SUPERMANIFOLDS OF CLASSICAL SOLUTIONS FOR LAGRANGIAN FIELD MODELS WITH GHOST AND FERMION FIELDS

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ABSTRACT. Using a supergeometric interpretation of field functionals, we show that for quite a large class of classical field models used for realistic quantum field theoretic models, an infinite-dimensional supermanifold (smf) of classical solutions in Minkowski space can be constructed. That is, we show that the smf of smooth Cauchy data with compact support is isomorphic with an smf of corresponding classical solutions of the model.

CONTENTS

1. Lagrangian field theories

1.1. **Introduction.** This paper continues the investigation of the space of classical solutions of geometric field models of quantum field theory started in [8], [9], [10], [11]. The interest in such spaces can be traced back to the sixties; cf. [12], [13], [14], [15]. The new feature in our investigations is the appropriate account of fermionic degrees of freedom by using supergeometric methods, and the actual mathematically rigorous construction of Poincaré invariant solution spaces.

We start with fixing in 1.2 a class of classical field models in Minkowski space $\mathbb{R}^{d+1}$ which contains a number of common models like e. g. the Φ^4 model, the Thirring model, and the Yukawa model of meson-nucleon scattering.

Yang-Mills models have their peculiarities and will be treated separately, in sections 3 and 4.

Also, σ models as well as models with a non-linearized gravitational field do not fit directly into our class; nevertheless, our methods are applicable also to them.

Using the framework of infinite-dimensional supergeometry constructed in [8], [9], and the results on non-linear field equations with anticommuting fields presented in [10], [11], we will solve the classical field equations for compactly supported smooth Cauchy data.

While in purely bosonic models, like e.g. the Φ^4 model (cf. [10] for a discussion), one constructs a map

$$\{\text{space of Cauchy data at } t=0\} \longrightarrow \{\text{space of configurations on space-time}\}, \tag{1.1.1}$$

this is no longer appropriate in the presence of fermionic, anticommuting fields, since both Cauchy data and configurations form no longer sets (cf. [9] for a discussion of the background). Instead of this, they are described by infinite-dimensional supermanifolds (smf's); thus, the correct generalization of (1.1.1) is a morphism of supermanifolds

(1.1.2)
$$\Xi^{\mathrm{sol}} : M^{\mathrm{Cau}} = \{\text{smf of Cauchy data at } t=0\} \longrightarrow \{\text{smf of configurations on space-time}\} = M.$$

A suitable calculus of infinite-dimensional smf's modelled on $\mathbb{Z}_2$-graded locally convex vector spaces has been constructed by the present author in [8] and [9] and will be used throughout in the rest of this paper.

Of course, both the map (1.1.1) and the morphism (1.1.2) make sense only after suitable functional-analytic qualities have been specified. A reasonable choice for the Cauchy data is the test function space $\mathcal{E}_{\mathrm{c}}^{\mathrm{Cau,V}} := \mathcal{D}(\mathbb{R}^d) \otimes (V \oplus \dot{V})$; here V is the target space for the fields, and $\dot{V}$ is the target space for the velocities of those fields the field equations of which are of order two, i. e. the bosonic fields and the Faddeev-Popov ghosts, if these are present. For the configurations we take the space $\mathcal{E}_{\mathrm{c}}^{\mathrm{V}}$ of all those $f \in C^\infty(\mathbb{R}^{d+1}) \otimes V$ the support of which on every time slice is compact and grows only with light velocity (cf. 1.3 for details).

Now we associate to a given model a *configuration supermanifold*, or more precisely, the *supermanifold of smooth configurations with causally growing spatially compact support*, which is the linear smf $\mathrm{L}(\mathcal{E}_{\mathrm{c}}^{\mathrm{V}})$ (or "affine smf") modelled over the "naive configuration space" $\mathcal{E}_{\mathrm{c}}^{\mathrm{V}}$

$$M = \mathrm{L}(\mathcal{E}_{\mathrm{c}}^{\mathrm{V}}) \tag{1.1.3}$$

with standard coordinate $\Xi \in \mathcal{M}^{\mathcal{E}_{\mathrm{c}}^{\mathrm{V}}}(M)$ (where $\mathcal{M}^{\mathcal{E}_{\mathrm{c}}^{\mathrm{V}}}(M)$ denotes the $\mathbb{Z}_2$-graded vector space of real, even $\mathcal{E}_{\mathrm{c}}^{\mathrm{V}}$-valued superfunctions on M). In particular, in the absence of fermionic fields, M is simply the locally convex space $\mathcal{E}_{\mathrm{c}}^{\mathrm{V}}$ viewed as infinite-dimensional manifold, and Ξ is the identity map $\mathcal{E}_{\mathrm{c}}^{\mathrm{V}} \to \mathcal{E}_{\mathrm{c}}^{\mathrm{V}}$.

Also, we need the *supermanifold of compactly supported smooth Cauchy data* which is the linear smf

$$M^{\mathrm{Cau}} = \mathrm{L}(\mathcal{E}_{\mathrm{c}}^{\mathrm{Cau,V}}) \tag{1.1.4}$$

with standard coordinate $(\Xi^{\mathrm{Cau}}, \dot{\Phi}^{\mathrm{Cau}}) \in \mathcal{M}^{\mathcal{E}_{\mathrm{c}}^{\mathrm{Cau,V}}}(M^{\mathrm{Cau}})$, where $\dot{\Phi}^{\mathrm{Cau}}$ are the initial velocities of the bosonic fields. (Here and in the following, we suppose for notational simplicity that the Pauli Theorem is valid, so that the second-order fields are exactly the bosonic ones. The modifications in the presence of ghost fields are obvious; cf. section 4.)

For proceeding, we have to suppose that the underlying bosonic model is *complete*, i. e. classically all-time solvable for smooth Cauchy data with compact support. As to be expected, the existence of some a priori estimate for the Sobolev norm of solutions of the bosonic field equations is sufficient for completeness (cf. Thm. 1.8.1). We do not need information on the continuous dependence of the solutions on the Cauchy data since our approach automatically provides real-analytic dependence.

We will show that, as a consequence of our previous investigations [10], [11] of non-linear wave equations involving anticommuting fields, there exists for a complete model a unique $\mathcal{E}_c^V$-valued superfunction on M^{Cau},

$$\Xi^{\mathrm{sol}} = \Xi^{\mathrm{sol}}[\Xi^{\mathrm{Cau}}, \dot{\Phi}^{\mathrm{Cau}}] \in \mathcal{M}^{\mathcal{E}_c^V}(M^{\mathrm{Cau}}), \tag{1.1.5}$$

which satisfies the field equations and has the correct Cauchy data:

$$\frac{\delta}{\delta \Xi_i}\mathcal{L}[\Xi^{\mathrm{sol}}] = 0 \quad \text{for } i = 1, \ldots, N, \tag{1.1.6}$$

$$\Xi^{\mathrm{sol}}[\Xi^{\mathrm{Cau}}, \dot{\Phi}^{\mathrm{Cau}}](0, \cdot) = \Xi^{\mathrm{Cau}}(\cdot), \qquad \partial_t \Phi^{\mathrm{sol}}[\Xi^{\mathrm{Cau}}, \dot{\Phi}^{\mathrm{Cau}}](0, \cdot) = \dot{\Phi}^{\mathrm{Cau}}(\cdot). \tag{1.1.7}$$

The superfunction Ξ^{sol} then determines the smf morphism (1.1.2) to be constructed. This morphism identifies M^{Cau} with a sub-smf M^{sol} of M which we call the *supermanifold of classical solutions*. The name is justified by the fact that given a morphism $\phi : Z \to M$, i. e. a Z-family of configurations, it factors through M^{sol} iff we have $\phi^*(\frac{\delta}{\delta \Xi_i}\mathcal{L}[\Xi]) = 0$, i. e. ϕ solves the field equations.

In particular, the underlying manifold of M^{sol} is just the set of all those solutions (in the usual sense) of the underlying bosonic equations which are smooth and compactly supported on every time slice.

If the Lagrangian is Poincaré invariant then the natural action of the Poincaré group on M restricts to M^{sol} and hence carries over onto M^{Cau} (however, the formal proof will be given only in the next part). In particular, translation in time direction determines a complete flow (i. e. a one parameter automorphism group) on the smf M^{Cau} which we call the *time evolution flow*.

In 1.9, we also consider smooth configurations and Cauchy data without any support and growth condition, getting a morphism $\Xi^{\mathrm{sol}} : M^{\mathrm{Cau}}_{C^\infty} \to M_{C^\infty}$ as a variant of (1.1.2). (In fact, in the case of Yang-Mills theory in temporal gauge, this is the only morphism we can construct since the arising constraint (3.2.2) is not sufficiently compatible with spatially compact support.)

Yet another variant arises by considering fluctuations around a fixed bosonic "background" configuration which solves the bosonic field equations; cf. 1.10.

1.2. **A class of classical models.**

Here we consider a class of Lagrangian field theories on Minkowski $\mathbb{R}^{d+1}$ for which our approach immediately leads to uniqueness theorems, local existence, and, if a non-blow-up of the local solutions can be guaranteed from the outside, to an smf of solutions.

The reduction onto first-order equations which was assumed in [10], [11] is not natural in a Lorentz-invariant context. Instead, we take the Lagrangians as they stand.

We will use $\mu, \nu, \ldots = 0, \ldots, d$ as Lorentz indices which are raised and lowered in the usual way: $T_\mu = g_{\mu\nu}T^\mu$, $(g_{\mu\nu}) := \mathrm{diag}(-1, 1, \ldots, 1)$. Also $a, b, \ldots = 1, \ldots, d$ are spatial indices, and i, j will number field components ranging from 1 to N^Φ or N^Ψ or N or ..., depending on the context. For all three types of indices, we will use the Einstein sum convention.

We consider a field-theoretical model in $\mathbb{R}^{d+1} = \mathbb{R} \times \mathbb{R}^d$ with N^Φ real bosonic, commuting second-order fields $\Phi_1, \ldots, \Phi_{N^\Phi}$ (e. g. bosonic matter fields, or Yang-Mills fields in the diagonal gauge), as well as N^Ψ real fermionic, anticommuting first-order fields $\Psi_1, \ldots, \Psi_{N^\Psi}$, (e.g. Dirac, Weyl, or Majorana spinors, or Rarita-Schwinger fields; of course, complex field components should be broken into real and imaginary part).

We collect all field components into one vector

$$\Xi = (\Xi_1, \ldots, \Xi_N) = (\Phi|\Psi);$$

this will be the standard coodinate on the configuration smf

$$M = \mathrm{L}(\mathcal{E}_c^V), \quad V := \mathbb{R}^{N^\Phi} \oplus \Pi\mathbb{R}^{N^\Psi}, \quad \mathcal{E}_c^V := \mathcal{E}_c \otimes V$$

(cf. 1.3 below for the definition of $\mathcal{E}_c$). The model is described by the *Lagrange density*, which is a real, even, entire differential power series of the form

$$\mathcal{L}[\Xi] = \mathcal{L}^{\mathrm{kin}}[\Xi] + \mathcal{V}[\Xi] \in \mathbb{C}[[(\Xi_i, \partial_\mu \Xi_i)_{i=1,\ldots,N,\ \ \mu=0,\ldots,d}]]_{\mathbf{0},\mathbb{R},\mathrm{ent}}.$$

(As usual, we call a power series in even and odd variables, $P[y|\eta] = \sum P_{\alpha\beta} y^\alpha \eta^\beta \in \mathbb{C}[[y_1, \ldots, y_m|\eta_1, \ldots, \eta_n]]$, *entire* iff for all $R > 0$ there exists $C > 0$ such that $|P_{\alpha\beta}| \leq CR^{-|\alpha|}$ for all α, β.)

Here $\mathcal{L}^{\mathrm{kin}}[\underline{\Xi}]$ is called the *kinetic Lagrangian*, and $\mathcal{V}[\underline{\Xi}]$ is the *interaction term*; we now specify our requirements onto these terms.

For the interaction term, we require that it does not contain derivatives of first-order fields, and is at most linear in the derivatives of second-order fields:

$$\mathcal{V}[\underline{\Xi}] = \mathcal{V}'(\underline{\Xi}) + \mathcal{V}^{\mu,i}(\underline{\Xi})\partial_\mu \underline{\Phi}_i$$

where $\mathcal{V}^{\mu,i}(\underline{\Xi})$ and $\mathcal{V}'(\underline{\Xi})$ are real, even, entire (non-differential) power series of lower degree ≥ 2 and ≥ 3, respectively. We get

$$\frac{\delta}{\delta \underline{\Phi}_i} \mathcal{V}[\underline{\Xi}] = \frac{\partial}{\partial \underline{\Phi}_i} \mathcal{V}'(\underline{\Xi}) + \mathcal{V}^{\mu,ij}(\underline{\Xi})\partial_\mu \underline{\Phi}_j \tag{1.2.1}$$

with

$$\mathcal{V}^{\mu,ij}(\underline{\Xi}) := \frac{\partial}{\partial \underline{\Phi}_i} \mathcal{V}^{\mu,j}(\underline{\Xi}) - \frac{\partial}{\partial \underline{\Phi}_j} \mathcal{V}^{\mu,i}(\underline{\Xi}).$$

Turning to the kinetic Lagrangian, we assume the standard forms which lead to the Klein-Gordon operator as kinetic operator for second-order fields, and to some Dirac-like operator for first-order fields:

$$\mathcal{L}^{\mathrm{kin}}[\underline{\Xi}] = \frac{1}{2}\left(-\partial_\mu \underline{\Phi}_i \partial^\mu \underline{\Phi}_i - m^\Phi_{ij} \underline{\Phi}_i \underline{\Phi}_j + \mathrm{i}\partial_\mu \underline{\Psi}_k \Gamma^\mu_{kl} \underline{\Psi}_l + \mathrm{i}\underline{\Psi}_k m^\Psi_{kl} \underline{\Psi}_l\right). \tag{1.2.2}$$

Usually, the eigenvalues of the symmetric matrix $m^\Phi \in \mathbb{R}^{N^\Phi \times N^\Phi}$ are the squared masses of the degrees of freedom; however, in view of the Higgs field, there is a reason not to put any requirement of positive definiteness onto m^Φ. For the matrices $m^\Psi, \Gamma^\mu \in \mathbb{R}^{N^\Psi \times N^\Psi}$ we require that $(\Gamma^\mu)^{\mathrm{T}} = \Gamma^\mu$ (any antisymmetric part would produce only a total derivative) and $(m^\Phi)^{\mathrm{T}} = -m^\Phi$. Also we suppose that for the arising "massive Dirac operator" $\not{D}$ with

$$\not{D}_{ij} := \mathrm{i}\left(-\Gamma^\mu_{ij}\partial_\mu + m^\Psi_{ij}\right) \tag{1.2.3}$$

there exists another first-order operator $K = K(\partial_t, \partial_a)$ with constant coefficients such that

$$K_{ij} \not{D}_{jk} = (\Box + M)\delta_{ik} \tag{1.2.4}$$

where, as usual, $\Box$ is the Klein-Gordon operator, and M is some real-valued mass matrix (no positivity condition is necessary).

Given such a model, we define the *underlying bosonic model* as having field content $\Phi_1, \ldots, \Phi_{N^\Phi}$ and Lagrangian $\mathcal{L}^{\mathrm{bos}}[\underline{\Phi}] := \mathcal{L}[\underline{\Xi}]|_{\underline{\Psi}=0}$.

1.3. **Configuration families and solution families.** We take over the notations of [10], [11]: For $r \geq 0$, let

$$\mathbb{V}_r := \{(t, x) \in \mathbb{R}^{d+1} : \quad |x| \leq r + |t|\}, \tag{1.3.1}$$

and denote temporarily by $C^\infty_{\mathbb{V}_r}(\mathbb{R}^{d+1})$ the closed subspace of $C^\infty(\mathbb{R}^{d+1})$ which consists of all those elements which have support in $\mathbb{V}_r$. Set

$$\mathcal{E}_{\mathrm{c}} = \bigcup_{r>0} C^\infty_{\mathbb{V}_r}(\mathbb{R}^{d+1})$$

and equip it with the inductive limit topology. This is a strict inductive limes of Frèchet spaces, and hence complete. Also, $\mathcal{D}(\mathbb{R}^{d+1})$ is dense in $\mathcal{E}_{\mathrm{c}}$; hence the latter space is admissible in the sense of [8]. Moreover, one easily shows that the subspace $\mathcal{E}_{\mathrm{c}}$ of $C^\infty(\mathbb{R}^{d+1})$ is invariant under the standard action of the Poincaré group $\mathfrak{P}$, and that the arising action is continuous.

We will call $V := \mathbb{R}^{N^\Phi|N^\Psi}$ the *field target space*. Also, we set $\dot{V} := \mathbb{R}^{N^\Phi}$; this will be the target space for the velocities of the second-order fields at the Cauchy hyperplane. Thus, setting

$$\mathcal{E}^{\mathrm{V}}_{\mathrm{c}} := \mathcal{E}_{\mathrm{c}} \otimes V, \qquad \mathcal{E}^{\mathrm{Cau,V}}_{\mathrm{c}} := \mathcal{D}(\mathbb{R}^d) \otimes (V \oplus \dot{V}),$$

the *smf's of Cauchy data and of configurations*, (1.1.4), (1.1.3) are now well-defined. Analogously, we set

$$\mathcal{E}^{\mathrm{Cau,V}} := C^\infty(\mathbb{R}^d) \otimes (V \oplus \dot{V}), \quad \mathcal{E}^{\mathrm{V}} := C^\infty(\mathbb{R}^{d+1}) \otimes V.$$

Let Z be an arbitrary smf. A *configuration family parametrized by* Z (or Z*-family*, for short) is an even, real superfunction Ξ' on Z with values in the locally convex space $\mathcal{E}^{\mathrm{V}}$:

$$\Xi' = (\Phi'|\Psi') \in \mathcal{M}^{\mathcal{E}^{\mathrm{V}}}(Z).$$

Thus, Ξ' encodes a N-tuple $\Xi' = (\Xi'_1, \dots, \Xi'_N)$ of $C^\infty(\mathbb{R}^{d+1})$-valued superfunctions on Z.

If Ξ' lies in the subspace $\mathcal{M}^{\mathcal{E}^{\mathrm{V}}_{\mathrm{c}}}(Z)$ we call Ξ' a Z*-family of quality* $\mathcal{E}_{\mathrm{c}}$; these are of main interest for us.

(In [10], [11] we have considered more general families which have values in Sobolev spaces and are defined over time intervals; however, for our purposes here, they are useless.)

Now, given an smf morphism $\pi : Z' \to Z$ we can assign as in [10] to every Z-family Ξ' its *pullback* $\Xi" := \pi^*(\Xi')$ which is a Z'-family. In fact, the process of passing from Ξ' to $\Xi"$ means in family language nothing but a *change of parametrization* (cf. [9, 1.11]).

One family of quality $\mathcal{E}_{\mathrm{c}}$ is given a priori, namely the M-family

$$\Xi = (\Phi|\Psi) \in \mathcal{M}^{\mathcal{E}^{\mathrm{V}}_{\mathrm{c}}}(M)$$

where, we recall, $M = \mathrm{L}(\mathcal{E}^{\mathrm{V}}_{\mathrm{c}})$ is the smf of configurations of quality $\mathcal{E}_{\mathrm{c}}$, and Ξ is the standard coordinate (cf. [9, 2.5, 2.6]).

Ξ is in fact the *universal family of quality* $\mathcal{E}_{\mathrm{c}}$: Given an arbitrary Z-family Ξ' of quality $\mathcal{E}_{\mathrm{c}}$, it defines by [9, 2.8.1] a *classifying morphism*

$$\Xi' : Z \to M, \quad \widehat{\Xi'} = \Xi'$$

and Ξ' arises from Ξ just by pullback: $\Xi' = \Xi'^*(\Xi)$.

