No-Go Theorem for "Free" Relativistic Anyons in $d = 2 + 1$

J. Mund *
Institut für theoretische Physik,
Freie Universität Berlin, Arnimallee 14, 14195 Berlin, Germany
e-mail: mund@physik.fu-berlin.de

July 17, 1997

Abstract

We show that a quantum field theoretic model of anyons cannot be "free" in the restrictive sense that the basic fields create only one-particle states out of the vacuum.
Mathematics Subject Classification (1991): 81T05, 32A10.
Keywords: Abelian braid group statistics, Jost-Schroer theorem, free field.

1 Introduction

In 2+1 dimensional spacetime, the Bose-Fermi alternative does not exhaust all possibilities for particle statistics. Rather, statistics may be described by a representation of the braid group. This has first been realized by Leinaas and Myrheim [1], and quantum mechanical models with an abelian representation of the braid group have first been discussed by F. Wilczek [2], who coined the name *anyons* for such particles. In the framework of algebraic quantum field theory, Buchholz and Fredenhagen have shown [3] that massive particle states might be localizable in spacelike cones only (rather than in bounded regions), which in $d = 2+1$ allows for the possibility of braid group statistics [4, 5]. The Hilbert spaces of scattering states for such theories are well known: they have been constructed by K. Fredenhagen et al. [6] and for the abelian case also by Fröhlich and Marchetti [7], and coincide with the ones proposed by R. Schrader and the author [8]. In particular, the one-particle space $\mathcal{H}^{(m,s)}$ is characterized by the mass m of the particle and its spin s, which may take any real value. The special cases $s \in \mathbb{Z}$ and $s \in \mathbb{Z}+\frac{1}{2}$ correspond

*Supported by DFG, Sfb 288 "Differentialgeometrie und Quantenphysik" and by Studienstiftung des Deutschen Volkes

to bosons and fermions, respectively. For these cases the well-known free fields establish a "second quantization functor", associating a quantum field to each particle type $\mathcal{H}^{(m,s)}$.

For $s \notin \frac{1}{2}\mathbb{Z}$, in contrast, free fields with braid group statistics and satisfying the requirements from relativistic quantum field theory have not been constructed yet[1]. The present paper gives a reason for this: It is shown that, under very mild assumptions, no free model with nontrivial abelian braid group statistics can exist. Here, "free" is meant in the restrictive sense that the field algebra is generated by operators creating only one-particle states out of the vacuum. We recall that in the Bose and Fermi cases ($s \in \frac{1}{2}\mathbb{Z}$) the Jost-Schroer theorem[2] asserts that this condition characterizes the free (Fock space) fields.

One may still hope to find models of anyons which, though not satisfying the above criterion, are "free" in the less restrictive sense that their S-matrix leads to a trivial cross section. Indeed, an analogous situation is encountered in the context of integrable massive models of anyons in $d = 1+1$, where one has models with a piecewise energy-independent S-matrix [14], but none which are free in the strict sense – even the models which are closest to being free show "virtual particle creation".

Recent investigations on a localization concept based on the Tomita-Takesaki modular theory point into the same direction as the result of the present paper. This localization concept, described e.g. in [16, 17], equips each one-particle Hilbert space $\mathcal{H}^{(m,s)}$ with a family of real subspaces $\mathcal{H}_R^{(m,s)}(W)$ indexed by wedge regions W in Minkowski space. For $s \in \frac{1}{2}\mathbb{Z}$, this family can be extended to a *net* of real subspaces $\mathcal{H}_R(\mathcal{O})$ indexed by the double cones $\mathcal{O}$ in such a way that the free fields establish a functor from nets of real subspaces $\mathcal{O} \mapsto \mathcal{H}_R(\mathcal{O})$ to nets of von Neumann algebras $\mathcal{O} \mapsto \mathcal{F}(\mathcal{O})$. For $s \notin \frac{1}{2}\mathbb{Z}$, on the other hand, the defect of $\mathcal{H}_R(W_1 \cap W_2) \subset \mathcal{H}_R(W_1) \cap \mathcal{H}_R(W_2)$ has been computed to be infinite (real) dimensional [18]. This implies that such a functor cannot exist for $s \notin \frac{1}{2}\mathbb{Z}$.

