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Diagonal Crossed Products by Duals of Quasi–Quantum Groups

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Abstract

A two-sided coaction $\delta : \mathcal{M} \to \mathcal{G} \otimes \mathcal{M} \otimes \mathcal{G}$ of a Hopf algebra $(\mathcal{G}, \Delta, \epsilon, S)$ on an associative algebra $\mathcal{M}$ is an algebra map of the form $\delta = (\lambda \otimes \mathrm{id}_{\mathcal{M}}) \circ \rho = (\mathrm{id}_{\mathcal{M}} \otimes \rho) \circ \lambda$, where (λ, ρ) is a commuting pair of left and right $\mathcal{G}$-coactions on $\mathcal{M}$, respectively. Denoting the associated commuting right and left actions of the dual Hopf algebra $\hat{\mathcal{G}}$ on $\mathcal{M}$ by $\triangleleft$ and $\triangleright$, respectively, we define the *diagonal crossed product* $\mathcal{M} \bowtie \hat{\mathcal{G}}$ to be the algebra generated by $\mathcal{M}$ and $\hat{\mathcal{G}}$ with relations given by

$$\varphi\, m = (\varphi_{(1)} \triangleright m \triangleleft \hat{S}^{-1}(\varphi_{(3)}))\, \varphi_{(2)}, \;\; m \in \mathcal{M},\; \varphi \in \hat{\mathcal{G}} \;.$$

We give a natural generalization of this construction to the case where $\mathcal{G}$ is a quasi–Hopf algebra in the sense of Drinfeld and, more generally, also in the sense of Mack and Schomerus (i.e., where the coproduct Δ is non-unital). In these cases our diagonal crossed product will still be an associative algebra structure on $\mathcal{M} \otimes \hat{\mathcal{G}}$ extending $\mathcal{M} \equiv \mathcal{M} \otimes \hat{\mathbf{1}}$, even though the analogue of an ordinary crossed product $\mathcal{M} \rtimes \hat{\mathcal{G}}$ in general is not well defined as an associative algebra.

Applications of our formalism include the field algebra constructions with quasi-quantum group symmetry given by Mack and Schomerus [MS,S] as well as the formulation of Hopf spin chains or lattice current algebras based on truncated quantum groups at roots of unity.

In the case $\mathcal{M} = \mathcal{G}$ and $\lambda = \rho = \Delta$ we obtain an explicit definition of the quantum double $\mathcal{D}(\mathcal{G})$ for quasi–Hopf algebras $\mathcal{G}$, which before had been described in the form of an implicit Tannaka-Krein reconstruction procedure by S. Majid [Ma2]. We prove that $\mathcal{D}(\mathcal{G})$ is itself a (weak) quasi-bialgebra and that any diagonal crossed product $\mathcal{M} \bowtie \hat{\mathcal{G}}$ naturally admits a two-sided $\mathcal{D}(\mathcal{G})$-coaction. In particular, the above mentioned lattice models always admit the quantum double $\mathcal{D}(\mathcal{G})$ as a localized cosymmetry, generalizing results of Nill and Szlachányi [NSz1]. A complete proof that $\mathcal{D}(\mathcal{G})$ is even a (weak) quasi-triangular quasi-Hopf algebra will be given in a separate paper [HN1].

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Contents

1 Introduction and summary of results

During the last decade quantum groups have become the most fashionable candidates describing the symmetry in low dimensional quantum field theories (QFT)[1] or lattice models[2]. Here, in an axiomatic approach by a "symmetry algebra" $\mathcal{G}$ one means a $*$-algebra acting on the Hilbert space of physical states $\mathcal{H}$, such that

- observables and space-time translations commute with $\mathcal{G}$,
- charge creating fields fall into multiplets transforming covariantly under the action of $\mathcal{G}$,
- equivalence classes of irreducible representations of $\mathcal{G}$ are in one-to-one correspondence with the Doplicher-Haag-Roberts (DHR) superselection sectors of the observable algebra $\mathcal{A}$, such that also the fusion rules of $\mathrm{Rep}_{\mathrm{DHR}}\,\mathcal{A}$ and $\mathrm{Rep}\,\mathcal{G}$ coincide.

It is well known that the results of Doplicher-Roberts [DR] characterizing $\mathcal{G}$ as a compact group (or the associated group algebra) break down in low dimensions due to the appearance of braid statistics. It was soon realized that at least for rational theories (i.e. with a finite number of sectors) quantum groups are also ruled out, unless all sectors have integer statistical dimensions (see e.g. [FrKe] for a review or [N2] for a specific discussion of q-dimensions in finite quantum groups).

Based on the theory of quasi-Hopf algebras introduced by Drinfel'd [Dr2], G. Mack and V. Schomerus [MS] have proposed the notion of *weak quasi-Hopf algebras* $\mathcal{G}$ as appropriate symmetry candidates, where "weak" means that the tensor product of two "physical" representations of $\mathcal{G}$ may also contain "unphysical" subrepresentations (i.e. of q-dimension ≤ 0), which have to be discarded. Examples are semisimple quotients of q-deformations of classical groups at $q=$ roots of unity.

