_1(x)$ being a quasydiagonal matrix with blocks $P_{jj}(x)$:

$$P_1(x) := \operatorname{diag}(P_{11}(x), \dots, P_{rr}(x)). \tag{4.9}$$

Since $BP_1(x) = P_1(x)B \ \forall x \in [0,1]$ (i.e. $P_1(x) \in \{B\}' \ \forall x \in [0,1]$) the matrix-functions $W_1(x) = BW(x)$ and $W_2(x) = W(x)B$ satisfy the equation (4.8) and the common initial conditions :

$$W_j'(x) = -BP_1(x)W_j(x) \ , \quad W_j(0) = B \quad (j = 1,2). \tag{4.10}$$

In view of the Cauchy Uniqueness Theorem $W_1(x) = W_2(x) \ \forall x \in [0,1]$, that is

$$[W(x), B] = W(x)B - BW(x) = 0 \qquad \forall x \in [0,1]. \tag{4.11}$$

Letting $L_0 = (I \otimes W^{-1})K^{-1}(I \otimes W)$ one can deduce from (4.6), (4.8) and (4.11) that

$$\begin{aligned} L_0 g &= W^{-1}(x)B^{-1}W(x)g'(x) + W^{-1}(x)B^{-1}W'(x)g+ \\ &+W^{-1}(x)P(x)W(x)g - \int_0^x W(x)^{-1}P_1(x,t)W(t)g(t)dt = \\ &= B^{-1}\frac{dg}{dx} + Q(x)g(x) - \int_0^x M(x,t)g(t)dt \,, \end{aligned} \tag{4.12}$$

where

$$\begin{aligned} Q(x) &= W^{-1}(x)(P(x) - P_1(x))W(x) \\ M(x,t) &= W(x)^{-1}P_1(x,t)W(t). \end{aligned} \tag{4.13}$$

It follows from (4.11) that $W(x)$ is quasidiagonal:

$$W(x) = \operatorname{diag}(W_{11}(x), \dots, W_{rr}(x)) \tag{4.14}$$

with $n_j \times n_j$ matrix blocks $W_{jj}(x)\,(1 \le j \le r)$. Consequently one concludes from (4.13) and (4.14) that $Q(x)$ is of the form

$$(4.15) \qquad Q(x) = (Q_{ij}(x))_{i,j=1}^{r}\ , \quad Q_{ii}(x) = 0\ \forall x \in [0,1],\ (1 \le i \le r).$$

Thus we have established that the operator K^{-1} with natural domain of definition $\mathcal{D}(K^{-1}) = W_{p,0}^{1} \otimes \mathbb{C}^n$ is similar in $L_p[0,1] \otimes \mathbb{C}^n$ to the operator

$$(4.16) \qquad L_0 : f \to B^{-1}\frac{df}{dx} + Q(x)f(x) - \int_0^x M(x,t)f(t)dt\ ,\ f(0) = 0$$

with matrix potential $Q(x)$ of the form (4.15) and the same domain of definition $\mathcal{D}(L) = W_{p,0}^{1} \otimes \mathbb{C}^n$. Here $W_{p,0}^{1} = W_{p,0}^{1}[0,1]$ means the following subspace of Sobolev space $W_p^1[0,1] : W_{p,0}^1 = \{f(x) : f(x) \in W_p^1[0,1], f(0) = 0\}$.

iii) Now we are ready to prove the similarity of the operators L_0 and $B^{-1} \otimes D_0$ with common domain of definition

$$W_{p,0}^1[0,1] \otimes \mathbb{C}^n := \{f(x) = (f_1(x), \dots , f_n(x)) \in W_p^1[0,1] \otimes \mathbb{C}^n : f_j(0) = 0, 1 \le j \le n\}$$

with D_0 being the restriction of the operator $D = \frac{d}{dx}$ on $W_{p,0}^1[0,1]$.

More precisely we will prove that there exists a triangular operator

$$(4.17) \qquad I + R : f(x) \to f(x) + \int_0^x R(x,t)f(t)dt$$

with $n \times n$ matrix kernel $R(x,t)$, intertwining the operators L_0 and $B^{-1} \otimes D_0$:

$$(4.18) \qquad L_0(I + R) = (I + R)B^{-1} \otimes D_0$$

Supposing that $R(x,t) \in W_1^1(\Omega) \otimes \mathbb{C}^{n^2}$ and substituting the expressions (4.16) and (4.17) into (4.18) one derives

(4.19)

$$[B^{-1}R(x,x) - R(x,x)B^{-1} + Q(x)]f(x) + \int_0^x [B^{-1}R_x'(x,t) + R_t'(x,t)B^{-1} +$$

$$+Q(x)R(x,t) + M(x,t) + \int_t^x M(x,s)R(s,t)ds]f(t)dt + R(x,0)B^{-1}f(0) = 0$$

Since $f(0) = 0$ and $f(x)$ is an arbitrary function from $W_{p,0}^1[0,1] \otimes \mathbb{C}^n$, this equation is equivalent to the system

(4.20)

$$B^{-1}D_xR(x,t) + D_tR(x,t)B^{-1} + Q(x)R(x,t) + M(x,t) + \int_t^x M(x,s)R(s,t)ds = 0$$

$$(4.21) \qquad R(x,x)B^{-1} - B^{-1}R(x,x) = Q(x).$$

In order to prove the existense of a solution of the system (4.20), (4.21) we take advantage of the block-matrix representation $R(x,t) = (R_{ij})_{i,j=1}^r$ and $M(x,t) = (M_{ij})_{i,j=1}^r$ of the matrices $R(x,t)$ and $M(x,t)$ with respect to the direct sum decomposition $\mathbb{C}^n = \oplus_{j=1}^r \ker(B - \lambda_j)$. Now the system (4.20), (4.21) may be rewritten in the form

$$\mu_i D_x R_{ij}(x,t) + \mu_j D_t R_{ij}(x,t) = \sum_{k=1}^{r} Q_{ik}(x) R_{kj}(x,t) + M_{ij}(x,t) + \\ + \sum_{k=1}^{r} \int_t^x M_{ik}(x,s) R_{kj}(s,t) ds, \ 1 \le i,j \le r \tag{4.22}$$

$$R_{ij}(x,x) = \frac{1}{\mu_i - \mu_j} Q_{ij}(x) \ , \ 1 \le i \ne j \le r, \tag{4.23}$$

where $\mu_j = \lambda_j^{-1}(1 \le j \le r)$.

It is clear that the system (4.22) is hyperbolic with real characteristics $l_{ij} : x = k_{ij}t + c$ with

$$k_{ij} = \mu_j \mu_i^{-1} \qquad (i,j \in \{1 \dots, r\}).$$

Thus in the triangle $\triangle_1 = \{0 \le t \le x \le 1\}$ we have an incomplete Cauchy problem (4.22), (4.23) with $n^2 - n_1^2 - \dots - n_r^2$ scalar conditions (4.23).

Note that each characteristic line with negative slope $k_{ij} < 0$ through an arbitrary point $P(x_0, t_0) \in \triangle_1$ intersects the diagonal $\Gamma = \{(x,x) : x \in \mathbb{R}\}$ inside the triangle. Nevertheless for the characteristic lines with positive slopes the situation is different (see pict.1). In order to overcome this difficulty we consider the isosceles triangle $\triangle ABC$(see pict.1) confined by the lines

$$AB : x = t, \quad AC : x - 1 = k_{\min} t \quad BC : x - 1 = k_{\max} t$$

where

$$k_{\min} = \max\{k_{ij} : k_{ij} \in (0,1), \ \ 1 \le i, \ j \le r\}$$
$$k_{\max} = k_{\min}^{-1} = \min\{k_{ij} : k_{ij} \in (1,\infty), \ \ 1 \le i, \ j \le r\}.$$

Let a and b be the abscissas of the points A and B respectively. It is obvious (see pict.1) that $a < 0$ and $b = 1 + |a|$.

Denote by $\widetilde{Q}(x)$ and $\widetilde{M}(x,t)$ norm preserving continuations of the functions $Q(x)$ and $M(x,t)$ onto the segment $[a,b](\in [0,1])$ and the triangle $\triangle(\supset \triangle_1)$ respectively. For the simplicity we keep for $\widetilde{Q}(x)$ and $\widetilde{M}(x,t)$ the previuos notations $Q(x)$ and $M(x,t)$.

Assume the conditions (4.23) to be satisfied for $x \in [a,b]$ and adjoin to the system (4.22), (4.23) the following $n^2 - n_1^2 - \dots - n_r^2$ conditions on the characteristic line AC:

$$R_{jj}(x, (x-1)k_{\min}) = 0, \quad \forall x \in [a,1], \ j \in \{1 \dots, r\}. \tag{4.24}$$

It is useful to note that $k_{\min}$ defined above satisfies

$$k_{\min} = a(a-1)^{-1}. \tag{4.25}$$

Thus we have the Goursat problem (4.23), (4.24) in the triangle $\triangle = ABC$ for the hyperbolic system (4.22).

iiii) Integrating the system (4.22) in the triangle $\triangle$ along the characteristics and using the conditions (4.23), (4.24) we obtain the following system of integral equations

(4.26)

$$\mu_i R_{ij}(x,t) = \frac{\mu_i Q_{ij}(x)}{\mu_i - \mu_j} + \int\limits_{\xi_{ij}(x,t)}^{x} \sum_{s=1}^{r} Q_{is}(\xi) R_{sj}(\xi,(\xi - x)k_{ij} + t)d\xi +$$
$$+ \int\limits_{\xi_{ij}(x,t)}^{x} M_{ij}(\xi,(\xi - x)k_{ij} + t)d\xi + \sum_{s=1}^{r} \int\limits_{\xi_{ij}(x,t)}^{x} (M_{is} * R_{sj})(\xi,(\xi - x)k_{ij} + t)d\xi$$

where

$$\xi_{ij}(x,t) = \begin{cases} (\mu_i x - \mu_j t)(\mu_j - \mu_i)^{-1} & \text{for } i \neq j; \\ a + (1-a)(x-t) & \text{for } i = j. \end{cases} \tag{4.27}$$

In (4.26) we assume that

$$\frac{\mu_i Q_{ij}(x)}{\mu_i - \mu_j} = 0 \qquad \text{for } i = j \tag{4.28}$$

and make use of the notation

$$(M_{is} * R_{sj})(x,t) = \int\limits_{t}^{x} M_{is}(x,\beta) R_{sj}(\beta,t)d\beta$$

It is clear that any solution of the Goursat's problem (4.22)-(4.24) is also a solution of the integral equation (4.25). The converse in also true if $Q(x)$ is sufficiently smooth, for example, if $Q(x) \in C^1[a,b] \otimes \mathbb{C}^{n^2}$.

We solve the equation (4.26) by the method of successive approximations, setting

$$R^0_{ij}(x,t) = \frac{Q_{ij}(x)}{\mu_i - \mu_j} + \frac{1}{\mu_i} \int\limits_{\xi_{ij}(x,t)}^{x} M_{ij}(\xi,(\xi - x)k_{ij} + t)d\xi \tag{4.29}$$

and for $m \geq 1$

$$\begin{aligned} \mu_i R^{(m)}_{ij}(x,t) = \sum_{s=1}^{r} \int\limits_{\xi_{ij}(x,t)}^{x} Q_{is}(\xi) R^{(m-1)}_{sj}(\xi,(\xi - x)k_{ij} + t)d\xi \\ + \sum_{s=1}^{r} \int\limits_{\xi_{ij}(x,t)}^{x} (M_{is} * R^{(m-1)}_{sj})(\xi,(\xi - x)k_{ij} + t)d\xi \end{aligned} \tag{4.30}$$

Let

(4.31)
$$c_1 = \max\left\{\|R^0_{ij}(x,t)\|_{L_\infty(\triangle)}:\ i,j\in\{1,\dots,r\}\right\}$$

(4.32)
$$c_2 = \max_{1\le i,j\le r}\left\{|\mu_i|^{-1}\cdot\|Q_{ij}(x)\|_{L_\infty[0,1]}, |\mu_i|^{-1}\cdot\|M_{ij}(x,t)\|_{L_\infty(\triangle)}\right\}$$

(4.33)
$$c_3 = \max\left\{k_{ij}(k_{ij}-1)^{-1} : k_{ij}>0, 1\le i\ne j\le r\right\}, \qquad c=c_2c_3.$$

Suppose by induction that the following inequalities hold:

$$\|R^{(m)}_{ij}(x,t)\| \le c_1(2rc)^m(b-a)^m\frac{(x-a)^m}{m!}\ ,\ m\in\mathbb{Z}_+. \tag{4.34}$$

For $m=0$ the estimate (4.34) is obvious. Having supposed (4.34) to hold for $m=k$, we prove it for $m=k+1$.
Since

$$a\le \xi_{ij}(x,t)\le(\xi-x)k_{ij}+t \tag{4.35}$$

the relations (4.32), (4.35) and the induction hypotesis (4.24) together yield

$$(4.36)\quad |\mu_i^{-1}|\cdot\int\limits_{(\xi-x)k_{ij}+t}^{\xi}\|M_{is}(\xi,\beta\cdot R^{(k)}_{sj}(\beta,(\xi-x)k_{ij}+t)\|d\beta$$
$$\le c_2 - c_1(2rc)^k(b-a)^k\int\limits_a^\xi\frac{(\beta-a)^k}{k!}d\beta \le c_1c_2(2rc)^k(b-a)^{k+1}\frac{(\xi-a)^k}{k!}\ .$$

The relations (4.30)-(4.33), (4.36) and the induction hypothesis (4.34) imply

$$\begin{aligned}\|R^{(k+1)}_{ij}(x,t)\| &\le rc_2c_1(2rc)^k(b-a)^k\left|\int\limits_{\xi_{ij}(x,t)}^{x}\frac{(\xi-a)^k}{k!}d\xi\right|\\ &+ rc_1c_2(2rc)^k(b-a)^{k+1}\left|\int\limits_{\xi_{ij}(x,t)}^{x}\frac{(\xi-a)^k}{k!}d\xi\right|\\ &\le 2c_1c_2r(2rc)^k(b-a)^{k+1}\left|\frac{(x-a)^{k+1}}{(k+1)!}-\frac{(\xi_{ij}(x,t)-a)^{k+1}}{(k+1)!}\right|.\end{aligned} \tag{4.37}$$

Let us consider three cases separately.

a) If $k_{ij}\in(-\infty,1)$ then $\xi_{ij}(x,t)\le x$ (see pict.2) and we derive from (4.37) the estimate

$$\|R^{(k+1)}_{ij}(x,t)\| \le 2rc_2c_1(2rc)^k(b-a)^{k+1}(x-a)^{k+1}/(k+1)!. \tag{4.38}$$

b) If $k_{ij} > 1$, then $\xi_{ij}(x,t) > x$ (see pict. 2) and we have from (4.27) and (4.33)

$$\xi_{ij}(x,t) - a = \frac{\mu_j(x-a) - \mu_i(t-a)}{\mu_j - \mu_i} = \frac{k_{ij}}{k_{ij}-1}[(x-a) - k_{ij}^{-1}(t-a)$$
$$\leq k_{ij}(k_{ij}-1)^{-1}(x-a) \leq c_3(x-a)$$

Therefore in this case

$$\|R_{ij}^{(k+1)}(x,t)\| \leq 2rc_1c_2(2rc)^k(b-a)^{k+1}\frac{(\xi_{ij}(x,t)-a)^{k+1}}{(k+1)!}$$
$$\leq c_1(2rc_2c_3)(2rc)^k(b-a)^{k+1}/(k+1)! \quad . \tag{4.39}$$

c) Finally, let $k_{ij} = 15c4vfgrt$, that is $i = j$. Since for all $(x,t) \in \triangle$, (4.27) yields that $\xi_{ij}(x,t) = a + (1-a)(x-t) > a$. Therefore the estimate (3.38) holds in this case.
Thus the inequalities (4.38), (4.39) prove the induction hypotesis (4.34).
It therefore follows from (4.34) that the matrix series

$$R_{ij}(x,t) := \sum_{m=0}^{\infty} R_{ij}^{(m)}(x,t) \qquad (1 \leq i,j \leq r)$$

converges absolutely and uniformly in $\triangle$ and defines there a solution $\{R_{ij}(x,t)\}_{i,j=1}^{r}$ of the system (4.26) of integral equations . It is not difficult to prove the uniqueness of this solution.

Having imposed the additional assumptions $Q(x) \in C^1[a,b]$ and $M(x,t) \in C^1(\triangle)$, one deduces in the just same way, that the series

$$\sum_{m=0}^{\infty} D_x R_{ij}^{(m)}(x,t), \qquad \sum_{m=0}^{\infty} D_t R_{ij}^{(m)}(x,t), \ (i,j \in \{1,\dots,r\})$$

also converge absolutely and consequently their sums coinside with $D_x R_{ij}^{(\infty)}(x,t)$ and $D_t R_{ij}^{(\infty)}(x,t)$ respectively.

Thus in this case the solution $\{R_{ij}(x,t)\}_{i,j=1}^{r}$ of the Goursat problem (4.22)-(4.24). Since the operator $I + R$ of the form (4.17) is invertible in $L_p[0,1] \otimes \mathbb{C}^n$ it follows from (4.18) that the operators L_0 and $B^{-1} \otimes D_0$ are similar in the spaces $L_p[0,1] \otimes \mathbb{C}^n$ for $p \in [1,\infty]$ and therefore the same is true for K and $J \otimes B$.

Thus under the additional assumptions $Q(x) \in C^1[a,b]$ and $M(x,t) \in C^1(\triangle)$ Theorem is proved.

iiiii) Going over to the general case $Q(x) \in C[a,b] \otimes \mathbb{C}^{n^2}$ $M(x,t) \in C(\triangle) \otimes \mathbb{C}^{n^2}$, we introduce the sequences $Q_n(x)$ and $M_n(x,t)$ of C^1-smooth $n \times n$-matrix functions unifomly approximating the matrix functions $Q(x)$ and $M(x,t)$ respectively. Let $R_n(x,t)$ be the solutions of the corresponding equations of the form (4.26) (with $Q_n(x)$ and $M_n(x,t)$ instead of $Q(x)$ and $M(x,t)$ respectively). It is not difficult to prove (compare with [M4,p.93]) that the sequence $R_n(x,t)$ converges in $C(\Omega)\otimes\mathbb{C}^{n^2}$ to $R(x,t)$. Therefore, setting

$$R_n : f \to \int_0^x R_n(x,t)f(t)dt\ , \quad R : f \to \int_0^x R(x,t)f(t)dt,$$

$$L_{n0} : f \to (B\otimes D_0)f + Q_n(x)f + \int_0^x M_n(x,t)f(t)dt,$$

one concludes that the operator R is the uniform limit in $L_p[0,1]\otimes\mathbb{C}^n(1\le p\le\infty)$ ofthe sequence of Volterra operators R_n. Since $Q_n(x)$ and $M_n(x,t)$ are sufficiently smooth, the identities

$$L_{n0}^{-1}(I+R_n) = (I+R_n)(B\otimes J) \tag{4.40}$$

hold (see (4.18)). To complete the proof it remains to pass to the limit in the last identities when $n\to\infty$.

Remark 4.1. For the case $B=(I_{n_1},-I_{n_2})$ Theorem 4.1 has been proved in [Sak2].

A.L. Sacknovich informed me after my lecture on IWOTA95 (Regensburg) that he had obtained a special case of Theorem 4.1. Namely, he has proven the similarity of K and $B\otimes J$ under the conditions

$$B = \mathrm{diag}(\lambda_1,\lambda_2,\dots,\lambda_n),\ \lambda_1>\lambda_2>\dots>\lambda_n>0,\ k(x,t)=B\beta(x)\beta^*(t),$$

with $\beta(x)$ being a $n\times n$ matrix function, such that $\|\beta'(x)\|\in L_\infty[0,1], \beta(x)\beta^*(x) = I_n$, $(\beta'(x)\beta^*(x))_{jj}=0\ \forall j\le m$.

4.2. The next result, which proof will be published elsewhere, may be considered as a complement to the previous one when the first condition of Theorem 4.1 is violated, i.e. a matrix B is allowed to have nonreal spectrum and elementary divisors of arbitrary multiplicities.

Theorem 4.2. *Let the $n\times n$ matrix kernel of the operator (4.1) satisfy the following conditions:*

1) $B=k(x,x)$ is a constant nonsingular matrix;

2) for all $t\in[0,1]$ the derivative $D_xk(x,t)\in C(\Omega)$;

3) for all $x\in[0,1]$ the derivative $k_1(x,t):=D_tD_xk(x,t)\in C(\Omega)$;

4) $\widetilde{k}_1(x,t):=k_1(x,x-t)$ is an entire analytic function with respect to x for all $t\in[0,1]$.

Then the operator K is similar to the operator $J\otimes B$ in $L_p[0,1]\otimes\mathbb{C}^n$ $(1\le p\le\infty)$.

§5. Similarity of Volterra Operators to the Operator $J^\alpha\otimes B$.

5.1. Here we present some sufficient conditions for the operator $K: L_p[0,1]\otimes\mathbb{C}^n\to L_p[0,1]\otimes\mathbb{C}^n$, given by

$$K : f\to \int_0^x k(x,t)f(t)dt, \qquad f=(f_1,\dots,f_n), \tag{5.1}$$

to be similar to the operator $J^\alpha\otimes B$. The proofs will be published elsewhere.

We start with the following proposition:

Proposition 5.1. *Let $\alpha \in \mathbb{C}$, $Re\alpha > 0$ and B_i be $n \times n$ nonsingular matrices $(i = 1, 2)$. Then the operators $J^\alpha \otimes B_1$ and $J^\alpha \otimes B_2$ are similar in $L_p[0,1] \otimes \mathbb{C}^n$ iff so are B_1 and B_2 in $\mathbb{C}^n$.*

Theorem 5.1. *Let the $n \times n$ matrix kernel $k(x,t)$ of an operator K satisfy the following conditions :*

1) $k(x,t)$ and $D_x k(x,t)$ are absolutely continuous with respect to x for almost all $t \in [0,1]$;

2) $D_x^2 k(x,t)$ and $D_t D_x^2 k(x,t)$ are absolutely continuous with respect to t for almost all $x \in [0,1]$ and $D_t^2 D_x^2 k(x,t) \in L_\infty(\Omega) \otimes \mathbb{C}^{n^2}, \Omega = \{0 \le t \le x \le 1\}$;

3) $k(x,x) = 0$ and $D_x k(x,t)\lceil_{t=x} =: B$ is a constant nonsingular matrix with positive eigenvalues and simple elementary divisors.

Then the operator K is similar to the operator $J^2 \otimes B$ in $L_p[0,1] \otimes \mathbb{C}^n$ $(1 \le p \le \infty)$.

5.2. In the case $\alpha \neq 1$, $\alpha \neq 2$ the similarity beween Volterra operators K and $J^\alpha \otimes B$ can be established under more restricting conditions, namely when the auxiliary kernel $\widetilde{k}(x,t) = k(x, x-t)$ is analytic with respect to x and is sufficiently smooth with respect to t.
To state the next result we introduce some notations.
Denote by $A_R[0,1]$ the set of functions$f(x)$ admitting an analytic continuation to the R-blow-up $G_R[0,1] \subset \mathbb{R}^2$ of the interval $[0,1]$, where $G_R[0,1] = \bigcup\limits_{x \in [0,1]} B(x,R)$ and $B(x,R)$ is the disk of radius R with the center at x.

Theorem 5.2. *Suppose that $Re\alpha = m - \varepsilon \ge 0$ with $\varepsilon \in [0,1)$, $m \in \mathbb{Z}_+ \setminus \{0\}$ and the $n \times n$ matrix kernel $k(x,t)$ of the Volterra operator (5.1) satisfies the following conditions:*

1) for all $t \in [0,1]$ the derivatives $D_x^{j-\varepsilon} k(x,t) \in C(\overline{\Omega})$ exist $(0 \le j \le m)$;

2) for all $x \in [0,1]$ the derivative $D_t^m k_1(x,t) \in C(\overline{\Omega})$ exists, where $k_1(x,t) = D_x^{m-\varepsilon} k(x,t)$;

3) $D_t^j k_1(x, x-t) \in A_R[t,1]$ for all $j \in \{1, \dots, m\}$ and for all $t \in [0,1]$ with $R \ge \max\{m(m-1), 1\}$;

4) $\left[D_x^{j-\varepsilon} k(x,t)\right]_{t=x} = 0$ $(0 \le j \le m-2)$ and

$$\left[D_x^{\alpha-1} k(x,t)\right]_{t=x} = I_n.$$

Then K is similar in $L_p[0,1] \otimes \mathbb{C}^n (1 \le p \le \infty)$ to the operator $J^\alpha \otimes I_n$.

