Parallel spinors and holonomy groups on pseudo-Riemannian spin manifolds

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Abstract

We describe the possible holonomy groups of simply connected irreducible non-locally symmetric pseudo-Riemannian spin manifolds which admit parallel spinors.

1 Introduction

In [Wa 89] McKenzie Y. Wang described the possible holonomy groups of complete simply connected irreducible non-flat Riemannian spin manifolds M^n which admit parallel spinors. Parallel spinors occur exactly for the holonomy representations $\mathrm{SU}(m)$, $n = 2m \geq 4$, $\mathrm{Sp}(m)$, $n = 4m \geq 8$, G_2 and $\mathrm{Spin}(7)$. A complete, simply connected locally symmetric Riemannian spin manifold with non-trivial parallel spinors is flat.

In the present paper we consider the same problem for pseudo-Riemannian spin manifolds. We describe the possible holonomy groups of simply connected irreducible non-locally symmetric pseudo-Riemannian spin manifolds $M^{r,s}$ of dimension $r+s$ and index r and study the chiral type and the causal type of these parallel spinors with respect to the indefinite scalar product on the spinor bundle.

The method to attack this problem is based on the fact that the space of parallel spinor fields can be identified with the vector space

$$V_{\tilde{H}_{\tilde{u}}} = \{v \in \Delta_{r,s} \mid \rho(\tilde{H}_{\tilde{u}})v = v\}$$

of all elements of the spinor module $\Delta_{r,s}$ which are invariant under the action of the holonomy group $\tilde{H}_{\tilde{u}} \subset \mathrm{Spin}(r,s)$ of the Levi-Civita connection $\tilde{A}$ on the spin structure Q of $M^{r,s}$ with respect to $\tilde{u} \in Q$. The first list of all possible holonomy groups H_u of simply connected irreducible non-locally symmetric semi-Riemannian manifolds was established by M. Berger in [Be 55]. Later on there were made some corrections and additions to Berger's original list by D. Alekseevskii [A 68], R. Brown and A. Gray [BG 72], R. McLean and by R. Bryant [Br 96]. Since $\lambda(\tilde{H}_{\tilde{u}}) = H_{\lambda(\tilde{u})}$, where $\lambda : \mathrm{Spin}(r,s) \to \mathrm{SO}(r,s)$ is the double covering of $\mathrm{SO}(r,s)$ by $\mathrm{Spin}(r,s)$, we have to check, for which of the holonomy groups of Berger's list $V_{\tilde{H}_{\tilde{u}}}$ is non-zero.

We obtain

Theorem. *Let $M^{r,s}$ be a simply connected, non-locally symmetric, irreducible semi-Riemannian spin manifold of dimension $n = r + s$ and index r and let $\mathfrak{P}$ be the space of parallel spinors of M. Then $\dim \mathfrak{P} = N > 0$ if and only if the holonomy representation H of M is (up to conjugacy in the full orthogonal group) one of the following:*

1. $H = \mathrm{SU}(p,q) \subset \mathrm{SO}(2p,2q)$, $\quad n = 2(p+q)$, $r = 2p$.
   Then $N = 2$.
2. $H = \mathrm{Sp}(p,q) \subset \mathrm{SO}(4p,4q)$, $\quad n = 4(p+q)$, $r = 4p$.
   Then $N = p+q+1$.
3. $H = \mathrm{G}_2 \subset \mathrm{SO}(7)$, $\quad n = 7$, $r = 0$.
   Then $N = 1$.
4. $H = \mathrm{G}^*_{2(2)} \subset \mathrm{SO}(4,3)$, $\quad n = 7$, $r = 4$.
   Then $N = 1$.
5. $H = \mathrm{G}_2^{\mathbb{C}} \subset \mathrm{SO}(7,7)$, $\quad n = 14$, $r = 7$.
   Then $N = 2$.
6. $H = \mathrm{Spin}(7) \subset \mathrm{SO}(8)$, $\quad n = 8$, $r = 0$.
   Then $N = 1$.
7. $H = \mathrm{Spin}(4,3) \subset \mathrm{SO}(4,4)$, $\quad n = 8$, $r = 4$.
   Then $N = 1$.
8. $H = \mathrm{Spin}(7)^{\mathbb{C}} \subset \mathrm{SO}(8,8)$, $\quad n = 16$, $r = 8$.
   Then $N = 1$.

This theorem shows that the possible holonomy group of a manifold with parallel spinors is uniquely determined (up to conjugacy) by the dimension and the index of M and the number of linearly independent parallel spinor fields.

Section 2 of this paper contains some necessary notations from the spinor calculus. In Section 3 we recall the complete Berger list of the holonomy representations of irreducible simply connected non-locally symmetric semi-Riemannian manifolds, explain the relation between parallel spinors and these holonomy groups and fix some notation concerning the appearing groups. In Section 4 we calculate the spaces $V_{\tilde{H}}$ for the holonomy groups of Berger's list explicitly and determine the chiral and the causal type of the corresponding parallel spinors.

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2 Spinor representations

For a more detailed explanation of the following facts see e.g. [LM 89], [Ba 81], [F 97].

Let $\mathcal{C}_{r,s}$ denote the Clifford algebra of $\mathbb{R}^{r,s} = (\mathbb{R}^{r+s}, \langle\ ,\ \rangle_{r,s})$, where $\langle\ ,\ \rangle_{r,s}$ is a scalar product of index r. It is generated by an orthonormal basis $e_1, \ldots, e_{r+s}$ of $\mathbb{R}^{r,s}$ with relations

$$e_i \cdot e_j + e_j \cdot e_i = -2\kappa_j \cdot \delta_{ij}\ ,$$

where

$$\kappa_j = \langle e_j\ ,\ e_j \rangle_{r,s} = \pm 1.$$

If $n = r+s$ is even, then the complexification $\mathcal{C}^{\mathbb{C}}_{r,s}$ of $\mathcal{C}_{r,s}$ is isomorphic to the matrix algebra $\mathbb{C}(2^{\frac{n}{2}})$. If $n = r+s$ is odd, then $\mathcal{C}^{\mathbb{C}}_{r,s}$ is isomorphic to $\mathbb{C}(2^{[\frac{n}{2}]}) \oplus \mathbb{C}(2^{[\frac{n}{2}]})$.

We will use the following isomorphisms:
Let $E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $T = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $U = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$, $V = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$ and

$$\tau_k = \begin{cases} i & \text{if} \quad \kappa_k = -1, \\ 1 & \text{if} \quad \kappa_k = 1. \end{cases}$$

In case $n = r+s$ even we define

$$\Phi_{r,s}\ :\ \mathcal{C}^{\mathbb{C}}_{r,s} \simeq \mathbb{C}(2^{\frac{n}{2}})$$

by

$$\begin{aligned} \Phi_{r,s}(e_{2k-1}) &= \tau_{2k-1}\, E \otimes \cdots \otimes E \otimes U \otimes \underbrace{T \otimes \cdots \otimes T}_{k-times} \\ \Phi_{r,s}(e_{2k}) &= \tau_{2k}\, E \otimes \cdots \otimes E \otimes V \otimes \underbrace{T \otimes \cdots \otimes T}_{k-times}, \end{aligned} \tag{1}$$

$k = 1, \ldots, \frac{n}{2}$. The restriction of $\Phi_{r,s}$ to $\mathrm{Spin}(r,s) \subset \mathcal{C}_{r,s}$ yields the spinor representation $\Delta_{r,s}$ which we will use here.

In case $n = r+s$ odd

$$\Phi_{r,s}\ :\ \mathcal{C}^{\mathbb{C}}_{r,s} \xrightarrow{\sim} \mathbb{C}(2^{[\frac{n}{2}]}) \oplus \mathbb{C}(2^{[\frac{n}{2}]})$$

is given by

$$\begin{aligned} \Phi_{r,s}(e_k) &= (\Phi_{r,s-1}(e_k)\ ,\ \Phi_{r,s-1}(e_k)), \quad k = 1, \ldots, n-1 \\ \Phi_{r,s}(e_n) &= \tau_n(i\, T \otimes \cdots \otimes T\ ,\ -i\, T \otimes \cdots \otimes T). \end{aligned} \tag{2}$$

If we restrict $\Phi_{r,s}$ to $\mathrm{Spin}(r,s) \subset \mathcal{C}_{r,s}$ and project onto the first component we obtain the spinor representation $\Delta_{r,s}$ in the odd-dimensional case.

The Lie algebra $\mathfrak{spin}(r,s)$ of $\mathrm{Spin}(r,s)$ is given by

$$\mathfrak{spin}(r,s) = \mathrm{span}(e_k \cdot e_l \,|\, 1 \le k < l \le n) \subset \mathcal{C}_{r,s}.$$

Let D_{kl} be the $(n \times n)$-matrix whose (k,l)-entry is 1 and all of whose others are 0 and let E_{kl} be the matrix

$$E_{kl} = -\kappa_l D_{kl} + \kappa_k D_{lk}\,. \tag{3}$$

The differential of the double covering $\lambda : \mathrm{Spin}(r,s) \to \mathrm{SO}(r,s)$ is given by

$$\lambda_*(e_k \cdot e_l) = 2E_{kl}\,.$$

Let $u(\varepsilon) \in \mathbb{C}^2$, $\varepsilon = \pm 1$, be the vectors $u(\varepsilon) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -\varepsilon \cdot i \end{pmatrix}$ of $\mathbb{C}^2$. For calculations we will use the orthonormal basis

$$\{u(\varepsilon_m, \ldots, \varepsilon_1) := u(\varepsilon_m) \otimes u(\varepsilon_{m-1}) \otimes \cdots \otimes u(\varepsilon_1) \mid \varepsilon_j = \pm 1\}$$

of $\Delta_{r,s} = \mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2 \quad (m = [\frac{n}{2}])$.

In case of even $n = r+s$ the splitting $\Delta_{r,s} = \Delta^+_{r,s} \oplus \Delta^-_{r,s}$ of $\Delta_{r,s}$ in positive and negative spinors is given by

$$\Delta^\pm_{r,s} = \left\{ u(\varepsilon_m, \ldots, \varepsilon_1) \,|\, \prod_{j=1}^m \varepsilon_j = \pm 1 \right\}.$$

Then, according to (1) and (2) we have for $l > k$, $2l-1 < n$

$$\begin{aligned} & e_{2k-1} \cdot e_{2l-1} \cdot u(\varepsilon_m, \ldots, \varepsilon_1) \\ = \;& -(-1)^{l-k} \tau_{2k-1} \cdot \tau_{2l-1}\, \varepsilon_{l-1} \cdot \varepsilon_{l-2} \cdots \varepsilon_{k+1} \cdot \varepsilon_k \\ & \cdot u(\varepsilon_m, \ldots, \varepsilon_{l+1}, -\varepsilon_l, \varepsilon_{l-1}, \ldots, \varepsilon_{k+1}, -\varepsilon_k, \varepsilon_{k-1}, \ldots, \varepsilon_1) \end{aligned} \tag{4}$$

$$\begin{aligned} & e_{2k} \cdot e_{2l} \cdot u(\varepsilon_m, \ldots, \varepsilon_1) \\ = \;& (-1)^{l-k} \tau_{2k} \tau_{2l} \cdot \varepsilon_l \cdot \varepsilon_{l-1} \cdots \varepsilon_{k+1} \\ & \cdot u(\varepsilon_m, \ldots, \varepsilon_{l+1}, -\varepsilon_l, \varepsilon_{l-1}, \ldots, \varepsilon_{k+1}, -\varepsilon_k, \varepsilon_{k-1}, \ldots, \varepsilon_1) \end{aligned} \tag{5}$$

$$\begin{aligned} & e_{2k-1} \cdot e_{2l} \cdot u(\varepsilon_m, \ldots, \varepsilon_1) \\ = \;& (-1)^{l-k} i \cdot \tau_{2k-1} \tau_{2l} \varepsilon_l \cdot \varepsilon_{l-1} \cdots \varepsilon_{k+1} \cdot \varepsilon_k \\ & \cdot u(\varepsilon_m, \ldots, \varepsilon_{l+1}, -\varepsilon_l, \varepsilon_{l-1}, \ldots, \varepsilon_{k+1}, -\varepsilon_k, \varepsilon_{k-1}, \ldots, \varepsilon_1) \end{aligned} \tag{6}$$

$$\begin{aligned} & e_{2k} \cdot e_{2l-1} \cdot u(\varepsilon_m, \ldots, \varepsilon_1) \\ = \;& (-1)^{l-k} i \cdot \tau_{2k} \tau_{2l-1} \varepsilon_{l-1} \cdots \varepsilon_{k+1} \\ & \cdot u(\varepsilon_m, \ldots, \varepsilon_{l+1}, -\varepsilon_l, \varepsilon_{l-1}, \ldots, \varepsilon_{k+1}, -\varepsilon_k, \varepsilon_{k-1}, \ldots, \varepsilon_1) \end{aligned} \tag{7}$$

$$e_{2k-1}\cdot e_{2k}\cdot u(\varepsilon_m,\,\ldots,\varepsilon_1)=i\tau_{2k-1}\tau_{2k}\varepsilon_k\cdot u(\varepsilon_m,\ldots,\varepsilon_1). \tag{8}$$

