Exponentially Small Corrections to Divergent Asymptotic Expansions of Solutions of the Fifth Painlevé Equation

F. V. Andreev and A. V. Kitaev *
Steklov Mathematical Institute, Fontanka 27, St.Petersburg, 191011, Russia

March 19, 1997

Abstract

We calculate the leading term of asymptotics for the coefficients of certain divergent asymptotic expansions of solutions to the fifth Painlevé equation (P_5) by using the isomonodromy deformation method and the Borel transform. Unexpectedly, these asymptotics appear to be periodic functions of the coefficients of P_5. We also show the relation of our results with some other facts already known in the theory of the Painlevé equations established by other methods: (1) a connection formula for the third Painlevé equation; (2) a condition for the existence of rational solutions of P_5; and (3) a numerical study of the τ-function for P_5.

Mathematics Subject Classification (1991). 34A20, 34E05, 30B40, 33E30, 40G10.

Short title: Divergent Asymptotic Expansions

*E-mail: andreev@pdmi.ras.ru, kitaev@pdmi.ras.ru

1 Instead of Introduction

There are several interesting asymptotic properties of the τ-function associated with a particular one-parameter family of solutions of the third Painlevé equation (P_3) with a special set of coefficients: this family can be also viewed as the fifth Painlevé transcendent for a very special choice of its parameters. Let us recall some of these properties, formulated in terms of the function $\sigma = \sigma(x;\lambda)$, which is the solution of the following ordinary differential equation (ODE),

$$(x\sigma'')^2 = -16(x\sigma' - \sigma)(x\sigma' - \sigma + \frac{1}{4}(\sigma')^2), \tag{1.1}$$

where $' = \frac{d}{dx}$, satisfying the boundary condition

$$\sigma(x;\lambda) \underset{x\to 0}{=} -2\lambda x - (2\lambda x)^2 - \dots . \tag{1.2}$$

If $\lambda = \frac{1}{\pi}$, then

$$\sigma(x;\frac{1}{\pi}) \underset{x\to +\infty}{=} -x^2 - \frac{1}{4} - \sum_{n=1}^{\infty} \frac{c_n}{x^{2n}} + O(x^{-\infty}), \tag{1.3}$$

where

$$c_1 = \frac{1}{16}, c_2 = \frac{5}{32}, c_3 = \frac{131}{128}, c_4 = \frac{6575}{512}, \dots, c_n, n \geq 5, \tag{1.4}$$

can be obtained recursively by substituting expression (1.3) into Eq. (1.1).

The asymptotic series (1.3) is divergent. E.L.Basor and C.A.Tracy [1] stated the following

Conjecture [1]:

$$c_n \underset{n\to\infty}{\sim} \text{const.} n!(n-1)!\,, \tag{1.5}$$

where const. ≈ 0.10132.

The function $\sigma(x;\lambda)$ and its mth–order derivatives at $\lambda = \frac{1}{\pi}$ define the function $E_2(m;s)|_{s=\frac{\pi x}{2}}$, which is the probability that a randomly chosen interval of the length s contains m eigenvalues in the Gaussian Unitary Ensemble (see [2], [3]). We are going to investigate the function $E_2(m;s)$ via the isomonodromy deformation method in a separate paper while here, we focus our attention on the calculation of the exponentially small correction term to the divergent expansion (1.3) and its intimate link with the asymptotics of the sequence c_n. We found that, among the functions $\sigma(x;\lambda)$ with $\lambda > 0$, only one, for $\lambda = \frac{1}{\pi}$, has the asymptotic expansion (1.3), (1.4) as $x \to \infty$, which is in agreement with the results obtained by a different method in [4]. Note that there is a one-parameter family of solutions to Eq. (1.1), other then $\sigma(x;\lambda)$, which has the asymptotics (1.3) as $x \to +\infty$. We calculated the explicit value of the const. in (1.5): it is equal to

$$\frac{1}{\pi^2} = 0.10132118364233777144387946320972763890435877467224654 88\dots . \tag{1.6}$$

Actually, in this work we consider a more general problem than that discussed above. Here, we investigate a one-parameter family of solutions of P_5 with the general set of

[1]As they told us later (June 1996, Mount Holyoke), this conjecture was based on a numerical study of the sequence c_n.

coefficients which has a (1.3)-like algebraic asymptotic expansion, so that the results announced above are just a particular case of the ones obtained for P_5. As another result of our study, we have found an unexpected and interesting property of the coefficients of the divergent algebraic asymptotic expansions which is general for all other Painlevé equations (P_2–P_5) possessing the solutions with such expansions: this property is that the large n asymptotics of the coefficients of the divergent asymptotic expansions are *periodic functions* of the coefficients of the Painlevé equations. Note that, being formally substituted into the Painlevé equations, the coefficients of the divergent algebraic series can be found via recurrence relations as *polynomials* of the coefficients of the Painlevé equations.

In the next section, we recall some basic facts concerning the isomonodromy approach to P_5. Our results are formulated in Section 3 where we also give a new derivation of the theorem on the classification of a special class of rational solutions to P_5, $y(x) : y(x) \underset{x\to\infty}{\to} -1$, which was first proved in [5] and by another method in [6]. In section 4, we apply the Borel transform to find the asymptotics of the coefficients of the divergent series for P_5. In the Appendix, we consider two applications of our results to P_3: (1) we prove asymptotic expansion (1.5), 1.6); and (2) verify that, for a very special choice of the coefficients of P_5, our result agrees with the connection formula for P_3 which was proved in [7].

2 The Fifth Painlevé Equation and Its Monodromy Manifold

Let us recall that, in [8], it was shown that isomonodromy deformations of the linear ODE

$$\frac{d}{d\lambda}\Psi = \Big(\frac{t}{2}\sigma_3 + \frac{A_0(t)}{\lambda} + \frac{A_1(t)}{\lambda-1}\Big)\Psi, \tag{2.1}$$

where $A_k(t) \in sl_2(\mathbb{C})$ and are independent of λ, are governed by the following system of nonlinear ODEs:

$$\begin{aligned}
t\frac{dy}{dt} &= ty - 2z(y-1)^2 - (y-1)\Big(\frac{\Theta_0-\Theta_1+\Theta_\infty}{2}y - \frac{3\Theta_0+\Theta_1+\Theta_\infty}{2}\Big), &(2.2)\\
t\frac{dz}{dt} &= yz\Big(z+\frac{\Theta_0-\Theta_1+\Theta_\infty}{2}\Big) - \frac{1}{y}(z+\Theta_0)\Big(z+\frac{\Theta_0+\Theta_1+\Theta_\infty}{2}\Big), &(2.3)\\
t\frac{d}{dt}\log u &= -2z-\Theta_0+y\Big(z+\frac{\Theta_0-\Theta_1+\Theta_\infty}{2}\Big)+\frac{1}{y}\Big(z+\frac{\Theta_0+\Theta_1+\Theta_\infty}{2}\Big), &(2.4)
\end{aligned}$$

where $\Theta_\nu, \nu = 0, 1, \infty$ are complex constants, namely, *the parameters of the formal monodromy*. Eliminating the function $z(t)$ from Eqs. (2.2) and (2.3), one finds that the function $y(t)$ satisfies P_5:

$$\frac{d^2y}{d^2t} = \Big(\frac{1}{2y}+\frac{1}{y-1}\Big)\Big(\frac{dy}{dt}\Big)^2 - \frac{dy}{tdt} + \frac{(y-1)^2}{t^2}\Big(\alpha y+\frac{\beta}{y}\Big) + \frac{\gamma y}{t} + \frac{\delta y(y+1)}{y-1}, \tag{2.5}$$

$$\alpha = \frac{1}{2}\Big(\frac{\Theta_0-\Theta_1+\Theta_\infty}{2}\Big)^2,\ \beta = -\frac{1}{2}\Big(\frac{\Theta_0-\Theta_1-\Theta_\infty}{2}\Big)^2,\ \gamma = 1-\Theta_0-\Theta_1,\ \delta = -\frac{1}{2}. \tag{2.6}$$

The so-called τ-function [8], [9] plays an important role in applications of P_5:

$$\frac{d}{dt}\log\tau_p(t) \equiv H_p \equiv -\frac{1}{2t}\operatorname*{Res}_{\lambda=p}\operatorname{tr}A^2(\lambda,\ t),\ p=0,1;\ H_0+H_1 = \frac{\Theta_\infty}{2},$$

where $A(\lambda,\ t)$ is the matrix in the parenthesis in Eq. (2.1). Using this definition, it is straightforward to write an explicit representation of the τ-function in terms of solutions $(y = y(t), z = z(t))$ of the system (2.2), (2.3) (see [8], [10]). It is also convenient to introduce an auxiliary function

$$\zeta(t) \equiv \zeta(\Theta_0, \Theta_1, \Theta_\infty; t) \equiv -tH_1 + \frac{t}{2}(\Theta_0 + \Theta_\infty) + \frac{\Theta_0}{2}(\Theta_0 + \Theta_\infty). \tag{2.7}$$

In particular, it is known [8] that the function $\zeta(t)$ solves the following ODE

$$\begin{aligned} \left(t\frac{d^2\zeta}{dt^2}\right)^2 &= \left(\zeta - t\frac{d\zeta}{dt} + 2\left(\frac{d\zeta}{dt}\right)^2 - (2\Theta_0 + \Theta_\infty)\frac{d\zeta}{dt}\right)^2 \\ &- 4\frac{d\zeta}{dt}\left(\frac{d\zeta}{dt} - \Theta_0\right)\left(\frac{d\zeta}{dt} - \frac{1}{2}(\Theta_0 - \Theta_1 + \Theta_\infty)\right)\left(\frac{d\zeta}{dt} - \frac{1}{2}(\Theta_0 + \Theta_1 + \Theta_\infty)\right), \end{aligned} \tag{2.8}$$

which, for $\Theta_0 = \Theta_1 = \Theta_\infty = 0$ and $t = 4ix$, coincides with Eq. (1.1).