Corollary 5.1. *Let $\alpha = m - \varepsilon + i\beta$ and let $k(x,t)$ be an $n \times n$ matrix kernel such that*

$$k(x,t) = \frac{(x-t)^{\alpha-1}}{\Gamma(\alpha)} P(x,t) \quad , \quad (\varepsilon \ge 0),$$

and $P(x,x) = I_n$. Suppose also that the derivatives $D_x^m P(x,t) = Q(x,t) \in C(\Omega)$ and $D_t^j Q(x,t) =: Q_j(x,t) \in C(\Omega)$ exist and $\widetilde{Q}_j(x,t) = Q_j(x, x-t) \in A_R[t,1]$ $(0 \le j \le m)$ for $R \ge \max\{m(m-1), 1\}$. Then the operator K of the form (5.1) is similar to $J^\alpha \otimes I_n$ in $L_p[0,1] \otimes \mathbb{C}^n$ $(1 \le p \le \infty)$.

Example. Let $\alpha = m - \varepsilon + i\beta,\ \varepsilon \in [0,1)$ and the $n \times n$ matrix kernel have the form

$$k(x,t) = \frac{(x-t)^{\alpha-1}}{\Gamma(\alpha)} \left[I_n + \sum_{j=1}^{N} \Phi_j(x)\Psi_j(x-t) \right]. \tag{5.2}$$

Here $\Phi_j(x)$ and $\Psi_j(x)$ are $n \times n$ matrix functions with entries from $C^{2m}[0,1]$ and from $A_R[0,1]$ respectively and such that

$$\sum_{j=1}^{N} \Phi_j(x)\Psi_j(0) = 0 \quad , \qquad x \in [0,1].$$

Then the kernel $k(x,t)$ satisfies the conditions of Theorem 5.2 and therefore K is similar to the operator $J^\alpha \otimes I_n$ in $L_p[0,1] \otimes \mathbb{C}^n$.

5.3. Let $\alpha = m - \varepsilon + i\beta,\ m \in \mathbb{Z}_+ \setminus \{0\}$ and $\varepsilon \geq 0$). We denote by $W_1^\alpha[0,1]$ the space of functions $f(x) \in L_1[0,1]$ having a fractional derivative $f^{(\alpha)}(x)$ of order $\mathrm{Re}\alpha = m - \varepsilon > 0$. The functions $f(x) \in W_1^\alpha[0,1]$ are characterized by the integral representation (see [Dj])

$$f(x) = \sum_{j=1}^{m} c_j \frac{x^{\alpha-j}}{\Gamma(\alpha-j+1)} + \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f_1(t)dt, \tag{5.3}$$

where $c_j = f^{(\alpha-j)}(0),\ f_1(x) = f^{(\alpha)}(x) = (D^\alpha f)(x) := (D^m J^{\varepsilon+i\beta} f)(x)$.

We denote the subspace of those $f(x) \in W_1[0,1]$, for which the polynomial in (5.3) is absent, that is $c_j = f^{(\alpha-j)}(0) = 0 (1 \leq j \leq m)$, by $\overset{0}{W}{}_1^\alpha[0,1]$.

Consider now the convolution operator $K : L_p[0,1] \otimes \mathbb{C}^n \to L_p[0,1] \otimes \mathbb{C}^n$ with the $n \times n$ matrix kernel $k(x-t) = (k_{ij}(x-t))$ given by

$$K : f \to \int_0^x k(x-t)f(t)dt, \quad f(t) = (f_1(t), \dots, f_n(t)). \tag{5.4}$$

It is clear that under the conditions

1) $k_{ij}(x) \in W_1^\alpha[0,1] \cap \overset{0}{W}{}_1^{\alpha-1}[0,1]\ (1 \leq i,j \leq n)$;
2) $k^{(\alpha-1)}(0) = B, \det B \neq 0$

an operator K of the form (5.4) admits the following factorization

$$K = (J^\alpha \otimes B)(I + K_1), \tag{5.5}$$

where K_1 is also a convolution operator of the form (5.3) with the $n \times n$ matrix kernel $k_1(x-t)$. Letting

$$F(K) = F(J^\alpha \otimes B) \cdot F(I + K_1), \tag{5.6}$$

we extend the functional calculus of an operator K to the set of functions $F(z)$ not necessary holomorphic in zero but such that the operators $F(J^\alpha \otimes B)$ and $F(I + K_1)$ are well defined. In particular we can set in (5.6) $F(z) = z^s (\Re s > 0)$ and $F(z) = \exp(-1/z^s)$.

Theorem 5.3. *Let $k(x)$ be an $n\times n$ matrix function with entries $k_{ij}(x) \in W_1^{\alpha+1}[0,1]$, $k^{(\alpha-j)}(0) = 0$ for $j \in \{2,\dots,m\}$ and let $k^{(\alpha-1)}(0) = B$ be a nonsingular $n\times n$ matrix. Then the convolution operator K of the form (5.4) is similar to the operator $J^\alpha \otimes B$ in the spaces $L_p[0,1]\otimes \mathbb{C}^n (1 \le p \le \infty)$.*

Sketch of proof (compare [M2] and [M4, p.89]). Since $k_{ij}(x) \in W_1^{\alpha+1} \cap W_1^{\alpha-1}$, the operator K admits the factorization (5.5) where K_1 is a convolution operator with $n\times n$ matrix kernel $k_1(x-t)$ and $k_1(x) \in W_1^1[0,1]\otimes \mathbb{C}^{n^2}$. Let $(1+z)^{1/\alpha}$ be the branch of multifunction $(1+z^{1/\alpha})$ in $\Pi_+ = \{z \in \mathbb{C} : \Re z > 0\}$ uniquelly determined by the condition $(1+x)^{1/\alpha} > 0$ for $x \in \mathbb{R}_+$. In view of the condition $\sigma(K_1) = \{0\}$, it follows from Riesz's representation, that the operator

$$K_2 = (I+K_1)^{1/\alpha} - I = \frac{1}{2\pi i}\int\limits_{|z|=1/2} (1+z)^{1/\alpha}(zI-K_1)^{-1}dz - I = \sum_{j=1}^{\infty}\binom{1/\alpha}{j}K_1^j \tag{5.7}$$

is well defined. It is easily seen from (5.7) that K_2 is a convolution Volterra operator of the form (5.4) with $n\times n$ matrix kernel $k_2(x) \in W_1^1[0,1]\otimes \mathbb{C}^{n^2}$. Setting

$$P = (J\otimes B^{1/\alpha})(I+K_2),$$

one observes, that the operator P satisfies the conditions of Theorem 4.2, since $P(t) \in W_1^2[0,1]\otimes \mathbb{C}^{n^2}$ and the auxiliary kernel $\widetilde{P}(x,t) = P(t)$ does not depend on x. Therefore the operator P is similar to the operator $J\otimes B^{1/\alpha}$ in $L_p[0,1]\otimes\mathbb{C}^n$, namely there exists an automrphism T of $L_p[0,1]\otimes\mathbb{C}^n$, such that $T^{-1}PT = J\otimes B^{1/\alpha}$. It is not difficult to prove, using the equality $P^\alpha = K$ and functional calculus (5.5), (5.6), that $T^{-1}KT = J^\alpha \otimes B$ (for integer $\alpha = m \in \mathbb{Z}_+$ this fact is obvious). □

Remark 5.2. When applied to the convolution Volterra operator K of the form (5.4) Theorem 5.2 yields a result weaker then Theorem 5.3. Namely, the similarity of K and $J^\alpha\otimes B$ follows from the Theorem 5.2 only if $k(x) \in W_1^{2m}[0,1]\otimes\mathbb{C}^{n^2} (\alpha = m-1-\varepsilon+i\beta)$, that is under some additional smoothness of $k(x)$.

Remark 5.3. In the scalar case , that is for $n=1$, Theorem 5.2 has been proved for positive $\alpha > 0$ by the author in [M4]. Previous papers on this theme were cited in [M4]. We observe only that a quite different approach to the spectral analysis of Volterra convolution operators, based on the complex analysis techniques [S2] was proposed by R.Frankfurt and J.Rovnyak [FR1]. In particular, for $n=1$ and $\alpha > 0$ they obtained a result slightely weaker than Theorem 5.3. An improvement of their result coinsiding with Theorem 5.3 for $n=1$ and $\alpha > 0$ has been obtained in [M1], [M2].

5.4. Since for similar operators the lattices of invariant and hyperinvariant subspaces are isomorphic, one can obtain from Theorems 4.1, 5.1-5.3 and 3.2 the following corollaries.

Corollary 5.2. *Let an operator K satisfy the conditions of one of Theorems 4.1, 5.1, 5.2 or 5.3 and B satisfy the conditions on Theorem 3.1. Then*

$$Hyplat K = Hyplat(J^\alpha \otimes B),$$

that is $Hyplat K$ has the form (3.32).

Corollary 5.3. *Let an operator K satisfy either the conditions of Theorem 5.2 or of Theorem 5.3 and let $B = diag(\lambda_1, \dots, \lambda_n)$ be a nonsingular diagonal matrix with eigenvalues of different arguments. Then the lattice $Lat K$ is anti-isomorphic to the unite cube $\square_n$:*

$$Lat K = \{\oplus_{i=1}^n L_2[a_i, 1] : 0 \le a_i \le 1\} = Lat(J^\alpha \otimes B).$$

Corollary 5.4. *Let an operator K satisfy the conditions of Theorem 4.1, where*

$$B = B_1 \oplus B_2, B_1 = diag(\lambda_1, \dots, \lambda_{n_1}), B_2 = diag(\mu_1, \dots, \mu_{n_2})$$

and

$$\lambda_i = \lambda_1 s_{1i} > 0, (1 \le i \le n_1), 1 = s_{11} \le s_{12} \le \dots \le s_{1n_1},$$
$$\mu_i = \mu_1 s_{2i} < 0, (1 \le i \le n_2), 1 = s_{21} \le s_{22} \le \dots \le s_{2n_2}$$

Then

$$Hyplat K = Hyplat(J \otimes B_1) \oplus Hyplat(J \otimes B_2) =$$
$$\oplus_{i=1}^2 \{\oplus_{j=1}^{n_i} E_{a_{ij}} : (a_{i1}, \dots, a_{in_i}) \in P(s_{i1}, \dots, s_{in_i})\}$$

where $P(s_{i1}, \dots, s_{in_i})$ is the parallelepiped defined by the relation (3.28).

§6. Cyclicity of Volterra operators and Gohberg-Krein Conjecture.

6.1. Here we study the stability of the property of a Volterra operator to be cyclic under a nilpotent perturbation.

At first we consider Volterra operators with kernels depending on the difference of arguments. In order to state the next result we remind the notation

$$E_a^p = \chi_{[a,1]} L_p[0, 1]$$

Theorem 6.1. *Let $\alpha > 0$, B be a diagonal matrix with eigenvalues of different arguments:*

$$B = diag(\lambda_1, \dots, \lambda_r), \quad \arg \lambda_i \ne \arg \lambda_j \quad (for\ i \ne j)$$

Let also K be a nilpotent operator in $L_p[0, 1]$ $(1 \le p < \infty)$ of the form

$$K : f \to \int_0^x k(x - t) f(t) dt, \qquad f \in L_p[0, 1] \tag{6.1}$$

and let $A = (J^\alpha + K) \otimes B$. *Then*

$$LatA = Lat(J^\alpha \otimes B) = \{\oplus_{j=1}^r E_{a_j}^p : \ a_j \in [0,1]\} \tag{6.2}$$

and, consequently, $LatA$ *is anti-isomorphic to the unit cube* $\square_r$.

Proof. i) Let $\pi_j (1 \le j \le r$ be a projection from $L_p[0,1] \otimes \mathbb{C}^r$ onto $L_p[0,1]$ defined by

$$\pi_j : L_p[0,1] \otimes \mathbb{C}^r \in f = (f_1, \dots, f_r) \to f_j \in L_p[0,1]$$

For an arbitrary invariant subspace $E(\subset L_p[0,1] \otimes \mathbb{C}^r)$ we introduce the subspaces

$$E_j = \operatorname{span}\{(J^\alpha + K)^n f_j : n \ge 0, f_j \in \pi_j E\}, \qquad 1 \le j \le r$$

and establish that $E = \oplus_{j=1}^r E_j$. Since the inclusion $E \subset \oplus_{j=1}^r E_j$ is obvious it suffices to prove the opposite one.

Let m be the least natural number, such that $K^m = 0$. Since J^α and K commute, we have

(6.3)

$$(I - \lambda A)^{-1} = (I - \lambda(J^\alpha \otimes B))^{-1} + \sum_{p=1}^{m} \lambda^p (K \otimes B)^p (I - \lambda(J^\alpha \otimes B))^{-p-1} =$$
$$= \oplus_{j=1}^r \left[\sum_{p=0}^{m-1} \lambda^p \lambda_j^p K^p (I - \lambda\lambda_j J^\alpha)^{-p-1} \right].$$

Using the obvious identity

$$(I - \lambda C)^{-p-1} = \frac{1}{p!} \frac{d^p}{d\lambda^p} [\lambda^p (1 - \lambda C)^{-1}], \qquad p \in \mathbb{Z}_+ \tag{6.4}$$

one concludes that the vector function $(I - \lambda\lambda_j J^\alpha)^{-p-1} f_j$ admits the representation

$$(I - \lambda\lambda_j J^\alpha)^{-p-1} f_j = f_j(x) + \int_0^x G_p(x-t; \lambda\lambda_j) f_j(t) dt \tag{6.5}$$

with

$$G_p(t;\lambda) := \frac{1}{p!} \frac{d^p}{d\lambda^p} [\lambda^p t^{\alpha-1} G_{1/\alpha}(\lambda t^\alpha; \alpha)], \qquad p \in \mathbb{Z}_+ \tag{6.6}$$

Let $g = \oplus_{j=1}^r g_j \in E^\perp \in L_q[0,1] \otimes \mathbb{C}^r$ and $k_p(x) = (\underbrace{k * \ldots * k}_{p})(x)$ be the kernel of the operator K^p. Introducing the notations

$$\begin{aligned} \varphi_j(t) &:= \varphi_{j0}(t) = \int_t^1 f_j(x-t)\overline{g_j(x)} dx, \qquad j \in \{1, \dots, r\} \\ \varphi_{jp}(t) &:= \int_t^1 k_p(s-t) \varphi_j(s) ds \qquad (1 \le p \le m-1) \end{aligned} \tag{6.7}$$

one obtains from (6.5) and (6.7):
(6.8)

$$(K^p(I-\lambda\lambda_j J^\alpha)^{-p-1}f_j, g_j) = \int_0^1 \overline{g_j(x)}dx \int_0^x k_p(x-t)f_j(t)dt +$$

$$+\int_0^1 \overline{g_j(x)}dx \int_0^x G_p(t;\lambda\lambda_j)dt \int_0^{x-t} k_p(x-t-\xi)f_j(\xi)d\xi$$

$$= \int_0^1 k_p(t)dt \int_t^1 f_j(x-t)\overline{g_j(x)}dx + \int_0^1 G_p(t;\lambda\lambda_j)dt \int_t^1 \overline{g_j(x)}dx \int_t^x k_p(s-t)f_j(x-s)ds$$

$$= \int_0^1 k_p(t)\varphi_j(t)dt + \int_0^1 G_p(t;\lambda\lambda_j)\varphi_{jp}(t)dt$$

We clarify that for $p=0$ the first summand in (6.8) should be replaced by the term $\varphi_j(0)$. Setting

$$(6.9)\qquad F_{jp}(\lambda) := \lambda^p\lambda_j^p \int_0^1 G_p(t;\lambda\lambda_j)\varphi_{jp}(t)dt, \qquad F_j(\lambda) := \sum_{p=0}^{m-1} F_{jp}(\lambda)$$

one concludes using (6.8) that for arbitrary vectors $f=(f_1,\dots,f_r)\in E$ and $g=(g_1,\dots,g_r)\in E^\perp$:

$$(6.10)\qquad 0 = ((I-\lambda A)^{-1}f, g) = \sum_{j=1}^r F_j(\lambda) + \sum_{j=1}^r \sum_{p=0}^{m-1} (k_p, \varphi_{jp}).$$

Let now

$$(6.11)\quad l_{jp} = \inf\{l\in[0,1] : \int_l^1 |\varphi_{jp}(t)|^2 dt = 0\},\ p\in\{0,\dots,m-1\},\ j\in\{1,\dots,r\}.$$

Since $K^m = 0$, the Titchmarsh convolution theorem yields, that $k(x)=0\ \forall x\in[0,1/m]$, whence

$$(6.12)\qquad k_p(x) = 0 \qquad \forall x\in[0,p/m],\ \ p\in\{1,\dots,m-1\}$$

and therefore

$$(6.13)\qquad l_{jp} \le \max\{l_{j0}-p/m, 0\} \qquad \forall p\in\{0,\dots,m-1\}.$$

Bearing in mind (6.9) and (6.11) one rewrites the entire functions $F_{jp}(\lambda)$ in the form

$$(6.14)\qquad F_{jp}(\lambda) = \lambda^p\lambda_j^p \int_0^{l_{jp}} G_p(t;\lambda\lambda_j)\varphi_{jp}(t)dt, p\in\{0,\dots,m-1\}.$$

When made use of, (6.6) implies, that the indicator functions of the entire functions $G_p(t,\lambda)$ coincide for $p \in \{0,1,\dots,m-1\}$. Therefore one obtains from (6.9), (6.14) and (6.13) that the indicator function $h_{F_j}(\theta)$ of the entire function $F_j(\lambda)$ $(1 \le j \le r)$ coincides with the indicator function $h_{F_{j0}}(\theta)$ of its first summand

$$F_{j0}(\lambda) = \int_0^{l_j} t^{\alpha-1} E_{1/\alpha}(\lambda\lambda_j t^\alpha;\alpha)\varphi_j(t)dt, \quad (l_j := l_{j0}). \tag{6.15}$$

In order to obtain the implication

$$\sum_{j=1}^{r} F_j(\lambda) = 0 \Longrightarrow F_j(\lambda) = 0 \qquad \forall j \in \{1,\dots,r\} \tag{6.16}$$

it suffices to repeat the arguments used in the proof of Proposition 2.1. Thus

$$F_j(\lambda) = ((I - \lambda\lambda_j(J^\alpha + K))^{-1} f_j, g_j) = 0 \ \ \forall j \in \{1,\dots,r\} \tag{6.17}$$

and consequently the implications $g \in E^\perp \Rightarrow g_j \in E_j^\perp (1 \le j \le r)$ are established. So, $E^\perp \subset \oplus_{j=1}^r E_j^\perp$, that is $E = \oplus_{j=1}^r E_j$. By other words the splitting of the lattice $\mathrm{Lat}A$, i.e., the equality

$$\mathrm{Lat}A = Lat((J^\alpha + K) \otimes B) = \oplus_{j=1}^r \mathrm{Lat}(\lambda_j(J^\alpha + K)) \tag{6.18}$$

is proven.

ii) Setting

$$a_j = \inf\{l \in [0,1] : \int_0^l |f_j(t)|^p dt = 0 \ \ \forall f_j \in E_j\} \tag{6.19}$$

we will show that $E_j^p = E_{a_j}^p$ for each $j \in \{1,\dots,r\}$. In order to prove this we note at first that there exists a function $f_j^0(t) \in E_j$ with $f_j^0(t - a_j)$ satisfying the ε-condition (1.4).

Futher, letting $f_j = f_j^0(t)$ in (6.7) we establish the equality $l_j = l_{j0} = 0 \ \forall j \in \{1,\dots,r\}$. Assuming the contrary one observes that $F_{j0}(\lambda)$ is an entire function of growth $p = 1/\alpha$ and type $l_j > 0$. On the other hand $F_{jp}(\lambda)$ (for $p \ge 1$) is an entire function of the same growth (if $l_{jp} \ne 0$) and type l_{jp}. This contradicts the equality $F_j(\lambda) = \sum_{i=0}^{m-1} F_{ji}(\lambda) = 0$.

Thus $l_j = 0$ that is

$$\varphi_j(t) = \int_t^1 f_j^0(x - t)\overline{g_j(x)}dx = 0 \qquad \forall t \in [0,1]. \tag{6.20}$$

Since $f_j^0(t - a_j)$ satisfies the ε-condition, it follows from (6.20) and the Titchmarsh convolution theorem that $g_j(x) = 0 \ \ \forall x \in [a_j, 1]$. It means that $(E_j^p)^\perp \subset (E_{a_j}^p)^\perp$, i.e., $E_j^p \supset E_{a_j}^p$. This completes the proof since the opposite inclusion $E_j^p \subset E_{a_j}^p$ is obvious. $\square$

Corollary 6.1. *Let the conditions of Theorem 6.1 be satisfied. Then the following are true:*

1) The operator A is cyclic in $L_p[0,1] \otimes \mathbb{C}^r$ $(1 \le p < \infty)$;

2) A vector $f = (f_1, \dots, f_r)$ is cyclic ($f \in Cyc(A)$) iff $f_j(x)$ satisfies the ε-condition (1.4) for each $j \in \{1, \dots, r\}$.

Remark 6.1. In the case $r = 1$ Theorem 6.1 means that the operator $A = J^\alpha + K$ is unicellular in $L_p[0,1]$ $(1 \le p < \infty)$ and $\mathrm{Lat}(J^\alpha + K) = \mathrm{Lat}\, J^\alpha$. A quite different proof of this fact for $\alpha = n \in \mathbb{Z}_+$ is contained in [Ka1]. For $\alpha \in \mathbb{R}_+$ the unicellularity of the operator $J^\alpha + K$ was announced in [Ki].

6.2. To prove the next result we need the following

Proposition 6.1. *Let K be a Volterra operator in $L_2[0,1]$ of the form*

$$K : f \to \int_0^x k(x,t) f(t) dt, \qquad f \in L_p[0,1] \tag{6.21}$$

with kernel $k(x,t) \in C(\Omega)$, $\Omega = \{0 \le t \le x \le 1\}$ and $k(x,t,\lambda)$ be its resolvent kernel. If

$$k(x,t) = \frac{(x-t)^{\alpha-1}}{\Gamma(\alpha)} + o((x-t)^\alpha), \qquad \alpha \ge 2$$

then

1) the asymptotic formula

$$k(x,t,\lambda) = \frac{1}{\alpha} \sum_n \left[\lambda^{1/\alpha}(x-t) e^{i2\pi n/\alpha}\right]^{1-\alpha} \exp(e^{i2\pi n/\alpha} \lambda^{1/\alpha}(x-t)) +$$
$$+ O(\exp \varepsilon \lambda^{1/\alpha}(x-t)).$$

holds as $|\lambda| \to \infty$.

2) for $0 < \alpha < 2$ and as $|\lambda| \to \infty$ we have the asymptotic formula

$$k(x,t,\lambda) = \frac{1}{\alpha} \lambda^{(1-\alpha)/\alpha} \exp(\lambda^{1/\alpha}(x-t)) + O(\frac{1}{|\lambda|}), \ |\arg \lambda| \le \beta$$

if $\pi\alpha/2 < \beta < \min\{\pi, \pi\alpha, \varphi + \pi\alpha/2\}$.

The proof will be published elsewhere.

Theorem 6.2. *Let K_1 and K_2 be Volterra operators in $L_2[0,1]$ of the form (6.21) with kernels $k_1(x,t)$and $k_2(x,t)$ such that*

a) $k_1(x,t)$ satisfies the conditions of one of Theorems 4.1 or 5.1-5.3 for $n = 1$;

b) $k_2(x,t) \in C(\Omega)$, where $\Omega = \{0 \le t \le x \le 1\}$ and

$$k_2(x,t) = 0 \qquad \forall (x,t) \in \Omega_\varepsilon := \{(x,t) \in \Omega : 0 \le x - t \le \varepsilon\}. \tag{6.22}$$

Then the Volterra operar $K = K_1 + K_2$ with kernel $k(x,t) = k_1(x,t) + k_2(x,t)$ is unicellular in $L_2[0,1]$ and

$$LatK = LatJ = \{E_a^p : 0 \le a \le 1\}. \tag{6.23}$$

Proof. i) It has been shown in the proof of Theorem 4.1 that

$$W^{-1}(I+R)^{-1}K_1(I+R)W = J^\alpha$$

where the operator $I+R$ is of the form (4.17) and W is a multiplication operator in $L_p[0,1]$:

$$W : f \to W(x)f(x), \qquad f(x) \in L_p[0,1]$$

It is not difficult to verify that the kernel $\widetilde{k}_2(x,t)$ of the operator $\widetilde{K}_2 = W^{-1}(I+R)^{-1}K_2(I+R)W$ satisfies the condition (6.22) simultaneously with the kernel $k_2(x,t)$. This fact permits us to deduce the case of general operator K_1 to the case $K_1 = J^\alpha$.

ii) Let $K_1 = J^\alpha$and $k_j(x,t,\lambda)$ be the resolvent kernel of the operator $K_j (1 \le j \le 2)$, that is

$$(I-\lambda K_j)^{-1}f = f(x) + \int_0^x k_j(x,t,\lambda)f(t)dt. \tag{6.24}$$

In accordance with (1.9), (1.10)

$$k_1(x,t,\lambda) = k_1(x-t,\lambda) = (x-t)^{\alpha-1}E_{\alpha/1}(\lambda(x-t)^\alpha;\alpha).$$

Since

$$(I-\lambda K)^{-1} = (I-\lambda K_1)^{-1} + \lambda(I-\lambda K_1)^{-1}K_2(I-\lambda K)^{-1}, \tag{6.25}$$

the resolvent kernel $k(x,t,\lambda)$ of the operator K satisfies the equation

$$k(x,t,\lambda) = k_1(x-t,\lambda) + G(x,t,\lambda) \tag{6.26}$$

with

$$\begin{gathered} G(x,t,\lambda) = \lambda k_2(x,t) + \lambda\int_t^x k_2(x,s)k(s,t,\lambda)ds+ \\ +\lambda\int_t^x k_1(x-s,\lambda)k_2(s,t)ds + \lambda\int_t^x k_1(x-\xi,\lambda)d\xi\int_t^\xi k_2(\xi,s)k(s,t,\lambda)ds. \end{gathered} \tag{6.27}$$

It is clear from (6.27) that

$$G(x,t,\lambda) = 0 \qquad \forall (x,t) \in \Omega_\varepsilon \tag{6.28}$$

because so is $k_2(x,t,\lambda)$.