If n is odd we obtain for $j\le m$

$$\begin{aligned} & e_{2j}\cdot e_n\cdot u(\varepsilon_m,\,\ldots,\varepsilon_1)\\ &\quad = (-1)^{m+j-1} i\tau_n\varepsilon_{j+1}\cdot\varepsilon_{j+2}\cdots\varepsilon_m\cdot u(\varepsilon_m,\ldots,\varepsilon_{j+1},-\varepsilon_j,\varepsilon_{j-1},\ldots,\varepsilon_1) \end{aligned}\tag{9}$$

$$\begin{aligned} & e_{2j-1}\cdot e_n\cdot u(\varepsilon_m,\ldots,\varepsilon_1)\\ &\quad = (-1)^{m+j}\tau_n\cdot\varepsilon_j\cdot\varepsilon_{j+1}\cdots\varepsilon_m\cdot u(\varepsilon_m,\ldots,\varepsilon_{j+1},-\varepsilon_j,\varepsilon_{j-1},\ldots,\varepsilon_1) \end{aligned}\tag{10}$$

If we are in the pseudo-Euclidean case ($0<r<r+s$), there exists an indefinite scalar product $\langle\,,\rangle$ on $\Delta_{r,s}$ which is invariant under the action of the connected component $\mathrm{Spin}_0(r,s)$ of $\mathrm{Spin}(r,s)$. $\langle\,,\rangle$ is given by

$$\langle u\,,\,v\rangle = i^{\frac{r(r-1)}{2}}(e_{i_r}\cdots e_{i_r}\cdot u\,,\,v)_{\mathbb{C}^{2^m}}\,,$$

where $e_{i_1},\ldots,e_{i_r}\in\mathbb{R}^{r+s}$ are the timelike vectors of the orthonormal basis $e_1,\ldots,e_n$ of $(\mathbb{R}^{r,s}\,,\ i_1<\cdots<i_r$.

In particular, if r and s are odd, then $\Delta^+_{r,s}$ and $\Delta^-_{r,s}$ are totally isotropic. If r and s are even, $\Delta^+_{r,s}$ is $\langle\,,\rangle$-orthogonal to $\Delta^-_{r,s}$.

In the case that

$$\langle x\,,\,y\rangle_{r,s}=-x_1y_1-\cdots-x_ry_r+x_{r+1}y_{r+1}+\cdots+x_{r+s}y_{r+s}$$

and that $e_1,\ldots,e_{r+s}$ is the standard basis of $\mathbb{R}^{r+s}$ we have for even index $r=2\hat r$

$$\begin{aligned} &\langle u(\varepsilon_m,\ldots,\varepsilon_1)\,,\,u(\delta_m,\ldots,\delta_1)\rangle\\ &\quad = \left\{\begin{array}{cl} 0 & \text{if}\quad (\varepsilon_m,\ldots,\varepsilon_1)\neq(\delta_m,\ldots,\delta_1)\\ \varepsilon_1\cdots\varepsilon_{\hat r} & \text{if}\quad (\varepsilon_m,\ldots,\varepsilon_1)=(\delta_m,\ldots,\delta_1) \end{array}\right. . \end{aligned}\tag{11}$$

If we consider the bilinear form

$$\langle x\,,\,y\rangle_{r,r}=-\sum_{j=1}^{r}x_{2j-1}\cdot y_{2j-1}+\sum_{j=1}^{r}x_{2j}\cdot y_{2j}$$

and the standard basis $(e_1,\ldots,e_{2r})$ of $\mathbb{R}^{2r}$ we obtain

$$\begin{aligned} &\langle u(\varepsilon_r,\ldots,\varepsilon_1)\,,\,u(\delta_r,\ldots,\delta_1)\rangle\\ &\quad = \left\{\begin{array}{ll} 0 & \text{if } (\varepsilon_r,\ldots,\varepsilon_1)\neq(-\delta_r,\ldots,-\delta_1)\\ (-i)^{\hat r}\,\varepsilon_1\cdot\varepsilon_3\cdots\varepsilon_{r-1} & \text{if } r=2\hat r \text{ and}\\ & (\varepsilon_r,\ldots,\varepsilon_1)=(-\delta_r,\ldots,-\delta_1)\\ -i^{\hat r}\,\varepsilon_2\cdot\varepsilon_4\cdots\varepsilon_{r-1} & \text{if } r=2\hat r+1 \text{ and}\\ & (\varepsilon_r,\ldots,\varepsilon_1)=(-\delta_r,\ldots,-\delta_1). \end{array}\right. \end{aligned}\tag{12}$$

3 Some facts on holonomy groups of semi-Riemannian spin manifolds

A connected semi-Riemannian manifold $(M^{r,s}, g)$ of dimension $r+s$ and index r is called *irreducible*, if its holonomy representation is irreducible. It is called *indecomposable* if its holonomy groups $H_u \subset \mathrm{O}(r,s)$ do not leave invariant any nondegenerate proper subspace of $\mathbb{R}^{r,s}$. Of course, in the Riemannian case, indecomposable manifolds are irreducible. In the pseudo-Riemannian case there exist holonomy groups whose holonomy representations are not irreducible but which are not decomposable into a direct sum of pseudo-Riemannian holonomy representations ([Wu 67], [BI 93]).

De Rham's splitting theorem reduces the study of complete simply connected semi-Riemannian manifolds to indecomposable ones.

Theorem 1 ([DR 52], [Wu 64]) *Let $(M^{r,s}, g)$ be a simply connected complete semi-Riemannian manifold. Then $(M^{r,s}, g)$ is isometric to a product of simply connected complete indecomposable semi-Riemannian manifolds.*

Up to now, there is no list of all indecomposable restricted holonomy groups in the pseudo-Riemannian setting, but the irreducible cases are known.

The irreducible pseudo-Riemannian symmetric spaces were classified by M. Berger in [Be 57]. The irreducible restricted holonomy groups of non-locally symmetric spaces are listed in the following (corrected) Berger list.

Theorem 2 ([Be 55], [S 62], [A 68], [BG 72], [Br 96]) *Let $(M^{r,s}, g)$ be a simply connected irreducible non-locally symmetric semi-Riemannian manifold of dimension $n = r+s$ and index r. Then its holonomy representation is (up to conjugacy in $\mathrm{O}(r,s)$) one of the following*

$\mathrm{SO}_0(p,q)$			$n = p+q \geq 2,$	$r = p$
$\mathrm{U}(p,q)$	$\subset$	$\mathrm{SO}(2p,2q)$	$n = 2(p+q) \geq 4,$	$r = 2p$
$\mathrm{SU}(p,q)$	$\subset$	$\mathrm{SO}(2p,2q)$		
$\mathrm{Sp}(p,q)$	$\subset$	$\mathrm{SO}(4p,4q)$	$n = 4(p+q) \geq 8,$	$r = 4p$
$\mathrm{Sp}(p,q)\cdot\mathrm{Sp}(1)$	$\subset$	$\mathrm{SO}(4p,4q)$		
$\mathrm{Sp}(p,\mathbb{R})\cdot\mathrm{SL}(2,\mathbb{R})$	$\subset$	$\mathrm{SO}(2p,2p)$	$n = 4p \geq 8,$	$r = 2p$
$\mathrm{Sp}(p,\mathbb{C})\cdot\mathrm{SL}(2,\mathbb{C})$	$\subset$	$\mathrm{SO}(4p,4p)$	$n = 8p \geq 16,$	$r = 4p$
$\mathrm{SO}(p,\mathbb{C})$	$\subset$	$\mathrm{SO}(p,p)$	$n = 2p \geq 4,$	$r = p$
G_2	$\subset$	$\mathrm{SO}(7)$	$n = 7$	
$\mathrm{G}^*_{2(2)}$	$\subset$	$\mathrm{SO}(4,3)$		
$\mathrm{G}_2^{\mathbb{C}}$	$\subset$	$\mathrm{SO}(7,7)$	$n = 14$	
$\mathrm{Spin}(7)$	$\subset$	$\mathrm{SO}(8)$	$n = 8$	
$\mathrm{Spin}(4,3)$	$\subset$	$\mathrm{SO}(4,4)$		
$\mathrm{Spin}(7)^{\mathbb{C}}$	$\subset$	$\mathrm{SO}(8,8)$	$n = 16$	

It is known for all groups H of the list in Theorem 2 that there exists a non-symmmetric semi-Riemannian manifold with holonomy H (see [Br 87], [Br 96]).

Now let $(M^{r,s}, g)$ be a semi-Riemannian spin manifold with spin structure (Q, f) and spinor bundle $S = Q \times_{\mathrm{Spin}(r,s)} \Delta_{r,s}$.

A spinor field $\varphi \in \Gamma(S)$ is called *parallel*, if $\nabla^S \varphi \equiv 0$, where ∇^S is the spinor derivative associated to the Levi-Civita connection $\tilde{A}$ on Q.

Let us denote by $\tau_\omega^{\tilde{A}} : Q_x \to Q_x$ the parallel transport in the spin structure Q with respect to $\tilde{A}$ along a closed curve ω starting at $x \in M$. Then for $\tilde{u} \in Q_x$

$$\begin{aligned} \tilde{H}_{\tilde{u}} \;:=\; \{ g \in \mathrm{Spin}(r,s) \mid & \text{ There exists a closed curve } \omega \\ & \text{ starting at } x \text{ such that } \tilde{u} \cdot g = \tau_\omega^{\tilde{A}}(\tilde{u}) \} \end{aligned}$$

denotes the holonomy group of $\tilde{A}$ with respect to $\tilde{u} \in Q_x$. If $\tilde{u}_1 \in Q_{x_1}$ is another point in Q and σ is a curve from x to x_1, then

$$\tilde{H}_{\tilde{u}_1} = a^{-1} \cdot \tilde{H}_{\tilde{u}} \cdot a,$$

where $a \in \mathrm{Spin}(r,s)$ is the element with $u_1 = \tau_\sigma^{\tilde{A}}(u) \cdot a$. Hence, all holonomy groups of $\tilde{A}$ are conjugated to each other.

Let

$$\begin{aligned} Q^{\tilde{A}}(\tilde{u}) \;:=\; \{ \hat{u} \in Q \mid & \text{ There exists an } \tilde{A}-\text{horizontal curve} \\ & \text{ connecting } \tilde{u} \text{ with } \hat{u} \} \end{aligned}$$

be the holonomy bundle of $(Q, \tilde{A})$ with respect to $\tilde{u} \in Q$. According to the reduction theorem, $(Q, \tilde{A})$ reduces to the $\tilde{H}_{\tilde{u}}$-principal bundle $Q^{\tilde{A}}(\tilde{u})$. Hence, the spinor bundle S of $M^{r,s}$ is given by

$$S = Q \times_{\mathrm{Spin}(r,s),\rho} \Delta_{r,s} = Q^{\tilde{A}}(\tilde{u}) \times_{\tilde{H}_{\tilde{u}},\rho} \Delta_{r,s}.$$

Then, we have a bijection between the space $\mathfrak{P}$ of all parallel spinor fields of the connected manifold $M^{r,s}$ and the space $V_{\tilde{H}_{\tilde{u}}}$ of all fixed spinors of $\Delta_{r,s}$ with respect to the holonomy group $\tilde{H}_{\tilde{u}}$ which is given by

$$\begin{array}{rcl} V_{\tilde{H}_{\tilde{u}}} := \{ v \in \Delta_{r,s} \mid \rho(\tilde{H}_{\tilde{u}}) v = v \} & \mapsto & \mathfrak{P} \\ v & \mapsto & \varphi_v \in \Gamma(S) \\ & & \varphi_v(y) = [\tau_\omega^{\tilde{A}}(\tilde{u}), v], \text{ where } \omega \\ & & \text{is a curve in } M \text{ from } x \text{ to } y. \end{array}$$

Now, let us suppose that $M^{r,s}$ is simply connected. Then the holonomy groups $\tilde{H}_{\tilde{u}}$ coincide with the restricted holonomy groups, in particular, they are connected. Therefore, the vector space $V_{\tilde{H}_{\tilde{u}}}$ is given by

$$V_{\tilde{H}_{\tilde{u}}} = \{ v \in \Delta_{r,s} \mid \rho_*(\tilde{\mathfrak{h}}_{\tilde{u}}) v = 0 \},$$

where $\tilde{\mathfrak{h}}_{\tilde{u}}$ is the Lie algebra of $\tilde{H}_{\tilde{u}}$. If $H_u \subset \mathrm{SO}(r,s)$ is the holonomy group of the Levi-Civita connection A on the bundle of positively oriented frames of $M^{r,s}$ with respect to a frame u in $x \in M$ and if $\tilde{u} \in Q_x$ denotes a lift of u into the spin structure Q, then $\lambda(\tilde{H}_{\tilde{u}}) = H_u$, where $\lambda : \mathrm{Spin}(r,s) \to \mathrm{SO}(r,s)$ is the double covering of $\mathrm{SO}(r,s)$ by $\mathrm{Spin}(r,s)$. Hence,

$$\tilde{\mathfrak{h}}_{\tilde{u}} = \lambda_*^{-1}(\mathfrak{h}_u),$$

where $\mathfrak{h}_u$ is the Lie algebra of H_u.

