SPECIAL FUNCTIONS OF THE ISOMONODROMY TYPE

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Abstract

We introduce a new notion, a special function of the isomonodromy type, and show that most of the functions known in applied mathematics and mathematical physics as special functions belong to this type. This definition provides a unified approach to the theories of "linear" special functions, i.e., classical higher transcendental functions, and "non-linear" special functions, i.e.,the functions of the Painlevé type. We also show that, our definition has not only a conceptual (methodological) value: many well-known properties of the single-variable special functions can be re-derived from the isomonodromy point of view, but the practical one too: (1) this approach is already known as the extremely useful one in the theory of the Painlevé functions, (2) many properties (some of them can be new ones) of the multi-variable special functions can be obtained on the regular basis, and (3) it gives rise to several interesting mathematical questions, which are discussed in the paper. We also make some remarks concerning the analogous description (via q-deformations) of the q-special functions.

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1 Introduction

In his Foreword to the book by W.Miller, Jr., Symmetry and Separation of Variables [1], R.Askey wrote: "One term which has not been defined so far is "special function". My definition is simple, but not time invariant. A function is a special function if it occurs often enough so that it gets a name." Although since that time (1977), there has been a very intensive and extensive development of the field of special functions, most of the people who apply special functions to the solution of various problems in mathematical physics and applied mathematics understand under the notion of special functions a set of the functions which can be found in one of the well-known reference books like [2, 3, 5, 6, 7] and many others. The most powerful and well-known now approach to the field of special functions, which allow one to introduce the new functions and to obtain many of their properties, is the group-theoretical approach presented in the fundamental monograph [5]. Nevertheless, this approach leaves aside a very large area of studies related with the special functions, namely, investigation of their, actually special, analytic and asymptotic properties. On the other hand, several thousands of works written about Painlevé equations and their multi-variable generalizations show that, the latter functions fit well the R.Askey's definition mentioned above, but these functions also stay aside from the group-theoretical approach.

The goal of this paper is to define a new notion, a special function of the isomonodromy type, and show that most of the special functions belong to this type, i.e., define isomonodromy deformations of linear ordinary differential equations (ODEs) with rational coefficients. We think that the latter fact is rather important: actually, one can put this property as the definition of the corresponding special functions! Moreover, pure methodologically, the definition we are going to introduce suggests a rather simple and clear picture for the general structure of the "field" of special functions and gives rise to a different (alternative to the group–theoretical) approach to the subject. By using this approach one can treat in a natural way not only the classical higher transcendental functions and the Painlevé functions of one and several variables but to obtain many of their basic properties. The lack of the latter approach is the so–called addition theorems for the classical higher transcendental functions of the hypergeometric type. It is not clear how to get them on the basis of the isomonodromy approach, while, as it is well known, in the group-theoretical approach such theorems arise in a very natural way: as a result of the group homomorphism property for matrix representations of corresponding groups. On the other hand, the addition theorems for the elliptic functions gain an interpretation in the isomonodromy approach as a kind of Bäcklund transformations, which are related with the Schlesinger transformations for the auxiliary linear ODEs [8]. Although, on the basis of the isomonodromy approach, one can hardly expect to get any new formula-type result for the basic special functions, especially of one variable, this approach is already known as the very efficient one for the "non-linear" special functions of the Painlevé type and could be quite useful for multi-variable "linear" special functions of the hypergeometric type. Before a description of the content of this paper we would like to draw attention of the reader to the following works where different features of an analogy between the "linear" and "non-linear" special functions are described from the isomonodromy point of view: [10, 11, 12, 13].

In Section 2 we give an elementary introduction to the isomonodromy deformations of ODEs. The purpose of this section is to make our presentation as self-contained as possible. A more sophisticated introduction, which covers not only many details omitted here and also includes many further references and different ramifications, can be found in review [9]. In the end of this section the basic definitions which constitutes the basis of the isomonodromy approach to the special functions are given.

In Section 3 we consider the famous Gamma-function to illustrate the general scheme of the isomonodromy approach to the special functions outlined in the previous section.

Section 4 is devoted to the other very important special function – the Gauß hypergeometric function. Presentation in this section is not as detailed as in the previous one, but we discuss here interesting, from our point of view, further ramifications and a generalization to the multi-variable case.

In Section 5 we consider the isomonodromy definition of the Bessel function.

In Section 6 we draw attention to the analogy between the classical special functions of the isomonodromy type and the Painlevé transcendents and discuss possible modifications of the isomonodromy deformations to the so-called q-deformations as an approach to the q-special functions.

This paper is written to retain the flavor of the conference talk and is, therefore, more informal in some parts than mathematical articles of the regular type.

2 Isomonodromy Deformations

The goal of this section is to recall some basic facts concerning the isomonodromy deformations of matrix linear ODEs and introduce the notions and notations which will be used in this paper. Throughout this paper we consider matrix linear ODEs of the first order,

$$\frac{d}{d\lambda}\Psi = A(\lambda)\Psi, \tag{2.1}$$

where $A(\lambda) \in sl_n(\mathbb{C})$ and $\Psi \in SL(n,\ \mathbb{C})$. For brevity we sometimes refer to $A(\lambda)$ as the *connection matrix*; it is always supposed to be a rational function of λ. In this introductory section we treat only the case in which the connection matrix $A(\lambda)$ has the poles only of the first order:

$$A(\lambda) = \sum_{k=1}^{N} \frac{A_k}{\lambda - t_k}, \quad i \neq j \Rightarrow t_i \neq t_j, \quad \frac{d}{d\lambda}A_k = 0. \tag{2.2}$$

Although in Secs. 3 and 5 the reader will meet examples with the poles of the second order, the following introduction should be enough to grasp the general ideas. In Secs. 3 and 5 we use the notion of the Stokes multipliers, which characterize the structure of the singular behavior of the $\Psi(\lambda)$ in a neighborhood of a higher order pole; the reader can find appropriate definitions in the papers [14] and [15].

Fix $\lambda_0 \in \mathbb{C} \setminus \{t_1, t_2, \ldots, t_N, \infty\}$ and impose the following *normalization condition* to fix the fundamental solution of Eq. (2.1):

$$\Psi(\lambda_0) = I. \tag{2.3}$$

Now one defines the *monodromy matrices*,

$$M_k = \Psi(\lambda_0)\,|_{\gamma_k}\,, \quad k = 1, \ldots, N, \infty,$$

as the analytic continuation of the fundamental solution normalized by condition (2.3) along the generators, γ_k, of the fundamental group $\pi_1(\mathbb{C} \setminus \{t_1, t_2, \ldots, t_N, \infty\}, \lambda_0)$ defined in Fig. 1. The monodromy matrices, M_k, satisfy the *cyclic relation*,

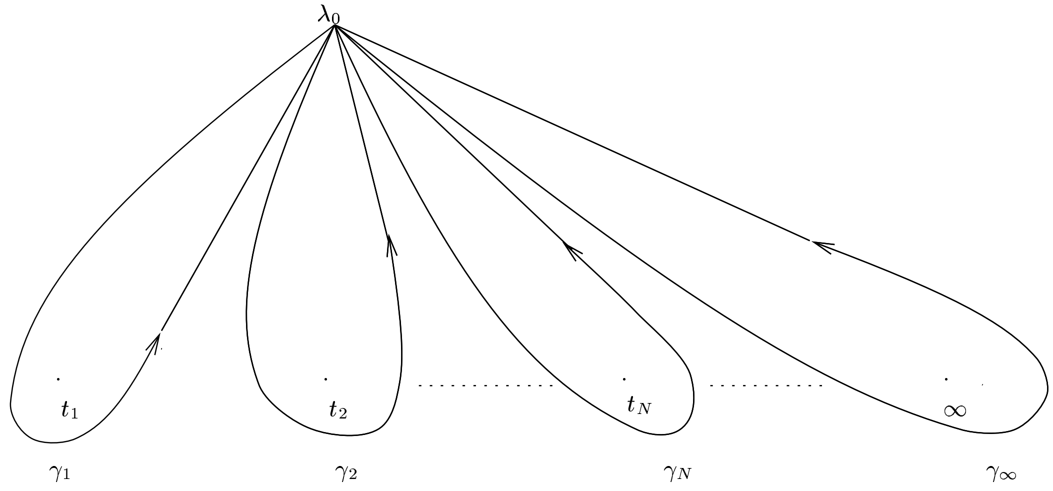

Figure 1: Generators of $\pi_1(\mathbb{C} \setminus \{t_1, t_2, \ldots, t_N, \infty\},\ \lambda_0)$.

$$M_\infty M_N \ldots M_1 = I. \tag{2.4}$$

The matrix subgroup of $SL(n, \mathbb{C})$ generated by the monodromy matrices, M_k, is called a monodromy representation of the fundamental group or just the *monodromy group*. In many different studies and, in particular, in our context related with the special functions, one arrives in a natural way to the *isomonodromy condition*:

$$\frac{d}{dt_i} M_k = \frac{d}{dt_i} M_k(A_1, \ldots, A_N; t_1, \ldots, t_N) = 0. \tag{2.5}$$

Using this this condition one proves that the function $\Psi(\lambda)$ along with the Eq. (2.1) satisfies the following system of ODEs:

$$\frac{\partial}{\partial t_i} \Psi = \left(\frac{A_i}{\lambda_0 - t_i} - \frac{A_i}{\lambda - t_i} \right) \Psi, \quad i = 1, \ldots, N. \tag{2.6}$$