Further, it easily follows from (6.28) that for $(x,t) \in \Omega \setminus \Omega_\varepsilon = \{(x,t) \in \Omega : x-t > \varepsilon\}$:

$$G(x,t,\lambda) = \lambda k_2(x,t) + \lambda \int_t^{x-\varepsilon} k_2(x,s)k(s,t,\lambda)ds + \tag{6.29}$$
$$+\lambda \int_t^{x-\varepsilon} k_1(x-s,\lambda)k_2(s,t)ds + \lambda \int_t^{x-\varepsilon} k_1(x-s,\lambda)ds \int_t^{s-\varepsilon} k_2(s,\xi)k(\xi,t,\lambda)d\xi$$

Let $\alpha \geq 2$. In this case the following estimate

$$k_1(x-t,r) = (x-t)^{\alpha-1}E_{1/\alpha}(r(x-t)^\alpha;\alpha) = O\left(r^{\frac{1-\alpha}{\alpha}}e^{(x-t)r^{1/\alpha}}\right) \tag{6.30}$$

on the ray $\{\lambda \in \mathbb{C} : \lambda = r > 0\}$ immediately follows from (2.8). At the same time, according to Proposition 6.1 we have asymptotically

$$k(x,t,r) = O\left(r^{\frac{1-\alpha}{\alpha}}e^{(x-t)r^{1/\alpha}}\right) \tag{6.31}$$

as $r \to \infty$.

In order to estimate $G(x,t,r)$ we evaluate each term on the right-hand side of the equality (6.29). Using (6.30) and (6.31), one obtains

(6.32)

$$\left| r \int_t^{x-\varepsilon} k_1(x-s,r)ds \int_t^{s-\varepsilon} k_2(s,\xi)k(\xi,t,r)d\xi \right| \leq$$
$$\leq cr^{\frac{2-\alpha}{\alpha}} \int_t^{x-\varepsilon} \exp((x-s)r^{1/\alpha})ds \int_t^{s-\varepsilon} \exp((\xi-t)r^{1/\alpha})d\xi \leq$$
$$\leq cr^{\frac{1-\alpha}{\alpha}} \int_t^{x-\varepsilon} \exp((x-s)r^{1/\alpha})\exp((s-t-\varepsilon)r^{1/\alpha})ds \leq cr^{\frac{1-\alpha}{\alpha}} \exp((x-t-\varepsilon)r^{1/\alpha}).$$

Other terms can be estimated similarly and even more simply. Making use of (6.29) and (6.32) we deduce that

$$G(x,t,r) = O(r^{1/\alpha}\exp((x-t-\varepsilon)r^{1/\alpha})) \tag{6.33}$$

as $r \to \infty$.

In the case $\alpha \in (0,2)$ the estimate (6.33) can be derived similarly from the asymptotic behavior of the resolvent kernel $k(x,t,\lambda)$ contained in Proposition 6.1.

iii) Continuing the proof we consider any invariant subspace E of the operator K. Denoting

$$a = \sup\{l \in [0,1] : \int_0^l |f(t)|^p dt = 0 \quad \forall f \in E\} \tag{6.34}$$

we show that $E = E_a^p$. Choosing a vector $f(x) \in E$ such that $f(x-a)$ satisfies the ε-condition (1.4) and an arbitrary vector $g \in E^{\perp}(\subset L_q[0,1])$ one obtains from (6.26)

$$\int_0^1 k_1(t,\lambda)dt \int_t^1 f(x-t)\overline{g(x)}dx + \int_0^1 dt \int_t^1 G(x,x-t,\lambda)f(x-t)\overline{g(x)}dx = 0 \tag{6.35}$$

Setting

$$\varphi_1(t) = \int_t^1 f(x-t)\overline{g(x)}dx, \quad \varphi_2(t) = \int_t^1 |f(x-t)\overline{g(x)}|dx, \tag{6.36}$$

we introduce two nonnegative numbers

$$b_i = \inf\{l \in [0,1] : \int_l^1 |\varphi_i(t)|^p dt = 0\}, \quad (i = 1,2). \tag{6.37}$$

The Titchmarsh convolution theorem immediately yields the equality $b_1 = b_2$. It will be established that $b := b_1 = b_2 = 0$.

Assuming the contrary (i.e $b > 0$) we denote by $F_1(\lambda)$ and $F_2(\lambda)$ respectively the first and the second summand on the left-hand side of the equation (6.35). Since $k_1(t,\lambda) = t^{\alpha-1}E_{1/\alpha}(\lambda t^\alpha;\alpha)$ the relations (6.36), (6.37) and (6.30) yield that

$$F_1(\lambda) = \int_0^b t^{\alpha-1}E_{1/\alpha}(\lambda t^\alpha;\alpha)dt \tag{6.38}$$

is an entire function of growth $1/\alpha$ and type b. Bearing in mind (2.16), (2.17) and (6.30) we have

$$\overline{\lim_{r\to\infty}} \frac{\ln|F_1(r)|}{r^{1/\alpha}} = b. \tag{6.39}$$

On the other hand, when combined, the relations (6.33), (6.35) and (6.36) yield the the estimate

$$\begin{aligned} F_2(r) &\le \int_0^1 dt \int_t^1 |G(x,x-t,r)| \cdot |f(x-t)\overline{g(x)}|dx \le \\ &\le cr^{1/\alpha} \int_0^1 \varphi_2(t)\exp((t-\varepsilon)r^{1/\alpha})dt. \end{aligned} \tag{6.40}$$

It is now clear, that the inequality

$$\overline{\lim_{r\to\infty}} r^{-1/\alpha} \ln|F_2(r)| \le \max\{b-\varepsilon, 0\}, \tag{6.41}$$

being an immediate consequence of (6.40), contradicts the equalities (6.35) and (6.39).

Thus $b = 0$, whence $\varphi_1(t) = 0 \ \ \forall t \in [0,1]$. In view of the Titchmarsh convolution theorem one obtains

$$f(x) = 0 \ \ \forall x \in [0,\beta], \quad g(x) = 0 \ \ \forall x \in [\beta, 1].$$

Since $f(x-a)$ satisfies the ε-condition (1.4) one obtains that $\beta \le \alpha$ and therefore $g(x) \in E_a^{\perp}$. So $E^{\perp} \subset E_a^{\perp}$, that is $E \supset E_a^{\perp}$. To complete the proof, it suffices to observe that the opposite including immediately follows from the definition (6.34). □

Remark 6.2. For $\alpha = n$ Proposition 6.1 has been proved in [Khr], [KhM]. It was also shown (for $\alpha = n$), that $g(x) = x^m/m! + o(x^m) \in \mathrm{Cyc}(K)$ that is the vectors $g(x)$ are cyclic for the operator K of the form (6.21) with sufficiently smooth kernel $k(x,t)$. The whole set $\mathrm{Cyc}(K)$ has not been described in [Khr], [KhM].

Combining the proofs of Theorems 6.1 and 6.2, one obtains the following result.

Theorem 6.3. *Let $\alpha > 0$, B be a diagonal matrix with eigenvalues of different arguments.*

$$B = diag(\lambda_1, \dots, \lambda_n), \quad \arg \lambda_i \ne \arg \lambda_j \ (1 \le i < j \le n).$$

Let also $K = K_1 + K_2$ be an operator satisfying the conditions of Theorem 6.2 and $A = (K_1 + K_2) \otimes B$. Then the operator A is cyclic in $L_p[0,1] \otimes \mathbb{C}^n$ and

$$LatA = \{\oplus_{j=1}^{n} E_{a_j}^{p} : a_j \in [0,1], \ 1 \le j \le n\}.$$

Remark 6.3. It is clear, that for a continuous kernel $k(x-t)$ Theorem 6.1 is a corollary of the last theorem . But we prefer a simple direct proof which doesn't rely on Proposition 6.1.

6.3. The Gohberg-Krein conjecture.

It is well known [GK], that each unicellular operator A in a separable Hilbert space $\mathfrak{H}$ is cyclic. I. Gohberg and M. Krein [GK,p.421] have presupposed that the opposite assertion holds true for a simple dissipative operator A. Futher we present simple counterexamples based on Proposition 3.2. Nevertheless the following theorem shows that this conjecture is really true under some additional assumptions.

Theorem 6.4. *Let A be a Volterra operator in a Hilbert space $\mathfrak{H}$ satisfying the conditions:*

1) $\ker A = \{0\}$;

2) $\pi\alpha_1 \le \arg(af, f) \le \pi\alpha_2 \ \ \forall f \in \mathfrak{h} \setminus \{0\}$ and $\alpha := \alpha_2 - \alpha_1 \le 1$;

3) the resolvent $(I - \lambda A)^{-1}$ is an entire function of growth $\rho = 1/\alpha$ and normal type.

Then the cyclicity of an operator A is equivalent to its unicellularity.

The proof will be published elsewhere.

Counterexamples. Choosing $\rho > 1/\alpha$ let us consider in $L_2[0,1]\otimes\mathbb{C}^2$ the operator

$$A = J^{1/p} \oplus e^{i(\alpha-1/\rho)}J^{1/p} \tag{6.41}$$

In view of Proposition 3.2 the operator A is cyclic but is not unicellular in $L_2[0,1]\otimes\mathbb{C}^2$. At the same time $|\arg(Af,f)| \le \pi\alpha/2$ and the resolvent $(I-\lambda A)^{-1}$ is an entire function of growth ρ ($> 1/\alpha$). For $\alpha = 1$ the operators iA present counterexamples to the Gohberg-Krein's conjecture.

Remark 6.5. It should be pointed out that the inequality $\rho < 1/\alpha$ does not hold for Volterra operators. This fact together with the above counterexamples yields the exactness of the conditions of Theorem 6.4.

6.4. Volterra operators J^α and $J^\alpha\otimes B$ are useful for many reasons. Making use of the operators J^α for $\alpha > 1$ we refine one result due to C. Foias and J. Williams [FW].

Proposition 6.2. *Let $\alpha\in(0,1)$. Then the operator $J^\alpha(I-J^a)^{-1}$ is not similar to the operator J^α in $L_p[0,1]$ for each $p\in[1,\infty]$.*

Proof. It is easily seen that

$$J^\alpha(I-J^\alpha)^{-1} : f \to \int_0^x (x-t)^{\alpha-1}E_{1/\alpha}((x-t)^\alpha;\alpha)f(t)dt$$

Here $E_{1/\alpha}(x^\alpha;\alpha)$ is the Mittag-Leffler function defined by (1.10). It is clear that

$$E_{1/\alpha}(x^\alpha;\alpha) = \frac{1}{\Gamma(\alpha)} + x^\alpha k_1(x), \qquad \alpha < 1,$$

where $k_1(0)\ne 0$ and $k_1(x) > 0$ for all $x\in[0,1]$. Since $\alpha < 1$ we obtain the proposition as an immediate consequence of Theorem 2 from [M2]. $\square$

Remark 6.6. Foias and Williams [FW] have proved that there exists a quasi-nilpotent operator Q in the commutant of J such that Q and $Q(I-Q)^{-1}$ are not similar in $L_2[0,1]$. Thus Proposition 6.2 refines their result. Observe also that $J^\alpha(I-J^\alpha)^{-1}\in\mathfrak{S}_p$ for all $p > 1/\alpha$.

§7. Transformation Operators and Inverse problems.

7.1. Let $B = \mathrm{diag}(\lambda_1 I_{n_1},\dots,\lambda_r I_{n_r})$ be a nonsingular diagonal $n\times n$ matrix with real eigenvalues $\lambda_j = \overline{\lambda}_j$ of multiplicities $n_j (1\le j\le r), Q(x)$ be an $n\times n$ matrix function such that

$$Q(x) = (Q_{ij}(x))_{i,j=1}^r, \qquad Q_{ii}(x) = 0 \quad \forall x\in[0,1] \qquad (1\le i\le r) \tag{7.1}$$

where $Q_{ij}(x)$ are $n_j\times n_j$ matrix functions with entries from $L_\infty[0,1]$.

We consider an operator

$$L : f \to B\otimes\frac{df}{dx} + Q(x)f + \int_0^x M(x,t)f(t)dt \tag{7.2}$$

acting in $H = L_2[0,1] \otimes \mathbb{C}^n$ with the domain of the definition $\mathcal{D}(L) = \mathcal{D}(B \otimes D) = W_2^1[0,1] \otimes \mathbb{C}^n$, where $D = d/dx$ and $M(x,t)$ is an $n \times n$ matrix function such that $M(x,t) \in L_\infty(\Omega) \otimes \mathbb{C}^{n^2}$, $\Omega = \{0 \le t \le x \le 1\}$.

Let for the definiteness $n_1 = \max\{n_j : 1 \le j \le r\}$. Setting $A_1 = I_{n_1}$ we introduce matrices $A_j \in [\mathbb{C}^{n_1}, \mathbb{C}^{n_j}]$ of maximal rank :

$$(7.3) \qquad A_j : \mathbb{C}^{n_1} \to \mathbb{C}^{n_j}, \qquad \operatorname{rank} A_j = n_j \quad (1 \le j \le r)$$

and the subspace $E(\{A_j\}_1^r)$ of $W_2^1[0,1] \otimes \mathbb{C}^n$ defined by

$$(7.4) \quad E(\{A_j\}_1^r) := \{f = (f_1, \dots, f_r) : f_j \in W_2^1[0,1] \otimes C^{n_j}, f_j = A_j f_1, 1 \le j \le r\}$$

Definition 7.1. Let $E = E(\{A_j\})_1^r$ be the subspace of $W_2^1[0,1] \otimes \mathbb{C}^n$ defined in (7.4). An operator $I + K \in [H]$ of triangular form :

$$(7.5) \qquad I + K : f(x) \to f(x) + \int_0^x K(x,t) f(t) dt, \qquad f = (f_1, \dots, f_n)$$

with $n \times n$ matrix kernel $K(x,t)$ will be called a transformation operator for the operator L of the form (7.2) if it maps E_1 onto itself and intertwines the operators $L|E_1$ and $(B \otimes D)|E_1$, that is

$$(7.6) \qquad L(I+K)f = (I+K)(B \otimes D)f \qquad \forall f \in E_1$$

Theorem 7.1. *Let $B = diag(\lambda_1 I_{n_1}, \dots, \lambda_r I_{n_r})$, $Q(x) = (Q_{ij}(x))_{i,j=1}^r$ and $M(x,t)$ be as above. Then there exists a transformation operator $I + K$ for the operator L.*

Proof. Let a transformation operator $I + K$ of the form (7.5) exist and its $n \times n$ matrix kernel $K(x,t)$ is sufficiently smooth. It was shown in the proof of Theorem 4.1 that the relation (7.6) is equivalent to (4.19) and consequently to the system (4.22) with boundary conditions

$$(7.7) \qquad K_{ij}(x,x) = (\lambda_i - \lambda_j)^{-1} Q_{ij}(x), \quad 1 \le i \ne j \le n$$

$$(7.8) \qquad K(x,0) B f(0) = 0$$

Here $f(0) = \operatorname{col}(f_1(0), \dots, f_r(0))$ and $K(x,t) = (K_{ij}(x,t))$ is the block representation of $K(x,t)$ with $n_i \times n_j$ blocks $K_{ij}(x,t)(1 \le i,j \le r)$.

The condition (7.8) was absent in the considerations of §4 because in (4.19) $f(0) = 0 \ \forall f \in \mathcal{D}(L_0) = \mathcal{D}(B \otimes D_0)$. One concludes by reversing the above arguments that the solution $k(x,t)$ of the Goursat problem (4.22), (7.7) and (7.8) is the $n \times n$ matrix kernel of a transformation operator for L, that is $I + K$ satisfies the equation (7.6). Therefore in order to prove the theorem it suffices to prove the solvability of the Goursat problem (4.22), (7.7) and (7.8).

According to the proof of Theorem 4.1 there exists the uniqueness solution $R(x,t) = (R_{ij}(x,t))$ of the Goursat problem (4.22)-(4.24). We use $R(x,t)$ for constructing the required solution of Goursat problem (4.22), (7.7), (7.8). To do this we introduce by means of a quasidiagonal $n \times n$ matrix function

$$\Phi(x) = \mathrm{diag}(\Phi_1(x), \dots, \Phi_r(x)) \tag{7.9}$$

with $n_j \times n_j$ matrix blocks $\Phi_j(x)(1 \le j \le r)$, the convolution operator

$$\Phi : f \to \int_0^x \Phi(x-t)f(t)dt, \qquad f(t) \in L_2[0,1] \otimes \mathbb{C}^n \tag{7.10}$$

and define the operator K by the equality $I + K = (I + R)(I + \Phi)$. It is clear that $K(x,t)$ satisfies the equation

$$K(x,t) = R(x,t) + \Phi(x-t) + \int_t^x R(x,s)\Phi(s-t)ds \tag{7.11}$$

Since Φ commutes with $J \otimes B = D_0^{-1} \otimes B$ one concludes from (4.18) that $I + K$ intertwines the operators L_0 and $D_0 \otimes B$, that is

$$L_0(I + K) = (I + K)(D_0 \otimes B) \tag{7.12}$$

Here the notations L_0 and $D_0 \otimes B$ remain valid for the restrictions on $\overset{0}{\mathrm{W}}{}_2^1[0,1] \otimes \mathbb{C}^n$ of the operators L and $D \otimes B$ respectively.

It follows from (7.12) that the kernel $K(x,t)$ satisfies the system (4.22) and the boundary conditions (7.6) (compare with (4.19) and (4.20)).

Using (7.4) one can rewrite the condition (7.8) in the form

$$[\lambda_1 K_{i1}(x,0)A_1 + \lambda_2 K_{i2}(x,0)A_2 + \ldots + \lambda_r K_{ir}(x,0)A_r]f_1(0) = 0, \quad (1 \le i \le r) \tag{7.13}$$

In order to satisfy the conditions (7.13), we must find $\Phi(x) = \mathrm{diag}(\Phi_1(x), \dots, \Phi_r(x))$ from the following system of Volterra equations of the second kind

$$\sum_{j=1}^r \lambda_j[\Phi_j(x) * \delta_{ij}(x) + \int_0^x R_{ij}(x,s)\Phi_j(s)ds + R_{ij}(x,0)]A_j f_1(0) = 0, \quad (1 \le i \le r) \tag{7.14}$$

As usual δ_{ij} in (7.14) stands for the Kronecker delta and consequently $\Phi_j * \delta_{ij} = 0$ for $i \neq j$ and $\Phi_j * \delta_{ij} = \Phi_j(x)$. Multiplying the i-th equation by A_j^* and bearing the condition $\mathrm{rank} A_j = n_j$ in mind (see (7.3)) one easily derives the existence and uniqueness of the solution $\{\Phi_j(x)\}_1^r$ of the system (7.14). $\square$

The following theorem is equivalent to the previous one.

Theorem 7.1$'$. *Let the diagonal matrix* $B = diag(\lambda_1 I_{n_1}, \dots, \lambda_r I_{n_r})$ *and* $n\times n$ *matrix functions* $Q(x)$ *and* $M(x,t)$ *be as above. Suppose that* $Y(x,\lambda) = \begin{pmatrix} Y_1(x,\lambda) \\ \dots \\ Y_r(x,\lambda) \end{pmatrix}$ *is an* $n \times n_1$ *matrix solution of the following Cauchy problem:*

$$LY = BY'(x,\lambda) + Q(x)Y(x,\lambda) + \int_0^x M(x,t)Y(t,\lambda)dt = \lambda Y(x,\lambda) \tag{7.15}$$

$$Y_j(0,\lambda) = A_j \ \ (A_1 = I_{n_1}, \ rank A_j = n_j, \ 1 \le j \le r) \tag{7.16}$$

where $Y_j(x,\lambda)$ *are* $n_j \times n_1$ *matrix functions* $(1 \le j \le r)$. *Then there exists a unique continous* $n \times n$ *matrix kernel* $K(x,t)$ *such that* $Y(x,\lambda)$ *admits the representation*

$$Y(x,\lambda) = \omega(x,\lambda) + \int_0^x K(x,t)\omega(t,\lambda)dt \tag{7.17}$$

where $\omega(x,\lambda) = \begin{pmatrix} \omega_1(x,\lambda) \\ \vdots \\ \omega_r(x,\lambda) \end{pmatrix}$ *is an* $n \times n_1$ *matrix solution of the Cauchy problem for the simplest equation*

$$B\omega'(x,\lambda) = \lambda\omega(x,\lambda) \tag{7.18}$$

with the same initial data $\omega_j(0,\lambda) = A_j \ (1 \le j \le r)$.

Corollary 7.1. *Let* $B = diag(I_{n_1}, -I_{n_2})$, $n_1 \ge n_2$ *and* $Q(x)$ *be an* $n\times n$ *summable matrix-function such that* $BQ(x) + Q(x)B = 0$, *that is*

$$B = \begin{pmatrix} I_{n_1} & 0 \\ 0 & -I_{n_2} \end{pmatrix}, \ Q(x) = \begin{pmatrix} 0 & Q_1(x) \\ Q_2(x) & 0 \end{pmatrix}, \ (n = n_1 + n_2) \tag{7.19}$$

Suppsose that $Y(x,\lambda) = \begin{pmatrix} Y_1(x,\lambda) \\ Y_2(x,\lambda) \end{pmatrix}$ *is an* $n \times n_1$ *matrix solution of the Cauchy problem*

$$BY'(x,\lambda) + Q(x)Y(x,\lambda) = i\lambda Y(x,\lambda), \tag{7.20}$$

$$Y_1(0,\lambda) = I_{n_1}, \ Y_2(0,\lambda) = A \ (rank A = n_2). \tag{7.21}$$

Then there exists a unique continuous $n\times n$ *matrix kernel* $K(x,t)$ *such that* $Y(x,\lambda)$ *admits the following triangular representation:*

$$\begin{pmatrix} Y_1(x,\lambda) \\ Y_2(x,\lambda) \end{pmatrix} = \begin{pmatrix} e^{i\lambda x} I_{n_1} \\ e^{-i\lambda x} A \end{pmatrix} + \int_0^x K(x,t) \begin{pmatrix} e^{i\lambda t} I_{n_1} \\ e^{-i\lambda t} A \end{pmatrix} dt \tag{7.22}$$

Remark 7.1. Setting $n_1 = n_2 = m, n = 2m$, $Z(x,\lambda) = UY(x,\lambda)$ and

$$U = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} I_m,\ \Omega(x) = \begin{pmatrix} p(x) & q(x) \\ q(x) & -p(x) \end{pmatrix},\ J = \begin{pmatrix} 0 & I_m \\ -I_m & 0 \end{pmatrix},$$

where $p(x) = (Q_1(x) + Q_2(x))/2, \quad q(x) = i(Q_1(x) - Q_2(x))/2$, and observing that $\Omega(x) = UQ(x)U^{-1}$, $J = UBU^{-1}$, one concludes that the system (7.20), (7.21) with $A = I_{n_2}$ is equivalent to the Dirac system

$$JZ'(x,\lambda) + \Omega(x)Z(x,\lambda) = \lambda Z(x,\lambda) \tag{7.23}$$

$$Z_1(0,\lambda) = I_m, \qquad Z_2(0,\lambda) = \mathbb{O}_m \tag{7.24}$$

It follows from (7.22) that $Z(x,\lambda)$ admits the representation

$$Z(x,\lambda) = \begin{pmatrix} \cos\lambda x \\ -\sin\lambda x \end{pmatrix} I_m + \int_0^x K_1(x,t)\begin{pmatrix} \cos\lambda t \\ -\sin\lambda t \end{pmatrix} dt \tag{7.25}$$

with $K_1(x,t) = UK(x,t)U^{-1}$. Different proofs of the representation (7.25) are contained in [Mar], [LS].

7.2. Now we are ready to investigate the inverse problem for the system (7.15) with $M(x,t) = 0$. For the simplicity we confine ourselves to examination on the Dirac system (7.20).