In Berger's list (Theorem 2) the holonomy groups are described up to conjugacy in the full orthogonal group $\mathrm{O}(r,s)$. Since we consider holonomy groups of oriented manifolds each of the conjugacy classes of the groups H of Theorem 2 gives rise to two different conjugacy classes in $\mathrm{SO}(r,s)$, generated by H and by $H' = T_1 \cdot H \cdot T_1^{-1}$, where $T_1 = \mathrm{diag}(-1,1,\ldots,1)$. It depends on the chosen orientation of M which conjugacy class appears. For the Lie algebras $\mathfrak{h}'$ and $\mathfrak{h}$ of H' and H we have $\mathfrak{h}' = \mathrm{Ad}(T)\mathfrak{h}$. For the Lie algebras $\tilde{\mathfrak{h}}'$ and $\tilde{\mathfrak{h}}$ of the corresponding holonomy groups $\tilde{H}'$ and $\tilde{H}$ of the spin structure $(Q, \tilde{A})$ it follows

$$\tilde{\mathfrak{h}}' = \mathrm{Ad}(\tilde{T})\tilde{\mathfrak{h}},$$

where $\lambda(\tilde{T}) = T$. Since $\tilde{T} = \pm e_1 \in \mathfrak{pin}(r,s) \subset \mathcal{C}_{r,s}$ and $\mathrm{Ad}(\tilde{T})X = \tilde{T}\cdot X \cdot \tilde{T}^{-1} \in \mathcal{C}_{r,s}$ for all $X \in \mathcal{C}_{r,s}$ it results $\tilde{\mathfrak{h}}' = -\kappa_1 e_1 \cdot \tilde{\mathfrak{h}} \cdot e_1 \subset \mathcal{C}_{r,s}$. Therefore we have

$$V_{\tilde{H}'} = e_1 \cdot V_{\tilde{H}}\,.$$

Hence, it is sufficient to calculate $V_{\tilde{H}}$. By Clifford multiplication of $v \in V_{\tilde{H}}$ with e_1 the chiral type of v is changed.

If we want to study the causal type of parallel spinors we have to restrict ourselves to space- and time-oriented pseudo-Riemannian manifolds in order to have an indefinite scalar product on the spinor bundle S. In that case to each conjugacy class of the group H of Theorem 2 correspond four different conjugacy classes in $\mathrm{SO}_0(r,s)$ generated by

$$H,\ H' = T_1 H T_1^{-1},\ H'' = T_2 H T_2^{-1},\ H''' = T_3 H T_3^{-1},$$

where $T_1 = \mathrm{diag}(-1,1,\ldots,1)$, $T_2 = \mathrm{diag}(1,\ldots,1,-1)$,
$T_3 = \mathrm{diag}(-1,1,\ldots,1,-1)$. With analogous notations as above we obtain

$$\begin{aligned} V_{\tilde{H}'} &= e_1 \cdot V_{\tilde{H}} \\ V_{\tilde{H}''} &= e_n \cdot V_{\tilde{H}} \\ V_{\tilde{H}'''} &= e_1 \cdot e_n \cdot V_{\tilde{H}}. \end{aligned}$$

(Here e_1 is timelike and e_n is spacelike.) In the first two cases the chiral character of $v \in V_{\tilde{H}}$ is changed, in the third case it remains the same. Since

$$\begin{aligned} \langle e_j \cdot v\,,\, e_j \cdot v\rangle &= (-1)^{r+1}\langle v\,,\, e_j \cdot e_j \cdot v\rangle \\ &= (-1)^r \kappa_j \langle v\,,\, v\rangle \qquad j = 1,\ldots,n,\ v \in \Delta_{r,s}, \end{aligned}$$

in case of even (odd) index r the causal type of $v \in V_{\tilde{H}}$ is changed (remains the same) by Clifford multiplication with e_1 or $e_1 \cdot e_n$ and remains the same (is changed) by Clifford multiplication with e_n.

Next we describe the groups which occur in Theorem 2 in more detail. We identify

$$\begin{array}{ccc} \mathbb{H}^n & \cong & \mathbb{C}^{2n} \\ \begin{pmatrix} a_1 \\ \\ \vdots \\ \\ a_n \end{pmatrix} & \mapsto & \begin{pmatrix} z_1 \\ \bar{w}_1 \\ \vdots \\ z_n \\ \bar{w}_n \end{pmatrix}, \end{array} \qquad \text{where } a_\nu = z_\nu + w_\nu \cdot j \qquad (\nu = 1, \ldots, n)$$

and

$$\begin{array}{ccc} \mathbb{C}^n & \cong & \mathbb{R}^{2n} \\ \begin{pmatrix} z_1 \\ \\ \vdots \\ \\ z_n \end{pmatrix} & \mapsto & \begin{pmatrix} x_1 \\ y_1 \\ \vdots \\ x_n \\ y_n \end{pmatrix}, \end{array} \qquad \text{where } z_\nu = x_\nu + i y_\nu \qquad (\nu = 1, \ldots, n).$$

We denote by $J_{\mathbb{R}}$ the matrix

$$J_{\mathbb{R}} = \begin{pmatrix} J & 0 & \cdots & & 0 \\ 0 & J & & & \\ \vdots & & \ddots & & \vdots \\ & & & & 0 \\ 0 & & \cdots & 0 & J \end{pmatrix} \in \mathrm{GL}(2n, \mathbb{R}),$$

where $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and by $J_{\mathbb{C}}$ the same matrix in $\mathrm{GL}(2n, \mathbb{C})$. Let $\mathbb{K}(n)$ be the space of $(n \times n)$-matrices with entries in the (skew-) field $\mathbb{K}$. We consider the embeddings $i_{\mathbb{C}}$ and $i_{\mathbb{R}}$ associated to the identifications $\mathbb{H}^n \cong \mathbb{C}^{2n}$ and $\mathbb{C}^n \cong \mathbb{R}^{2n}$:

$$i_{\mathbb{C}} \; : \; \mathbb{H}(n) \hookrightarrow \mathbb{C}(2n)$$

$$i_{\mathbb{C}}\left((a_{\mu\nu})_{\mu,\nu=1,\ldots,n}\right) = \left(\begin{pmatrix} z_{\mu\nu} & -w_{\mu\nu} \\ \bar{w}_{\mu\nu} & \bar{z}_{\mu\nu} \end{pmatrix} \right)_{\mu,\nu=1,\ldots,n},$$

where $a_{\mu\nu} = z_{\mu\nu} + w_{\mu\nu} \cdot j$,

$$i_{\mathbb{R}} \; : \; \mathbb{C}(n) \hookrightarrow \mathbb{R}(2n)$$

$$i_{\mathbb{R}}\left((z_{\mu\nu})_{\mu,\nu=1,\dots,n}\right) = \left(\begin{pmatrix} x_{\mu\nu} & -y_{\mu\nu} \\ y_{\mu\nu} & x_{\mu\nu} \end{pmatrix}\right)_{\mu,\nu=1,\dots,n},$$

where $z_{\mu\nu} = x_{\mu\nu} + iy_{\mu\nu}$.

Then we have

$$\begin{aligned} i_{\mathbb{C}}(\mathbb{H}(n)) &= \{A \in \mathbb{C}(2n) \mid \bar{A}J_{\mathbb{C}} = J_{\mathbb{C}}A\} \\ i_{\mathbb{R}}(\mathbb{C}(n)) &= \{A \in \mathbb{R}(2n) \mid AJ_{\mathbb{R}} = J_{\mathbb{R}}A\}. \end{aligned}$$

Furthermore, let $I^{\mathbb{K}}_{p,q}$ be the matrix

$$I^{\mathbb{K}}_{p,q} = \begin{pmatrix} -E_p & 0 \\ 0 & E_q \end{pmatrix} \in \mathrm{GL}(p+q, \mathbb{K}),$$

where E_r denotes the unity matrix in $\mathbb{K}(r)$.

The special pseudo-orthogonal group

$$\mathrm{SO}(p,q) = \{A \in \mathrm{SL}(p+q,\mathbb{R}) \mid A^T I^{\mathbb{R}}_{p,q} A = I^{\mathbb{R}}_{p,q}\}$$

(in its standard form) is the invariance group of the bilinear form

$$\langle x\,,\, y\rangle = -x_1y_1 - \cdots - x_py_p + x_{p+1}y_{p+1} + \cdots + x_{p+q}y_{p+q}$$

on $\mathbb{R}^{p+q}$.

We identify the pseudo-unitary group

$$U(p,q) = \{A \in \mathbb{C}(p+q) \mid \bar{A}^T I^{\mathbb{C}}_{p,q} A = I^{\mathbb{C}}_{p,q}\}$$

with the subgroup $i_{\mathbb{R}}(U(p,q)) = \mathrm{SO}(2p,2q) \cap i_{\mathbb{R}}\mathbb{C}(p+q)$ of $\mathrm{SO}(2p,2q)$ and the symplectic group

$$\mathrm{Sp}(p,q) = \{A \in \mathbb{H}(p+q) \mid \bar{A}^T I^{\mathbb{H}}_{p,q} A = I^{\mathbb{H}}_{p,q}\}$$

with the subgroup $i_{\mathbb{C}}\mathrm{Sp}(p,q) = U(2p,2q) \cap i_{\mathbb{C}}(\mathbb{H}(p+q))$ of $U(2p,2q)$.

Now we are going to describe the subgroup $\mathrm{Sp}(p,q)\cdot\mathrm{Sp}(1)$ of $\mathrm{SO}(4p,4q)$. Each quaternion $a \in \mathrm{Sp}(1)$ defines an orthogonal map $R_a : \mathbb{R}^{4p+4q} \to \mathbb{R}^{4p+4q}$ by right multiplication with a on $\mathbb{H}^{p+q} = \mathbb{R}^{4p+4q}$. In particular, if $a = x_0 + iy_0 + (x_1 + iy_1)j$ and

$$r_a := \begin{pmatrix} x_0 & -y_0 & -x_1 & y_1 \\ y_0 & x_0 & y_1 & x_1 \\ x_1 & -y_1 & x_0 & -y_0 \\ -y_1 & -x_1 & y_0 & x_0 \end{pmatrix}$$

then we have

$$R_a = \begin{pmatrix} r_a & 0 & \cdots & & 0 \\ 0 & r_a & & & \\ \vdots & & \ddots & & \vdots \\ & & & & 0 \\ 0 & & \cdots & 0 & r_a \end{pmatrix} \in \mathrm{SO}(4p,4q).$$

The group $\mathrm{Sp}(p,q)\cdot\mathrm{Sp}(1)$ equals

$$Sp(p,q)\cdot\mathrm{Sp}(1) = \{A\cdot R_a \mid A\in i_{\mathbb{R}}i_{\mathbb{C}}\mathrm{Sp}(p,q),\ a\in\mathrm{Sp}(1)\}.$$

Let $\omega_{\mathbb{K}}$ be a non-degenerate skew-symmetric bilinear form on $\mathbb{K}^{2p}$, $\mathbb{K}=\mathbb{R}$, $\mathbb{C}$. By $\mathrm{Sp}(p,\mathbb{K})$ we denote the group of all automorphisms of $(\mathbb{K}^{2p},\omega_{\mathbb{K}})$. The bilinear form g on $\mathbb{R}^{4p}=\mathbb{R}^{2p}\otimes\mathbb{R}^2$ defined by

$$g(x\otimes a, y\otimes b) := \det(a,b)\cdot\omega_{\mathbb{R}}(x,y)$$

is a metric of signature $(2p,2p)$ on $\mathbb{R}^{4p}$. Hence, $\mathrm{Sp}(p,\mathbb{R})\cdot\mathrm{SL}(2,\mathbb{R})$ can be considered as a subgroup of all automorphisms of $(\mathbb{R}^{4p},g)$, where $A\cdot B\in\mathrm{Sp}(p,\mathbb{R})\cdot\mathrm{SL}(2,\mathbb{R})$ acts on $\mathbb{R}^{4p}=\mathbb{R}^{2p}\otimes\mathbb{R}^2$ by

$$A\cdot B(x\otimes a) = Ax\otimes Ba.$$

In the same way, the bilinear form h on $\mathbb{R}^{8p}=\mathbb{C}^{4p}=\mathbb{C}^{2p}\otimes\mathbb{C}^2$ defined by

$$h(z\otimes a, w\otimes b) = \mathrm{Re}(\det(a,b)\cdot\omega_{\mathbb{C}}(z,w))$$

gives an embedding of $\mathrm{Sp}(p,\mathbb{C})\cdot\mathrm{SL}(2,\mathbb{C})$ into $\mathrm{SO}(4p,4p)$.

