Generalized Knizhnik-Zamolodchikov Equations and Isomonodromy Quantization of the Equations Integrable via the Inverse Scattering Transform: Maxwell-Bloch System with Pumping

H. M. Babujian*
Institut für Theoretische Physik
Freie Universität Berlin, Arnimallee 14, D-14195 Berlin, Germany

A. V. Kitaev**
Steklov Mathematical Institute, Fontanka 27, St.-Petersburg, 191011, Russia

July 9, 1997

Abstract

Canonical quantization of the isomonodromy solutions of equations integrable via the Inverse Scattering Transform leads to generalized Knizhnik-Zamolodchikov equations. One can solve these equations by the Off-shell Bethe Ansatz method provided the Knizhnik-Zamolodchikov equations are related with the highest weight representations of the corresponding Lie algebras: These solutions can be written in terms of multi-variable generalizations of special functions of the hypergeometric type. In this work, we consider a realization of the above scheme for the Maxwell–Bloch system with pumping: quantum states for this system are found in terms of the multi-variable confluent hypergeometric function.

PACS (1996) numbers: 02.30.Gp, 03.65.Ge, 11.10.Lm, 42.65.Tg

Mathematics Subject Classification (1991): 33D70, 35Q58, 35Q60, 81Q05, 81S99.

Short title: Maxwell–Bloch system with pumping

* Permanent address: Yerevan Physics Institute, Alichanian Brothers 2, Yerevan 375036, Armenia, E-mail: babujian@lx2.yerphi.am . Partially supported by the grant 211-5291 YPI of the German Bundesministerium für Forshung und Technologie and Sonderforschungsbereich 288

**E-mail: kitaev@pdmi.ras.ru . Supported by Sonderforschungsbereich 288 and Alexander von Humboldt Foundation

1 Introduction

Some time ago N. Reshetikhin [1] introduced generalized Knizhnik-Zamolodchikov equations and noticed that they can be viewed as a quantization of the non-linear system of ordinary differential equations (ODEs) defining isomonodromy deformations of linear ODEs. In particular, the original system of the Knizhnik-Zamolodchikov [2] equations can be viewed as a quantization of the classical Schlesinger equations [3]. Later J. Harnad [4] gave a more "physical" interpretation of this quantization procedure: by considering the Schlesinger equations as a multi-time Hamiltonian system, he, using the standard quantum mechanical language, has shown that the system of Knizhnik-Zamolodchikov equations can be viewed as a multi-time non-autonomous system of Schrödinger equations.

An attractive feature of this quantization is that the Knizhnik-Zamolodchikov equations on the Lie algebras with the finite weight representations can be solved in terms of the multi-variable hypergeometric functions [5], so that one can get explicit formulas for the wave-functions, correlators, and so on. To make this quantization procedure closer to applications one has to find physical models which can be described in terms of solutions to the system of Schlesinger equations and/or their generalizations. One regular procedure is well-known: The system of Schlesinger equation and their multi-pole generalizations govern the so-called isomonodromy solutions of the equations integrable via the Inverse Scattering Transform (IST) [6]. In particular, there are special classes of the isomonodromy solutions to the self-dual Yang-Mills equations, Einstein-Maxwell equations, and Ernst equation, which can be written in terms of the solutions to the system of the Schlesinger equations [7, 8]. In general, to fix a class of the isomonodromy solutions of a partial differential equation integrable via IST means to pick out a finite dimensional non-autonomous Hamiltonian system of its solutions which defines an isomonodromy deformation of some auxiliary linear ODE (see Sec. 3 details). Most of the important solutions which are studied in the IST framework; solitons, finite-gap, or similarity, belong to the isomonodromy classes of solutions. The quantization of the system of Schlesinger equations (or their generalizations) which governs one of such isomonodromy subsystems means to associate with the classical system a spin-like non-autonomous finite dimensional quantum system. These systems usually contain some parameters, in particular the number of "particles (waves)", N, so that further one can study different scaling limits $N \to \infty$.