Very often, one practical modification, namely $\lambda_0 = \infty$, is used to remove the auxiliary parameter λ_0. Suppose, that in general situation the eigenvalues $-\theta^1_\infty, \ldots, -\theta^n_\infty$ of the matrix $A_1 + \ldots + A_N$ are such that for all $i,\ k \in \{1, \ldots, n\}$ and the differences, $\theta^i_\infty - \theta^k_\infty \notin \mathbb{Z}$, then, possibly after a gauge transformation of the form $\Psi \to G\Psi$, one uses the following condition:

$$\Psi \underset{\lambda \to \infty}{\sim} \lambda^{-\mathrm{diag}(\theta^1_\infty, \ldots, \theta^n_\infty)}, \tag{2.7}$$

instead of the normalization (2.3), Of course, to use the normalization (2.7) one has to fix a branch of the $\log \lambda$. For the normalization (2.7) the monodromy matrix $M_\infty = \exp\{2\pi i\, \mathrm{diag}(\theta^1_\infty, \ldots, \theta^n_\infty)\}$. The compatibility condition of the system of differential Eqs. (2.6) for $\lambda_0 = \infty$ and Eq. (2.1) is known (since 1912) as the Schlesinger equations or the system of isomonodromy deformations of Eq. (2.1):

$$k \neq i: \qquad \frac{\partial A_k}{\partial t_i} = \frac{[A_i, A_k]}{t_i - t_k}, \tag{2.8}$$

$$k = i: \qquad \frac{\partial A_i}{\partial t_i} = -\sum_{\substack{k=1 \\ k \neq i}}^{N} \frac{[A_i, A_k]}{t_i - t_k}. \tag{2.9}$$

Note that due to the condition (2.7)

$$\sum_{k=1}^{N} A_k = -\mathrm{diag}(\theta_\infty^1, \ldots, \theta_\infty^n), \tag{2.10}$$

and consequently, by using Eqs. (2.8) one finds that Eqs. (2.9) are equivalent to the following system:

$$\partial_{t_i} \theta_\infty^1 = \ldots = \partial_{t_i} \theta_\infty^n = 0, \quad i = 1, \ldots, n. \tag{2.11}$$

In this paragraph we briefly outline some modifications which are needed in the case of the higher order pole of $A(\lambda)$ at $\lambda = \infty$. We suppose that $A(\lambda) \in sl_2(\mathbb{C})$ and has the following asymptotic behavior

$$A(\lambda) \underset{\lambda \to \infty}{=} \tau_M \sigma_3 \lambda^M + \mathcal{O}\left(\lambda^{M-1}\right), \quad M \in \mathbb{Z}, \quad \tau_M \in \mathbb{C}, \quad \sigma_3 = \left(\begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right).$$

In this case instead of the normalization condition (2.7) usually the following one is used

$$\Psi \underset{\lambda \to \infty}{=} \left(I + \mathcal{O}\left(\frac{1}{\lambda}\right)\right) \exp\left\{ \left(\sum_{k=1}^{M} \tau_k \frac{\lambda^{k+1}}{k+1} + \theta_\infty \log \lambda \right) \sigma_3 \right\}. \tag{2.12}$$

We call the parameters τ_k together with the parameters t_k as the *pole parameters*. Further, where it can not be the course of a confusion, we denote the set of the pole parameters simply as $t_1, \ldots, t_N$ having in mind that in the case of the higher order poles one has to add to the latter parameters the set of the parameters τ_k. Actually, the higher order poles can be viewed as a coalescence of several single poles and after some regularizing procedure one can even trace: how the τ_k-like parameters originated from the several parameters t_j corresponding to the merging single poles. Some simplest examples of this sort were considered in the work [17]. As a function of the parameters τ_k the function Ψ satisfies the following system of equations:

$$\frac{\partial}{\partial \tau_k} \Psi = B_k(\lambda) \Psi, \quad \textit{where} \quad B_k(\lambda) \underset{\lambda \to \infty}{=} \frac{\lambda^{k+1}}{k+1} \sigma_3 + \mathcal{O}\left(\lambda^k\right) \tag{2.13}$$

is a polynomial of λ. The system of isomonodromy deformations in this case is the overdetermined system of partial differential equations which represent the compatibility condition (for arbitrary λ) of system (2.13) and Eq. (2.1).

Remark 1 If $A(\lambda)$ as a rational function of λ has only one pole at $\lambda = \infty$, then $M_\infty \equiv I$; so that for a definition of the isomonodromy deformations (2.5) we need a more detailed description of the asymptotic behavior of $\Psi(\lambda)$ in a neighborhood of the infinity point (the *Stokes phenomenon*). Taking into account this more complicated situation, we will sometimes use the term *monodromy data* to denote the complete set of the monodromy parameters which include: (1) the matrix elements of M_k, (2) the eigenvalues of the matrices A_k, and (3) the so-called Stokes multipliers (see [9, 14, 15, 16, 17]).

The *Schlesinger transformations – the discrete isomonodromy deformations* play an important role in many questions related with the isomonodromy deformations. For a general introduction refs. [15, 16] could be recommended; for a presentation more specific to this work see [17]. Some concrete examples of these transformations the reader will meet in Secs. 3 and 5 below, here we write down one of the possible types of these transformations just to give a general idea about them:

$$\hat{\Psi}(\lambda) = L\Psi(\lambda), \quad L = \sqrt{\lambda - t_k}\sigma^{\pm} + \frac{1}{\sqrt{\lambda - t_k}}J_k^{\pm}, \quad \sigma^+ = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \sigma^- = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \tag{2.14}$$

Here: (1) t_k is one of the poles of the matrix $A(\lambda)$ (2.2); (2) the function $\hat{\Psi}(\lambda)$ solves an equation of the form (2.1), (2.2) but with new matrices $\hat{A}_i$ instead of A_i, for $i \neq k$ the matrices $\hat{A}_i$ and A_i have equal eigenvalues (note that here $n = 2$), while for $i = k$ the eigenvalues differ by $\pm\frac{1}{2}$; (3) $\det L \equiv 1$; and (4) the matrices $J_k^{\pm}$ have the vanishing determinants ($\det J_k^{\pm} = 0$) and can be specified in terms of the matrices A_k (see [17]). The monodromy matrices $\hat{M}_i$ corresponding to the function $\hat{\Psi}(\lambda)$ are related with the ones for the function Ψ, M_i in the following way:

$$\hat{M}_i = M_i, \text{ for } i \neq \infty, k; \; \hat{M}_\infty = -M_\infty, \; \hat{M}_k = -M_k.$$

It is important to notice that, if $A_(\lambda)$ is of the triangular form, $A(\lambda)_{ij} = 0$ for $j > i$, then there exists transformation (2.14) which preserves its triangular structure. The compatibility condition of Eqs. (2.1) and (2.14), which is a system of difference (in general case non-linear) equations for the matrix elements of A_k, defines the *discrete isomonodromy deformations*

Now we are ready to formulate our basic definitions:

Definition 1 *A function $f(t_1, \; t_2, \ldots, t_N)$ is called a special function of the isomonodromy type if it defines the general solution to the system of isomonodromy deformations for some linear matrix ODE with the pole parameters: $t_1, \; t_2, \ldots, t_N$.*

Definition 2 *We call a special function the linear one if it defines an isomonodromy deformation of a linear ODE with a connection matrix $A(\lambda)$ of the triangular form, i.e., $A(\lambda)_{ij} = 0$ for $j > i$.*

Definition 3 *We call a special function g a non-linear analog of a linear special function f if they: 1) define isomonodromy deformations of matrix ODEs with the expansions on the elementary fractions of the same type, i.e., with the same positions: $t_1, \; t_2, \ldots, t_N$, and orders of the poles at these positions; 2) the function f defines isomonodromy deformations related with the general triangular monodromy representation of $\pi_1(\mathbb{C} \setminus \{t_1, \; t_2, \ldots, t_N\})$; and 3) the function g – related with the general (non-triangular) monodromy group.*

3 The Gamma–function

The Gamma–function is the function which plays, possibly, the most important role in analytic and asymptotic methods in the theory of functions, so that most of the books devoted to the description of analytic properties of the special functions (see, [2, 6, 7]) begin with the definition and properties of this function. We follow this tradition here, although it forces us to start with the discrete isomonodromy deformations.

Let us consider the following linear ODE,

$$\frac{d}{d\lambda}\Psi(\lambda,\ x)=\left(-\frac{\sigma_3}{2}+\frac{x}{2\lambda}\sigma_3+\frac{1}{\lambda}\begin{pmatrix}0 & 0\\ \frac{1}{\hat{\Gamma}(x)} & 0\end{pmatrix}\right)\Psi(\lambda,\ x). \tag{3.1}$$

The corresponding Schlesinger transformation (2.14) reads

$$\Psi(\lambda,\ x+1)=\left(\lambda^{\frac{1}{2}}\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}+\lambda^{-\frac{1}{2}}\begin{pmatrix}0 & 0\\ \frac{1}{\hat{\Gamma}(x)} & 1\end{pmatrix}\right)\Psi(\lambda,\ x). \tag{3.2}$$

The compatibility condition of Eqs. (3.1) and (3.2) is as follows

$$\hat{\Gamma}(x+1)=x\hat{\Gamma}(x), \tag{3.3}$$

and thus it is clear that the general solution can be written as $\hat{\Gamma}(x)=v(x)\Gamma(x)$, where $v(x)$ is a periodic function, $v(x+1)=v(x)$ and $\Gamma(x)$ is the standard notation for the Gamma–function. Since we would like to consider the system (3.1), (3.2) as the definition of the Gamma–function , we can proceed in the following two ways: (1) we can use the theorem by Bohr and Mollerup to define the Gamma-function as the convex for $x>0$ solution of the Eq. (3.3) satisfying the well-known initial condition, $\Gamma(1)=1$, or (2) we can exploit "a solubility" of Eq. (3.1) and isomonodromy condition (3.2) to get a more explicit definition of the Gamma–function.