The article is organized as follows. Section 2 sets up our framework and formulates the assumptions. We first describe what is understood by relativistic anyons in algebraic quantum field theory, and to which class of such models we restrict. Then these assumptions are collected in (A0) to (A5). The field algebra is assumed to be generated by "free fields" (A6) localized in spacelike cones[3], whose asymptotic directions are equipped with "winding numbers". Our main technical assumption, reminiscent of the Wightman axioms, is that for two fields φ_1, φ_2 with spacelike separated localization regions, the norm of $\varphi_1 \, U(x) \, \varphi_2 \, \Omega$ is polynomially bounded in x (A7). Here $U(x)$ represents the translation by $x \in \mathbb{R}^3$ and Ω denotes the vacuum vector.

In section 3 we arrive at the no-go result in two steps: If the asymptotic *directions* of the localization regions of two fields φ_1 and φ_2 are spacelike separated, their "twisted

[1] D.R.Grigore has constructed free fields in $d = 2+1$ for any spin [9], but in contradiction to the generalized spin statistics connection holding in algebraic quantum field theory [4, 5] they have bosonic statistics. Presumably, this is due to the fields having infinitely many components.

[2] This theorem is due to B. Schroer [10] and has been elaborated by R. Jost [11] and further by K. Pohlmeyer [12]. For a didactic account, see [13, Thm. 4-15]. O. Steinmann has extended it to string-localized fields satisfying modified Wightman assumptions and Bose or Fermi statistics [14].

[3] We speak of "fields" although these operators are *not* pointlike localized like Wightman distributions (not even stringlike, like in [14]).

commutator" is a c-number function, even if the localization regions overlap (Proposition 1). This is completely analogous to the (first part of the) well known Jost-Schroer theorem. On the other hand, these commutation relations are consistent only in the case of permutation group statistics (Proposition 2).

2 Assumptions

In algebraic quantum field theory a model is specified by an observable algebra $\mathcal{A}$ containing a family $\mathcal{A}(\mathcal{O})$ of a von Neumann algebras labelled by the open bounded regions $\mathcal{O}$ in Minkowski space, which acts in a vacuum Hilbert space $\mathcal{H}_0$ carrying a representation of the Poincaré group, and which satisfies the Haag Kastler axioms [19]. According to the Doplicher-Haag-Roberts theory, the set Δ of inequivalent irreducible representations of $\mathcal{A}$ (superselection sectors) satisfying certain physical criteria, is in one-to-one correspondence with the set of inequivalent DHR endomorphisms of $\mathcal{A}$, and the composition endowes Δ with the structure of an abelian semigroup [20, 3]. If it is actually a group, the model is called abelian. We will restrict to abelian models whose superselection sectors are generated by exactly one *auto*morphism γ of the observable algebra, so that in the present context Δ is assumed to be isomorphic to $\mathbb{Z}$, or, if there is a natural number N s.t. $\gamma^N \cong id$, to $\mathbb{Z}_N$. Admitting the more general case of a finitely generated abelian group would not change the result of this article. In addition we assume γ to be a covariant massive one particle representation of $\mathcal{A}$. This means that the representation of $\mathcal{A}$ corresponding to γ intertwines the vacuum representation of the Poincaré group with a representation in which the energy-momentum spectrum contains an isolated mass shell as its lower boundary. The automorphism γ^q inherits from γ the property of being an irreducible covariant massive representation [3]. Buchholz and Fredenhagen have shown [3], that such representations are localizable in spacelike cones, i.e. equivalent to the vacuum representation when restricted to the observable algebra of the causal complement of any spacelike cone. This entails an intrinsic notion of statistics associating to each sector a representation of the braid group, which for abelian models is one dimensional and characterized by the so called statistics phase $\exp(2\pi i s)$. The values $s = 0$ and $s = \frac{1}{2}$ correspond to bosons and fermions, respectively. In the case of a $\mathbb{Z}_N$ superselection structure, i.e. if there is a natural number N s.t. $\gamma^N \cong id$, we further assume the statistics phase of γ to be an N^{th} root of unity. Thus we restrict to models not exhibiting the obstruction $\exp(2\pi i s N) = -1$ analysed by K.-H. Rehren in [21, section 3.1].

A model of anyons satisfying the above requirements can equivalently be described (see, e.g. [21, section 3.1]) by an algebra $\mathcal{F}$ of unobservable charged field operators transforming under the global gauge group $\hat{\Delta}$ (the dual of the abelian group Δ, i.e. in our case $U(1)$ or $\mathbb{Z}_N$, respectively). Essentially, $\mathcal{F}$ is the crossed product of $\mathcal{A}$ with Δ. Within this setting, the assumptions made so far are made precise in the following conditions (A0) to (A5).