D. Korotkin and H. Nicolai [9] used the above ideas for the quantization of a special class of the isomonodromy solutions to the Ernst equation - the class which is governed by the classical Schlesinger equations. They found that for the further interpretation of the quantum model obtained as the model of quantum gravity one has to demand the wave function belongs to the coset space $SL(2,\mathbb{R})/SO(2)$ and satisfies an additional constraint: like the vanishing of a total spin of the system. There are no (non-trivial) solutions of the corresponding Knizhnik-Zamolodchikov equations on the representations of $sl_2(\mathbb{R})$ with the highest weights which satisfy this constraint. There are two possibilities of continuation of these studies for the Ernst equation: 1) Instead of the principal discrete series of representations of $sl_2(\mathbb{R})$ one can try to use the principal continuous series. But the corresponding system of Knizhnik-Zamolodchikov equations becomes a rather complicated system of partial differential equations which is not investigated at the present time. 2) To study an application of this quantization scheme to other classes of isomonodromy solutions to the Ernst equation. In the last case, of course, we change the objects under quantization. The corresponding systems of Knizhnik-Zamolodchikov equations also change; probably they belong to the class of the generalized Knizhnik-Zamolodchikov equations which are introduced in [1].

In this paper we would like to draw attention to the fact that the quantization procedure briefly described above is applicable to isomonodromy solutions of any IST integrable (KdV-like) differential equation. The corresponding system of the generalized Knizhnik-Zamolodchikov equations can be solved via the off-shell Bethe ansatz method [10, 11]. Since, there is a rather big variety of the IST integrable equations, each of these equations possesses (infinitely) many classes of the isomonodromy solutions, and the same equation very often serves for a description of different physical models, so that the different classes of the isomonodromy solutions are really involved, we hope that this quantization procedure can appear to be quite reasonable for some physical models and therefore believe that it requires more wide attention and further studies.

In this work we consider the Maxwell-Bloch system with pumping (Sec. 2). From the IST point of view this is an intermediate case between the systems integrable via IST with the variable spectral parameter and IST with the spectral parameter independent of the dynamical variables. The spectral parameter in this case depends on one (space) variable and independent of time. It is worth mentioning that the wave function, the solution of a generalized system of Knizhnik-Zamolodchikov equations, is obtained in Sec. 6 in terms of the multi-variable confluent hyper-geometric function.

Probably, the Maxwell-Bloch system without pumping (Sec. 2) would be physically the most reasonable example for further studies. The associated auxiliary ODE with respect to the spectral parameter, which plays a key role in the definition of the isomonodromy solutions (Sec. 3), has an additional pole of the second order compared to the corresponding ODE studied in this work and is therefore the next complicated case to the one considered here.

2 Maxwell–Bloch System with Pumping

The Maxwell–Bloch system (without pumping) [12, 13] describes the propagation of a sequence of electromagnetic pulses along the positive direction of the x-axis in the two level atomic medium. It is known to be integrable via the Inverse Scattering Transform [14, 15, 16]. The Maxwell–Bloch system with pumping was introduced by S. P. Burtsev, V. E. Zakharov, and A. V. Mikhailov [17] in the framework of the Inverse Scattering Transform with a variable spectral parameter as an integrable deformation (with the deformation parameter s) of the reduced Maxwell–Bloch system (RMB). More exactly, they considered the following system of linear differential equations:

$$\partial_\xi \psi = (z\sigma_3 + U_0)\psi, \qquad \left(\partial_\eta + \frac{s}{z}\partial_z\right)\psi = \frac{\hat{\rho}}{4z}\psi, \tag{2.1}$$

where $z \in \mathbb{C}$ is the auxiliary (spectral) parameter and the 2×2–matrices σ_3, U_0, and $i\hat{\rho} \in su_2$ have the following form:

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad U_0 = \frac{1}{2}\begin{pmatrix} 0 & E \\ -E^* & 0 \end{pmatrix}, \quad \hat{\rho} = \begin{pmatrix} N & -\rho \\ -\rho^* & -N \end{pmatrix}, \tag{2.2}$$

with the functions: $E = E(\xi,\ \eta), N = N(\xi,\ \eta)$, and $\rho = \rho(\xi,\ \eta)$ independent of z and (*) denoting the complex conjugation. The compatibility condition for the system (2.1) reads:

$$\partial_\eta E = \rho, \qquad \partial_\xi \rho = NE, \qquad \partial_\xi N + \frac{1}{2}(E\rho^* + E^*\rho) = 4s. \tag{2.3}$$