Remark. In the language of category theory, this means that the cofunctor

$$\{\text{supermanifolds}\} \to \{\text{sets}\}, \qquad Z \mapsto \mathcal{M}^{\mathcal{E}^{\mathrm{V}}_{\mathrm{c}}}(Z),$$

is represented by the object M with the universal element Ξ.

Fixing a Z-family Ξ' of quality $\mathcal{E}_{\mathrm{c}}$, the field strengthes $\Xi'_i(t,x) = \delta_{(t,x)} \circ \Xi'_i$ for $(t,x) \in \mathbb{R}^{d+1}$ are scalar superfunctions on Z. More generally, we define the *value at* Ξ' of any superfunctional $K \in \mathcal{O}^F(M)$ as the pullback of K along Ξ':

$$K[\Xi'] := \Xi'^*(K) \in \mathcal{O}^F(Z).$$

For instance, in case Z is a point, the value $K[\Xi']$ of an F-valued superfunctional K at a Z-family Ξ' is an element of $F_{\mathbb{C}}$; thus, for a scalar functional $K \in \mathcal{O}(M)$, it is simply a complex number (which, however, is zero for all odd K, and, in particular, for the fermionic field strengthes).

1.4. **Solution families and action principle.** A Z*-family of solutions*, or *solution family* for short, is a Z-family $\Xi' \in \mathcal{M}^{\mathcal{E}^{\mathrm{V}}}(Z)$ which satisfies

$$(\delta/\delta\Xi_i)\mathcal{L}[\Xi'] = 0 \tag{1.4.1}$$

for all i (the l. .h. s. is obviously well-defined in $\mathcal{O}^{\mathcal{E}^{\mathrm{V}}}(Z)$). Trivially, every pullback of a solution family is a solution family.

Of course, the universal family Ξ is not a family of solutions. However, we will show in Thm. 1.8.3 that in the case of a complete model, there exists a family of solutions Ξ^{sol} of quality $\mathcal{E}_{\mathrm{c}}$ which is universal for this quality, i.e. every other solution family of quality $\mathcal{E}_{\mathrm{c}}$ will be a pullback of Ξ^{sol}.

If $Z = P$ is a point then a Z-family of solutions is just an element $\phi \in \mathcal{E}^{\mathrm{V}}_{\mathbf{0}} = C^\infty(\mathbb{R}^{d+1}) \otimes V_{\mathbf{0}}$ which solves the field equations of the underlying bosonic model in the usual sense.

The *Cauchy data* of a family $\Xi' \in \mathcal{M}^{\mathcal{E}^{\mathrm{V}}}(Z)$ is the element

$$(\Xi'(0), \partial_0\Phi'(0)) \in \mathcal{M}^{\mathcal{E}^{\mathrm{Cau,V}}}(Z).$$

Up to now, solution families are characterized rather formally, as making the variational derivatives of the differential polynomial $\mathcal{L}$ vanish. We now look for the action principle.

Inserting the coordinate superfunctions $\Xi_i \in \mathcal{O}^{\mathcal{E}_c}(M)$ into the Lagrangian, we get the *action density* $\mathcal{L}[\Xi] \in \mathcal{M}^{\mathcal{E}_c}(M)$ as an $\mathcal{E}_c$-valued superfunction on the configuration smf M; however, its space-time integral,

$$S := \int dx \mathcal{L}[\Xi](x), \tag{1.4.2}$$

is ill-defined. What we can do is to define the action over any closed subset $\Omega \subseteq \mathbb{R}^d$ which has the property that $\Omega \cap \mathbb{V}_r$ is compact for all $r > 0$ (cf. (1.3.1)):

$$S_\Omega[\Xi] := \int_\Omega dx \mathcal{L}[\Xi](x) \in \mathcal{M}(M).$$

In particular, the action within a time interval $[t_0, t_1]$ is well-defined:

$$S_{t_0,t_1}[\Xi] = \int_{t_0}^{t_1} dt \int_{\mathbb{R}^d} dx \mathcal{L}[\Xi](t,x) \in \mathcal{M}(M).$$

Note that on the smf of smooth configurations $M_{C^\infty} = \mathrm{L}(\mathcal{E}^{\mathrm{V}})$ (cf. 1.9 below), the action $S_\Omega[\Xi] \in \mathcal{M}(M_{C^\infty})$ is defined only if Ω is compact; in particular, $S_{t_0,t_1}[\Xi]$ is not well-defined as superfunction on M_{C^∞}.

For a superfunction with values in continuous functions on $\mathbb{R}^{d+1}$, i.e. $K \in \mathcal{O}^{C(\mathbb{R}^{d+1})}(Z)$, let, as in [10], the *target support of* K be defined as

$$\text{t-supp}\, K := \text{Closure}\Big(\{x \in \mathbb{R}^{d+1} : \quad K(x) \neq 0\}\Big),$$

where, of course, $K(x) = \delta_x \circ K$. This should not be confused with the support of a power series as defined in [8, 3.11].

Now Ξ' is a solution family iff $S_\Omega[\Xi']$ remains stationary with respect to infinitesimal increments of Ξ' which have their target support in the interior Ω^0:

Proposition 1.4.1. *For a Z-family of configurations $\Xi' \in \mathcal{M}^{\mathcal{E}^{\mathrm{V}}}(Z)$ the following conditions are equivalent:*

(i) Ξ' is a Z-family of solutions;

(ii) We have $(\partial_\xi S_\Omega)[\Xi'] = 0$ for all Ω as above and $\xi \in \mathcal{D}(\mathbb{R}^{d+1}) \otimes V$ with $\mathrm{supp}\xi \subseteq \Omega^0$. *Here ∂_ξ is the directional derivative, cf.* [8, 3.9, 3.12]*;*

(iii) We have $\chi(S_\Omega)[\Xi'] = 0$ for all Ω as above and $\chi \in \mathcal{X}_{\mathcal{E}_c^{\mathrm{V}}}((\mathcal{E}_c^{\mathrm{V}})_{\mathbf{0}})$ with $\text{t-supp}\,\chi(\Xi_i) \subseteq \Omega^0$ *for all i. Here $\mathcal{X}_{\mathcal{E}_c^{\mathrm{V}}}((\mathcal{E}_c^{\mathrm{V}})_{\mathbf{0}})$ denotes the space of all infinitesimal transformations over $(\mathcal{E}_c^{\mathrm{V}})_{\mathbf{0}}$, cf.* [8, 3.12].

Proof. We first need a formula for $\chi(S_\Omega)[\Xi']$ where χ is as in (iii): Mimicking the proof of [8, Thm. 2.8], we have

$$\chi(S_\Omega)[\Xi'] = \sum_i \int_\Omega dx \ \left(\chi(\Xi_i)[\Xi'](x)\,\frac{\delta}{\delta \Xi_i}\mathcal{L}[\Xi'](x) + \partial_\mu\left(\chi(\Xi_i)[\Xi'](x)\,\frac{\partial}{\partial(\partial_\mu \Xi_i)}\mathcal{L}[\Xi'](x)\right)\right). \tag{1.4.3}$$

Now if Ξ' is a solution family then the first term of the sum on the r. h. s. vanishes by (1.4.1) while the second one is a total differential with target support in the interior of Ω; hence its integral vanishes, too, proving (i)$\Rightarrow$(iii).

(iii)$\Rightarrow$(ii) is obvious.

To show (ii)$\Rightarrow$(i), we specialize (1.4.3) to $\chi = \partial_\xi$, getting $0 = \sum_i \int_\Omega dx\ \xi_i(x) \frac{\delta}{\delta \Xi_i}\mathcal{L}[\Xi'](x)$ for all Ω, ξ as in (ii); this implies (1.4.1). $\square$

Remark. One can give (1.4.2) sense as scalar superfunction $S \in \mathcal{M}(M_{C_0^\infty})$ on the smaller configuration smf $M_{C_0^\infty} := \mathrm{L}(C_0^\infty(\mathbb{R}^{d+1}) \otimes V)$ (which, of course, is too small to contain non-trivial solutions of the field equations). Taking its exterior differential, alias total variation (cf. [8, 3.8]),

$$\delta S = \delta S[\Xi, \delta\Xi] = \sum_{i=1}^N \int_{\mathbb{R}^{d+1}} dx\ \delta\Xi_i(x)\,\frac{\delta}{\delta \Xi_i(x)} S,$$

and using the notations of [7, 5.5, 6.6], we have

$$\delta S \in \Omega^1(M_{C_0^\infty}) \;=\; {}^{C_0^\infty(\mathbb{R}^{d+1})\otimes V}\mathcal{M}^{\mathbb{R}}(M_{C_0^\infty}).$$

It is easy to see that, with respect to the embedding of smf's $M_{C_0^\infty} \xrightarrow{\subseteq} M \xrightarrow{\subseteq} M_{C^\infty}$, this lifts to unique elements

$$\delta S \in {}^{C_0^\infty(\mathbb{R}^{d+1})\otimes V}\mathcal{M}^{\mathbb{R}}(M), \quad \delta S \in {}^{C_0^\infty(\mathbb{R}^{d+1})\otimes V}\mathcal{M}^{\mathbb{R}}(M_{C^\infty}).$$

Note, however, that δS does not lie in the subspace ${}^{\mathcal{E}_c}\mathcal{M}^{\mathbb{R}}(M) = \Omega^1(M)$, i. e., it is not a one form on M (and the less on M_{C^∞}), and thus there is no contradiction to the Poincaré Lemma given in [7, 6.9].

Now a Z-family of configurations $\Xi' \in \mathcal{M}^{\mathcal{E}^{\mathrm{V}}}(Z)$ is a Z-family of solutions iff the pullback $\Xi'^*(\delta S) = \delta S[\Xi', \delta\Xi] \in {}^{C_0^\infty(\mathbb{R}^{d+1})\otimes V}\mathcal{M}^{\mathbb{R}}(Z)$ vanishes.

1.5. **Reduction to first-order equations.** The relevant variational derivatives of the Lagrangian take the form

$$\frac{\delta}{\delta\underline{\Phi}_i}\mathcal{L}[\Xi] = \Box\Phi_i - m^{\Phi}_{ij}\Phi_j + \frac{\partial}{\partial\underline{\Phi}_i}\mathcal{V}'(\Xi) + \mathcal{V}^{\mu,ij}(\Xi)\partial_\mu\Phi_j, \qquad (i = 1,\dots,N^{\Phi})$$

$$\frac{\delta}{\delta\underline{\Psi}_i}\mathcal{L}[\Xi] = \not{D}_{ij}\Psi_j + \frac{\partial}{\partial\overline{\Psi}_i}\mathcal{V}[\Xi] \qquad (i = 1,\dots,N^{\Psi})$$

as identities in $\mathcal{O}^{\mathcal{E}_c}(M)$. In a standard way, we write the field equations as first-order evolution system:

$$\partial_t\Phi_i = \dot{\Phi}_i, \tag{1.5.1a}$$

$$\partial_t\dot{\Phi}_i = \Delta\Phi_i - m^{\Phi}_{ij}\Phi_j + \frac{\partial}{\partial\underline{\Phi}_i}\mathcal{V}'(\Xi) + \mathcal{V}^{\mu,ij}(\Xi)\partial_\mu\Phi_j, \tag{1.5.1b}$$

$$\partial_t\Psi_i = -(\Gamma^0)^{-1}{}_{ij}\Big(\Gamma^a_{jk}\partial_a\Psi_k + m^{\Psi}_{jk}\Psi_k + \mathrm{i}\frac{\partial}{\partial\underline{\Psi}_j}\mathcal{V}[\Xi]\Big) \tag{1.5.1c}$$

(for legibility, we have suppressed the apostrophes which indicate that these are conditions for a configuration family). Thus, Ξ' is a solution family of our model iff $(\Xi', \partial_t\Phi')$ is a solution family of the system (1.5.1a)–(1.5.1c).

We get a system of field equations for the first-order field components

$$(\Xi^{\mathrm{fo}}_1,\dots,\Xi^{\mathrm{fo}}_N) = (\Phi_i, \dot{\Phi}_i|\Psi_j), \quad N^{\mathrm{fo}} := 2N^{\Phi} + N^{\Psi}.$$

Lemma 1.5.1. *The system* (1.5.1a)–(1.5.1c) *belongs to the class of systems of field equations considered in* [10], [11]*, and is causal.*

Proof. First we have to look for the spatially Fourier-transformed influence function $\hat{A} = (\hat{A}_{ij}) = \hat{A}(t,p)$, i.e. the solution of the linearized and spatially Fourier-transformed equations (1.5.1a)–(1.5.1c) with the initial data

$$\hat{A}(0,p) = (2\pi)^{-d/2}1_{N\times N}.$$

For notational convenience, we work with the vector $(\Phi_i, \dot{\Phi}_i|\Psi_j)$ of real and complex field components. Thus, $\hat{A} = \mathrm{diag}(\hat{A}^{\Phi},\ \hat{A}^{\Psi})$ where

$$\hat{A}^{\Phi}(t,p) = \begin{pmatrix} \partial_t\hat{\mathcal{A}}^{\Phi} & \hat{\mathcal{A}}^{\Phi} \\ \partial_t{}^2\hat{\mathcal{A}}^{\Phi} & \partial_t\hat{\mathcal{A}}^{\Phi} \end{pmatrix}, \tag{1.5.2}$$

and $\mathcal{A}^{\Phi}$ is the solution of

$$(\partial_t^2 + p^2 + m^{\Phi})\hat{\mathcal{A}}^{\Phi}(t,p) = 0, \quad \hat{\mathcal{A}}^{\Phi}(0,p) = (2\pi)^{-d/2}1_{N^{\Phi}\times N^{\Phi}}.$$

Explicitly,

$$\hat{\mathcal{A}}^{\Phi}(t,p) = (2\pi)^{-d/2}f_t(p^2 + m^{\Phi}), \qquad f_t(z) := \frac{\sin t\sqrt{z}}{\sqrt{z}}.$$

Thus, if $m^{\Phi} = \mathrm{diag}(\mu_1,\dots,\mu_{N^{\Phi}})$ then the inverse spatial Fourier transform of $\hat{\mathcal{A}}^{\Phi}$ becomes $\mathcal{A}^{\Phi}(t,x) = \mathrm{diag}(D_{\mu_1}(t,x),\dots,D_{\mu_{N^{\Phi}}}(t,x))$ where $D_{\mu_i}(t,x)$ is the *Pauli-Jordan exchange function* with mass μ_i.

In the first-order sector, if writing $\not{D} = \not{D}(\partial_t, \partial_x)$ then $\hat{A}^\Psi$ is determined by

$$\not{D}(\partial_t, \mathrm{i}p)\hat{A}^\Psi(t,p) = 0, \quad \hat{A}^\Psi(0,p) = (2\pi)^{-d/2} 1_{N^\Psi \times N^\Psi}.$$

We claim that the solution is explicitly given by

$$\hat{A}^\Psi(t,p) := (2\pi)^{-d/2} K(\partial_t, \mathrm{i}p_a) \cos t\sqrt{p^2 + m^\Psi} \cdot \not{D}(0, \mathrm{i}p_a)/(p^2 + M) \tag{1.5.3}$$

where $K(\partial_t, \partial_a)$ is the operator from (1.2.4). Indeed,

$$\begin{aligned}\hat{A}^\Psi(0,p) &= (2\pi)^{-d/2} K(\partial_t, \mathrm{i}p_a)(1_{N^\Psi \times N^\Psi})\,\not{D}(0, \mathrm{i}p_a)/(p^2 + M) \\ &= (2\pi)^{-d/2} K(0, \mathrm{i}p_a)\,\not{D}(0, \mathrm{i}p_a)/(p^2 + M) 1_{N^\Psi \times N^\Psi} = (2\pi)^{-d/2} 1_{N^\Psi \times N^\Psi}.\end{aligned}$$

Having constructed $\hat{A}$, one now verifies the estimates

$$\left\| \hat{A}^\Phi(t,p) \right\| \le \frac{K}{1 + |p|}, \tag{1.5.4a}$$

$$\left\| \frac{d}{dt}\hat{A}^\Phi(t,p) \right\| \le K, \tag{1.5.4b}$$

$$\left\| \frac{d^2}{dt^2}\hat{A}^\Phi(t,p) \right\| \le K(1 + |p|), \tag{1.5.4c}$$

$$\left\| \hat{A}^\Psi(t,p) \right\| \le K \tag{1.5.4d}$$

valid for (say) $t \in (-1,1)$ and suitable $K > 0$. This allows to take the vector of smoothness offsets τ which assigns 1 to the second-order field components Φ_i, and 0 to their velocities $\dot{\Phi}_i$ as well as to the first-order fields Ψ_j.

Also, using the Paley-Wiener Theorem, we have for the inverse Fourier transform A of $\hat{A}$

$$\operatorname{supp} A \subseteq \{(t,x) \in \mathbb{R}^{d+1} : \quad |x| \le |t|\}.$$

so that our system is causal. □

Remark. In view of the expectation that particles with imaginary rest mass should move tachyonically, it is somewhat astonishing that the scalar fields Φ_i have causal propagation even if m^Φ has negative eigenvalues. Of course, the corresponding components (after diagonalization) cannot describe a physically senseful free field; attempting to canonically quantize it would lead to a state space with indefinite metric. (Complex eigenvalues cannot occur due to symmetry of m^Φ.)

1.6. **The formal solution.** In the following, we look at the solution of the Cauchy problem for the field equations on the formal power series level. Essentially, one only needs to repeat the approach of [11], [10], with adapting it to our context which contains first as well as second-order fields. The result is:

Corollary 1.6.1. *There exists a uniquely determined formal power series (in the sense of* [8, 2.3]*),*

$$\Xi^{\mathrm{sol}} = \Xi^{\mathrm{sol}}[\Xi^{\mathrm{Cau}}, \dot{\Phi}^{\mathrm{Cau}}] \in \mathcal{P}_f(\mathcal{E}_{\mathrm{c}}^{\mathrm{Cau,V}}; \mathcal{E}_{\mathrm{c}}^{\mathrm{V}})_{\mathbf{0},\mathbb{R}}$$

which solves the Cauchy problem (1.1.6), (1.1.7)*. Letting* $\Xi^{\mathrm{sol}} = \sum_{m>0} \Xi^{\mathrm{sol}}_{(m)}$ *denote its splitting into homogeneous components, the first one,* $\Xi^{\mathrm{sol}}_{(1)}$*, is the formal solution of the corresponding free model:*

$$(\Phi^{\mathrm{sol}}_{(1)})_i[\Xi^{\mathrm{Cau}}, \dot{\Phi}^{\mathrm{Cau}}](t,y) = \int dx\, \left(\partial_t \mathcal{A}^\Phi_{ij}(t, y-x)\Phi^{\mathrm{Cau}}_j(x) + \mathcal{A}^\Phi_{ij}(t, y-x)\dot{\Phi}^{\mathrm{Cau}}_j(x)\right),$$

$$(\Psi^{\mathrm{sol}}_{(1)})_i[\Xi^{\mathrm{Cau}}, \dot{\Phi}^{\mathrm{Cau}}](t,y) = \int dx A^\Psi_{ij}(t, y-x)\Psi^{\mathrm{Cau}}_j(x),$$

while the higher components are recursively given by

$$(\Xi^{\mathrm{sol}}_{(m)})_i(t,y) = \int_{\mathbb{R}\times\mathbb{R}^d} ds dx G_{ij}(t,s,y-x)\frac{\delta}{\delta \Xi_j}\mathcal{V}[\Xi^{\mathrm{sol}}_{(<m)}]_{(m)}(s,x)$$

for $m \ge 2$ *where, of course,* $\Xi^{\mathrm{sol}}_{(<m)} = \sum_{n=1}^{m-1} \Xi^{\mathrm{sol}}_{(n)}$*, and*

$$G(t,s,x) := \big(\theta(t-s) - \theta(s)\big)\, \mathrm{diag}\Big(\mathcal{A}^\Phi(t-s,x),\ A^\Psi(t-s,x)\cdot(\Gamma^0)^{-1}\Big)$$

where $\theta(s)$ *is* $1,\ 1/2,\ 0$ *for* $t>0,\ \ t=0,\ \ t<0$, *respectively.* □

Remark. The Green function matrix $G := (G_{ij}) = \mathrm{diag}(G^{\Phi},\ G^{\Psi})$ is characterized by

$$\mathrm{diag}(\Box,\ \not{D})_{t,x}G(t,s,x) = -\delta(t-s)\delta(x)\cdot 1_{N^{\Phi}+N^{\Psi}}, \qquad G(0,s,x)=0, \quad \partial_t G^{\Phi}(0,s,x)=0.$$

The formal power series Ξ^{sol}, which we call the *formal solution of the classical model*, will be the power series expansion at the zero configuration of the solution of the "analytic Cauchy problem" (1.1.5), (1.1.6), (1.1.7).