The remaining classical group of the list in Theorem 2 is

$$\mathrm{SO}(n,\mathbb{C}) = \mathrm{SL}(n,\mathbb{C})\cap\{A\in\mathbb{C}(n)\mid A^TA=E_n\}.$$

Let us come to the exceptional cases. G_2, $\mathrm{G}^*_{2(2)}$, $\mathrm{Spin}(7)$, $\mathrm{Spin}(4,3)$ are subgroups of $\mathrm{SO}(7)$, $\mathrm{SO}(4,3)$, $\mathrm{SO}(8)$ and $\mathrm{SO}(4,4)$, respectively. They are defined as invariance groups of so-called *nice* 3-*forms* and 4-*forms*, respectively. Let w_0, $w_1\in\Lambda^3(\mathbb{R}^7)$, σ_0, $\sigma_1\in\Lambda^4(\mathbb{R}^8)$ be the forms

$$\begin{aligned}
w_0 &= w^{127}+w^{135}-w^{146}-w^{236}-w^{245}+w^{347}+w^{567}\\
w_1 &= -w^{127}-w^{135}+w^{146}+w^{236}+w^{245}-w^{347}+w^{567}\\
\sigma_0 &= \sigma^{1234}+\sigma^{1256}-\sigma^{1278}+\sigma^{1357}+\sigma^{1368}+\sigma^{1458}-\sigma^{1467}\\
&\quad -\sigma^{2358}+\sigma^{2367}+\sigma^{2457}+\sigma^{2468}-\sigma^{3456}+\sigma^{3478}+\sigma^{5678}\\
\sigma_1 &= \sigma^{1234}-\sigma^{1256}+\sigma^{1278}-\sigma^{1357}-\sigma^{1368}-\sigma^{1458}+\sigma^{1467}\\
&\quad +\sigma^{2358}-\sigma^{2367}-\sigma^{2457}-\sigma^{2468}+\sigma^{3456}-\sigma^{3478}+\sigma^{5678},
\end{aligned}$$

where $w^{\alpha\beta\gamma} = w^\alpha \wedge w^\beta \wedge w^\gamma$ and $\sigma^{\alpha\beta\gamma\delta} = \sigma^\alpha \wedge \sigma^\beta \wedge \sigma^\gamma \wedge \sigma^\delta$ with respect to the dual bases $w^1, \ldots, w^7$ of $e_1, \ldots, e_7 \in \mathbb{R}^7$ and $\sigma^1, \ldots, \sigma^8$ of $e_1, \ldots, e_8 \in \mathbb{R}^8$.

Then

$$\begin{aligned}
&\mathrm{G}_2 = \{A \in \mathrm{SO}(7) \mid A^* w_0 = w_0\}\\
&\mathrm{G}^*_{2(2)} = \{A \in \mathrm{SO}(4,3) \mid A^* w_1 = w_1\}\\
&\mathrm{Spin}(7) = \{A \in \mathrm{SO}(8) \mid A^* \sigma_0 = \sigma_0\}\\
&\mathrm{Spin}(4,3) = \{A \in \mathrm{SO}(4,4) \mid A^* \sigma_1 = \sigma_1\}
\end{aligned}$$

(compare [Br 87]).

Finally, the groups $\mathrm{G}_2^{\mathbb{C}} \subset \mathrm{SO}(7, \mathbb{C})$ and $\mathrm{Spin}(7, \mathbb{C}) \subset \mathrm{SO}(8, \mathbb{C})$ are the complexifications of $\mathrm{G}_2 \subset \mathrm{SO}(7)$ and $\mathrm{Spin}(7) \subset \mathrm{SO}(8)$, respectively.

4 The fixed spinors of the holonomy representation

Let $M^{r,s}$ be a simply connected semi-Riemannian spin manifold of index r and dimension $r+s = n$ with holonomy representation H and let $\mathfrak{h}$ be the Lie algebra of H. Then parallel spinors are given by the kernel of the representation of $\tilde{\mathfrak{h}} := \lambda_*^{-1}(\mathfrak{h})$ on the spinor module $\Delta_{r,s}$. Hence, we have to check the groups in the Berger-Simons list and to determine the dimension of the kernel

$$V_H = \{v \in \Delta_{r,s} \mid \tilde{X} \cdot v = 0 \text{ for any } \tilde{X} \in \tilde{\mathfrak{h}}\}.$$

4.1 $H = \mathrm{SO}_0(p,q)$ $(p = r,\ q = s)$

The Lie algebra $\tilde{\mathfrak{h}} = \mathfrak{so}(p,q)$ of $\mathrm{SO}(p,q)$ is spanned by the matrices E_{kl} $(1 \leq k < l \leq p+q)$ defined in (3). Hence, $\tilde{\mathfrak{h}} = \mathfrak{spin}(p,q) = \mathrm{span}\{e_k\, e_l \mid 1 \leq k < l \leq p+q\}$. From (8) we see that $V_{\mathrm{SO}_0(p,q)} = \{0\}$.

Consequently, there is no parallel spinor on $M^{p,q}$.

4.2 $\mathrm{U}(p,q)$ $(r = 2p,\ s = 2q)$

$i_{\mathbb{R}^*}\mathfrak{u}(p,q) \subset \mathfrak{so}(2p, 2q)$ is spanned by

$$\begin{aligned}
X_{kl} &:= E_{2k-1\,2l-1} + E_{2k\,2l} && (1 \leq k < l \leq p+q)\\
Y_{kl} &:= E_{2k-1\,2l} - E_{2k\,2l-1} && (1 \leq k < l \leq p+q)\\
D_k &:= E_{2k-1\,2k} && (1 \leq k \leq p+q).
\end{aligned}$$

Hence,

$$\begin{aligned}
\tilde{X}_{kl} &:= e_{2k-1}e_{2l-1} + e_{2k}e_{2l} && (1 \leq k < l \leq p+q)\\
\tilde{Y}_{kl} &:= e_{2k-1}e_{2l} - e_{2k}e_{2l-1} && (1 \leq k < l \leq p+q)\\
\tilde{D}_k &:= e_{2k-1}e_{2k} && (1 \leq k \leq p+q).
\end{aligned}$$

is a basis of $\tilde{\mathfrak{h}}$. (8) shows that $V_{\mathrm{U}(p,q)} = \{0\}$.

4.3 $\mathrm{SU}(p,q)$ $(r = 2p,\ s = 2q)$

$i_{\mathbb{R}^*}\mathfrak{su}(p,q) \subset \mathfrak{so}(2p,2q)$ is spanned by X_{kl}, Y_{kl} $(1 \le k < l \le n)$ and $\kappa_2 D_1 - \kappa_{2k} D_k$ $(2 \le k \le p+q)$. Consequently, $\tilde{X}_{kl}$, $\tilde{Y}_{kl}$ $(1 \le k < l \le p+q)$, $\kappa_2 \tilde{D}_1 - \kappa_{2k}\tilde{D}_k$ $(2 \le k \le p+q)$ is a basis of $\tilde{\mathfrak{h}}$.

It acts on $\Delta_{2p,2q}$ in the following way.

$$
\begin{aligned}
\tilde{X}_{kl} u(\varepsilon_{p+q}, \ldots, \varepsilon_1) &= (-1)^{l-k}\tau_{2k}\tau_{2l}(\varepsilon_l - \varepsilon_k)\varepsilon_{l-1}\cdots\varepsilon_{k+1} \\
&\quad \cdot u(\varepsilon_{p+q}, \ldots, -\varepsilon_l, \ldots, -\varepsilon_k, \ldots, \varepsilon_1) \qquad (13)\\
\tilde{Y}_{kl} u(\varepsilon_{p+q}, \ldots, \varepsilon_1) &= (-1)^{l-k} i\tau_{2k}\tau_{2l}(\varepsilon_l\varepsilon_k - 1)\varepsilon_{l-1}\cdots\varepsilon_{k+1} \\
&\quad \cdot u(\varepsilon_{p+q}, \ldots, -\varepsilon_l, \ldots, -\varepsilon_k, \ldots, \varepsilon_1) \qquad (14)
\end{aligned}
$$

$$
(\kappa_2\tilde{D}_1 - \kappa_{2k}\tilde{D}_k)u(\varepsilon_{p+q}, \ldots, \varepsilon_1) = i(\varepsilon_1 - \varepsilon_k)u(\varepsilon_{p+q}, \ldots, \varepsilon_1). \qquad (15)
$$

(15) and (8) imply $V_{\mathrm{SU}(p,q)} \subset \mathrm{span}\{u(1,\ldots,1),\, u(-1,\ldots,-1)\}$.

On the other hand (13) and (14) show

$$
\tilde{X}_{kl} u(\varepsilon, \varepsilon, \ldots, \varepsilon) = \tilde{Y}_{kl} u(\varepsilon, \varepsilon, \ldots, \varepsilon) = 0 \quad (1 \le k < l \le p+q)
$$

for $\varepsilon = \pm 1$. Hence,

$$
V_{\mathrm{SU}(p,q)} = \mathrm{span}\{u(1,1,\ldots,1)\,,\, u(-1,-1,\ldots,-1)\}.
$$

If $p + q = \frac{1}{2}\dim M$ is even then $V_{\mathrm{SU}(p,q)} \subset \Delta^+_{2p,2q}$. If $p+q = \frac{1}{2}\dim M$ is odd, we have an 1-dimensional space of parallel spinors in S^+ as well as in S^-. According to (11) the quadratic length of $u(\varepsilon, \ldots, \varepsilon)$ is

$$
\langle u(\varepsilon, \ldots, \varepsilon)\,,\, u(\varepsilon, \ldots, \varepsilon)\rangle = \varepsilon^p.
$$

Hence $u(1,\ldots,1)$ and $u(-1,\ldots,-1)$ are of the same causal type (spacelike) if p is even and of opposite causal type (timelike) if p is odd.