This non-linear system of partial differential equations is called the Maxwell–Bloch system with pumping (PMB). The physical meaning of the notations used is as follows: dimensionless variables η and ξ are related to the space, x, and time, t, variables:

$$\eta = \Omega\frac{x}{c}, \qquad \xi = \Omega\left(t - \frac{x}{c}\right); \tag{2.4}$$

E and ρ are slowly varying complex amplitudes of the electric field and polarization respectively; N is the population inversion, therefore $-1 \le N \le 1$; $\Omega > 0$ is a parameter characterizing the medium; and $s > 0$ represents the "pumping", i.e., an additional excitation of the active medium as it was interpreted by the authors [17]. Note that the standard RMB equations are given by (2.3) and (2.4) with $s = 0$. From the mathematical viewpoint all the techniques related with IST, including the quantization which we present in this work, can be developed for the complex function N, and the parameters ω and s. However, here we follow the restrictions mentioned above. Further details about the system (2.3), (2.4) can be found in the works [18, 19, 20, 21, 22].

3 Isomonodromy solutions

The general notion of the isomonodromy solutions of the equations integrable via IST was formulated in the work [6]. It is applicable also to the equations integrable via IST with a variable spectral parameter. The ψ-function which corresponds to such solutions, possibly after some gauge-like transformations, solves together with the system (2.1) one more linear ODE, the ODE with respect to the spectral parameter,

$$\partial_z \psi = A(z)\psi, \tag{3.1}$$

where $A(z) = A(z, \xi, \eta) \in su_2$ is a rational function of the spectral parameter z. The compatibility condition of Eq. (3.1) with the system (2.1) does not uniquely define the matrix $A(z, \xi, \eta)$. This freedom is such that all the most important solutions to the equations integrable via IST (multi-soliton, finite-gap, similarity) can be specified as the solutions of the isomonodromy class: the ψ-functions corresponding to them solve Eq. (3.1) with different matrices $A(z, \xi, \eta)$ for the solutions of the different types. Some solutions, which do not belong to the isomonodromy class, e.g. rapidly decaying solutions, also show an isomonodromy behavior in many interesting asymptotic regimes.

In our case the simple possible choice of $A(z, \xi, \eta)$ which is consistent with the system (2.1) reads

$$A(z, \xi, \eta) = (\xi - \xi_0)\sigma_3 + \sum_{j=1}^{N} \frac{A_j}{z - z_j}, \quad z_j = \sqrt{2s\eta - k_j^2}, \tag{3.2}$$

where k_j^2, $\xi_0 \in \mathbb{R}$ are the parameters and the functions $A_j = A_j(\xi, \eta)$ are independent of z. In the following, to avoid technical details, we suppose that our system has a finite size, i.e.,

$$0 \le \eta \le L, \;\; k_j^2 > 2sL \quad \Longrightarrow \quad z_j^* = -z_j. \tag{3.3}$$

Proposition 1 *If for $j = 1, \ldots, N$ the matrix-functions $A_j \in su_2$ are such that the set $\{A_j\}_{j=1,\ldots,N}$ is a solution of the system:*

$$\partial_\xi A_j = [z_j\sigma_3 + U_0,\, A_j], \quad \partial_\eta A_j = [\tfrac{\hat\rho}{4z_j},\, A_j], \tag{3.4}$$

$$U_0 = \tfrac{1}{\xi-\xi_0}\left(\sum_{l=1}^{N} A_l - i\theta_\infty\sigma_3\right), \;\; i\theta_\infty\sigma_3 \equiv \operatorname{diag}\sum_{l=1}^{N} A_l, \;\; \hat\rho = 4s\left((\xi-\xi_0)\sigma_3 - \sum_{l=1}^{N}\tfrac{A_l}{z_l}\right), \tag{3.5}$$

where the brackets,$[\cdot,\,\cdot]$, denote the commutator of two matrices; then $\theta_\infty \in \mathbb{R}$, $\partial_\xi\theta_\infty = \partial_\eta\theta_\infty = 0$, the matrix U_0 is off-diagonal, and the functions: E, ρ, and N, which are defined by the Eqs. (2.2) and (3.5), are the solutions of PMB (2.3).

The system (3.4), (3.5) defines the deformation (with the deformation parameters ξ and η) of the coefficients, A_j, of the Eq. (3.1) with the matrix $A(z)$ given by (3.2), which preserves the monodromy group or, more generally, the monodromy data [25] of this equation. This comment "explains" the name, isomonodromy solutions.