1.7. **Cauchy uniqueness and causality.** The first thing to observe is that solution families are uniquely determined by their Cauchy data. Before giving a "localized" variant of this fact which also expresses the causality of the propagation of perturbations, we need some notations:

For $(s,x),(t,y)\in\mathbb{R}^{d+1}$ we will write $(s,x)\prec(t,y)$ iff (t,y) lies in the forward light cone of (s,x), i. e. $(t-s)^2\geq(y-x)^2$.

Given a point $p=(s,x)\in\mathbb{R}^{d+1}$ with $s\neq 0$, write

$$\Omega(p) := \begin{cases}\{(s',x')\in\mathbb{R}^{d+1}:\ (s',x')\prec(s,x),\ 0<s'\} & \text{if } s>0,\\ \{(s',x')\in\mathbb{R}^{d+1}:\ (s,x)\prec(s',x'),\ s'<0\} & \text{if } s<0,\end{cases}$$

$$\mathcal{J}(p) := \{x'\in\mathbb{R}^d:\ |x'-x|<|s|\}.$$

By the results of [10], [11], causality implies that perturbations of solution families propagate within the light cone, as to be expected:

Theorem 1.7.1. *Let be given a point* $p=(s,x)\in\mathbb{R}^{d+1}$ *with* $s\neq 0$, *and let be given two* Z*-families* $\Xi',\Xi"\in\mathcal{M}^{\mathcal{E}^{\mathrm{V}}}(Z)$. *Suppose that*

$$\frac{\delta}{\delta\underline{\Xi}_i}\mathcal{L}[\Xi']|_{\Omega(p)} = \frac{\delta}{\delta\underline{\Xi}_i}\mathcal{L}[\Xi"]|_{\Omega(p)} = 0 \quad (i=1,\ldots,N),$$
$$((\Xi'-\Xi")(0),\partial_0(\Psi'-\Psi")(0))\,|_{\mathcal{J}(p)} = 0.$$

Then $(\Xi'-\Xi")|_{\Omega(p)}=0$. □

1.8. **Completeness, universal solution family, and the smf of classical solutions.** Loosely said, we call the model complete iff the underlying bosonic model is globally solvable:

Theorem 1.8.1. *The following conditions are equivalent:*

(i) For every smooth solution $\phi\in C^{\infty}((a,b)\times\mathbb{R}^d)\otimes V_{\mathbf{0}}$ *of the underlying bosonic field equations*

$$\frac{\delta}{\delta\underline{\Phi}_i(x)}\mathcal{L}^{\mathrm{bos}}[\phi]=0 \tag{1.8.1}$$

on a bounded open time interval (a,b) *such that* $\mathrm{supp}\phi(t)$ *is compact for some (and hence all) all* $t\in(a,b)$, *there exists a Sobolev index* $k>d/2$ *such that*

$$\sup_{t\in(a,b)}\ \max\left\{||\phi_i(t)||_{H_{k+1}(\mathbb{R}^d)},\ ||\partial_t\phi_i(t)||_{H_k(\mathbb{R}^d)}\right\}<\infty \tag{1.8.2}$$

for all $i=1,\ldots,N^{\Phi}$.

(ii) The underlying bosonic equations are all-time solvable with quality $\mathcal{E}_{\mathrm{c}}$*:*

Given bosonic Cauchy data $(\phi^{\mathrm{Cau}},\dot{\phi}^{\mathrm{Cau}})\in(\mathcal{E}_{\mathrm{c}}^{\mathrm{Cau,V}})_{\mathbf{0}}$ *there exists an element* $\phi\in(\mathcal{E}_{\mathrm{c}}^{\mathrm{V}})_{\mathbf{0}}$ *with these Cauchy data which solves* (1.8.1).

If these conditions are satisfied we call the model complete.

Remark 1.8.2. In the absence of derivative couplings, $\mathcal{V}^{\mu,i}=0$, it is sufficient to have (1.8.2) for some $k>d/2-1$.

Proof. In view of Lemma 1.5.1, the Theorem immediately follows from the corresponding results of [10], [11]. □

For more comment, cf. [10].

The central result of this section is:

Theorem 1.8.3. *Suppose that the model is complete.*

(i) The formal solution $\Xi^{\rm sol}$ *is the Taylor expansion at zero of a unique superfunctional*

$$\Xi^{\rm sol}[\Xi^{\rm Cau}, \dot\Phi^{\rm Cau}] \in \mathcal{M}^{\mathcal{E}_c^{\rm V}}(M^{\rm Cau}), \tag{1.8.3}$$

and (1.8.3) *is an* $M^{\rm Cau}$*-family of solutions of quality* $\mathcal{E}_c$.

(ii) The image of the arising smf morphism (cf. [9, 2.8,2.12]*)* $\Xi^{\rm sol} : M^{\rm Cau} \to M$ *is a split sub-smf which we call the* smf of classical solutions*, or, more exactly, the* smf of smooth classical solutions with spatially compact support*, and denote by* $M^{\rm sol} \subseteq M$.

(iii) $M^{\rm sol}$ *has the following universal property: Recall that fixing an smf* Z *we have a bijection between* Z*-families* Ξ' *of configurations of quality* $\mathcal{E}_c$*, and morphisms* $\Xi' : Z \to M$*. Now* Ξ' *is a solution family iff* Ξ' *factors to* $\Xi' : Z \to M^{\rm sol} \subseteq M$.

In this way, we get a bijection between Z*-families* Ξ' *of solutions of quality* $\mathcal{E}_c$ *and morphisms* $\Xi' : Z \to M^{\rm sol}$.

Proof. In view of Lemma 1.5.1, this follows from the corresponding results of [10], [11]. □

Remarks. (1) In the language of category theory, assertion (v) means that the cofunctor

$$\{\text{supermanifolds}\} \to \{\text{sets}\}, \qquad Z \mapsto \{\text{supermanifolds}\}$$

is represented by the object M with the universal element $\Xi^{\rm sol}$. Thus, (1.8.3) is the *universal family of solutions of quality* $\mathcal{E}_c$: Every other family of solutions of this quality arises uniquely as pullback from (1.8.3).

(2) Taking in (iii) P-families where P is a single point we get that the underlying manifold $\widetilde{M^{\rm sol}}$ is just the set of all bosonic configurations $\phi \in (\mathcal{E}_c^{\rm V})_{\bar 0}$ which satisfy the underlying bosonic field equations (1.8.1). The underlying map of $\Xi^{\rm sol} : M^{\rm Cau} \to M$ assigns to each bosonic Cauchy datum $(\phi^{\rm Cau}, \dot\phi^{\rm Cau})$ the unique solution ϕ of (1.8.1) with $\phi(0) = \phi^{\rm Cau}$, $\partial_t \phi(0) = \dot\phi^{\rm Cau}$.

(3) Note that $M^{\rm sol}$ is still a linear smf which is, however, in a non-linear way embedded into M. In a forthcoming paper we will show that, once the action of the Lorentz group on V has been fixed, the sub-smf $M^{\rm sol}$ is invariant under the arising action of the Poincaré group on M; the other data α, $\Xi^{\rm free}$, $\Xi^{\rm sol}$ are not (they are only invariant under the Euclidian group of $\mathbb{R}^d$).

(4) The superfunction (1.8.3) is uniquely characterized by the conditions (1.1.6), (1.1.7). The latter property can be recoded supergeometrically to the fact that the composite morphism

$$M^{\rm Cau} \xrightarrow{\Xi^{\rm sol}} M \xrightarrow{\pi} M^{\rm Cau}$$

is the identity; here π is the projection onto the Cauchy data: $\hat\pi[\Xi] = (\Xi(0), \partial_0\Phi(0))$.

(5) In qft slang, the homomorphism

$$\mathcal{O}^F(M) \to \mathcal{O}^F(M^{\rm sol}), \qquad K[\Xi] \mapsto K[\Xi^{\rm sol}] \tag{1.8.4}$$

is called *restriction of classical observables onto the mass shell* (the latter term comes from free field theory). It follows from ass. (iii) that (1.8.4) is surjective.

(6) Let us comment on the fact that completeness depends only on the underlying bosonic model: Mathematically, this is an analogon of several theorems in supergeometry that differential-geometric tasks, like trivializing a fibre bundle, or presenting a closed form as differential, are solvable iff the underlying smooth tasks are solvable.

Physically, our interpretation is somewhat speculative: In the bosonic sector, the classical field theory approximates the behaviour of coherent states, and completeness excludes that "too many" particles may eventually assemble at a space-time point, making the state non-normable. On the fermionic side, apart from the non-existence of genuine coherent states, it is the Pauli principle which automatically prevents such an assembly.

1.9. **The smf of classical solutions without support restriction.** It takes not much additional effort to lift the constraints on the supports of solution families, considering arbitrary smooth solution families. The appropriate smf's of Cauchy data and configurations are

$$M^{\rm Cau}_{C^\infty} := {\rm L}(\mathcal{E}^{\rm Cau,V}), \quad M_{C^\infty} := {\rm L}(\mathcal{E}^{\rm V}).$$

In view of Lemma 1.5.1, the results of [10], [11] yield:

Theorem 1.9.1. *Suppose the model is complete. Then* Ξ^{sol} *extends uniquely to a superfunctional*

$$\Xi^{\mathrm{sol}} \in \mathcal{M}^{\mathcal{E}^{\mathrm{V}}}(M_{C^\infty}^{\mathrm{Cau,V}}).$$

Moreover, the image of the arising smf morphism

$$\Xi^{\mathrm{sol}} : M_{C^\infty}^{\mathrm{Cau}} = \mathrm{L}(\mathcal{E}^{\mathrm{Cau,V}}) \to \mathrm{L}(\mathcal{E}^{\mathrm{V}}) = M_{C^\infty}$$

is a split sub-smf which we call the smf of smooth classical solutions (without support restriction), *and denote by* $M_{C^\infty}^{\mathrm{sol}} \subseteq M_{C^\infty}$. □

Also, the remaining properties of $M_{C^\infty}^{\mathrm{sol}}$ are completely analogous to that of M^{sol}. We get a commutative diagram of smf's

$$\begin{array}{ccccc} M^{\mathrm{Cau}} & \xrightarrow{\Xi^{\mathrm{sol}}} & M^{\mathrm{sol}} & \xrightarrow{\subseteq} & M \\ \subseteq\downarrow & & \subseteq\downarrow & & \subseteq\downarrow \\ M_{C^\infty}^{\mathrm{Cau}} & \xrightarrow{\Xi^{\mathrm{sol}}} & M_{C^\infty}^{\mathrm{sol}} & \xrightarrow{\subseteq} & M_{C^\infty}. \end{array}$$

1.10. **Local excitations.** A further variant arises by considering compactly supported excitations of a classical solution; in particular, it is applicable for situations with spontaneous symmetry breaking, like the Higgs mechanism. In view of Lemma 1.5.1, the results of [10] yield:

Theorem 1.10.1. *Suppose that the model is complete, and fix a solution* $\phi \in \mathcal{E}^{\mathrm{V}}{}_{\mathbf{0}}$ *of the underlying bosonic field equations* (1.8.1)*; let* $(\phi^{\mathrm{Cau}}, \dot\phi^{\mathrm{Cau}})$ *be its Cauchy data.*

(i) The superfunctional

$$\Xi_\phi^{\mathrm{exc}}[\Xi^{\mathrm{Cau}}, \dot\Phi^{\mathrm{Cau}}] := \Xi^{\mathrm{sol}}[\Xi^{\mathrm{Cau}} + \phi^{\mathrm{Cau}}, \dot\Phi^{\mathrm{Cau}} + \dot\phi^{\mathrm{Cau}}] - \phi, \tag{1.10.1}$$

which lies a priori in $\mathcal{M}^{\mathcal{E}^{\mathrm{V}}}(M_{C^\infty}^{\mathrm{Cau}})$*, restricts to a superfunctional*

$$\Xi_\phi^{\mathrm{exc}}[\Xi^{\mathrm{Cau}}, \dot\Phi^{\mathrm{Cau}}] \in \mathcal{M}^{\mathcal{E}_c^{\mathrm{V}}}(M^{\mathrm{Cau}}).$$

(ii) The image of the arising smf morphism $\Xi_\phi^{\mathrm{exc}} : M^{\mathrm{Cau}} \to M$ *is a split sub-smf which we call the* smf of excitations around ϕ, *and denote by* $M_\phi^{\mathrm{exc}} \subseteq M$.

(iii) M_ϕ^{exc} *has the following universal property: Given a* Z*-family* $\Xi' \in \mathcal{M}^{\mathcal{E}_c^{\mathrm{V}}}(Z)$*, the corresponding morphism* $\Xi' : Z \to M$ *factors through* M_ϕ^{exc} *iff the* Z*-family* $\Xi' + \phi \in \mathcal{M}^{\mathcal{E}^{\mathrm{V}}}(Z)$ *is a solution family.* □

Remark. This theorem yields new information only if the $(\phi^{\mathrm{Cau}}, \dot\phi^{\mathrm{Cau}})$ are not compactly carried. If they are, i. e. $(\phi^{\mathrm{Cau}}, \dot\phi^{\mathrm{Cau}}) \in (\mathcal{E}_c^{\mathrm{Cau,V}})_{\mathbf{0}}$, then (1.10.1) is already a priori defined as element of $\mathcal{M}^{\mathcal{E}_c^{\mathrm{V}}}(M^{\mathrm{Cau}})$, and M_ϕ^{exc} can be identified with M^{sol}.

1.11. **Simple sufficient completeness criteria.** We call a measurable function $W : \mathbb{R}^m \to \mathbb{R}$ *of polynomial growth* if it obeys an estimate

$$|W(y)| \le C(1 + \|y\|^N) \tag{1.11.1}$$

with some $C, N > 0$.

Proposition 1.11.1. *Suppose that for every solution* $\phi \in C^\infty((a,b) \times \mathbb{R}^d) \otimes V_{\mathbf{0}}$ *of the underlying bosonic field equations* (1.8.1) *for which* $\mathrm{supp}\phi(t)$ *is compact for all* $t \in (a,b)$*, we have*

$$\sup_{t\in(a,b)} \left(\|\partial_t \phi_i(t)\|_{L_2(\mathbb{R}^d)} + \sum_{a=1}^{d} \|\partial_a \phi_i(t)\|_{L_2(\mathbb{R}^d)} \right) < \infty \tag{1.11.2}$$

for $i = 1, \dots, N^\Phi$*. Also, suppose that either*

(i) $d \le 1$*, or*

(ii) $d = 2$*, the interaction of the underlying bosonic model has the form* $\mathcal{V}[\underline{\Phi}|0] = V(\underline{\Phi})$ *with an entire power series* $V(\underline{\Phi})$ *(thus, derivative couplings are not allowed), and the functions* $\partial_{\underline{\Phi}_i} V(\underline{\Phi})$*,* $\partial_{\underline{\Phi}_i}\partial_{\underline{\Phi}_j} V(\underline{\Phi})$ *are all of polynomial growth, or*

(iii) $d=3$, and the interaction of the underlying bosonic model has the form

$$\mathcal{V}[\underline{\Phi}|0]=\sum_{i,j,k} c_{ijk}\underline{\Phi}_i\underline{\Phi}_j\underline{\Phi}_k+\sum_{i,j,k,l} c_{ijkl}\underline{\Phi}_i\underline{\Phi}_j\underline{\Phi}_k\underline{\Phi}_l$$

with real numbers c_{ijk}, c_{ijkl}.

Then the model is complete.

Remarks. (1) Thus, in $d=3$, couplings of at most fourth order are allowed. Remarkably, this is at the same time the condition for power-counting renormalizability. (2) The hypothesis (1.11.2) is usually guaranteed by energy conservation.

Proof. For notational convenience, we assume from the beginning that fermionic fields are absent. As in [10], [11], we use the Sobolev norm

$$\left\|(\phi^{\mathrm{Cau}},\dot{\phi}^{\mathrm{Cau}})\right\|_{\mathcal{H}_k^{\mathrm{Cau,V}}}:=\sum_{i=1}^{N^\Phi}\left(\left\|\phi_i^{\mathrm{Cau}}\right\|_{H_{k+1}}+\left\|\dot{\phi}_i^{\mathrm{Cau}}\right\|_{H_k}\right).$$

In view of Thm. 1.8.1 and Remark 1.8.2, it is in all three cases sufficient to show

$$\sup_{t\in(a,b)}\|(\phi(t),\partial_t\phi(t))\|_{\mathcal{H}_k^{\mathrm{Cau,V}}}<\infty. \tag{1.11.3}$$

for $k=1$.

First we show (1.11.3) for $k=0$: We have $\partial_t\|\phi_i(t)\|_{L_2(\mathbb{R}^d)}\le\|\partial_t\phi_i(t)\|_{L_2(\mathbb{R}^d)}<C$ for all i with some $C>0$, and hence

$$\|\phi_i(t)\|_{L_2(\mathbb{R}^d)}\le\|\phi_i((a+b)/2)\|_{L_2(\mathbb{R}^d)}+C\,|t-(a+b)/2|$$

is bounded, which implies the assertion.

For the step to $k=1$, we apply [11, Lemma 4.2.3] with respect to the norms $\|\cdot\|_{\mathcal{H}_1^{\mathrm{Cau,V}}}\le\|\cdot\|_{\mathcal{H}_0^{\mathrm{Cau,V}}}$ within (say) the Banach space $(\mathcal{H}_2^{\mathrm{Cau,V}})_{\mathbf{0}}$ which is the completion of $(\mathcal{E}_{\mathrm{c}}^{\mathrm{Cau,V}})_{\mathbf{0}}$ with respect to $\|\cdot\|_{\mathcal{H}_2^{\mathrm{Cau,V}}}$. Thus, it is sufficient to show that there exists a monotonously increasing function $F:\mathbb{R}_+\to\mathbb{R}_+$ with

$$\left\|\frac{\delta}{\delta\underline{\Phi}_i}\mathcal{V}[\phi](t)\right\|_{H_1}\le\left(1+\|(\phi(t),\partial_t\phi(t))\|_{\mathcal{H}_1^{\mathrm{Cau,V}}}\right)F(\|(\phi(t),\partial_t\phi(t))\|_{\mathcal{H}_0^{\mathrm{Cau,V}}}) \tag{1.11.4}$$

for all $t\in(a,b)$, $i=1,\dots,N^\Phi$.

Ad (i). In $d\le 1$, we have a continuous embedding $H_1(\mathbb{R}^d)\subseteq L_\infty(\mathbb{R}^d)$. Obviously, pointwise multiplication makes $L_\infty(\mathbb{R}^d)$ a Banach algebra, and yields a continuous pairing $L_2(\mathbb{R}^d)\times L_\infty(\mathbb{R}^d)\to L_2(\mathbb{R}^d)$.