(A0) *Framework.* The Hilbert space $\mathcal{H}$ of the model carries a representation of the global gauge group $\hat{\Delta} = U(1)$ or $\mathbb{Z}_N$. The corresponding decomposition into charge superselection sectors is written as

$$\mathcal{H} = \bigoplus_{q\in\Delta} \mathcal{H}_q \,. \quad \text{Here } \Delta = \mathbb{Z} \text{ or } \mathbb{Z}_N, \text{ respectively,} \tag{2.1}$$

is the character group of $\hat{\Delta}$. $\mathcal{H}$ also carries a unitary representation U of the universal covering group $\tilde{P}_3^\uparrow$ of the Poincaré group s.t. the joint spectrum of the generators P_μ of the translations $U(x)$ contains $\{0\}$ and an isolated mass shell $H_m^+ := \{p \in \mathbb{R}^3 \,/\, p^2 = m^2, p_0 > 0\}$, where $m > 0$ is the particle mass:

$$\{0\} \cup H_m^+ \subseteq \text{spec}P \subseteq \{0\} \cup H_m^+ \cup \{p \in \mathbb{R}^3 \,/\, p^2 > M^2, p_0 > 0\} \text{ for some } M > m. \tag{2.2}$$

The eigenspace to 0 is one dimensional and is spanned by the vacuum vector $\Omega \in \mathcal{H}_{q=0}$. The subspace of $\mathcal{H}$ belonging to the H_m^+ part of the spectrum will be referred to as the one particle Hilbert space $\mathcal{H}^{(1)}$ and is assumed to contain only states with charge $q = 1$ and -1 (particles and antiparticles):

$$\mathcal{H}^{(1)} = \{\phi \in \mathcal{H} \,/\, P^2\phi = m^2\phi\} \subset \mathcal{H}_1 \oplus \mathcal{H}_{-1} \,. \tag{2.3}$$

Charge carrying anyonic fields are not localizable in compact spacetime regions. Rather, the localization of a field operator is characterized by a path in the set $\mathcal{K}$ of spacelike cones and their causal complements. We denote by $\widetilde{\mathcal{K}}$ the set of homotopy classes of such paths. This concept is described in detail in [6, see equ.(2.23)]; here we will only have to compare paths $\tilde{I}, \tilde{J} \in \widetilde{\mathcal{K}}$ ending at regions $I, J \in \mathcal{K}$ with either $I \subset J$ or $I \subset J'$ (the spacelike complement of J). Then the relevant information of the paths is just the "cumulated" angle, which can be unambigously compared in these cases. In this sense we understand the relations

$$\tilde{I} \subset \tilde{J}, \quad \tilde{I} = \tilde{J} + 2\pi \quad \text{and} \quad \tilde{I} < \tilde{J} \,. \tag{2.4}$$

The field algebra is a net assigning to every $\tilde{I} \in \widetilde{\mathcal{K}}$ a von Neumann algebra $\mathcal{F}(\tilde{I})$ of bounded operators in $\mathcal{H}$ such that the following properties are satisfied:

(A1) *Isotony:* $\mathcal{F}(\tilde{I}) \subset \mathcal{F}(\tilde{J})$ if $\tilde{I} \subset \tilde{J}$ in the sense of (2.4).

(A2) *Twisted locality:* $Z(\tilde{I},\tilde{J})\,\mathcal{F}(\tilde{J})\,Z(\tilde{I},\tilde{J})^{-1} \subset \mathcal{F}(\tilde{I})'$ if $J \subset I'$.
Here $\mathcal{F}(\tilde{I})'$ denotes the commutant of $\mathcal{F}(\tilde{I})$, and $Z(\tilde{I},\tilde{J})$ is the "twist operator" defined (for the first time in [22]) by

$$Z(\tilde{I},\tilde{J}) \,|\, \mathcal{H}_q := e^{-i\pi s q^2(2n+1)} \quad \text{if } \tilde{I} + 2\pi n < \tilde{J} < \tilde{I} + 2\pi(n+1) \tag{2.5}$$

(A3) *Covariance under translations:* $\text{Ad}U(x)\,\mathcal{F}(\tilde{I}) = \mathcal{F}(\tilde{I} + x)$ for all $x \in \mathbb{R}^3$.

(A4) *Internal symmetry:* $\mathcal{F}(\tilde{I})$ is mapped onto itself under the action of the global gauge group $\hat{\Delta}$.

(A5) *Reeh - Schlieder property:* The vacuum vector Ω is cyclic for each $\mathcal{F}(\tilde{I})$.