The solutions of system (2.2)-(2.4) are in one-to-one correspondence with the points of the manifold of monodromy data, $\mathcal{M}_5(\Theta_0, \Theta_1, \Theta_\infty)$. The detailed definition of $\mathcal{M}_5(\Theta_0, \Theta_1, \Theta_\infty)$ can be found in our previous works [10, 11]. Here, we recall that this manifold is defined by the matrix elements of the monodromy matrices M_0 and M_1, representing, respectively, the monodromy of the canonical solutions to Eq. (2.1) around the regular singularities $\lambda = 0$ and $\lambda = 1$. Denoting the matrix elements of M_p as $m^p_{ij}, p = 0, 1;\ i, j = 1, 2$, we can define $\mathcal{M}_5(\Theta_0, \Theta_1, \Theta_\infty)$ by the following algebraic equations in $\mathbb{C}^8$:

$$\begin{gathered} m^0_{11}m^0_{22} - m^0_{12}m^0_{21} = 1, \quad m^1_{11}m^1_{22} - m^1_{12}m^1_{21} = 1, \\ m^0_{11} + m^0_{22} = 2\cos\pi\Theta_0, \quad m^1_{11} + m^1_{22} = 2\cos\pi\Theta_1, \\ m^1_{11}m^0_{11} + m^1_{12}m^0_{21} = e^{-\pi i\Theta_\infty}. \end{gathered}$$

Note that, $\dim_{\mathbb{C}} M_5(\Theta_0, \Theta_1, \Theta_\infty) = 3$. The Stokes matrices, S_{p+1}, $p = 0, 1$ (for their definition see [10]), also play an essential role in the description of analytical properties of the solutions to Eq. (2.1). These matrices define the monodromy properties of the canonical solutions to Eq. (2.1) near the irregular singularity $\lambda = \infty$. The matrices S_{p+1} have the following structure:

$$S_1 = \begin{pmatrix} 1 & 0 \\ s_1 & 1 \end{pmatrix}, \quad S_2 = \begin{pmatrix} 1 & s_2 \\ 0 & 1 \end{pmatrix},$$

where the complex parameters s_{p+1}, $p = 0, 1$, are called the Stokes multipliers. Further, we call m^p_{ij}, s_{p+1}, $p = 0, 1$, $i, j = 1, 2$, the *monodromy coordinates.* Although, it is easy to express explicitly s_{p+1} in terms of m^p_{ij}, hereafter, we write them down for the reader's convenience.

The nonlinear system (2.2)-(2.4) has two singular points, $t = 0$ and ∞, so that solutions of this system are defined on the universal covering of $\mathbb{C}\backslash\{0\}$. In the previous works [10, 11] we investigated the asymptotics of the solutions to the system (2.2)-(2.4) as $t \to +0$ and $t \to +\infty$, therefore, it is natural for us to consider the analytic continuation of these solutions from $(+0, +\infty)$. Furthermore, we understand the functions $y(t)$, $z(t)$, $u(t)$, and $\zeta(t)$, respectively, for $t \in \mathbb{C}$, as the ones analytically continued from $\mathbb{R}_+$. Since the system (2.2)-(2.4) possesses the Painlevé property, single-valued solutions can be defined on $\mathbb{C}\backslash[-0, -\infty)$. We call the ray $[-0, -\infty)$ the cut and denote by $y_\pm(t)$, $z_\pm(t)$,

$u_\pm(t)$, and $\zeta_\pm(t)$, respectively, the functions $y(t)$, $z(t)$, $u(t)$, and $\zeta(t)$ on the upper/lower sides of the cut, i.e., for $\arg t = \pm\pi$. Note that, the $\pm$-functions correspond to, i.e., define via Eq. (2.1), the same point of $\mathcal{M}_5(\Theta_0, \Theta_1, \Theta_\infty)$. Let us denote by $y^-(t)$, $z^-(t)$, $u^-(t)$, and $\zeta^-(t)$, respectively, a solution of system (2.2)-(2.4) and the corresponding function (2.7) with Θ_∞ changed to $-\Theta_\infty$: the corresponding manifold of the monodromy data is $\mathcal{M}_5(\Theta_0, \Theta_1, -\Theta_\infty)$. As soon as the functions $y^-(t)$, $z^-(t)$, $u^-(t)$, and $\zeta^-(t)$, respectively, are known for real positive t and an arbitrary point of $\mathcal{M}_5(\Theta_0, \Theta_1, -\Theta_\infty)$, one can find the functions $y_\pm(t)$, $z_\pm(t)$, $u_\pm(t)$, and $\zeta_\pm(t)$ via the following transformation,

$$u_\pm(t) = \tfrac{1}{u^-(e^{\mp\pi\imath}t)}, \quad y_\pm(t) = \tfrac{1}{y^-(e^{\mp\pi\imath}t)}, \tag{2.9}$$

$$z_\pm(t) = -z^-(e^{\mp\pi\imath}t) - \Theta_0, \quad \zeta_\pm(t) = \zeta^-(e^{\mp\pi\imath}t) + \Theta_0 t + \Theta_0\Theta_\infty, \tag{2.10}$$

and the corresponding action of this transformation on the manifold of monodromy data:

$$M_p^+ = \sigma_1 \hat{S}_1 \hat{M}_p \hat{S}_1^{-1} \sigma_1, \quad S_1^+ = \sigma_1 e^{\pi\imath\Theta_\infty\sigma_3} \hat{S}_2 e^{-\pi\imath\Theta_\infty\sigma_3} \sigma_1, \quad S_2^+ = \sigma_1 \hat{S}_1 \sigma_1, \tag{2.11}$$

$$M_p^- = \sigma_1 \hat{S}_2^{-1} \hat{M}_p \hat{S}_2 \sigma_1, \quad S_1^- = \sigma_1 \hat{S}_2 \sigma_1, \quad S_2^- = \sigma_1 e^{-\pi\imath\Theta_\infty\sigma_3} \hat{S}_1 e^{\pi\imath\Theta_\infty\sigma_3} \sigma_1. \tag{2.12}$$

Here, $M_p^\pm$ and $S_{p+1}^\pm$, $p = 0, 1$, are the monodromy and Stokes matrices corresponding to the functions $y_\pm(t)$, $z_\pm(t)$, $u_\pm(t)$, and $\zeta_\pm(t)$, $\hat{M}_p$ and $\hat{S}_{p+1}$ are the monodromy objects for $y^-(t)$, $z^-(t)$, $u^-(t)$, and $\zeta^-(t)$, respectively, i.e., their matrix elements define a point on $\mathcal{M}_5(\Theta_0, \Theta_1, -\Theta_\infty)$. According to the above definition, we have $M_p^+ = M_p^-$, $S_{p+1}^+ = S_{p+1}^-$. It means that, depending of what we want to find: the +- or, respectively, −-functions, we have to take in Eqs. (2.9) and (2.10), the solutions $y^-(\cdot)$, $z^-(\cdot)$, $u^-(\cdot)$, and $\zeta^-(\cdot)$ which correspond (in the general case) to different points on $\mathcal{M}_5(\Theta_0, \Theta_1, -\Theta_\infty)$, i.e., the matrices $\hat{M}_p$ and $\hat{S}_{p+1}$ in Eqs. (2.11) are (in the general case) different from those in Eqs. (2.12).