To the N-soliton solution of PMB (2.3) corresponds a particular solution of the system (3.4), (3.5): the eigenvalues of all the matrices A_j are equal to $\pm 1/2$. To each soliton corresponds one (and only one) eigenvalue z_l. So, intuitively, one can think of the isomonodromy solutions defined by the system (3.4), (3.5) as an N-wave-like object, although the behavior of the general solution is much more complicated than the N-soliton train.

4 Hamiltonian structure

It is known that the equations which define isomonodromy deformations of the linear ODEs (3.1) where the matrix $A(z)$ is a rational function of z admit a Hamiltonian structure. The Poisson brackets between the matrix elements can be written in the following compact form [23, 24]

$$\{A(z) \overset{\otimes}{,} A(w)\} = [r(z-w),\, A(z)\otimes I + I\otimes A(w)], \tag{4.1}$$

where $r(z-w) = P/(z-w)$ with P the permutation operator on $\mathbb{C}^2\otimes\mathbb{C}^2$. The corresponding Hamiltonians are generated by the so-called τ-function [25]. In our case Eq. (4.1) for the matrix elements, $(A_k)_{ab}$, reads [26]:

$$\{(A_k)_{ab},\, (A_j)_{cd}\} = \delta_{kj}((A_k)_{ad}\delta_{bc} - \delta_{ad}(A_k)_{cb}). \tag{4.2}$$

Proposition 2 *The system (3.4), (3.5) can be written in the Hamiltonian form:*

$$\partial_\xi A_j = \{A_j,\, H_\xi\}, \qquad \partial_\eta A_j = \{A_j,\, H_\eta\}, \tag{4.3}$$

where the dynamics with respect to the variables ξ and η is defined by the following non-autonomous Hamiltonians:

$$H_\xi = \sum_{k=1}^{N} z_k \operatorname{tr}(A_k\sigma_3) + \frac{S^-S^+ + S^+S^-}{\xi-\xi_0}, \quad S^+ = \left(\sum_{k=1}^{N} A_k\right)_{12}, \quad S^- = \left(\sum_{k=1}^{N} A_k\right)_{21}, \tag{4.4}$$

$$H_\eta = \sum_{k=1}^{N} H_k\frac{\partial z_k}{\partial\eta} = \sum_{k=1}^{N}\frac{H_k}{z_k}, \quad H_k = (\xi-\xi_0)\operatorname{tr}(A_k\sigma_3) + \sum_{i\neq k}^{N}\frac{\operatorname{tr}(A_kA_i)}{z_k - z_i}, \; k = 1,\ldots,N. \tag{4.5}$$

The compatibility of the system (4.3) follows from

Proposition 3 $\{H_\xi,\, H_\eta\} = 0, \quad \frac{\partial H_\xi}{\partial\eta} = \frac{\partial H_\eta}{\partial\xi}.$

5 Quantization

We consider the N-particle sector of the Fock space,

$$\mathcal{H}_N = \otimes_{k=1}^N V_k, \tag{5.1}$$

where V_k are left su_2-modules corresponding to the spin m_k representations of su_2, i.e., V_k are isomorphic to $\mathbb{C}^{2m_k+1}$, $m_k \in \mathbb{Z}/2$. Denote by s_k^a, $a = 1, 2, 3$, the linear operators, which generate in V_k these representations ($s_k^a \in su_2$):

$$[s_k^a, s_j^b] = i\delta_{kj}\epsilon^{abc}s_k^c, \quad a, b, c \in \{1, 2, 3\}, \qquad s_k^3\omega_k = m_k\omega_k, \tag{5.2}$$

where ω_k is the highest weight vector. In particular, for the spin 1/2 representation $s_k^a = \frac{1}{2}\sigma_k^a$, where $\sigma_k^1, \sigma_k^2, \sigma_k^3$ are multiplications on the Pauli matrices. The natural extension of s_k^a to $\mathcal{H}_\mathcal{N}$ is denoted by $\hat{s}_k^a$,

$$\hat{s}_k^a(g) = I \otimes \cdots \underbrace{s_k^a(g)}_{k-th} \cdots \otimes I \in \mathrm{End}(\mathcal{H}), \qquad g \in su_2.$$