The next assumptions express our definition of a free field and a temperedness condition.

(A6) *Free field:* Each $\mathcal{F}(\tilde{I})$ is generated by a star stable set $\Phi(\tilde{I})$ of closed operators in $\mathcal{H}$ (the free fields localized in $\tilde{I}$) creating only one particle states out of the vacuum:

$$\varphi\,\Omega \in \mathcal{H}^{(1)} \quad \text{for all } \varphi \in \Phi(\tilde{I})\,. \tag{2.6}$$

(A7) *Temperedness condition:* Let $\varphi_1 \in \Phi(\tilde{I}_1)$ and $\varphi_2 \in \Phi(\tilde{I}_2)$ with $I_1 \subset I_2'$. Then $U(x)\,\varphi_2\,\Omega$ is in the domain of φ_1 for all $x \in \mathbb{R}^3$ and, furthermore, the function

$$x \mapsto \|\,\varphi_1\,U(x)\,\varphi_2\,\Omega\,\| \tag{2.7}$$

is locally integrable and polynomially bounded for large x. (Note that this can be violated only if the fields are unbounded operators.)

Properties (A4) and (A5) imply a grading of the algebras $\mathcal{F}(\tilde{I})$: every $F \in \mathcal{F}(\tilde{I})$ has a unique decomposition

$$F = \sum_{q\in\Delta} F_q \text{ , where } F_q \text{ carries charge } q\,, \tag{2.8}$$

i.e. $F_q : \mathcal{H}_{q'} \to \mathcal{H}_{q'+q}$ for all $q' \in \hat{\Delta}$. The same holds for the generating fields. In terms of this grading, twisted locality (A2) entails the more familiar commutation relations for spacelike separated fields: Let $\varphi_1 \in \Phi(\tilde{I}_1)$ and $\varphi_2 \in \Phi(\tilde{I}_2)$ carry charges q_1 and q_2, respectively. Let further $I_2 \subset I_1'$. Then

$$\varphi_1\,\varphi_2\,\Omega \;=\; R(\tilde{I}_1,\tilde{I}_2)^{q_1\cdot q_2}\,\varphi_2\,\varphi_1\,\Omega\,. \tag{2.9}$$

Here, $R(\tilde{I}_1,\tilde{I}_2)$ is defined by

$$R(\tilde{I}_1,\tilde{I}_2) := e^{-2\pi i s(2n+1)} \quad \text{if } \tilde{I}_1 + 2\pi n < \tilde{I}_2 < \tilde{I}_1 + 2\pi(n+1) \tag{2.10}$$

To prove equation (2.9), one approximates the positive part of the polar decomposition of φ_1 by operators in $\mathcal{F}(\tilde{I}_1)$ to show $\varphi_1\,Z(\tilde{I}_1,\tilde{I}_2)\,\varphi_2\,\Omega = Z(\tilde{I}_1,\tilde{I}_2)\,\varphi_2\,Z(\tilde{I}_1,\tilde{I}_2)^{-1}\,\varphi_1\,\Omega$ which implies (2.9).

3 Results

We have chosen (A7) so that the free fields can be decomposed into creation and annihilation parts at least on vectors of the form $\varphi\,\Omega$ (Lemma 1), which is enough to prove the (first part of the) Jost-Schroer theorem: If in the situation of equation (2.9), we translate the localization regions such that they are not spacelike seperated any more, only a multiple of Ω is added to the r.h.s. of that equation (Proposition 1).

For $\varphi \in \Phi(\tilde{I})$, let $\varphi(x) := U(x)\,\varphi\,U(x)^{-1}$ be the translated field, which is affiliated to $\mathcal{F}(\tilde{I}+x)$,[4] and let $H_m^- := -H_m^+$ be the negative mass shell.

[4] (and *not* localized at x)

Lemma 1 *Let φ_1 and φ_2 be space like seperated fields as in (A7) above. Then the $\mathcal{H}$-valued function $\varphi_1(x)\,\varphi_2(y)\,\Omega$ is a tempered distribution, whose Fourier transform has support contained in $(H_m^- \cup H_m^+) \times H_m^+$. Let F^+ and F^- be defined by the corresponding decomposition, i.e.*

$$\varphi_1(x)\,\varphi_2(y)\,\Omega \;=\; F^+(x,y) + F^-(x,y)\,, \qquad \textit{with} \quad \mathrm{supp}\,\widetilde{F^\pm} \subset H_m^\pm \times H_m^+ . \tag{3.11}$$

Then

$$F^-(x,y) \;=\; \langle\,\Omega\,,\,\varphi_1(x)\,\varphi_2(y)\,\Omega\,\rangle\,\Omega, \qquad \textit{and} \tag{3.12}$$

$$\mathrm{sp}_P F^+(x,y) \;\subseteq\; H_m^+ + H_m^+ . \tag{3.13}$$

Here $\mathrm{sp}_P\psi$ denotes the spectral support of ψ w.r.t. the energy momentum operators[5].