We consider here the second possibility. By substituting for Ψ into Eq. (3.1) the column $(\psi_1,\ \psi_2)^T$ one gets the following system of equations:

$$\frac{\partial}{\partial\lambda}\psi_1=\left(-\frac{1}{2}+\frac{x}{2\lambda}\right)\psi_1,\qquad \frac{\partial}{\partial\lambda}\psi_2=\frac{1}{\lambda\hat{\Gamma}(x)}\psi_1+\left(\frac{1}{2}-\frac{x}{2\lambda}\right)\psi_2. \tag{3.4}$$

The the general solution of system (3.4) for $\Re x<1$ can be written as follows:

$$\psi_1=c_1e^{-\frac{\lambda}{2}+\frac{x}{2}\ln\lambda},\psi_2=e^{\frac{\lambda}{2}-\frac{x}{2}\ln\lambda}\left(c_2+\frac{c_1}{\hat{\Gamma}(x)}\int\limits_{+i\infty}^{\lambda}e^{-\mu+(x-1)\ln\mu}d\mu\right),$$

so that the matrix solution of system (3.1), (3.2) with the right normalization (2.12) on the upper imaginary semi-axis reads

$$\Psi=e^{\left(-\frac{\lambda}{2}+\frac{x}{2}\ln\lambda\right)\sigma_3}\begin{pmatrix}1 & 0\\ \frac{1}{\hat{\Gamma}(x)}\int\limits_{+i\infty}^{\lambda}e^{-\mu+(x-1)\ln\mu}d\mu & 1\end{pmatrix} \tag{3.5}$$

The monodromy matrices, $M_0(x)$ and $M_1(x)$, are defined in the usual way:

$$\Psi(\infty^+)|_{\gamma_I}=\Psi(\infty^+)M_0(x),\qquad M_0(x)M_\infty(x)=I, \tag{3.6}$$

where the path γ_I is defined in Fig. 2. The isomonodromy condition (3.2) means that $M_k(x+1)=M_k(x)$. Now, using (3.6) and (3.5) one finds that

$$M_0(x)\equiv\begin{pmatrix}e^{\pi ix} & 0\\ m_{21}(x) & e^{-\pi ix}\end{pmatrix}=e^{\pi ix\sigma_3}\begin{pmatrix}1 & 0\\ \frac{1}{\hat{\Gamma}(x)}\int\limits_{\gamma_R}e^{-\mu+(x-1)\ln\mu}d\mu & 1\end{pmatrix}, \tag{3.7}$$

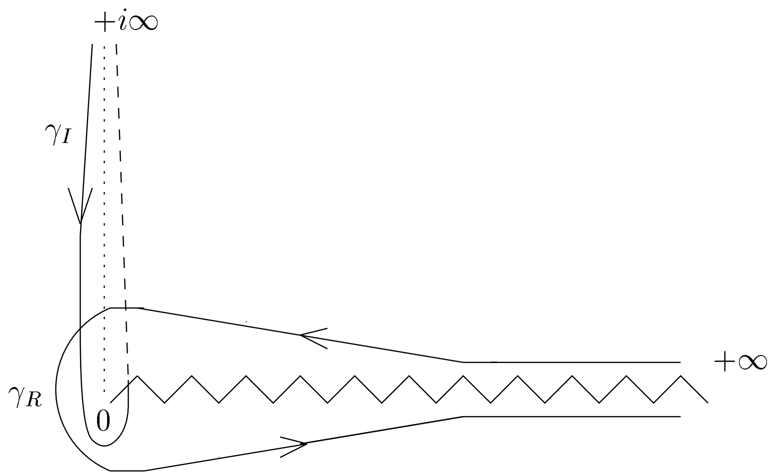

Figure 2: The paths γ_I and γ_R.

where we have changed the path γ_I to the path γ_R which doesn't influence on the value of the integral in (3.7) and allows one to omit the convergence condition $\Re x < 1$. Thus, the relation between the function $\hat{\Gamma}(x)$ and the monodromy parameter, $m_{21}(x)$, reads

$$\hat{\Gamma}(x) = \frac{2i\sin\pi x}{m_{21}(x)} \int_0^{+\infty} e^{-\mu+(x-1)\ln\mu} d\mu, \qquad \Re x > 0. \tag{3.8}$$

Theorem 1 *Gamma-function, $\Gamma(x)$, is the function which being substituted instead of $\hat{\Gamma}(x)$ defines the discrete isomonodromy deformation of Eq. (3.1) with the monodromy matrix*

$$M_0(x) = \begin{pmatrix} e^{\pi i x} & 0 \\ 2i\sin\pi x & e^{-\pi i x} \end{pmatrix}. \tag{3.9}$$

Proof. Iff $m_{21}(x) = 2i\sin\pi x$, then

$$\hat{\Gamma}(x) = \Gamma(x) = \int_0^{+\infty} e^{-\mu+(x-1)\ln\mu} d\mu, \tag{3.10}$$

which is the well-known integral representation for the Gamma-function [2]. Since from Eq. (3.9) immediately follows that the discrete isomonodromy condition is satisfied we proof that $\Gamma(x)$ is a particular solution to the difference Eq. (3.3).

In particular, we proved the following

Corollary 1 *The Gamma-function is a special function of the isomonodromy type.*

Remark 2 Theorem 1 can serve as an alternative definition of the Gamma-function, which immediately allow one to get two of its basic properties, Eqs. (3.3) and (3.10).

Let us now look for a non-linear analog of the Gamma-function. According to the definition 3 we have to consider the discrete isomonodromy deformations of the following

ODE
$$\frac{d}{d\lambda}\Psi(\lambda,\ x)=\left(-\frac{\sigma_3}{2}+\frac{x\sigma_3}{2\lambda}+\frac{1}{2\lambda}\begin{pmatrix} a(x) & c(x)\\ b(x) & d(x)\end{pmatrix}\right)\Psi(\lambda,\ x), \tag{3.11}$$
which are governed (see definition 2.14) by the following Schlesinger transformation
$$\Psi(\lambda,\ x+1)=\left(\lambda^{\frac{1}{2}}\begin{pmatrix} 1 & 0\\ 0 & 0\end{pmatrix}+\lambda^{-\frac{1}{2}}\begin{pmatrix} r(x)q(x) & q(x)\\ r(x) & 1\end{pmatrix}\right)\Psi(\lambda,\ x). \tag{3.12}$$
The compatibility condition of the system (3.11), (3.12) reads as the system of the following non-linear difference equations:
$$a(x+1)=a(x)\equiv a,\ \ 2(x+1)q(x)+a(x)q(x)+c(x+1)=q(x)(r(x)c(x)+d(x)),$$
$$d(x+1)=d(x)\equiv d,\ \ 2xr(x)+a(x)r(x)+b(x)=r(x)(q(x)b(x+1)+d(x+1)),$$
$$b(x+1)=-2r(x),\ \ c(x)=-2q(x),\ \ c(x)b(x+1=c(x+1)b(x+1)-c(x)b(x).$$
This system can be rewritten as follows:
$$c(x+1)\ =\ c(x)\left(x+1+\frac{a-d}{2}+\frac{c(x)b(x+1)}{4}\right), \tag{3.13}$$
$$b(x)\ =\ b(x+1)\left(x+\frac{a-d}{2}+\frac{c(x)b(x+1)}{4}\right), \tag{3.14}$$
where a and d can be considered as the parameters. By making the shift, $x\to x+\frac{a-d}{2}$, together with the corresponding re-notation of the functions $b(x)$ and $c(x)$ one finds that, the general solution of system (3.13), (3.13) can be obtained from the general solution this system with $a=d=0$ by the shift of x. The latter system can be rewritten as:
$$c(x)=\frac{4}{b(x+1)}\left(\frac{b(x)}{b(x+1)}-x\right),$$
$$\frac{b(x+1)}{b(x+2)}\left(\frac{b(x+1)}{b(x+2)}-x-1\right)=\left(\frac{b(x)}{b(x+1)}-x\right)\left(\frac{b(x)}{b(x+1)}+1\right). \tag{3.15}$$
The last equation is clearly a non-linear counterpart of the linear difference Eq. (3.3) and its solutions are the non-linear analogs of the functions $v(x)\Gamma(x)$. The interesting question here is: does Eq. (3.15) actually define any "new transcendental functions" or not? We will not need to discuss the issue of the " new transcendental functions", since in our case the answer on the last question is negative in the most naive sense: the general solution of Eq. (3.15) can be written in terms of the Gamma-function and parameterized via two arbitrary periodic functions, which, for difference equations, are just the counterparts of constants for the differential ones. To prove the above assertion let us introduce an auxiliary function $y(x)=\frac{b(x)}{b(x+1)}-x$, then Eq. (3.15) can be rewritten as follows:
$$y^2(x+1)-y^2(x)=(x+1)(y(x)-y(x+1)),$$
so that one arrives at the following two possibilities:
$$1)\ y(x+1)=y(x)\qquad\text{and}\qquad 2)\ \mathrm{y(x+1)+y(x)=-x-1}.$$
These equations imply that the function $b(x)$ must satisfy one of the following conditions:
$$1)\ \frac{b(x)}{b(x+1)}=x+v(x),\ \ 2)\ \frac{b(x)}{b(x+1)}=\frac{1}{2}\left(x-\frac{1}{2}+e^{\pi ix}v(x)\right), \tag{3.16}$$

where $v(x)$ is a periodic function of x. Conversely, one proves that for arbitrary periodic (with a period 1) function $v(x)$, the function b(x) which solves one of the Eqs. (3.16) also solves Eq. (3.15). After simple analysis of Eqs. (3.16) one obtains

Proposition 1 *The functions $b(x)$ and $c(x)$ solve the system (3.13), (3.13) iff they can be presented in one of the following forms:*

$$1) \quad b(x) = \frac{v_1(x)}{\Gamma(x + v_1(x)v_2(x))}, \qquad c(x) = 4v_2(x)\Gamma(x + 1 + v_1(x)v_2(x)),$$

$$2) \quad b(x) = \frac{v_1(x)}{\Gamma\left(\frac{1}{2}(x - \frac{1}{2} + e^{\pi i x}v_2(x))\right)\Gamma\left(\frac{1}{2}(x + \frac{1}{2} - e^{\pi i x}v_2(x))\right)},$$

$$c(x) = -\frac{4}{v_1(x)}\Gamma\left(\frac{1}{2}(x + \frac{3}{2} + e^{\pi i x}v_2(x))\right)\Gamma\left(\frac{1}{2}(x + \frac{5}{2} - e^{\pi i x}v_2(x))\right),$$

where $v_1(x)$ and $v_2(x)$ are arbitrary periodic functions.