Alongside the system (7.20) we consider the analogous Dirac system with continuous $n \times n$ matrix potential $\widetilde{Q}(x)$ and denote by $\widetilde{Y}(x,\lambda)$ the $n \times n_1$ matrix solution of the Cauchy problem :

$$B\widetilde{Y}'(x,\lambda) + \widetilde{Q}(x)\widetilde{Y}(x,\lambda) = i\lambda\widetilde{Y}(x,\lambda) \tag{7.26}$$

$$\widetilde{Y}(0,\lambda) = \begin{pmatrix} I_{n_1} \\ \widetilde{A} \end{pmatrix},\ \operatorname{rank}\widetilde{A} = n_2 \tag{7.27}$$

Theorem 7.2. *Let $Y(x,\lambda)$ and $\widetilde{Y}(x,\lambda)$ be the $n \times n_1$ matrix solutions of the Cauchy problems (7.20), (7.21) and (7.26), (7.27) respectively. If $Y(1,\lambda) = \widetilde{Y}(1,\lambda)$ $\forall\lambda \in \mathbb{R}$ then $A = \widetilde{A}$ and $Q(x) = \widetilde{Q}(x)$ for all $x \in [0,1]$.*

Proof. 1. For the simplicity we assume that $A = \widetilde{A}$. According to the corollary 7.1 the solution $Y(x,\lambda)$ admits the representation (7.22). A similar representation with continuous $n \times n$ matrix kernel $\widetilde{K}(x,t)$ also holds for the solution $\widetilde{Y}(x,\lambda)$. Therefore $\widetilde{Y}(x,\lambda)$ and $Y(x,\lambda)$ are connected via the triangular transformation

$$\widetilde{Y}(x,\lambda) = (I+P)Y(x,\lambda) = Y(x,\lambda) + \int_0^x P(x,t)Y(t,\lambda)dt \tag{7.28}$$

Here $P(x,t) = (P_{ij})_{i,j=1}^2$ is a continuous $n \times n$ matrix kernel of the Volterra operator P defined by

$$I + P = (I + \widetilde{K})(I + K)^{-1} \tag{7.29}$$

Repeating the arguments used in the proof of Theorem 4.1 one obtains from (7.28) the following boundary problem for $P(x,t)$:

$$BD_xP(x,t) + D_tP(x,t)B = -Q(x)P(x,t) + P(x,t)\widetilde{Q}(t) \tag{7.30}$$

$$BP(x,x) - P(x,x)B = \widetilde{Q}(x) - Q(x) \tag{7.31}$$

$$P_{11}(x,0) - P_{12}(x,0)A = P_{21}(x,0) - P_{22}(x,0)A = 0 \tag{7.32}$$

One more boundary condition for $P(x,t)$ can be easily deduced from the equality $Y(1,\lambda) = \widetilde{Y}(1,\lambda)$. Indeed. using the block partition $K(x,t) = (K_{ij}(x,t))_{i,j=1}^2$ of the kernel $K(x,t)$ with $n_j \times n_j$ blocks $K_{ij}(x,t)$ we deduce from (7.22):

$$Y_1(1,\lambda) = e^{i\lambda}I_{n_1} + \int_0^1 (K_{11}(1,t)e^{i\lambda t}I_{n_1} + K_{12}(1,t)e^{-i\lambda t}I_{n_2})dt$$
$$Y_2(1,\lambda) = e^{-i\lambda}I'_{n_1} + \int_0^1 (K_{21}(1,t)e^{i\lambda t}I_{n_1} + K_{22}(1,t)e^{-i\lambda t}I_{n_2})dt$$

An elemntary fact about the indicator diagram of a sum of entire functions of exponential type (see remark 2.1) yields that $K_{ij}(1,t)(j = 1,2)$ are uniquely determined by $Y_i(1,\lambda)$ for each $i = 1,2)$. Applying the same arguments to the representation (7.23) written for $\widetilde{Y}(x,\lambda)$ with $n \times n$ matrix kernel $\widetilde{K}(x,t) = (\widetilde{K}_{ij}(x,t))_{i,j=1}^2$ and using the condition $Y(1,\lambda) = \widetilde{Y}(1,\lambda)$ one concludes that

$$K(1,t) = \widetilde{K}(1,t)(\Leftrightarrow K_{ij}(1,t) = \widetilde{K}_{ij}(1,t), \qquad i,j \in \{1,2\}) \tag{7.33}$$

It is clear that $K_1 := (I + K)^{-1} - I$ is a Volterra operator of the form (4.1) with $n \times n$ matrix kernels $K_1(x,t)$. Moreover $K_1(x,t)$ and $K(x,t)$ are related by the formula

$$K(x,t) + K_1(x,t) + \int_t^x K(x,s)K_1(s,t)ds = 0 \tag{7.34}$$

Futher the relation (7.29) can be rewritten as

$$P(x,t) = \widetilde{K}(x,t) + K_1(x,t) + \int_t^x \widetilde{K}(x,s)K_1(s,t)ds \tag{7.35}$$

It follows now from (7.33)-(7.35) that

$$(7.36) \qquad P(1,t) = 0 \quad (\Leftrightarrow P_{ij}(1,t) = 0) \quad \forall t \in [0,1]$$

2. Thus for $R(x,t)$ we have the boundary problem (7.30)-(7.32), (7.36). We claim that the problem (7.30),(7.32),(7.36) has the sole solution $P(x,t) = 0$ in $\Omega = \{0 \le t \le x \le 1\}$.

Since

$$Q(x) = \begin{pmatrix} 0 & Q_1(x) \\ Q_2(x) & 0 \end{pmatrix}, \qquad \widetilde{Q}(x) = \begin{pmatrix} 0 & \widetilde{Q}_1(x) \\ \widetilde{Q}_2(x) & 0 \end{pmatrix},$$

one can rewrite the system (7.30) in the form

$$(7.37) \qquad (D_x + D_t)P_{11}(x,t) = -Q_1(x)P_{21}(x,t) + P_{12}(x,t)\widetilde{Q}_2(t)$$

$$(7.38) \qquad (D_x - D_t)P_{12}(x,t) = -Q_1(x)P_{22}(x,t) + P_{11}(x,t)\widetilde{Q}_1(t)$$

$$(7.39) \qquad (D_x - D_t)P_{21}(x,t) = Q_2(x)P_{11}(x,t) - P_{22}(x,t)\widetilde{Q}_2(t)$$

$$(7.40) \qquad (D_x + D_t)P_{22}(x,t) = Q_2(x)P_{12}(x,t) - P_{21}(x,t)\widetilde{Q}_1(t)$$

The characteristic matrix of this sytem is diagonal with real eigenvalues $\lambda_j = \pm 1$. So this system is hyperbolic with characteristic lines $l_C^{\pm} = \{(\xi,\eta) : \xi \pm \eta = C\}$. At first we consider the system (7.37)-(7.40) in the chracteristic triangle $\triangle_1$ confined by the lines

$$1)AB: \ x = 1; \qquad 2)AC: \ x = t; \qquad 3)BC: \ x + t = 1.$$

It is clear that $P(x,t) = (P_{ij}(x,t)) = 0 \ \forall (x,t) \in \triangle_1$, because the problem (7.37)-(7.40), (7.36) is a nonchracteristic Cauchy problem in the chracteristic triangle $\triangle_1$ with zero data on the characteristic base $AB = \{(1,t) : 0 \le t \le 1\}$.

However this fact can be easily derived by integrating the equations (7.37)-(7.40) along the characteristics and taking (7.36) into account. We arrive at the following Volterra-type system of the second kind:

$$(7.41) \quad P_{ii}(x,t) = \int_x^1 [(-1)^j Q_i(\xi)P_{ji}(\xi,\xi-x+t) + (-1)^i P_{ij}(\xi,\xi-x+t)\widetilde{Q}_j(\xi-x+t)]d\xi$$

(7.42)

$$P_{ij}(x,t) = \int_x^1 [(-1)^j Q_i(\xi)P_{jj}(\xi,x+t-\xi) + (-1)^i P_{ii}(\xi,x+t-\xi)\widetilde{Q}_i(x+t-\xi)]d\xi$$

$$i,j \in \{1,2\}, \ i \ne j$$

which has the only trivial solution $P(x,t)=0$ in $\triangle_1$.

3. Let now $\triangle_2=\Omega\setminus\triangle_1$ be the triangle OBC, where $O(0,0)$ is the origin and $(x,t)\in\triangle_2$. Bearing in mind (7.32) and taking the equality $P(x,t)=0\ \forall(x,t)\in\triangle_1$ into account, we obtain the Goursat problem

$$P_{11}(x,1-x)=P_{22}(x,1-x)=0 \tag{7.43}$$

$$P_{12}(x,0)A=P_{11}(x,0),\qquad P_{21}(x,0)=P_{22}(x,0)A \tag{7.44}$$

for the system (7.37)-(7.40).

Since rank$A=n_2$ there exists an $n_1\times n_2$ (not unique) matrix A_1 such that $A\cdot A_1=I_{n_2}$. Therefore one rewrites the first of the conditions (7.44) in the form:

$$P_{12}(x,0)=P_{11}(x,0)A_1. \tag{7.44'}$$

It is clear that the equations (7.41) are valid for all $(x,t)\in\Omega$ (not only in $\triangle_1$). Since $P(x,t)=0\quad\forall(x,t)\in\Delta_1$ it follows from (7.41) that in the triangle $\triangle_2$ they can be rewritten as:

(7.45)

$$P_{ii}(x,t)=\int\limits_{x}^{\frac{1+x-t}{2}}[(-1)^jQ_i(\xi)P_{ji}(\xi,\xi-x+t)+(-1)^iP_{ij}(\xi,\xi-x+t)\widetilde{Q}_j(\xi-x+t)]d\xi$$

Setting $t=0$ in (7.45) and taking (7.44) and (7.44') into account we obtain the expressions for $P_{12}(x,0)$ and $P_{21}(x,0)$ by means of $P_{ij}(x,t)$ with $i\neq j$. Integrating the equations (7.38), (7.39) along the characteristics and using the expressions for $P_{12}(x,0)$ and $P_{21}(x,0)$ which have just been obtained , we deduce the following integral equations in $\triangle_2$:

(7.46)

$$P_{ij}(x,t)=\int\limits_{x+t}^{\frac{1+x+t}{2}}[(-1)^jQ_i(\xi)P_{ji}(\xi,\xi-x-t)+(-1)^iP_{ij}(\xi,\xi-x-t)\widetilde{Q}_j(\xi-x-t)]d\xi A_1$$

$$+\int\limits_{x}^{x+t}[(-1)^iQ_i(\xi)P_{jj}(\xi,x+t-\xi)+(-1)^jP_{ii}(\xi,x+t-\xi)\widetilde{Q}_i(x+t-\xi)]d\xi\qquad i,j\in\{1,2\},\ i\neq j$$

Thus, in $\triangle_2$ we have a homogeneous Volterra-type system (7.46), (7.47) of the second kind. Letting

$$2^{-1}c_1=\max\{\|Q_i(x)\|_{L_\infty[0,1]},\|\widetilde{Q}_i(x)\|_{L_\infty[0,1]}\} \tag{7.47}$$

$$c_2=\max\{\|P_{ij}(x,t)\|_{L_\infty(\Omega)}:1\le i,\ j\le 2\} \tag{7.47'}$$

we prove step by step that for $P_{ij}(x,t)$ with $(x,t)\in\triangle_2$ the following estimates hold:

$$\max_{1\le i\le 2}\{\|P_{ii}(x,t)\|\}\le\frac{c_1^n(1+c_1)^{n-1}c_2}{2^n}\cdot\frac{(1-x-t)^n}{n!} \tag{7.48}$$

$$(7.48') \qquad \max_{1\le i,j<2}\{\|P_{ij}(x,t)\| : i\neq j\} \le \frac{c_1^n(1+c_1)^n c_2}{2^n}\cdot\frac{(1-x-t)^n}{n!}.$$

Really, for $n=1$ these estimates are obvious. We suppose by induction that the inequalities (7.48)-(7.48') hold for $n=m$. Then (7.45) and the induction hypothesis together yield

$$(7.49) \qquad \begin{aligned}\max_{1\le i\le 2}\{\|P_{ii}(x,t)\|\} &\le \int_x^{\frac{1+x+t}{2}} 2\frac{A}{2}\frac{c_1^m(1+c_1)^m c_2}{2^m}\frac{(1-2\xi+x-t)^m}{m!}d\xi = \\ &= \frac{c_1^{m+1}(1+c_1)^{m+1}c_2}{2^{m+1}}\frac{(1-x-t)^{m+1}}{(m+1)!}\end{aligned}$$

Similarly, using the estimate (7.49) and the induction hypothesis (7.51) for $n=m$ we obtain from (7.46) the following estimate for $n=m+1$:

$$\begin{aligned}\max\{\|P_{12}(x,t)\|,\|P_{21}(x,t)\|\} &\le \int_{x+t}^{\frac{1+x+t}{2}} 2\frac{c_1c_1^m(1+c_1)^m c_2}{2^{m+1}}\times \\ \times\frac{(1-2\xi+x+t)^m}{m!}d\xi + \int_x^{x+t} 2\frac{c_1}{2}\frac{c_1^{m+1}(1+c_1)^m c_2}{2^{m+1}}\frac{(1-x-t)^{m+1}}{(m+1)!}d\xi &= \\ = \frac{c_1^{m+1}(1+c_1)^m c_2}{2^{m+1}}\frac{(1-x-t)^{m+1}}{(m+1)!}[1+c_1t] \le \frac{c_1^{m+1}(1+c_1)^{m+1}c_2(1-x-t)^{m+1}}{2^{m+1}(m+1)!}.\end{aligned}$$

Thus the inequalities (7.48), (7.48') are proved and consequently $P(x,t)=0\ \forall(x,t)\in\triangle_2$. Bearing in mind the condition (7.31) we conclude that $Q(x)=\widetilde{Q}(x)$ for all $x\in[0,1]$. $\square$

Corollary 7.2. *Let $Y(x,\lambda)$ and $\widetilde{Y}(x,\lambda)$ be the same as in Theorem 7.2 and $A=\widetilde{A}$. If $Y(1,\lambda)=T\widetilde{Y}(1,\lambda)$ with some $T\in[\mathbb{C}^n]$ then $T=I_n$ and $Q(x)=\widetilde{Q}(x)$ for a.a. $x\in[0,1]$.*

Proof. It follows from (7.22) (with $x=1$) and the corresponding formula for $\widetilde{Y}(x,\lambda)$, that

$$(T-I_n)\begin{pmatrix} e^{i\lambda}I_{n_1} \\ e^{-i\lambda}A\end{pmatrix}\in L_2(\mathbb{R})\otimes\mathbb{C}^{n\times n_1}$$

since

$$\int_0^1 TK(1,t)\begin{pmatrix} e^{i\lambda}I_{n_1} \\ e^{-i\lambda}A\end{pmatrix}dt, \quad \int_0^1 \widetilde{K}(1,t)\begin{pmatrix} e^{i\lambda}I_{n_1} \\ e^{-i\lambda}A\end{pmatrix}dt\in L_2[0,1]\otimes\mathbb{C}^{n\times n_1}.$$

Setting $T=(T_{ij})_{i,j=1}^2$ with T_{ij} being $n_i\times n_j$ matrices, one concludes that

$$T_{11}-I_{n_1}=T_{21}=T_{12}A=(T_{22}-I_{n_2})A=0.$$

Since $\mathrm{rank}A=n_2=n-n_1$ one deduces the desired equality $T=I_n$. $\square$

In the case of even $n=2n_1$ we obtain the following refinement of the previous result.

Proposition 7.1. *Let $n = 2n_1$ and let $Y = \binom{Y_1(x,\lambda)}{Y_2(x,\lambda)}$ and $\widetilde{Y} = \binom{\widetilde{Y}_1(x,\lambda)}{\widetilde{Y}_2(x,\lambda)}$ be the $2n_1 \times n_1$ matrix solutions of the Cuchy problems (7.20), (7.21) and (7.26), (7.27) respectively. If $A = \widetilde{A}$ and*

$$(7.50) \qquad Y_1(1,\lambda)Y_2^{-1}(1,\lambda) = \widetilde{Y}_1(1,\lambda)\widetilde{Y}_2^{-1}(1,\lambda) \qquad \forall\lambda \in \mathbb{R}$$

then $Q(x) = \widetilde{Q}(x)$ for a.a. $x \in [0,1]$.

Proof. For the simplicity we confine ourselves to the case of commuting matrices $Y_1(1,\lambda)$ and $Y_2(1,\lambda)$. In this case one rewrites the condition (7.50) in the form $Y_1(1,\lambda)\widetilde{Y}_2(1,\lambda) = Y_2(1,\lambda)\widetilde{Y}_1(1,\lambda)$ and then, making use of the representation (7.28), we deduce

$$(7.51) \quad Y_2(1,\lambda)\cdot\int_0^1 \left[\sum_{j=1}^{2} P_{1j}(1,t)Y_j(t,\lambda)\right] dt = Y_1(1,\lambda)\cdot\int_0^1 \left[\sum_{j=1}^{2} P_{2j}(1,t)Y_j(t,\lambda)\right] dt$$

It follows from (7.22) and the dominated convergence theorem that, as $|\Im\lambda| \to \infty$

$$(7.52) \qquad Y_1(1,\lambda) = e^{i\lambda}I_{n_1}(1+o(1)), \qquad Y_2(1,\lambda) = e^{-i\lambda}I_{n_1}(1+o(1))$$

In view of (7.52) the left-hand side of the equation (7.51) is bounded in the lower half-plane $\mathbb{C}_-$ and so does the right-hand side in the upper half-plane. Thus by Liouville theorem both sides of the equation (7.51) are equal to zero. It follows now from the simple property of the indicator diagrams of entire functions of exponential type(compare with Remark 2.1) that $P_{ij}(1,t) = 0$ for $i,j \in \{1,2\}$. Therefore by (7.22) $\widetilde{Y}(1,\lambda) = Y(1,\lambda)$ $\forall\lambda \in \mathbb{R}$. One completes the proof applying Theorem 7.2. $\square$

Remark 7.2. Note that for the matrices $Y_1(1,\lambda)$ and $Y_2(1,\lambda)$ to commute it is sufficient that either $n_1 = 1$ or the matrix potential $Q(x)$ be selfadjoint and real. In the opposite case one completes the proof making use of the represenations for $Y_2^{-1}(1,\lambda)$ and $\widetilde{Y}_2^{-1}(1,\lambda)$ similar to the represenations (7.22) (with $x = 1$) for $Y(1,\lambda)$ and $\widetilde{Y}(1,\lambda)$.

Corollary 7.3. *Let $J_n = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$ and let $W_i(x,\lambda)$ $(i = 1,2)$ be the fundamental matrices of the solutions for the system*

$$J_n\frac{dY(x,\lambda)}{dx} + Q_i(x)Y(x,\lambda) = \lambda Y(x,\lambda), \qquad Q_i(x) = \begin{pmatrix} p_i & q_i \\ q_i & -p_i \end{pmatrix},$$

satisfying the initial condition $W_i(0,\lambda) = I_{2n}$ $(i = 1,2)$. If $W_1(1,\lambda) = W_2(1,\lambda)$ for all $\lambda \in \mathbb{R}$, then $Q_1(x) = Q_2(x)$ for a.a. $x \in [0,1]$ that is the potential matrix of the Dirac system is uniquely determined by its Wronskian.

Remark 7.3. The problem of the unique reconstruction of the pootential matrix of a system of ordinary differential euqations from its Wronskian was considered in the papers [B1], [BL], [Sak1], [Sak3], [Lei]. So for $n = 1$ Corollary 7.2 has been obtained in [Lei] and for $Q = Q^*$ it goes as far back as in [Sak1]. In the case $n > 1$ Corollary 7.2 is not covered by the works, mentioned above, even in the slefadjoint case. Nevertheless, it should be pointed out, that Corollary 7.2 may be obtained as an extension of the result from [Lei]. to the case of multieigenvalues of the matrix B with silght modifications in the proof.

7.3. If some additional information on the potential matrices $Q(x)$ and $\widetilde{Q}(x)$ is known, the conclusion that they are equal (close) may be derived if only a linear combination of two blocks of the monodromy matrix $W(\lambda)$ is equal (close) to the corresponding linear combination of two blocks of the matrix $\widetilde{W}(\lambda)$.

Theorem 7.3. *Let $n = 2n_1$ and let $Y(x,\lambda) = \begin{pmatrix} Y_1(x,\lambda) \\ Y_2(x,\lambda) \end{pmatrix}$ and $\widetilde{Y}(x,\lambda) = \begin{pmatrix} \widetilde{Y}_1(x,\lambda) \\ \widetilde{Y}_2(x,\lambda) \end{pmatrix}$ be the $n \times n_1$ matrix solutions of the Cauchy problems (7.20), (7.21) and (7.26), (7.27) for the Dirac systems (7.20) and (7.26) with summable matrix potentials $Q(x)$ and $\widetilde{Q}(x)$ respectively and the same initial data $Y(0,\lambda) = \widetilde{Y}(o,\lambda)$. Let also*

$$Y_1(1,\lambda) - HY_2(1,\lambda) = \widetilde{Y}_1(1,\lambda) - H\widetilde{Y}_2(1,\lambda) \qquad \forall \lambda \in \mathbb{R} \tag{7.53}$$

with some invertible $n_1 \times n_1$ matrix H. If $Q(x) = \widetilde{Q}(x)$ for a.e. $x \in [1/2,1]$, then $Q(x) = \widetilde{Q}(x)$ for a.e. $x \in [0,1]$.

Proof. As in the proof of Theorem 7.2 we start with the representation (7.28). According to it the solutions $\widetilde{Y}(x,\lambda)$ and $Y(x,\lambda)$ are related by the equality (7.28) with the matrix kernel $P(x,t) = (P_{ij}(x,t))_{i,j=1}^2$ of the transformation operator satisfying the equation (7.30) and the conditions (7.31), (7.32).

Since $Q(x) = \widetilde{Q}(x)$ for all $x \in [1/2,1]$ we deduce from (7.31) that

$$P_{12}(x,x) = P_{21}(x,x) = 0 \qquad \forall x \in [1/2,1] \tag{7.54}$$

Further it follows from the representation (7.28) that the relation (7.53) is equaivalent to the equalities

$$P_{11}(1,t) - HP_{21}(1,t) = 0, \tag{7.55}$$

$$P_{12}(1,t) - HP_{22}(1,t) = 0. \tag{7.56}$$

Integrating the equations (7.38), (7.39) along the characteristics and taking (7.54) into account, we arrive at the following integral equations in the triangle $\triangle_1 = \triangle_{ABC}$ (with vertices $A(1;1)$, $B(1;0)$, $C(1/2,1/2)$):

(7.57)

$$P_{12}(x,t) = \int_{(x+t)/2}^{x} [-Q_1(\xi)P_{22}(\xi, x+t-\xi) + P_{11}(\xi, x+t-\xi)\widetilde{Q}_1(x+t-\xi)]d\xi,$$

(7.58)

$$P_{21}(x,t) = \int_{(x+t)/2}^{x} [Q_2(\xi)P_{11}(\xi, x+t-\xi) - P_{22}(\xi, x+t-\xi)\widetilde{Q}_2(x+t-\xi)]d\xi.$$

Let for the brevity $A = \widetilde{A} = I_{n_1}$ in the conditions (7.20) and (7.26). Now making use of the conditions (7.32) we may set

$$(7.59) \qquad \varphi_1(x) = P_{11}(x,0) = P_{12}(x,0), \quad \varphi_2(x) := P_{21}(x,0) = P_{22}(x,0).$$

By virtue of these conditions we may integrate the equations (7.37) and (7.40) in the triangle $\triangle = \triangle_{OAB}$ (with $O(0;0)$). We have

(7.60)

$$P_{11}(x,t) = \varphi_1(x-t) + \int\limits_{x-t}^{x} [-Q_1(\xi)P_{21}(\xi,\xi-x+t) + P_{12}(\xi,\xi-x+t)\widetilde{Q}_2(\xi-x+t)]d\xi$$

(7.61)

$$P_{22}(x,t) = \varphi_2(x-t) + \int\limits_{x-t}^{x} [Q_2(\xi)P_{12}(\xi,\xi-x+t) - P_{21}(\xi,\xi-x+t)\widetilde{Q}_1(\xi-x+t)]d\xi$$

Comparing the equations (7.58) and (7.60) and taking the conditions (7.55) into account we derive the following expression for the function $\varphi_1(t)$:

(7.62)

$$\varphi_1(t) = \int\limits_{t}^{1} [Q_1(\xi)P_{21}(\xi,\xi-t) - P_{12}(\xi,\xi-t)\widetilde{Q}_2(\xi-t)]d\xi +$$

$$+ H \int\limits_{(2-t)/2}^{1} [Q_2(\xi)P_{11}(\xi,2-t-\xi) - P_{22}(\xi,2-t-\xi)\widetilde{Q}_2(2-t-\xi)]d\xi$$

In a similar fahion comparing (7.57) and (7.61) and bearing (7.56) in mind we obtain ($H_1 := H^{-1}$) :

(7.63)

$$\varphi_2(t) = -\int\limits_{t}^{1} [Q_2(\xi)P_{12}(\xi,\xi-t) - P_{21}(\xi,\xi-t)\widetilde{Q}_1(\xi-t)]d\xi -$$

$$- H_1 \int\limits_{(2-t)/2}^{1} [Q_1(\xi)P_{11}(\xi,2-t-\xi) - P_{11}(\xi,2-t-\xi)\widetilde{Q}_1(2-t-\xi)]d\xi.$$

Taking (7.62) and (7.63) into account one rewrites the equations (7.60), (7.61) as

(7.64)

$$P_{11}(x,t) = \int\limits_{x}^{1} [Q_1(\xi)P_{21}(\xi,\xi-x+t) + P_{12}(\xi,\xi-x+t)\widetilde{Q}_2(\xi-x+t)]d\xi +$$

$$+ H \int\limits_{(2-x+t)/2}^{1} [Q_2(\xi)P_{11}(\xi,2-x+t-\xi) - P_{22}(\xi,2-x+t-\xi)\widetilde{Q}_2(2-x+t-\xi)]d\xi$$

(7.65)

$$P_{22}(x,t) = -\int\limits_x^1 [Q_2(\xi)P_{12}(\xi,\xi - x + t) - P_{21}(\xi,\xi - x + t)\widetilde{Q}_1(\xi - x + t)]d\xi +$$

$$- H_1 \int\limits_{(2-x+t)/2}^1 [Q_1(\xi)P_{22}(\xi, 2 - x + t - \xi) - P_{11}(\xi, 2 - x + t - \xi)\widetilde{Q}_1(2 - x + t - \xi)]d\xi$$

Thus we obtain the homogeneous system (7.57), (7.58), (7.64), (7.65) of Volterra - type equations of the second kind. Bearing in mind the notations (7.48) and (7.49), one can prove by induction the following extimates:

$$\|P_{ij}(x,t)\|_{L_\infty(\Omega)} \le 2c_2(2c_1)^n \frac{(1-t)^n}{n!}, \qquad n \in \mathbb{Z}_+ \tag{7.66}$$

Actually one obtains from (7.64) - (7.66)

$$\begin{aligned} \|P_{ii}(x,t)\|_{L_\infty(\Omega)} &\le \int\limits_x^1 c_1 \cdot 2c_2(2c_1)^n \frac{(1-\xi+x-t)^n}{n!} d\xi + \\ &+ \int\limits_{(2-x+t)/2}^1 c_1 \cdot 2c_2(2c_1)^n \frac{(\xi+x-t-1)^n}{n!} d\xi \le \\ &\le 2c_2(2c_1)^{n+1}\frac{(1-t)^{n+1}}{(n+1)!} \qquad (i = 1,2) \end{aligned} \tag{7.67}$$

Similar arguments applied to the equations (7.57), (7.58) yield the estimates (7.67) for $\|P_{ij}(x,t)\|_{L_\infty(\Omega)}$ with $i \ne j$.