3 Properties of the Truncated Solutions

Proposition 1 *The system (2.2), (2.3) and Eq. (2.8) admit the following formal solutions:*

$$y_F(t) = -1 + \sum_{n=1}^{\infty} \frac{\alpha_n(\Theta_0, \Theta_1, \Theta_\infty)}{t^n}, \tag{3.1}$$

$$z_F(t) = -\frac{t}{8} - \frac{2\Theta_0 + \Theta_\infty}{4} + \sum_{n=1}^{\infty} \frac{\beta_n(\Theta_0, \Theta_1, \Theta_\infty)}{t^n}, \tag{3.2}$$

$$\begin{aligned} \zeta_F(t) &= \frac{t^2}{16} + (2\Theta_0 + \Theta_\infty)\frac{t}{4} + \frac{3\Theta_0^2}{4} + \\ &+ \frac{\Theta_1^2}{4} + \frac{\Theta_\infty^2}{4} + \frac{\Theta_0\Theta_\infty}{2} - \frac{1}{4} + \sum_{n=1}^{\infty} \frac{\gamma_n(\Theta_0, \Theta_1, \Theta_\infty)}{t^n}, \end{aligned} \tag{3.3}$$

where the coefficients, $\alpha_n = \alpha_n(\Theta_0,\ \Theta_1,\ \Theta_\infty)$, $\beta_n = \beta_n(\Theta_0,\ \Theta_1,\ \Theta_\infty)$, and $\gamma_n = \gamma_n(\Theta_0,\ \Theta_1,\ \Theta_\infty)$, are uniquely determined polynomials of their arguments of the degrees:

$$\deg \alpha_n(\Theta_0,\ \Theta_1,\ \Theta_\infty) = n, \ \deg \beta_n(\Theta_0,\ \Theta_1,\ \Theta_\infty) = n + 1, \ \deg \gamma_n(\Theta_0,\ \Theta_1,\ \Theta_\infty) = n + 2.$$

Proof The proof of Proposition 1 is just the straightforward substitution of the expansions (3.1), (3.2) and (3.3) into the system (2.2), (2.3) and Eq. (2.8) correspondingly. As the result one finds recurrence relations for the coefficients α_n, β_n, and γ_n, which enable one to make the conclusions stated above.

Remark 1 The recurrence relations are rather complicated. We don't use them directly and that is why we don't write them down here. The following is the list of the first few polynomials:

$$\alpha_1 = 4(\Theta_0 + \Theta_1 - 1), \qquad \alpha_2 = 8\Theta_\infty(\Theta_1 - \Theta_0) - 8(\Theta_0 + \Theta_1 - 1)^2,$$
$$\alpha_3 = 16(\Theta_0 + \Theta_1 - 1)(\Theta_0 - \Theta_1 + \Theta_\infty)^2 + 16(\Theta_0 + \Theta_1 - 3)(\Theta_0 + \Theta_1)^2 + 32(\Theta_0 + \Theta_1),$$
$$\beta_1 = \ 0, \quad \beta_2 = \Theta_\infty(\Theta_1^2 - \Theta_0^2), \quad \beta_3 = 2(\Theta_1^2 - \Theta_0^2)^2 + 2({\Theta_\infty}^2 - 1)(2\Theta_0^2 + 2\Theta_1^2 - 1),$$
$$\gamma_1 = \beta_2, \quad \gamma_2 = \tfrac{1}{2}\beta_3, \qquad \gamma_3 = 4\Theta_\infty(\Theta_1^2 - \Theta_0^2)(2\Theta_0^2 + 2\Theta_1^2 + \Theta_\infty^2 - 5),$$
$$\gamma_4 = 8(\Theta_0^6 + \Theta_1^6 - \Theta_0^2\Theta_1^4 - \Theta_1^2\Theta_0^4 + \Theta_0^2\Theta_\infty^4 + \Theta_1^2\Theta_\infty^4 + 5\Theta_1^4\Theta_\infty^2 + 5\Theta_0^4\Theta_\infty^2 - 2\Theta_0^2\Theta_1^2\Theta_\infty^2$$
$$- 7\Theta_0^4 - 7\Theta_1^4 - \tfrac{1}{2}\Theta_\infty^4 - 12\Theta_1^2\Theta_\infty^2 - 12\Theta_0^2\Theta_\infty^2 + 6\Theta_0^2\Theta_1^2 + 11\Theta_0^2 + 11\Theta_1^2 + \tfrac{11}{2}\Theta_\infty^2 - 5).$$

Although it is not evident from definition (2.7), one proves that $\zeta'(t) = -z(t)$; this means that $\beta_{n+1}(\Theta_0,\ \Theta_1,\ \Theta_\infty) = n\gamma_n(\Theta_0,\ \Theta_1,\ \Theta_\infty)$. Another property of the polynomials α_n, β_n, and γ_n is that they can be rewritten as polynomials of the coefficients α, β, and γ (see Eq. (2.6)) of P_5 (2.5), without any square roots! This is more or less clear for the polynomials α_n since they can be obtained recursively by substituting expansion (3.1) into Eq. (2.5); but than, the polynomials β_n should be obtained from Eq. (2.2), which obviously contains square roots of the coefficients, (2.6). Eq. (2.8), being rewritten with the coefficients, α, β, and γ, by using formulas (2.6), also contains the square roots. The explanation is that: by substituting expansions (3.1)-(3.3) into the system (2.2), (2.3) and Eq. (2.8), respectively, one gets, for the sums in (3.1)-(3.3), the equations which can be written with integer powers of α, β, and γ.

Proposition 2 *There exist three special (***doubly truncated***) solutions of the system (2.2), (2.3) which, along with the corresponding functions $\zeta(t)$ (see Eq. (2.7)), we denote as follows:*
(1) $y_0(t)$, $z_0(t)$, *and* $\zeta_0(t)$*;*
(2) $y_{+1}(t)$, $z_{+1}(t)$, *and* $\zeta_{+1}(t)$*; and*
(3) $y_{-1}(t)$, $z_{-1}(t)$, *and* $\zeta_{-1}(t)$*.*
These solutions are uniquely defined by their asymptotic expansions as $|t| \to +\infty$:

$$y_k(t) = y_F(t) + \mathcal{O}\left(t^{-\infty}\right), \ z_k(t) = z_F(t) + \mathcal{O}\left(t^{-\infty}\right), \ \zeta_k(t) = \zeta_F(t) + \mathcal{O}\left(t^{-\infty}\right), \quad (3.4)$$

in the sectors

$$-\pi + \pi k < \arg t < \pi + \pi k, \quad k = -1, 0, +1. \quad (3.5)$$

Proof The existence and uniqueness of the solutions for $\arg t = 0$ is proved in our previous work [11] (see Sec. 3 of [11]). To prove the fact that these solutions can be analytically continued with the same asymptotics (3.4) into the whole sector (3.5) one should notice that for each asymptotic expansion (3.1)–(3.3) there is only one function analytic in the sector (3.5) with this asymptotics, so that, it must coincide with the corresponding function: $y(t)$, $z(t)$, or $\zeta(t)$, which has the asymptotics (3.4) for $\arg t = 0$.

Remark 2 By integrating Eq. (2.4) one finds that the corresponding functions $u_k(t)$ have the following asymptotic behavior:

$$u_k(t) = \hat{u}e^{\frac{t}{2}}(1 + o(1)), \quad k = -1,\ 0,\ +1, \quad (3.6)$$

where $\hat{u} \in \mathbb{C} \setminus \{0\}$ is the constant of integration.

Proposition 3 *The monodromy coordinates of the solutions, $y_k(t), z_k(t), u_k(t)$, and $\zeta_k(t)$ are as follows:*

$$
\begin{aligned}
(1)\ k = \ \ 0 \ \Rightarrow \ \ & m_{11}^0 = 0,\ m_{21}^0 = \frac{2^{2\Theta_\infty}}{\imath\hat{u}}e^{-\pi\imath\Theta_\infty},\ \ m_{11}^1 = 0,\ \ m_{12}^1 = \imath\hat{u}2^{-2\Theta_\infty},\\
s_2 \ \ = \ \ & -2\imath\hat{u}e^{\pi\imath\Theta_\infty}2^{-2\Theta_\infty}\cos\pi\Theta_0, \qquad s_1 = \frac{2\imath}{\hat{u}}2^{2\Theta_\infty}\cos\pi\Theta_1;\\
(2)\ k = +1 \ \Rightarrow \ \ & m_{11}^0 = 2(\cos\pi\Theta_0 + e^{-\pi\imath\Theta_\infty}\cos\pi\Theta_1),\ m_{21}^0 = -\frac{\imath}{\hat{u}}e^{-\pi\imath\Theta_\infty}2^{2\Theta_\infty},\ m_{11}^1 = 0,\\
m_{12}^1 \ \ = \ \ & \imath 2^{-2\Theta_\infty}\hat{u}, \quad s_2 = 2\imath\hat{u}2^{-2\Theta_\infty}\cos\pi\Theta_1, \quad s_1 = -\frac{2\imath}{\hat{u}}e^{\pi\imath\Theta_\infty}2^{2\Theta_\infty}\cos\pi\Theta_0;\\
(3)\ k = -1 \ \Rightarrow \ \ & m_{11}^0 = 0,\ \ m_{21}^0 = \frac{2^{2\Theta_\infty}}{\imath\hat{u}}e^{-\pi\imath\Theta_\infty},\ \ m_{11}^1 = 2(\cos\pi\Theta_1 + e^{-\pi\imath\Theta_\infty}\cos\pi\Theta_0),\\
m_{12}^1 \ \ = \ \ & \imath\hat{u}2^{-2\Theta_\infty}, s_2 = 2\imath\hat{u}2^{-2\Theta_\infty}e^{2\pi\imath\Theta_\infty}\cos\pi\Theta_1, s_1 = -\frac{2\imath}{\hat{u}}2^{2\Theta_\infty}e^{-\pi\imath\Theta_\infty}\cos\pi\Theta_0.
\end{aligned}
$$

Proof The formulas in item (1) were proved in Corollary 3.1 of [11]; the formulas in items (2) and (3) are special cases of the ones obtained in Corollaries 3.1, and 3.2 of [11].