Remark 3 The fact that we could hardly obtain any new transcendental functions by considering isomonodromy deformations of Eq. (3.11) could be expected *a priori*, since it is known that the fundamental solution of Eq. (3.11) can be written in terms of confluent hypergeometric functions which monodromy data are explicitely expressible in terms of the Gamma-function. Using this fact the periodic functions $v_1(x)$ and $v_2(x)$ can be related with the monodromy data of the corresponding solution of Eq. (3.11). The result of Proposition 1, of course, doesn't mean that there are no any other non-linear analogs of the Gamma-function in the sense of definition 3 which are new transcendental transcendental functions.

4 The Gauß Hypergeometric Function

Before speaking about the Gauß hypergeometric function, let us notice that in the case of the Gamma-function which is considered in the previous section the matrix $A(\lambda)$ has two poles: at the origin of the first order and at $\lambda = \infty$ of the second order (to check the latter statement introduce the local variable $\mu = 1/\lambda$ in a neighborhood of the infinity point). So, the basic equation of the previous section, Eq. (3.1), can be viewed as a coalescence limit of Eq. (2.1) with three first order poles at $\lambda := t_1 = 0, t_2 \to \infty$, and ∞. It looks natural to begin with the non-degenerate case, i.e., ODE (2.1) which matrix (2.2) has three first order poles fixed at $t_1 = 0$, $t_2 = 1$, and ∞. The special function related with the isomonodromy deformations of the latter equation seems to be a more fundamental object than the Gamma-function. As an exercise, the reader can, in the manner analogous to that of the previous section, show that the special function associated with the discrete isomonodromy deformations of the triangular ODE (2.1) which residue matrices are subjected to the condition (2.7) is nothing but the famous Euler Beta-function, $B(x,\ y)$, where $\pm x$ and $\pm y$ are the eigenvalues of the matrices A_0 and A_1 correspondingly (note that in this case: $n = 2$ and $\mathrm{tr}\, A_k = 0$). The coalescing process mentioned above resulted in the following two limits:

$$\lim_{y\to+\infty}(x+y)^x B(x,\ y) = \Gamma(x), \quad \lim_{y\to+\infty}\frac{1}{\pi}\sin(\pi(x-y))(y-x)^{-x+1}B(x-y,\ y) = \frac{1}{\Gamma(x)}.$$

We will not comment these limits here, a general discussion with some (more complicated) examples can be found in [17]. An interesting question would be to find a proof based on the isomonodromy approach, i.e., related with the transformations of linear ODEs preserving their monodromy groups and without any reference to the integral representations for the Gamma- and Beta-functions, of the following well-known identity

$$B(x,\ y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}. \tag{4.1}$$

Of course, the integral representations is a natural part of the isomonodromy approach, but usually the formulas like (4.1) represent some interesting properties of the associated ODEs.

The next natural step in the above setting is to consider the special function which is defined by the discrete isomonodromy deformations of the ODE whose connection matrix $A(\lambda)$ has four poles of the first order. As before, we can fix via the Möbius transformation three of this poles at $\lambda := 0,\, 1,\, \infty$ while the position of the fourth pole remains a parameter which we denote just by t. So, system (2.1) in this case looks as follows:

$$\frac{d}{d\lambda}\Psi = \left(\frac{A_0}{\lambda} + \frac{A_1}{\lambda - 1} + \frac{A_t}{\lambda - t}\right)\Psi, \quad A_k = \begin{pmatrix} 0 & 0 \\ u_k & 0 \end{pmatrix} + \theta_k\sigma_3. \tag{4.2}$$

The normalization condition (2.10) in this notation reads

$$u_0 + u_1 + u_t = 0. \tag{4.3}$$

Following the previous routine we must consider the discrete isomonodromy deformations of the quantities u_k considered as functions of the variables θ_k's– the eigenvalues the matrices A_k. These deformations result in a linear system of difference equations on the functions u_k which general solution, according definition 1, should be called as some (new?) special function. The example of the Gamma-function shows that the special functions, i.e., the functions which define the general solutions of the difference equations parametrically depend of the monodromy data. For the linear special functions (see definition 2) one can make a further "specialization" of the special functions in the sense of our definition 1 and extract a function which depends on the parameters of formal monodromy and independent of the other monodromy data, nevertheless, using it and, possibly some other elementary and periodic functions (recall $\sin(\cdot)$ and $m_{21}(\cdot)$ in Eq. (3.8)), one can still define the corresponding general solution. As for the non-linear analogs (in the sense of definition 3) of these functions, such procedure is not any more possible: so in general we must think of special functions as the ones parametrized by complete sets of the associated monodromy data. In the situation related with Eq. (4.2), the corresponding special function should also depend on one more parameter, t. In this context, it seems natural*to consider all the parameters as the ones independent of each other*, i.e., we arrive at the *isomonodromy deformations of Eq. (4.2) on the parameter t*. The previous discussion shows that after the "specialization" our special functions, $u_k \equiv u_k(t;\ \theta_0,\, \theta_1,\, \theta_t)$, where the variables t and θ_k are independent of each other as it follows from the isomonodromy condition. The system of isomonodromy deformations (2.8) in our case: $t_1 = 0,\ t_2 = 1,\ t_3 = \infty,\ t_4 = t$, and A_k are given by Eq. (4.2), is equivalent, together with the conditions (4.3) and (2.11), to the following system:

$$\frac{du_0}{dt} = \frac{1}{t}(2\theta_0 u_t - 2\theta_t u_0), \quad \frac{du_1}{dt} = \frac{1}{t-1}(2\theta_1 u_t - 2\theta_t u_1). \tag{4.4}$$

The solution of system (4.3), (4.4) is equivalent to the solution of the Euler differential equation,

$$t(1-t)u'' + [c - (a+b+1)t]u' - abu = 0, \tag{4.5}$$

where the primes denote the differentiation by t. For example, the function u_0 can be obtained as the general solution of Eq. (4.5) with the coefficients:

$$a = 2\theta_t, \quad b = 2\theta_0 + 2\theta_1 + 2\theta_t, \quad c = 2\theta_0 + 2\theta_t + 1, \tag{4.6}$$

while the functions u_1 and u_t can be calculated via the first equation in (4.4) and Eq. (4.3) correspondingly. Thus as the functions of the parameter t, our special function(s) $u_k(t;\ \theta_0,\ \theta_1,\ \theta_t)$ are (general) solutions of the Euler ODE (4.5). To follow the standard framework we have to consider t as the "basic" independent variable and the variables θ_k's as some auxiliary parameters defining the coefficients (4.6) of the Euler equation. One of the standard definitions of the Gauß hypergeometric function, ${}_2F_1(a,\ b;\ c;\ t)$, is as follows: *it is the solution of Eq. (4.5) which is holomorphic at t_0 and such that*

$${}_2F_1(a,\ b;\ c;\ 0) = 1 \tag{4.7}$$

The isomonodromy definition of the hypergeometric function in the sense of definition 1 goes in the following way: *It is the function which defines the isomonodromy deformations of the 2×2 matrix triangular linear ODE whose only singularities are four poles of the first order (see Eq. (4.2).* Of course, according the definition 2, it is a linear special function. This definition names the general solution of the Euler equation (4.5) as the hypergeometric function. To define the function ${}_2F_1(a,\ b;\ c;\ t)$ we must be more specific, namely, ${}_2F_1(a,\ b;\ c;\ t)$ *is equal to the function which should be substituted for* $\{21\}$*-element of the matrix A_0 in Eq. (4.2) with the eigenvalues θ_k's fixed according to Eq. (4.6), such that: (1) the monodromy group of Eq. (4.2) doesn't change when t varies within the ball of the radius* 1 *centered at the origin of the complex plane* $\mathbb{C}$ *and (2) satisfies Eq. (4.7).* One can also specify the function ${}_2F_1(a,\ b;\ c;\ t)$ by pointing out the explicit values of the monodromy data of Eq. (4.2), like it is done in the previous section for the Gamma-function.