4.4 $\mathrm{Sp}(p,q)$ $(r = 4p,\ s = 4q)$

$i_{\mathbb{R}_*} i_{\mathbb{C}_*}\mathfrak{sp}(p,q) \subset \mathfrak{so}(4p,4q)$ is spanned by

$$
\begin{aligned}
& X_{2k-1\,2l-1} + X_{2k\,2l},\ Y_{2k-1\,2l-1} - Y_{2k\,2l}, \\
& X_{2k-1\,2l} - X_{2k\,2l-1},\ Y_{2k-1\,2l} + Y_{2k\,2l-1}, \quad (1 \le k < l \le p+q) \\
& X_{2k-1\,2k},\ Y_{2k-1\,2k} \ \ (1 \le k \le p+q) \\
& D_{2k-1} - D_{2k} \ \ (1 \le k \le p+q).
\end{aligned}
$$

$\tilde{\mathfrak{h}}$ is spanned by the corresponding combinations of $\tilde{X}_{kl}$, $\tilde{Y}_{kl}$, and $\tilde{D}_k$. From

$$(D_{2k-1} - D_{2k})u(\varepsilon_{2(p+q)}, \ldots, \varepsilon_1) = \kappa_{2k} i(\varepsilon_{2k-1} - \varepsilon_{2k})u(\varepsilon_{2(p+q)}, \ldots, \varepsilon_1)$$

we see that

$$V_{\mathrm{Sp}(p,q)} \subset \mathrm{span}\{u(\varepsilon_{p+q}, \varepsilon_{p+q}, \ldots, \varepsilon_2, \varepsilon_2, \varepsilon_1, \varepsilon_1) \mid \varepsilon_\nu = \pm 1\}.$$

Furthermore, (13) and (14) imply

$$\begin{aligned} \tilde{X}_{2k-1\,2k} u(\varepsilon_{p+q}, \varepsilon_{p+q}, \ldots, \varepsilon_1, \varepsilon_1) &= 0, \\ \tilde{Y}_{2k-1\,2k} u(\varepsilon_{p+q}, \varepsilon_{p+q}, \ldots, \varepsilon_1, \varepsilon_1) &= 0. \end{aligned}$$

Using

$$\begin{aligned} &(\tilde{X}_{2k-1\,2l-1} + \tilde{X}_{2k\,2l})u(\varepsilon_{p+q}, \varepsilon_{p+q}, \ldots, \varepsilon_l, \varepsilon_l, \ldots, \varepsilon_k, \varepsilon_k, \ldots, \varepsilon_1, \varepsilon_1) \\ &\quad = \tau_{4k}\tau_{4l}(\varepsilon_l - \varepsilon_k)\{\varepsilon_k u(\varepsilon_{p+q}, \varepsilon_{p+q}, \ldots, \varepsilon_l, -\varepsilon_l, \ldots, \varepsilon_k, -\varepsilon_k, \ldots, \varepsilon_1, \varepsilon_1) \\ &\qquad + \varepsilon_l u(\varepsilon_{p+q}, \varepsilon_{p+q}, \ldots, -\varepsilon_l, \varepsilon_l, \ldots, -\varepsilon_k, \varepsilon_k, \ldots, \varepsilon_1, \varepsilon_1)\} \end{aligned}$$

and

$$\begin{aligned} &(\tilde{X}_{2k-1\,2l} - \tilde{X}_{2k\,2l-1})u(\varepsilon_{p+q}, \varepsilon_{p+q}, \ldots, \varepsilon_l, \varepsilon_l, \ldots, \varepsilon_k, \varepsilon_k, \ldots, \varepsilon_1, \varepsilon_1) \\ &\quad = -\tau_{4k}\tau_{4l}(\varepsilon_l - \varepsilon_k)\{\varepsilon_k \varepsilon_l u(\varepsilon_{p+q}, \varepsilon_{p+q}, \ldots, -\varepsilon_l, \varepsilon_l, \ldots, \varepsilon_k, -\varepsilon_k, \ldots, \varepsilon_1, \varepsilon_1) \\ &\qquad - u(\varepsilon_{p+q}, \varepsilon_{p+q}, \ldots, \varepsilon_l, -\varepsilon_l, \ldots, -\varepsilon_k, \varepsilon_k, \ldots, \varepsilon_1, \varepsilon_1)\} \end{aligned}$$

we obtain for $v = \sum_{\varepsilon = (\varepsilon_{p+q}, \ldots, \varepsilon_1)} \lambda_\varepsilon u(\varepsilon_{p+q}, \varepsilon_{p+q}, \ldots, \varepsilon_2, \varepsilon_2, \varepsilon_1, \varepsilon_1) \in V_{\mathrm{Sp}(p,q)}$ the condition

$$\lambda_\varepsilon = \lambda_{(\varepsilon_{p+q}, \ldots, -\varepsilon_l, \ldots, -\varepsilon_k, \ldots, \varepsilon_1)} \text{ in case } \varepsilon_l \neq \varepsilon_k.$$

Consequently $V_{\mathrm{Sp}(p,q)}$ is contained in $\mathrm{span}\{\varphi_0, \varphi_1, \ldots, \varphi_{p+q}\}$, where

$$\varphi_k = \sum_{\substack{\varepsilon_1, \ldots, \varepsilon_{p+q} \\ \varepsilon_i = -1 \text{ exactly } k \text{ times}}} u(\varepsilon_{p+q}, \varepsilon_{p+q}, \ldots, \varepsilon_1, \varepsilon_1).$$

Furthermore, the remaining generators of $\tilde{\mathfrak{h}}$ annihilate $\varphi_0, \varphi_1, \ldots, \varphi_{p+q}$. Consequently, $\varphi_0, \varphi_1, \ldots, \varphi_{p+q}$ is a basis of $V_{\mathrm{Sp}(p,q)}$. Obviously $V_{\mathrm{Sp}(p,q)} \subset \Delta^+$. Furthermore all φ_k ($k = 0, \ldots, p+q$) have the same causal type since $\langle \varphi_k, \varphi_k \rangle = \binom{p+q}{k} > 0$.

4.5 $\mathrm{Sp}(p,q)\cdot\mathrm{Sp}(1)$ $(r=4p,\ s=4q)$

The Lie algebra of $\mathrm{Sp}(p,q)\cdot\mathrm{Sp}(1)$ equals

$$\begin{aligned} i_{\mathbb{R}^*}i_{\mathbb{C}^*}\,\mathfrak{sp}(p,q)\oplus\operatorname{span}\Big\{X_1 &= \sum_{\nu=1}^{p+q}\kappa_{4\nu}(E_{4\nu-3\,4\nu-2}+E_{4\nu-1\,4\nu}),\\ X_2 &= \sum_{\nu=1}^{p+q}\kappa_{4\nu}(E_{4\nu-3\,4\nu-1}-E_{4\nu-2\,4\nu}),\\ X_3 &= \sum_{\nu=1}^{p+q}\kappa_{4\nu}(E_{4\nu-3\,4\nu}+E_{4\nu-2\,4\nu-1})\Big\}. \end{aligned}$$

Denote by $\tilde{X}_1$, $\tilde{X}_2$, $\tilde{X}_3$ the double of the lifts of X_1, X_2, X_3 into $\mathfrak{spin}(4p,4q)$. From

$$\tilde{X}_1 u(\varepsilon_{p+q},\varepsilon_{p+q},\ldots,\varepsilon_2,\varepsilon_2,\varepsilon_1,\varepsilon_1)=2i\left(\sum_{k=1}^{p+q}\varepsilon_k\right)u(\varepsilon_{p+q},\varepsilon_{p+q},\ldots,\varepsilon_1,\varepsilon_1)$$

we obtain $\tilde{X}_1\varphi_\nu=2i(p+q-2\nu)\varphi_\nu$.

Hence, $V=V_{\mathrm{Sp}(p,q)\cdot\mathrm{Sp}(1)}=\{0\}$ if $p+q$ is odd and $V\subset\mathbb{C}\cdot\varphi_{\frac{p+q}{2}}$ if $p+q$ is even. On the other hand $\tilde{X}_3\cdot\varphi_{\frac{p+q}{2}}$ does not equal 0. Finally,

$$V_{\mathrm{Sp}(p,q)\cdot\mathrm{Sp}(1)}=\{0\}.$$

4.6 $H=\mathrm{G}_2,\ G^*_{2(2)}$

We define $\kappa=1$ in case $H=\mathrm{G}_2$ and $\kappa=-1$ in case $\mathrm{G}^*_{2(2)}$. Then $\mathfrak{g}_2\subset\mathfrak{so}(7)$ and $\mathfrak{g}^*_{2(2)}\subset\mathfrak{so}(4,3)$ are spanned by

$$\begin{aligned} &E_{12}-E_{34},\ \kappa E_{12}-E_{56},\ E_{13}+E_{24},\ \kappa E_{13}-E_{67},\ E_{14}-E_{23},\\ &\kappa E_{14}-E_{57},\ E_{15}+E_{26},\ E_{15}+E_{47},\ E_{16}-E_{25},\ E_{16}+E_{37},\\ &E_{17}-E_{36},\ E_{17}-E_{45},\ E_{27}-E_{35},\ E_{27}+E_{46}, \end{aligned}$$

where $E_{ij}\in\mathfrak{so}(7)$ in case of G_2 and $E_{ij}\in\mathfrak{so}(4,3)$ in case of $\mathrm{G}^*_{2(2)}$ are defined as in (3).

First we see that for $v\in\Delta_7$ $(\Delta_{4,3})$

$$(e_1e_2-e_3e_4)v=(\kappa e_1e_2-e_5e_6)v=0$$

holds if and only if $v\in\operatorname{span}\{u(1,1,1),\ u(-1,-1,-1)\}$. The equation

$$(\kappa e_1e_3-e_6e_7)(\alpha u(1,1,1)+\beta u(-1,-1,-1))=0$$

implies $\beta = i\alpha$. Furthermore, all remaining generators of $\mathfrak{g}_2$ ($\mathfrak{g}^*_{2(2)}$) annihilate $u(1,1,1) + iu(-1,-1,-1)$. Hence,

$$\begin{aligned} V_{\mathrm{G}_2} &= \mathrm{span}\{u(1,1,1) + iu(-1,-1,-1)\} \subset \Delta_7 \\ V_{\mathrm{G}^*_{2(2)}} &= \mathrm{span}\{u(1,1,1) + iu(-1,-1,-1)\} \subset \Delta_{4,3}. \end{aligned}$$

From $\langle u(1,1,1)+iu(-1,-1,-1)\,,\, u(1,1,1)+iu(-1,-1,-1)\rangle = 2$ we see that all non-zero spinors in $V_{\mathrm{G}^*_{2(2)}}$ are spacelike.

4.7 $H = \mathrm{Spin}(7)$, $\mathrm{Spin}(4,3)$

As above we use $\kappa = 1$ in the Riemannian case and $\kappa = -1$ in the pseudo-Riemannian one. Then $\mathfrak{spin}(7) \subset \mathfrak{so}(8)$ and $\mathfrak{spin}(4,3) \subset \mathfrak{so}(4,4)$ are spanned by the following (linearly dependent) generators:

$$\begin{aligned} &E_{12} + E_{34},\ E_{13} - E_{24},\ E_{14} + E_{23},\ E_{56} + E_{78},\ -E_{57} + E_{68},\ E_{58} + E_{67}, \\ &-E_{15} + E_{26},\ \kappa E_{12} + E_{56},\ E_{16} + E_{25},\ E_{37} - E_{48},\ \kappa E_{34} + E_{78},\ E_{38} + E_{47}, \\ &\kappa E_{12} - E_{78},\ E_{17} + E_{28},\ E_{18} - E_{27},\ \kappa E_{34} - E_{56},\ E_{35} + E_{46},\ E_{36} - E_{45}, \\ &E_{18} + E_{36},\ E_{17} + E_{35},\ E_{26} - E_{48},\ E_{25} + E_{38},\ E_{23} + E_{67},\ E_{24} + E_{57}. \end{aligned}$$

By (8) we obtain from

$$\begin{aligned} (e_1e_2 + e_3e_4)v &= (e_5e_6 + e_7e_8)v = (\kappa e_1e_2 + e_5e_6)v \\ &= (\kappa e_3e_4 + e_7e_8)v = 0 \quad (v \in \Delta_8\ (\Delta_{4,4})) \end{aligned}$$

the condition $v \in \mathrm{span}\{u(1,-1,-1,1),\, u(-1,1,1,-1)\}$. On the other hand

$$(e_2e_7 + e_3e_6)(\alpha u(1,-1,-1,1) + \beta u(-1,1,1,-1)) = 0$$

implies $\alpha = -\beta$. Furthermore, all remaining generators annihilate $u(1,-1,-1,1) - u(-1,1,1,-1)$. Consequently,

$$\begin{aligned} V_{\mathrm{Spin}(7)} &= \mathrm{span}\{u(1,-1,-1,1) - u(-1,1,1,-1)\} \subset \Delta_8 \\ V_{\mathrm{Spin}(4,3)} &= \mathrm{span}\{u(1,-1,-1,1) - u(-1,1,1,-1)\} \subset \Delta_{4,4}. \end{aligned}$$

We have $u(1,-1,-1,1) - u(-1,1,1,-1) \in \Delta^+_8$ ($\Delta^+_{4,4}$). Furthermore, in case of $\Delta^+_{4,3}$ this spinor is timelike.