Proposition 4 *The following asymptotic formulas as $|t| \to +\infty$ and $k = \pm 1$:*

$$y_k(t) \ = \ y_0(t) + 2\Xi_k \exp\left(ki\frac{\pi}{4} - \frac{i}{2}\arg t\right)\sqrt{\frac{2}{|t|}}\,e^{k\frac{it}{2}}(1 + o(1)), \tag{3.7}$$

$$z_k(t) \ = \ z_0(t) + \frac{\Xi_k}{4}\exp\left(-ki\frac{\pi}{4} + \frac{i}{2}\arg t\right)\sqrt{\frac{|t|}{2}}\,e^{k\frac{it}{2}}(1 + o(1)), \tag{3.8}$$

$$\zeta_k(t) \ = \ \zeta_0(t) + \frac{\Xi_k}{2}\exp\left(-ki\frac{\pi}{4} + \frac{i}{2}\arg t\right)\sqrt{\frac{|t|}{2}}\,e^{k\frac{it}{2}}(1 + o(1)), \tag{3.9}$$

where

$$\Xi_k = \frac{2}{\sqrt{2\pi}}\left(\cos(\pi\Theta_0)e^{k\frac{\pi i\Theta_\infty}{2}} + \cos(\pi\Theta_1)e^{-k\frac{\pi i\Theta_\infty}{2}}\right) \tag{3.10}$$

are valid for the doubly truncated solutions in the sectors:

$$k = +1 \ \Rightarrow \ 0 \le \arg t \le \pi,\ k = -1 \ \Rightarrow \ -\pi \le \arg t \le 0. \tag{3.11}$$

Proof For $\arg t = 0,\ \ \pm\pi$ these asymptotic expansions follow from Corollaries 3.1–3.3 of the paper [11] and Eqs. (2.9)–(2.12) of Sec. 2. Their extensions to the whole sectors (3.11) can be proved either by linearization of the corresponding differential equations or by noting that, being restricted to the solutions $y_k(t)$, $z_k(t)$, and $\zeta_k(t)$, the proof of the asymptotic expansions in [10], [11] is also valid for complex t in these sectors.

Corollary 1 *The (analytic continuation of the) solutions $y_0(t), z_0(t)$, and $\zeta_0(t)$ have asymptotic expansions (3.1)–(3.3) in the domain $-\pi < \arg t < +\infty$ and, in particular, $y_{+1}(t) = y_0(t)$, $z_{+1}(t) = z_0(t)$, and $\zeta_{+1}(t) = \zeta_0(t)$ iff $\Xi_{+1} = 0$. Similarlyly, the (analytic continuation of the) solutions $y_0(t), z_0(t)$, and $\zeta_0(t)$ have asymptotic expansions (3.1)–(3.3) in the domain $-\infty < \arg t < +\pi$ and, in particular: $y_{-1}(t) = y_0(t)$, $z_{-1}(t) = z_0(t)$, and $\zeta_{-1}(t) = \zeta_0(t)$ iff $\Xi_{-1} = 0$.*

Proof Induction on positive (for the proof of the first statement) or negative (for the proof of the second one) $k \in \mathbb{Z}$: $|\arg t - \pi k| < \pi$, by making use of Eqs. (2.9)–(2.12). The basis of the induction follows from Proposition 4.

Corollary 2 *Expansions (3.1)–(3.3) are convergent iff* $\Xi_{+1} = \Xi_{-1} = 0$.

Proof The condition $\Xi_{+1} = \Xi_{-1} = 0$ means that all the solutions $y_k(t)$, $z_k(t)$, and $\zeta_k(t)$ are rational functions.

Corollary 3 *The system (2.2), (2.3) and, consequently, the fifth Painlevé equation (2.5) have rational solutions with the property* $\lim_{t\to\infty} y(t) = -1$ *iff at least one of the following conditions:*

$$
\begin{aligned}
(1)\ \Theta_\infty &= 2l, & \Theta_0 &= \Theta_1 + 1 + 2m,\\
(2)\ \Theta_\infty &= 2l, & \Theta_0 &= -\Theta_1 + 1 + 2m,\\
(3)\ \Theta_\infty &= 1+2l, & \Theta_0 &= \Theta_1 + 2m,\\
(4)\ \Theta_\infty &= 1+2l, & \Theta_0 &= -\Theta_1 + 2m,\\
(5)\ \ \Theta_0 &= \frac{1}{2}+l, & \Theta_1 &= \frac{1}{2}+m,
\end{aligned}
$$

where $l, m \in \mathbb{Z}$, *is satisfied. This solution is unique, if the coefficients of the system (2.2), (2.3) are set as shown above, and coincides with the doubly truncated solution,* $(y_0(t), z_0(t))$. *In terms of the coefficients:* α, β, *and* γ *(see Eq. (2.6)), of the fifth Painlevé equation, the conditions (1) and (3) can be rewritten as follows:*

$$
\sqrt{2\alpha} = \frac{1}{2} + n_1, \quad \sqrt{-2\beta} = \frac{1}{2} + n_2, \quad n_1, n_2 \in \mathbb{Z}, \quad \gamma \in \mathbb{C}, \tag{3.12}
$$

and the conditions (2), (4), and (5) as:

$$
\sqrt{2\alpha} + \sqrt{-2\beta} = n_1, \quad \gamma = 2n_2 - n_1, \quad n_1, n_2 \in \mathbb{Z}. \tag{3.13}
$$

For the fixed set of α, β, *and* γ *satisfying Eqs. (3.12) and (3.13) for some choice of the branches of the square roots, this rational solution of the fifth Painlevé equation (2.5) is unique and coincides with* $y_0(t)$. *In this case the functions* $z_0(t)$ *and* $\zeta_0(t)$ *are also rational.*

Proof The proof follows from Corollary 2 and solution of the equation $\Xi_{-1} = \Xi_{+1}$.

Proposition 5 *The solution of the system (2.2)-(2.4) with the following monodromy coordinates (the upper-truncated solution):*

$$
\begin{aligned}
m^0_{11} &= \sqrt{2\pi}\hat{w}e^{-\frac{\pi\imath\Theta_\infty}{2}}, \quad m^0_{21} = \frac{2^{2\Theta_\infty}}{\imath u}e^{-\pi\imath\Theta_\infty}, \quad s_1 = \frac{2^{2\Theta_\infty}}{\imath\hat{u}}(\sqrt{2\pi}\hat{w}e^{\frac{\pi\imath\Theta_\infty}{2}} - 2\cos\pi\Theta_1),\\
m^1_{11} &= 0, \quad m^1_{12} = \imath\hat{u}2^{-2\Theta_\infty}, \quad s_2 = \imath\hat{u}2^{-2\Theta_\infty}e^{\pi\imath\Theta_\infty}(\sqrt{2\pi}\hat{w}e^{-\frac{\pi\imath\Theta_\infty}{2}} - 2\cos\pi\Theta_0),
\end{aligned}
$$

where $\hat{w} \in \mathbb{C}$, *has in the sector* $0 \le \arg t \le \pi$ *the following asymptotic expansion as* $|t| \to +\infty$:

$$
\begin{aligned}
y(t) &= y_0(t) + 2\hat{w}\sqrt{\frac{2}{|t|}}e^{\frac{i}{2}(\frac{\pi}{2}-\arg t)}e^{\frac{it}{2}} + O(\frac{1}{t}e^{\frac{it}{2}}),\\
z(t) &= z_0(t) + \frac{\hat{w}}{4}\sqrt{\frac{|t|}{2}}e^{-\frac{i}{2}(\frac{\pi}{2}-\arg t)}e^{\frac{it}{2}} + O(t^{-\frac{1}{2}}e^{\frac{it}{2}}),\\
u(t) &= u_0(t)(1+o(1)),\\
\zeta(t) &= \zeta_0(t) + \frac{\hat{w}}{2}\sqrt{\frac{|t|}{2}}e^{\frac{i}{2}(\frac{\pi}{2}+\arg t)}e^{\frac{it}{2}} + O(t^{-\frac{1}{2}}e^{\frac{it}{2}}).
\end{aligned}
$$

Proof The proof is based on the same arguments as the proof of Propositions 3 and 4.