To feel better the difference between the "old" and "new" definitions of the hypergeometric function, let us rewrite Eq. (4.5) in the matrix notation with the variable t changed to λ:

$$\frac{d}{d\lambda}\Psi = \left(\frac{A_0}{\lambda} + \frac{A_1}{\lambda - 1}\right)\Psi, \tag{4.8}$$

where the matrices A_0 and A_1 are as follows:

$$A_0 = \begin{pmatrix} -\theta_0 - \theta_t & -2\theta_0 \\ 0 & \theta_0 + \theta_t \end{pmatrix}, \quad A_1 = \begin{pmatrix} \theta_1 + \theta_t & 0 \\ -2\theta_1 & -\theta_1 - \theta_t \end{pmatrix}. \tag{4.9}$$

To obtain Eqs. (4.8) and (4.9) the reader can write ODE for a column, ψ, of the matrix solution to Eq. (4.8), Ψ; the column can be constructed as $\psi = \lambda^{\theta_0+\theta_t}(\lambda - 1)^{\theta_1+\theta_t}(u_0(\lambda),\ u_1(\lambda))^T$. Note, that an arbitrary 2×2 matrix linear ODE with a rational connection matrix $A(\lambda)$ having three poles of the first order can be reduced to the form (4.8), (4.9) by means of a Möbius transformation of λ and "gauge" transformation, $\Psi \to f(\lambda)G\Psi$, where $f(\lambda)$ is a scalar function and the matrix G is independent of λ. Three is the maximal number of poles in Eq. (2.1) when the coefficients of the connection matrix $A(\lambda)$ (2.2) can be fixed via the latter transformation in terms of eigenvalues of the matrices: $A_1, \ldots, A_N$, $A_\infty \equiv -(A_1 + \ldots + A_N)$. Recall that, these eigenvalues are called the parameters of formal monodromy: they define the branching of Ψ in a neighborhood of the poles, $t_1, \ldots, t_N, \infty$. As soon as the connection matrix is fixed in terms of the formal monodromy the monodromy group is also fixed in terms of the formal monodromy. In our case, the three matrices: A_0, A_1, A_∞, have the parameters of formal monodromy: $-\theta_0 - \theta_t$, $\theta_1 + \theta_t$, $\theta_0 + \theta_1$ correspondingly, and the elements of the monodromy matrices can be written via the Gamma-functions depending of these parameters. So, the classical definition can be viewed as the one based on the general type 2×2 matrix ODE with three poles of the first order, Eq. (4.8), while the isomonodromy definition – on the general triangular 2×2 matrix ODE with four poles of the first order, Eq. (4.2).

In the next section we consider in some more detail the isomonodromy definition of the Bessel function, therefore in the next two paragraphs we only briefly mention how one can reproduce some of the most important results for the Gauß hypergeometric function on the basis of its isomonodromy definition. These new proofs of the well-known properties are very often useful for the derivation of their non-linear analogs for the Painlevé equations.

The difference equations for the functions u_k's, which are related with the discrete isomonodromy deformation (2.14) with $t_k \equiv t$ can be rewritten in terms of ${}_2F_1(a,\ b,\ c;\ t)$ as the following well-known relation:

$$\frac{d}{dt}F(a,\ b,\ c;\ t) = \frac{ab}{c}F(a+1,\ b+1,\ c+1;\ t). \tag{4.10}$$

By considering the discrete isomonodromy deformations which are related with the other shifts of the variables $\theta_k \to \pm 1$ the reader can obtain all the other analogous elementary relations cited in [2], say, by considering discrete isomonodromy deformation which corresponds to the shifts: $\theta_0 \to \theta_0 - 1$ and $\theta_t \to \theta_t + 1$,

$$\hat{\Psi} = L\psi, \quad L = \frac{\sqrt{\lambda - t_i}}{\sqrt{\lambda - t_k}}J_{ik} + \frac{\sqrt{\lambda - t_k}}{\sqrt{\lambda - t_i}}J_{ki}, \quad t_i,\ t_k \in 0,\ 1,\ t, \tag{4.11}$$

where J_{ik} are some degenerate matrices (see for details [15, 17]), one finds that

$$t\frac{d}{dt}F(a,\ b,\ c;\ t) = a(F(a+1,\ b,\ c;\ t) - F(a,\ b,\ c;\ t)). \tag{4.12}$$

Eqs. (4.10) and (4.12) are the direct analogs of the difference equation for the Gamma-function (3.3). The derivatives in Eqs. (4.10) and (4.12) appear due to the use of Eqs. (4.4): in terms of the variables u_k, Eqs. (4.10) and (4.12) have a pure difference form.

The compatibility condition of two discrete isomonodromy deformations of the type (4.11) and/or (2.14) leads to the so-called Gauß contiguity relations [2]. Each of the discrete isomonodromy deformations shifts two of the three parameters θ_k on ± 1, so that all in all we have $C_2^6 = 15$ different contiguity relations. Using the isomonodromy definition based on Eq. (4.2) one can also reproduce the transformations of the hypergeometric series related with the Möbius and quadratic transformations of t. A method to obtain them on the basis of Eq. (4.2) is via corresponding (Möbius and quadratic) transformations of the auxiliary parameter λ.

The isomonodromy deformations of the general (non-triangular) 2×2 matrix ODE with four poles of the first order are known to be defined by the sixth Painlevé equation

[15], so that the non-linear analog, in the sense of definition 3, of the Gauß hypergeometric function is the sixth Painlevé equation. The non-linear analog of the quadratic transformations for the sixth Painlevé equation was derived in [18]. By considering the triangular specialization of the construction of the last work one can reproduce the quadratic transformations for the Gauß hypergeometric functions.

Isomonodromy deformations of a natural generalization of Eq. (4.2) with the triangular connection matrix $A \in sl_n(\mathbb{C})$ for $n > 2$ are defined by the Meijer G-functions. It would be interesting to derive properties of these functions and their non-linear analogs, in particular quadratic transformations, by making use of the isomonodromy definition.

Whilst by applying the isomonodromy ideas one can hardly get any new result for the classical higher transcendental functions of a single variable, that is why, the main emphasis here is made on the non-linear analogs of these functions, the Painlevé transcendents. The isomonodromy approach can be quite fruitful in the theory of the multi-variable special functions, in particular, for the multi-variable hypergeometric functions. An application of the isomonodromy approach to these functions is under investigation now; therefore, we make only some general comments below.
Let us consider the hypergeometric series with two variables. According the Horn list [2] there are 14 basic hypergeometric series of such type which are denoted as: F_1, F_2, F_3, F_4, G_1, G_2, G_3, $H_1,\ldots,H_7$, and also 20 other series which can be viewed as their different degenerations. The series F_2, F_3, H_2 depend on 5, F_1, F_4, G_2, H_1, H_4, H_7 – 4, G_1, H_3, H_5, H_6 – 3, and G_3 – 2 parameters correspondingly. The first interesting question from our point of view would be to specify: which of these series has the isomonodromy type? According to what was written above, to answer this question, it is natural to consider the following system of the matrix ODEs of the triangular type:

$$\tfrac{d}{d\lambda}\Psi = \left(\tfrac{A_0}{\lambda} + \tfrac{A_1}{\lambda-1} + \tfrac{A_{t_1}}{\lambda-t_1} + \tfrac{A_{t_2}}{\lambda-t_2}\right)\Psi, \tag{4.13}$$

$$\tfrac{d}{dt_1}\Psi = -\tfrac{A_{t_1}}{\lambda-t_1}\Psi, \qquad \tfrac{d}{dt_2}\Psi = -\tfrac{A_{t_2}}{\lambda-t_2}\Psi, \tag{4.14}$$

where the matrices A_k are of the form (4.2) with the evident modification of the notation related with the change $t \to t_1,\, t_2$. Without any calculations, it is clear that, the functions: u_1, u_2, u_{t_1}, u_{t_2}, our candidates for the two-variable hypergeometric functions, depend only of four parameters: θ_0, θ_1, θ_{t_1}, θ_{t_2}; have one-dimensional integral representation in terms of the elementary functions; and due to the normalization condition:

$$u_1 + u_2 + u_{t_1} + u_{t_2} = 0, \tag{4.15}$$

only three of them are linearly independent: hence that the overdetermined system of the partial differential equations for these functions [2] has only three independent non-trivial solutions. These properties suit the following six functions: F_1, G_1, G_2, G_3, H_3, H_6. So, definitely, the other eight basic hypergeometric functions can not be obtained. Trying to find an isomonodromy presentations for the hypergeometric functions depending on five additional parameters, one has two basic opportunities: to insert into Eq. (4.13) an additional pole with a fixed dependence on the variables t_1, t_2 or to increase the matrix dimension of the Eq. (4.13) keeping, say, the same number of poles. The first opportunity will lead to some special reduction of n-variable (with $n \geq 3$) hypergeometric functions, which also have one-dimensional integral representations in terms of the elementary functions. Since for the functions F_2, F_3, H_2 such representations are not known, this possibility seems not to be successful. The other opportunity related with the extension of the matrix dimension of the ODEs under consideration also does not lead immediately to the desired result. To get an idea what is going on in this situation, let us consider the case when the coefficient matrices of Eq. (4.13), A_k, belong to the solvable subgroup of the low-triangular matrices of $sl_3(\mathbb{C})$:

$$A_k = \begin{pmatrix} \theta_k^1 & 0 & 0 \\ u_k^1 & \theta_k^2 & 0 \\ v_k & u_k^2 & -\theta_k^1 - \theta_k^2 \end{pmatrix}, \qquad k = 0,\, 1,\, t_1,\, t_2. \tag{4.16}$$

The compatibility condition of Eqs. (4.13) and (4.14) for the matrices A_k given by (4.16) reads as follows:

$$\frac{\partial}{\partial t_l} u_k^m = \frac{1}{t_l - k}(\Theta_{t_l}^m u_k^m - \Theta_k^m u_{t_l}^m), \tag{4.17}$$

$$\frac{\partial}{\partial t_l} v_k = \frac{1}{t_l - k}(\Theta_{t_l}^3 v_k - \Theta_k^3 v_{t_l} + u_{t_l}^1 u_k^2 - u_k^1 u_{t_l}^2), \tag{4.18}$$

where

$$l,\, m \in \{1,\, 2\}, \quad k \in \{0,\, 1,\, t_1,\, t_2\} \setminus \{t_l\},$$

and

$$\Theta_k^1 = \theta_k^1 - \theta_k^2, \quad \Theta_k^2 = \theta_k^1 + 2\theta_k^2, \quad \Theta_k^3 = 2\theta_k^1 + \theta_k^2.$$