In this way non-integer dimensions could successfully be incorporated. The price to pay was that now commutation relations of $\mathcal{G}$-covariant charged fields involve operator valued R-matrices and, more drastically, the operator product expansion for $\mathcal{G}$-covariant multiplets of charged fields involves non-scalar coefficients with values in $\mathcal{G}$. Thus, the analogue of the "would-be" DHR-field algebra $\mathcal{F}$ is no longer algebraically closed. Instead, Mack and Schomerus have proposed a new "covariant product" for charged fields, which does not lead outside of $\mathcal{F}$, but which is no longer associative. In [S] Schomerus has analyzed this scenario somewhat more systematically in the framework of DHR-theory, showing that a weak quasi-Hopf algebra $\mathcal{G}$ and a field "algebra" $\mathcal{F}$ may always be constructed such that the combined algebra $\mathcal{F}\vee\mathcal{G}$ is associative and satisfies all desired properties, except that $\mathcal{F}\subset\mathcal{F}\vee\mathcal{G}$ is only a linear subspace but not a subalgebra. Technically, the reason for this lies in the fact that the dual $\hat{\mathcal{G}}$ of a quasi-Hopf algebra is not an associative algebra.

One should also remark at this point that the above reconstruction of $\mathcal{G}$ from the category of DHR-endomorphisms is not unique. Also, in a more mathematical framework a general Tannaka-Krein like reconstruction theorem for quasi-Hopf algebras has been obtained by S. Majid [M1] and for weak quasi–Hopf algebras by [Hä].

To study quantum symmetries on the lattice in an axiomatic approach, K. Szlachányi and P. Vecsernyes [SzV] have proposed an "amplified" version of the DHR-theory, which also applies to locally finite dimensional lattice models. This setting has been further developed

[1] see [BaWi], [BL], [G], [DPR], [FGV], [FrKe], [MS1,2], [MoRe], [Mü], [PSa1], [ReSm], [Sz,V]

[2] see [AFFS], [AFSV], [AFS], [ByS], [Fa], [FG], [KaS], [NSz1,2], [P], [PSa2], [SzV]

by [NSz1,2], where based on the example of Hopf spin chains the authors proposed the notion of a *universal localized cosymmetry* $\rho : \mathcal{A} \to \mathcal{A} \otimes \mathcal{G}$, incorporating all sectors ρ_I of $\mathcal{A}$ via $\rho_I = (\mathrm{id}_{\mathcal{A}} \otimes \pi_I) \circ \rho,\ \pi_I \in \mathrm{Rep}\,\mathcal{G}$. In the specific example studied by [NSz1,2] $\mathcal{G}$ was given by a quantum double and the cosymmetry ρ was given by a *coaction* of $\mathcal{G}$ on $\mathcal{A}$. Related results have later been obtained for lattice current algebras [AFFS], the later actually being a special case of the Hopf spin chains of [NSz1,2] (see [N1] and Sect. 11.3). The analogue of a DHR–field algebra for these models is now given by the standard crossed product $\mathcal{F} \equiv \mathcal{A} \rtimes \hat{\mathcal{G}}$ [NSz2], where $\hat{\mathcal{G}}$ is the Hopf algebra dual to $\mathcal{G}$.

Now the methods and results of these works were still restricted to ordinary Hopf algebras and therefore to integer dimensions. To formulate lattice current algebras at roots of unity one may of course identify them with the boundary part of lattice Chern-Simons algebras [AGS,AS] defined on a disk. Nevertheless, it remains unclear whether and how for q =root of unity the structural results of [AFFS] survive the truncation to the semi-simple ("physical") quotients. Similarly, the generalizations of the model, the methods and the results of [NSz] to weak quasi quantum groups are by no means obvious. In particular one would like to know whether and in what sense in such models universal localized cosymmetries $\rho : \mathcal{A} \to \mathcal{A} \otimes \mathcal{G}$ still provide coactions and whether $\mathcal{G}$ would still be (an analogue of) a quantum double of a quasi-Hopf algebra, possibly in the sense recently described by Majid [M2].

In this work we present a theory of left, right and two-sided coactions of a (weak) quasi-Hopf algebra $\mathcal{G}$ on an associative algebra $\mathcal{M}$. Based on these structures we then provide a new construction of what we call the *diagonal crossed product* $\mathcal{M} \bowtie \hat{\mathcal{G}}$, which we will show to be the appropriate mathematical structure underlying all constructions discussed above. In particular, $\mathcal{M} \bowtie \hat{\mathcal{G}}$ will always be an associative algebra extending $\mathcal{M} \equiv \mathcal{M} \bowtie \hat{\mathbf{1}}$. On the other hand, the linear subspace $\mathbf{1}_{\mathcal{M}} \bowtie \hat{\mathcal{G}}$ will in general not be a subalgebra of $\mathcal{M} \bowtie \hat{\mathcal{G}}$, unless $\mathcal{G}$ is an ordinary (i.e. coassociative) Hopf algebra.