From (1.2.1) we have for $a=1,\dots,d$

(1.11.5)

$$\partial_a\frac{\delta}{\delta\underline{\Phi}_i}\mathcal{V}[\phi](t)=\partial_a\phi_l(t)\left(\frac{\partial^2}{\partial\underline{\Phi}_i\partial\underline{\Phi}_l}\mathcal{V}'(\phi(t))+\frac{\partial}{\partial\underline{\Phi}_l}\mathcal{V}^{\mu,ij}(\phi(t))\partial_\mu\phi_j(t)\right)+\mathcal{V}^{\mu,ij}(\phi(t))\partial_a\partial_\mu\phi_j(t)$$

(note that the sum over μ ranges $0,\dots,d$). For $s>0$, let

$$F'(s):=\sup_{(\xi,\dot{\xi})\in\mathcal{H}_0^{\mathrm{Cau,V}};\ \|(\xi,\dot{\xi})\|\le s}\ \max_{\mu,i,l}\ \max\left\{\left\|\frac{\partial^2}{\partial\underline{\Phi}_i\partial\underline{\Phi}_l}\mathcal{V}'(\xi)\right\|_{L_\infty},\left\|\frac{\partial}{\partial\underline{\Phi}_l}\mathcal{V}^{c,ij}(\xi)\partial_c\xi_j+\frac{\partial}{\partial\underline{\Phi}_l}\mathcal{V}^{0,ij}(\xi)\dot{\xi}_j\right\|_{L_2},\left\|\mathcal{V}^{\mu,il}(\xi)\right\|_{L_\infty}\right\}.$$

By our remarks above, this is finite. From (1.11.5) we have

$$\begin{aligned}\left\|\partial_a\frac{\delta}{\delta\underline{\Phi}_i}\mathcal{V}[\phi](t)\right\|_{L_2}&\le CF'(\|(\phi(t),\partial_t\phi(t))\|_{\mathcal{H}_0^{\mathrm{Cau,V}}})\left(\sum_a\|\partial_a\phi_k(t)\|_{L_2}+\sum_{a,\mu,j}\|\partial_a\partial_\mu\phi_j(t)\|_{L_2}\right)\\&\le CF'(\|(\phi(t),\partial_t\phi(t))\|_{\mathcal{H}_0^{\mathrm{Cau,V}}})\,\|(\phi(t),\partial_t\phi(t))\|_{\mathcal{H}_1^{\mathrm{Cau,V}}},\end{aligned}$$

yielding the estimate (1.11.4) wanted.

Ad (ii). Recall that we have a continuous embedding

$$H_1(\mathbb{R}^2) \subseteq L_p(\mathbb{R}^2) \tag{1.11.6}$$

for all $p \geq 2$. Also, we note that if W is of polynomial growth and satisfies $W(0) = 0$, the estimate (1.11.1) implies that for all $p > 1$ there exist C_1, C_2 such that

$$\|W(\xi_1, \ldots, \xi_m)\|_{L_p}^p \leq C_1 + C_2 \sum_i \|\xi_i\|_{L_{Np}}^{Np}.$$

for $\xi_1, \ldots, \xi_m \in H_1$. Combining this with (1.11.6) we get that there exist C_3, C_4 such that

$$\|W(\xi_1, \ldots, \xi_m)\|_{L_p} \leq C_3 + C_4 \sum_i \|\xi_i\|_{H_1}^N. \tag{1.11.7}$$

Applying (1.11.6) and (1.11.7), we get an estimate

$$\|\partial_a(\partial_i V(\phi(t)))\|_{L_2} \leq \sum_j \|\partial_a \phi_j(t)\|_{L_4} \|\partial_i \partial_j V(\phi(t))\|_{L_4} \leq \sum_j \|\phi_j(t)\|_{H_2} \left(C_5 + C_6 \sum_k \|\phi_k(t)\|_{H_1}^N\right).$$

Applying also (1.11.7) with $W := \partial_i V$, we get the estimate (1.11.4) needed:

$$\|\partial_i V(\phi(t))\|_{H_1} \leq \left(1 + \sum_j \|\phi_j(t)\|_{H_2}\right)\left(C_7 + C_8 \sum_j \|\phi_j(t)\|_{H_1}^N\right).$$

Ad (iii). Choose $R > 0$ such that $\mathrm{supp}\phi(t) \subseteq \{x : \|x\| < R\}$ for all $t \in (a, b)$. It is a standard fact that we have a continuous embedding

$$H_1(\mathbb{R}^3) \subseteq L_6(\mathbb{R}^3),$$

Also, for each $p \leq 6$ we have a continuous embedding

$$\left\{\xi \in L_6(\mathbb{R}^3) : \ \mathrm{supp}\xi \subseteq \{x : \|x\| < R\}\right\} \subseteq L_p(\mathbb{R}^3),$$

Finally, we recall that pointwise multiplication yields a continuous bilinear map

$$L_p(\mathbb{R}^3) \times L_q(\mathbb{R}^3) \to L_{1/(1/p+1/q)}(\mathbb{R}^3).$$

It follows that

$$\|\partial_i \mathcal{V}(\phi))\|_{H_1} \leq \|\partial_i \mathcal{V}(\phi))\|_{L_2} + \sum_a \|\partial_a(\partial_i \mathcal{V}(\phi)))\|_{L_2} \leq C_R\left(\|\phi\|_{H_1}^3 + \|\phi\|_{H_2} \|\phi\|_{H_1}^2\right)$$

with suitable $C_R > 0$, which is the estimate (1.11.4) needed. □

2. Example models

2.1. Scalar models. The simplest examples of classical models are the scalar models

$$\mathcal{L}[\underline{\Phi}] = -\frac{1}{2}\left(\partial_\mu \underline{\Phi} \partial^\mu \underline{\Phi} + m^2 \underline{\Phi}^2\right) - \mathcal{V}(\underline{\Phi})$$

where $\mathcal{V}(\cdot)$ is an entire function on $\mathbb{C}$ which has an at least triple zero at the origin (derivative couplings are physically excluded by Lorentz invariance).

Here is nothing "super", and M^{sol}, if defined, is simply a real-analytic manifold.

The case of the Φ^4 model, i. e. $\mathcal{V}(\underline{\Phi}) = q\underline{\Phi}^4$ with $q > 0$ and $d = 3$, has already been discussed in [10]. Using energy conservation, as in the proof of Cor. 2.4.1 below, the (well-known) completeness of the Φ^4 model follows from Prop. 1.11.1(iii).

2.2. Purely fermionic models. In the absence of bosonic fields, $N^\Phi = 0$, the completeness condition is void, and Thm. 1.8.3 immediately provides a sub-smf of classical solutions $M^{\mathrm{sol}} \subseteq M$. In particular, this applies to the Gross-Neveu models, and to the Thirring model, which admits a closed formula for Ξ^{sol} (cf. [10]). Note that both M^{sol} and M are simply points equipped with infinite-dimensional Grassmann algebras, and the embedding is given by a surjection $\mathcal{O}(M) \to \mathcal{O}(M^{\mathrm{sol}})$ ("restriction of field functionals onto the mass shell").

2.3. **A simple boson-fermion model.** As a more realistic example, we consider the Yukawa model of meson-nucleon scattering: The field components are Φ_i $(i = 1, 2, 3)$ for the three isospin components of the meson field on the bosonic side and, using complex components, $\Psi_{\alpha,k}, \overline{\Psi}_{\alpha,k}$ $(k = 1, 2,\ \alpha = 1, \ldots, 4)$ for the isospin dublet of the nucleon field on the fermionic side. Thus, the setup is $(d, V) = (4, \mathbb{R}^{3|16})$. Using matrix writing for the spinor indices, the Lagrangian is

$$\mathcal{L}[\underline{\Phi}|\underline{\Psi}] := -\frac{\mathrm{i}}{2}\left(\underline{\overline{\Psi}}_k\gamma^\mu\partial_\mu\underline{\Psi}_k - \partial_\mu\underline{\overline{\Psi}}_k\gamma^\mu\underline{\Psi}_k\right) - \mathrm{i}m^\Psi\underline{\overline{\Psi}}_k\underline{\Psi}_k - \frac{1}{2}\partial_\mu\underline{\Phi}_i\partial^\mu\underline{\Phi}_i - \frac{1}{2}(m^\Phi)^2\underline{\Phi}_i^2 - \mathrm{i}g\underline{\overline{\Psi}}_k\gamma_5\tau^i_{kl}\underline{\Psi}_l\underline{\Phi}_i$$

where the isospin matrices τ^i $(i = 1, 2, 3)$ are $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{smallmatrix}\right)$, and $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$, respectively, $\underline{\overline{\Psi}}_{\alpha,k} := \underline{\overline{\Psi}}_{\beta,k}\gamma^0_{\beta\alpha}$ is the Dirac conjugate, and $g \in \mathbb{R}$ is the coupling constant. The operator $\not{D}$ from (1.2.3) is here $\not{D} = \mathrm{i}(\gamma^\mu\partial_\mu - m^\Psi)$, so that $K := -\mathrm{i}(\gamma^\mu\partial_\mu + m^\Psi)$ satisfies (1.2.4), and the fermionic influence function $\hat{\mathcal{A}}^\Psi$ is now given by (1.5.3).

Since the underlying bosonic model describes three free scalar fields, the model is complete.

2.4. **A supersymmetric model.** Let us consider the most general $d = 3$, simply supersymmetric renormalizable Lagrangian with only chiral superfields, taken from [16, (5.12)]. Of course, we use the component field formulation, with the auxiliary field being eliminated.

The model contains on the bosonic side complex scalar fields $A_1, \ldots, A_n$, and on the fermionic one Weyl spinor fields $\Psi_1, \ldots, \Psi_n$. The parameters are the "masses" $(m_{ik}) \in \mathbb{C}^{n\times n}$ which are symmetric, the coupling constants $(g_{ijk}) \in \mathbb{C}^{n\times n\times n}$ which are symmetric in all indices, and the complex parameters $(\lambda_k) \in \mathbb{C}^n$.

For shortness, we set

$$F_k[\underline{A}] := -\lambda_k - m_{ik}\underline{A}_i - g_{ijk}\underline{A}_i\underline{A}_j, \qquad \mathcal{V}[\underline{\overline{A}}, \underline{A}] := \overline{F_k[\underline{A}]}F_k[\underline{A}].$$

The Lagrange density is

$$\mathcal{L}[\underline{\overline{A}}, \underline{A}|\,\underline{\overline{\Psi}}, \underline{\Psi}] = \mathrm{i}\partial_\mu\underline{\overline{\Psi}}_i\overline{\sigma}^\mu\underline{\Psi}_i - \partial_\mu\underline{\overline{A}}_i\partial^\mu\underline{A}_i - \mathrm{Re}\, m_{ik}\underline{\Psi}_i\underline{\Psi}_k - 2\mathrm{Re}\, g_{ijk}\underline{\Psi}_i\underline{\Psi}_j\underline{A}_k - \mathcal{V}[\underline{\overline{A}}, \underline{A}].$$

Unfortunately, the λ_k spoil the immediate fitting of this Lagrangian into our model class, since they produce linear terms in the $\underline{A}_i$. However, if $c \in \mathbb{R}^n$ is a critical point of $\mathcal{V}$, i. e. $\partial_{\underline{A}_i}\mathcal{V}[\overline{c}, c] = \partial_{\underline{\overline{A}}_i}\mathcal{V}[\overline{c}, c] = 0$, then we may pass to new variables $\underline{A}'_i := \underline{A}_i - c_i$, getting a new Lagrangian

$$\mathcal{L}'[\underline{\overline{A}}', \underline{A}'|\,\underline{\overline{\Psi}}, \underline{\Psi}] := \mathcal{L}[\underline{\overline{A}}' + \overline{c}, \underline{A}' + c|\,\underline{\overline{\Psi}}, \underline{\Psi}_i] - \mathcal{V}[\overline{c}, c]$$

which is easily seen to fit into our model class. (If $\mathcal{V}[\overline{c}, c] \neq 0$ then supersymmetry is spontaneously broken.)

Corollary 2.4.1. *This model is complete.*

Proof. By standard conclusions of QFT, the energy of the underlying bosonic model,

$$\mathcal{H}[\underline{A}'] := \sum\nolimits_i(\partial_0\underline{A}'_i)^2 - \mathcal{L}'[\underline{\overline{A}}', \underline{A}'|\,0, 0] = \sum\nolimits_i(\partial_0\underline{A}'_i)^2 + \sum\nolimits_{a,i}(\partial_a\underline{A}'_i)^2 + \mathcal{V}[\underline{\overline{A}}' + \overline{c}, \underline{A}' + c],$$

satisfies $\partial_0\mathcal{H}[A'] = 0$ for each solution A' of the underlying bosonic equations. Since $\mathcal{V}[\overline{A}' + \overline{c}, A' + c] \geq 0$, this implies that the hypotheses of Prop. 1.11.1(iii) are satisfied. $\square$

In [8], we have shown that the usual supersymmetry operators act as odd vector fields on the configuration smf M. In the successor paper, we will show much more: The solution smf M^{sol} has a natural symplectic structure, the supersymmetry vector fields restrict to odd vector fields $Q_\alpha, \overline{Q}_{\dot\alpha} \in \mathcal{X}(M^{\mathrm{sol}})_{\mathbf{1}}$ which are just the hamiltonian vector fields generated by the Noether charges associated to supersymmetry.

3. Yang-Mills-Dirac-Higgs models in the temporal gauge

3.1. **The general setting.** In the following, we assume $d \geq 1$. Let G be a compact Lie group. We consider a full Yang-Mills theory coupled to Dirac and Higgs fields, with the following field components:

– the gauge potential $A = (A^i_\mu)^{i=1,\ldots,N^{\mathfrak{g}}}_{\mu=0,\ldots,d}$ where $N^{\mathfrak{g}}$ is the dimension of the Lie algebra $\mathfrak{g} := \mathrm{lie}\, G$, and the Lie algebra index i here and in the following always refers to an orthonormal basis of $\mathfrak{g}$ with respect to a fixed invariant positively definite scalar product on $\mathfrak{g}$,

– the Higgs field $\Phi = (\Phi_j)_{j=1,\dots,N^\Phi}$, where N^Φ is the dimension of a real orthogonal representation space V^Φ of G,

– the Dirac field $\Psi = (\Psi_{k\alpha})_{\alpha=1,\dots,N^{\mathrm{D}}}^{k=1,\dots,N^\Psi}$, where N^Ψ is the dimension of a unitary representation space V^Ψ of G, and $N^{\mathrm{D}} = 2^{1+[d/2]}$ is the dimension of the Dirac representation $V^{\mathrm{D}} = \mathbb{C}^{N^{\mathrm{D}}}$ of the Lorentz group. (We take Dirac spinors for definiteness; everything we do can be carried over to Weyl, Majorana, and Majorana-Weyl spinors, as far as they are defined in space dimension d.)

Thus, we are given elements $\gamma_\mu \in \mathrm{End}(V^{\mathrm{D}})$ with $\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 2g_{\mu\nu}$, which we may assume to satisfy $\overline{\gamma}_0^{\mathrm{T}} = -\gamma_0$, $\overline{\gamma}_a^{\mathrm{T}} = \gamma_a$ (of course, this condition is not Lorentz invariant). As usual, we write the element of V^{D} as columns, and we define the *Dirac conjugate* of $\psi \in V^{\mathrm{D}}$ as the row $\overline{\psi} \in \overline{V}^{\mathrm{D}}$ given by $\overline{\psi}_\alpha := \overline{\psi_\beta}\gamma^0_{\beta\alpha}$. We get a Lorentz invariant indefinite pairing $\overline{V}^{\mathrm{D}} \times V^{\mathrm{D}} \to \mathbb{C}$, $(\overline{\psi}, \chi) \mapsto \overline{\psi}\chi = \overline{\psi_\beta}\gamma^0_{\beta\alpha}\chi_\alpha$.

We allow the special cases $N^\Phi = 0$, $N^\Psi = 0$.

We will denote the action of the corresponding infinitesimal representations of $\mathfrak{g}$ by $\times$, so that

$$(A_\mu \times \Phi)_k := (f^\Phi)^l_{ik} A^i_\mu \Phi_l, \quad (A_\mu \times \Psi)_k := (f^\Psi)^l_{ik} A^i_\mu \Psi_l,$$

where $(f^\Phi)^l_{ik}, (f^\Psi)^l_{ik}$ are the structure constants of the corresponding infinitesimal representations. If $\Phi', \Phi"$ are field monomials with values in V^Φ, and $\overline{\Psi}', \Psi"$ are field monomials with values in $\overline{V^\Psi} \otimes V^{\mathrm{D}}$ and $V^\Psi \otimes V^{\mathrm{D}}$, respectively, we set

$$(\Phi' \times \Phi")_i := (f^\Phi)^l_{ik} \Phi'_k \Phi"_l, \quad (\overline{\Psi}' \times \Psi")_i := (f^\Psi)^l_{ik} \overline{\Psi'_k} \Psi"_l,$$

where it is understood that orthonormal bases have been used in the corresponding representation spaces. Clearly, both field monomials take values in $\mathfrak{g}$. We will use the usual covariant derivatives

$$D_\mu\Phi := \partial_\mu\Phi + A_\mu \times \Phi, \quad D_\mu\Psi := \partial_\mu\Psi + A_\mu \times \Psi.$$

The standard Lagrangian is

$$\mathcal{L}^{\mathrm{YMDH}}[\underline{A}, \underline{\Phi}|\underline{\Psi}] := \mathcal{L}^{\mathrm{YM}}[\underline{A}] + \mathcal{L}^{\mathrm{DH}}[\underline{A}, \underline{\Phi}|\underline{\Psi}] \tag{3.1.1}$$

where

$$\mathcal{L}^{\mathrm{YM}}[\underline{A}] = -\frac{1}{4} F_{\mu\nu}[\underline{A}] F^{\mu\nu}[\underline{A}], \qquad F_{\mu\nu}[\underline{A}] := \partial_\mu \underline{A}_\nu - \partial_\nu \underline{A}_\mu + [\underline{A}_\mu, \underline{A}_\nu]$$

is the Lagrangian of pure Yang-Mills theory, and

$$\mathcal{L}^{\mathrm{DH}}[\underline{A}, \underline{\Phi}|\underline{\Psi}] := -\frac{\mathrm{i}}{2}\left(\overline{\underline{\Psi}}\gamma^\mu D_\mu \underline{\Psi} - \overline{D_\mu \underline{\Psi}}\gamma^\mu \underline{\Psi}\right) - \mathrm{i}\overline{\underline{\Psi}} m^\Psi \underline{\Psi} - \frac{1}{2} D_\mu \underline{\Phi}_i D^\mu \underline{\Phi}_i + \mathcal{V}(\underline{\Phi})$$

is the Lagrangian of the Higgs field and of minimally coupled Dirac matter. Here $m = (m_{ij})_{i,j=1}^{N^\Psi}$ is supposed to be real and G-invariant. (Many textbooks use the opposite Lorentz metric $(1, -1, \dots, -1)$; since then $\overline{\gamma}_0^{\mathrm{T}} = \gamma_0$, they do not have an imaginary unit in front of the fermionic mass term).

For the moment, we suppose for $\mathcal{V}$ only that it is a gauge-invariant entire power series with lower degree ≥ 2.

Remark. The most popular choice is

$$\mathcal{V}(\underline{\Phi}) = \frac{1}{2}\left(-m^\Phi \underline{\Phi}_i^2 + h^2(\underline{\Phi}_i^2)^2\right).$$

Note that in the underlying free model, the "mass square" m^Φ is usually negative, in order to make $\Phi = 0$ not even a local minimum of the potential. Thus, the "underlying free model" is here a purely mathematical notion; the free approximation is physically senseful only at the minima of $\mathcal{V}$. Fortunately, we had found before that scalar fields with negative mass square still behave classically well.

For later use, we recall the Bianchi identity:

$$D^\mu F^{\nu\lambda} + D^\nu F^{\lambda\mu} + D^\lambda F^{\mu\nu} = 0. \tag{3.1.2}$$

The field equations resulting from (3.1.1) are

(3.1.3a) $$D^\mu F_{\mu\nu} + J_\nu = 0,$$

(3.1.3b) $$\left(\gamma^\mu D_\mu + m^\Psi\right)\Psi = 0,$$

(3.1.3c) $$D^\mu D_\mu \Phi_i + \mathcal{V}_i(\Phi) = 0, \quad \mathcal{V}_i(\Phi) := \frac{\partial}{\partial \underline{\Phi}_i}\mathcal{V}(\Phi),$$

with the matter current J_μ given by

$$J_\mu = \frac{\delta}{\delta \underline{A}^\mu}\mathcal{L}^{\mathrm{DH}}[A,\Phi|\Psi] = -\mathrm{i}\overline{\Psi}\times\gamma_\mu\Psi + \Phi\times D_\mu\Phi.$$

Of course, the corresponding Cauchy problem for the field equations is not well-posed. There are essentially two main approaches to get a well-posed Cauchy problem: The simplest possibility is to impose an explicit gauge condition which diminishes the effective number of degrees of freedom, like the Lorentz gauge $\partial^\mu A_\mu = 0$, or the *temporal gauge* $A_0 = 0$ (which, however, sacrifices Poincaré invariance).