Proof . Let $f, g \in \mathcal{S}(\mathbb{R}^3)$, the space of Schwartz functions. Due to the temperedness condition (A7), Riesz' theorem asserts the existence of a unique vector $F(f,g) \in \mathcal{H}$ satisfying

$$\langle\,\phi\,,F(f,g)\,\rangle \;=\; \int dxdy\, f(x)\,g(y)\,\langle\,\phi\,,\,\varphi_1(x)\,\varphi_2(y)\,\Omega\,\rangle \;\text{ for all } \phi\in\mathcal{H}\,,$$

and whose norm can be estimated by

$$F(f,g)\| \leq \int dxdy\,|f(x)g(y)|\,\|\varphi_1(x)\varphi_2(y)\Omega\|\,.$$

By (A7), this shows that the linear map $F\,:\,(f,g)\;\mapsto\;F(f,g)$ is continuous in both entries w.r.t. the usual locally convex topology on Schwartz space, i.e. F is a vector valued tempered distribution.

To prove the statement on the support of its Fourier transform, let f_1 be in $C_0^\infty(\mathbb{R}^3)$, i.e. a smooth function with compact support, and $g \in \mathcal{S}(\mathbb{R}^3)$. For all $A \in \mathcal{F}(\tilde{I}_1 + \mathrm{supp} f_1)'$ one verifies that

$$\langle\,A\,\Omega,\;F(f_1,g)\,\rangle \;=\; \langle\,A\,\bar{\tilde{f}}_1(P)\,\varphi_1^*\,\Omega\,,\;\tilde{g}(P)\,\varphi_2\,\Omega\,\rangle\,.$$

Since $\varphi_2\,\Omega$ and $\varphi_1^*\,\Omega$ are in $\mathcal{H}^{(1)}$, the scalar product vanishes if $\mathrm{supp}\tilde{g}\cap H_m^+ = \emptyset$, or if f_1 is of the form $f_1 = (\Box + m^2)f$ for some function $f \in C_0^\infty(\mathbb{R}^3)$. Taking into account the fact that $\mathcal{F}(\tilde{I}_1 + \mathrm{supp} f_1)'\,\Omega$ is dense in $\mathcal{H}$, we conclude that

$$F((\Box + m^2)f, g) \;=\; 0 \text{ for all } f\in C_0^\infty(\mathbb{R}^3), g \in \mathcal{S}(\mathbb{R}^3)\,, \qquad \text{and}$$

$$F(f,g) \;=\; 0 \text{ for all } f\in C_0^\infty(\mathbb{R}^3) \;\text{ and } g\in\mathcal{S}(\mathbb{R}^3) \text{ with } \mathrm{supp}\tilde{g}\cap H_m^+ = \emptyset\,.$$

By continuity of F, these two properties extend to all $f \in \mathcal{S}(\mathbb{R}^3)$. Now we can proceed as in the case of Wightman fields [13]:

The support of the Fourier transform of F consists of two disjoint sets contained in $H_m^+ \times H_m^+$ and $H_m^- \times H_m^+$, respectively, thus defining the decomposition (3.11). To analyse

[5] i.e. the set of points $p \in \mathrm{spec}P$ such that for any neighbourhood V of p, the spectral projector $E_V(P)$ does not map ψ to zero.

the energy momentum supports, we extend F via the Schwartz nuclear theorem to a continuous linear map from $\mathcal{S}(\mathbb{R}^3 \times \mathbb{R}^3)$ into $\mathcal{H}$, and note that for all $f, g \in \mathcal{S}(\mathbb{R}^3)$ we have $e^{ix\cdot P}\, F(f \otimes g) \;=\; \widetilde{F}(\, e^{ix\cdot(p_1+p_2)}\; \tilde{f} \otimes \tilde{g}\,)$. By linearity and continuity this yields

$$h(P)\, F(f \otimes g) \;=\; \widetilde{F}\,\Big(h(p_1 + p_2) \cdot\, \tilde{f} \otimes \tilde{g} \Big) \quad \text{for all } h \in \mathcal{S}(\mathbb{R}^3). \tag{3.14}$$