The basic idea for this construction comes from generalizing the relations defining the quantum double. To this end we start from an algebra $\mathcal{M}$ equipped with a (quasi-)commuting pair of right and left $\mathcal{G}$-coactions, $\rho : \mathcal{M} \to \mathcal{M} \otimes \mathcal{G}$ and $\lambda : \mathcal{M} \to \mathcal{G} \otimes \mathcal{M}$ and denote $\delta_\ell := (\lambda \otimes \mathrm{id}) \circ \rho$ and $\delta_r := (\mathrm{id} \otimes \rho) \circ \lambda$ as the associated equivalent *two-sided coactions.* In the simplest case of $\mathcal{G}$ being an ordinary Hopf algebra and (λ, ρ) being strictly commuting (i.e. $\delta_\ell = \delta_r$) this amounts to providing a commuting pair of left and right Hopf module actions $\triangleright : \hat{\mathcal{G}} \otimes \mathcal{M} \to \mathcal{M}$ (dual to ρ) and $\triangleleft : \mathcal{M} \otimes \hat{\mathcal{G}} \to \mathcal{M}$ (dual to λ) of the dual Hopf algebra $\hat{\mathcal{G}}$ on $\mathcal{M}$. In this case our diagonal crossed product $\mathcal{M} \bowtie \hat{\mathcal{G}}$ is defined to be generated by $\mathcal{M}$ and $\hat{\mathcal{G}}$ as unital subalgebras with commutation relations given by ($\hat{S} : \hat{\mathcal{G}} \to \hat{\mathcal{G}}$ being the antipode)

$$\varphi\, m = (\varphi_{(1)} \triangleright m \triangleleft \hat{S}^{-1}(\varphi_{(3)}))\, \varphi_{(2)},\ \ m \in \mathcal{M},\ \varphi \in \hat{\mathcal{G}}\ . \tag{1.1}$$

Note that for $\mathcal{M} = \mathcal{G}$ and $\rho = \lambda = \Delta$ the coproduct on $\mathcal{G}$, these are the defining relations of the quantum double $\mathcal{D}(\mathcal{G})$ [Dr1], and therefore $\mathcal{G} \bowtie \hat{\mathcal{G}} = \mathcal{D}(\mathcal{G})$. Introducing the "generating matrix"

$$\mathbf{\Gamma} := \sum_\mu e_\mu \otimes e^\mu \in \mathcal{G} \otimes \hat{\mathcal{G}} \subset \mathcal{G} \otimes (\mathcal{M} \bowtie \hat{\mathcal{G}}),$$

where $e_\mu \in \mathcal{G}$ is a basis with dual basis $e^\mu \in \hat{\mathcal{G}}$, Eq. (1.1) is equivalent to

$$\mathbf{\Gamma}\lambda(m) = \rho^{op}(m)\mathbf{\Gamma}, \quad \forall m \in \mathcal{M}. \tag{1.2}$$

Moreover, in this case $\hat{\mathcal{G}} \subset \mathcal{M} \bowtie \hat{\mathcal{G}}$ being a unital subalgebra is equivalent to

$$\begin{aligned} (\epsilon \otimes \mathrm{id})(\mathbf{\Gamma}) &= \mathbf{1} && (1.3)\\ \mathbf{\Gamma}^{13}\mathbf{\Gamma}^{23} &= (\Delta \otimes \mathrm{id})(\mathbf{\Gamma}) && (1.4) \end{aligned}$$

where (1.4) is an identity in $\mathcal{G} \otimes \mathcal{G} \otimes (\mathcal{M} \bowtie \hat{\mathcal{G}})$, the indices denoting the embeddings of tensor factors. We call $\mathbf{\Gamma}$ the *universal normal and coherent $\lambda\rho$-intertwiner* in $\mathcal{G} \otimes (\mathcal{M} \bowtie \hat{\mathcal{G}})$, where normality is the property (1.3) and coherence is the property (1.4). Again, for $\mathcal{M} = \mathcal{G}$ and $\mathcal{M} \bowtie \mathcal{G} = \mathcal{D}(\mathcal{G})$ Eqs. (1.2)-(1.4) are precisely the defining relations for the generating matrix $\mathbf{D} \equiv \mathbf{\Gamma}_{\mathcal{D}(\mathcal{G})}$ of the quantum double (see e.g. [N1, Lem.5.2]).

Inspired by the techniques of [AGS, AS] we show in the main body of this work how to generalize the notion of coherent $\lambda\rho$-intertwiners to the case of (weak) quasi-Hopf algebras $\mathcal{G}$, such that analogues of the Eqs. (1.2)-(1.4) still serve as the defining relations of an associative algebra extending $\mathcal{M} \equiv \mathcal{M} \bowtie \hat{\mathbf{1}}$. We also show that diagonal crossed products may equivalently be modeled on the linear spaces $\mathcal{M} \otimes \hat{\mathcal{G}}$ or $\hat{\mathcal{G}} \otimes \mathcal{M}$ (or – in the weak case – certain subspaces thereof).

The basic model for this generalization is again given by $\mathcal{M} = \mathcal{G}$ with its natural two-sided $\mathcal{G}$-coactions $\delta_\ell := (\Delta \otimes \mathrm{id}) \circ \Delta$ and $\delta_r := (\mathrm{id} \otimes \Delta) \circ \Delta$. In this case our construction provides a definition of the quantum double $\mathcal{D}(\mathcal{G})$ for (weak) quasi-Hopf algebras $\mathcal{G}$. In fact, we show that $\mathrm{Rep}\,\mathcal{D}(\mathcal{G})$ coincides with what has been called the "double of the category" $\mathrm{Rep}\,\mathcal{G}$ in [M2]. Hence our definition provides a concrete realization of the abstract Tannaka-Krein like reconstruction of the quantum double given by [M2]. We also give a proof that $\mathcal{D}(\mathcal{G})$ is a (weak) quasi-bialgebra. In [HN1] we will show, that $\mathcal{D}(\mathcal{G})$ is in fact a (weak) quasi-triangular quasi-Hopf algebra, and there we will also visualize many of our (otherwise almost untraceable) algebraic identities in terms of graphical proofs.