4.8 $H = \mathrm{SO}(p,\mathbb{C})$ $\quad (r = p,\ s = p)$

Now let (in difference to the previous sections) $\mathrm{SO}(p,p) \subset \mathrm{GL}(2p,\mathbb{R})$ be the invariance group of the inner product $\langle x\,,\,y\rangle_{p,p} = -x_1y_1 + x_2y_2 - x_3y_3 + \cdots + x_{2p}y_{2p}$. According to that we have

$$\kappa_k = \begin{cases} 1 & k \text{ even} \\ -1 & k \text{ odd} \end{cases} \qquad \tau_k = \begin{cases} 1 & k \text{ even} \\ i & k \text{ odd.} \end{cases}$$

Then $i_{\mathbb{R}}(\mathrm{SO}(p,\mathbb{C}))$ is contained in $\mathrm{SO}(p,p)$ and $i_{\mathbb{R}^*}\mathfrak{so}(p,\mathbb{C}) \subset \mathfrak{so}(p,p)$ is spanned by

$$\begin{aligned} \xi_{kl} &= -E_{2k-1\,2l-1} + E_{2k\,2l} && (1 \le k < l \le p) \\ \eta_{kl} &= -E_{2k-1\,2l} - E_{2k\,2l-1} && (1 \le k < l \le p). \end{aligned}$$

We denote the double of the lifts of ξ_{kl}, η_{kl} into $\mathfrak{spin}(p,p)$ by $\tilde{\xi}_{kl}$, $\tilde{\eta}_{kl}$.

Equations (4) – (7) now imply

$$\begin{aligned} \tilde{\xi}_{kl} u(\varepsilon_p,\ldots,\varepsilon_1) &= (-1)^{l-k}(-\varepsilon_k+\varepsilon_l)\varepsilon_{l-1}\cdots\varepsilon_{k+1} \\ &\quad \cdot u(\varepsilon_p,\ldots,-\varepsilon_l,\ldots-\varepsilon_k,\ldots,\varepsilon_1) \qquad (16) \\ \tilde{\eta}_{kl} u(\varepsilon_p,\ldots,\varepsilon_1) &= (-1)^{l-k}(\varepsilon_l\varepsilon_k+1)\varepsilon_{l-1}\cdots\varepsilon_{k+1} \\ &\quad \cdot u(\varepsilon_p,\ldots,-\varepsilon_l,\ldots-\varepsilon_k,\ldots,\varepsilon_1). \qquad (17) \end{aligned}$$

In particular,

$$V_{\mathrm{SO}(p,\mathbb{C})} = \{0\}.$$

4.9 $H = \mathrm{G}_2^{\mathbb{C}}$ $(r = 7,\ s = 7)$

$i_{\mathbb{R}}\mathfrak{g}_2^{\mathbb{C}} \subset i_{\mathbb{R}}\mathfrak{so}(7,\mathbb{C}) \subset \mathfrak{so}(7,7)$ is spanned by

$$\begin{aligned} &\xi_{12}-\xi_{34},\ \xi_{13}+\xi_{24},\ \xi_{14}-\xi_{23},\ \xi_{12}-\xi_{56},\ \xi_{16}-\xi_{25},\ \xi_{15}+\xi_{26},\ \xi_{36}-\xi_{45}, \\ &\xi_{35}+\xi_{46},\ -\xi_{13}+\xi_{67},\ -\xi_{14}+\xi_{57},\ \xi_{16}+\xi_{37},\ \xi_{15}+\xi_{47},\ \xi_{36}-\xi_{17},\ \xi_{35}-\xi_{27}, \\ &\eta_{12}-\eta_{34},\ \eta_{13}+\eta_{24},\ \eta_{14}-\eta_{23},\ \eta_{12}-\eta_{56},\ \eta_{16}-\eta_{25},\ \eta_{15}+\eta_{26},\ \eta_{36}-\eta_{45}, \\ &\eta_{35}+\eta_{46},\ -\eta_{13}+\eta_{67},\ -\eta_{14}+\eta_{57},\ \eta_{16}+\eta_{37},\ \eta_{15}+\eta_{47},\ \eta_{36}-\eta_{17},\ \eta_{35}-\eta_{27}. \end{aligned}$$

Let $v = \sum_{\varepsilon_7,\ldots,\varepsilon_1} \lambda(\varepsilon_7,\ldots,\varepsilon_1)u(\varepsilon_7,\ldots,\varepsilon_1) \in \Delta_{7,7}$ be a spinor satisfying $(\tilde{\xi}_{12}-\tilde{\xi}_{34})v = 0$. Using

$$\begin{aligned} &(\tilde{\xi}_{12}-\tilde{\xi}_{34})u(\varepsilon_7,\ldots,\varepsilon_1) \\ &\quad = (\varepsilon_1-\varepsilon_2)u(\varepsilon_7,\ldots,-\varepsilon_2,-\varepsilon_1) - (\varepsilon_3-\varepsilon_4)u(\varepsilon_7,\ldots,-\varepsilon_4,-\varepsilon_3,\ldots,\varepsilon_1) \end{aligned}$$

which follows from (16) we obtain

$$\begin{aligned} &(\tilde{\xi}_{12}-\tilde{\xi}_{34})v \\ &\quad = \sum_{\varepsilon_7,\ldots,\varepsilon_1} \left(\lambda(\varepsilon_7,\ldots,\varepsilon_1)(\varepsilon_1-\varepsilon_2) + \lambda(\varepsilon_7,\ldots,-\varepsilon_4,-\varepsilon_3,-\varepsilon_2,-\varepsilon_1)(\varepsilon_3-\varepsilon_4)\right) \\ &\qquad\qquad \cdot u(\varepsilon_7,\ldots,-\varepsilon_2,-\varepsilon_1) \\ &\quad = 0 \end{aligned}$$

Hence,

$$\begin{aligned} \lambda(\varepsilon_7,\ldots,\varepsilon_1) = 0 \quad &\text{if} \quad \varepsilon_1 = \varepsilon_2 \ \text{ and } \ \varepsilon_3 \ne \varepsilon_4 \\ &\text{or} \quad \varepsilon_1 \ne \varepsilon_2 \ \text{ and } \ \varepsilon_3 = \varepsilon_4. \end{aligned}$$

and

$$\begin{aligned}\lambda(\varepsilon_7,\ldots,\varepsilon_1) &= -\varepsilon_1\varepsilon_3\lambda(\varepsilon_7,\ldots,-\varepsilon_4,-\varepsilon_3,-\varepsilon_2,-\varepsilon_1)\\ &\qquad \text{if } \varepsilon_1\neq\varepsilon_2 \text{ and } \varepsilon_3\neq\varepsilon_4.\end{aligned}$$

Therefore

$$\begin{aligned}\lambda(\varepsilon_7,\varepsilon_6,\varepsilon_5,-1,1,-1,1) &= -\lambda(\varepsilon_7,\varepsilon_6,\varepsilon_5,1,-1,1,-1)\\ \lambda(\varepsilon_7,\varepsilon_6,\varepsilon_5,-1,1,1,-1) &= \lambda(\varepsilon_7,\varepsilon_6,\varepsilon_5,1,-1,-1,1).\end{aligned}$$

In the same way $(\tilde\eta_{12}-\tilde\eta_{34})v=0$ yields the conditions

$$\begin{aligned}\lambda(\varepsilon_7,\varepsilon_6,\varepsilon_5,1,1,1,1) &= \lambda(\varepsilon_7,\varepsilon_6,\varepsilon_5,-1,-1,-1,-1)\\ \lambda(\varepsilon_7,\varepsilon_6,\varepsilon_5,1,1,-1,-1) &= \lambda(\varepsilon_7,\varepsilon_6,\varepsilon_5,-1,-1,1,1).\end{aligned}$$

Analogously, $(\tilde\xi_{12}-\tilde\xi_{56})v=0$ and $(\tilde\eta_{12}-\tilde\eta_{56})v=0$ imply

$$\begin{aligned}\lambda(\varepsilon_7,-1,1,\varepsilon_4,\varepsilon_3,-1,1) &= -\lambda(\varepsilon_7,1,-1,\varepsilon_4,\varepsilon_3,1,-1)\\ \lambda(\varepsilon_7,-1,1,\varepsilon_4,\varepsilon_3,1,-1) &= \lambda(\varepsilon_7,1,-1,\varepsilon_4,\varepsilon_3,-1,1)\\ \lambda(\varepsilon_7,1,1,\varepsilon_4,\varepsilon_3,1,1) &= \lambda(\varepsilon_7,-1,-1,\varepsilon_4,\varepsilon_3,-1,-1)\\ \lambda(\varepsilon_7,1,1,\varepsilon_4,\varepsilon_3,-1,-1) &= \lambda(\varepsilon_7,-1,-1,\varepsilon_4,\varepsilon_3,1,1).\end{aligned}$$

Consequently, $V_{G_2^{\mathbb{C}}}\cap\Delta^+_{7,7}$ is contained in $\mathrm{span}\{\psi_1^\varepsilon,\psi_2^\varepsilon,\psi_3^\varepsilon,\psi_4^\varepsilon\}$ for $\varepsilon=1$ and $V_{G_2^{\mathbb{C}}}\cap\Delta^-_{7,7}\subset\mathrm{span}\{\psi_1^\varepsilon,\ldots,\psi_4^\varepsilon\}$ for $\varepsilon=-1$, where

$$\begin{aligned}\psi_1^\varepsilon &= u(\varepsilon,1,1,1,1,1,1)+u(\varepsilon,1,1,-1,-1,-1,-1)\\ &\quad +u(\varepsilon,-1,-1,-1,-1,1,1)+u(\varepsilon,-1,-1,1,1,-1,-1)\\ \psi_2^\varepsilon &= u(\varepsilon,1,1,1,1,-1,-1)+u(\varepsilon,1,1,-1,-1,1,1)\\ &\quad +u(\varepsilon,-1,-1,1,1,1,1)+u(\varepsilon,-1,-1,-1,-1,-1,-1)\\ \psi_3^\varepsilon &= u(-\varepsilon,1,-1,1,-1,1,-1)-u(-\varepsilon,-1,1,-1,1,1,-1)\\ &\quad -u(-\varepsilon,-1,1,1,-1,-1,1)-u(-\varepsilon,1,-1,-1,1,-1,1)\\ \psi_4^\varepsilon &= u(-\varepsilon,-1,1,-1,1,-1,1)-u(-\varepsilon,1,-1,1,-1,-1,1)\\ &\quad -u(-\varepsilon,1,-1,-1,1,1,-1)-u(-\varepsilon,-1,1,1,-1,1,-1).\end{aligned}$$

Next we investigate the action of $(-\tilde\xi_{13}+\tilde\xi_{67})$. Because of

$$\begin{aligned}(-\tilde\xi_{13}+\tilde\xi_{67})u(\varepsilon_7,\ldots,\varepsilon_1) &= (\varepsilon_1-\varepsilon_3)\varepsilon_2 u(\varepsilon_7,\ldots,-\varepsilon_3,\ldots,-\varepsilon_1)\\ &\quad +(\varepsilon_6-\varepsilon_7)u(-\varepsilon_7,-\varepsilon_6,\varepsilon_5,\ldots,\varepsilon_1)\end{aligned}$$

we get

$$\begin{aligned}&(-\tilde\xi_{13}+\tilde\xi_{67})\psi_1^1\\ &\quad = 2\{u(1,-1,-1,-1,1,1,-1)+u(1,-1,-1,1,-1,-1,1)\\ &\qquad -u(-1,1,-1,-1,-1,1,1)-u(-1,1,-1,1,1,-1,-1)\}\\ &\quad = -(-\tilde\xi_{13}+\tilde\xi_{67})\psi_4^1\end{aligned}$$

$$
\begin{aligned}
&(-\tilde{\xi}_{13}+\tilde{\xi}_{67})\psi_2^1 \\
&\quad = \quad 2\{u(1,1,1,1,-1,-1,1)+u(1,1,1,-1,1,1,-1) \\
&\qquad\quad -u(-1,1,-1,1,1,1,1)-u(-1,1,-1,-1,-1,-1,-1)\}
\end{aligned}
$$

$$
\begin{aligned}
&(-\tilde{\xi}_{13}+\tilde{\xi}_{67})\psi_3^1 \\
&\quad = \quad 2\{u(-1,1,-1,1,1,-1,-1)+u(-1,1,-1,-1,-1,1,1) \\
&\qquad\quad -u(1,-1,-1,1,-1,-1,1)-u(1,-1,-1,-1,1,1,-1)\}.
\end{aligned}
$$

We conclude $V_{\mathrm{G}_2^{\mathbb{C}}}\cap\Delta_{7,7}^{+}\subset\mathbb{C}\cdot(\psi_1^1+\psi_4^1)$ and similarly $V_{\mathrm{G}_2^{\mathbb{C}}}\cap\Delta_{7,7}^{-}\subset\mathbb{C}\cdot(\psi_2^{-1}-\psi_3^{-1})$.