From this equation and from the support properties of $\tilde{F}$ we conclude that $h(P)\, F^{\pm}(f, g) = 0$ if $\mathrm{supp} h \cap H^{\pm}_m + H^{+}_m = \emptyset$, respectively. This shows that $\mathrm{sp}_P\, F^{\pm}(f, g) \subset H^{\pm}_m + H^{+}_m$. Since $H^{-}_m + H^{+}_m$ intersects the energy momentum spectrum (2.2) only in $\{0\}$, $F^{-}(f, g)$ is a multiple of the vacuum vector Ω, the factor being $\langle \Omega\,,\, F^{-}(f, g)\,\rangle \;=\; \langle \Omega\,,\, F(f, g)\,\rangle$. This shows that $F^{\pm}(x, y)$ are, like $F(x, y)$, well defined as *functions*, and have the properties (3.12) and (3.13), q.e.d.

Now we are ready to establish an analogon to the (first part of the) Jost-Schroer theorem: If in the situation of equation (2.9), we translate the localization regions such that they are not spacelike separated any more, only a multiple of Ω is added to the r.h.s of the commutation relation (2.9). More precisely:

Proposition 1 *Let $\varphi_1 \in \Phi(\tilde{I}_1)$ and $\varphi_2 \in \Phi(\tilde{I}_2)$ carry charges q_1 and q_2, respectively. Let further $I_1 \subset I_2'$. Then the fields satisfy the commutation relations*

$$\varphi_1(x)\, \varphi_2\, \Omega - R(\tilde{I}_1, \tilde{I}_2)^{q_1\cdot q_2}\, \varphi_2\, \varphi_1(x)\, \Omega \;=\; c_{\varphi_1,\varphi_2}(x)\, \Omega \quad \textit{for all } x \in \mathbb{R}^3\ . \tag{3.15}$$

Here $c_{\varphi_1,\varphi_2}(x)$ is the scalar product of the vacuum with the l.h.s. of equation (3.15).

Note that these commutation relations extend from Ω to $\bigcap_{i=1,2} \mathcal{F}(\tilde{I}_i + x_i)'\, \Omega$, which is dense in $\mathcal{H}$ if $\tilde{I}_1$ and $\tilde{I}_2$ have "equal winding numbers" as explained before equation (3.17) below. The
Proof of this proposition is a straightforward adaption of the proof of theorem 4-15 in [13] to the present anyonic case: Let $F^{+}_{1,2}(x, y)$ be the component of $\varphi_1(x)\varphi_2(y)\, \Omega$ whose Fourier transform has support in $H^{+}_m + H^{+}_m$ according to Lemma 1, and $F^{+}_{2,1}(x, y)$ that of $\varphi_2(x)\varphi_1(y)\, \Omega$. Let further $R := R(\tilde{I}_1, \tilde{I}_2)^{q_1\cdot q_2}$ be as defined in equation (2.10). Lemma 1 asserts that

$$\varphi_1(x)\varphi_2\, \Omega - R\, \varphi_2\varphi_1(x)\, \Omega = c_{\varphi_1,\varphi_2}(x)\, \Omega + F^{+}_{1,2}(x, 0) - R\, F^{+}_{2,1}(0, x). \tag{3.16}$$

We have to show that the last two terms add up to zero. For all $\psi \in \mathcal{H}$, the distribution $F_\psi(x) \;=\; \big\langle\, \psi\,,\, F^{+}_{1,2}(x, 0) - R\, F^{+}_{2,1}(0, x)\,\big\rangle$ is the boundary value of an analytic function, since its Fourier transform has support in the cone V_+ according to Lemma 1 [23, Thm. IX.16]. Further, equations (3.16) and (2.9) imply that F_ψ vanishes on the real open set of points satisfying $I_1 + x \subset I_2'$. Due to the edge of the wedge theorem, this forces $\hat{F}_\psi$ to vanish identically as a distribution [13, Thm. 2-17], and hence as a function. q.e.d.

From this proposition we obtain our main result, the no-go theorem for free anyons:

Proposition 2 *Assume, some of the commutator functions $c_{\varphi_1,\varphi_2}(x)$ appearing in equation (3.15) of Proposition 1 do* not *vanish identically in x. Then the commutation relations (3.15) are consistent only if the statistics parameter s is half integer, i.e. only in the case of permutation group statistics.*

Remark. The additional assumption of the proposition does not seem to be a severe restriction: if it where violated, we only needed a criterion allowing us to deduce commutation relations for the field algebra elements from those of the fields (like an energy bound satisfied by the fields), to conclude that the local observable algebras are commutative – in contradiction to our general framework (A0) to (A5).