It is convenient to introduce also the following notations:

$$\hat{s}_k^{\pm} = \hat{s}_k^1 \pm i\hat{s}_k^2, \qquad \hat{S}^3 = \sum_{k=1}^N \hat{s}_k^3, \quad \hat{S}^{\pm} = \sum_{k=1}^N \hat{s}_k^{\pm}.$$

In the Schrödinger picture quantization of the classical observables A_k reads

$$A_k \quad \Longrightarrow \quad \hat{A}_k = i\hbar\begin{pmatrix} \hat{s}_k^3 & \hat{s}_k^+ \\ \hat{s}_k^- & -\hat{s}_k^3 \end{pmatrix}, \qquad (\xi - \xi_0)\sigma_3 \quad \Longrightarrow \quad i\hbar(\xi - \xi_0)\sigma_3 \otimes I$$

The generator of the quantum observables (cf. (3.2)),

$$\hat{A}(z,\xi,\eta) = i\hbar(\xi - \xi_0)\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} + \sum_{j=1}^N \frac{\hat{A}_j}{z - z_j}, \quad z_j = \sqrt{2s\eta - k_j^2}, \tag{5.3}$$

satisfies the following commutator relation (cf. (4.1))

$$[\hat{A}(z) \overset{\otimes}{,} \hat{A}(w)] = i\hbar[r(z-w), \hat{A}(z) \otimes \hat{I} + \hat{I} \otimes \hat{A}(w)], \tag{5.4}$$

where $\hat{I}$ is the matrix with the identity operators on the main diagonal and the classical r-matrix above is the same as in Poisson structure (4.1). The quantum Hamiltonians in the above notations can be written as follows:

$$\hat{H}_\xi = 2(i\hbar)^2 \sum_{j=1}^N z_j \hat{s}_j^3 + \frac{(i\hbar)^2}{\xi - \xi_0}\left(\hat{S}^-\hat{S}^+ + \hat{S}^3\right),$$

$$\hat{H}_k = 2(i\hbar)^2(\xi - \xi_0)\hat{s}_k^3 + 2(i\hbar)^2 \sum_{j\neq k}^N \frac{\hat{s}_k^a \hat{s}_j^a}{z_k - z_j},$$

where in the last equation the summation on the upper-script a is supposed.

The time evolution of the state-vector Ψ is governed by the Schrödinger equation,

$$i\hbar\kappa\frac{\partial\Psi}{\partial\xi} = \hat{H}_\xi\Psi, \tag{5.5}$$

where $\kappa \in \mathbb{R}$ plays a role of the dimension constant, note that our "time-variable" ξ is dimensionless! To be consistent with the classical picture we have to impose on the vector Ψ the system of N more "Schrödinger" equations,

$$i\hbar\kappa\frac{\partial\Psi}{\partial z_k} = \hat{H}_k\Psi, \quad k = 1, \ldots, N, \tag{5.6}$$

which govern the dependence of the state-vector on the parameters z_k and so define the space-like dependence of the state vector. The necessity of the appearance of additional equations, system (5.6)), is clear since we are quantizing the (soliton-) wave-like objects (see the last paragraph in Sec. 3), whilst the Schrödinger equation (5.5) defines only the time dependence of the state-vector.
The system (5.5), (5.6) is compatible. It follows from

Proposition 4 $[\hat{H}_\xi,\, \hat{H}_\eta] = 0, \qquad \partial_\eta \hat{H}_\xi = \partial_\xi \hat{H}_\eta.$

The system (5.5), (5.6) is a generalized system of the Knizhnik-Zamolodchikov equations. In particular, the subsystem (5.6) for $\xi - \xi_0 = 0$ coincides with the classical Knizhnik-Zamolodchikov equations [2].

6 Generalized Knizhnik-Zamolodchikov equations

In previous section we have obtained the following system of generalized Knizhnik-Zamolodchikov equations:

$$\kappa\frac{\partial\Psi}{\partial\xi} = i\hbar\left(2\sum_{j=1}^{N} z_j\hat{s}_j^3 + \frac{\hat{S}^-\hat{S}^+ + \hat{S}^3}{\xi-\xi_0}\right)\Psi, \tag{6.1}$$

$$\kappa\frac{\partial\Psi}{\partial z_k} = i\hbar\left(2(\xi-\xi_0)\hat{s}_k^3 + 2\sum_{j\neq k}^{N}\frac{\hat{s}_k^a\,\hat{s}_j^a}{z_k - z_j}\right)\Psi. \tag{6.2}$$

We have applied the off-shell Bethe ansatz method [10, 11] to construct the general solution of this system. This method has a pure algebraic nature that is why it works not only for the real values of the parameters: ξ, ξ_0, $\hbar$, iz_k, as is assumed in the setting related with the PMB, but also in the case when one allows arbitrary complex values ($z_k \neq z_j$ for $k \neq j$) for these parameters. Thus, the following formulae are valid for complex values of the parameters too.