However, such an explicit gauge breaking is not well adapted to the needs of quantization. In the physical literature, it is more common to use a "soft" gauge breaking where the longitudinal degrees of freedom are not constrained but damped by an additional term in the Lagrangian, and the correct quantum dynamics is restored by the introduction of Faddeev-Popov ghost fields. We will follow this approach in section 4, concentrating in the rest of this section onto the temporal gauge (despite of its disadvantages, we treat it simply because for it the completeness result needed is available from [4], while for the Faddeev-Popov approach the corresponding result is not yet proven).

In the temporal gauge, the field target space becomes

$$V := \mathbb{R}^d\otimes\mathfrak{g} \;\oplus\; V^\Phi \;\oplus\; \Pi(V^\Psi\otimes_{\mathbb{C}} V^{\mathrm{D}}),$$

and we will look for the sub-smf of classical solutions within the smf of smooth configurations

$$M_{C^\infty} := \mathrm{L}(\mathcal{E}^{\mathrm{V}}), \quad \mathcal{E}^{\mathrm{V}} := C^\infty(\mathbb{R}^{d+1})\otimes V.$$

(Unfortunately, we cannot treat our favorite quality $\mathcal{E}_{\mathrm{c}}$. Cf. Rem. 3.2.2(2) below for the reason why.)

3.2. **Constraint, Cauchy data, and results.** In the temporal gauge $A_0 = 0$, the remaining gauge potentials are $(A_a^i)_{a=1,\dots,d}^{i=1,\dots,N^{\mathfrak{g}}}$. As usual, we will use the magnetic field strengthes

(3.2.1) $$B_{ab} := F_{ab} = \partial_a A_b - \partial_b A_a - [A_a, A_b]$$

and the electric field strengthes

$$E_a := F_{0a} = \partial_0 A_a.$$

(3.1.3a) for $\nu = 0$ turns into the constraint

(3.2.2) $$-D^a E_a + J_0 = 0, \quad \text{i. e.} \quad \partial^a E_a = -[A^a, E_a] - \mathrm{i}\overline{\Psi}\times\gamma_0\Psi + \Phi\times\partial_0\Phi,$$

while for $\nu = a$ it becomes

(3.2.3) $$\partial_t E_a = \partial^b B_{ba} + \left[A^b, B_{ba}\right] - \mathrm{i}\overline{\Psi}\times\gamma_a\Psi + \Phi\times D_a\Phi.$$

Remark. (3.2.3) belongs to the Lagrangian which arises from (3.1.1) by setting $\underline{A}_0 := 0$:

(3.2.4) $$\mathcal{L}^{\text{YMDH-tg}}[\underline{A}] = -\frac{1}{4}F_{ab}[\underline{A}]F^{ab}[\underline{A}] + \frac{1}{2}(\partial_0\underline{A}_a)^2 + \mathcal{L}^{\mathrm{DH}}[0,\underline{A}_1,\dots,\underline{A}_d,\underline{\Phi}|\underline{\Psi}].$$

However, from this point of view, (3.2.2) is an "external" constraint.

An obvious idea to solve (3.2.2) on the Cauchy hyperplane is to represent (say) $E_1(0)$ as functional of its restriction E_1^0 onto $x^1 = 0$ and the remaining Cauchy data. Thus, we take as smf of smooth Cauchy data

(3.2.5)

$$\begin{aligned} M^{\mathrm{par}}_{C^\infty} :=& \mathrm{L}(\mathcal{E}^{\mathrm{Cau}}\otimes\mathbb{R}^d\otimes\mathfrak{g})_{A_1^{\mathrm{Cau}},\dots,A_d^{\mathrm{Cau}}}\times\mathrm{L}(C^\infty(\mathbb{R}^{d-1})\otimes\mathfrak{g})_{E_1^0}\times\mathrm{L}(\mathcal{E}^{\mathrm{Cau}}\otimes\mathbb{R}^{d-1}\otimes\mathfrak{g})_{E_2^{\mathrm{Cau}},\dots,E_d^{\mathrm{Cau}}}\times\\ &\times\mathrm{L}(\mathcal{E}^{\mathrm{Cau}}\otimes(V^\Phi\oplus V^\Phi))_{\Phi^{\mathrm{Cau}},\dot{\Phi}^{\mathrm{Cau}}}\times\mathrm{L}(\mathcal{E}^{\mathrm{Cau}}\otimes\Pi(V^\Psi\otimes_{\mathbb{C}}V^{\mathrm{D}}))_{\Psi^{\mathrm{Cau}}}, \end{aligned}$$

where $\mathcal{E}^{\mathrm{Cau}} := C^\infty(\mathbb{R}^d)$.

With the methods of [10], [11] one shows:

Lemma 3.2.1. *There exists a unique element*

$$E_1^{\mathrm{cst}} = E_1^{\mathrm{cst}}[A^{\mathrm{Cau}}, E_1^0, E_2^{\mathrm{Cau}}, \dots, E_d^{\mathrm{Cau}}, \Phi^{\mathrm{Cau}}, \dot\Phi^{\mathrm{Cau}} |\ \Psi^{\mathrm{Cau}}] \in \mathcal{M}^{\mathcal{E}^{\mathrm{Cau}}\otimes\mathfrak{g}}(M^{\mathrm{par}}_{C^\infty})$$

such that

$$\partial^1 E_1^{\mathrm{cst}} + [A_1^{\mathrm{Cau}}, E_1^{\mathrm{cst}}] + \sum_{b=2}^{d}\big(\partial^b E_b^{\mathrm{Cau}} + \big[A_b^{\mathrm{Cau}}, E_b^{\mathrm{Cau}}\big]\big) + \mathrm{i}\overline{\Psi}^{\mathrm{Cau}} \times \gamma_0 \Psi^{\mathrm{Cau}} - \Phi^{\mathrm{Cau}} \times \dot\Phi^{\mathrm{Cau}} = 0,$$

and $E_1^{\mathrm{cst}}|_{x_1=0} = E_1^0$. □

Remarks 3.2.2. (1) It easily follows that the element

$$\partial^a E_a^{\mathrm{Cau}} + [A_a^{\mathrm{Cau}}, E_a^{\mathrm{Cau}}] + \mathrm{i}\overline{\Psi}^{\mathrm{Cau}} \times \gamma_0 \Psi^{\mathrm{Cau}} - \Phi^{\mathrm{Cau}} \times \dot\Phi^{\mathrm{Cau}} \in \mathcal{M}^{\mathcal{E}^{\mathrm{Cau}}\otimes\mathfrak{g}}(M^{\mathrm{Cau}}_{C^\infty}) \tag{3.2.6}$$

cuts out a sub-supermanifold (cf. [9, 2.12]) $M^{\mathrm{Cau,cst}}_{C^\infty} \subseteq M^{\mathrm{Cau}}_{C^\infty}$ in the smf of unconstrained smooth Cauchy data

$$\begin{aligned} M^{\mathrm{Cau}}_{C^\infty} :=& \mathrm{L}(\mathcal{E}^{\mathrm{Cau}} \otimes \mathbb{R}^d \otimes \mathfrak{g})_{A^{\mathrm{Cau}}} \times \mathrm{L}(\mathcal{E}^{\mathrm{Cau}} \otimes \mathbb{R}^d \otimes \mathfrak{g})_{E^{\mathrm{Cau}}} \times \\ &\times \mathrm{L}(\mathcal{E}^{\mathrm{Cau}} \otimes (V^\Phi \oplus V^\Phi))_{\Phi^{\mathrm{Cau}},\dot\Phi^{\mathrm{Cau}}} \times \mathrm{L}(\mathcal{E}^{\mathrm{Cau}} \otimes \Pi(V^\Psi \otimes_{\mathbb{C}} V^{\mathrm{D}}))_{\Psi^{\mathrm{Cau}}}. \end{aligned}$$

Indeed, the sub-smf sought is just the image of the morphism $M^{\mathrm{par}}_{C^\infty} \to M^{\mathrm{Cau}}_{C^\infty}$, the pullback of which maps E_1^{Cau} to E_1^{cst}, and all other coordinate components to the coordinate components with the same name.

(2) Unfortunately, the analogous assertion for our favorite quality $\mathcal{E}_{\mathrm{c}}$ seems to be false; at any rate, we will have in general $E_1^{\mathrm{cst}} \notin \mathcal{M}^{\mathcal{E}_{\mathrm{c}}^{\mathrm{Cau}}\otimes\mathfrak{g}}(M^{\mathrm{par}})$ where M^{par} is formed like (3.2.5) but using $\mathcal{E}_{\mathrm{c}}^{\mathrm{Cau}} := \mathcal{D}(\mathbb{R}^d)$ instead of $\mathcal{E}^{\mathrm{Cau}}$, and $C_0^\infty(\mathbb{R}^{d-1})$ instead of $C^\infty(\mathbb{R}^{d-1})$.

The main result of this section is:

Theorem 3.2.3. (i) *Let be given a point* $p = (s, x) \in \mathbb{R}^{d+1}$ *with* $s \neq 0$ *and two* Z*-families* $(A', \Phi'|\Psi')$, $(A", \Phi"|\Psi")$ *of* (3.1.3a)–(3.1.3c) *which satisfy* $A'_0 = A"_0 = 0$. *Suppose that, with the notations of 1.7,*

$$(D^\mu_{A'} F_{\mu\nu}[A'] + J_\nu[A', \Phi'|\Psi'])\,|_{\Omega(p)} = 0, \qquad \big(\gamma^\mu (D_{A'})_\mu + m^\Psi\big)\,\Psi'|_{\Omega(p)} = 0,$$
$$(D^\mu_{A'}(D_{A'})_\mu \Phi'_i + \mathcal{V}_i(\Phi'))\,|_{\Omega(p)} = 0,$$

and analogously for $(A", \Phi"|\Psi")$, *and that, writing* $\Delta = (A' - A", \Phi' - \Phi")$, *we have*

$$\Delta(0)|_{\mathcal{J}(p)} = 0, \quad \partial_0\Delta(0)|_{\mathcal{J}(p)} = 0, \quad (\Psi' - \Psi")(0)|_{\mathcal{J}(p)} = 0.$$

Then

$$(A' - A", \Phi' - \Phi"|\Psi' - \Psi")\,|_{\Omega(p)} = 0.$$

(ii) *Suppose that* $d \geq 2$, *and that there exists a Sobolev index* $k > d/2$ *such that for every solution*

$$(a, \phi) \in C^\infty((t_0, t_1) \times \mathbb{R}^d) \otimes (\mathbb{R}^{d+1} \otimes \mathfrak{g} \oplus V^\Phi)$$

of the Yang-Mills-Higgs equations in the temporal gauge,

$$D^\mu F_{\mu\nu}[a] + \phi \times D_\nu \phi = 0, \quad D^\mu D_\mu \phi_i + \mathcal{V}_i(\phi) = 0, \quad a_0 = 0,$$

on an open time interval (t_0, t_1) *such that* $\mathrm{supp}(a, \phi)(t)$ *is compact for some (and hence all)* $t \in (t_0, t_1)$, *we have*

$$\sup_{t\in(t_0,t_1)} \max\left\{\|a^i_\mu(t)\|_{H_{k+1}(\mathbb{R}^d)},\ \|\phi_j(t)\|_{H_{k+1}(\mathbb{R}^d)},\ \|\partial_t a^i_\mu(t)\|_{H_k(\mathbb{R}^d)},\ \|\partial_t\phi_j(t)\|_{H_k(\mathbb{R}^d)}\right\} < \infty$$

where $i = 1, \dots, N^{\mathfrak{g}}$, $\mu = 1, \dots, d$, $j = 1, \dots, N^\Phi$. *Then the following assertions are valid.*

(1) *There exists a unique solution family*

$$\begin{aligned}\Xi^{\mathrm{sol}} &= (A^{\mathrm{sol}}, \Phi^{\mathrm{sol}}|\Psi^{\mathrm{sol}}) \\ &= \Xi^{\mathrm{sol}}[A^{\mathrm{Cau}}, E^0_1, E^{\mathrm{Cau}}_2, \ldots, E^{\mathrm{Cau}}_d, \Phi^{\mathrm{Cau}}, \dot\Phi^{\mathrm{Cau}}|\Psi^{\mathrm{Cau}}] \in \mathcal{M}^{\mathcal{E}^{\mathrm{V}}}(M^{\mathrm{par}}_{C^\infty})\end{aligned} \tag{3.2.7}$$

of (3.1.3a)–(3.1.3c) *which satisfies* $A^{\mathrm{sol}}_0 = 0$ *with the initial data*

$$\begin{aligned} A^{\mathrm{sol}}_a(0,\cdot) &= A^{\mathrm{Cau}}_a(\cdot), & a = 1, \ldots, d,\\ \partial_t A^{\mathrm{sol}}_1(0,0,x_2,\ldots,x_d) &= E^0_1(x_2,\ldots,x_d), & \\ \partial_t A^{\mathrm{sol}}_a(0,\cdot) &= E^{\mathrm{Cau}}_a(\cdot), & a = 2,\ldots,d,\\ (\Phi^{\mathrm{sol}}, \partial_t\Phi^{\mathrm{sol}}|\Psi^{\mathrm{sol}})(0,\cdot) &= (\Phi^{\mathrm{Cau}}, \dot\Phi^{\mathrm{Cau}}|\Psi^{\mathrm{Cau}})(\cdot). &\end{aligned}$$

(2) *This solution family is universal, i. e. any other smooth solution family* $(A', \Phi'|\Psi') \in \mathcal{M}^{\mathcal{E}^{\mathrm{V}}}(Z)$ *of* (3.1.3a)–(3.1.3c) *which satisfies* $A'_0 = 0$ *is in a unique way a pullback of* (3.2.7).

(3) *The image of* $M^{\mathrm{par}}_{C^\infty} \xrightarrow{\Xi^{\mathrm{sol}}} M_{C^\infty}$ *is a sub-smf.*

(iii) *In particular, suppose that either*

(4) $d = 2$, *and* $\mathcal{V} = V(\underline{\Phi}_i\underline{\Phi}_i)$ *with an entire power series* $V(r)$ *of polynomial growth which satisfies* $V(0) = 0$ *and* $V(r) \geq -Cr$ *for suitable* $C > 0$, *or*

(4) $d = 3$, *and* $\mathcal{V}$ *is a* G*-invariant polynomial in* $\underline{\Phi}$ *of degree* ≤ 4.

Then the assertions (1), (2), (3) are true.

We postpone the proof of the Theorem to the next section.

3.3. **The first-order system.** Let us shortly discuss the obstackles to a naive approach. The quadratic part of (3.2.4) has not the required form (1.2.2). A straightforward generalization of our previous approach to (3.2.4) fails for the following reason: The linearized equation of motion for A is

$$K^{ai,bj}A_{aj} = 0, \quad K^{ai,bj} = \delta_{ij}(-\delta^{ab}\Box + \partial^a\partial^b).$$

The corresponding Fourier-transformed influence function $\hat{\mathcal{A}}(t,p)$ is given as $\hat{\mathcal{A}}(t,p) = 1_{m\times m} \otimes \hat{\mathcal{A}}'(t,p)$ where $\hat{\mathcal{A}}'(t,p)$ is determined by

$$0 = \partial_t^2\hat{\mathcal{A}}' + (p^2 1_{d\times d} - pp^{\mathrm{T}})\hat{\mathcal{A}}', \quad \hat{\mathcal{A}}'(0) = 0, \quad \partial_t\hat{\mathcal{A}}'(0) = (2\pi)^{-d/2}1_{d\times d}.$$

Explicitly,

$$(2\pi)^{d/2}\hat{\mathcal{A}}'(t,p)^a_c = \frac{\sin|p|\,t}{|p|}\left(\delta^a_c - \frac{p^a p_c}{|p|^2}\right) + t\frac{p^a p_c}{|p|^2}$$

Thus, (1.5.4a) (with $\hat{\mathcal{A}}^\Phi := \hat{\mathcal{A}}$) is violated, since the spatially longitudinal direction is not sufficiently smoothened, and we cannot assign to A the smoothness offset 1, which we need because of the derivative coupling implicitly arising in (3.1.3a). (In another language, this obstackle to a naive approach was observed earlier, cf. e.g. [4].)

In order to get a first-order system which fits into the class considered in [11], we follow Segal's idea and temporarily forget the definition of the magnetic field strengthes (3.2.1), considering them as independent $\mathfrak{g}$-valued antisymmetric tensor field B_{ab} instead. Also, we introduce a new $\mathfrak{g}$-valued scalar field L for the longitudinal component

$$L = \partial^a A_a \tag{3.3.1}$$

(otherwise, the term $(\partial^a A_a)\Phi$ would lead to smoothness difficulties). Thus, we consider the following system in the commuting fields $A_a, E_a, B_{ab}, L, \Phi, \dot{\Phi}$ and the anticommuting fields $\mathrm{Re}\Psi, \mathrm{Im}\Psi$:

(3.3.2a) $\quad \partial_t A_a = E_a,$

(3.3.2b) $\quad \partial_t E_a = \partial_b B_{ba} + [A_b, B_{ba}] + J_a,$

(3.3.2c) $\quad \partial_t B_{ab} = \partial_a E_b - \partial_b E_a + [A_a, E_b] - [A_b, E_a]$ (by (3.1.2)),

(3.3.2d) $\quad \partial_t L = -[A_a, E_a] + J_0$ (by (3.2.2)),

(3.3.2e) $\quad \partial_t \Phi = \dot{\Phi},$

(3.3.2f) $\quad \partial_t \dot{\Phi} = \Delta\Phi + L \times \Phi + 2A_a \times \partial_a \Phi + A_a \times (A_a \times \Phi) + \mathcal{V}_*(\Phi)$ (by (3.1.3c), (3.3.1)),

(3.3.2g) $\quad \partial_t \Psi = -(\gamma^0)^{-1}\left(\gamma^a D_a + m^\Psi\right)\Psi$ (by (3.1.3b)),

where we have abbreviated

$$\mathcal{V}_*(\Phi) := (\mathcal{V}_i(\Phi))_{i=1}^{N^\Phi}, \quad J_0 := -\mathrm{i}\overline{\Psi} \times \gamma_0 \Psi + \Phi \times \dot{\Phi}, \quad J_a := -\mathrm{i}\overline{\Psi} \times \gamma_a \Psi + \Phi \times D_a \Phi.$$

We claim that this system together with the constraints (3.2.1), (3.2.2), (3.3.1) is equivalent with the original system (3.1.3a)–(3.1.3c) together with the constraint $A_0 = 0$:

Lemma 3.3.1. *(i) Let be given a solution family* $(A, \Phi|\Psi)$ *of* (3.1.3a)–(3.1.3c) *which satisfies* $A_0 = 0$. *Then*

$$(A_a, E_a, B_{ab}, L, \Phi, \dot{\Phi}|\ \mathrm{Re}\Psi, \mathrm{Im}\Psi) \tag{3.3.3}$$

where $E_a := \partial_t A_a,\ \ B_{ab} := F_{ab}[0, A_1, \ldots, A_d],\ \ L := \partial^a A_a,$ *and* $\dot{\Phi} := \partial_0 \Phi$, *is a solution family of the system* (3.3.2a)–(3.3.2g) *which satisfies the constraints* (3.2.1), (3.2.2), *and* (3.3.1).