Furthermore, all remaining generators of $\mathfrak{g}_2^{\mathbb{C}}$ annihilate $\psi_1^1+\psi_4^1$ and $\psi_2^{-1}-\psi_3^{-1}$. Finally,

$$
V_{\mathrm{G}_2^{\mathbb{C}}}=\mathrm{span}\{\psi_1^1+\psi_4^1\,,\,\psi_2^{-1}-\psi_3^{-1}\}.
$$

From (12) we calculate

$$
\begin{aligned}
&\langle\psi_1^1+\psi_4^1\,,\,\psi_1^1+\psi_4^1\rangle=0,\quad\langle\psi_2^{-1}-\psi_3^{-1}\,,\,\psi_2^{-1}-\psi_3^{-1}\rangle=0,\\
&\langle\psi_1^1+\psi_4^1\,,\psi_2^{-1}+\psi_3^{-1}\rangle=8i.
\end{aligned}
$$

Furthermore, we have already seen

$$
\psi_1^1+\psi_4^1\in\Delta_{7,7}^{+},\quad\psi_2^{-1}-\psi_3^{-1}\in\Delta_{7,7}^{-}.
$$

4.10 $H=\mathrm{Spin}(7)^{\mathbb{C}}$ $(r=8,\,s=8)$

$i_{\mathbb{R}^*}\mathfrak{spin}(7,\mathbb{C})\subset i_{\mathbb{R}^*}\mathfrak{so}(8,\mathbb{C})\subset\mathfrak{so}(8,8)$ is generated by

$$
\begin{aligned}
&\xi_{12}+\xi_{34},\ \xi_{13}-\xi_{24},\ \xi_{14}+\xi_{23},\ \xi_{56}+\xi_{78},\ -\xi_{57}+\xi_{68},\ \xi_{58}+\xi_{67},\ -\xi_{15}+\xi_{26},\\
&\xi_{12}+\xi_{56},\ \xi_{16}+\xi_{25},\ \xi_{37}-\xi_{48},\ \xi_{38}+\xi_{47},\ \xi_{17}+\xi_{28},\ \xi_{18}-\xi_{27},\ \xi_{35}+\xi_{46},\\
&\xi_{36}-\xi_{45},\ \xi_{18}+\xi_{36},\ \xi_{17}+\xi_{35},\ \xi_{26}-\xi_{48},\ \xi_{25}+\xi_{38},\ \xi_{23}+\xi_{67},\ \xi_{24}+\xi_{57},\\
&\eta_{12}+\eta_{34},\ \eta_{13}-\eta_{24},\ \eta_{14}+\eta_{23},\ \eta_{56}+\eta_{78},\ -\eta_{57}+\eta_{68},\ \eta_{58}+\eta_{67},\ -\eta_{15}+\eta_{26},\\
&\eta_{12}+\eta_{56},\ \eta_{16}+\eta_{25},\ \eta_{37}-\eta_{48},\ \eta_{38}+\eta_{47},\ \eta_{17}+\eta_{28},\ \eta_{18}-\eta_{27},\ \eta_{35}+\eta_{46},\\
&\eta_{36}-\eta_{45},\ \eta_{18}+\eta_{36},\ \eta_{17}+\eta_{35},\ \eta_{26}-\eta_{48},\ \eta_{25}+\eta_{38},\ \eta_{23}+\eta_{67},\ \eta_{24}+\eta_{57}.
\end{aligned}
$$

Let $v=\sum_{\varepsilon_8,\ldots,\varepsilon_1}\lambda(\varepsilon_8,\ldots,\varepsilon_1)u(\varepsilon_8,\ldots,\varepsilon_1)\in\Delta_{8,8}$ be a spinor annihilated by $\tilde{\xi}_{12}+\tilde{\xi}_{34}$, $\tilde{\eta}_{12}+\tilde{\eta}_{34}$, $\tilde{\xi}_{12}+\tilde{\xi}_{56}$, $\tilde{\eta}_{12}+\tilde{\eta}_{56}$, $\tilde{\xi}_{56}+\tilde{\xi}_{78}$, $\tilde{\eta}_{56}+\tilde{\eta}_{78}$. Then we can show as in the case of $\mathrm{G}_2^{\mathbb{C}}$

$$
\begin{aligned}
\lambda(\varepsilon_8,\ldots,\varepsilon_1)=0\quad&\text{if}\quad\varepsilon_1\cdot\varepsilon_2\cdot\varepsilon_3\cdot\varepsilon_4=-1\\
&\text{or if}\quad\varepsilon_1\cdot\varepsilon_2\cdot\varepsilon_5\cdot\varepsilon_6=-1\\
&\text{or if}\quad\varepsilon_5\cdot\varepsilon_6\cdot\varepsilon_7\cdot\varepsilon_8=-1,
\end{aligned}
$$

and

$$
\begin{aligned}
\lambda(\varepsilon_8,\dots,\varepsilon_5,1,-1,1,-1) &= \lambda(\varepsilon_8,\dots,\varepsilon_5,-1,1,-1,1)\\
\lambda(\varepsilon_8,\dots,\varepsilon_5,1,-1,-1,1) &= -\lambda(\varepsilon_8,\dots,\varepsilon_5,-1,1,1,-1)\\
\lambda(\varepsilon_8,\dots,\varepsilon_5,1,1,1,1) &= -\lambda(\varepsilon_8,\dots,\varepsilon_5,-1,-1,-1,-1)\\
\lambda(\varepsilon_8,\dots,\varepsilon_5,1,1,-1,-1) &= -\lambda(\varepsilon_8,\dots,\varepsilon_5,-1,-1,1,1)
\end{aligned}
$$

$$
\begin{aligned}
\lambda(\varepsilon_8,\varepsilon_7,1,-1,\varepsilon_4,\varepsilon_3,1,-1) &= \lambda(\varepsilon_8,\varepsilon_7,-1,1,\varepsilon_4,\varepsilon_3,-1,1)\\
\lambda(\varepsilon_8,\varepsilon_7,1,-1,\varepsilon_4,\varepsilon_3,-1,1) &= -\lambda(\varepsilon_8,\varepsilon_7,-1,1,\varepsilon_4,\varepsilon_3,1,-1)\\
\lambda(\varepsilon_8,\varepsilon_7,1,1,\varepsilon_4,\varepsilon_3,1,1) &= -\lambda(\varepsilon_8,\varepsilon_7,-1,-1,\varepsilon_4,\varepsilon_3,-1,-1)\\
\lambda(\varepsilon_8,\varepsilon_7,1,1,\varepsilon_4,\varepsilon_3,-1,-1) &= -\lambda(\varepsilon_8,\varepsilon_7,-1,-1,\varepsilon_4,\varepsilon_3,1,1)
\end{aligned}
$$

$$
\begin{aligned}
\lambda(1,-1,1,-1,\varepsilon_4,\dots,\varepsilon_1) &= \lambda(-1,1,-1,1,\varepsilon_4,\dots,\varepsilon_1)\\
\lambda(1,-1,-1,1,\varepsilon_4,\dots,\varepsilon_1) &= -\lambda(-1,1,1,-1,\varepsilon_4,\dots,\varepsilon_1)\\
\lambda(1,1,1,1,\varepsilon_4,\dots,\varepsilon_1) &= -\lambda(-1,-1,-1,-1,\varepsilon_4,\dots,\varepsilon_1)\\
\lambda(1,1,-1,-1,\varepsilon_4,\dots,\varepsilon_1) &= -\lambda(-1,-1,1,1,\varepsilon_4,\dots,\varepsilon_1).
\end{aligned}
$$

Consequently, $\mathrm{V}_{\mathrm{Spin}(7,\mathbb{C})} \subset \mathrm{span}\{\psi_1,\psi_2,\psi_3,\psi_4\}$, where

$$
\begin{aligned}
\psi_1 \;=\;& u(1,1,1,1,1,1,1,1) - u(1,1,1,1,-1,-1,-1,-1)\\
&-u(-1,-1,-1,-1,1,1,1,1) + u(-1,-1,-1,-1,-1,-1,-1,-1)\\
&-u(1,1,-1,-1,1,1,-1,-1) - u(-1,-1,1,1,-1,-1,1,1)\\
&+u(1,1,-1,-1,-1,-1,1,1,) + u(-1,-1,1,1,1,1,-1,-1)
\end{aligned}
$$

$$
\begin{aligned}
\psi_2 \;=\;& u(1,1,1,1,1,1,-1,-1) - u(1,1,1,1,-1,-1,1,1)\\
&-u(-1,-1,-1,-1,1,1,-1,-1) + u(-1,-1,-1,-1,-1,-1,1,1)\\
&+u(-1,-1,1,1,1,1,1,1) + u(1,1,-1,-1,-1,-1,-1,-1)\\
&-u(-1,-1,1,1,-1,-1,-1,-1) - u(1,1,-1,-1,1,1,1,1)
\end{aligned}
$$

$$
\begin{aligned}
\psi_3 \;=\;& u(1,-1,1,-1,1,-1,1,-1) + u(-1,1,-1,1,1,-1,1,-1)\\
&+u(1,-1,1,-1,-1,1,-1,1) + u(-1,1,-1,1,-1,1,-1,1)\\
&+u(1,-1,-1,1,1,-1,-1,1) + u(-1,1,1,-1,-1,1,1,-1)\\
&-u(1,-1,-1,1,-1,1,1,-1) - u(-1,1,1,-1,1,-1,-1,1)
\end{aligned}
$$

$$
\begin{aligned}
\psi_4 \;=\;& u(1,-1,1,-1,1,-1,-1,1) + u(-1,1,-1,1,1,-1,-1,1)\\
&-u(1,-1,-1,1,1,-1,1,-1) - u(1,-1,-1,1,-1,1,-1,1)\\
&-u(1,-1,1,-1,-1,1,1,-1) - u(-1,1,-1,1,-1,1,1,-1)\\
&+u(-1,1,1,-1,-1,1,-1,1) + u(-1,1,1,-1,1,-1,1,-1).
\end{aligned}
$$

Using

$$\begin{aligned}
&(\tilde{\xi}_{18}+\tilde{\xi}_{36})u(\varepsilon_8,\ldots,\varepsilon_1)\\
&\quad= (\varepsilon_1-\varepsilon_8)\varepsilon_2\cdot\varepsilon_3\cdot\varepsilon_4\cdot\varepsilon_5\cdot\varepsilon_6\cdot\varepsilon_7 u(-\varepsilon_8,\varepsilon_7,\ldots,\varepsilon_2,-\varepsilon_1)\\
&\qquad +(\varepsilon_3-\varepsilon_6)\cdot\varepsilon_4\cdot\varepsilon_5 u(\varepsilon_8,\ldots,-\varepsilon_6,\ldots,-\varepsilon_3,\ldots,\varepsilon_1)
\end{aligned}$$

we get

$$\begin{aligned}
&(\tilde{\xi}_{18}+\tilde{\xi}_{36})\psi_1\\
&\quad= 2\{-u(-1,1,1,1,-1,-1,-1,1)+u(1,-1,-1,-1,1,1,1,-1)\\
&\qquad -u(-1,1,-1,-1,1,1,-1,1)+u(1,-1,1,1,-1,-1,1,-1)\\
&\qquad -u(1,1,-1,1,-1,1,-1,-1)+u(-1,-1,1,-1,1,-1,1,1)\\
&\qquad +u(1,1,1,-1,1,-1,-1,-1)-u(-1,-1,-1,1,-1,1,1,1)\}\\
&\quad= (\tilde{\xi}_{18}+\tilde{\xi}_{36})\psi_3
\end{aligned}$$

$$\begin{aligned}
&(\tilde{\xi}_{18}+\tilde{\xi}_{36})\psi_2\\
&\quad= 2\{u(-1,1,1,1,1,1,-1,1)-u(1,-1,-1,-1,-1,-1,1,-1)\\
&\qquad -u(1,-1,1,1,1,1,1,-1)+u(-1,1,-1,-1,-1,-1,-1,1)\\
&\qquad -u(1,1,-1,1,-1,1,1,1)+u(-1,-1,1,-1,1,-1,-1,-1)\\
&\qquad -u(-1,-1,-1,1,-1,1,-1,-1)+u(1,1,1,-1,1,-1,1,1)\}
\end{aligned}$$

$$\begin{aligned}
&(\tilde{\xi}_{18}+\tilde{\xi}_{36})\psi_4\\
&\quad= 2\{-u(1,1,-1,1,1,-1,-1,-1)-u(-1,-1,-1,1,1,-1,1,1)\\
&\qquad -u(-1,-1,1,-1,-1,1,1,1)-u(1,1,1,-1,-1,1,-1,-1)\\
&\qquad +u(1,-1,-1,-1,1,1,-1,1)+u(1,-1,1,1,-1,-1,-1,1)\\
&\qquad +u(-1,1,1,1,-1,-1,1,-1)+u(-1,1,-1,-1,1,1,1,-1)\}.
\end{aligned}$$

Hence $\mathrm{V}_{\mathrm{Spin}(7,\mathbb{C})}\subset\mathbb{C}(\psi_1-\psi_3)$. A direct calculation shows that all other generators of $\mathfrak{spin}(7,\mathbb{C})$ annihilate $\psi_1-\psi_3$. We conclude

$$\mathrm{V}_{\mathrm{Spin}(7)^{\mathbb{C}}}=\mathbb{C}(\psi_1-\psi_3).$$

In particular we have $\mathrm{V}_{\mathrm{Spin}(7)^{\mathbb{C}}}\subset\Delta^+_{8,8}$. According to (12) we obtain for the quadratic length of the generator $\langle\psi_1-\psi_3\,,\,\psi_1-\psi_3\rangle=16$, i.e. all non-zero parallel spinors are spacelike.