Thus the inequalities (7.66) are proved and therefore $P(x,t) = 0 \quad \forall (x,t) \in \triangle_1$. To complete the proof, that is to establish the equality $P(x,t) = 0 \quad \forall (x,t) \in \triangle_2 = \triangle_{ABC}$, it suffices to repeat the previous arguments used in the proof of the second part of Theorem 7.2. $\square$

7.4. Throughout this section a matrix B is assumed to be an $n \times n$ diagonal matrix with two real eigenvalues of opposite signs, that is $B = \mathrm{diag}(\lambda_1 I_{n_1}, \lambda_2 I_{n_2})$ (with $n_1 + n_2 = n$ and $n_1 \ge n_2$) and $\lambda_1\lambda_2 < 0$. We present some generalizations of Theorems 7.2 and 7.3 considering the equations (7.20) and (7.26) with such a matrix B. Their proofs, being completely similar, are based on the existence of a transformation operator ontained in Theorem 7.1.

Theorem 7.4. *Let* $B = \mathrm{diag}(\lambda_1 I_{n_1}, \lambda_2 I_{n_2})$ *be a diagonal matrix with* $\lambda_i = \bar{\lambda}_i$ *and* $\lambda_1\lambda_2 < 0$. *Let also* $Y(x,\lambda) = \begin{pmatrix} Y_1(x,\lambda) \\ Y_2(x,\lambda) \end{pmatrix}$ *and* $\widetilde{Y}(x,\lambda) = \begin{pmatrix} \widetilde{Y}_1(x,\lambda) \\ \widetilde{Y}_2(x,\lambda) \end{pmatrix}$ *be the* $n \times n_1$ *matrix solutions of the Cauchy problems (7.20), (7.21) and (7.26), (7.27) respectively. If* $Y(1,\lambda) = \widetilde{Y}(1,\lambda) \quad \forall \lambda \in \mathbb{R}$ *then* $Q(x) = \widetilde{Q}(x)$ *for a.a.* $x \in [0,1]$.

Proposition 7.2. *Let under the conditions of Theorem 7.4* $n_1 = n_2$. *If* $Y_2(0,\lambda) = \widetilde{Y}_2(0,\lambda)$ $(\Longleftrightarrow A = \widetilde{A})$ *and*

$$Y_1(1,\lambda)Y_2^{-1}(1,\lambda) = \widetilde{Y}_1(1,\lambda)\widetilde{Y}_2^{-1}(1,\lambda) \qquad \forall \lambda \in \mathbb{R}$$

then $Q(x) = \widetilde{Q}(x)$ *for a.a.* $x \in [0,1]$.

Theorem 7.5. *Let* B *be as above with* $n = 2n_1$ *and let* $Y(x,\lambda) = \begin{pmatrix} Y_1(x,\lambda) \\ Y_2(x,\lambda) \end{pmatrix}$ *and* $\widetilde{Y}(x,\lambda) = \begin{pmatrix} \widetilde{Y}_1(x,\lambda) \\ \widetilde{Y}_2(x,\lambda) \end{pmatrix}$ *be the* $n \times n_1$ *matrix solutions of the Cauchy problems (7.20), (7.21) and (7.26), (7.27) respectively, satisfying the same initial conditions* $Y(0,\lambda) = \widetilde{Y}(0,\lambda)$ $(\Longleftrightarrow A = \widetilde{A})$. *Suppose that*

$$Y_1(1,\lambda) - HY_2(1,\lambda) = \widetilde{Y}_1(1,\lambda) - H\widetilde{Y}_2(1,\lambda) \qquad \forall \lambda \in \mathbb{R}$$

for some invertible $H \in [\mathbb{C}^n]$ *and* $Q(x) = \widetilde{Q}(x)$ *for a.a.* $x \in [x_0, 1]$ *with*

$$x_0 = \max\left\{ \frac{|\lambda_1|}{|\lambda_1| + |\lambda_2|}, \frac{|\lambda_2|}{|\lambda_1| + |\lambda_2|} \right\}.$$

Then $Q(x) = \widetilde{Q}(x)$ *for a.a.* $x \in [0,1]$.

Going over to the case $n = 2$ we consider the system

$$\frac{1}{i} B \otimes \frac{dy}{dx} + Q(x)y = \lambda y, \qquad Q(x) = \begin{pmatrix} 0 & q_1(x) \\ q_2(x) & 0 \end{pmatrix}, \; y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \tag{7.68}$$

with $B = \operatorname{diag}(\lambda_1, \lambda_2)$ and $\lambda_1 \lambda_2 < 0$.

Adjoining the boundary conditions

$$y_2(0,\lambda) - hy_1(0,\lambda) = 0, \qquad y_2(1,\lambda) + h_1 y(1,\lambda) = 0, \tag{7.69}$$

we denote by $S(Q; h, h_1)$ the spectrum (accounting multiplicities of eigenvalues) of the problem (7.68), (7.69).

Alongside the system (7.68) we consider the analogous system with the matrix potential $\widetilde{Q}(x)$ and the same B:

$$\frac{1}{i} B \otimes \frac{d\widetilde{y}}{dx} + \widetilde{Q}(x)\widetilde{y} = \lambda \widetilde{y}, \qquad \widetilde{Q}(x) = \begin{pmatrix} 0 & \widetilde{q}_1(x) \\ \widetilde{q}_2(x) & 0 \end{pmatrix}, \; \widetilde{y} = \begin{pmatrix} \widetilde{y}_1 \\ \widetilde{y}_2 \end{pmatrix} \tag{7.70}$$

subject to the boundary conditions

$$\widetilde{y}_2(0,\lambda) - h\widetilde{y}_1(0,\lambda) = 0, \qquad \widetilde{y}_2(1,\lambda) + \widetilde{h}_1 \widetilde{y}(1,\lambda) = 0, \tag{7.69`}$$

One may consider the following corollary as a generalization of the well-known Borg's result.

Corollary 7.4. *Suppose that $q_i, \widetilde{q}_i \in L_1[0,1]$ $(i = 1,2)$ and the systems (7.68) and (7.70) have the same two spectra*

$$S(Q;h,h_1) = S(\widetilde{Q};h,\widetilde{h}_1), \ \ S(Q;h,h_2) = S(\widetilde{Q};h,\widetilde{h}_2) \tag{7.72}$$

where $h, \ h_i, \widetilde{h}_i \in \mathbb{C}\setminus\{0\}$ $(i = 1,2)$ and $(h_1 - h_2)(\widetilde{h}_1 - \widetilde{h}_2) \neq 0$. Then $h_i = \widetilde{h}_i$ for $i = 1,2$ and $Q(x) = \widetilde{Q}(x)$ a.e. on $[0,1]$.

Proof. Let $y = \begin{pmatrix} y_1(x,\lambda) \\ y_2(x,\lambda) \end{pmatrix}$ and $\widetilde{y} = \begin{pmatrix} \widetilde{y}_1(x,\lambda) \\ \widetilde{y}_2(x,\lambda) \end{pmatrix}$ be the solutions of the equations (7.68) and (7.70) respectively, satisfying the initial conditions

$$\widetilde{y}_1(0,\lambda) = y_1(0,\lambda) = 1, \qquad y_2(0,\lambda) = \widetilde{y}_2(0,\lambda) = h. \tag{7.73}$$

In accordance with Corollary 7.1 $y(x,\lambda)$ and $\widetilde{y}(x,\lambda)$ admit the represenations (7.74)

$$y(x,\lambda) = w(x,\lambda) + \int_0^x k(x,t)w(x,\lambda)dt, \ \widetilde{y}(x,\lambda) = w(x,\lambda) + \int_0^x \widetilde{k}(x,t)w(x,\lambda)dt$$

where (7.75)

$$w(x,\lambda) = \begin{pmatrix} e^{i\mu_1\lambda x} \\ he^{i\mu_2\lambda x} \end{pmatrix}, \qquad k(x,t) = (k_{ij}(x,t))_{i,j=1}^2, \qquad \widetilde{k}(x,t) = (\widetilde{k}_{ij}(x,t))_{i,j=1}^2$$

and $\mu_j = \lambda_j^{-1}$ $(j = 1,2)$. It follows from (7.74) and (7.75) that the spectrum $S(Q;h,h_j)$ is the same as the zeroes of the function

$$\begin{aligned} F_j(\lambda) &:= y_2(1,\lambda) + h_j y_1(1,\lambda) = e^{i\mu_2\lambda} + h_j e^{i\mu_1\lambda} + \\ &+ \int_0^1 [k_{21}(1,t) + h_j k_{11}(1,t)]e^{i\mu_1\lambda t}dt + \int_0^1 [k_{22}(1,t) + h_j k_{12}(1,t)]e^{i\mu_2\lambda t}dt \end{aligned} \tag{7.76}$$

Naturally, the same connection exists between the spectrum $S(\widetilde{Q};h,\widetilde{h}_j)$ and the function $\widetilde{F}_j(\lambda) = \widetilde{y}_2(1,\lambda) + \widetilde{h}_j\widetilde{y}_1(1,\lambda)$ admitting the represenation (7.76) with $\widetilde{k}_{ij}(1,t)$ and $\widetilde{h}_j$ $(i,j \in \{1,2\})$ instead of $k_{ij}(1,t)$ and h_j respectively.

It may be shown in a standart fashion that the dimension of the root subspace corresponding to $\lambda_0 \in S(Q;h,h_j)$ coincides with the muliplicity of the root λ_0 of $F_j(\lambda)$ $(j = 1,2)$. One easily derives from (7.76) that $F_j(\lambda)$ as well as $\widetilde{F}_j(\lambda)$ satisfies the asymptotic relation

$$F_j(\lambda) = (e^{i\mu_2\lambda} + h_j e^{i\mu_1\lambda})[1 + o(1)], \qquad (j = 1,2) \tag{7.77}$$

$$\widetilde{F}_j(\lambda) = (e^{i\mu_2\lambda} + \widetilde{h}_j e^{i\mu_1\lambda})[1 + o(1)], \qquad (j = 1,2) \tag{7.77'}$$

outside some exceptional disks. Denoting by $\lambda_k^{(j)}$ and $\widetilde{\lambda}_k^{(j)}$ the zeroes of F_j and $\widetilde{F}_j$ respectively we immediately obtain from (7.77) that the formula

$$\lambda_k^{(j)} = \left(\frac{i\ln h_j}{\mu_1 - \mu_2} + \frac{2\pi k}{\mu_1 - \mu_2}\right)\left(1 + O\left(\frac{1}{k}\right)\right), \qquad k \in \mathbb{Z}. \tag{7.78}$$

holds true. It follows from (7.72), (7.78) and the coresponding formula for zeroes $\widetilde{\lambda}_k^{(j)}$ of $\widetilde{F}_j$ that $h_j = \widetilde{h}_j$ $(j = 1, 2)$.

Further the functions $F_j(\lambda)$ and $\widetilde{F}_j(\lambda)$ being the entire functions of the exponential type $\sigma = 1$ may differ from each other only by an exponential factor. Since $h_j = \widetilde{h}_j$ $(j = 1, 2)$ it folowss from the asymptotic formulas (7.77), (7.77') that this factor equals to one.

Thus the characteristic function $F_j(\lambda)$ is uniquely determined by the spectrum $S(Q; h, h_j)$. Since $h_1 \neq h_2$ it allows us to uniquely reconstruct $y(1, \lambda)$ and $y'(1, \lambda)$ from two spectra and to complete the proof applying Theorem 7.5. □

Remark 7.4. It follows from (7.76) that $F_j(\lambda)$ is bounded on the real axis and consequently is of completely regular growth (see [L]).Moreover, (see [L, p.324]) $F_j(\lambda)$ admits the representation

$$F_j(\lambda) = c_j e^{ia_j\lambda} f_j(\lambda), \qquad f_j(\lambda) = \lim_{R\to\infty} \prod_{|\lambda_n^{(j)}|<R} \left(1 - \frac{\lambda}{\lambda_n^{(j)}}\right) \tag{7.79}$$

with real a_j and its indicator diagram is the segment $[i\mu_2, -i\mu_1]$ (if $\mu_1 > 0$ and $\mu_2 < 0$).

It may be proven that in the case $\mu_1 = -\mu_2 = 1$ the exponential factor in the representation (7.79) equals one, that is $a_j = 0$. Note also that in this case $(\mu_1 = -\mu_2 = 1)$ Corollary 7.3 seems to be known to specialists.

Corollary 7.5. *Suppose that $Q(x)$ and $\widetilde{Q}(x)$ are two summable complex-valued matrix potentials of the form (7.68) and (7.70) respectively and $h, h_1 \in \mathbb{C} \setminus \{0\}$. If $Q(x) = \widetilde{Q}(x)$ for a.a. $x \in [x_0, 1]$ with x_0 defined in Theorem 7.5 and*

$$S(Q; h, h_1) = S(\widetilde{Q}; h, h_1),$$

then $Q(x) = \widetilde{Q}(x)$ for a.a. $x \in [0, 1]$.

Proof. Since $F(\lambda) = y_2(1, \lambda) - h_1 y_1(1, \lambda)$ is uniquely determined by the spectrum $S(Q; h, h_1)$ the required assertion immediately follows from Theorem 7.6. □

Remark 7.5. a) Corollary 7.5 remains valid if one replaces the condition $Q(x) = \widetilde{Q}(x)$ $\forall x \in [x_0, 1]$ by $Q(x) = \widetilde{Q}(x)$ $\forall x \in [0, 1 - x_0]$. To do this it suffices to make use of the transformation $f(x) \to f(1 - x)$.

b) The Dirichlet boundary conditions in Theorem 7.3 and Corollary 7.5 may be considered with slight modification.

7.5. Here we consider the matrix Sturm - Liouville operator

$$LY = -Y'' + Q(x)Y = \lambda^2 Y \tag{7.80}$$

with the boundary conditions

$$Y'(0) - H_1 Y(0) = 0, \qquad Y'(1) + H_2 Y(1) = 0. \tag{7.81}$$

Here $Q(x)$ is a (generally) non-selfadjoint potential, H_1 and H_2 are (generally) non-selfadjoint matrices. As usual $H_1 = \infty$ $(H_2 = \infty)$ corresponds to the Dirichlet boundary condition $Y(0) = 0$ $(Y(1) = 0)$.

The following result may be obtained as a corollary of Theorem 7.3 but we prefer the simple direct proof, close to the author's proof of the well - known Borg's theorem, contained in [M4,p.115].

Alongside the equation (7.80) we consider the analogous equation with $n \times n$ matrix potential $\widetilde{Q}(x)$:

$$\widetilde{L}\widetilde{Y} = -\widetilde{Y}'' + \widetilde{Q}(x)\widetilde{Y} = \lambda^2 \widetilde{Y} \tag{7.82}$$

and the boundary conditions

$$\widetilde{Y}'(0) - \widetilde{H}_1 Y(0) = 0, \qquad \widetilde{Y}'(1) + H_2 \widetilde{Y}(1) = 0$$

Theorem 7.6. *Let Q, $\widetilde{Q} \in L_1[0,1] \otimes \mathbb{C}^{n^2}$ and $Q(x) = \widetilde{Q}(x)$ for a.a. $x \in [0,1/2]$. Let also $Y(x,\lambda)$ and $\widetilde{Y}(x,\lambda)$ be the solutions of the Cauchy problems for the equations (7.80) and (7.82) respectively satisfying the initial conditions*

$$Y(0,\lambda) = \widetilde{Y}(0,\lambda) = I,\ Y'(0,\lambda) = H_1,\ \widetilde{Y}'(0,\lambda) = \widetilde{H}_1. \tag{7.83}$$

If

$$Y'(1,\lambda) + H_2 Y(1,\lambda) = \widetilde{Y}'(1,\lambda) + H_2 \widetilde{Y}(1,\lambda) \quad \forall \lambda \in \mathbb{R} \tag{7.84}$$

then $Q(x) = \widetilde{Q}(x)$ for almost all $x \in [0,1]$.

Proof. It is well - known that the soluions $\widetilde{Y}(x,\lambda)$ and $Y(x,\lambda)$ are related (see [Mar]) by the equality

$$\widetilde{Y}(x,\lambda) = Y(x,\lambda) + \int_0^x R(x,t) Y(t,\lambda)dt \tag{7.85}$$

with the $n \times n$ matrix kernel $R(x,t)$ satisfying the equation

$$(D_1^2 - D_2^2)R(x,t) = [\widetilde{Q}(x) - Q(t)]R(x,t) \tag{7.86}$$

and the conditions

$$[D_2 R(x,t) - H_1 R(x,t)]|_{t=0} = 0, \quad R(x,x) = \widetilde{H}_1 - H_1 + \frac{1}{2}\int_0^x [\widetilde{Q}(s) - Q(s)]ds.$$

Making the substituting $u = x+t,\ v = x-t$ we see that $P(u,v) := R((u+v)/2,(u-v)/2)$ is a solution of the problem

$$4D_1 D_2 P(u,v) = \left[\widetilde{Q}\left(\frac{u+v}{2}\right) - Q\left(\frac{u-v}{2}\right)\right] P(u,v) \tag{7.87}$$

$$[P(u,v)H_1 - (D_1 - D_2)P(u,v)]|_{u=v} = 0 \tag{7.88}$$

$$4P(u,0) = 4(H_1 - \widetilde{H}_1) + \int_0^u [\widetilde{Q}(s/2) - Q(s/2)]ds \tag{7.89}$$

in the triangle $\triangle = \triangle_{OAB}$ with vertices $O(0;0)$, $A(1;1)$ and $B(2;0)$. It follows from (7.84) and (7.85) that $[D_1R(x,t) + H_2R(x,t)]|_{x=1} = 0$ and $R(1,1) = 0$ that is

$$(7.90) \quad [D_1 + D_2)P(u,v) + H_2P(u,v)]|_{v=2-u} = 0 \quad \forall u \in [1,2], \qquad P(2,0) = 0$$

Rewriting these conditions in the form

$$\frac{dP(u,2-u)}{du} - H_2P(u,2-u) = 2D_1P(u,2-u), \qquad P(2,0) = 0$$

one derives

$$(7.91) \qquad P(u,2-u) = 2\int_2^u e^{H_2(u-\alpha)}D_1P(\alpha,2-\alpha)d\alpha \qquad \forall u \in [1,2]$$

Since $Q(x) = \widetilde{Q}(x)$ for a.a. $x \in [1/2,1]$ we obtain from (7.89)

$$(7.92) \qquad D_1P(u,0) = 0 \qquad \forall u \in [1,2]$$

Integrating now (7.87) with respect to the second variable and taking (7.92) into account one deduces

$$(7.93) \quad D_1P(\alpha,v) = \frac{1}{4}\int_0^v \left[\widetilde{Q}\left(\frac{\alpha+\beta}{2}\right) - Q\left(\frac{\alpha-\beta}{2}\right)\right] P(\alpha,\beta)d\beta, \ (\alpha \in [1,2]).$$

Integrating (7.93) with regard to (7.91) and setting $G(\alpha,\beta) := \widetilde{Q}((\alpha+\beta)/2) - Q((\alpha-\beta)/2)$, we arrive at the equation
(7.94)

$$P(u,v) = -\frac{1}{4}\int_u^{2-v} d\alpha \int_0^v G(\alpha,\beta)P(\alpha,\beta)d\beta - \frac{1}{2}\int_{2-v}^2 e^{H_2(2-v-\alpha)}d\alpha \int_0^{2-\alpha} G(\alpha,\beta)P(\alpha,\beta)d\beta$$

in the triangle $\triangle_1 = \triangle_{ABC}$ (with vertices $A(1;1)$, $B(2;0)$, $C(1;0)$). Assuming for the simplicity that $Q(x)$ and $\widetilde{Q}(x)$ are essentially bounded on $[0,1]$ and taking into account that $2-v \geq \alpha \geq u$ in $\triangle_1$ one derives from (7.94) that the following inequality holds:

$$(7.95) \qquad |P(u,v)| \leq c_1 \int_u^2 d\alpha \int_0^{2-\alpha} |P(\alpha,\beta)|d\beta \qquad \forall (u,v) \in \triangle_1$$

with $c_1 = c(1/4 + 1/2\max_{u\in[0,1]}|\exp(-H_2u)|)$ and $c = \operatorname{vraisup}_{(\alpha,\beta)\in\triangle}|G(\alpha,\beta)|$. Here $|\cdot|$ stands for the matrix norm.
Applying the Gronwall's lemma to (7.95) one concludes that $P(u,v) = 0$ $\forall (u,v) \in \triangle_1$.

On the second step, making use of the just proved equalities $P(1,v) = D_2P(1,v) = 0$ $\forall v \in [0,1]$ and (7.88) we easily obtain that $P(u,v) = 0$ in the triangle $\triangle_2 = \triangle_{OAC}$. It was done in the authors proof of Borg's theorem (see [M4, p.115]).

After returning to the condition (7.89) one deduces that $H_1 = \widetilde{H}_1$ and $\widetilde{Q}(x) = Q(x)$ for a.a. $x \in [0,1]$. $\square$

Remark 7.6. a) The assertion of Theorem 7.6 remains valid if one replaces the condition (7.84) by the condition $Y(1,\lambda) = \widetilde{Y}(1,\lambda)$ corresponding to $H_2 = \infty \cdot I_n$. The proof is the same and is even simpler. Indeed in this case one obtains the condition $P(u, 2-u) = 0$ $\forall u \in [1,2]$ instead of (7.90) and then arrives at the equation

$$P(u,v) = \frac{1}{4}\int_u^{2-v} d\alpha \int_0^v \left[\widetilde{Q}\left(\frac{\alpha+\beta}{2}\right) - Q\left(\frac{\alpha-\beta}{2}\right)\right] P(\alpha,\beta)d\beta$$

instead of (7.94). To complete the proof it saffices to repeat the previous arguments.

b) The case on the another initial data

$$Y(0,\lambda) = \widetilde{Y}(0,\lambda) = 0, \qquad Y'(0,\lambda) = H_1,\ \widetilde{Y}'(0,\lambda) = \widetilde{H}_1$$

may be considered similarly.

Corollary 7.6. *Let $n = 1$ and let $Q(x), \widetilde{Q}(x) \in L_1[0,1]$ be two complex - valued potentials with $Q(x) = \widetilde{Q}(x)$ for a.a. $x \in [1/2, 1]$. If*

$$S(Q; H_1, H_2) = S(\widetilde{Q}; \widetilde{H}_1, H_2)$$

then $H_1 = \widetilde{H}_1$ and $Q(x) = \widetilde{Q}(x)$ for almost all $x \in [0,1]$.