(ii) Conversely, let be given a solution family (3.3.3) *of* (3.3.2a)–(3.3.2g) *which satisfies the constraints* (3.2.1), (3.2.2), *and* (3.3.1) *at* $t = 0$. *Then these constraints will be satisfied at all times, and* $(A, \Phi|\Psi)$ *will be a solution family of* (3.1.3a)–(3.1.3c).

(iii) The system (3.3.2a)–(3.3.2g) *belongs to the class considered in* [10], [11], *with assigning the smoothness offset* 1 *to* Φ_i, *and* 0 *to all remaining field components. Moreover, this system is causal in the sense of* [10], [11].

Proof. Ad (i). This follows from the construction.

Ad (ii). Using (3.3.2c) and (3.1.2), we have

$$\partial_t\left(B_{ab} - F_{ab}[0, A_a]\right) = 0,$$

which implies that (3.2.1) is satisfied for all times. Using (3.3.2b) we find

$$\begin{aligned}\partial_t\left(D^a E_a - J_0\right) &= D^a(D^b B_{ba} + J_a) - \partial_t J_0 = D^\mu J_\mu \\ &= \mathrm{i}\overline{D^\mu \gamma_\mu \Psi} \times \Psi - \mathrm{i}\overline{\Psi} \times D^\mu \gamma_\mu \Psi + D^\mu \Phi \times D_\mu \Phi + \Phi \times D^\mu D_\mu \Phi.\end{aligned}$$

Now (3.3.2f) yields $D^\mu D_\mu \Phi + (L - \partial^a A_a) \times \Phi + \mathcal{V}_*(\Phi) = 0$; on the other hand, infinitesimal gauge invariance of $\mathcal{V}$ is equivalent with $\underline{\Phi} \times \mathcal{V}_*(\underline{\Phi}) = 0$. Since G acts orthogonally on V^Φ we have $(f^\Phi)^l_{ik} = -(f^\Phi)^k_{il}$ and hence $D^\mu\Phi \times D^\mu \Phi = 0$. Investing all this and also (3.3.2g), we get

$$\begin{aligned}\partial_t\left(D^a E_a - J_0\right) &= -\mathrm{i}\overline{m^\Psi \Psi} \times \Psi + \mathrm{i}\overline{\Psi} \times m^\Psi \Psi + \Phi \times \left(\left((\partial^a A_a - L) \times \Phi\right) - \mathcal{V}_*(\Phi)\right) \\ &= \Phi \times \left((\partial^a A_a - L) \times \Phi\right).\end{aligned} \tag{3.3.4}$$

Using (3.3.2d), (3.3.2a) we have

$$\partial_t\left(L - \partial^a A_a\right) = J_0 - D^a E_a, \tag{3.3.5}$$

and hence

$$\partial_t^2\left(L - \partial^a A_a\right) = \Phi \times \left((L - \partial^a A_a) \times \Phi\right). \tag{3.3.6}$$

By hypothesis, $(L - \partial^a A_a)(0) = 0$ and $(J_0 - D^a E_a)(0) = 0$, which by (3.3.5) yields $\partial_t (L - \partial^a A_a)(0) = 0$. Therefore, (3.3.6) implies that (3.3.1) is satisfied for all times. Now (3.3.4) together with the initial data yields that the constraint (3.2.2) is satisfied at all times, too. The remaining field equations are now obviously satisfied.

Ad (iii). For finding the influence function $\hat{\mathcal{A}}$ of the free evolution, we can, of course, consider the sectors separately: $\hat{\mathcal{A}} = \operatorname{diag}(\hat{\mathcal{A}}^{AEB}, \hat{\mathcal{A}}^L, \hat{\mathcal{A}}^\Phi, \hat{\mathcal{A}}^\Psi)$. Obviously, $\hat{\mathcal{A}}^L = (2\pi)^{-d/2}$ is constant (here and below, we do not indicate the Kronecker deltas for the colour indices). $\hat{\mathcal{A}}^\Phi$ is given by (1.5.2) with $m^\Phi_{ij} := \frac{\partial^2}{\partial \underline{\Phi}_i \partial \underline{\Phi}_j} \mathcal{V}(0)/2$, and $\hat{\mathcal{A}}^\Psi = A^\Psi$ is the same as in 2.3. Finally, the entries of

$$\hat{\mathcal{A}}^{AEB} = \begin{pmatrix} (2\pi)^{-d/2}\delta^a_c & \hat{\mathcal{X}}^a_c & \hat{\mathcal{X}}^{ab}_c \\ 0 & \hat{\mathcal{A}}^a_c & \hat{\mathcal{A}}^{ab}_c \\ 0 & \hat{\mathcal{A}}^a_{cd} & \hat{\mathcal{A}}^{ab}_{cd} \end{pmatrix}$$

are determined by the spatially Fourier-transformed free field equations

$$\partial_t \hat{\mathcal{X}}^I_c = \hat{\mathcal{A}}^I_c, \quad \partial_t \hat{\mathcal{A}}^I_c = \mathrm{i} p_d \hat{\mathcal{A}}^I_{dc}, \quad \partial_t \hat{\mathcal{A}}^I_{cd} = \mathrm{i} p_c \hat{\mathcal{A}}^I_d - \mathrm{i} p_d \hat{\mathcal{A}}^I_c,$$

with I standing for a or ab, and the initial values $\hat{\mathcal{A}}^{AEB}(0) = (2\pi)^{-d/2} \operatorname{diag}(\delta^a_c, \delta^a_c, \delta^a_{[c}\delta^b_{d]})$ (here and in the following, we use Bach's antisymmetrizing brackets). By direct verification, we have

(3.3.7)
$$\hat{\mathcal{A}}^{AEB} = (2\pi)^{-d/2} \begin{pmatrix} \delta^a_c & \frac{\sin|p|t}{|p|}\left(\delta^a_c - \frac{p^a p_c}{|p|^2}\right) + t\frac{p^a p_c}{|p|^2} & \mathrm{i}\frac{1-\cos|p|t}{|p|^2} p^{[a}\delta^{b]}_c \\ 0 & \cos|p|t\left(\delta^a_c - \frac{p^a p_c}{|p|^2}\right) + \frac{p^a p_c}{|p|^2} & \mathrm{i}\frac{\sin|p|t}{|p|} p^{[a}\delta^{b]}_c \\ 0 & \mathrm{i}\frac{\sin|p|t}{|p|} p_{[c}\delta^a_{d]} & \delta^a_{[c}\delta^b_{d]} + \frac{\cos|p|t - 1}{|p|^2}\left(p^a p_{[c}\delta^b_{d]} - p^b p_{[c}\delta^a_{d]}\right) \end{pmatrix}$$

(with the continuous extension at $p = 0$ understood). Since each entry remains bounded for $|p| \to \infty$, the smoothness conditions are satisfied.

The causality assertion follows from the Paley-Wiener Theorem. □

Proof of Thm. 3.2.3. Ad (i). This follows from the Lemma and the results of [11].

Ad (ii). Let

$$V^{\mathrm{ext}} := \mathfrak{g} \otimes \left(\mathbb{R}^d \oplus \mathbb{R}^d \oplus \Lambda^2\mathbb{R}^d \oplus \mathbb{R}\right) \oplus V^\Phi \oplus V^\Phi \ \oplus\ \Pi(V^\Psi \otimes_{\mathbb{C}} V^{\mathrm{D}})$$

be the field target space for the extended system (3.3.2a)–(3.3.2g), so that the corresponding smf of smooth Cauchy data is $M^{\mathrm{extCau}}_{C^\infty} = \mathrm{L}(\mathcal{E}^{\mathrm{Cau}} \otimes V^{\mathrm{ext}})$. Consider the smf morphism

$$(3.3.8) \qquad M^{\mathrm{par}}_{C^\infty} \xrightarrow{(A^{\mathrm{Cau}}_a, E^0_1, E^{\mathrm{Cau}}_2, \ldots, E^{\mathrm{Cau}}_d, B^{\mathrm{Cau}}_{ab} := F_{ab}[0, A^{\mathrm{Cau}}_1, \ldots, A^{\mathrm{Cau}}_d], L := \partial^a A^{\mathrm{Cau}}_a, \Phi^{\mathrm{Cau}}, \dot{\Phi}^{\mathrm{Cau}} | \Psi^{\mathrm{Cau}})} M^{\mathrm{extCau}}_{C^\infty}.$$

Let Y denote temporarily the image of its underlying map, i. e. the set of all bosonic Cauchy data

$$(a_c, e_c, b_{cd}, l, \phi, \dot{\phi}) \in \mathcal{E}_{\mathbf{0}}^{\mathrm{V}^{\mathrm{ext}}}$$

which satisfy the constraints

$$b_{cd} = \partial_c a_d - \partial_d a_c - [a_c, a_d], \quad l = \partial^c a_c,$$
$$(3.3.9) \qquad \partial^c e_c = -[a_c, e_c] + \phi \times \dot{\phi}.$$

Now the hypothesis just says that the system (3.3.2a)–(3.3.2g) is $(\mathcal{H}^{\mathrm{V}^{\mathrm{ext}}_k}, Y \cap (\mathcal{E}_c^{\mathrm{V}^{\mathrm{ext}}})_{\mathbf{0}})$-complete, where we use the Sobolev space $\mathcal{H}^{\mathrm{V}^{\mathrm{ext}}_k}$ as defined in [10], [11].

On the other hand, in order to apply [10, Thm. 3.4.5], we have to show that each bosonic Cauchy datum from Y is approximable in the sense of [10, 3.4], i.e. for each ball $n\mathbb{B}$ (where $\mathbb{B} := \{x \in \mathbb{R}^d : |x| \le 1\}$) there exist bosonic Cauchy data $(a'_c, e'_c, b'_{cd}, l', \phi', \dot{\phi}') \in Y \cap (\mathcal{E}_c^{\mathrm{V}^{\mathrm{ext}}})_{\mathbf{0}}$ which coincide on $n\mathbb{B}$ with the Cauchy data given, and which have infinite lifetime. Indeed, this follows easily from the hypothesis and the following Lemma:

Lemma 3.3.2. *Suppose $d \ge 2$, and let be given C^∞ bosonic initial data $(a, e, \phi, \dot{\phi}) \in \mathcal{E}_{\mathbf{0}}^{\mathrm{V}}$ which satisfy the constraint* (3.3.9)*. For given $n > 0$, there exist compactly supported bosonic initial data*

$$(3.3.10) \qquad (a', e', \phi', \dot{\phi}') \in (\mathcal{E}_c^{\mathrm{V}})_{\mathbf{0}}$$

which satisfy

$$(a', e', \phi', \dot\phi')|_{n\mathbb{B}} = (a, e, \phi, \dot\phi)|_{n\mathbb{B}}, \tag{3.3.11}$$

$$\partial^c e'_c = -[a'_c, e'_c] + \phi' \times \dot\phi'. \tag{3.3.12}$$

Proof. Choose a buffer function $g \in C^\infty(\mathbb{R}^d)$ such that $g|_{n\mathbb{B}} = 1$ and $g|_{\mathbb{R}^d\setminus(n+1)\mathbb{B}} = 0$. The idea is to get rid of the charge $(-1)^c \int_{n\mathbb{S}} dx_1 \ldots dx_{c-1} dx_{c+1} \ldots dx_d g e_c$ by adding the opposite charge on a of the ball $n\mathbb{B}$. Let $p := (0, 3n, 0, \ldots, 0) \in \mathbb{R}^d$, and let

$$a'(x) := (ga)(x) + (ga)(x-p),$$
$$e"_2(x) := (ge_2)(x) - (ge_2)(x-p), \qquad e'_c(x) := (ge_c)(x) - (ge_c)(x-p) \quad \text{for } c = 3, \ldots, d,$$
$$\phi'(x) := (g\phi)(x) - (g\phi)(x-p)), \qquad \dot\phi'(x) := (g\dot\phi)(x) + (g\dot\phi)(x-p).$$

In order to determine e'_1, e'_2, choose another buffer function $h \in C^\infty(\mathbb{R})$ with $h|_{[-n-1,n+1]} = 1$, $h|_{\mathbb{R}\setminus[-n-2,n+2]} = 0$. Let $f \in C^\infty(\mathbb{R}^d) \otimes V^{\mathfrak{g}}$ be the solution of the ODE

$$\partial_1 f = -[a'_1, f] - \partial_2 e"_2 - [a'_2, e"_2] - \sum_{c=3}^{d} (\partial_c e'_c + [a'_c, e'_c]) + \phi' \times \dot\phi' \tag{3.3.13}$$

with the initial data ($x = (x_1, x_2, \underline{x})$)

$$f(0, x_2, \underline{x}) := (ge_1)(0, x_2, \underline{x}) - (ge_1)(0, x_2 - 3n, \underline{x})$$

and let $e'_1(x) := h(x_1) f(x)$. Let

$$e'_2(x) := e"_2(x) - \partial_1 h(x_1) \int_{-\infty}^{x_2} d\xi_2 f(x_1, \xi_2, \underline{x}) \tag{3.3.14}$$

Except for the component e'_1, the validity of (3.3.11) is clear by construction, while $(e'_1 - e_1)|_{n\mathbb{B}} = 0$ follows since e'_1, e_1 satisfy on $n\mathbb{B}$ the same ordinary differential equation with the same initial data; this proves (3.3.11).

With a similar argument, we have $f(x) = -f(x-p)$ for $x_2 \in [2n-1, 4n+1]$. Note also that $f(x) = 0$ if $x \in \mathbb{R} \setminus ([-n-1, n+1] \cup [2n-1, 4n+1])$, so the integral in (3.3.14) is well-defined, and becomes zero for $x_2 \in \mathbb{R} \setminus [-n-1, 4n+1]$. This guarantees that e'_2 has compact support, and (3.3.10) is now easily verified.

It remains to show (3.3.12). For $x_1 \in [-n-1, n+1]$ we have $e'_1 = f$ and $e'_2 = e"_2$, so that (3.3.13) yields (3.3.12). For $x_1 \in \mathbb{R} \setminus [-n-1, n+1]$, (3.3.12) simplifies to $\partial_1 e'_1 + \partial_2 e'_2 = 0$, which is fulfilled by construction. □

By [10, Thm. 3.4.5], there now exists a solution family

$$\Xi^{\mathrm{extsol}} = (A_a^{\mathrm{extsol}}, E_a^{\mathrm{extsol}}, B_{ab}^{\mathrm{extsol}}, L^{\mathrm{extsol}}, \Phi^{\mathrm{extsol}}, \dot\Phi^{\mathrm{extsol}} | \Psi^{\mathrm{extsol}})$$

parametrized by $M_{C^\infty}^{\mathrm{par}}$ which has (3.3.8) as its Cauchy data. Now $(A_a^{\mathrm{extsol}}, \Phi^{\mathrm{extsol}} | \Psi^{\mathrm{extsol}}) \in \mathcal{M}^{\mathcal{E}^{\mathrm{V}}}(M_{C^\infty}^{\mathrm{par}})$ will be the universal solution family wanted. For showing that its image will be a sub-smf, the proof of [11, Thm. 2.4.1] carries over.

Ad (iii). This follows from the results of Ginebre/Velo [5] and Eardley/Moncrief [4]. □

Remarks. (1) Cf. also [3], where, however, the global existence is proved only for sufficiently small Cauchy data and coupling constants.

(2) In $d = 1$, Lemma 3.3.2 may become wrong (take e.g. $G = U(1)$ with $V^\Phi = 0$; then the constraint is simply $\partial_1 e_1 = 0$). The best we can say is:

Suppose that there exists some $k > 1/2$ such that for every $n > 0$ and every choice of smooth bosonic Cauchy data $\xi = (a_1, e_1, \phi, \dot\phi)$ which satisfy the constraint $D^1 e_1 = \phi \times \dot\phi$, there exists a bosonic solution $(a'_1, \phi) \in C^0((-n, n),\ H_{k+1}(\mathbb{R}) \otimes (\mathfrak{g} \oplus V^\Phi)) \cap C^1((-n, n),\ H_k(\mathbb{R}) \otimes (\mathfrak{g} \oplus V^\Phi))$ of

$$D^1 e_1 = \phi' \times \partial_0 \phi', \quad \partial^0 e_1 = \phi' \times D_1 \phi', \quad D^\mu D_\mu \phi'_i + \mathcal{V}_i(\phi') = 0,$$

the Cauchy data of which agree with ξ on the ball $n\mathbb{B}$. Then the assertions of Thm. 3.2.3.(ii).(1-3) are valid.

4. Yang-Mills-Dirac-Higgs models in the Faddeev-Popov approach

4.1. **The Lagrangian.** The approach of explicitly stating the gauge condition like $A_0 = 0$ is known to have some defects, like the uncertainty whether the results will be still Poincaré invariant. In fact, in almost all calculations of experimentally verifiable predictions, one introduces a gauge symmetry breaking term, and Faddeev-Popov ghosts. That is, one introduces new fermionic fields Υ_i, Λ_i (*Faddeev-Popov ghost and antighost fields*) with $i = 1, \ldots, N^{\mathfrak{g}}$ which transform under the adjoint representation of the gauge group and are scalars with respect to the Poincaré group (so they are the "ghost drivers" with respect to the Pauli Theorem on the connection between spin and statistics). Thus

$$D^\mu \Upsilon = \partial^\mu \Upsilon + [A^\mu, \Upsilon], \quad D^\mu \Lambda = \partial^\mu \Lambda + [A^\mu, \Lambda].$$

One now works with the Lagrangian

$$\mathcal{L}^{\text{YMDH-gb}}[\underline{A}, \underline{\Phi}|\underline{\Lambda}, \underline{\Upsilon}, \underline{\Psi}] := \mathcal{L}^{\text{YMDH}}[\underline{A}, \underline{\Phi}|\underline{\Psi}] - \frac{\zeta}{2}\Big(\partial^\mu \underline{A}_\mu\Big)^2 + \mathrm{i}\partial_\mu \underline{\Lambda}_i D^\mu \underline{\Upsilon}_i \tag{4.1.1}$$

(cf. (3.1.1)) where the parameter $\zeta \in \mathbb{R} \setminus 0$ is arbitrary.

The field equations resulting from (4.1.1) are

$$D^\mu F_{\mu\nu} + \zeta \partial_\nu(\partial^\mu A_\mu) + J_\nu = 0, \tag{4.1.2a}$$

$$\partial^\mu D_\mu \Upsilon = 0, \tag{4.1.2b}$$

$$D_\mu \partial^\mu \Lambda = 0, \tag{4.1.2c}$$

$$\left(\gamma^\mu D_\mu + m^\Psi\right)\Psi = 0, \tag{4.1.2d}$$

$$D^\mu D_\mu \Phi + \mathcal{V}_*(\Phi) = 0, \tag{4.1.2e}$$

with the matter current now given by

$$J_\mu = \frac{\delta}{\delta \underline{A}^\mu}\left(\partial_\nu \underline{\Lambda}_i D^\nu \underline{\Upsilon}_i + \mathcal{L}^{\text{DH}}\right)[A, \Phi|\Psi] = -\mathrm{i}\,[\partial_\mu \Lambda, \Upsilon] + \Phi \times D_\mu \Phi - \mathrm{i}\overline{\Psi} \times \gamma_\mu \Psi. \tag{4.1.3}$$

In this approach, all the difficulties connected with constraints disappear: The smfs of Cauchy data and the smf of the configuration are given by

$$\begin{aligned} M^{\text{Cau}} :=& \mathrm{L}(\mathcal{E}_{\text{c}}^{\text{Cau}} \otimes \mathbb{R}^{d+1} \otimes \mathfrak{g})_{A_0^{\text{Cau}}, \ldots, A_d^{\text{Cau}}} \times \mathrm{L}(\mathcal{E}_{\text{c}}^{\text{Cau}} \otimes \mathbb{R}^{d+1} \otimes \mathfrak{g})_{\dot{A}_0^{\text{Cau}}, \ldots, \dot{A}_d^{\text{Cau}}} \times \\ & \times \mathrm{L}(\mathcal{E}_{\text{c}}^{\text{Cau}} \otimes (V^\Phi \oplus V^\Phi))_{\Phi^{\text{Cau}}, \dot{\Phi}^{\text{Cau}}} \times \mathrm{L}(\mathcal{E}_{\text{c}}^{\text{Cau}} \otimes \Pi(\mathfrak{g} \oplus \mathfrak{g} \oplus \mathfrak{g} \oplus \mathfrak{g}))_{\Lambda^{\text{Cau}}, \dot{\Lambda}^{\text{Cau}}, \Upsilon^{\text{Cau}}, \dot{\Upsilon}^{\text{Cau}}} \times \\ & \times \mathrm{L}(\mathcal{E}_{\text{c}}^{\text{Cau}} \otimes \Pi(V^\Psi \otimes_{\mathbb{C}} V^{\text{D}}))_{\Psi^{\text{Cau}}}, \\ M :=& \mathrm{L}(\mathcal{E}_{\text{c}}^{\text{V}}) = \mathrm{L}(\mathcal{E}_{\text{c}} \otimes \mathbb{R}^{d+1} \otimes \mathfrak{g})_{A, \ldots, A_d} \times \mathrm{L}(\mathcal{E}_{\text{c}} \otimes V^\Phi)_{\Phi^{\text{Cau}}} \times \\ & \times \mathrm{L}(\mathcal{E}_{\text{c}} \otimes \Pi(\mathfrak{g} \oplus \mathfrak{g}))_{\Lambda, \Upsilon} \times \mathrm{L}(\mathcal{E}_{\text{c}} \otimes \Pi(V^\Psi \otimes_{\mathbb{C}} V^{\text{D}}))_{\Psi^{\text{Cau}}}. \end{aligned}$$

Remarks. (1) The resulting description of the configuration space of such a model as supermanifold was first given in [1] (although in another technical setting).