The field algebra construction of [MS,S] may also be described as a diagonal crossed product $\mathcal{M} \bowtie \hat{\mathcal{G}}$ within our formalism by putting $\mathcal{M} = \mathcal{A} \otimes \mathcal{G}$, where $\mathcal{A}$ is the observable algebra. In this case the right $\mathcal{G}$-coaction $\rho : \mathcal{M} \to \mathcal{M} \otimes \mathcal{G}$ is a localized cosymmetry acting only on $\mathcal{A}$, whereas the left $\mathcal{G}$-coaction $\lambda : \mathcal{M} \to \mathcal{G} \otimes \mathcal{M}$ only acts on $\mathcal{G}$, where it is given by the coproduct Δ, see Sect. 11.4 for a rough sketch. A more detailed account of this within an appropriate von-Neumann algebraic framework will be given elsewhere.

The application of our formalism to $\mathcal{G}$-spin quantum chains is given by putting in the previous example also $\mathcal{A} = \mathcal{G}$ and $\rho = \Delta$, in which case $\mathcal{M} \bowtie \hat{\mathcal{G}} \cong \mathcal{G} \rtimes \hat{\mathcal{G}} \ltimes \mathcal{G}$ becomes a *two–sided crossed product*. We take this construction as building block of a quantum chain living on two neighboring sites (carrying the copies of $\mathcal{G}$) joined by a link (carrying the copy of $\hat{\mathcal{G}}$). We show how this construction iterates to provide a local net of associative algebras $\mathcal{A}(I)$ for any lattice interval I bounded by sites. Generalizing the methods of [NSz] we also construct localized coactions of the quantum double $\mathcal{D}(\mathcal{G})$ on such (weak) quasi-Hopf spin chains. Periodic boundary conditions for these models are again described as a diagonal crossed product of the open chain by a copy of $\hat{\mathcal{G}}$ sitting on the link joining the end points. In this way we arrive at a formulation of lattice current algebras at roots of unity by adjusting the transformation rules of [N1] to the quasi-coassociative setting. We expect to have more detailed results also on these models in the near future.

We now describe the plan of this paper. In Part I we prepare our language and develop our basic ideas within a strictly coassociative setting, putting emphasis on a pedagogical presentation of the "read thread". In Sect. 2 we start with reviewing the notions of left and right coactions and crossed products. In Sect. 3 we introduce two-sided coactions $\delta : \mathcal{M} \to \mathcal{G} \otimes \mathcal{M} \otimes \mathcal{G}$ and identify them with pairs (λ, ρ) of commuting left and right coactions. We then define the diagonal crossed product $\mathcal{M}\,{}_\lambda\!\bowtie_\rho \hat{\mathcal{G}}$ and show that it may be identified with the subalgebra of $\hat{\mathcal{G}}\,{}_\lambda\!\ltimes \mathcal{M} \rtimes_\rho \hat{\mathcal{G}}$ generated by $\mathcal{M}$ and the "diagonal" $\hat{\Delta}_{op}(\hat{\mathcal{G}}) \subset \hat{\mathcal{G}} \otimes \hat{\mathcal{G}}$. For $\mathcal{M} = \mathcal{G}$ this gives the quantum double $\mathcal{D}(\mathcal{G})$. Moreover any diagonal crossed product $\mathcal{M} \bowtie \hat{\mathcal{G}}$ canonically admits a two-sided $\mathcal{D}(\mathcal{G})$-coaction.

In Sect. 4 we give examples and applications of our formalism. Section 4.1 discusses the relation of our diagonal crossed products with Majid's notion of *double crossed products* [M3,4]. In Section 4.2 we introduce *two-sided crossed products* $\mathcal{A} \rtimes \hat{\mathcal{G}} \ltimes \mathcal{B}$ as special examples obtained for the case $\mathcal{M} = \mathcal{A} \otimes \mathcal{B}$, where ρ acts trivially on $\mathcal{B}$ and λ acts trivially on $\mathcal{A}$. By putting $\mathcal{A} = \mathcal{B} = \mathcal{G}$ and iterating, these become the building structures of Hopf spin chains described in Section 4.3.

In Sect. 5 we reformulate our constructions using the generating matrix formalism. This leads to the description of diagonal crossed products as being generated by $\mathcal{M}$ and the "matrix entries" of a normal coherent $\lambda\rho$-intertwiner $\mathbf{\Gamma} \in \mathcal{G} \otimes (\mathcal{M} \bowtie \hat{\mathcal{G}})$ subjected to the relations (1.2)-(1.4).

This is the appropriate language to be generalized to the quasi-coassociative setting in Part II. Sect. 6 starts with a short review of Drinfeld's axioms for quasi-Hopf algebras $\mathcal{G}$ and some of their properties. In Sects. 7 and 8 we provide a generalization of the notions of left, right and two-sided $\mathcal{G}$-coactions on an associative algebra $\mathcal{M}$. We also show that up to twist equivalence two-sided $\mathcal{G}$-coactions are in one-to-one correspondence with *quasi-commuting* pairs of (left and right) $\mathcal{G}$-coactions (λ, ρ).

In Sect. 9 we give a representation theoretic interpretation of these concepts by showing that they give rise to left, right and two-sided "actions" of $\mathrm{Rep}\,\mathcal{G}$ on $\mathrm{Rep}\,\mathcal{M}$, respectively. By this we mean that representations γ of $\mathcal{M}$ may be "tensored" with representations π of $\mathcal{G}$ to yield again representations of $\mathcal{M}$, such that an analogue of McLane's natural associativity constraints for monoidal categories is satisfied. This formalism allows to motivate and explain many otherwise almost untraceable algebraic identities in terms of commuting diagrams. Yet, the most technical parts of these proofs are deferred to an Appendix.