Proposition 6 *The solution of system (2.2)-(2.4) with the following monodromy coordinates (the lower-truncated solution):*

$$
\begin{aligned}
m^0_{11} &= 0,\ m^0_{21} = \frac{2^{2\Theta_\infty}}{\imath\hat{u}} e^{-\pi\imath\Theta_\infty},\ \ s_2 = -\hat{u}2^{-2\Theta_\infty} e^{\pi\imath\Theta_\infty}(\sqrt{2\pi}\hat{v}e^{\frac{\pi\imath\Theta_\infty}{2}} + 2\imath\cos\pi\Theta_0),\\
m^1_{11} &= \imath\sqrt{2\pi}\hat{v}e^{-\frac{\pi\imath\Theta_\infty}{2}},\ \ m^1_{12} = \imath\hat{u}2^{-2\Theta_\infty},\ \ s_1 = \frac{2^{2\Theta_\infty}}{\hat{u}}(\sqrt{2\pi}\hat{v}e^{-\frac{\pi\imath\Theta_\infty}{2}} + 2\imath\cos\pi\Theta_1),
\end{aligned}
$$

where $\hat{v} \in \mathbb{C}$, has in the sector $-\pi \leq \arg t \leq 0$ the following asymptotic expansion as $|t| \to +\infty$:

$$
\begin{aligned}
y(t) &= y_0(t) + 2\hat{v}\sqrt{\frac{2}{|t|}} e^{\frac{\imath}{2}(\frac{\pi}{2}-\arg t)} e^{-\frac{\imath t}{2}} + O(\frac{1}{t}e^{-\frac{\imath t}{2}}),\\
z(t) &= z_0(t) - \frac{\hat{v}}{4}\sqrt{\frac{|t|}{2}} e^{-\frac{\imath}{2}(\frac{\pi}{2}-\arg t)} e^{-\frac{\imath t}{2}} + O(t^{-\frac{1}{2}}e^{-\frac{\imath t}{2}}),\\
u(t) &= u_0(t)(1+o(1)),\\
\zeta(t) &= \zeta_0(t) + \frac{\hat{v}}{2}\sqrt{\frac{|t|}{2}} e^{\frac{\imath}{2}(\frac{\pi}{2}+\arg t)} e^{-\frac{\imath t}{2}} + O(t^{-\frac{1}{2}}e^{-\frac{\imath t}{2}}).
\end{aligned}
$$

Proof The proof is based on the same arguments as the proof of Propositions 3 and 4.

Theorem 1 *The polynomials: $\alpha_n(\Theta_0,\ \Theta_1,\ \Theta_\infty)$, $\beta_n(\Theta_0,\ \Theta_1,\ \Theta_\infty)$, and $\gamma_n(\Theta_0,\Theta_1,\Theta_\infty)$ have the following asymptotics as $n \to \infty (\equiv p \to \infty)$:*

$$
\begin{aligned}
&\alpha_{2p-1}(\Theta_0,\ \Theta_1,\ \Theta_\infty) = \mathcal{O}(C^p) +\\
&(-1)^p\frac{4^{2p}}{2\pi^2}\cos\frac{\pi\Theta_\infty}{2}\cos\frac{\pi}{2}(\Theta_0-\Theta_1)\cos\frac{\pi}{2}(\Theta_0+\Theta_1)\,\Gamma\Big(p-\frac{3}{4}\Big)\,\Gamma\Big(p-\frac{1}{4}\Big)\Big(1+\mathcal{O}\Big(\frac{1}{p}\Big)\Big),\\
&\alpha_{2p}(\Theta_0,\ \Theta_1,\ \Theta_\infty) = \mathcal{O}(C^p) +\\
&(-1)^p\frac{4^{2p+1}}{2\pi^2}\sin\frac{\pi\Theta_\infty}{2}\sin\frac{\pi}{2}(\Theta_0-\Theta_1)\sin\frac{\pi}{2}(\Theta_0+\Theta_1)\,\Gamma\Big(p-\frac{1}{4}\Big)\,\Gamma\Big(p+\frac{1}{4}\Big)\Big(1+\mathcal{O}\Big(\frac{1}{p}\Big)\Big),\\
&\beta_{2p-1}(\Theta_0,\ \Theta_1,\ \Theta_\infty) = \mathcal{O}(C^p) +\\
&(-1)^p\frac{4^{2p-1}}{2\pi^2}\cos\frac{\pi\Theta_\infty}{2}\cos\frac{\pi}{2}(\Theta_0-\Theta_1)\cos\frac{\pi}{2}(\Theta_0+\Theta_1)\,\Gamma\Big(p-\frac{1}{4}\Big)\,\Gamma\Big(p+\frac{1}{4}\Big)\Big(1+\mathcal{O}\Big(\frac{1}{p}\Big)\Big),\\
&\beta_{2p}(\Theta_0,\ \Theta_1,\ \Theta_\infty) = \mathcal{O}(C^p) +\\
&(-1)^p\frac{4^{2p}}{2\pi^2}\sin\frac{\pi\Theta_\infty}{2}\sin\frac{\pi}{2}(\Theta_0-\Theta_1)\sin\frac{\pi}{2}(\Theta_0+\Theta_1)\,\Gamma\Big(p+\frac{1}{4}\Big)\,\Gamma\Big(p+\frac{3}{4}\Big)\Big(1+\mathcal{O}\Big(\frac{1}{p}\Big)\Big),\\
&\gamma_{2p-1}(\Theta_0,\ \Theta_1,\ \Theta_\infty) = \mathcal{O}(C^p) +\\
&(-1)^p\frac{4^{2p-1}}{\pi^2}\sin\frac{\pi\Theta_\infty}{2}\sin\frac{\pi}{2}(\Theta_0-\Theta_1)\sin\frac{\pi}{2}(\Theta_0+\Theta_1)\,\Gamma\Big(p-\frac{1}{4}\Big)\,\Gamma\Big(p+\frac{1}{4}\Big)\Big(1+\mathcal{O}\Big(\frac{1}{p}\Big)\Big),\\
&\gamma_{2p}(\Theta_0,\ \Theta_1,\ \Theta_\infty) = \mathcal{O}(C^p) +
\end{aligned}
$$

$$-(-1)^p\frac{4^{2p}}{\pi^2}\cos\frac{\pi\Theta_\infty}{2}\,\cos\frac{\pi}{2}(\Theta_0-\Theta_1)\,\cos\frac{\pi}{2}(\Theta_0+\Theta_1)\,\Gamma\Big(p+\frac{1}{4}\Big)\,\Gamma\Big(p+\frac{3}{4}\Big)\,\Big(1+\mathcal{O}\Big(\frac{1}{p}\Big)\Big),$$

where $\Gamma(\,\cdot\,)$ *is the gamma-function* [13] *and* $C>0$ *is independent of* p.

Proof The proof is given in the next Section.

Corollary 4 *All the leading terms of the above asymptotics' vanish iff the parameters* Θ_0, Θ_1, *and* Θ_∞ *satisfy one of the conditions of Corollary 3. In this case, at least one, odd or even, subsequence of each sequence of the coefficients has the following asymptotic expansion:*

$$\begin{aligned}|\alpha_n(\Theta_0,\ \Theta_1,\ \Theta_\infty)| &\underset{n\to\infty}{=} a^n_{\Theta_0,\Theta_1,\Theta_\infty}(1+o(1)),\\ |\beta_n(\Theta_0,\ \Theta_1,\ \Theta_\infty)| &\underset{n\to\infty}{=} b^n_{\Theta_0,\Theta_1,\Theta_\infty}(1+o(1)),\\ |\gamma_n(\Theta_0,\ \Theta_1,\ \Theta_\infty)| &\underset{n\to\infty}{=} c^n_{\Theta_0,\Theta_1,\Theta_\infty}(1+o(1)),\end{aligned} \tag{3.14}$$

where $a_{\Theta_0,\Theta_1,\Theta_\infty}$, $b_{\Theta_0,\Theta_1,\Theta_\infty}$, *and* $c_{\Theta_0,\Theta_1,\Theta_\infty}$ *are numbers equal to the maxima of absolute values of poles of the rational solutions to system (2.2), (2.3) and Eq. (2.8) which are described in Corollary 3. If the asymptotics of one of the subsequences is not given by the corresponding equation in (3.14), then the absolute values of its members, at least for a rather large n, are less than the r.-h.s. of one of the corresponding equations in (3.14).*

Proof The statement follows from Theorem 1 with the help of Corollaries 2 and 3.

4 Derivation

To derive the large n asymptotics of the polynomials $\alpha_n(\Theta_0,\ \Theta_1,\ \Theta_\infty)$, $\beta_n(\Theta_0,\Theta_1,\Theta_\infty)$, and $\gamma_n(\Theta_0,\ \Theta_1,\ \Theta_\infty)$ (3.1)-(3.3), we use ideas based on the Borel transform. Here, we summarize some facts concerning the Borel transform, which are needed for the derivation of the results stated in Theorem 1. The proof of these results can be found in the monograph [12]: the following presentation is specific to the concrete problem under consideration.