The derivatives $\partial_{t_l} u_{t_l}^m$ and $\partial_{t_l} v_{t_l}$ can be calculated via Eq. (4.15) and the analogous one for v_k. Eqs. (4.17) actually forms two independent systems of equations: the first one for $m = 1$ and the second for $m = 2$. Each of these systems possess four independent parameters Θ_k^m, since $k \in \{0,\, 1,\, t_1,\, t_2\}$: consequently, the functions u_k^m coincide with the special functions which are defined by the triangular Eq. (4.13) in 2×2 matrices. One can hope to get one of the hypergeometric functions depending on five parameters as the solutions of Eq. (4.18) with the especially chosen solutions u_k^m, since in general the functions v_k depend on 8 arbitrary parameters. For example, the systems (4.17) admit the following special solution:$u_k^m = c^m \Theta_k^m$. After substitution of the last solution into Eq. (4.18) and scaling: $v_k \to c^1 c^2 v_k$, one has to take care that the remaining constants: $\Theta_{t_l}^1 \Theta_k^2 - \Theta_{t_l}^2 \Theta_k^1$ would be not removable via the shifts: $v_k \to v_k + \delta_k$ with the appropriately chosen parameters δ_k. The last condition results in the additional restriction on the parameters θ_k^m (the corresponding determinant formed of Θ_k^3's should be equal to zero). As the next step, we would like to get an overdetermined system of two second order partial differential equations for the functions v_k and compare it with the systems for the two-variable hypergeometric functions given in the list presented in [2]. Since the latter systems are the homogeneous ones, while the equations (4.18) do not, one can expect the appearance of two additional constraints for parameters Θ_k^3 which are equivalent to some additional constraints for θ_k^m. So, hopefully, we get only three additional constraints for the eight parameters, θ_k^m, together with a proper system of partial differential equations which, therefore, defines one of the hypergeometric functions depending on five parameters. Some other more complicated special solutions of the systems (4.17) could also fit this procedure and/or(!) generate the desired hypergeometric functions.At the moment these possibilities are under investigation.

As in the case of the Gauß hypergeometric function, it is interesting to rewrite the scalar differential equations for the multi-variable isomonodromy hypergeometric functions into the matrix form. Let us do it, for example, for the two variable isomonodromy hypergeometric function which solves the system (4.17):

$$\frac{d}{d\lambda}\Psi = \left(\frac{A_0}{\lambda} + \frac{A_1}{\lambda - 1} + \frac{A_t}{\lambda - t}\right)\Psi, \tag{4.19}$$

where the 3×3 matrices A_k are as follows:

$$(A_k)_{ij} = \begin{cases} 0, & i \neq k \\ \Theta_k^m, & i = k,\ j \neq k \\ \Theta_\lambda^m + \Theta_k^m, & i = j = k \end{cases}, \qquad i,\ j,\ k \in \{0,\ 1,\ t\}.$$

To get Eq. (4.19) we have changed the variables t_1 and t_2 in Eqs. (4.17) to λ and t and written equation for the vector-column, $\psi = (u_0^m,\ u_1^m,\ u_t^m)^T$, which can be taken as one of the columns of the fundamental solution Ψ. By using the formal Laplace transform, $\hat{\Psi}(\lambda) = \int e^{-\hat{\lambda}\lambda}\Psi(\hat{\lambda})\, d\hat{\lambda}$, Eq. (4.19) can be rewritten as

$$\frac{d}{d\lambda}\Phi = -\left(\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & t \end{pmatrix} + \frac{A}{\lambda}\right)\Phi, \quad \Phi = \lambda\hat{\Psi}, \quad A = \sum_{k \in \{0,\,1,\,t\}} A_k. \tag{4.20}$$

Analogous equations can be obtained for the isomonodromy hypergeometric functions with any number of variables. The relation between Eqs. (4.19)) and (4.20) can be used to derive a multi-variable generalization of the well-known integral representation of the confluent hypergeometric function in terms of the Gauß one [2]. The latter integral representation can be treated in the analogous manner as the Laplace transform of Eq. (4.8).

5 The Bessel function

Consider the ODE whose (2×2) connection matrix has two poles of the second order. We can fix positions of the poles at the origin and the infinity point of the independent variable λ:

$$\frac{d}{d\lambda}\Psi = \left(\hat{A}_\infty + \frac{\hat{A}_1}{\lambda} + \frac{\hat{A}_0}{\lambda^2}\right)\Psi. \tag{5.1}$$

We consider the general situation when the matrices $\hat{A}_k \in sl_2(\mathbb{C})$ and have different eigenvalues: $\pm x$ and $\pm\theta/2$ for $k = \infty$ and 0 correspondingly. After an appropriate gauge, $\Psi \to G\Psi$, and the scaling, $\lambda \to x\lambda$, transformations, Eq. (5.1) can be rewritten in the following form:

$$\frac{d}{d\lambda}\Psi = \left(-i\sigma_3 - \frac{A_1}{\lambda} - \frac{ix}{2\lambda^2}A_0\right)\Psi, \tag{5.2}$$

where $A_0,\ A_1 \in sl_2(\mathbb{C})$, $\det A_0 = -\theta^2 \neq 0$, and x is the pole parameter, i.e., with respect to it the isomonodromy deformations are considered. The parameter θ is really not essential, it can be fixed equal, say, to 1 by a redefinition of x, $x \to \theta x$.

Let us define the monodromy data for Eq. (5.2). The canonical solutions, X_k and Y_k, $k = 1,\ 2$ are defined in the following sectors respectively:

$$-2\pi + \pi k < \arg\lambda < \pi k,$$
$$-2\pi + \pi k + \arg x < \arg\lambda < \pi k + \arg x,$$

by their asymptotic expansions:

$$X_k \underset{\lambda\to\infty}{=} \left(I+\mathcal{O}\left(\frac{1}{\lambda}\right)\right)\exp\left(-i\lambda\sigma_3+\theta_\infty\ln\lambda\sigma_3\right), \tag{5.3}$$

$$Y_k \underset{\lambda\to 0}{=} \left(\Phi_0+\mathcal{O}(\lambda)\right)\exp\left(\frac{ix}{2\lambda}\sigma_3+\theta_0\ln\lambda\sigma_3\right), \quad \det\Phi_0=1, \tag{5.4}$$

where $\theta_0,\,\theta_\infty\in\mathbb{C}$ are the parameters of the formal monodromy and the branch of ln is fixed by the cut along the negative imaginary semi-axis, $[-i0,\ -i\infty]$. Now we can define the Stokes matrices, S_k:

$$X_2(\lambda)=X_1(\lambda)S_1, \qquad X_1\left(\lambda e^{-2\pi i}\right)e^{2\pi i\theta_\infty\sigma_3}=X_2(\lambda)S_2,$$
$$Y_2(\lambda)=Y_1(\lambda)S_3, \qquad Y_1\left(\lambda e^{-2\pi i}\right)e^{2\pi i\theta_0\sigma_3}=Y_2(\lambda)S_4,$$

and the connection matrices, G_1 and G_2:

$$Y_1(\lambda)=X_1(\lambda)G_1, \qquad Y_2(\lambda)=X_2(\lambda)G_2. \tag{5.5}$$

The above definitions imply the following relations between the monodromy data:

$$G_1S_3=S_1G_2, \qquad G_2S_4=S_2e^{-2\pi i\theta_\infty\sigma_3}G_1e^{2\pi i\theta_0\sigma_3}.$$

These equations can be rewritten as the cyclic relation:

$$S_1S_2e^{-2\pi i\theta_\infty\sigma_3}=G_1S_3S_4e^{-2\pi i\theta_0\sigma_3}G_1^{-1}.$$

Note that the Stokes matrices have the following structure:

$$S_{2l-1}=\begin{pmatrix}1 & 0\\ s_{2l-1} & 1\end{pmatrix}, S_{2l}=\begin{pmatrix}1 & s_{2l}\\ 0 & 1\end{pmatrix}, \quad l=1,\,2,$$

where the complex parameters s_k are called the *Stokes multipliers*. We denote the matrix elements of G_k as g^k_{ij}, $i,\,j,\,k\in\{1,\,2\}$. According Eq. (5.5)

$$\det G_k=1 \ \Leftrightarrow\ g^k_{11}g^k_{22}-g^k_{12}g^k_{21}=1.$$

By definition the monodromy data: $\theta_0,\,\theta_\infty,\,s_k,\,g^k_{ij}$, are independent of λ. By demanding that these data are independent also of x, one arrives at the following additional equation for the function Ψ,

$$\frac{d}{d\lambda}\Psi=\frac{iA_0}{2\lambda}\Psi. \tag{5.6}$$

The compatibility condition of Eqs. (5.2) and (5.6) are given by:

$$x\frac{dA_0}{dx}=[A_0,\ A_1], \qquad \frac{dA_1}{dx}=\frac{1}{2}[\sigma_3,\ A_0]. \tag{5.7}$$

In the non-triangular case Eqs. (5.7) are known to be equivalent to the third Painlevé equation [15]. Here, we consider in details the triangular reduction:

$$A_0=\theta\sigma_3+\begin{pmatrix}0 & 0\\ v_1 & 0\end{pmatrix}, \qquad A_1=-\theta_\infty\sigma_3-\begin{pmatrix}0 & 0\\ v & 0\end{pmatrix}. \tag{5.8}$$

The isomonodromy deformations (5.7) in the triangular case (5.8) reads:

$$v \equiv v(x), \quad v_1 = v^{'}(x), \quad xv^{''}(x) + 2\theta_\infty v^{'}(x) - 2\theta v(x) = 0. \tag{5.9}$$

Henceforth the prime denotes differentiation with respect to the indicated argument of the functions. Note that, say, v is considered here also as the function of the monodromy data, but not as an arbitrarily fixed general solution of Eq. (5.9). Eq. (5.9) can be rewritten as the Bessel equation of order ν,

$$y^2 w^{''}(y) + yw^{'}(y) + (y^2 - \nu^2)w(y) = 0, \tag{5.10}$$

where

$$w(y) = \theta \left(\frac{y}{4}\right)^{-\nu} v\left(-\frac{y^2}{8\theta}\right), \quad \nu = 1 - 2\theta_\infty. \tag{5.11}$$