In Sect. 10 we proceed to the generalized construction of diagonal crossed products associated with any two-sided $\mathcal{G}$-coaction $\delta : \mathcal{M} \to \mathcal{G} \otimes \mathcal{M} \otimes \mathcal{G}$. Sect. 10.1 gives the associativity proof and states the equivalence of a "left" and a "right" convention for this construction. In Sect. 10.2 we describe these two conventions by the associated generating matrices $\mathbf{L}$ and $\mathbf{R}$, called *left (right) diagonal δ-implementers*. In Sect 10.3 we generalize the notion of coherent $\lambda\rho$-intertwiners $\mathbf{\Gamma}$ to the quasi-coassociative setting and show that they are always in one-to-one correspondence with the above coherent left (or right) diagonal δ-implementers. This provides a proof of our main result in Sect. 10.4, showing that diagonal crossed products are uniquely (up to equivalence) described by a quasi-coassociative version of the relations (1.2)-(1.4), see Theorem II on page 24. In Sect. 11 we provide examples and applications by discussing the quantum double $\mathcal{D}(\mathcal{G})$, the two-sided crossed products $\mathcal{A} \rtimes \hat{\mathcal{G}} \ltimes \mathcal{B}$ and their appearance in quasi-Hopf spin chains and lattice current algebras, and finally a reformulation of the Mack-Schomerus scenario within our formalism.

Finally, in Part III we generalize all results to weak quasi-Hopf algebras $\mathcal{G}$.

In conclusion we point out that most of our algebraic constructions are based on representation categorical concepts and may therefore also be visualized by graphical proofs. We will put more emphasis on this technique in [HN1] when providing further results on the quantum double $\mathcal{D}(\mathcal{G})$.

We also remark that without mentioning explicitly at every instance the (weak, quasi) Hopf algebras $\mathcal{G}$ are always supposed to be finite dimensional. Although we believe that many aspects of our formalism would also carry over to an infinite dimensional setting, we don't consider it worthwhile to discuss this complication at present.

More importantly, in applications to quantum physics one should extend our formalism to incorporate C^*- or von-Neumann algebraic structures, which we will come back to in the near future when discussing our examples in more detail.

Part I

The Coassociative Setting

To strip off all technicalities from the main ideas, in this first part we restrict ourselves to strictly coassociative Hopf algebras $\mathcal{G}$. Throughout by an algebra we will mean an associative unital algebra over $\mathbb{C}$ and unless stated differently all algebra morphisms are supposed to be unit preserving. Given an algebra $\mathcal{M}$, two algebra extensions $\mathcal{M}_1 \supset \mathcal{M}$ and $\mathcal{M}_2 \supset \mathcal{M}$ are called *equivalent*, if there exists an isomorphism $\mathcal{M}_1 \cong \mathcal{M}_2$ restricting to the identity on $\mathcal{M}$. The main result we are going for is given by

Theorem I *Let $(\mathcal{G}, \Delta, \epsilon, S)$ be a finite dimensional Hopf algebra and let (λ, ρ) be a commuting pair of (left and right) $\mathcal{G}$–coactions on an associative algebra $\mathcal{M}$.*

1. *Then there exists a unital associative algebra extension $\mathcal{M}_1 \supset \mathcal{M}$ together with a linear map $\Gamma : \hat{\mathcal{G}} \longrightarrow \mathcal{M}_1$ satisfying the following universal property:*
   $\mathcal{M}_1$ is algebraically generated by $\mathcal{M}$ and $\Gamma(\hat{\mathcal{G}})$ and for any algebra map $\gamma : \mathcal{M} \longrightarrow \mathcal{A}$ into some target algebra $\mathcal{A}$ the relation
   $$\gamma_T(\Gamma(\varphi)) = (\varphi \otimes \mathrm{id})(\mathbf{T}), \quad \varphi \in \hat{\mathcal{G}} \tag{1.5}$$
   provides a one–to–one correspondence between algebra maps $\gamma_T : \mathcal{M}_1 \longrightarrow \mathcal{A}$ extending γ and elements $\mathbf{T} \in \mathcal{G} \otimes \mathcal{A}$ satisfying $(\epsilon \otimes \mathrm{id}_\mathcal{A})(\mathbf{T}) = \mathbf{1}_\mathcal{A}$ and
   $$\mathbf{T}\,\lambda_\mathcal{A}(m) = \rho_\mathcal{A}^{op}(m)\,\mathbf{T}, \quad \forall m \in \mathcal{M} \tag{1.6}$$
   $$\mathbf{T}^{13}\mathbf{T}^{23} = (\Delta \otimes \mathrm{id}_\mathcal{A})(\mathbf{T}), \tag{1.7}$$
   where $\lambda_\mathcal{A}(m) := (\mathrm{id}_\mathcal{G} \otimes \gamma)(\lambda(m))$ and $\rho_\mathcal{A}(m) := (\gamma \otimes \mathrm{id}_\mathcal{G})(\rho(m))$.