Proof . We choose three mutually spacelike seperated spacelike cones [6] $I_i \in \mathcal{K}$ $(i = 1, 2, 3)$, small enough that for all $x \in \mathbb{R}^3$ there is some $I_x \in \mathcal{K}$ s.t. $I_1 + x \cup I_2 \cup I_3 \subset I_x$, and three paths $\tilde{I}_i \in \widetilde{\mathcal{K}}$ ending at I_i which satisfy $\tilde{I}_1 < \tilde{I}_3 < \tilde{I}_2 < \tilde{I}_1 + 2\pi$. Then we choose two fields $\varphi_1 \in \Phi(\tilde{I}_1)$, $\varphi_2 \in \Phi(\tilde{I}_2)$ and a translation vector $x \in \mathbb{R}^3$ s.t. $c_{\varphi_1,\varphi_2}(x) \neq 0$. This implies that the charges q_1 and q_2 of the fields add up to zero. Now we choose a field $\varphi_3 \in \Phi(\tilde{I}_3)$ with charge $q_3 = -q_2$, which entails $\varphi_3\,\Omega \perp \varphi_2\,\Omega$. We have chosen the localisation regions so that there is a path $\tilde{I}_x \in \widetilde{\mathcal{K}}$ containing $\tilde{I}_1 + x, \tilde{I}_2$ and $\tilde{I}_3$ in the sense of (2.4). Using isotony (A1), this implies that the subspace

$$D := \mathcal{F}(\tilde{I}_1 + x)'\,\Omega \cap \bigcap_{i=2,3} \mathcal{F}(\tilde{I}_i)'\,\Omega \tag{3.17}$$

contains $\mathcal{F}(\tilde{I}_x)'\,\Omega$, which is dense in $\mathcal{H}$ due to locality (A2) and cyclicity (A5). Thus, D is a dense subspace. Now let $\psi \in D$. Denoting $R_{ij} := R(\tilde{I}_i, \tilde{I}_j)^{q_i \cdot q_j}$ and $c_{1j} := c_{\varphi_1,\varphi_j}(x)$ as in Proposition 1, we get from Proposition 1 and from equation (2.9)

$$\begin{aligned} &\langle\, \varphi_1(x)^*\,\psi\,,\, \varphi_2\,\varphi_3\,\Omega\,\rangle \;= \\ &\quad = \langle\, \psi\,, (R_{12}c_{13}\,\varphi_2 + c_{12}\,\varphi_3\,)\,\Omega\,\rangle + R_{12}R_{13}R_{23}\,\langle\, \varphi_3^*\,\psi\,, \varphi_2\,\varphi_1(x)\,\Omega\,\rangle && (3.18)\\ &\quad = \langle\, \psi\,, (R_{23}c_{13}\,\varphi_2 + R_{23}R_{13}c_{12}\,\varphi_3\,)\,\Omega\rangle + R_{12}R_{13}R_{23}\,\langle\, \varphi_3^*\,\psi\,, \varphi_2\,\varphi_1(x)\,\Omega\,\rangle\,. && (3.19)\end{aligned}$$

In equation (3.18) we have first commuted φ_2^* with $\varphi_1(x)^*$ and then $\varphi_1(x)$ with φ_3, and in (3.19) first φ_3 with φ_2, and then φ_3^* with $\varphi_1(x)^*$. Note that all of the twofold products $\varphi_2^*\,\varphi_1(x)^*$ etc. are well defined on D. Since D is dense, we conclude that

$$(R_{12}c_{13}\,\varphi_2 + c_{12}\,\varphi_3\,)\,\Omega = (R_{23}c_{13}\,\varphi_2 + R_{23}R_{13}c_{12}\,\varphi_3\,)\,\Omega\,.$$

By assumption, $\varphi_2\,\Omega$ and $\varphi_3\,\Omega$ are linearly independent and $c_{12} \neq 0$, so that $R_{23}\,R_{13} = 1$ follows. On the other hand, according to equation (2.10) we compute $R_{13} = \exp(-2\pi i s q_1 q_3) = R_{23}$, which implies $\exp(-4\pi i s q_1 q_3) = 1$. This is only possible if $s \in \frac{1}{2}\mathbb{Z}$, since the charges q_1, q_3 only take the values ± 1 according to (2.3), q.e.d.