The specification of the monodromy data corresponding to the "triangular" isomonodromy deformations is as follows:

$$g^1_{12} = g^2_{12} = s_2 = s_3 = 0, \quad g^1_{11} = g^2_{11} = \theta, \quad g^1_{22} = g^2_{22} = \frac{1}{\theta},$$
$$\theta_\infty = \theta_0, \quad g^1_{11} s_1 = g^1_{21} - g^2_{21}, \quad g^2_{22} s_4 = g^1_{21} e^{4\pi i\theta_\infty} - g^2_{21}.$$

A comment is required, possibly, only for the formulas for g^k_{11}. The explanation is as follows: the canonical solution (5.4) is fixed as soon as the matrix Φ_0 is fixed; the latter matrix is defined up to multiplication on an arbitrarily chosen complex parameter. One can choose the value of this parameter to fix the value g^k_{11}. Our choice of the matrix Φ_0 below corresponds to the values of g^k_{11} written above.
The canonical solutions (5.3) and (5.4) of Eq. (5.2) in the triangular case (5.8) can be presented in the following way:

$$X_{1,2} = e^{-\phi\sigma_3}\begin{pmatrix} 1 & 0 \\ \int\limits_{\pm\infty}^{\lambda} \chi_\infty e^{-2\phi} d\lambda & 1 \end{pmatrix}, \quad Y_{1,2} = \begin{pmatrix} \theta & 0 \\ \frac{1}{2}v^{'} & \frac{1}{\theta} \end{pmatrix} e^{-\phi\sigma_3}\begin{pmatrix} 1 & 0 \\ \int\limits_{\pm 0}^{\lambda} \chi_0 e^{-2\phi} d\lambda & 1 \end{pmatrix}, \tag{5.12}$$

where

$$\chi_0 = \theta\left(iv^{'} - \frac{\theta_\infty v^{'}}{\lambda} + \frac{\theta v}{\lambda}\right), \quad \chi_\infty = \frac{v}{\lambda} - \frac{ixv^{'}}{2\lambda^2}, \quad \phi = i\left(\lambda - \frac{x\theta}{2\lambda} + i\theta_\infty \ln\lambda\right),$$

and integration is taken along the real axis. To get the canonical solution with the subscript 1 (2) one has to take the upper (lower) signs in the integration limits in the r.-h.s.'s of (5.12). Note that to guarantee the convergence of the integrals in Eqs. (5.12), one has to impose the condition

$$|\Re\theta_\infty| < 1/2. \tag{5.13}$$

Later on we remove this restriction by transformation of the contour of integration. Now, from definition (5.5) and explicit expressions for the canonical solutions (5.12), one finds the following equations for the monodromy data:

$$g^k_{21} = \frac{v^{'}}{2} e^{-2\phi} + \frac{1}{\theta}\int\limits_{\pm 0}^{\lambda} \chi_0 e^{-2\phi} d\lambda - \theta \int\limits_{\pm\infty}^{\lambda} \chi_\infty e^{-2\phi} d\lambda, \tag{5.14}$$

where for $k=1$ one has to take the upper signs in the lower limits of integration (the positive semi-axis) and for $k=2$ – the lower signs (integration along the negative semi-axis). Eqs. (5.14) can be rewritten in the following way:

$$g_{21}^{k} = v\theta \int\limits_{\mathcal{L}_k} \frac{e^{-2\phi}}{\lambda} d\lambda + \frac{ixv'\theta}{2} \int\limits_{-\mathcal{L}_k} \frac{e^{-2\phi}}{\lambda^2} d\lambda, \tag{5.15}$$

where the contours $\mathcal{L}_k$ are indicated in Fig. 3. For $k=1,2$ we get two equations (5.15)

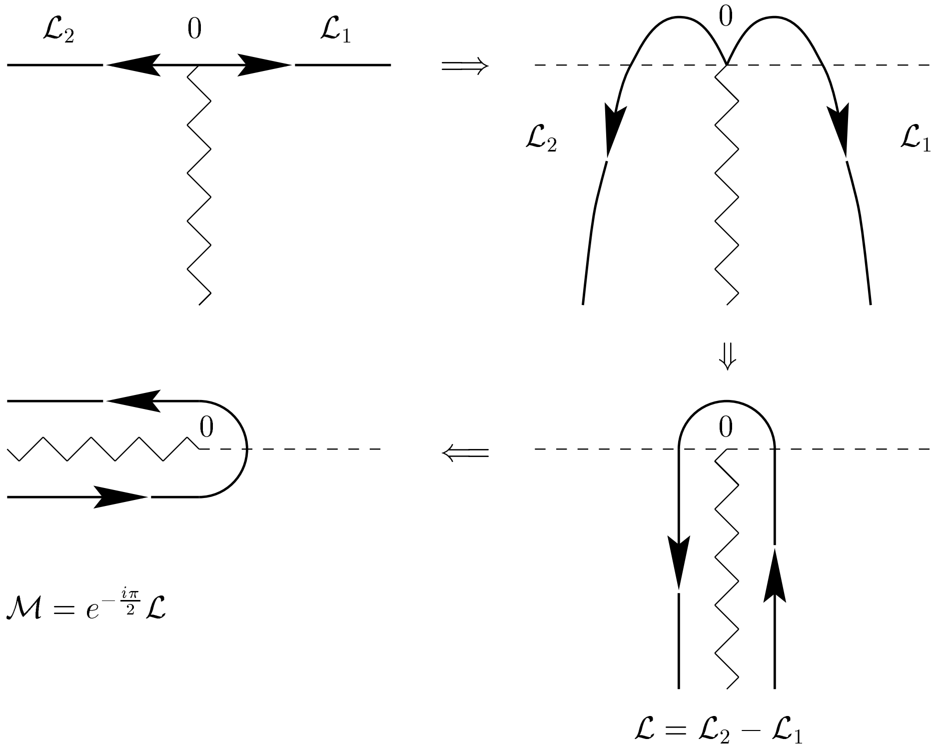

Figure 3: The transformations of the contours of integration.

for the monodromy data, g_{21}^{k}, which can be viewed as the system of two linear algebraic equations for the functions $v = v(x)$ and $v' = v'(x)$. Solving them via the Cramer's rule one obtains:

$$v = \frac{1}{\Delta}(g_{21}^2 J_1^\nu - g_{21}^1 J_2^\nu), \qquad \frac{i}{2}xv' = \frac{1}{\Delta}(g_{21}^2 I_1^\nu - g_{21}^1 I_2^\nu), \tag{5.16}$$
$$\Delta = \theta(J_1^\nu I_2^\nu - J_2^\nu I_1^\nu), \qquad I_k^\nu = \int\limits_{\mathcal{L}_k} \frac{e^{-2\phi}}{\lambda} d\lambda, \ \ J_k^\nu = \int\limits_{\mathcal{L}_k} \frac{e^{-2\phi}}{\lambda^2} d\lambda,$$

The function Δ can be calculated explicitly:

$$\Delta = const.(x\theta)^{-\nu} = -\pi e^{-i\pi\nu}\left(-\frac{x\theta}{2}\right)^{-\nu}. \tag{5.17}$$

The first equation in (5.17) follows from the relation of Δ with the Wronskian of two solutions to Eq. (5.9). Alternatively, this result can be found from the self-consistency of the first and the second equations in (5.16). The explicite value of the *const.* in (5.17) is calculated via the steepest descent method (note, that $x \in \mathbb{C}$). Now, transforming the contours of the integration as it is shown in Fig. 3 and introducing the notation,

$$J^{\nu} = \int\limits_{\mathcal{L}} e^{-2i\left(\lambda - \frac{x\theta}{2\lambda}\right)} \lambda^{-\nu-1} d\lambda = J_2^{\nu} - J_1^{\nu}, \tag{5.18}$$

one finds the following integral representation for the general solution of Eq. (5.9):

$$v = \frac{1}{2i\pi\theta \sin(\pi\nu)} \left(g_{11}^1 s_1 J^{-\nu} - g_{22}^1 s_4 e^{2i\pi\nu} \left(-\frac{x\theta}{2} \right)^{\nu} J^{\nu} \right). \tag{5.19}$$

Note, that after the transformation of the contours shown in Fig. 3 convergence of the corresponding integrals becomes better (absolute). Thus we can omit the restriction for θ_∞, (5.13), in the integral representations (5.18) and (5.19). Let us rewrite integral representation (5.19) for the general solution, $w(y)$, of the canonical Bessel equation (5.10) in terms of the canonical Bessel function,

$$J_{\nu} = \frac{1}{2\pi i} \left(\frac{y}{2} \right)^{\nu} \int\limits_{\mathcal{M}} e^{\mu - \frac{y^2}{4\mu}} \mu^{-\nu-1} d\mu,$$

where the contour $\mathcal{M}$ is defined in Fig. 3. By employing the following substitution $-\frac{x\theta}{2} = \frac{y^2}{16}$, $\lambda = \frac{1}{2} e^{\frac{i\pi}{2}} \mu$ one finds

$$J^{\nu}(y) = 2i\pi e^{-\frac{i\pi\nu}{2}} \left(\frac{4}{y} \right)^{\nu} J_{\nu}(y). \tag{5.20}$$

Finally, Eqs. (5.11), (5.16), and (5.20) imply that

$$w(y) = \frac{e^{i\pi\nu}}{\sin(\pi\nu)} \left(s_1 g_{11}^1 e^{-\frac{i\pi\nu}{2}} J_{-\nu}(y) - s_4 g_{22}^1 e^{\frac{i\pi\nu}{2}} J_{\nu}(y) \right). \tag{5.21}$$

Now, like it is done in the previous sections for the Gamma-function and Gauß hypergeometric function, one can define the Bessel function as an isomonodromy deformation of the coefficients of Eq. (5.2) related to the special choice of the monodromy data (which is evident from Eq. (5.21)).