2. *If $\mathcal{M} \subset \tilde{\mathcal{M}}_1$ and $\tilde{\Gamma} : \hat{\mathcal{G}} \longrightarrow \tilde{\mathcal{M}}_1$ satisfy the same universality property as in part 1.), then there exists a unique algebra isomorphism $f : \mathcal{M}_1 \longrightarrow \tilde{\mathcal{M}}_1$ restricting to the identity on $\mathcal{M}$, such that $\tilde{\Gamma} = f \circ \Gamma$*

3. *The linear maps*
   $$\mu_L : \hat{\mathcal{G}} \otimes \mathcal{M} \ni (\varphi \otimes m) \mapsto \Gamma(\varphi)m \, \in \mathcal{M}_1 \tag{1.8}$$
   $$\mu_R : \mathcal{M} \otimes \hat{\mathcal{G}} \ni (m \otimes \varphi) \mapsto m\Gamma(\varphi) \, \in \mathcal{M}_1 \tag{1.9}$$
   provide isomorphisms of vector spaces.

Putting $\mathbf{\Gamma} := e_\mu \otimes \Gamma(e^\mu) \in \mathcal{G} \otimes \mathcal{M}_1$ Theorem II implies that $\mathbf{\Gamma}$ itself satisfies the defining relations (1.6) and (1.7), see also (1.2) and (1.4). We call $\mathbf{\Gamma}$ the *universal $\lambda\rho$–intertwiner* in $\mathcal{M}_1$. We emphasize that once being stated Theorem I almost appears trivial. Its true power only arises when generalized to the quasi-coassociative setting in Part II. Note that part 2. of

Theorem I implies that the algebraic structures induced on $\hat{\mathcal{G}} \otimes \mathcal{M}$ and $\mathcal{M} \otimes \hat{\mathcal{G}}$ via $\mu_{L/R}^{-1}$ from $\mathcal{M}_1$ are uniquely fixed. We denote them as

$$\hat{\mathcal{G}}\,{}_\lambda\!\bowtie_\rho\, \mathcal{M} \equiv \mu_L^{-1}(\mathcal{M}_1) \tag{1.10}$$

$$\mathcal{M}\,{}_\lambda\!\bowtie_\rho\, \hat{\mathcal{G}} \equiv \mu_R^{-1}(\mathcal{M}_1). \tag{1.11}$$

and call them *left (right) diagonal crossed products*, respectively. (The right diagonal version (1.11) will be shown to be the one described by Eq. (1.1)).

To actually prove Theorem I we will first construct the algebras $\mathcal{M}\,{}_\lambda\!\bowtie_\rho\, \hat{\mathcal{G}}$ and $\hat{\mathcal{G}}\,{}_\lambda\!\bowtie_\rho\, \mathcal{M}$ explicitly as equivalent extensions of $\mathcal{M}$ in Proposition 3.3 and Corollary 3.7, respectively. The description in terms of (λ, ρ)-intertwiners will then be given in Sect. 5. In order to carefully prepare the much more complicated quasi-coassociative scenario we deliberately present these arguments in rather elementary steps.

2 Coactions and crossed products

To fix our conventions and notations we start with shortly reviewing some basic notions on Hopf module actions, coactions and crossed products. For full textbook treatments see e.g. [A,M3,Sw].

Let $\mathcal{G}$ and $\hat{\mathcal{G}}$ be a dual pair of finite dimensional Hopf algebras. We denote elements of $\mathcal{G}$ by Roman letters $a, b, c, \ldots$ and elements of $\hat{\mathcal{G}}$ by Greek letters $\varphi, \psi, \xi, \ldots$. The units are denoted by $\mathbf{1} \in \mathcal{G}$ and $\hat{\mathbf{1}} \in \hat{\mathcal{G}}$. Identifying $\hat{\hat{\mathcal{G}}} = \mathcal{G}$, the dual pairing $\mathcal{G} \otimes \hat{\mathcal{G}} \to \mathbb{C}$ is written as

$$\langle a|\psi\rangle \equiv \langle \psi|a\rangle \in \mathbb{C}, \quad a \in \mathcal{G}, \psi \in \hat{\mathcal{G}}\ .$$

We denote $\Delta : \mathcal{G} \to \mathcal{G} \otimes \mathcal{G}$ the coproduct, $\epsilon : \mathcal{G} \to \mathbb{C}$ the counit and $S : \mathcal{G} \to \mathcal{G}$ the antipode. Similarly, $\hat{\Delta}, \hat{\epsilon}$ and $\hat{S}$ are the structural maps on $\hat{\mathcal{G}}$. We will use the Sweedler notation $\Delta(a) = a_{(1)} \otimes a_{(2)}$, $(\Delta \otimes \mathrm{id})(\Delta(a)) \equiv (\mathrm{id} \otimes \Delta)(\Delta(a)) = a_{(1)} \otimes a_{(2)} \otimes a_{(3)}$, etc. where the summation symbol and the summation indices are suppressed. Together with $\mathcal{G}$ we have the Hopf algebras $\mathcal{G}_{op}$, $\mathcal{G}^{cop}$ and $\mathcal{G}_{op}^{cop}$, where "op" refers to opposite multiplication and "cop" to opposite comultiplication. Note that the antipode of $\mathcal{G}_{op}$ and $\mathcal{G}^{cop}$ is given by S^{-1} and the antipode of $\mathcal{G}_{op}^{cop}$ by S. Also, $\widehat{\mathcal{G}_{op}} = (\hat{\mathcal{G}})^{cop}$, $\widehat{\mathcal{G}^{cop}} = (\hat{\mathcal{G}})_{op}$ and $\widehat{\mathcal{G}_{op}^{cop}} = (\hat{\mathcal{G}})_{op}^{cop}$.