Proof. Since $y'(1,\lambda) + H_2y(1,\lambda)$ is uniquely dertermined by its zeros, that is by the spectrum $S(Q; H_1, H_2)$, one completes the proof applying Theorem 7.6. $\square$

Corollary 7.7. *Let $n = 1$ and let $Q(x)$, $\widetilde{Q}(x) \in L_1[0,1]$ with $Q(x) = \widetilde{Q}(x)$ for a.a. $x \in [0,1/2]$. If*

$$S(Q; H_1, H_2) = S(\widetilde{Q}; H_1, \widetilde{H}_2)$$

then $H_2 = \widetilde{H}_2$ and $Q(x) = \widetilde{Q}(x)$ for almost all $x \in [0,1]$.

Remark 7.7. Cororllaries 7.6 and 7.7 were obtained first by Hochstadt-Liberman [HL]. Analogs of their result for certain Shrodinger operators were considered in [Kh] and some other refinements were obtained in [Hal], [Su], [Rio].

Quite recently Gesztesy and Simon [GS2] (see also [GS1] and [GS3]) discovered a new approach to this result as well as to its further developements. It allowed them to obtain some interesting results on the unique recovery of the potential from a subset of spectrum and a part of the potential on $[0, 1/2+\varepsilon]$.

Note also that Theorems 7.3, 7.5 and Corollary 7.5 ,may be considrered as further developement of the mentioned above result from [HL].

7.6. Let $B = \operatorname{diag}(\lambda_1 I_{k_1}, \dots, \lambda_n I_{k_n})$ be a block - diagonal matrix and $Q(x) = (q_{ij})_{i,j=1}^n$ be a block - matrix function consisting of $k_i \times k_j$ blocks $q_{ij}(x)$ with zero diagonal: $q_{ii}(x) = 0$ $\forall x \in [0,1]$ $(1 \le j \le n)$. Let for the definiteness $k_1 = \max_{1\le j\le n} k_j$.

Theorem 7.7. *Let $Y(x,\lambda)$ be an $nk_1 \times k_1$ matrix solution of the Cauchy problem for the system*

$$B \otimes \frac{dY(x,\lambda)}{dx} + Q(x)Y(x,\lambda) = \lambda Y(x,\lambda), \qquad Y(0,\lambda) = \begin{pmatrix} A_1 \\ \dots \\ A_n \end{pmatrix}$$

with A_j $(1 \le j \le n)$ being $k_j \times k_1$ matrices with $\mathrm{rank} A_j = k_j$. If all $\arg\lambda_j (\in [0,2\pi])$ are distinct for $j \in \{1,\dots,n\}$ and $Q(x)$ is an analytic entire matrix function, then $Y(1,\lambda)$ uniquely determines $Q(x)$.

For $k_1 = \ldots = k_n = 1$ we adjoint the boundary conditions

$$U_i y = \sum_{1 \le j \le n} \alpha_{ij} y_j(0) \ (0 \le i \le n-1), \qquad V_r y = y_r(1) = 0.$$

Denote by $S_r(Q;\{U_i\}_1^{n-1})$ the spectrum of the problem (1), (2) assuming the forms U_i to be linearly independent. Let also Δ_i be the minor of order $n-1$ of the matrix (α_{ij}) obtained by deleting its i-th column.

Theorem 7.8. *Let under the hypothesis of Theorem 7.7 $k_1 = \ldots = k_n = 1$ and $\Delta_i \neq 0$ $(1 \le i \le n)$. Then $Q(x)$ is uniquely determined by n spectra $S_r(Q;\{U_i\}_1^{n-1})$.*

For the case of a matrix $B = \mathrm{diag}(\varepsilon_1,\dots,\varepsilon_n)$ with $\varepsilon_p = \exp(i2\pi p/n)$, $p \in \{1,\dots,n\}$ one easily derives from Theorem 7.8 a theorem on the unique determination of an ordinary differential equation of n-th order with analytic coefficients from its n spectra (see [M4]).

The proof of Theorem 7.7 is based on the existence of a trianglular transformation operator, which proof may be obtained from Theorem 4.2.

§8. On the stability of inverse spectral problems

8.1 Our goal in this section is to consider the stability of the recovery of the Sturm - Liouville operator on the finite interval [0,1] from the functions $Y(1,\lambda)$ and $Y'(1,\lambda)$ and related questions.
Let us consider two Sturm - Liouville equations

$$-Y''(x,\lambda) + Q_i(x)Y(x,\lambda) = \lambda^2 Y(x,\lambda) \qquad (i = 1,2) \tag{8.1}$$

with essentially bounded $n \times n$ matrix potentials $Q_i(x) \in L_\infty^{n\times n} := L_\infty[0,1] \otimes \mathbb{C}^{n\times n}$ $(i = 1,2)$.

In the sequel $|T|$ stands for the matrix norm of an $n \times n$ matrix $T \in [\mathbb{C}^n]$. Let also

$$B(Q_1;r) = \{Q(x) \in L_\infty^{n\times n} : \operatorname*{vraisup}_{x\in[0,1]} |Q(x) - Q_1(x)| \le r\}, \tag{8.2}$$

that is, $B(Q_1;r)$ is the set of all $n \times n$ matrix potentials $Q(x) \in L_\infty^{n\times n}$ satisfying the inequality (8.2).

Theorem 8.1. *Let $Y_i(x,\lambda)$ be the $n \times n$ matrix solution of the Cauchy problem for the equation (8.1) satisfying the initial conditions*

$$Y_i(0,\lambda) = I_n, \qquad Y_i'(0,\lambda) = H_i \qquad (i = 1,2) \tag{8.3}$$

and let

$$A := H_2 - H_1 + \frac{1}{2}\int_0^1 [Q_2(x) - Q_1(x)]dx \quad (\in [\mathbb{C}^n]). \tag{8.4}$$

Then the estimate

$$\begin{aligned} \|Q_2 - Q_1\|_{L_2[0,1]} \le c_0\{\|\lambda(Y_2(1,\lambda) - Y_1(1,\lambda)) - A\sin\lambda\|_{L_2[\mathbb{R})} + \\ + \|Y_2(1,\lambda) - Y_1(1,\lambda)\|_{L_2(\mathbb{R})} + \|Y_2'(1,\lambda) - Y_1'(1,\lambda) - A\cos\lambda\|_{L_2(\mathbb{R})}\} \end{aligned} \tag{8.5}$$

holds for each $Q_2 \in B(Q_1;r_1)$ with constant $c_0 = c_0(Q_1,H_1,r_1)$ independent of $Q_2 \in B(Q_1;r_1)$ and $H_2 \in [\mathbb{C}^n]$.

Proof. 1) It is well - known (see [Mar]) that $Y_i(x,\lambda)$ admits the representation

$$Y_i(x,\lambda) = \cos\lambda x \cdot I_n + \int_0^x R_i(x,t)\cos\lambda t dt = (I + R_i)\cos\lambda t \tag{8.6}$$

with C^1 - smooth $n \times n$ matrix kernel $R_i(x,t)$ $(i = 1,2)$. It follows from (8.6) and the Parseval equality that

$$\frac{2}{\pi}\int_{\mathbb{R}} |Y_2(1,\lambda) - Y_1(1,\lambda)|^2 d\lambda = \int_0^1 |R_2(1,t) - R_1(1,t)|^2 dt. \tag{8.7}$$

Further, since $R_2(1,1) - R_1(1,1) = A$ one obtains from (8.6)

$$\frac{2}{\pi}\int_{\mathbb{R}} |Y_2'(1,\lambda) - Y_1'(1,\lambda) - A\cos\lambda|^2 d\lambda = \int_0^1 |D_1R_2(1,t) - D_1R_1(1,t)|^2 dt. \tag{8.8}$$

Finally, integrating by parts in (8.6) and applying the Parseval equality we deduce

$$\frac{2}{\pi}\|\lambda[Y_2(1,\lambda) - Y_1(1,\lambda)] - A\sin\lambda\|^2_{L_2(\mathbb{R})} = \|D_2R_2(1,t) - D_2R_1(1,t)\|^2_{L_2[0,1]} \tag{8.9}$$

Setting $R_{21}(x,t) := R_2(x,t) - R_1(x,t)$ and taking (8.7)-(8.9) into account, one rewrites the desired estimate in the form

$$\|Q_2(t) - Q_1(t)\|_{L_2[0,1]} \le c_0' \left\{ \|R_{21}(1,t)\|_{L_2[0,1]} + \sum_{j=1}^2 \|D_jR_{21}(1,t)\|_{L_2[0,1]} \right\} \tag{8.10}$$

2) It is clear that the inverse operator $(I+R_1)^{-1}$ is of the form $I+\widetilde{R}_1$ with $\widetilde{R}_1$ also being a Volterra operator. Its kernel $\widetilde{R}_1(x,t)$ is connected with the kernel $R_1(x,t)$ by the evident equality

$$R_1(x,t)+\widetilde{R}_1(x,t)+\int_t^x R_1(x,s)\widetilde{R}_1(s,t)ds=0. \tag{8.11}$$

The following connection between the solutions $Y_i(x,\lambda)$ $(i=1,2)$ immediately follows from (8.6):

$$Y_2(x,\lambda)=(I+R_2)(I+\widetilde{R}_1)Y_1(x,\lambda)=Y_1(x,\lambda)+\int_0^x R(x,t)Y_1(t,\lambda)dt \tag{8.12}$$

with

$$R(x,t)=R_2(x,t)+\widetilde{R}_1(x,t)+\int_t^x R_2(x,s)\widetilde{R}_1(s,t)ds. \tag{8.13}$$

Letting

$$\varphi_0(t):=R(1,t),\quad \varphi_1(t):=D_1R(1,t)=D_1R(x,t)|_{x=1} \tag{8.14}$$

we derive from (8.11) and (8.13)

$$\varphi_0(t)=R_{21}(1,t)+\int_t^1 R_{21}(1,s)\widetilde{R}_1(s,t)ds, \tag{8.15}$$

$$\varphi_1(t)=D_1R_{21}(1,t)+\varphi_0(1)\widetilde{R}_1(1,t)+\int_t^1 (D_1R_{21}(1,s))\widetilde{R}_1(s,t)ds, \tag{8.16}$$

Since $|\varphi_0(1)|\le\sqrt{2}\|\varphi_0\|_{W_2^1[0,1]}$, the relations (8.15), (8.16) imply the inequalities

$$\|\varphi_0\|_{W_2^1[0,1]}\le c_1\|R_{21}(1,t)\|_{W_2^1[0,1]} \tag{8.17}$$

$$\|\varphi_1\|_{L_2[0,1]}\le c_1(\|R_{21}(1,t)\|_{W_2^1[0,1]}+\|D_1R_{21}(1,t)\|_{L_2[0,1]}) \tag{8.18}$$

in which the constant c_1 depends only on $\widetilde{R}_1(x,t)$ and, consequently, only on $Q_1(x)$ and $H_1:\ c_1=c_1(Q_1,H_1)$.

Thus to complete the proof it suffices to establish the estimate

$$\|Q_1(t)-Q_2(t)\|_{L_2[0,1]}\le c_2(\|\varphi_0\|_{W_2^1[0,1]}+\|\varphi_1\|_{L_2[0,1]}) \tag{8.19}$$

3) To obtain this inequality we recall that $P(u,v) := R((u+v)/2, (u-v)/2)$ satisfies the equation

$$4D_1D_2P(u,v) = [Q_2((u+v)/2) - Q_1((u-v)/2)] \tag{8.20}$$

and the conditions

$$[P(u,v)H_1 - (D_1 - D_2)P(u,v)]_{u=v} = 0, \tag{8.21}$$

$$4P(u,0) = 4(H_2 - H_1) + \int\limits_0^u [Q_2(s/2) - Q_1(s/2)]ds \tag{8.22}$$

in the triangle $\triangle = \triangle_{OAB}$ with vertices $O(0;0)$, $A(1;1)$ and $B(2;0)$. Bearing in mind (8.14), we observe that

$$P(u, 2-u) = R(u, u-1) = \varphi_0(u-1) \qquad \forall u \in [1,2] \tag{8.23}$$

and therefore

$$\psi_1(u-1) := D_1P(u,v)|_{v=2-u} = [\varphi_1(u-1) + \varphi_0'(u-1)]/2, \tag{8.24}$$

$$\psi_2(u-1) := D_2P(u,v)|_{v=2-u} = [\varphi_1(u-1) - \varphi_0'(u-1)]/2, \tag{8.25}$$

4) At first we derive an estimate for $P(u,v)$ in the triangle $\triangle_1 = \triangle_{ABC}$ (with $C = C(1;0)$).

Let us set $G(\alpha,\beta) := Q_2\left(\frac{\alpha+\beta}{2}\right) - Q_1\left(\frac{\alpha-\beta}{2}\right)$. By successive integration of the equation (8.20) with regard to (8.23)—(8.25) we obtain

(8.26)

$$D_1P(\alpha,v) = -\frac{1}{4}\int\limits_v^{2-\alpha} G(\alpha,\beta)P(\alpha,\beta)d\beta + \psi_1(\alpha-1)$$

(8.27)

$$P(u,v) = \frac{1}{4}\int\limits_u^{2-v} d\alpha \int\limits_v^{2-\alpha} G(\alpha,\beta)P(\alpha,\beta)d\beta + \int\limits_u^{2-v} \psi_1(\alpha-1)d\alpha + \varphi_0(1-v).$$

Since $Q_2 \in B(Q_1; r_1)$, one observes that $|Q(x) - Q(y)| \le r := r_1 + 2\|Q_1\|_{L_\infty}$ $\forall x, y \in [0,1]$ and then derives from (8.27)

$$|P(u,v)|^2 \le \frac{3r^2}{16}\int\limits_u^2 d\alpha \int\limits_0^{2-\alpha} |P(\alpha,\beta)d\beta + 3\int\limits_1^2 |\psi_1(\alpha-1)|^2 d\alpha + 3|\varphi_0(1-v)|^2. \tag{8.28}$$

Letting

$$I_1(u) = \int_0^{2-u} |P(u,v)|^2 dv \tag{8.29}$$

we deduce from (8.28)

$$I_1(u) \leq \frac{3r^2}{16} \int_u^2 I_1(\alpha) d\alpha + 3\|\psi_1\|^2_{L_2[0,1]} + 3\|\varphi_0\|^2_{L_2[0,1]} \tag{8.30}$$

Applying the Gronwall's lemma to (8.30) one obtains the inequality

$$I_1(u) \leq 3(\|\psi_1\|^2_{L_2[0,1]} + \|\varphi_0\|^2_{L_2[0,1]}) \exp\left(\frac{3r^2}{16}(2-u)\right). \tag{8.31}$$

Combining the inequalities (8.28) and (8.31) we arrive at the uniform estimate for the solution $P(u,v)$ in $\triangle_1$:

$$\begin{aligned} |P(u,v)|^2 \leq 3(\|\psi_1\|^2_{L_2[0,1]} + \|\varphi_0\|^2_{L_2[0,1]}) \exp\left(\frac{3r^2}{16}(2-u)\right) + \\ + 3\|\psi_1\|^2_{L_2[0,1]} + 3|\varphi_0(1-v)|^2 \qquad \forall (u,v) \in \triangle_1. \end{aligned} \tag{8.32}$$

Observing that $\|\varphi_0\|_{C[0,1]} \leq \sqrt{2}\|\varphi_0\|_{W_2^1[0,1]}$ and bearing (8.24) in mind one deduces from (8.32) that

$$\|P(u,v)\|^2_{C(\triangle_1)} \leq c_3(\|\varphi_0\|^2_{W_2^1[0,1]} + \|\varphi_1\|^2_{L_2[0,1]}) \tag{8.33}$$

with $c_3 = 6(\exp(3r^2/16) + 1)$.

5) Now we are in position to establish the first "half" of the required estimate (8.19). Indeed, it follws from (8.22), (8.26), (8.31) and (8.33) that

(8.34)

$$\int_{1/2}^1 |Q_2(x) - Q_1(x)|^2 dx = \frac{1}{2} \int_1^2 |Q_2(u/2) - Q_1(u/2)|^2 du = 8 \int_1^2 |D_1 P(u,0)|^2 du$$

$$\leq r^2 \int_1^2 du \int_0^{2-u} |P(u,\beta)|^2 d\beta + 16 \int_1^2 |\psi_1(u-1)|^2 du = r^2 \int_1^2 I_1(u) du$$

$$+ 16\|\psi_1\|^2_{L_2[0,1]} \leq 6c_3(\|\varphi_0\|^2_{W_2^1[0,1]} + \|\varphi_1\|^2_{L_2[0,1]}).$$

In order to obtain an estimate for the solution $P(u,v)$ in the triangle $\triangle_1 = \triangle_{OAC}$ we also need an integral estimate for $D_2 P(u,v)$ in $\triangle_1$. It is clear from (8.27) that

$$D_2 P(u,v) = -\frac{1}{4} \int_u^{2-v} G(\alpha,v) P(\alpha,v) d\alpha - \psi_1(1-v) - \varphi_0'(1-v). \tag{8.35}$$

Hence, combining (8.29) and (8.31) we arrive at the inequality

$$\begin{aligned} \int_0^{2-u} |D_2 P(u,v)|^2 dv \leq \frac{3r^2}{16} \int_0^{2-u} dv \int_u^{2-v} |P(\alpha,v)|^2 d\alpha + 3(\|\psi_1\|^2_{L_2} + \|\varphi_0'\|^2_{L_2}) = \\ = \frac{3r^2}{16} \int_u^2 I_1(\alpha) du + 3(\|\psi_1\|^2_{L_2} + \|\varphi_0'\|^2_{L_2}) \leq 2c_3(\|\varphi_0\|^2_{W_2^1} + \|\varphi_1\|^2_{L_2}) \end{aligned} \tag{8.36}$$

with $u \in [1,2]$.

6) Going over to the triangle $\triangle_2 = \triangle_{OAC} = \{0 \le v \le u \le 1\}$, we use the condition (8.21) rewritten in the form

$$\frac{dP(u,u)}{du} - H_1 P(u,u) = 2D_2 P(u,v)|_{v=u}, \qquad u \in [0,1].$$

The solution of this equation is

$$(8.37) \qquad P(u,u) = 2\int_1^u D_2 P(\beta,\beta) e^{H_1(u-\beta)} d\beta + P(1,1) e^{H_1(u-1)}.$$

Integrating (8.20) in $\triangle_2$ with resepct to the first variable we obtain

$$(8.38) \qquad D_2 P(u,\beta) = -\frac{1}{4}\int_u^1 G(\alpha,\beta)P(\alpha,\beta)d\alpha + D_2 P(1,\beta).$$

Having integrated (8.38) with respect to the second variable from v to u and taking (8.37) into account we deduce

(8.39)

$$P(u,v) = \frac{1}{4}\int_u^1 d\alpha \int_v^u G(\alpha,\beta)P(\alpha,\beta)d\beta + \frac{1}{2}\int_u^1 d\alpha \int_u^\alpha e^{H_1(u-\beta)} G(\alpha,\beta)P(\alpha,\beta)d\beta -$$
$$- \int_v^u D_2 P(1,\beta)d\beta - 2\int_u^1 e^{H_1(u-\beta)} D_2 P(1,\beta)d\beta + P(1,1)e^{H_1(u-1)}.$$

Since $|P(1,1)| = |\varphi_0(0)| \le \sqrt{2}\|\varphi_0\|_{W_2^1[0,1]}$, it follows the estimate

(8.40)

$$|P(u,v)|^2 \le 25k_1 \left[\frac{r^2}{16}\int_v^1 d\alpha \int_v^\alpha |P(\alpha,\beta)|^2 d\beta + \int_0^1 |D_2 P(1,\beta)|^2 d\beta + \|\varphi_0\|^2_{W_2^1}\right]$$

with $k_1 = \max\{1, \|\exp(-H_1)\|^2\}$. Further setting

$$(8.41) \qquad I_2(v) = \int_v^1 |P(u,v)|^2 du$$

and bearing in mind (8.36) (with $u = 1$) we obtain after integration of (8.40) with respect to the first variable from v to 1:

$$(8.42) \qquad I_2(v) \le \frac{25r^2k_1}{16}\int_v^1 I_2(\beta)d\beta + c_4(\|\varphi_0\|^2_{W_2^1} + \|\varphi_1\|^2_{L_2})$$

Here we make use of the notation $c_4 = 10(c_3 + k_1 + 4k_1c_3)$. By the Gronwall's inequality we conclude from (8.42) that

$$I_2(v) \le c_4(\|\varphi_0\|^2_{W_2^1} + \|\varphi_1\|^2_{L_2}) \exp\left(\frac{25r^2k_1}{16}(1-v)\right) \tag{8.43}$$

Combining the inequalities (8.40) and (8.43) and again taking (8.36) with $u = 1$ into account we arrive at the uniform estimate for $P(u,v)$:

$$|P(u,v)|^2 \le c_5(\|\varphi_0\|^2_{W_2^1} + \|\varphi_1\|^2_{L_2}) \qquad \forall (u,v) \in \triangle_2, \tag{8.44}$$

where $c_5 = c_4(1 + \exp(2^{-1}r^2))$.

7) Here we establish an estimate for $D_1P(u,v)$ in $\triangle_2$, with will allow us to complete the proof. Differentiating (8.39) with respect to u we obtain after simple calculations

$$\begin{aligned} |D_1P(u,v)|^2 \le \frac{6r^2}{16}\int_v^u |P(u,\beta)|^2 d\beta + \frac{6r^2}{16} I_2(u) + \frac{3}{2}|H_1|^2 k_1 r^2 \int_u^1 I_2(\beta) d\beta \\ + 12k_1 \int_u^1 |D_2P(1,\beta)|^2 d\beta + 6|D_2P(1,u)|^2 + 6k_1|H_1P(1,1)|^2. \end{aligned} \tag{8.45}$$

Now one derives from (8.22), (8.43) — (8.45) and (8.36) (with $u = 1$) that

(8.46)

$$\begin{aligned} \int_0^{1/2} |Q_2(x) - Q_1(x)|^2 dx = \int_0^1 |Q_2(u/2) - Q_1(u/2)|^2 du = \int_0^1 |D_1P(u,v)|^2 du \le \\ \le \left(\frac{3}{4} + \frac{3}{2}|H_1|^2 k_1\right) r^2 \int_0^1 I_2(u) du + (12|H_1|k_1 + 6) \int_0^1 |D_2P(1,\beta)|^2 d\beta \\ + 6k_1|H_1P(1,1)|^2 \le c_6(\|\varphi_0\|^2_{W_2^1[0,1]} + \|\varphi_1\|^2_{L_2[0,1]}). \end{aligned}$$

with constant c_6 depending only on r and H_1. Combining the estimates (8.34) and (8.46) we obtain the desired inequality (8.19), which completes the proof. $\square$

Corollary 8.1. *Let under the hypothesis of Theorem 8.1 $A = 0$. Then the inequality*

(8.47)

$$\|Q_2 - Q_1\|_{L_2[0,1]} \le c_0\{\|Y_2'(1,\lambda) - Y_1'(1,\lambda)\|_{L_2(\mathbb{R})} + \|\lambda(Y_2(1,\lambda) - Y_1(1,\lambda))\|_{L_2(\mathbb{R})}\}$$

takes place with constant $c_0 = c_0(Q_1, H_1, r)$ not depending on $Q_2 \in B(Q_1; r)$.

Remark 8.1. One immediately obtains from Theorem 8.1 (or Corollary 8.1)the following generalization of Borg's result: if $Y_1(1,\lambda) = Y_2(1,\lambda)$ and $Y'(1,\lambda) = Y_2'(1,\lambda)$ $\forall\lambda \in \mathbb{R}$, then $Q_1(x) = Q_2(x)$ a.e. on $[0,1]$. Note also, that the author's proof of the Borg's result remains valid for the matrix case without any changes.

8.2 Here we consider the unique recovery of the Sturm-Liouville operator from one spectrum and a half of the potential.

In the proof of the following theorem we retain the notations used in the proof of the previous result.

Theorem 8.2. *Let $Y_i(x,\lambda)$ $(i=1,2)$ be the same as in Theorem 8.1 and $H_3 \in [\mathbb{C}^n]$. Then the estimate*

$$
\begin{aligned}
&\|Q_2 - Q_1\|_{L_2[0,1]} \le c_0\{\|Q_2 - Q_1\|_{L_2[1/2,1]} + \\
&\|Y_2'(1,\lambda) - Y_1'(1,\lambda) + H_3(Y_2(1,\lambda) - Y_1(1,\lambda)) - A\cos\lambda\|_{L_2(\mathbb{R})}\}
\end{aligned}
\tag{8.48}
$$

holds for each $Q_2 \in B(Q_1;r)$ with constant $c_0 = c_0(Q_1, r, H_1, H_3)$ independent of $Q_2 \in B(Q_2;r)$ and $H_2 \in [\mathbb{C}^n]$.