[6] It is a necessary condition for the proposition that this geometric situation can be achieved. This is not the case, e.g. if the “free fields” are only localizable in wedge regions.

Acknowledgements. It is a pleasure to thank B. Schroer, to whom the basic idea leading to this article is due, him and H.-W. Wiesbrock for the discussion of the modular localization concept, and also R. Schrader, B. Schroer, H.-W. Wiesbrock and M. U. Schmidt for numerous helpful and stimulating discussions.

References

[1] Leinaas, J.M., Myrheim, J.: On the Theory of Identical Particles, *Il Nuovo Cimento* **37 b** (1977) 1-23.

[2] Wilczek, F.: Quantum Mechanics of Fractional-Spin Particles,*Phys.Rev.Lett.* **49** No.14 (1982) 957-1149.

[3] Buchholz D., Fredenhagen, K.: Locality and the Structure of Particle States, *Commun. Math. Phys.* **84** (1982) 1-54.

[4] Fredenhagen, K.: Structure of Superselection Sectors in Low Dimensional Quantum Field Theory, in: Chau, L.L., Nahm, W. (eds): Proceedings, Lake Tahoe City 1989.

[5] Fröhlich, J., Marchetti, P.A.: Quantum Field Theories of Vortices and Anyons, *Commun. Math. Phys.***121** (1989) 177-223.

[6] Fredenhagen, K., Gaberdiel, M., Rüger, S.M.: Scattering States of Plektons (Particles with Braid Group Statistics) in 2+1 Dimensional Field Theory, *Commun. Math. Phys.* **175** (1996) 319-355.

[7] Fröhlich, J., Marchetti, P.A.: Spin-Statistics Theorem and Scattering in Planar Quantum Field Theories with Braid Statistics, *Nucl. Phys. B* **356** (1991) 533-573.

[8] Mund, J., Schrader, R.: Hilbert Spaces for Nonrelativistic and Relativistic "Free" Plektons (Particles with Braid Group Statistics), in: S.Albeverio, R.Figari, E.Orlandi and A.Teta (eds.), *Proceedings of the Conference on Advances in Dynamical Systems and Quantum Physics*, World Scientific, Singapore 1995, pp. 235-259; and hep-th/9310054.

[9] Grigore, D.R.: Free Fields for any Spin in 1+2 Dimensions, *J. Math. Phys.* **35** (1994) 6304-6331.

[10] Schroer, B.: Diploma thesis (1958), unpublished.

[11] Jost, R.: Properties of Wightman Functions, in: E.R.Caianello (ed.), *Lectures on Field Theory: The many Body Problem* , Academic Press, New York 1961.

[12] Pohlmeyer, K.: The Jost-Schroer Theorem for Zero-Mass Fields, *Commun. Math. Phys.* **12** (1969) 204.

[13] Streater, R.F., Wightman, A.S.: *PCT, Spin and Statistics, and all that* , W.A.Benjamin Inc., New York 1964.

[14] Steinmann, O.: A Jost-Schroer Theorem for String Fields, *Commun .Math. Phys.* **87** (1982) 259-264.

[14] Schroer, B.: Scattering Properties of Anyons and Plektons, *Nucl. Phys.* **B 369** (1992) 478-498.

[16] Brunetti, R., Guido, D., Longo, R.: to be published.

[17] Schroer, B.: Wigner Representation Theory of the Poincaré Group, Localization, Statistics and the S-Matrix, to appear in *Nucl. Phys.* **B 499** (1997) [PM].

[18] Mund., J.: PhD thesis, in preparation.

[19] Haag, R., Kastler, D.: An Algebraic Approach to Quantum Field Theory, *J. Math. Phys.***5** (1964) 848.

[20] Doplicher, S., Haag, R., Roberts, J.E.: Local Observables and Particle Statistics I, *Commun. Math. Phys.* **23** (1971) 199.

[21] Rehren, K.-H.: Field Operators for Anyons and Plektons, *Commun.Math.Phys.* **145** (1992) 123.

[22] Schroer, B.: Modular Theory and Symmetry in QFT, in: Sen, R.N., Gersten, A. (eds.), *Mathematical Physics towards the* 21^{st} *Century*, Ben-Gurion of the Negev Press, Beer-Sheva 1994.

[23] Reed, M., Simon, B.: *Methods of Modern Mathematical Physics II*, Academic Press, New York 1975.