4.11 $H = \mathrm{Sp}(p,\mathbb{R}) \cdot \mathrm{SL}(2,\mathbb{R})$ $(r = 2p,\ s = 2p)$

For the calculations we consider the elements of $H = \mathrm{Sp}(p,\mathbb{R}) \cdot \mathrm{SL}(2,\mathbb{R})$ as matrices with respect to the following basis $(e_1, \dots, e_{4p})$ of $\mathbb{R}^{4p}$: Let $(v_1, \dots, v_{2p})$ be a symplectic basis of $(\mathbb{R}^{2p}, \omega_{\mathbb{R}})$, i.e. $(\omega_{\mathbb{R}}(v_k, v_l)) = J_{\mathbb{R}}$ and let $a_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $a_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \in \mathbb{R}^2$. Then

$$\begin{aligned}
e_{4j-3} &= \frac{1}{\sqrt{2}}(v_{2j-1} \otimes a_1 + v_{2j} \otimes a_2) \\
e_{4j-2} &= \frac{1}{\sqrt{2}}(v_{2j-1} \otimes a_1 - v_{2j} \otimes a_2) \\
e_{4j-1} &= \frac{1}{\sqrt{2}}(v_{2j} \otimes a_1 - v_{2j-1} \otimes a_2) \\
e_{4j} &= \frac{1}{\sqrt{2}}(v_{2j} \otimes a_1 + v_{2j-1} \otimes a_2),
\end{aligned}$$

$j = 1, \dots, p$. Then $H = \mathrm{Sp}(p,\mathbb{R}) \cdot \mathrm{SL}(2,\mathbb{R})$ is a subgroup of $\mathrm{SO}(2p,2p)$, where $\mathrm{SO}(2p,2p) \subset \mathrm{GL}(4p,\mathbb{R})$ is (as in 4.8 – 4.10) the invariance group of the inner product

$$\langle x, y \rangle_{2p,2p} = -\sum_{j=1}^{2p} x_{2j-1} \cdot y_{2j-1} + \sum_{j=1}^{2p} x_{2j} \cdot y_{2j}.$$

The Lie algebra $\mathfrak{h} = \mathfrak{sp}(p,\mathbb{R}) \oplus \mathfrak{sl}(2,\mathbb{R}) \subset \mathfrak{so}(2p,2p)$ is spanned by the following matrices of $\mathfrak{so}(2p,2p)$

$$\begin{aligned}
H_k &= E_{4k-3\,4k-2} - E_{4k-1\,4k} \\
I_k &= E_{4k-3\,4k} - E_{4k-2\,4k-1} \\
J_k &= -E_{4k-3\,4k-1} + E_{4k-2\,4k} && k = 1, \dots, p \\
K_{kl} &= E_{4k-3\,4l} - E_{4k-2\,4l-1} + E_{4k-1\,4l-2} - E_{4k\,4l-3} \\
L_{kl} &= -E_{4k-3\,4l-1} + E_{4k-2\,4l} + E_{4k-1\,4l-3} - E_{4k\,4l-2} \\
M_{kl} &= -E_{4k-3\,4l-3} + E_{4k-2\,4l-2} - E_{4k-1\,4l-1} + E_{4k\,4l} \\
N_{kl} &= -E_{4k-3\,4l-2} + E_{4k-2\,4l-3} + E_{4k-1\,4l} - E_{4k\,4l-1} && 1 \le k < l \le p \\
P_1 &= -\sum_{j=1}^{p}(E_{4j-3\,4j-2} + E_{4j-1\,4j}) \\
P_2 &= -\sum_{j=1}^{p}(E_{4j-3\,4j-1} + E_{4j-2\,4j}) \\
P_3 &= +\sum_{j=1}^{p}(E_{4j-3\,4j} + E_{4j-2\,4j-1}).
\end{aligned}$$

As above, if $X \in \mathfrak{so}(2p,2p)$, then $\tilde{X}$ denotes the element $\tilde{X} = 2\lambda_*^{-1}(X)$ of $\mathfrak{spin}(2p,2p)$. If we compare the action of

$$\lambda_*^{-1}(\mathfrak{sp}(p,\mathbb{R}) \cdot \mathfrak{sl}(2,\mathbb{R})) \subset \mathfrak{spin}(2p,2p)$$

on $\Delta_{2p,2p} \simeq \mathbb{C}^{2^{2p}}$ with that of $\lambda_*^{-1}(\mathfrak{sp}(p) \cdot \mathfrak{sp}(1)) \subset \mathfrak{spin}(4p)$ on $\Delta_{4p,0} \simeq \mathbb{C}^{2^{2p}}$ we obtain the same formulas for the following elements (see 4.4, 4.5)

$\lambda_*^{-1}(\mathfrak{sp}(p,\mathbb{R}) \cdot \mathfrak{sl}(2,\mathbb{R}))$	$\lambda_*^{-1}(\mathfrak{sp}(p) \cdot \mathfrak{sp}(1))$
$-i\tilde{H}_k$	$\tilde{D}_{2k-1} - \tilde{D}_{2k}$
$-i\tilde{I}_k$	$\tilde{Y}_{2k-1\,2k}$
$\tilde{J}_k$	$\tilde{X}_{2k-1\,2k}$
$-i\tilde{K}_{kl}$	$\tilde{Y}_{2k-1\,2l} + \tilde{Y}_{2k\,2l-1}$
$\tilde{L}_{kl}$	$\tilde{X}_{2k-1\,2l} - \tilde{X}_{2k\,2l-1}$
$\tilde{M}_{kl}$	$\tilde{X}_{2k-1\,2l-1} + \tilde{X}_{2k\,2l}$
$i\tilde{N}_{kl}$	$\tilde{Y}_{2k-1\,2l-1} - \tilde{Y}_{2k\,2l}$
$i\tilde{P}_1$	$\tilde{X}_1$
$\tilde{P}_2$	$\tilde{X}_2$
$-i\tilde{P}_3$	$\tilde{X}_3$

($\mathfrak{sp}(p,\mathbb{R}) \cdot \mathfrak{sl}(2,\mathbb{R})$ and $\mathfrak{sp}(p) \cdot \mathfrak{sp}(1)$ are different real forms of the complex Lie algebra $\mathfrak{sp}(p,\mathbb{C}) \cdot \mathfrak{sl}(2,\mathbb{C})$).

Hence, using the results of 4.5 we obtain

$$V_{\mathrm{Sp}(p,\mathbb{R})\cdot\mathrm{SL}(2,\mathbb{R})} = V_{\mathrm{Sp}(p)\cdot\mathrm{Sp}(1)} = \{0\}.$$

4.12 $H = \mathrm{Sp}(p,\mathbb{C}) \cdot \mathrm{SL}(2,\mathbb{C})$ $(r = 4p,\ s = 4p)$

According to 4.11 the Lie algebra $\mathfrak{sp}(p,\mathbb{C}) \oplus \mathfrak{sl}(2,\mathbb{C}) = (\mathfrak{sp}(p,\mathbb{R}) \oplus \mathfrak{sl}(2,\mathbb{R}))^{\mathbb{C}}$ is contained in $\mathfrak{so}(2p,2p)^{\mathbb{C}}$. For the embedding

$$i_{\mathbb{R}} : \mathbb{C}(4p) \supset \mathfrak{so}(2p,2p)^{\mathbb{C}} \to \mathfrak{so}(4p,4p) \subset \mathbb{R}(8p),$$

where in this case $\mathfrak{so}(4p,4p)$ has to be considered as the Lie algebra of the invariance group of the bilinear form

$$\langle x,y\rangle_{4p,4p} = \sum_{j=1}^{2p} \Big(- x_{4j-3}y_{4j-3} + x_{4j-2}y_{4j-2} + x_{4j-1}y_{4j-1} - x_{4j}y_{4j} \Big)$$

on $\mathbb{R}^{8p}$, we obtain

$$\begin{aligned} i_{\mathbb{R}}(E_{kl}) &= E_{2k-1\,2l-1} - E_{2k\,2l} \\ i_{\mathbb{R}}(iE_{kl}) &= E_{2k-1\,2l} + E_{2k\,2l-1}. \end{aligned} \qquad (*)$$

(Here the E_{kl} are the basis elements of the corresponding Lie algebras, where the normal forms of the metric are choosen in the above mentioned way.)

Using $(*)$ one can derive a basis of $i_{\mathbb{R}}(\mathfrak{sp}(p,\mathbb{C}) \oplus \mathfrak{sl}(2,\mathbb{C})) \subset \mathfrak{so}(4p,4p)$ easily from the basis of $\mathfrak{sp}(p,\mathbb{R}) \oplus \mathfrak{sl}(2,\mathbb{R})$ given in 4.11. Then, a direct (but rather long) calculation shows that

$$V_{\mathrm{Sp}(p,\mathbb{C})\cdot \mathrm{SL}(2,\mathbb{C})} = \{0\}.$$

Summarizing the previous calculations we obtain:

Theorem. *Let $M^{r,s}$ be a simply connected, non-locally symmetric irreducible semi-Riemannian space and time oriented spin manifold of dimension $n = r + s$ and index r. Let $\mathfrak{P}$ be the space of parallel spinors of M. Then $\dim \mathfrak{P} = N > 0$ if and only if the holonomy representation H of M is (up to conjugacy in the full orthogonal group) one of the following:*

1. *$H = \mathrm{SU}(p,q) \subset \mathrm{SO}(2p,2q)$, $n = 2(p+q)$, $r = 2p$.*
   Then $N = 2$. There exists a basis $\{\varphi_1, \varphi_2\}$ of $\mathfrak{P}$ such that
   $\varphi_1,\ \varphi_2 \in \Gamma(S^+)$ or $\varphi_1,\ \varphi_2 \in \Gamma(S^-)$ if $p+q$ is even,
   $\varphi_1 \in \Gamma(S^+)$, $\varphi_2 \in \Gamma(S^-)$ if $p+q$ is odd,
   φ_1, φ_2 have the same causality if p is even
   φ_1, φ_2 have different causality if p is odd.

2. *$H = \mathrm{Sp}(p,q) \subset \mathrm{SO}(4p,4q)$, $n = 4(p+q)$, $r = 4p$.*
   Then $N = p+q+1$, $\mathfrak{P} \subset \Gamma(S^+)$ or $\mathfrak{P} \subset \Gamma(S^-)$. All non-trivial parallel spinors have the same causal type.

3. *$H = \mathrm{G}_2 \subset \mathrm{SO}(7)$, $n = 7$, $r = 0$.*
   Then $N = 1$.

4. *$H = \mathrm{G}^*_{2(2)} \subset \mathrm{SO}(4,3)$, $n = 7$, $r = 4$.*
   Then $N = 1$. All non-trivial parallel spinors are non-isotropic.

5. *$H = \mathrm{G}_2^{\mathbb{C}} \subset \mathrm{SO}(7,7)$, $n = 14$, $r = 7$.*
   Then $N = 2$. There exists a basis $\{\varphi_1,\ \varphi_2\}$ of $\mathfrak{P}$ such that $\varphi_1 \in \Gamma(S^+)$, $\varphi_2 \in \Gamma(S^-)$, φ_1, φ_2 are isotropic but not orthogonal to each other.

6. *$H = \mathrm{Spin}(7) \subset \mathrm{SO}(8)$, $n = 8$, $r = 0$.*
   Then $N = 1$.

7. *$H = \mathrm{Spin}(4,3) \subset \mathrm{SO}(4,4)$, $n = 8$, $r = 4$.*
   Then $N = 1$. All non-trivial parallel spinors are non-isotropic.

8. *$H = \mathrm{Spin}(7)^{\mathbb{C}} \subset \mathrm{SO}(8,8)$, $n = 16$, $r = 8$.*
   Then $N = 1$. All non-trivial parallel spinors are non-isotropic.

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