Proof. 1) Using the representations (8.6) one derives, just like it was done in the proof of Theorem 8.1 that

$$
\begin{aligned}
\frac{2}{\pi}\|Y_2'(1,\lambda) - Y_1'(1,\lambda) + H_3(Y_2(1,\lambda) - Y_1(1,\lambda)) - A\cos\lambda\|^2_{L_2(\mathbb{R})} = \\
\|D_2R_{21}(1,t) + H_3R_{21}(1,t)\|_{L_2[0,1]}
\end{aligned}
\tag{8.49}
$$

where as before $R_{21}(x,t) = R_2(x,t) - R_1(x,t)$. Repeating the arguments used on the second step of the previous proof we deduce

$$
\|D_1R(1,t) + H_3R(1,t)\|_{L_2[0,1]} \le c_1(\|D_1R_{21}(1,t) + H_3R_{21}(1,t)\|_{L_2[0,1]})
\tag{8.50}
$$

Combining the inequalities (8.49) and (8.50) one concludes that in order to obtain (8.48) it suffices to prove the estimate

$$
\|Q_2 - Q_1\|_{L_2[0,1/2]} \le c_2\{\|Q_2 - Q_1\|_{L_2[1/2,1]} + \|D_1R(1,t) + H_2R(1,t)\|_{L_20,1]}\}
\tag{8.51}
$$

with constant $c_2 = c_2(Q_1, r, H_1, H_3)$ independent of $Q_2 \in B(Q_1;r)$ and $H_2 \in [\mathbb{C}^n]$.

2) We establish the desired estimate (8.51) considering the boundary problem (8.20)-(8.22) for the kernel $P(u,v) = R((u+v)/2, (u-v)/2)$. Setting

$$
\varphi(t) := [D_1R(x,t) + H_3R(x,t)]|_{x=1}
\tag{8.52}
$$

one finds $P(u, 2-u)$ from the equation

$$
[(D_1 + D_2)P(u,v) + H_3P(u,v)]|_{v=2-u} = \varphi(u-1), \quad P(2,0) = A.
$$

We have

$$
\begin{aligned}
P(u,2-u) = 2\int_2^u e^{H_3(u-\alpha)}D_1P(\alpha, 2-\alpha)d\alpha - \int_2^u e^{H_3(u-\alpha)}\varphi(\alpha-1)d\alpha + \\
+ \exp(H_3(u-2))\cdot A.
\end{aligned}
\tag{8.53}
$$

Letting for the brevity $A = 0$ and

$$
G(u,v) := Q_2((u+v)/2) - Q_1((u-v)/2), \qquad Q(x) := Q_2(x) - Q_1(x)
$$

we obtain, having integrated the equation (8.20)

$$
D_1P(\alpha, v) = \frac{1}{4}\int_0^v G(\alpha,\beta)P(\alpha,\beta)d\beta + \frac{1}{4}Q\left(\frac{\alpha}{2}\right), \ (\alpha \in [1,2])
\tag{8.54}
$$

Hence making use of (8.53) we deduce

(8.55)

$$P(u,v) = -\frac{1}{4}\int_u^{2-v} d\alpha \int_0^v G(\alpha,\beta)P(\alpha,\beta)d\beta - \frac{1}{2}\int_{2-v}^{2} e^{H_3(2-v-\alpha)}d\alpha \int_0^{2-\alpha} G(\alpha,\beta)P(\alpha,\beta)d\beta$$

$$-\frac{1}{2}\int_{2-v}^{2} e^{H_3(2-v-\alpha)}Q\left(\frac{\alpha}{2}\right)d\alpha + \frac{1}{4}\int_{2-v}^{2} e^{H_3(2-v-\alpha)}\varphi(\alpha-1)d\alpha - \frac{1}{4}\int_u^{2-v} Q(\alpha/2)d\alpha$$

Hence bearing in mind the notation (8.29) we arrive at the inequality

$$\begin{aligned}|P(u,v)|^2 \le c_3 \Bigg\{ &\int_u^{2-v} d\alpha \int_0^{2-\alpha} |P(\alpha,\beta)|^2 d\beta + \int_u^{2} d\alpha \int_0^{2-\alpha} |P(\alpha,\beta)|^2 d\beta + \\ &+\|\varphi\|^2_{L_2[0,1]} + \|Q\|^2_{L_2[1/2,1]}\Bigg\} \le 2c_3 \int_u^2 I_1(\alpha)d\alpha + c_3(\|\varphi\|^2_{L_2} + \|Q\|^2_{L_2[1/2,1]})\end{aligned} \tag{8.56}$$

with constant $c_3 = c_3(Q_1, r, H_3)$ not depending on $Q_2 \in B(Q_1; r)$. Integrating (8.56) with respect to v from $v = 0$ to $v = 2-u$ and applying the Gonwall's lemma one concludes that

$$I_1(u) \le c_3\left(\|\varphi\|^2_{L_2[0,1]} + \|Q\|^2_{L_2[1/2,1]}\right)\exp(2c_3(2-u)). \tag{8.57}$$

Combining the inequalities (8.56) and (8.57) we obtain the uniform estimate for $P(u,v)$ in the triangle $\triangle_1 = \triangle_{ABC}$:

$$|P(u,v)|^2 \le c_3(\|\varphi\|^2_{L_2[0,1]} + \|Q_1 - Q_2\|^2_{L_2[1/2,1]}\exp(2c_3(2-u)) \tag{8.58}$$

3) As well as in the proof of Theorem 8.1 we need an integral estimate for $D_2P(u,v)$. Differentiating (8.55) with respect to v and taking (8.57) into account we derive after simple calculations

(8.59)

$$\begin{aligned}\int_0^{2-u} |D_2P(u,v)|^2 dv \le c_4 \Bigg\{ &\int_0^{2-u} dv \int_0^v |P(2-v,\beta)|^2 d\beta + \int_0^{2-u} dv \int_u^{2-v} |P(\alpha,v)|^2 d\alpha \\ &+ \int_0^{2-u} dv \int_{2-v}^{2} d\alpha \int_0^{2-\alpha} |P(\alpha,\beta)|^2 d\beta + \int_0^{2-u} dv \int_{2-v}^{2} [|Q(\alpha/2)|^2 + |\varphi(\alpha-1)|^2]d\alpha \\ &+ \int_0^{2-u} \left[|\varphi(1-v)|^2 + |Q(1-v/2)|^2\right] dv \Bigg\} \le 3c_4 \int_u^2 d\alpha \int_0^{2-\alpha} |P(\alpha,\beta)|^2 d\beta \\ &+ 2c_4(\|\varphi\|^2_{L_2[0,1]} + \|Q\|^2_{L_2[1/2,1]}) \le c_5(\|\varphi\|^2_{L_2[0,1]} + \|Q_1 - Q_2\|^2_{L_2[1/2,1]}) \qquad \forall u \in [1,2]\end{aligned}$$

Here the constants c_4 and $c_5 = c_5(Q_1, r, H_3)$ don't depend on $Q_2 \in B(Q_1; r)$ and $H_2 \in [\mathbb{C}^n]$.

4) Going over to the triangle $\triangle_2 = \triangle_{OAC}$ we start with the equation (8.39) for $P(u,v)$ as in the proof of Theorem 8.1. But now, as distinquished from the proof of Theorem 8.1, we make use of the estimates (8.59) (with $u=1$) and (8.58) (with $u=v=1$) instead of (8.36) and (8.33). It allows us to arrive at the inequality

$$|P(u,v)|^2 \le c_6 \int_v^1 \int_v^\alpha |P(\alpha,\beta)|^2 d\beta + c_6(\|\varphi\|^2_{L_2[0,1]} + \|Q\|^2_{L_2[1/2,1]})$$

instead of (8.40) and then

$$(8.60)\quad I_2(v) := \int_v^1 |P(u,v)|^2 du \le c_7(\|\varphi\|^2_{L_2[0,1]} + \|Q_2 - Q_1\|^2_{L_2[1/2,1]}) \exp(c_7(1-v))$$

with constants c_6 and c_7 independent of $Q_2 \in B(Q_1;r)$ and $H_2 \in [\mathbb{C}^n]$. Combining the inequalities (8.45) and (8.58)-(8.60) and bearing in mind the relation (8.22) and the notation (8.52), we obtain the desired estimate (8.51), which is what had to be proved. $\square$

Remark 8.2. Theorem 8.4 is an obvious consequence of Theorem 8.2. Indeed the condition (7.84) implies with regard to (7.85) the equality $A = R(1,1) = 0$. But the direct proof seems to be more preferable in view of its brevity and simplicity.

8.3. In the case $n=1$ we present an estimate for the difference $Q_2 - Q_1$ in terms of the eigenvalues.

Let $B_0(Q_1;r)$ be the subset of those $Q_2 \in B(Q_1;r)$ which satisfy the condition $\int_0^1 [Q_2(x) - Q_1(x)]dx = 0$.

Proposition 8.1. *Let $\{\lambda^0_{kj}\}_{k=1}^\infty = S(Q_0; h^0_1, h_j)$ be the spectrum of the problem (7.80), (7.81) with $Q_0 = Q_1$, $H_1 = h^0_1$ and $H_2 = h_j$ $(j=1,2)$. Then there exist $\varepsilon > 0$ and $c > 0$ such that for each $Q \in B^0(Q_1;r)$ and $\{\lambda_{kj}\}_{k=1}^\infty = S(Q; h^0_1, h_j)$ we have*

$$(8.61)\quad \begin{aligned} \|Q - Q_0\|^2_{L_2[0,1]} \le c \Bigg[&\sum_{j=1}^2 \sum_{k=1}^\infty |\lambda_{kj} - \lambda^0_{kj}|^2 + \\ &+ \sum_{k=1}^\infty |k(\lambda_{k1} - \lambda^0_{k1}) - k(\lambda_{k2} - \lambda^0_{k2})|^2 \Bigg] \end{aligned}$$

if only the expression in the square brackets in (8.61) is less then ε.

Proposition 8.2. *Let $\{\lambda^0_k\}_1^\infty = S(Q_0; h_1, h_3)$ be the spectrum of the problem (7.80), (7.81) with $Q = Q_0$, $H_1 = h_1$ and $H_2 = h_3$. Then there exist $\varepsilon > 0$ and $c > 0$ such that for each $Q \in B^0(Q_0;r)$ and $\{\lambda_k\}_1^\infty = S(Q; h_1, h_3)$ with $\sum_1^\infty |\lambda_k - \lambda^0_k|^2 \le \varepsilon$ the estimate*

$$\|Q - Q_0\|^2_{L_2[0,1]} \le c \left\{ \|Q - Q_0\|^2_{L_2[1/2,1]} + \sum_1^\infty |\lambda_k - \lambda^0_k|^2 \right\}$$

takes place.

The proofs of this propositions will be published elsewhere.

8.4 Our goal in this section is to consider the stablity of the reovery of a Dirac system on the interval $[0,1]$ from $Y(1,\lambda)=\begin{pmatrix}Y_1(1,\lambda)\\ Y_2(1,\lambda)\end{pmatrix}$. Let

$$B_2(r) := \{\widetilde{Q}=\begin{pmatrix}0 & \widetilde{Q}_1(x)\\ \widetilde{Q}_2(x) & 0\end{pmatrix}\in L_2[0,1]\otimes\mathbb{C}^{n\times n}:\ \|\widetilde{Q}\|_{L_2^{n\times n}}\le r\}. \tag{8.62}$$

Theorem 8.3. *Let* $Y=\begin{pmatrix}Y_1(x,\lambda)\\ Y_2(x,\lambda)\end{pmatrix}$ *and* $\widetilde{Y}=\begin{pmatrix}\widetilde{Y}_1(x,\lambda)\\ \widetilde{Y}_2(x,\lambda)\end{pmatrix}$ *be the* $n\times n_1$ *matrix solutions of the Cauchy problems (7.20), (7.21) and (7.26), (7.27) respectively with matrix potentials* $Q(x),\ \widetilde{Q}(x)\in B_2(r)$. *Then the following estimate*

$$\|Q-\widetilde{Q}\|^2_{L_2[0,1]\otimes\mathbb{C}^{n^2}}\le c_0\sum_{i=1}^{2}\|\widetilde{Y}_i(1,\lambda)-Y_i(1,\lambda)\|^2_{L_2(\mathbb{R})} \tag{8.63}$$

holds with constant $c_0=c_0(Q,r)$ *not depending on* $\widetilde{Q}\in B_2(r)$.

Proof. 1) As in the proof of Thorem 8.1 we start with the representation (7.22) for $Y(x,\lambda)$ and similar representation for $\widetilde{Y}(x,\lambda)$. Taking the difference on these representations at $x=1$ and applying the Parseval equality one derives $(i=1,2)$:

$$\frac{2}{\pi}\int_{\mathbb{R}}|Y_i(1,\lambda)-\widetilde{Y}_i(1,\lambda)|^2d\lambda=\sum_{j=1}^{2}\int_0^1|K_{ij}(1,t)-\widetilde{K}_{ij}(1,t)|^2dt \tag{8.64}$$

Further we consider the triangular representation (7.28) with the Volterra operator $P:\ f\to\int_0^x P(x,t)f(t)dt$, defined by (7.29). As it was done on the second step of the proof of Theorem 8.1, we easily deduce from (7.29) the estimate

$$\sum_{i,j=1}^{2}\|P_{ij}(1,t)\|^2_{L_2[0,1]}\le c_1\sum_{i,j=1}^{2}\|K_{ij}(1,t)-\widetilde{K}_{ij}(1,t)\|^2_{L_2[0,1]} \tag{8.65}$$

with constant c_1 depending only on $K(x,t)$ and consequently only on $Q(x):\ c_1=c_1(Q)$.

Combining (8.64) and (8.65) one concludes that to establish the inequality (8.63) it suffices to prove the estimate

$$\|Q-\widetilde{Q}\|^2_{L_2[0,1]\otimes\mathbb{C}^{n^2}}\le c_2\sum_{i,j=1}^{2}\|P_{ij}(1,t)\|^2_{L_2[0,1]}. \tag{8.66}$$

2) To obtain (8.66) we will consider the boundary value problem (7.30)-(7.32) for the kernel $P(x,t)$ of the transformation operator. Setting $\psi_{ij}(t):=P_{ij}(1,t)$ $(i,j=$

1, 2) and integrating the system (7.37)-(7.40) in the triangle $\triangle_1 = \triangle_{ABC}$ we obtain the following system of the integral equations :

(8.67)

$$P_{ii}(x,t) = \int_x^1 [(-1)^j Q_i(\xi) P_{ji}(\xi, \xi - x + t) + (-1)^i P_{ij}(\xi, \xi - x + t) \widetilde{Q}_j(\xi - x + t)] d\xi + \psi_{ii}(1 - x + t)$$

(8.68)

$$P_{ij}(x,t) = \int_x^1 [(-1)^j Q_i(\xi) P_{jj}(\xi, \xi - x + t) + (-1)^i P_{ii}(\xi, \xi - x + t) \widetilde{Q}_i(\xi - x + t)] d\xi + \psi_{ij}(x + t - 1)$$

Having changed ξ for η in (8.67) and substituting (8.67) into (8.68) we arrive, after the change of variables $\alpha = \eta,\ \beta = \eta - 2\xi + x + t$, at the equations (with $1 \leq i \neq j \leq 2$)

(8.69)

$$\begin{aligned} P_{ij}(x,t) &= \psi_{ij}(x + t - 1) \\ &+ \int_x^1 [(-1)^j Q_i(\xi) \psi_{jj}(1 - x + t) + (-1)^i \psi_{ii}(1 - x + t) \widetilde{Q}_i(x + t - \xi)] d\xi \\ &+ \int_x^1 \int_{x+t-\alpha}^{\alpha - x + t} Q_i\left(\frac{\alpha - \beta + x + t}{2}\right) [(-1)^{i+j} Q_j(\alpha) P_{ij}(\alpha, \beta) + P_{ji}(\alpha, \beta) \widetilde{Q}_i(\beta)] d\beta d\alpha \\ &+ \int_x^1 \int_{x+t-\alpha}^{\alpha - x + t} [(-1)^{i+j} Q_i(\alpha) P_{ji}(\alpha, \beta) + P_{ij}(\alpha, \beta) \widetilde{Q}_j(\beta)] d\beta \cdot \widetilde{Q}_i\left(\frac{x + t - \alpha + \beta}{2}\right) d\alpha. \end{aligned}$$

Each term on the right-hand side of (8.69) may be estimated by means of the Schwartz inequality. For example we have

$$\begin{aligned} &\left(\int_x^1 |Q_j(\alpha)| d\alpha \int_{x+t-\alpha}^{\alpha - x + t} \left|Q_i\left(\frac{\alpha - \beta + x + t}{2}\right)\right| \cdot |P_{ij}(\alpha, \beta)| d\beta\right)^2 \\ &\leq \int_x^1 d\alpha \int_{1-\alpha}^{\alpha} |P_{ij}(\alpha, \beta)|^2 d\beta \cdot \int_x^1 |Q_j(\alpha)|^2 d\alpha \int_{x+t-\alpha}^{\alpha - x + t} \left|Q_i\left(\frac{\alpha - \beta + x + t}{2}\right)\right|^2 d\beta \\ &\leq \|Q_j\|_{L_2}^2 \cdot \|Q_i\|_{L_2}^2 \cdot \int_x^1 d\alpha \int_{1-\alpha}^{\alpha} |P_{ij}(\alpha, \beta)|^2 d\beta. \end{aligned}$$

Thus, setting

$$I(x) := \int_{1-x}^{x} (|P_{12}(x,t)|^2 + |P_{21}(x,t)|^2) dt \tag{8.70}$$

one obtains from (8.69)

$$
\begin{aligned}
(8.71)\qquad |P_{12}(x,t)|^2+|P_{21}(x,t)|^2 &\le \sum_{i\neq j}|\psi_{ij}(x+t-1)|^2+c\sum_{i=1}^{2}|\psi_{ii}(1-x+t)|^2+\\
&+c\int\limits_x^1 I(\alpha)d\alpha \qquad \forall x\ge 1/2
\end{aligned}
$$

and therefore

$$
(8.72)\qquad I(x)\le c\sum_{i,j=1}^{2}\|\psi_{ij}\|_{L_2[0,1]}+c\int\limits_x^1 I(\alpha)d\alpha
$$

Applying the Gronwall's inequality to (8.72) one concludes, that

$$
(8.73)\qquad I(x)\le c\left(\sum_{i,j=1}^{2}\|\psi_{ij}\|_{L_2[0,1]}\right)\exp c(1-x),\qquad (x\ge 1/2).
$$

Letting in (8.71) $t=x$ and bearing in mind (7.31) one derives, making use of (8.73):
(8.74)

$$
\begin{aligned}
&\int\limits_{1/2}^{1}|Q(x)-\widetilde{Q}(x)|^2dx=\int\limits_{1/2}^{1}|BP(x,x)-P(x,x)B|^2dx\\
&=\int\limits_{1/2}^{1}(|P_{12}(x,x)|^2+|P_{21}(x,x)|^2)dx\le c\sum_{i,j=1}^{2}\|\psi_{ij}\|^2_{L_2[0,1]}=c\sum_{i,j=1}^{2}\|P_{ij}(1,t)\|^2_{L_2[0,1]}.
\end{aligned}
$$

The inequality (8.74) proves the first "half" of the required estimate (8.66). The second one may be proven similarly. □

The following theorem may be considered as an analog of Theorem 8.2.

Theorem 8.4. *Let Q, $\widetilde{Q}\in B_2(r)$ and $Y=\binom{Y_1}{Y_2}$, $\widetilde{Y}=\binom{\widetilde{Y}_1}{\widetilde{Y}_2}$ be the same as in Theorem 8.3 and $H_3\in[\mathbb{C}^{n_1},\mathbb{C}^{n_2}]$. Then the estimate*

$$
\begin{aligned}
(8.75)\qquad \|Q-\widetilde{Q}\|^2_{L_2[0,1]} &\le c\{\|Q-\widetilde{Q}\|^2_{L_2[1/2,1]}\\
&+\|Y_2(1,\lambda)-\widetilde{Y}_2(1,\lambda)+H_3(Y_1(1,\lambda)-\widetilde{Y}_1(1,\lambda))\|^2_{L_2(\mathbb{R})}\}
\end{aligned}
$$

holds for all $\widetilde{Q}\in B_2(r)$ with constant $c_0=c_0(Q,r,H_3)$ independent of $\widetilde{Q}\in B_2(r)$.

This result may be proven in just the same manner as Theorem 8.3 or Theorem 8.2.

In the case $n=1$ one may evaluate the difference $Q-\widetilde{Q}$, making use of the spectra of the boundary value problems. Namely, the analogs of Propositions 8.1 and 8.2 are also valid. We state here only one of them.

Proposition 8.3. *Let $\{\lambda_k^0\}_{-\infty}^{\infty} = S(Q; h, h_1)$ be the spectrum of the problem (7.68), (7.69). Then there exist $\varepsilon > 0$ and $c > 0$ such that for $\widetilde{Q} \in B_2(r)$ and $\{\lambda_k\}_{-\infty}^{\infty} = S(\widetilde{Q}; h, h_1)$ with $\sum_{-\infty}^{\infty} |k(\lambda_k - \lambda_k^0)|^2 \le \varepsilon$ the following estimate holds with constant c not depending on $\widetilde{Q} \in B_2(r)$:*

$$\|Q - \widetilde{Q}\|_{L_2[0,1/2]}^2 \le c\{\|Q - \widetilde{Q}\|_{L_2[1/2,1]}^2 + \sum_{-\infty}^{\infty} |k(\lambda_k - \lambda_k^0)|^2\}.$$

8.5. Our goal in this section is to estimate the difference $Q(x) - \widetilde{Q}(x)$ in the Sobolev W_2^l - norm instead of the L_2 - norm like in Theorems 8.3 and 8.4.

As usual, $W_2^l[0,1] \otimes \mathbb{C}^n$ stands for the Sobolev space of vector functions $(f_1, \dots, f_n)$ with $f_j \in W_2^l[0,1]$ $(1 \le j \le n)$.

At first we need the following lemma.

Lemma 8.1. *Let $Y = \begin{pmatrix} Y_1(x,\lambda) \\ Y_2(x,\lambda) \end{pmatrix}$ be the $n \times n_1$ matrix solution of the Cauchy problem (7.20), (7.21) and $l \in \mathbb{Z}_+$. If $Q(x) \in B_2(r)$ then the following implication holds:*

$$\lambda^l \begin{pmatrix} Y_1(1,\lambda) - e^{i\lambda} I_{n_1} \\ Y_2(1,\lambda) - e^{-i\lambda} A \end{pmatrix} \in L_2^{n \times n_1}[0,1] \Rightarrow Q(x) \in W_2^l[0,1] \otimes \mathbb{C}^{n^2} \tag{8.76}$$

Proof. 1) Let us make use of the representation (7.22) for the solution $Y(x,\lambda)$. One immediately obtains from (7.22) the implication

$$\lambda^l \begin{pmatrix} Y_1(1,\lambda) - e^{i\lambda} I_{n_1} \\ Y_2(1,\lambda) - e^{-i\lambda} A \end{pmatrix} \in L_2^{n \times n_1}[0,1] \Rightarrow K(1,t) \in W_2^l[0,1] \otimes \mathbb{C}^{n^2} \tag{8.77}$$

2) On the second step we prove the implication

$$K(1,t) \in W_2^l[0,1] \otimes \mathbb{C}^{n^2} \Rightarrow Q(x) \in W_2^l[0,1] \otimes \mathbb{C}^{n^2} \tag{8.78}$$

It is clear that $K_{ij}(x,t)$ satisfy the equations (8.67) and (8.68) with $\widetilde{Q}_j(x) = 0$ $(i,j = 1,2)$ as well as the equations (8.69) with $\widetilde{Q}_j(x) = 0$. Returnign to the old variables $\eta = \alpha,\ \xi = (x + t + \alpha - \beta)/2$ one rewrites the last equations in the form

$$\begin{aligned} K_{ij}(x,t) &= \psi_{ij}(x+t-1) + \int_x^1 (-1)^j Q_i(\xi) \psi_{jj}(1-x+t) d\xi \\ &\quad + \int_x^1 Q_i(\xi) d\xi \int_\xi^1 (-1)^{i+j} Q_j(\eta) K_{ij}(\eta, \eta - 2\xi + x + t) d\eta \quad (i \ne j). \end{aligned} \tag{8.79}$$

We solve (8.79) by the method of successive approximations, setting

$$K_{ij0}(x,t) = \psi_{ij}(x+t-1) + \int_x^1 (-1)^j Q_i(\xi) d\xi \psi_{jj}(1-x+t),$$

and for $m \geq 1$

$$(8.80) \qquad K_{ijm}(x,t) = \int\limits_x^1 Q_i(\xi)d\xi \int\limits_\xi^1 (-1)^{i+j} Q_j(\eta) K_{ij(m-1)}(\eta, \eta - 2\xi + x + t)d\eta.$$

Since $\psi_{ij}(t) = K_{ij}(1,t) \in W_2^l[0,1] \otimes \mathbb{C}^{n_i \times n_j}$ $(i,j = 1,2)$ we may set

$$(8.81) \qquad I_l(\alpha) = \max_{1 \leq i \neq j \leq 2} \int\limits_{1-\alpha}^{\alpha} |D_2^l K_{ij0}(\alpha,\beta)|^2 d\beta$$

and suppose by induction that the inequalities

$$(8.82) \qquad |D_2^l K_{ijm}(x,t)|^2 \leq r^{4m} \int\limits_x^1 \frac{(\alpha - x)^{2m-2}}{(2m-2)!} I_l(\alpha)d\alpha, \qquad m \geq 1$$

hold with $Q(x) \in B_2(r)$ (see (8.62)).