Let $F(t)$ be one of the divergent series $y_F(t)$, $z_F(t)$, or $\zeta_F(t)$. Let us denote the coefficients of these expansions by f_n, i.e.,

$$F(t)=\sum f_n t^{-n}. \tag{4.1}$$

It is known that, the divergent expansions (3.1)–(3.3) for systems like (2.2)–(2.3) are Borel summable. In particular, in our case, this means that for some positive numbers: M, A, and a, the following inequality is valid, $|f_n/\Gamma(an+1)|<MA^{-n}$, where $\Gamma(\,\cdot\,)$ is the gamma function [13]. Actually, in our case $a=1$, but we use this fact only to make our derivation shorter: the fact which is essential for the following is that such a number $a>0$ exists. The Borel transform, $B_F(p)$, is defined by the following, convergent in some neighborhood of $p=0$, series:

$$B_F(p)=\sum\frac{f_n}{\Gamma(an+1)}p^n. \tag{4.2}$$

The analytic continuation of the function $B_F(p)$ into the sectors): $|\arg p+\pi k|<\pi/2, k=0,\pm1$ ($a=1$!) has a finite exponential type T. The Laplace transform of $B_F(\tau/t)$ with respect to τ is therefore convergent in the domain $|t| > T, |argt + \pi k| < \pi/2$ and yields the function $f(t)$ (the Borel sum of $F(t)$):

$$f(t) = \int_0^\infty d\tau e^{-\tau} B_F(t^{-1}\tau^a), \tag{4.3}$$

whose analytic continuation into the domain $|t| > T, |\arg t + \pi k| < \pi$ has the asymptotic expansion (4.1). Since the analytic function with the asymptotic expansion (4.1) in the sector $|\arg t + \pi k| < \pi$ is unique, the function f(t) (4.3) coincides with the corresponding doubly truncated solution, $f_k(t)$, i.e., with one of the functions: $y_k(t), z_k(t)$, or $\zeta_k(t), k = 0, \pm 1$.

The function $B_F(p)$ may have poles and cuts running from its singularities to the infinity. The previous discussion shows that the only possibility for such singularities (and cuts) is along the imaginary axis. The indications of these cuts are ruptures of the function $f(t)$: in the sectors $|\arg t + \pi k| < \pi/2$ this function coincides with $f_k(t)$. Thus in our case, there are two such ruptures: along the positive and negative imaginary semi-axes. To get the asymptotic formulas announced in Theorem 1 for f_n, we proceed as follows:

$$f_n = \Gamma(an+1)\tfrac{f_n}{\Gamma(an+1)} = \int\limits_0^{+\infty} d\tau e^{-\tau}\tau^{an}\tfrac{f_n}{\Gamma(an+1)} = \tfrac{1}{2\pi\imath}\int\limits_0^{+\infty} d\tau e^{-\tau} \oint\limits_{\Upsilon(\tau)} \tfrac{B_F(p\tau^a)}{p^{n+1}}\,dp \underset{n\to\infty}{=} \tfrac{1}{2\pi\imath}\times$$
$$\left(1+\mathcal{O}\left(\tfrac{1}{n}\right)\right)\int\limits_0^{+\infty} d\tau e^{-\tau}\left(\int\limits_{+i\infty}^{iA/\tau^a} + \int\limits_{-iA/\tau^a}^{-i\infty}\right)\tfrac{\Delta B_F(p\tau^a)}{p^{n+1}}\,dp\,\Big|_{p=\frac{1}{is}} = \tfrac{i^n}{2\pi}\left(1+\mathcal{O}\left(\tfrac{1}{n}\right)\right)\int\limits_0^{+\infty} d\tau e^{-\tau}\times$$
$$\left(\int\limits_{+0}^{\tau^a/A} + \int\limits_{-\tau^a/A}^{-0}\right) s^{n-1}\Delta B_F(\tfrac{\tau^a}{is})\,ds = \tfrac{i^n}{2\pi}\left(1+\mathcal{O}\left(\tfrac{1}{n}\right)\right)\left(\int\limits_{+0}^{+\infty} + \int\limits_{-\imath\infty}^{-0}\right) s^{n-1}\,ds\int\limits_0^{+\infty}\Delta B_F(\tfrac{\tau^a}{is})e^{-\tau}d\tau,$$

where $\Upsilon(\tau)$ is a positively oriented contour around 0 in the complex p-plane and $\Delta B_F(\cdot)$ is the difference of values of the Borel transform on the cut along the imaginary axis (r.h.s. minus l.h.s.). In the last integral we took into account the fact that $\Delta B_F(\tau^a/is) = 0$ when $|\tau^a|$ is varying from 0 to As. Finally, using Eq. (4.3), we arrive at the following asymptotic formula for f_n,

$$f_n \underset{n\to\infty}{=} \mathcal{O}(C^n) + \tag{4.4}$$
$$\frac{i^{n-1}}{2\pi}\left(1+\mathcal{O}\left(\frac{1}{n}\right)\right)\left(\int_0^{+\infty} s^{n-1}ds(f_0(is) - f_{+1}(is) - (-1)^n(f_0(-is) - f_{-1}(-is))\right),$$

where $C > 0$ and independent of n. The power-like error estimation in the last equation appears due to the fact that the integral representation (4.3) is valid only for $s > T$. Now, consequently substituting for f_k in the last equation the asymptotic results from Proposition 4 for the doubly truncated solutions, $y_k(t)$, $z_k(t)$, and $\zeta_k(t)$, defined in Proposition 2, we obtain the asymptotic expansions stated in the Theorem 1. Note, that by using the asymptotics in Eq. (4.4) instead of the exact solutions, we get an additional error which can also be estimated as $\mathcal{O}(C^n)$, where $C > 0$ is independent of n. The error estimates indicated in the asymptotic expansions given in Proposition 4 contribute to $\mathcal{O}(1/p)$-correction terms to f_n in Theorem 1.

Acknowledgment. A.V.K. is grateful to L.Vinet and P.Winternitz for the invitation to the Workshop "Theory of Nonlinear Special Functions: The Painlevé Transcendents" (May 13-18, 1996) at CRM Université de Montréal, where these results were announced, and P.M.Bleher and A.R.Its for the invitation to the joint AMS-SIAM-IMS summer

research conference "Random Matrices, Statistical Mechanics, and Painlevé Transcendents" (June 23-27, 1996) in Mount Holyoke, where this work was presented.
F.V.A. was partially supported by the grant PRO-MATH and the Soros Foundation under the grant ME-430. During the preparation of the final version of this paper A.V.K. was supported by Alexander von Humboldt Foundation.

Appendix: Applications to the Third Painlevé Transcendent

Here, we consider two special cases of our results: the first one ($\Theta_0 = \Theta_1 = \Theta_\infty = 0$) concerns the function $\sigma(x;\ \lambda)$, which is discussed in the Sec. 1, the second example ($\Theta_0 = \Theta_1 = 1/2, \Theta_\infty = 0$) shows how one can reproduce one of the first connection formulas for the Painlevé equations [7] (in our notations it is a connection formula for asymptotics of P_5 along the imaginary axis) by using the connection results for asymptotics on the real axis [10] and analytical continuation of Sec. 3.

Proposition 7 *A solution of the system (2.2), (2.3) with $\Theta_0 = \Theta_1 = \Theta_\infty = 0$ is holomorphic at $t = 0$ iff for some $z_0 \in \mathbb{C}$, $y(t)$ and $z(t)$ have the following asymptotic expansions as $t = 0$:*

$$
\begin{aligned}
y(t) = {} & 1 + t + (\frac{1}{2} - z_0)t^2 + (\frac{1}{6} - z_0)t^3 + (\frac{1}{4!} - \frac{13}{24}z_0 + \frac{z_0^2}{2})t^4 + (\frac{1}{5!} - \frac{5}{24}z_0 + \frac{5}{9}z_0^2)t^5 \\
& + (\frac{1}{6!} - \frac{137}{2160}z_0 + \frac{25}{72}z_0^2 - \frac{3}{16}z_0^3)t^6 + (\frac{1}{7!} - \frac{7}{432}z_0 + \frac{847}{5400}z_0^2 - \frac{35}{144}z_0^3)t^7 + (\frac{z_0^4}{16} - \frac{19}{108}z_0^3 \\
& + \frac{1}{8!} - \frac{871}{241920}z_0 + \frac{4907}{86400}z_0^2)t^8 + (\frac{1}{9!} - \frac{517}{725760}z_0 + \frac{8189}{470400}z_0^2 - \frac{497}{5400}z_0^3 + \frac{3}{32}z_0^4)t^9 \\
& + (\frac{1}{10!} - \frac{1543}{12096000}z_0 + \frac{989}{211680}z_0^2 - \frac{48019}{1244160}z_0^3 + \frac{805}{10368}z_0^4 - \frac{5}{256}z_0^5)t^{10} \ldots,
\end{aligned} \tag{A.1}
$$

$$
\begin{aligned}
z(t) = {} & z_0 + 2z_0^2 t + 3z_0^3 t^2 + (\frac{z_0^2}{9} + 4z_0^4)t^3 + (\frac{25}{144}z_0^3 + 5z_0^5)t^4 + (\frac{z_0^2}{300} + \frac{z_0^4}{4} + 6z_0^6)t^5 \\
& + (\frac{49}{10800}z_0^3 + \frac{49}{144}z_0^5 + 7z_0^7)t^6 + (\frac{1}{17640}z_0^2 + \frac{121}{16200}z_0^4 + \frac{4}{9}z_0^6 + 8z_0^8)t^7 + (\frac{761}{11289600}z_0^3 \\
& + \frac{73}{6400}z_0^5 + \frac{9}{16}z_0^7 + 9z_0^9)t^8 + (\frac{1}{1632960}z_0^2 + \frac{1349}{9144576}z_0^4 + \frac{19}{1152}z_0^6 + \frac{25}{36}z_0^8 + 10z_0^{10})t^9 \\
& + (\frac{7381}{11430720000}z_0^3 + \frac{234377}{914457600}z_0^5 + \frac{5929}{259200}z_0^7 + \frac{121}{144}z_0^9 + 11z_0^{11})t^{10} \ldots .
\end{aligned} \tag{A.2}
$$

In that case, expansions (A.1) and (A.2), of course, coincide with the Taylor expansions of the functions $y(t)$ and $z(t)$ at $t = 0$; moreover, the parameter z_0 characterizes them uniquely.