Since the off-shell Bethe ansatz method has been already presented in the literature [10, 11] and its application to system (6.1), (6.2) requires no any essential modification, here we only write down the final answer.
The vacuum solution reads

$$\Psi_0 = \chi_0\,|0>, \qquad |0>= \otimes_{k=1}^{N}\omega_k,$$

$$\chi_0 = (\xi-\xi_0)^{i\frac{\hbar}{\kappa}\sum\limits_{k=1}^{N} m_k}\prod_{1\leq k<j\leq N}(z_k - z_j)^{2i\frac{\hbar}{\kappa}m_k m_j}\,\exp\left(2i\frac{\hbar}{\kappa}(\xi-\xi_0)\sum_{k=1}^{N} m_k z_k\right),$$

where m_k and ω_k are the highest weights and the highest weight vectors correspondingly (5.2). For each $m \in \mathbb{Z}$ satisfying the inequality, $1 \leq m \leq 2\sum\limits_{k=1}^{N} m_k$, the function

$$\Psi_m(\xi, \vec{z}\,) = \oint\limits_{\Delta_m} \chi_m(\vec{\mu}_m|\xi, \vec{z})\,\hat{\varphi}_m(\vec{\mu}_m|\xi, \vec{z})\,d\mu_1, \ldots, d\mu_m,$$

where $\vec{z} \equiv (z_1, \ldots, z_N)$, $\vec{\mu}_m \equiv (\mu_1, \ldots, \mu_m)$,

$$\chi_m(\vec{\mu}_m|\xi,\vec{z}) = (\xi-\xi_0)^{\frac{i\hbar}{\kappa}\left(\sum\limits_{k=1}^{N} m_k - m\right)} \exp\!\left(\frac{2i\hbar}{\kappa}(\xi-\xi_0)\sum_{k=1}^{N} z_k m_k\right) \prod_{1\le k<j\le N} (z_k - z_j)^{\frac{2i\hbar}{\kappa} m_k m_j}$$

$$\prod_{\substack{1\le\alpha\le m\\ 1\le k\le N}} (z_k-\mu_\alpha)^{-\frac{i\hbar}{2\kappa}m_k} \prod_{1\le\alpha<\beta\le m} (\mu_\alpha-\mu_\beta)^{\frac{i\hbar}{2\kappa}} \exp\!\left(-\frac{i\hbar}{\kappa}(\xi-\xi_0)\sum_{\alpha=1}^{m}\mu_\alpha\right),$$

$$\hat{\varphi}_m(\vec{\mu}_m|\xi,\vec{z}) = \prod_{k=1}^{m} \hat{S}^-(\mu_k|\xi,\vec{z})|0>, \qquad \hat{S}^-(\mu|\xi,\vec{z}) = \sum_{k=1}^{N} \frac{\hat{s}_k^-}{\mu - z_k},$$

and the cycles Δ_m are defined by the following equation

$$\sum_{k=1}^{m} \oint_{\Delta_m} \frac{d}{d\mu_k} \frac{\chi_m(\vec{\mu}_m|\xi,\vec{z})\,\hat{s}_j^-\,\hat{\varphi}_m^k(\vec{\mu}_m|\xi,\vec{z})}{z_j-\mu_k}\, d\mu_1,\ldots,d\mu_m = 0, \;\; \hat{\varphi}_m^k(\vec{\mu}_m|\xi,\vec{z}) = \prod_{j\neq k}^{N} \hat{S}^-(\mu_j|\xi,\vec{z}).$$

The solution, Ψ_m, also should satisfy the quantum version of the second equation in (3.4),

$$(\hbar\hat{S}^3 - \theta_\infty)\Psi_m = 0 \quad\Longrightarrow\quad \hbar(m - \sum_{k=1}^{N} m_k) = \theta_\infty. \tag{6.3}$$