A right coaction of $\mathcal{G}$ on an algebra $\mathcal{M}$ is an algebra map $\rho : \mathcal{M} \to \mathcal{M} \otimes \mathcal{G}$ satisfying

$$(\rho \otimes \mathrm{id}) \circ \rho \ = \ (\mathrm{id} \otimes \Delta) \circ \rho \tag{2.1}$$

$$(\mathrm{id} \otimes \epsilon) \circ \rho \ = \ \mathrm{id} \tag{2.2}$$

Similarly, a left coaction λ is an algebra map $\lambda : \mathcal{M} \to \mathcal{G} \otimes \mathcal{M}$ satisfying

$$(\mathrm{id} \otimes \lambda) \circ \lambda \ = \ (\Delta \otimes \mathrm{id}) \circ \lambda \tag{2.3}$$

$$(\epsilon \otimes \mathrm{id}) \circ \lambda \ = \ \mathrm{id} \tag{2.4}$$

Obviously, after a permutation of tensor factors $\mathcal{M} \otimes \mathcal{G} \leftrightarrow \mathcal{G} \otimes \mathcal{M}$ a left coaction of $\mathcal{G}$ may always be viewed as a right coaction by $\mathcal{G}^{cop}$ and vice versa. Similarly as for coproducts we will also use the suggestive notations

$$\rho(m) \ = \ m_{(0)} \otimes m_{(1)} \tag{2.5}$$

$$(\rho \otimes \mathrm{id})(\rho(m)) \equiv (\mathrm{id} \otimes \Delta)(\rho(m)) \ = \ m_{(0)} \otimes m_{(1)} \otimes m_{(2)} \tag{2.6}$$

$$\lambda(m) \ = \ m_{(-1)} \otimes m_{(0)} \tag{2.7}$$

$$(\Delta \otimes \mathrm{id}) \circ \lambda \equiv (\mathrm{id} \otimes \lambda)(\lambda(m)) \ = \ m_{(-2)} \otimes m_{(1)} \otimes m_{(0)} \tag{2.8}$$

etc., where again summation indices and a summation symbol are suppressed. In this way we will always have $m_{(i)} \in \mathcal{G}$ for $i \neq 0$ and $m_{(0)} \in \mathcal{M}$. Next, we recall that there is a one-to-one correspondence between right (left) coactions of $\mathcal{G}$ on $\mathcal{M}$ and left (right) Hopf module actions, respectively, of $\hat{\mathcal{G}}$ on $\mathcal{M}$ given for $\psi \in \hat{\mathcal{G}}$ and $m \in \mathcal{M}$ by

$$\begin{aligned} \psi \triangleright m &:= (\mathrm{id} \otimes \psi)(\rho(m)) && (2.9)\\ m \triangleleft \psi &:= (\psi \otimes \mathrm{id})(\lambda(m)) && (2.10) \end{aligned}$$

One easily verifies the defining properties of Hopf module actions, i.e.

$$\begin{array}{rclcrcl} \varphi \triangleright (\psi \triangleright m) &=& (\varphi\psi) \triangleright m &,& (m \triangleleft \varphi) \triangleleft \psi &=& m \triangleleft (\varphi\psi) \\ \hat{\mathbf{1}} \triangleright m &=& m &,& m \triangleleft \hat{\mathbf{1}} &=& m \\ \varphi \triangleright (mn) &=& (\varphi_{(1)} \triangleright m)(\varphi_{(2)} \triangleright n) &,& (mn) \triangleleft \varphi &=& (m \triangleleft \varphi_{(1)})(n \triangleleft \varphi_{(2)}) \\ \varphi \triangleright \mathbf{1}_{\mathcal{M}} &=& \hat{\epsilon}(\varphi)\mathbf{1}_{\mathcal{M}} &,& \mathbf{1}_{\mathcal{M}} \triangleleft \varphi &=& \mathbf{1}_{\mathcal{M}}\hat{\epsilon}(\varphi) \end{array} \qquad (2.11)$$

where $\varphi, \psi \in \hat{\mathcal{G}}$ and $m, n \in \mathcal{M}$. As a particular example we recall the case $\mathcal{M} = \mathcal{G}$ with $\rho = \lambda = \Delta$. In this case we denote the associated left and right actions of $\psi \in \hat{\mathcal{G}}$ on $a \in \mathcal{G}$ by $\psi \rightharpoonup a$ and $a \leftharpoonup \psi$, respectively.

Given a right coaction $\rho : \mathcal{M} \to \mathcal{M} \otimes \mathcal{G}$ with dual left $\hat{\mathcal{G}}$-action $\triangleright$ one defines the (untwisted) crossed product (also called smash product) $\mathcal{M} \rtimes \hat{\mathcal{G}}$ to be the vector space $\mathcal{M} \otimes \hat{\mathcal{G}}$ with associative algebra structure given for $m, n \in \mathcal{M}$ and $\varphi, \psi \in \hat{\mathcal{G}}$ by

$$(m \rtimes \varphi)(n \rtimes \psi) = (m(\varphi_{(1)} \triangleright n) \rtimes \varphi_{(2)}\psi) \qquad (2.12)$$