Proof By substituting the formal power series for $y(t)$ and $z(t)$ into the system (2.2), (2.3) one gets a recursion relation which allow one to uniquely determine all the coefficients as soon as $z_0 \in \mathbb{C}$ is given and prove the convergence in the standard manner.

Proposition 8 *In the case $\Theta_0 = \Theta_1 = \Theta_\infty = 0$, the function $\zeta(t)$ (2.7) is holomorphic at $t = 0$ iff the corresponding solution $(y(t), z(t))$ of the system (2.2), (2.3) is holomorphic at $t = 0$.*

Proof The proof follows from Proposition 7 and the equations:

$$
\zeta = -tz + \frac{z^2}{y}(1-y)^2, \qquad z = -\zeta', \quad y = 1 + \frac{1}{2z^2}(tz' + \zeta + tz), \tag{A.3}
$$

where the prime denotes differentiation with respect to t. Eqs. (A.3) result from system (2.2), (2.3) and definition (2.7). If $\zeta(t)$ is a holomorphic function at $t=0$, then according to the last equation in (A.3), the function $y(t)$ can, theoretically, have a pole at $t=0$, but this contradicts system (2.2), (2.3).

Remark 3 It is easy to see that the coefficient z_n before t^n in the Taylor series (A.2) is a polynomial of order $n+1$ of the parameter z_0. For $n \geq 2$ the senior coefficients have the following structure: $z_n = (n+1)z_0^{n+1} + \frac{(n+1)^2}{3^2 2^4} z_0^n + \ldots$, while the structure of the junior coefficients is not evident. Denoting by y_n the coefficient before t^n in the Taylor series (A.1): $y_1 = 1, y_2 = 1/2 - z_0, \ldots$, we find (see Eq. (2.9)) that

$$\sum_{k+l=m} (-)^l p_l(-z_0) p_k(z_0) + (-)^m p_m(-z_0) + p_m(z_0) = 0.$$

The function $y(t)$ (A.1) can be presented in the form: $y(t) = e^t + z_0 f(t,\ z_0)$, where the function $f(t,\ z_0)$ is holomorphic at $(t=0, z_0=0)$. The Tailor expansion for the function $\zeta(t)$ can be obtained by integration of the expansion (A.2), $\zeta(t) = -\int_0^t z(t)\,dt$. We write it in terms of the function $\sigma(x;\ \lambda)$(see Eq. (1.2)) by using the relation $\sigma(x;\ \lambda) = \zeta(0,\ 0,\ 0;\ 4ix)|_{z0=-i\lambda/2}$ and introducing the parameter $\mu = 2\lambda$:

$$\begin{aligned}\sigma(x;\ \frac{\mu}{2}) &= -\mu x - \mu^2 x^2 - \mu^3 x^3 - (\mu^4 - \frac{4}{9}\mu^2)x^4 - (\mu^5 - \frac{5}{9}\mu^3)x^5 \qquad \text{(A.4)}\\ &-(\mu^6 - \frac{2}{3}\mu^4 + \frac{32}{225}\mu^2)x^6 - (\mu^7 - \frac{7}{9}\mu^5 + \frac{112}{675}\mu^3)x^7 - (\mu^8 - \frac{8}{9}\mu^6 + \frac{484}{2025}\mu^4 - \frac{64}{2205}\mu^2)x^8\\ &-(\mu^9 - \mu^7 + \frac{73}{225}\mu^5 - \frac{3044}{99225}\mu^3)x^9 - (\mu^{10} - \frac{10}{9}\mu^8 + \frac{19}{45}\mu^6 - \frac{10792}{178605}\mu^4 + \frac{512}{127575}\mu^2)x^{10}\\ &-(\mu^{11} - \frac{11}{9}\mu^9 + \frac{1078}{2025}\mu^7 - \frac{85228}{893025}\mu^5 + \frac{85888}{22325625}\mu^3)x^{11} \ldots .\end{aligned}$$

Proposition 9 *The solution of system (2.2), (2.3) with $\Theta_0 = \Theta_1 = \Theta_\infty = 0$ holomorphic at $t=0$ has the truncated behavior as $t \to \infty$ iff the parameter $z_0 = \pm\frac{1}{2\pi\imath}$. The solution with the expansions (A.1)–(A.2) with $z_0 = \frac{1}{2\pi\imath}$, $\zeta_u(0,\ 0,\ 0;t)$, is the upper-truncated (see Proposition 5) and can be characterized by the monodromy coordinates:*

$$m_{11}^0 = 2,\ \ m_{21}^0 = (iu)^{-1},\ \ m_{11}^1 = 0,\ \ m_{12}^1 = iu,\ \ s_1 = s_2 = 0. \qquad \text{(A.5)}$$

The solution with the expansions (A.1)–(A.2) with $z_0 = -\frac{1}{2\pi\imath}$, $\zeta_l(0,\ 0,\ 0;t)$, is the lower-truncated (see Proposition 6) and can be characterized by the monodromy coordinates:

$$m_{11}^0 = 0,\ \ m_{21}^0 = (iu)^{-1},\ \ m_{11}^1 = 2,\ \ m_{12}^1 = iu,\ \ s_1 = s_2 = 0. \qquad \text{(A.6)}$$

Proof The assertion of the proposition follows from our previous work (see Theorems 4.1–4.3 [10]), where the monodromy coordinates of the general solution of system (2.2)–(2.4) were calculated. In these theorems the small t asymptotics of the general solution of system (2.2)–(2.4) is characterized by the complex parameters σ and $\hat{z}$. From Theorems 4.1 and 4.3 of [10] we see that the solution can be holomorphic at $t=0$ only if the parameter $\sigma = 0$ (in the case $\sigma = 1$ one of the functions $y(t)$ or $z(t)$ has a pole at the origin. Substitute into the formulas (4.1) $\Theta_0 = \Theta_1 = \Theta_\infty = 0$ and then consider the limit $\sigma \to 0$ to find that $\hat{s} \underset{\sigma\to 0}{=} \sigma/(8z_0) + O(\sigma)$. On the other hand, Theorem 4.2 shows that in the case $\Theta_0 = \Theta_1 = \Theta_\infty = \sigma = 0$ the Stokes multipliers $s_1 = s_2 = 0$ and the limits as $\sigma \to 0$ of $m_{11}^0 = 1 - 2\pi\imath z_0$ and $m_{22}^0 = 1 + 2\pi\imath z_0$. Comparing these formulas with the monodromy coordinates of the upper/lower-truncated solutions (see Propositions 5 and 6) we get that, necessarily, $z_0 = \pm 1/(2\pi\imath)$. Finally, calculating the remaining monodromy coordinates we arrive at Eqs. (A.5)–(A.6).

Corollary 5 *For $\lambda = 1/\pi$ the function $\sigma(x;\lambda)$ defined by Eqs. (1.1) and (1.2) satisfies the following equality,*

$$\sigma(x;\frac{1}{\pi}) = \zeta_u(0,\ 0,\ 0; 4ix) = \zeta_l(0,\ 0,\ 0; -4ix), \tag{A.7}$$

where $\zeta_u(0,\ 0,\ 0;\ \cdot)$ and $\zeta_l(0,\ 0,\ 0;\ \cdot)$ are the solutions of Eq. (2.8) which are specified in Proposition 9. The asymptotics of the sequence c_n (see Eq. (1.3)) is as follows:

$$c_n = -\frac{\gamma_{2n}(0,\ 0,\ 0)}{(4i)^{2n}} \underset{n\to\infty}{=} \frac{1}{\pi^2}\Gamma\left(n+\frac{1}{4}\right)\Gamma\left(n+\frac{3}{4}\right)\left(1+\mathcal{O}\left(\frac{1}{n}\right)\right). \tag{A.8}$$

Proof The first equality in (A.7) follows from the definition of $\sigma(x;\lambda)$ and Proposition 9. The second equality results from Proposition 9 and formulas (2.10)–(2.12). Equation (A.8) is a special case of the last formula in Theorem 1.