For $m = 1$ (8.82) immediately follows from (8.80) after the change $\alpha = \eta$, $\beta = \eta - 2\xi + x + t$ (compare with (8.69), when $\widetilde{Q}_i(x) = 0$). When made use of, the induction hypothesis (8.82) implies

$$\begin{aligned}
&|D_2^l K_{ijm}(x,t)|^2 \\
&\leq \int\limits_x^1 d\xi \int\limits_\xi^1 |Q_i(\xi)|^2 \cdot |Q_j(\eta)|^2 d\eta \cdot \int\limits_x^1 d\xi \int\limits_\xi^1 |D_2^l K_{ijm}(\eta, \eta - 2\xi + x + t)|^2 d\eta \\
&\leq r^4 \int\limits_x^1 d\xi \int\limits_\xi^1 r^{4m} d\eta \int\limits_\eta^1 \frac{(\alpha - \eta)^{2m-2}}{(2m-2)!} I_l(\alpha)d\alpha = r^{4(m+1)} \int\limits_x^1 \frac{(\alpha - x)^{2m}}{(2m)!} I_l(\alpha)d\alpha.
\end{aligned}$$

This inequality proves the induction step.
In view of (8.82) the series $K_{ij}(x,t) = \sum_{m=0}^\infty K_{ijm}(x,t)$ admits termwise differentiation with respect to the second variable and therefore

$$|D_t^l K_{ij}(x,t)|^2 \leq 2|\psi_{ij}(x+t-1)|^2 + 2r^2 |\psi_{jj}(1-x+t)^2 + r^4 \int\limits_x^1 \operatorname{ch} r^2(\alpha - x) I_l(\alpha)d\alpha.$$

In particular it means that $D_2^l K_{ij}(x,t) \in L_2(\triangle_1)$ for $i \neq j$. Setting now $t = x$ in (8.79) and bearing in mind (7.31) one derives: $Q_i(x) = K_{ij}(x,x) \in W_2^1[1/2,1] \otimes \mathbb{C}^{n_i^2}$ $(i = 1,2)$.

Repeating the previous arguments l times we obtain $Q_i(x) \in W_2^l[1/2,1] \otimes \mathbb{C}^{n_i^2}$ $(i = 1,2)$. The including $Q_i(x) \in W_2^l[0,1/2]$ may be proven similarly. $\square$

Remark 8.1. In fact, the implication (8.78) may be replaced by the equivalence

$$(8.83) \qquad Q(x) \in W_2^l[0,1] \otimes \mathbb{C}^{n^2} \Longleftrightarrow K(1,t) \in W_2^l[0,1] \otimes \mathbb{C}^{n^2}.$$

Now we are in a position to state the main result of the section. Let

$$B_2^l(r) = \{\widetilde{Q} \in W_2^l \otimes \mathbb{C}^{n^2} : \ \|\widetilde{Q}\|_{W_2^l[0,1]} \leq r\}, \ l \in \mathbb{Z}_+.$$

Theorem 8.5. *Let* $Q = \begin{pmatrix} 0 & Q_1 \\ Q_2 & 0 \end{pmatrix} \in B_2^l(r)$ *and* $\widetilde{Q} = \begin{pmatrix} 0 & \widetilde{Q}_1 \\ \widetilde{Q}_2 & 0 \end{pmatrix} \in B_2(r)$. *Let also* $Y = \binom{Y_1}{Y_2}$ *and* $\widetilde{Y} = \binom{\widetilde{Y}_1}{\widetilde{Y}_2}$ *be the* $n \times n_1$ *matrix solutions of the Cauchy problems (7.20), (7.21) and (7.26), (7.27) respectively with* $A = \widetilde{A}$. *If*

$$\lambda^l(Y_j(1,\lambda) - \widetilde{Y}_j(1,\lambda)) \in L_2(\mathbb{R}) \otimes \mathbb{C}^{n_j \times n_1} \ (j = 1, 2) \tag{8.84}$$

then $\widetilde{Q}(x) \in W_2^l[0,1] \otimes \mathbb{C}^{n^2}$ *and the estimate*

$$\|\widetilde{Q} - Q\|_{W_2^l[0,1]\otimes\mathbb{C}^{n^2}} \le c \sum_{j=1}^{2} \|\lambda^l(Y_j(1,\lambda) - \widetilde{Y}_j(1,\lambda))\|_{L_2(\mathbb{R})\otimes\mathbb{C}^{n_j^2}} \tag{8.85}$$

holds with the constant $c = c(Q, r)$ *not depending on* $\widetilde{Q} \in B_2^l(r)$.

Proof. 1) Since $Q \in W_2^l[0,1]$ one concludes from (8.83) that $K(1,t) \in W_2^l[0,1]$. It follows from (8.84) and the representation (7.22) for $Y(x,\lambda)$ and a similar representation (with kernel $\widetilde{K}(x,t)$) for $\widetilde{Y}(x,\lambda)$ that $D_2^p(K(1,1) - \widetilde{K}(1,1)) = D_2^p(K(1,0) - \widetilde{K}(1,0)) = 0$ for each $p \in \{0, \dots, l-1\}$ and $K(1,t) - \widetilde{K}(1,t) \in W_2^l[0,1]$. Thus $\widetilde{K}(1,t) \in W_2^l[0,1]$ and, by Lemma 8.1, (see implication (8.78)) $\widetilde{Q} \in W_2^l[0,1]$.

2) Applying the Parseval equality one derives

$$\frac{2}{\pi} \sum_{j=1}^{2} \|\lambda^l(Y_j(1,\lambda) - \widetilde{Y}_j(1,\lambda)\|^2_{L_2(\mathbb{R})} = \sum_{i,j=1}^{2} \|K_{ij}(1,t) - \widetilde{K}_{ij}(1,t)\|^2_{W_2^l[0,1]}$$

instead of (8.64). Then, similarly to (8.65) one obtains

$$\sum_{i,j=1}^{2} \|P_{ij}(1,t)\|^2_{W_2^l[0,1]} \le c_1 \sum_{i,j=1}^{2} \|K_{ij}(1,t) - \widetilde{K}_{ij}(1,t)\|^2_{W_2^l[0,1]}.$$

So, to complete the proof it suffices to establish the estimate

$$\|Q - \widetilde{Q}\|_{W_2^l[0,1]} \le c_2 \sum_{i,j=1}^{2} \|P_{ij}(1,t)\|^2_{W_2^l[0,1]}$$

It may be done with slight modification of the arguments used on the second step of the proof of Theorem 8.3. □

The following proposition may be considered as a complement to Theorem 8.4.

Proposition 8.4. *Let* $Q, \widetilde{Q} \in B_2^l(r)$ *and let* $Y = \binom{Y_1}{Y_2}$, $\widetilde{Y} = \binom{\widetilde{Y}_1}{\widetilde{Y}_2}$ *be the same as in Theorem 8.5 and* $H_3 \in [\mathbb{C}^{n_1}, \mathbb{C}^{n_2}]$. *Then the estimate*

$$\|Q - \widetilde{Q}\|^2_{W_2^l[0,1]} \le c\{\|Q - \widetilde{Q}\|^2_{W_2^l[1/2,1]} + \|\lambda^l[Y_2(1,\lambda) - \widetilde{Y}_2(1,\lambda) + H_3(Y_1(1,\lambda) - \widetilde{Y}_1(1,\lambda))]\|^2_{L_2(\mathbb{R})}\}$$

holds for all $\widetilde{Q} \in B_2^l(r)$ with constant $c_0 = c_0(Q, r, H_3)$ independent of $\widetilde{Q} \in B_2^l(r)$.

Remark 8.2. All the results of this and previous sections remain also valid for systems (7.20), (7.21) with B being a diagonal matrix: $B = \operatorname{diag}(\lambda_1 I_{n_1}, \lambda_2 I_{n_2})$ with real eigenvalues of the oposite signs ($\lambda_1\lambda_2 < 0$).

8.6 In Theorems 8.3-8.5 a potential matrix is not assumed to be selfadjoint. If in addition $n_1 = n_2$ and $Q(x)$ is selfadjoint and real, that is $Q(x) = Q^*(x) = Q^\tau(x)$ (as usual Q^τ stands for the transpose matrix), the above results may be applied to the canonical systems

$$J_n \frac{dz(t,\lambda)}{dt} - \lambda \mathcal{H}(t) z(t,\lambda) = 0, \qquad J_n = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} \tag{8.86}$$

with positive definite Hamiltonian $\mathcal{H}(t)$.

Actually under some assumptions on the Hamiltonian $\mathcal{H}(t)$ the system (8.86) may be reduced to the Dirac system with real selfadjoint summable potential matrix $Q(t) = Q^\tau(t) = Q^*(t)$:

$$J_n \frac{dy(t,\lambda)}{dt} + Q(t)y(t,\lambda) = \lambda y, \quad Q(t) = \begin{pmatrix} p(t) & q(t) \\ q(t) & -p(t) \end{pmatrix} \tag{8.87}$$

Let $W(t,\lambda)$ be the fundamental $2n \times 2n$ matrix solution of (8.87) satisfying the initial condition $W(0,\lambda) = I_{2n}$. Since $Q(t) = Q^\tau(t)$ the matrix $W(t,\lambda)$ is symplectic for $t \in [0,1]$ and $\lambda \in \mathbb{C}$, i.e.,

$$W^\tau(t,\lambda) J_n W(t,\lambda) = W(t,\lambda) J_n W^\tau(t,\lambda) = J_n \tag{8.88}$$

Further, since $Q(t)$ is real, so is $W(t,\lambda)$ with $\lambda \in \mathbb{R}$, that is $W^\tau(t,\lambda) = W^*(t,\lambda) \quad \forall \lambda \in \mathbb{R}$. Therefore making use of the substitution (see [GK], ch. VI)

$$y(t,\lambda) = W_0(t) z(t,\lambda) \tag{8.89}$$

where $W_0(t) := W(t,0)$ and bearing in mind (8.88) (for $t = 0$) one arrives at the equation (8.86) with the Hamiltonian

$$\mathcal{H}(t) := W_0^\tau(t) W_0(t) = W_0^*(t) W_0(t) = \mathcal{H}^\tau(t) = \mathcal{H}^*(t) \tag{8.90}$$

Observe also that the relations (8.90) are implied by the equality $W_0^\tau(t) = W_0^*(t) \quad \forall t \in [0,1]$.

It follows from (8.90) that $\mathcal{H}(t)$ posesses the following properties:

a) $\mathcal{H}(t) \in W_1^1[0,1] \otimes \mathbb{C}^{2n \times 2n}$; b) $\mathcal{H}(t)$ is positive definite for $t \in [0,1]$.;

c) $\mathcal{H}(t)$ is symplectic for $t \in [0,1]$, that is $\mathcal{H}(t) J_n \mathcal{H}(t) = J_n$ for $t \in [0,1]$.

We clarify that the property b) is implied by the represenntation (8.90) with regard to the relation $\det \mathcal{H}(t) = 1$, which follows at once from c) and (8.90).

Note also that $\mathcal{H}(0) = I_{2n}$, since $W_0(0) = I_{2n}$. But for an arbitrary positive definite matrix $C > 0$ we can make $\mathcal{H}(0) = C$ by choosing another fundamental solution $W_0(t)$.

Theorem 8.6. *Let $\mathcal{H}(t)$ be an absolutely continuous positive definite and symplectic Hamiltonian on $[0,1]$. Let also $Z(t,\lambda)=\begin{pmatrix} z_1(t,\lambda) \\ z_2(t,\lambda)\end{pmatrix}$ be the $2n\times n$ matrix solution of the system (8.86), satisfying the initial condition $Z(0,\lambda)=\begin{pmatrix} I_n \\ 0\end{pmatrix}$. Then:*

1) The Hamiltonian is uniquely determined by the $n \times n$ matrix function $z_1(1,\lambda)z_2^{-1}(1,\lambda)$;

2) The following implication holds ($l \in \mathbb{Z}_+$) :

$$\lambda^l\begin{pmatrix} z_1(1,\lambda)-\cos\lambda\cdot I_n \\ z_2(1,\lambda)+\sin\lambda\cdot I_n\end{pmatrix} \in L_2(\mathbb{R})\otimes\mathbb{C}^{2n\times n} \Rightarrow \mathcal{H}(t)\in W_2^{l+1}[0,1]\otimes\mathbb{C}^{2n\times 2n}.$$

Proof. i) Since $\mathcal{H}(t)$ posesses the properties a), b) and c) (see above), it admits a factorization $\mathcal{H}(t)=G_1^\tau(t)G_1(t)$ with real symplectic positive definite and absolutely continuous matrix function $G_1(t)$ ($\in W_1^1[0,1]\otimes\mathbb{C}^{2n\times 2n}$). In the sequel this factorization will be referred to as symplectic factorization. Setting $Z(t,\lambda)=G_1(t)^{-1}X(t,\lambda)$, one arrives at the equation

$$\text{(8.92)}\qquad J_n\frac{dX(t,\lambda)}{dt}-\Omega_1(t)X(t,\lambda)=\lambda X(t,\lambda),\quad X(0,\lambda)=G_1(0)\begin{pmatrix} I_n \\ 0\end{pmatrix}$$

with $\Omega_1(t)=-(G_1^\tau)^{-1}(G_1^{-1})'=J_nG'(t)G^{-1}(t)$.

Assuming without loss of generality that $\mathcal{H}(0)=I_{2n}$, one concludes that $G(0)$ is a unitary operator. Since $G(t)$ is symplectic, $\Omega(t)$ is selfadjoint and real: $\Omega(t)=\Omega^\tau(t)=\Omega^*(t)$ $\forall t\in[0,1]$. It is clear also that $\Omega(t)\in L_1[0,1]\otimes\mathbb{C}^{n\times n}$. But, generally speaking, $\Omega_1(t)$ doesn't satisfy the commutation relation $\Omega_1(t)J_n+J_n\Omega_1(t)=0$.

ii) We will show that for a special choice of the symplectic factorization of $\mathcal{H}(t)$ the corresponding system (8.92) will be of Dirac type.

Note first that each pair of matrix functions $G_i(t)$ ($i=1,2$) in the symplectic factorization of the Hamiltonian $\mathcal{H}(t)=G_i^\tau(t)G_i(t)$ are connected by the relation $G_2(t)=U(t)G_1(t)$ with $U(t)$ being a real unitary and symplectic operator for $t\in[0,1]$. Conversely, starting with a C^1-smooth family $U(t)(t\in[0,1]$ of unitary and symplectic operators and setting $G_2(t)=U(t)G_1(t)$, one arrives at the symplectic factorization $\mathcal{H}(t)=G_2^\tau(t)G_2(t)$.

Starting with the potential matrix $\Omega_1(t)$ we introduce the C^1-smooth family $U(t)$ as the solution of the following Cauchy problem:

$$\text{(8.93)}\qquad \frac{dU(t)}{dt}=U(t)\Omega_2(t),\quad \Omega_2(t):=\frac{1}{2}(J_n\Omega_1(t)+\Omega_1(t)J_n),\ U(0)=G_1^*(0).$$

It is clear that $U(t)$ is real ($U^\tau(t)=U^*(t)$ $\forall t\in[0,1]$) because so is $\Omega_2(t)$. Further $U(t)$ is unitary because of $\Omega_2^\tau(t)=-\Omega_2(t)$ and $U(t)$ is symplectic for $t\in[0,1]$ since $[J_n,\Omega_2(t)]=0$. Setting $G_2(t):=U(t)G_1(t)$ and $Y(t,\lambda)=U(t)X(t,\lambda)$, we arrive at the equation

$$\text{(8.94)}\qquad \frac{dY(t,\lambda)}{dt}-Q(t)Y(t,\lambda)=\lambda Y(t,\lambda)$$

with $Q(t) := J_n G_2'(t) G_2^{-1}(t)$.

Now one easily proves that the system (8.94) is of Dirac type. Indeed, transforming the expression for $Q(x)$ with regard to (8.93) and taking the relation $[J_n, U(t)] = 0$ into account, one derives

$$Q(t) = J_n G_2'(t) G_2^{-1}(t) = J_n (U'(t) G_1(t) + U(t) G_1'(t)) G_1^{-1}(t) U^{-1}(t) = \frac{1}{2} U(t)(\Omega_1(t) + J_n \Omega_1(t) J_n) U^*(t). \tag{8.95}$$

The requeired property $J_n Q(t) + Q(t) J_n = 0$ is implied by (8.95).

iii) Thus, $Y(1,\lambda)$ is the $2n \times n$ matrix solution of the Cauchy problem for the DiracΦsystem (8.94) with the initial data $Y(0,\lambda) = U(0)X(0,\lambda) = G_1^*(0)G_1(0)Z(0,\lambda) = \begin{pmatrix} I_n \\ 0 \end{pmatrix}$ and $Y(1,\lambda) = G_2(1)Z(1,\lambda)$.

Let now $\widetilde{\mathcal{H}(t)}$ be another Hamiltonian, satisfying the conditions of Theorem 8.6. Starting with $\widetilde{\mathcal{H}(t)}$ one constructs the functions $\widetilde{G}_i(t)$, $\widetilde{Q}(t)$, $\widetilde{Z}(t,\lambda)$ and $\widetilde{Y}(t,\lambda)$ corresponding to $\widetilde{\mathcal{H}(t)}$ in just the same way, as the functions $G_i(t)$, $Q(t)$, $Z(t,\lambda)$, $Y(t,\lambda)$ corresponding to $\mathcal{H}(t)$ and then arrives at the relation $\widetilde{Y}(1,\lambda) = \widetilde{G}_2(1)\widetilde{Z}(1,\lambda)$.

If now $Z(1,\lambda) = \widetilde{Z}(1,\lambda)$ then $\widetilde{Y}(1,\lambda) = TY(1,\lambda)$ with $T = \widetilde{G}_2(1)G_2^{-1}(1)$. By Corollary 7.2 $Q(t) = \widetilde{Q}(t)$ for a.e. $t \in [0,1]$. In turn $Q(t)$ uniquely determines the $2n \times 2n$ matrix function $G_2(t)$ as the solution of the Cauchy problem

$$J_n G_2'(t) = Q(t) G_2(t), \qquad G_2(0) = I_{2n}, \tag{8.96}$$

Thus $G_2(t) = \widetilde{G}_2(t)$ for all $t \in [0,1]$ and consequently $\mathcal{H}(t) = \widetilde{\mathcal{H}(t)}$, that is $\mathcal{H}(t)$ is uniquely determined by $Z(1,\lambda)$ being the "half" of the monodromy matrix. The implication $z(1,\lambda)z_2^{-1}(1,\lambda) = \widetilde{z}_1(1,\lambda)\widetilde{z}_2^{-1}(1,\lambda) \Rightarrow \mathcal{H}(t) = \widetilde{\mathcal{H}(t)}$ may be considered with slight modification of the previous arguments by applying Proposition 7.1.

iiii) Let us prove the implication (8.91). As it has just been proven, $Y(t,\lambda) = G_2^{-1}(t)Z(t,\lambda)$ is a solution of the Cauchy problem for the Dirac sysytem (8.94), satisfying the initial condition $Y(0,\lambda) = \begin{pmatrix} I_n \\ 0 \end{pmatrix}$. According to (7.25), $Y(x,\lambda)$ admits the representation

$$Y(x,\lambda) = \begin{pmatrix} \cos \lambda x \cdot I_n \\ -\sin \lambda x \cdot I_n \end{pmatrix} + \int_0^x K(x,t) \begin{pmatrix} \cos \lambda t \\ -\sin \lambda t \end{pmatrix} dt. \tag{8.97}$$

Setting $T = G_2^{-1}(1)$ and observing that

$$\int_0^1 TK(1,t) \begin{pmatrix} \cos \lambda t \\ -\sin \lambda t \end{pmatrix} dt \in L_2(\mathbb{R}) \otimes \mathbb{C}^{2n \times n}$$

one concludes from (8.91) and (8.97) (with $x = 1$) that

$$(I_{2n} - T) \begin{pmatrix} \cos \lambda \\ -\sin \lambda \end{pmatrix} \in L_2[0,1] \otimes \mathbb{C}^{2n \times n}. \tag{8.98}$$

Thus $T = I_{2n}$, $Z(1,\lambda) = Y(1,\lambda)$ and the condition (8.91) remains valid if one replaces $z_j(1,\lambda)$ by $Y_j(1,\lambda)$ $(j = 1,2)$. Applying now Lemma 8.1 to the system (8.94) one derives that $Q(t) \in W_2^l[0,1] \otimes \mathbb{C}^{2n\times 2n}$.

Finally, it follows from (8.96) that $G_2(t) \in W_2^l[0,1] \otimes \mathbb{C}^{2n\times 2n}$ and consequently $\mathcal{H}(t) = G_2^\tau(t)G_2(t) \in W_2^l[0,1] \otimes \mathbb{C}^{2n\times 2n}$. $\square$

Remark 8.3. Note, that there exists the sole special symplectic factorization of $\mathcal{H}(t)$ which allows to reduce the Hamiltonian equation (8.86) to the Dirac system (8.94).

Remark 8.4. Let $U(x,\lambda)$ be the $2n \times 2n$ fundamental solution of the Hamiltonian equation (8.86), satisfying the initial condition $U(0,\lambda) = I_{2n}$. The matrix function $U(\lambda) := U(1,\lambda)$ is called the monodromy matrix. For $n = 1$ the problem of the unique recovery of the normed Hamiltonian $\mathcal{H}(t)$ ($\mathrm{Tr}\mathcal{H}(t) = 1$) from the monodromy matrix $U(\lambda)$ is completely solved by L. de Branges [Bra1], [Bra2]. Using a quite different approach he established (for $n = 1$) that each normed Hamiltonian $\mathcal{H}(t)$ of positive type is uniquely reconstructed from $U(\lambda)$. Note that for $n = 1$ the symplecticity of $\mathcal{H}(t)$ is equivalent to $\det\mathcal{H}(t) = 1$. For $n > 1$ the unique recovery of the Hamiltonian $\mathcal{H}(t)$, satisfying the conditions a), b), c) from the monodromy matrix $U(\lambda)$ has been annonced in [GK].

The following theorem complemets the previous result and contains it as a special case.

To present the statement we consider the Hamiltonian equation

$$J_n \frac{d\widetilde{Z}(t,\lambda)}{dt} - \lambda\widetilde{\mathcal{H}(t)}\widetilde{Z}(t,\lambda) = 0, \qquad \widetilde{Z}(0,\lambda) = \begin{pmatrix} I_n \\ 0 \end{pmatrix} \tag{8.99}$$

anongside the equation (8.86) and introduce the set $B_2^l(r,r^{-1})$ $(l \in \mathbb{Z}_+)$ of Hamiltonians, letting

$$B_2^l(r,r^{-1}) = \{\mathcal{H}(t) \in W_2^l[0,1] \otimes \mathbb{C}^{2n\times 2n} : \ r^{-1} \le \|\mathcal{H}(t)\|_{W_2^l} \le r\}.$$

Theorem 8.7. *Let $\mathcal{H}(t)$ and $\widetilde{\mathcal{H}(t)}$ satisfy the conditions of Theorem 8.6 and $\mathcal{H}(t) \in W_2^l[0,1] \otimes \mathbb{C}^{2n\times 2n}$ with $l \in \mathbb{Z}_+$. Then*

1) The following implication holds:

$$\lambda^l(\widetilde{Z}(1,\lambda) - Z(1,\lambda)) \in L_2(\mathbb{R}) \otimes \mathbb{C}^{2n\times n} \Rightarrow \widetilde{\mathcal{H}(t)} \in W_2^{l+1}[0,1] \otimes \mathbb{C}^{2n\times 2n}$$

2) For each $\widetilde{\mathcal{H}} \in B_2^{l+1}(r,r^{-1})$ the estimate

$$\|\mathcal{H}(t) - \widetilde{\mathcal{H}(t)}\|_{W_2^{l+1}[0,1]} \le c_0 \sum_{j=1}^{2} \|\lambda^l(z_j(1,\lambda) - \widetilde{z}_j(1,\lambda))\|_{L_2(\mathbb{R})} \tag{8.100}$$

holds with constant $c_0 = c_0(\mathcal{H};r)$ independent of $\widetilde{\mathcal{H}} \in B_2^{l+1}(r,r^{-1})$.

One obtains a proof of this result, combining the above considerations with Theorem 8.5 and Proposition 8.4.

Setting $\mathcal{H}(t) = I_{2n}$ one observes that the first assertion implies the second one of Theorem 8.6. Besides, the inequality (8.100) with $l = 0$ yields the first assertion of Theorem 8.6.

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