(2) Let us comment about the imaginary unit in front of the ghost term, which is absent in most textbook presentations, but which in our approach is necessary in order to ensure the reality of $\mathcal{L}^{\text{YMDH-gb}}$. In fact, there is some confusion in literature about whether Λ and Υ should be mutually conjugated complex fields or independent real (or, in textbook language, hermitian) fields. One can work with either of both interpretations since if $\Delta(A)$ denotes the Faddeev-Popov determinant the formal path integral representation

$$\Delta(A) = \int [d\Lambda][d\Upsilon] \exp\Big(\mathrm{i}\int d^{d+1}x \partial_\mu \Lambda_i(x) D^\mu \Upsilon_i(x)\Big)$$

is valid for both interpretations. However, the next step in the textbooks is to include the term $\partial_\mu \underline{\Lambda}_i D^\mu \underline{\Upsilon}_i$ into the Lagrangian, without bothering about its reality.

Here we settle for treating Λ and Υ as independent real fields. The additional i introduces an inessential constant factor i^∞ into the functional integral. Also, the perturbation theory remains essentially unchanged: both the $\partial_\mu \Lambda_i A^\mu \Upsilon_i$ vertex and the kinetic operators for Λ, Υ get an additional factor i, and hence the Green functions of Λ, Υ get a factor $1/\mathrm{i}$. Since ghosts appear only in closed loops, these additional factors cancel away. (Cf. also [2] as one of the very few exceptions in the textbook literature; this book considers also the canonical quantization of the ghosts and therefore uses the factor i, too.)

4.2. **The main result.**

Theorem 4.2.1. (i) *Let be given a point* $p = (s, x) \in \mathbb{R}^{d+1}$ *with* $s \neq 0$ *and two* Z*-families* $(A', \Phi' \,|\, \Lambda', \Upsilon', \Psi'), (A", \Phi" \,|\, \Lambda", \Upsilon", \Psi")$. *Suppose that*

$$\left(D^\mu_{A'} F_{\mu\nu}[A'] + \zeta\partial_\nu(\partial^\mu A'_\mu) + J_\nu[A', \Phi'|\Psi']\right)|_{\Omega(p)} = 0, \qquad (\partial_\mu D^\mu_{A'}\Upsilon')|_{\Omega(p)} = (D^\mu_{A'}\partial_\mu\Lambda')|_{\Omega(p)} = 0,$$
$$\left(D^\mu_{A'}(D_{A'})_\mu \Phi'_i + \mathcal{V}_i(\Phi')\right)|_{\Omega(p)} = 0, \qquad \left(\gamma^\mu(D_{A'})_\mu + m^\Psi\right)\Psi'|_{\Omega(p)} = 0,$$

and analogously for $(A", \Phi" \,|\, \Lambda", \Upsilon", \Psi")$, *and that, setting* $\Delta = (A' - A", \Phi' - \Phi" \,|\, \Lambda' - \Lambda", \Upsilon' - \Upsilon")$, *we have*

$$\Delta(0)|_{\mathcal{J}(p)} = 0, \quad \partial_0\Delta(0)|_{\mathcal{J}(p)} = 0, \quad (\Psi' - \Psi")(0)|_{\mathcal{J}(p)} = 0.$$

Then $(A' - A", \Phi' - \Phi" \,|\, \Lambda' - \Lambda", \Upsilon' - \Upsilon", \Psi' - \Psi")\,|_{\Omega(p)} = 0$.

(ii) *Suppose there exists a Sobolev index* $k > d/2$ *such that for every bosonic solution* $(a, \phi) \in C^\infty((t_0, t_1) \times \mathbb{R}^d) \otimes (\mathfrak{g} \otimes \mathbb{R}^{d+1} \oplus V^\Phi)$ *of*

$$D^\mu F_{\mu\nu}[a] + \zeta\partial_\nu(\partial^\mu a_\mu) + \phi \times D_\nu\phi = 0, \quad D^\mu D_\mu\phi + \mathcal{V}_*(\phi) = 0,$$

on an open time interval (t_0, t_1) *such that* $\operatorname{supp}(a, \phi)(t)$ *is compact for some (and hence all) all* $t \in (t_0, t_1)$, *we have*

$$\sup_{t \in (t_0, t_1)} \max\left\{\|a^i_\mu(t)\|_{H_{k+1}(\mathbb{R}^d)},\ \|\phi_j(t)\|_{H_{k+1}(\mathbb{R}^d)},\ \|\partial_t a^i_\mu(t)\|_{H_k(\mathbb{R}^d)},\ \|\partial_t\phi_j(t)\|_{H_k(\mathbb{R}^d)}\right\} < \infty$$

where $i = 1, \ldots, N^{\mathfrak{g}}$, $\mu = 0, \ldots, d$, $j = 1, \ldots, N^\Phi$. *Then the following assertions are valid.*

(1) *There exists a unique solution family*

$$\begin{aligned} \Xi^{\mathrm{sol}} &= (A^{\mathrm{sol}}, \Phi^{\mathrm{sol}} \,|\, \Lambda^{\mathrm{sol}}, \Upsilon^{\mathrm{sol}}, \Psi^{\mathrm{sol}}) \\ &= \Xi^{\mathrm{sol}}[A^{\mathrm{Cau}}, \dot{A}^{\mathrm{Cau}}, \Phi^{\mathrm{Cau}}, \dot{\Phi}^{\mathrm{Cau}} \,|\, \Lambda^{\mathrm{Cau}}, \dot{\Lambda}^{\mathrm{Cau}}, \Upsilon^{\mathrm{Cau}}, \dot{\Upsilon}^{\mathrm{Cau}}, \Psi^{\mathrm{Cau}}] \in \mathcal{M}^{\mathcal{E}^{\mathrm{V}}}(M^{\mathrm{Cau}}_{C^\infty}) \end{aligned} \tag{4.2.1}$$

of (4.1.2a)–(4.1.2e) *with the initial data*

$$\begin{aligned} &(A^{\mathrm{sol}}, \partial_t A^{\mathrm{sol}}, \Phi^{\mathrm{sol}}, \partial_t\Phi^{\mathrm{sol}} \,|\, \Lambda^{\mathrm{sol}}, \partial_t\Lambda^{\mathrm{sol}}, \Upsilon^{\mathrm{sol}}, \partial_t\Upsilon^{\mathrm{sol}}, \Psi^{\mathrm{sol}})(0) \\ =\ &(A^{\mathrm{Cau}}, \dot{A}^{\mathrm{Cau}}, \Phi^{\mathrm{Cau}}, \dot{\Phi}^{\mathrm{Cau}} \,|\, \Lambda^{\mathrm{Cau}}, \dot{\Lambda}^{\mathrm{Cau}}, \Upsilon^{\mathrm{Cau}}, \dot{\Upsilon}^{\mathrm{Cau}}, \Psi^{\mathrm{Cau}}). \end{aligned}$$

(2) *This solution family is universal for the quality* $\mathcal{E}$, *i. e. any other smooth solution family* $(A', \Phi'|\Lambda', \Upsilon', \Psi') \in \mathcal{M}^{\mathcal{E}^{\mathrm{V}}}(Z)$ *of* (4.1.2a)–(4.1.2e) *is in a unique way a pullback of* (4.2.1).

(3) *The image of* $M^{\mathrm{sol}}_{C^\infty} \subseteq M_{C^\infty}$ *is a sub-smf.*

(4) *Assume that* $\phi \in V^\Phi$ *satisfies* $\mathcal{V}_i(\phi) = 0$ *for all* i *(and hence that* $(A, \Phi) := (0, \phi)$ *is a solution of the underlying bosonic equations). Then the superfunctional*

$$\begin{aligned} &\Xi^{\mathrm{exc}}_\phi[A^{\mathrm{Cau}}, \dot{A}^{\mathrm{Cau}}, \Phi^{\mathrm{Cau}}, \dot{\Phi}^{\mathrm{Cau}} \,|\, \Lambda^{\mathrm{Cau}}, \dot{\Lambda}^{\mathrm{Cau}}, \Upsilon^{\mathrm{Cau}}, \dot{\Upsilon}^{\mathrm{Cau}}, \Psi^{\mathrm{Cau}}] \\ &:= \Xi^{\mathrm{sol}}[A^{\mathrm{Cau}}, \dot{A}^{\mathrm{Cau}}, \Phi^{\mathrm{Cau}} + \phi^{\mathrm{Cau}}, \dot{\Phi}^{\mathrm{Cau}} \,|\, \Lambda^{\mathrm{Cau}}, \dot{\Lambda}^{\mathrm{Cau}}, \Upsilon^{\mathrm{Cau}}, \dot{\Upsilon}^{\mathrm{Cau}}, \Psi^{\mathrm{Cau}}] \end{aligned}$$

which lies a priori in $\mathcal{M}^{\mathcal{E}^{\mathrm{V}}}(M^{\mathrm{Cau}}_{C^\infty})$, *restricts to a superfunctional*

$$\Xi^{\mathrm{exc}}_\phi \in \mathcal{M}^{\mathcal{E}^{\mathrm{V}}_c}(M^{\mathrm{Cau}}).$$

Moreover, the image of the arising smf morphism $\Xi^{\mathrm{exc}}_\phi : M^{\mathrm{Cau}} \to M$ *is again a split sub-smf* $M^{\mathrm{exc}}_\phi \subseteq M$ *called the* smf of excitations around ϕ. *It has the following universal property: Given a* Z*-family* $\Xi' := (A', \Phi'|\Lambda', \Upsilon', \Psi') \in \mathcal{M}^{\mathcal{E}^{\mathrm{V}}_c}(Z)$, *the corresponding morphism* $\Xi' : Z \to M$ *factors through* M^{exc}_ϕ *iff the* Z*-family* $(A', \Phi' + \phi|\Lambda', \Upsilon', \Psi') \in \mathcal{M}^{\mathcal{E}^{\mathrm{V}}}(Z)$ *is a solution family.*

Remarks. (1) The simplest case in which the Theorem applies is the abelian case, i. e. that of quantum electrodynamics (where, however, the ghost field is decoupled from the rest of the system). Here $N^\Phi = 0$, and hence the underlying bosonic theory is free, so that the hypothesis of (ii) is obviously satisfied.

(2) Unfortunately, this Theorem hangs for the non-abelian case somewhat in the air, because, at the time being, there is no result which assures the validity of the hypothesis of (ii). Indeed, the gauge breaking term extends the dynamics of the underlying bosonic model, so that the results of Ginebre/Velo and Eardley/Moncrief can no longer be applied.

Clearly, a bosonic configuration $(a, \phi) \in \mathrm{space}(M)$ which satisfies the Lorentz gauge condition

$$\partial^\mu a_\mu = 0 \tag{4.2.2}$$

is a solution of the underlying bosonic equations of (4.1.1) iff it is a solution of the unmodified Yang-Mills equation

$$D^\mu F_{\mu\nu}[a] + \phi \times D_\mu \phi = 0; \tag{4.2.3}$$

and, conversely, given a solution of the latter, one can achieve by a gauge transformation that the constraint (4.2.2) is fulfilled. However, a solution of the field equations of $\mathcal{L}^{\text{YM-GB}}[\underline{A}]$ which does not satisfy (4.2.2), does not solve (4.2.3). Note also that the energy belonging to (4.1.1) is not positively defined, since $(\partial_\mu A_0)^2$ enters the Lagrangian, and hence the energy, with the "wrong" sign. Thus, there is no guaranty even for boundedness of the H_1 norms.

Thus, in the physically most interesting case, the completeness question is still open. Once it is settled positively, our results will immediately apply to the standard model, which unifies the Salam-Weinberg electroweak model with quantum chromodynamics.

(3) Of course, the hypothesis of (4) on ϕ is always satisfied for $\phi := 0$. However, in models with spontanuous gauge breaking, it is from the beginning more senseful to consider classical solutions around a minumum of the potential $\mathcal{V}$.

(4) In the diagonal gauge, $\zeta = 1$, we have modulo a total derivative

$$L^{\text{kin}}[\underline{A}] \equiv -\frac{1}{2}\partial_\mu \underline{A}_\nu \partial^\mu \underline{A}^\nu,$$

i. e. it describes dm scalar fields, and we may use a straightforward reduction to first order, with ascribing to the A_μ the smoothness offsets 1. Unfortunately, this ceases to be true for arbitrary $\zeta \neq 0$, and we face a situation analogous to that arising in the temporal gauge: The kinetic operator arising from (4.1.1) is

$$K^{\mu\nu} = \delta^{\mu\nu}\Box + (\zeta - 1)\partial^\mu \partial^\nu.$$

Setting $p := (0, p_1, \ldots, p_d)^{\mathrm{T}}$, $e := (1, 0, \ldots, 0)^{\mathrm{T}}$ and

$$\alpha = \mathrm{i}(\zeta - 1)(\frac{1}{\zeta}ep^{\mathrm{T}} - pe^{\mathrm{T}}), \quad \beta = p^2 \cdot 1_{(d+1)\times(d+1)} + p^2\frac{1-\zeta}{\zeta}ee^{\mathrm{T}} + (\zeta - 1)pp^{\mathrm{T}},$$

the influence function $\mathcal{A}$ is given by $\hat{\mathcal{A}}(t,p) = (2\pi)^{-d/2} 1_{m\times m} \otimes \hat{\mathcal{A}}(t,p)$ where $\hat{\mathcal{A}}(t,p)$ is determined by

$$\frac{d^2}{dt^2}\hat{\mathcal{A}}(t,p) + \alpha\frac{d}{dt}\hat{\mathcal{A}}(t,p) + \beta\hat{\mathcal{A}}(t,p) = 0, \qquad \hat{\mathcal{A}}(0,p) = 0, \quad \frac{d}{dt}\hat{\mathcal{A}}(0,p) = 1_{(d+1)\times(d+1)}.$$

The solution is given by

$$\hat{\mathcal{A}}(t,p) = g_0(t,p)(1 - ee^{\mathrm{T}} - \frac{1}{p^2}pp^{\mathrm{T}}) + g_{ee}(t,p)ee^{\mathrm{T}} + \frac{g_{ep}(t,p)}{|p|}(ep^{\mathrm{T}} - \frac{1}{\zeta}pe^{\mathrm{T}}) + \frac{g_{pp}(t,p)}{p^2}pp^{\mathrm{T}}$$

with

$$g_0(t,p) = \frac{\sin(|p|\,t)}{|p|}, \qquad g_{ee}(t,p) = \frac{(1-\zeta)}{2}t\cos(|p|\,t) + \frac{(1+\zeta)}{2\,|p|}\sin(|p|\,t),$$

$$g_{ep}(t,p) = \frac{\mathrm{i}(\zeta-1)}{2}t\sin(|p|\,t), \qquad g_{pp}(t,p) = \frac{\zeta - 1}{2\zeta}t\cos(|p|\,t) + \frac{\zeta+1}{2\zeta\,|p|}\sin(|p|\,t).$$

Unfortunately, the terms proportional to $\zeta - 1$ spoil the decay of $\hat{\mathcal{A}}(t,p)$ with $p \to \infty$, i. e. we cannot assign to the A_μ the smoothness degree 1. Therefore, we will have to use a more complicated reduction to first order, as in (3.3).

4.3. **The first-order system.** Besides the magnetic field strengthes (3.2.1), we will use the electric field strengthes

$$E_a = \partial_0 A_a - \partial_a A_0 + [A_0, A_a]\,. \tag{4.3.1}$$

Introducing also the longitudinal component

$$L = \partial^\mu A_\mu, \tag{4.3.2}$$

(4.1.2a) takes the form

$$-\,D^a E_a + \zeta \partial_0 L + J_0 = 0, \tag{4.3.3}$$

$$D^0 E_b + D^a B_{ab} + \zeta \partial_b L + J_b = 0. \tag{4.3.4}$$

The computation of the covariant divergence of the matter current is similar as in the derivation of (3.3.4), using this time the matter field equations (4.1.2c), (4.1.2d), (4.1.2e); we find for any solution family

$$D^\mu J_\mu = -\mathrm{i}\,[D^\mu \partial_\mu \Lambda, \Upsilon] - \mathrm{i}\,[\partial_\mu \Lambda, D^\mu \Upsilon] = -\mathrm{i}\,[\partial_\mu \Lambda, D^\mu \Upsilon]\,. \tag{4.3.5}$$

Taking the covariant derivative D^ν of (4.1.2a) we get, using (4.3.5), an evolution equation for L:

$$\zeta D^\mu \partial_\mu L = \mathrm{i}\,[\partial_\mu \Lambda, D^\mu \Upsilon]\,. \tag{4.3.6}$$