Remark 4 Since (see [13])

$$\Gamma\left(n+\frac{1}{4}\right)\Gamma\left(n+\frac{3}{4}\right) \underset{n\to\infty}{=} n!(n-1)!\left(1-\frac{0.1875}{n}+\mathcal{O}\left(\frac{1}{n^2}\right)\right), \tag{A.9}$$

Eq. (A.8) proves the conjecture (1.5) by E.L.Basor and C.A.Tracy. More exactly we can write

$$c_n \underset{n\to\infty}{=} \frac{1}{\pi^2} n!(n-1)!\left(1+\mathcal{O}\left(\frac{1}{n}\right)\right). \tag{A.10}$$

Although, formally, the error estimates in formulas (A.8) and (A.10) are of the same order, the leading term in our formula (A.8) more exactly approximates the numbers c_n, since the term $\frac{3}{16n}$ in Eq. (A.9) add the systematic error equal to $\frac{3}{16\pi^2}(n-1)!^2$. In the last two columns of the following table, we present the results of calculations of the numbers c_n for $n = 5, 10$, and 15 via the leading terms of asymptotic expansions (A.8) and (A.10).

c_n	Exact value	Eq. (A.8)	Eq. (A.10)
c_5	$\frac{1080091}{4096} = 263.694091796875$	$281.0779\ldots$	$291.8050\ldots$
c_{10}	$\frac{266126511253407125}{2097152} = 0.12689\ldots\cdot 10^{12}$	$0.1309\ldots\cdot 10^{12}$	$0.1334\ldots\cdot 10^{12}$
c_{15}	$\frac{2399091646222867044694815448942 1}{2147483648} = 0.1117\ldots\cdot 10^{23}$	$0.1140\ldots\cdot 10^{23}$	$0.1155\ldots\cdot 10^{23}$

Our numerical calculations show that the error term in Eq. (A.8) can be evaluated as $\approx -0.3105/n$. To calculate it explicitly or, more generally, to develop the asymptotic expansions obtained in Theorem 1 into complete asymptotic series, one has to study the so-called trans-series for system (2.2), (2.3) and Eq. (2.8).

Proposition 10 *If $\Theta_0 = \Theta_1 = 1/2$ and $\Theta_\infty = 0$, then the upper-truncated solutions, $y(i\tau)$, (see Proposition 5) with the parameter $\hat{w} = \frac{2i}{\sqrt{2\pi}}\sin\frac{\pi\sigma}{2}$, where $0 < \Re\sigma < 1$, have the following asymptotics:*

$$y(i\tau) \underset{\tau\to+0}{=} 1+4iK\left(\frac{\tau}{2}\right)^{\sigma}+\mathcal{O}\left(\tau^{2\sigma}\right)+\mathcal{O}(\tau),\quad y(i\tau) \underset{\tau\to+\infty}{=} -1-\frac{4i\sin\frac{\pi\sigma}{2}}{\sqrt{\pi\tau}}e^{-\frac{\tau}{2}}\left(1+\mathcal{O}\left(\frac{e^{-\frac{\tau}{2}}}{\sqrt{\tau}}\right)\right),$$

$$z(i\tau) \underset{\tau\to+0}{=} \frac{(1+\sigma)}{8} iK \left(\frac{\tau}{2}\right)^{\sigma} - \frac{1}{4} + \frac{(1-\sigma)}{8iK} \left(\frac{\tau}{2}\right)^{-\sigma} + \mathcal{O}(\tau^{1-\sigma}),$$
$$\zeta(i\tau) \underset{\tau\to+0}{=} \frac{\sigma^2}{4} - \frac{is}{8}\left((iK)^{\frac{1}{2}}\left(\frac{\tau}{2}\right)^{\frac{\sigma}{2}} - (iK)^{-\frac{1}{2}}\left(\frac{\tau}{2}\right)^{-\frac{\sigma}{2}}\right)^2 + \mathcal{O}(\tau^{2-2\sigma}),$$
where $K = 2^{-3\sigma}\frac{\Gamma\left(\frac{1}{2}-\frac{\sigma}{2}\right)}{\Gamma\left(\frac{1}{2}+\frac{\sigma}{2}\right)}$.

Proof These results are a consequence of Theorems 4.1 and 4.2 of our previous work [10]. More exactly, one has to put $\Theta_0 = \Theta_1 = 1/2$ and $\Theta_\infty = 0$ into formulas (4.3) and (4.4) of Theorem 4.2: this considerably simplifies them, in particular, parameters $\hat{m}^p_{11} = \hat{m}^p_{22} = 0$. After that, by equating the monodromy coordinates of the upper-truncated solutions given in Proposition 5 with the ones given in Theorem 4.2, one finds the relations between the parameter w, which defines the asymptotics as $t \to \infty$, and the parameters σ and $\hat{s}$ (see formulas (4.1) and (4.2)), which define the asymptotics of the general solution of system (2.2), (2.3) as $t \to 0$. More precisely, one gets the equations relating w with σ (see above) and $\hat{u}$ with $r, \hat{u} = ir$, where the parameter r defines the asymptotics of the function $u(t)$ (see Eq. (2.4)): $u(t) \underset{t\to0}{=} -r(1+\mathcal{O}(t^\sigma))$, by comparing formulas for the Stokes multipliers, s_1 and s_2. All the remaining equations are equivalent to $(\sigma-1)\hat{s} = i(1+\sigma)K$, where K is given above. Finally, one finds the asymptotics of the functions $y(t), z(t)$, and $\zeta(t)$ as $t \to 0$ by substituting $\hat{s}$ into Eqs. (4.1) and (4.2) of the work [10].

Corollary 6 [7] *For each $\sigma \in \mathbb{C}, 0 < \Re\sigma < 1$, there exists a unique solution, $\eta = \eta(\tau)$, of the third Painlevé equation,*

$$\eta'' = \frac{\eta'^2}{\eta} - \frac{\eta'}{\tau} + \frac{1}{16}\left(\eta^3 - \eta^{-1}\right),$$

where the primes denote differentiation with respect to τ, with the following asymptotics:

$$\eta(\tau) \underset{\tau\to+0}{=} K\left(\frac{\tau}{2}\right)^{\sigma} + \mathcal{O}\left(\tau^{2\sigma}\right) + \mathcal{O}(\tau), \qquad \eta(\tau) \underset{\tau\to+\infty}{=} 1 - \frac{2\sin\frac{\pi\sigma}{2}}{\sqrt{\pi\tau}} e^{-\frac{\tau}{2}}\left(1 + \mathcal{O}\left(\frac{e^{-\frac{\tau}{2}}}{\sqrt{\tau}}\right)\right),$$

in which K is given in Proposition 10.

Proof $\eta(\tau) = -i\frac{\sqrt{y(i\tau)}-1}{\sqrt{y(i\tau)}+1}$.

References

[1] E. L. Basor and C. A. Tracy, Some Problems Associated with the Asymptotics of τ-functions, Preprint RIMS-845, Kyoto University, November 1991.

[2] M. L. Mehta, Random Matrices, 2nd edition, Academic Press, San Diego (1991).

[3] C. A. Tracy and H. Widom, Introduction to Random Matrices, In: Helminck, G. F. (ed.), Geometric and quantum aspects of integrable systems. Proceedings of the eighth Scheveningen conference, Scheveningen, The Netherlands, August 16-21, 1992. Berlin: Springer-Verlag, (ISBN 3- 540-57365-8/hbk). Lect. Notes Phys. 424, 103-130 (1993); hep-th 9210073.

[4] B. M. McCoy and Sh. Tang, Connection formulae for Painlevé V functions, Physica **19D**, 42–72 (1986); **20D**, 187–216 (1986).

[5] L. A. Bordag and A. V. Kitaev, Algebraic and rational solutions of the fifth Painlevé equation (Russian), Comm. JINR, Dubna, (1986).

[6] A. V. Kitaev, C. K. Law, and J. B. McLeod, Rational solutions of the fifth Painlevé equation, DIE **7**, 967–1000 (1994).

[7] B. M. McCoy, C. A. Tracy, and T. T. Wu, Painlevé functions of the third kind, J.Math.Phys.**18**, 1058–1092 (1977).

[8] M. Jimbo and T. Miwa, Monodromy preserving deformation of linear ordinary differential equations with rational coefficients II, Physica **2D**, 407–448 (1981).

[9] K. Okamoto, On the τ-function of the Painlevé equations, Physica **2D**, 525–535 (1981)

[10] F. V. Andreev and A. V. Kitaev, Connection formulas for the asymptotics of the fifth Painlevé transcendent on the real axis I, Sfb 288 Preprint 160, TU-Berlin (1995) (http://www-sfb288.math.tu-berlin.de).

[11] F. V. Andreev and A. V. Kitaev, Connection formulas for the asymptotics of the fifth Painlevé transcendent on the real axis II, Sfb 288 Preprint 196, TU-Berlin (1996) (http://www-sfb288.math.tu-berlin.de).

[12] G. Doetsch, Handbuch der Laplace-Transformation I, II, Birkhäuser Verlag Basel (1950), (1955).

[13] H. Bateman and A. Erdélyi, Higher Transcendental Functions, I, II, McGraw–Hill Book Company Inc. (1953).