Consider the discrete isomonodromy deformations for Eq. (5.2) in the triangular case (5.8),

$$\hat{\Psi} = \left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \lambda^{\frac{1}{2}} + \begin{pmatrix} 0 & 0 \\ -v'(x)/2 & 0 \end{pmatrix} \lambda^{-\frac{1}{2}} \right) \Psi. \tag{5.22}$$

The "$(\hat{\cdot})$-notation" is explained in Sec. 2 (see Eq. (2.14)). The form of the coefficient in Eq. (5.22), $-v'$, follows from the compatibility of Eq. (5.22) with the continuous isomonodromy deformation (5.6). The compatibility condition of Eqs. (5.22) and (5.2) in the triangular case (5.8) reads:

$$\hat{v} = iv', \quad x\hat{v}' = 2i(v - \theta_\infty v'), \quad \hat{\theta}_\infty = \theta_\infty + 1/2, \quad \hat{\theta} = \theta. \tag{5.23}$$

where $\hat{v} = \hat{v}(x)$ and $\hat{v}' = \hat{v}'(x)$ are the coefficients of ODE with respect to λ for the function $\hat{\Psi}$. The function $v = \hat{v}(x)$ solves Eq. (5.9) with $\hat{\theta}$ and $\hat{\theta}_\infty$ substituted for θ and

θ_∞ correspondingly. Now, using the relation of the function v with the Bessel function (5.11) and (5.21) one finds that the first equation in (5.23) can be rewritten as follows:

$$y^{\nu-1} J_{\nu-1}(y) = \frac{d}{ydy} y^{\nu} J_{\nu}(y). \tag{5.24}$$

The second equation in (5.23) also represents the well-known relation for the Bessel functions,

$$2\nu y^{-1} J_\nu(y) = J_{\nu-1}(y) + J_{\nu+1}(y). \tag{5.25}$$

To get Eq. (5.25) from the second Eq. (5.23) one has to employ not only Eqs. (5.11) and (5.21) but also Eq. (5.24) and the substitution $\nu \to \nu + 1$.

6 Concluding Remarks

Of course, in this short article we were not able to discuss all topics related with the isomonodromy approach to the theory of special functions. In particular, it is worth to mention that elliptic functions and their multi-variable generalizations, the theta-functions of the compact Riemann surfaces, also belong to the isomonodromy type: the corresponding matrix linear ODEs whose isomonodromy deformations are defined by these functions can be obtained [15] via the construction of the Baker-Akhiezer function which is well-known in the theory of solitons. The content of the paper together with the remark above show that the most important special functions of one variable as well as some multi-variable special functions belong to the isomonodromy class.

In the same time it is clear that there are some special functions of one variable which are not of the isomonodromy type: the Riemann ζ-function is, possibly, the most famous example. One can conjecture that, it is related to the isomonodromy deformations of the linear ODEs with the meromorphic connection matrices.

An important point of this paper is a presentation of the Painlevé transcendents as the non-linear generalizations of the classical higher transcendental functions. It can be summarized as the following

Proposition 2 *The non-linear analog in the sense of definition 3 of,*
(1) the Airy function is the second Painlevé transcendent,
(2) the Bessel function is the third Painlevé transcendent,
(3) the parabolic cylinder function (the Weber-Hermite function) is the fourth Painlevé transcendent,
(4) the confluent hypergeometric function (the Whittaker function) is the fifth Painlevé transcendent, and
(5) the Gauß hypergeometric function is the sixth Painlevé transcendent.
Different Painlevé transcendents are not the non-linear analogs of the same classical higher transcendental function.

Proof. The proof is quite analogous to the one for the Bessel function. The corresponding matrix linear ODEs can be found in [15].
By applying to any triangular matrix linear ODE the Schlesinger transformations of a non-triangular type one can generate special families of solutions to the non-linear ODEs whose general solutions are given by the corresponding non-linear special functions. It means that non-linear special functions and their linear counterparts define solutions

to the same non-linear ODEs: the non-linear special function defines the general solution, the linear one – some special solution with a smaller number of the parameters. The last comment proves that two different Painlevé transcendents can not be the non-linear analogs to the same classical higher transcendental function. Otherwise, different Painlevé equations would have special solutions in terms of the same (linear) higher transcendental functions, which is not the case, since the proof given in the works: [19, 20, 21].

Remark 4 *The first Painlevé transcendent is the exceptional one and has no linear counterparts among the classical transcendental functions, since the first Painlevé equation has no classical solutions [22],[23].*

Analogies of different types between Painlevé transcendents and some (linear) higher transcendental functions were pointed out by many authors:
(1) One of the most popular idea is an asymptotic linearization of the corresponding Painlevé equation and a description of the asymptotic properties of the Painlevé transcendent (usually for real values of its argument in the neighborhood of the infinity) in terms of special solutions to the linearized equation. The latter solutions can be explicitly found in terms of the linear higher transcendental functions. This analogy leads to the same conclusions as stated in Proposition 2.
(2) It is well-known that many (linear) special functions can be obtained by separation of variables in the linear "integrable" equations [1]: the wave equation and its different reductions, the Schrödinger equation with different model potentials, etc. The occurrence of the Painlevé transcendents as similarity solutions to the non-linear equations integrable via Inverse Scattering Transform [24, 25] is the other type of the analogy between the linear higher transcendental functions and the Painlevé equations.
(3) In different asymptotic methods the linear special functions are used as the model functions describing different critical phenomenon: coalescences of singularities and saddle points, near resonance behaviors, and caustic-like solutions to some linear equations describing wave propagation. The Painlevé functions are also known to be applicable in analogous situations for non-linear wave equations integrable via Inverse Scattering Transform [13] as well as for some non-integrable non-linear equations [26, 27].
It would be interesting to develop the last two types of the analogy further to the level of Proposition 2.

In conclusion let us make some remarks concerning q-special functions. The natural approach to these functions from the point of view developed in this paper is to consider q-difference equations which can be obtained by replacing of the derivatives, $\frac{d}{d\lambda}$ and $\frac{d}{dt_i}$, in Eqs. (2.1) and (2.6) by the q-differentiation symbol:

$$\begin{aligned}\Psi(q\lambda,\ t_1,\ldots,t_N) &= A(\lambda,\ t_1,\ldots,t_N)\Psi(\lambda,\ t_1,\ldots,t_N), &(6.1)\\ \Psi(\lambda,\ t_1,\ldots,qt_l,\ldots,qt_N) &= B_l(\lambda,\ t_1,\ldots,t_l,\ldots,t_N)\Psi(\lambda,\ t_1,\ldots,t_l,\ldots,t_N), &(6.2)\end{aligned}$$

where $A(\lambda,\ t_1,\ldots,t_N)$ and $B_l(\lambda,\ t_1,\ldots,t_N)$ are rational matrix functions of λ and $q \neq 1$ is a complex parameter. The classical q-difference special functions, possibly, could be obtained as the solutions of the system of q-difference equations representing the compatibility condition of Eqs. (6.1) for the triangular reductions of the matrices A and B. In the work [28] the above mentioned framework ($N = 1$ or $N = 2$ and $t_1/t_2 = const.$) was suggested as an approach to the derivation of q-difference analogs of the Painlevé equations. In particular, a q-difference analog of the sixth Painlevé equation was obtained. It is also shown, that the q-hypergeometric function defines a non-triangular q-deformations of a system (6.1), where $N = 2$, A is a quadratic polynomial of λ, and B_1 is

a rational matrix with two poles, t_1 and t_2 such that $t_1/t_2 = const$. But the representation for the q-hypergeometric function was obtained in an *ad hoc* manner, so that it is not clear how to apply this idea to get the q-special functions of the other types. That is why, it would be interesting to obtain representation of the q-hypergeometric function as the function which defines q-deformation of a triangular matrix q-difference equation.
We can give also another interpretation of the q-difference system considered in the work [28]. Notice that the even iterations of the transformations (2.14) and (4.11) are the rational functions of λ, in particular, four successive transformations of the type (2.14) can be chosen such that their action is equivalent to the multiplication on a quadratic polynomial, and successive iteration of two transformations of the type (4.11) – is a rational function of λ with two poles. Let us fix three sequences of complex numbers: a_l, b_l, c_l, $\ l \in \mathbb{Z}$, which will be used instead of variables t_k in Eq. (2.14) and t_k, t_i in Eq. (4.11). These sequences can coincide and be a constant beginning from some l, that is why the number of the parameters involved can be finite. Now, starting from arbitrary function Ψ which is not any more assumed to be a solution of any auxiliary differential (or any other) equation, say, starting from $\Psi = I$, we generate the sequence of the functions $\Psi_{m,n}$ via the following equations:

$$\Psi_{m+1,n} = A_{m,n}(\lambda)\Psi_{m,n}, \quad \Psi_{m,n+1} = B_{m,n}(\lambda)\Psi_{m,n}, \tag{6.3}$$

where $A_{m,n}(\lambda)$ is the quadratic polynomial and $B_{m,n}(\lambda)$ is the rational functions with two poles mentioned above; A and B depend also of the parameters a_l, b_l, c_l. The compatibility condition of Eqs. (6.3) is a non-linear system of lattice equations. We can now impose different constraints on the Ψ-function, for example, require that $\Psi_{m,n}$ for all m, $n \in \mathbb{Z}$ satisfy differential equations on λ which connection matrices have the same structure as the rational functions of λ or we can require that

$$\Psi_{m+1,n}(\lambda,\ t) = \Psi_{m,n}(q\lambda,\ t), \quad \Psi_{m,n+1}(\lambda,\ t) = \Psi_{m,n}(\lambda,\ qt), \tag{6.4}$$

where t some additional parameter, say, one of the parameters, a_l, b_l, c_l. These additional constraints make restrictions also on the set of the parameters a_l, b_l, c_l. The paper [28] can be viewed as giving an important explicit example of the constraint (6.4).

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