As in (3.3), we forget the definition (3.2.1) of the magnetic field strengthes and consider them as independent antisymmetric tensor field B_{ab}. Also, we will use for convenience the E_a instead of $\dot{A}_a$. Finally, we introduce again a new $\mathfrak{g}$-valued scalar field L for the longitudinal component. Thus, we consider the following system in the commuting fields $A_0, \dot{A}_0, A_a, E_a, B_{ab}, L, \dot{L}, \Phi, \dot{\Phi}$ and the anticommuting fields $\Lambda, \dot{\Lambda}, \Upsilon, \dot{\Upsilon}, \mathrm{Re}\Psi, \mathrm{Im}\Psi$:

$$\partial_t A_0 = \dot{A}_0, \tag{4.3.7a}$$

$$\begin{aligned}\partial_t \dot{A}_0 = {}& \Delta A_0 + (\zeta - 1)\dot{L} - \left[A_0, L + \dot{A}_0\right] - [\partial_a A_0, A_a] \\ & - [A_a, E_a] + J_0 \qquad \text{(see below)},\end{aligned} \tag{4.3.7b}$$

$$\partial_t A_a = E_a + \partial_a A_0 - [A_0, A_a] \qquad \text{(by (4.3.1))}, \tag{4.3.7c}$$

$$\partial_t E_a = \partial_b B_{ba} - [A_0, E_a] + [A_b, B_{ba}] + \zeta \partial_a L + J_a \qquad \text{(by (4.3.4), (4.1.3))}, \tag{4.3.7d}$$

$$\partial_t B_{ab} = \partial_a E_b - \partial_b E_a - [A_0, B_{ab}] + [A_a, E_b] - [A_b, E_a] \qquad \text{(Bianchi id.)}, \tag{4.3.7e}$$

$$\partial_t L = \dot{L}, \tag{4.3.7f}$$

$$\begin{aligned}\partial_t \dot{L} = {}& \Delta L + [A_a, \partial_a L] - \left[A_0, \dot{L}\right] \\ & + \frac{\mathrm{i}}{\zeta}\left(\left[\dot{\Lambda}, \dot{\Upsilon} + [A_0, \Upsilon]\right] - [\partial_a \Lambda, \partial_a \Upsilon + [A_a, \Upsilon]]\right) \qquad \text{(by (4.3.6))},\end{aligned} \tag{4.3.7g}$$

$$\partial_t \Phi = \dot{\Phi}, \tag{4.3.7h}$$

$$\begin{aligned}\partial_t \dot{\Phi} = {}& \Delta \Phi + L \times \Phi - 2A_0 \times \dot{\Phi} + 2A_a \times \partial_a \Phi \\ & - A_0 \times (A_0 \times \Phi) + A_a \times (A_a \times \Phi) + \mathcal{V}_*(\Phi) \qquad \text{(by (4.1.2e))},\end{aligned} \tag{4.3.7i}$$

$$\partial_t \Lambda = \dot{\Lambda}, \tag{4.3.7j}$$

$$\partial_t \dot{\Lambda} = \Delta \Lambda + [A_a, \partial_a \Lambda] - \left[A_0, \dot{\Lambda}\right] \qquad \text{(by (4.1.2c))}, \tag{4.3.7k}$$

$$\partial_t \Upsilon = \dot{\Upsilon}, \tag{4.3.7l}$$

$$\partial_t \dot{\Upsilon} = \Delta \Upsilon + [L, \Upsilon] - \left[A_0, \dot{\Upsilon}\right] + [A_a, \partial_a \Upsilon] \qquad \text{(by (4.1.2b))}, \tag{4.3.7m}$$

$$\partial_t \Psi = -(\gamma^0)^{-1}\left(\gamma^a \partial_a \Psi + \gamma^a A_a \times \Psi + m^\Psi \Psi\right) - A_0 \times \Psi \qquad \text{(by (3.1.3b))}. \tag{4.3.7n}$$

where we have abbreviated

$$\begin{aligned} J_0 &:= -\mathrm{i}\left[\dot{\Lambda}, \Upsilon\right] + \Phi \times (\dot{\Phi} + A_0 \times \Phi) - \mathrm{i}\overline{\Psi} \times \gamma_0 \Psi, \\ J_a &:= -\mathrm{i}\,[\partial_a \Lambda, \Upsilon] + \Phi \times (\partial_a \Phi + A_a \times \Phi) - \mathrm{i}\overline{\Psi} \times \gamma_a \Psi. \end{aligned}$$

For the derivation of (4.3.7b) we note that (4.3.3) yields

$$-\partial^a(\partial_0 A_a - \partial_a A_0 + [A_0, A_a]) - [A^a, E_a] + \zeta\partial_0 L + J_0 = 0,$$
$$-\partial_0\partial^a A_a + \Delta A_0 - [A_0, \partial^a A_a] - [\partial^a A_0, A_a] - [A^a, E_a] + \zeta\partial_0 L + J_0 = 0,$$
$$-D_0(L + \partial_0 A_0) + \Delta A_0 - [\partial^a A_0, A_a] - [A^a, E_a] + \zeta\partial_0 L + J_0 = 0,$$

which now rewrites to (4.3.7b).

From (4.3.2), (4.3.3) we get constraints for $L, \dot{L}$:

$$L = -\dot{A}_0 + \partial_a A_a, \tag{4.3.8}$$

$$\dot{L} = \frac{1}{\zeta}(\partial_a E_a + [A_a, E_a] - J_0) \tag{4.3.9}$$

(note that trying to use the latter relation as evolution equation would result in smoothness problems).

Now the system (4.3.7a)–(4.3.7n) together with the constraints (3.2.1), (4.3.8), (4.3.9) is equivalent with the original system (4.1.2a)–(4.1.2e):

Lemma 4.3.1. *(i) A configuration family* $(A, \Phi|\Lambda, \Upsilon, \Psi)$ *is a solution family of* (4.1.2a)–(4.1.2e) *iff*

$$(A_0, \partial_t A_0, A_a, E_a, B_{ab}, L, \partial_t L, \Phi, \partial_t\Phi|\Lambda, \partial_t\Lambda, \Upsilon, \partial_t\Upsilon, \mathrm{Re}\Psi, \mathrm{Im}\Psi) \tag{4.3.10}$$

where E_a, B_{ab}, *and* L *are defined by* (4.3.1), (3.2.1), *and* (4.3.2), *respectively, is a solution family of the system* (4.3.7a)–(4.3.7n). *In that case, the constraints* (4.3.8), (4.3.9) *are satisfied.*

(ii) Conversely, let given a solution family (4.3.10) *of* (4.3.7a)–(4.3.7n) *which satisfies the constraints* (3.2.1), (4.3.8), (4.3.9) *at* $t = 0$. *Then these are satisfied at all times,* (4.3.2) *holds, and* $(A, \Phi|\Upsilon, \Psi)$ *will be a solution family of* (4.1.2a)–(4.1.2e).

(iii) The system (4.3.7a)–(4.3.7n) *belongs to the class considered in* [10], [11], *with assigning the smoothness offset* 1 *to* A_0, Υ, Φ, L, *and* 0 *to all other components. Moreover, the system* (4.3.7a)–(4.3.7n) *is causal in the sense of* [10], [11].

Remark. The Lemma and the results of [11] yield a short-time solvability result for (4.1.2a)–(4.1.2e).

Proof. Ad (i). This is clear by construction. Ad (ii). (4.1.2d) is clear. (4.3.7c) shows that (4.3.1) is satisfied. Now the Bianchi identity (3.1.2) yields

$$\partial_t(B_{ab} - F_{ab}[A]) = [A_0, B_{ab} - F_{ab}[A]].$$

Therefore, the vanishing of the initial value implies $B_{ab} = F_{ab}[A]$, i. e. the validity of (3.2.1).

Now we have to reconstruct L. Setting for shortness $\lambda := \partial_a A_a - \dot{A}_0 - L$, we get from (4.3.7b) and (4.3.7c)

$$\begin{aligned}\partial_t\lambda &= \partial_a(E_a + \partial_a A_0 - [A_0, A_a]) - \Delta A_0 - \zeta\dot{L} + \left[A_0, L + \dot{A}_0\right] + [\partial_a A_0, A_a] + [A_a, E_a] - J_0 \\ &= \partial_a E_a + [A_0, \partial_a A_a] - \zeta\dot{L} + \left[A_0, L + \dot{A}_0\right] + [A_a, E_a] - J_0\end{aligned}$$

i. e.

$$D_0\lambda = D_a E_a - \zeta\dot{L} - J_0 \tag{4.3.11}$$

(essentially, this is the derivation of (4.3.7b) read backwards). Hence, using (4.3.7d),

$$\begin{aligned}D_0 D_0\lambda &= D_a D_0 E_a - D_0(\zeta\dot{L} + J_0) \\ &= D_a D_b B_{ba} + D_a(\zeta\partial_a L + J_a) - D_0(\zeta\dot{L} + J_0) = \zeta D^\mu\partial_\mu L + D^\mu J_\mu.\end{aligned}$$

For evaluating the r.h.s., we rewrite the matter field equations given by (4.3.7g), (4.3.7i), (4.3.7k), (4.3.7m), (4.3.7n) to

$$\begin{aligned}&\zeta D^\mu\partial_\mu L - \mathrm{i}\,[\partial^\mu\Lambda, D_\mu\Upsilon] = 0, \qquad &&\partial^\mu D_\mu\Upsilon - [\lambda, \Upsilon] = 0, \qquad D_\mu\partial^\mu\Lambda = 0, \\ &\left(\gamma^\mu D_\mu + m^\Psi\right)\Psi = 0, &&D^\mu D_\mu\Phi + \mathcal{V}_*(\Phi) - \lambda\times\Phi = 0.\end{aligned}$$

With a similar computation as in the derivation of (3.3.4), we get

$$D_0 D_0\lambda = \Phi\times(\lambda\times\Phi). \tag{4.3.12}$$

Now the constraint (4.3.8) at $t = 0$ says that $\lambda(0) = 0$, and (4.3.9) at $t = 0$ says that the r. h. s. of (4.3.11) vanishes at $t = 0$, so that $\partial_t \lambda(0) = 0$. Therefore (4.3.12) now implies $\lambda = 0$ for all times, i. e. (4.3.8) is satisfied for all times. Now (4.3.11) implies the validity of (4.3.9) for all times as well as (4.3.3), while (4.3.7d) yields (4.3.4). The remaining equations are now obviously satisfied.

Ad (iii). For finding the influence function $\hat{\mathcal{A}}$ of the free evolution, we may consider again the sectors separately: We have

$$\hat{\mathcal{A}} = \operatorname{diag}(\hat{\mathcal{A}}^{AEBL}, \hat{\mathcal{A}}^{\Phi}, \hat{\mathcal{A}}^{\Lambda}, \hat{\mathcal{A}}^{\Upsilon}, \hat{\mathcal{A}}^{\Psi}).$$

Here $\hat{\mathcal{A}}^{\Lambda}, \hat{\mathcal{A}}^{\Upsilon}$ are both given by (1.5.2) with $m^{\Phi} = 0$, $\hat{\mathcal{A}}^{\Phi}$ is given by (1.5.2) with $m^{\Phi}_{ij} := \frac{\partial^2}{\partial \underline{\Phi}_i \partial \underline{\Phi}_j} \mathcal{V}(0)/2$ again, and $\hat{\mathcal{A}}^{\Psi} = A^{\Psi}$ is again the same as in 2.3.

The linearized evolution of both L and A_0 is given by the massless Klein-Gordon equation, with A_0 having $\dot{L}$ as source term. Also, the linearized evolution of A_a, E_a, B_{ab} is the same as in the proof of Lemma 3.3.1, apart from the source terms containing A_0 and L; the latter affect only the last two columns of the matrix below. These observations lead to the following structure for the influence function for $A_0, \dot{A}_0, A_a, E_a, B_{ab}, L, \dot{L}$:

$$\hat{\mathcal{A}}^{AEBL} = \begin{pmatrix} \partial_t \hat{\mathcal{P}}_0 & \hat{\mathcal{P}}_0 & 0 & 0 & 0 & \hat{\mathcal{X}}^L_0 & \hat{\mathcal{X}}^{\dot{L}}_0 \\ \partial_t^2 \hat{\mathcal{P}}_0 & \partial_t \hat{\mathcal{P}}_0 & 0 & 0 & 0 & \partial_t \hat{\mathcal{X}}^L_0 & \partial_t \hat{\mathcal{X}}^{\dot{L}}_0 \\ \mathrm{i} p_c \hat{\mathcal{P}}_0 & \mathrm{i} p_c \int_0^t d\tau \hat{\mathcal{P}}_0(\tau, p) & & & & \hat{\mathcal{X}}^L_c & \hat{\mathcal{X}}^{\dot{L}}_c \\ 0 & 0 & & \hat{\mathcal{A}}^{AEB} & & \hat{\mathcal{A}}^L_c & \hat{\mathcal{A}}^{\dot{L}}_c \\ 0 & 0 & & & & \hat{\mathcal{A}}^L_{cd} & \hat{\mathcal{A}}^{\dot{L}}_{cd} \\ 0 & 0 & 0 & 0 & 0 & \partial_t \hat{\mathcal{P}}_0 & \hat{\mathcal{P}}_0 \\ 0 & 0 & 0 & 0 & 0 & \partial_t^2 \hat{\mathcal{P}}_0 & \partial_t \hat{\mathcal{P}}_0 \end{pmatrix} \tag{4.3.13}$$

where $\hat{\mathcal{A}}^{AEB}$ is as in (3.3.7), $\hat{\mathcal{P}}_0 := (2\pi)^{-d/2} \sin |p| t / |p|$, and the first five lines in the last two columns are yet to be determined. Letting I stand for either L or $\dot{L}$, they are determined by

$$\hat{\mathcal{X}}^I_0(0) = \partial_t \hat{\mathcal{X}}^I_0(0) = \hat{\mathcal{X}}^I_c(0) = \hat{\mathcal{A}}^I_c(0) = \hat{\mathcal{A}}^I_{cd}(0) = 0,$$

$$\partial_t^2 \hat{\mathcal{X}}^I_0 = -p^2 \hat{\mathcal{X}}^I_0 + (\zeta - 1) \times \begin{cases} \partial_t^2 \hat{\mathcal{P}}_0 & \text{for } I = L \\ \partial_t \hat{\mathcal{P}}_0 & \text{for } I = \dot{L} \end{cases},$$

$$\partial_t \hat{\mathcal{X}}^I_c = \hat{\mathcal{A}}^I_c + \mathrm{i} p_c \hat{\mathcal{X}}^I_0, \qquad \partial_t \hat{\mathcal{A}}^I_{cd} = \mathrm{i} p_c \hat{\mathcal{A}}^I_d - \mathrm{i} p_d \hat{\mathcal{A}}^I_c,$$

$$\partial_t \hat{\mathcal{A}}^I_c = \mathrm{i} p_d \hat{\mathcal{A}}^I_{dc} + \mathrm{i} \zeta p_c \begin{cases} \partial_t \hat{\mathcal{P}}_0 & \text{for } I = L \\ \hat{\mathcal{P}}_0 & \text{for } I = \dot{L} \end{cases}.$$

Abbreviating $C := \cos |p| t$, $S := \sin |p| t$, and omitting the colour indices, one checks that the solution is given by $(2\pi)^{d/2} \hat{\mathcal{A}}^{AEBL} =$

$$\begin{pmatrix} C & \frac{1}{|p|} S & \dots & \frac{\zeta-1}{2|p|} (|p| t C - S) & \frac{\zeta-1}{2|p|} t S \\ -|p| S & C & \dots & \frac{(1-\zeta)}{2} |p| t S & \frac{\zeta-1}{2|p|} (S + t |p| C) \\ \frac{\mathrm{i}}{|p|} p_c S & \frac{\mathrm{i}(1-C)}{p^2} p_c & \dots & \frac{\mathrm{i} p_c}{2|p|^2} (-2C + (\zeta - 1) |p| t S) & \frac{\mathrm{i} p_c}{2|p|^3} (|p| t (2\zeta + (1-\zeta) C) - (1+\zeta) S) \\ 0 & 0 & \dots & \frac{\mathrm{i}\zeta}{|p|} p_c S & \frac{\mathrm{i}\zeta}{|p|^2} p_c (1 - C) \\ 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & C & \frac{1}{|p|} S \\ 0 & 0 & \dots & -|p| S & C \end{pmatrix}$$

where the three middle columns, which are already known from (4.3.13), (3.3.7), have been omitted.

The assertion (iii) is now easy to check. □

Proof of Thm. 4.2.1. Ad (i). This follows from Lemma 4.3.1 and the results of [11].

Ad (ii). Let

$$V^{\mathrm{ext}} := \mathfrak{g} \otimes \left(\mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R}^d \oplus \mathbb{R}^d \oplus \Lambda^2 \mathbb{R}^d \oplus \mathbb{R} \oplus \mathbb{R} \right) \oplus V^{\Phi} \oplus V^{\Phi} \oplus \Pi(\mathfrak{g} \oplus \mathfrak{g} \oplus \mathfrak{g} \oplus \mathfrak{g}) \oplus \Pi(V^{\Psi} \otimes_{\mathbb{C}} V^{\mathrm{D}})$$

be the field target space for the extended system (4.3.7a)–(4.3.7n). Thus, the corresponding smf of smooth Cauchy data is $M_{C^\infty}^{\mathrm{extCau}} = \mathrm{L}(\mathcal{E}^{\mathrm{Cau}} \otimes V^{\mathrm{ext}})$. Consider the smf morphism

$$M_{C^\infty}^{\mathrm{Cau}} \xrightarrow{(A_0^{\mathrm{Cau}}, \dot{A}_0^{\mathrm{Cau}}, A_a^{\mathrm{Cau}}, E_a^{\mathrm{Cau}}, B_{ab}^{\mathrm{Cau}}, L^{\mathrm{Cau}}, \dot{L}^{\mathrm{Cau}}, \Phi^{\mathrm{Cau}}, \dot{\Phi}^{\mathrm{Cau}} | \Lambda^{\mathrm{Cau}}, \dot{\Lambda}^{\mathrm{Cau}}, \Upsilon^{\mathrm{Cau}}, \dot{\Upsilon}^{\mathrm{Cau}}, \Psi^{\mathrm{Cau}})} M_{C^\infty}^{\mathrm{extCau}}, \tag{4.3.14}$$

where

$$\begin{aligned}
&E_a^{\mathrm{Cau}} := \dot{A}_a^{\mathrm{Cau}} - \partial_a A_0^{\mathrm{Cau}} + \left[A_0^{\mathrm{Cau}}, A_a^{\mathrm{Cau}}\right], \quad B_{ab}^{\mathrm{Cau}} := \partial_a A_b^{\mathrm{Cau}} - \partial_b A_a^{\mathrm{Cau}} + \left[A_a^{\mathrm{Cau}}, A_b^{\mathrm{Cau}}\right],\\
&L^{\mathrm{Cau}} := -\dot{A}_0^{\mathrm{Cau}} + \partial^a A_a^{\mathrm{Cau}},\\
&\dot{L}^{\mathrm{Cau}} := \frac{1}{\zeta}\Big(\mathrm{i}\left[\dot{\Lambda}^{\mathrm{Cau}}, \Upsilon^{\mathrm{Cau}}\right] - \Phi^{\mathrm{Cau}} \times (\dot{\Phi}^{\mathrm{Cau}} + A_0^{\mathrm{Cau}} \times \Phi^{\mathrm{Cau}}) + \mathrm{i}\overline{\Psi}^{\mathrm{Cau}} \times \gamma_0 \Psi^{\mathrm{Cau}}\\
&\qquad + \partial_a E_a^{\mathrm{Cau}} + \left[A_a^{\mathrm{Cau}}, E_a^{\mathrm{Cau}}\right]\Big).
\end{aligned}$$

Letting Y denote temporarily the image of its underlying map, the hypothesis just says that the system (4.3.7a)–(4.3.7n) is $(\mathcal{H}^{\mathrm{V}_k^{\mathrm{ext}}}, Y \cap \mathcal{H}^{\mathrm{V}_k^{\mathrm{ext}}})$-complete, where we use the Sobolev space $\mathcal{H}^{\mathrm{V}_k^{\mathrm{ext}}}$ as defined in [10], [11]. Hence, by [10, Thm. 3.4.3], there exists a solution family

$$\begin{aligned}
\Xi^{\mathrm{extsol}} = (&A_0^{\mathrm{extsol}}, \dot{A}_0^{\mathrm{extsol}}, A_a^{\mathrm{extsol}}, E_a^{\mathrm{extsol}}, B_{ab}^{\mathrm{extsol}}, L^{\mathrm{extsol}}, \dot{L}^{\mathrm{extsol}}, \Phi^{\mathrm{extsol}}, \dot{\Phi}^{\mathrm{extsol}} |\\
&\Lambda^{\mathrm{extsol}}, \dot{\Lambda}^{\mathrm{extsol}}, \Upsilon^{\mathrm{extsol}}, \dot{\Upsilon}^{\mathrm{extsol}}, \Psi^{\mathrm{extsol}}) \in \mathcal{M}^{\mathcal{E} \otimes V^{\mathrm{ext}}}(M_{C^\infty}^{\mathrm{Cau}})
\end{aligned}$$

which has (4.3.14) as its Cauchy data. Now $\Xi^{\mathrm{sol}} := (A^{\mathrm{extsol}}, \Phi^{\mathrm{extsol}} | \Lambda^{\mathrm{extsol}}, \Upsilon^{\mathrm{extsol}}, \Psi^{\mathrm{extsol}})$ will be the universal solution family wanted. For showing that its image is a sub-smf, the proof of [11, Thm. 2.4.1] carries over again. □

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