Acknowledgment. This work was done during the authors' stay at Sonderforshungsbereich 288 in May 1995. Since than different versions were presented at seminars in FU Berlin, POMI (St. Petersburg), and INFN (Padova). It is our pleasure to thank A. I. Bobenko, L. D. Faddeev, I. R. Gabitov, M. Karowski, D. A. Korotkin, I. M. Krichever, P. A. Marchetti, M. Matone, H. Nicolai, R. Seiler, R. Schrader, and F. Toppan for discussions and useful comments. One of us (A. V. K.) is grateful to the Max-Planck-Insitut für Gravitationsphysik (Potsdam) for hospitality during the preparation of this work for publication.

References

[1] N. Reshetikhin, *Lett. Math. Phys.* **26**, 167 (1992).

[2] V. G. Knizhnik and A. B. Zamolodchikov, *Nucl. Phys.* **B 247**, 83 (1984).

[3] L. Schlesinger, *J. Reine Angew. Math.* **141**, 96 (1912).

[4] J. Harnad, *Symmetries and integrability of difference equations*, Levi, Decio (ed.) et al., *CRM Proc. Lect. Notes.* **9**, 155 (1996), hep-th/9406078.

[5] V. V. Schechtman and A. N. Varchenko, *Lett. Math. Phys.* **20**, 279 (1990), *Invent. Math.* **106**, 139 (1991).

[6] A. R. Its, *Math. USSR Izv.* **26**, 497 (1986).

[7] A. V. Kitaev, *Zap. Nauchn. Semin. LOMI* **181**, 65 (1990), English translation in *J. Sov. Math., Plenum* **62**, No.2, 2646 (1992).

[8] D. Korotkin and H. Nicolai, *Phys. Rev. Lett.* **74**, 1272 (1995).

[9] D. Korotkin and H. Nicolai, *Nucl. Phys.* **B 475**, 397 (1996).

[10] H. M. Babujian, Correlation function in WZNW model as a Bethe wave function for the Gaudin magnetics, in *Proc. XXIV Int. Symp. Ahrenshoop*, Zeuthen, 1990; *J. Phys. A: Math. Gen.* **26**, 6981 (1993); *J. Phys. A: Math. Gen.* **27**, 7753 (1994).

[11] H. M. Babujian and R. Flume, *Mod. Phys. Lett. A* **9**, 2029 (1994).

[12] S. L. McCall and E. L. Hahn, *Phys. Rev. Lett.* **18**, 908 (1967).

[13] S. L. McCall and E. L. Hahn, *Phys. Rev.* **183**, 457 (1969).

[14] G. L. Lamb, Jr., *Rev. Mod. Phys.* **43**, 99 (1971).

[15] M. J. Ablowitz, D. J. Kaup, and A. C. Newell, *J. Math. Phys.* **15**, 1852 (1974).

[16] I. R. Gabitov, V. E. Zakharov, and A. V. Mikhailov, *Theor. Math. Phys.* **63**, 328 (1985).

[17] S. P. Burtsev, V. E. Zakharov, and A. V. Mikhailov, *Theor. Math. Phys.* **70**, 227 (1987).

[18] D. A. Korotkin and V. B. Matveev, *Leningr. Math. J.* **1**, 379 (1990).

[19] A. V. Rybin, *J. Phys. A: Math. Gen.* **24**, 5235 (1991).

[20] A. V. Kitaev, A. V. Rybin, and J. Timonen, *J. Phys. A: Math. Gen.* **26**, 3583 (1993).

[21] S. P. Burtsev, *Inverse Probl.* **10**, 837 (1994).

[22] S. P. Burtsev and I. R. Gabitov, *Phys. Rev. A* **49**, 2065 (1994).

[23] M. A. Semenov-Tian-Shansky, *Funct. Anal. Appl.* **14**, 146 (1980).

[24] L. D. Faddeev, L. A. Takhtajan, *Hamiltonian Methods in the Theory of Solitons*, Springer-Verlag, Berlin, 1986.

[25] M. Jimbo, T. Miwa, and K. Ueno, *Physica* **2D**, 306 (1981).

[26] M. Jimbo, T. Miwa, Y. Môri, and M. Sato, *Physica* **1D**, 80 (1980).