where we use the notation $m \rtimes \psi$ in place of $m \otimes \psi$ to emphasize the new algebraic structure. Then $\mathbf{1}_{\mathcal{M}} \rtimes \hat{\mathbf{1}}$ is the unit in $\mathcal{M} \rtimes \hat{\mathcal{G}}$ and $m \mapsto (m \rtimes \hat{\mathbf{1}})$, $\varphi \mapsto (\mathbf{1}_{\mathcal{M}} \rtimes \varphi)$ provide unital inclusions $\mathcal{M} \to \mathcal{M} \rtimes \hat{\mathcal{G}}$ and $\hat{\mathcal{G}} \to \mathcal{M} \rtimes \hat{\mathcal{G}}$, respectively. Similarly if $\lambda : \mathcal{M} \to \mathcal{G} \otimes \mathcal{M}$ is a left coaction with dual right action $\triangleleft$ then $\hat{\mathcal{G}} \ltimes \mathcal{M}$ denotes the associative algebra structure on $\hat{\mathcal{G}} \otimes \mathcal{M}$ given by

$$(\varphi \ltimes m)(\psi \ltimes n) = (\varphi\psi_{(1)} \ltimes (m \triangleleft \psi_{(2)})n) \qquad (2.13)$$

containing again $\mathcal{M}$ and $\hat{\mathcal{G}}$ as unital subalgebras. If there are several coactions under consideration we will also write $\mathcal{M} \rtimes_\rho \hat{\mathcal{G}}$ and $\hat{\mathcal{G}} \ltimes_\lambda \mathcal{M}$, respectively. We note that Eq. (2.12) implies that as an algebra

$$\mathcal{M} \rtimes \hat{\mathcal{G}} = \mathcal{M}\hat{\mathcal{G}} = \hat{\mathcal{G}}\mathcal{M} \qquad (2.14)$$

where we have identified $\mathcal{M} \equiv \mathcal{M} \rtimes \hat{\mathbf{1}}$ and $\hat{\mathcal{G}} \equiv \mathbf{1}_{\mathcal{M}} \rtimes \hat{\mathcal{G}}$. In fact using the antipode axioms one easily verifies from (2.12)

$$m \rtimes \varphi = (m \rtimes \hat{\mathbf{1}})(\mathbf{1}_{\mathcal{M}} \rtimes \varphi) = (\mathbf{1}_{\mathcal{M}} \rtimes \varphi_{(2)})((\hat{S}^{-1}(\varphi_{(1)}) \triangleright m) \rtimes \hat{\mathbf{1}}) \qquad (2.15)$$

Similar statements hold in $\hat{\mathcal{G}} \ltimes \mathcal{M}$. More generally we have

Lemma 2.1. *Let $\triangleright : \hat{\mathcal{G}} \otimes \mathcal{M} \to \mathcal{M}$ be a left Hopf module action and let $\mathcal{A}$ be an algebra containing $\mathcal{M}$ and $\hat{\mathcal{G}}$ as unital subalgebras. Then in $\mathcal{A}$ the relation*

$$\varphi m = (\varphi_{(1)} \triangleright m)\varphi_{(2)}, \quad \forall \varphi \in \hat{\mathcal{G}}, \forall m \in \mathcal{M} \qquad (2.16)$$

is equivalent to

$$m\varphi = \varphi_{(2)}(\hat{S}^{-1}(\varphi_{(1)}) \triangleright m), \quad \forall \varphi \in \hat{\mathcal{G}}, \forall m \in \mathcal{M} \qquad (2.17)$$

and if these hold then $\mathcal{M}\hat{\mathcal{G}} = \hat{\mathcal{G}}\mathcal{M} \subset \mathcal{A}$ *is a subalgebra and*

$$\mathcal{M} \rtimes \hat{\mathcal{G}} \ni (m \rtimes \varphi) \mapsto m\varphi \in \mathcal{M}\hat{\mathcal{G}}, \tag{2.18}$$

is an algebra epimorphism.

The proof of Lemma 2.1 is obvious from the antipode axioms and therefore omitted. A similar statement of course holds for the crossed product $\hat{\mathcal{G}} \ltimes \mathcal{M}$. As an application we remark that any left crossed product $\hat{\mathcal{G}} \ltimes_\lambda \mathcal{M}$ can be identified with an associated right crossed product $\mathcal{M} \rtimes_\rho \hat{\mathcal{G}}^{cop}$, where $\hat{\mathcal{G}}^{cop} \equiv \widehat{(\mathcal{G}_{op})}$ and where $\rho : \mathcal{M} \to \mathcal{M} \otimes \mathcal{G}_{op}$ is the right coaction given by

$$\rho = (\mathrm{id} \otimes S^{-1}) \circ \tau_{\mathcal{G},\mathcal{M}} \circ \lambda \tag{2.19}$$

$\tau_{\mathcal{G},\mathcal{M}} : \mathcal{G} \otimes \mathcal{M} \to \mathcal{M} \otimes \mathcal{G}$ being the permutation of tensor factors. In fact we have

Lemma 2.2. *Let* $\lambda : \mathcal{M} \to \mathcal{G} \otimes \mathcal{M}$ *and* $\rho : \mathcal{M} \to \mathcal{M} \otimes \mathcal{G}_{op}$ *be a pair of left and right coactions, respectively, related by (2.19). Then*

$$f(m \rtimes \varphi) := (\hat{\mathbf{1}} \ltimes m)(\varphi \ltimes \mathbf{1}_{\mathcal{M}}) \tag{2.20}$$

defines an algebra isomorphism $f : \mathcal{M} \rtimes_\rho \hat{\mathcal{G}}^{cop} \to \hat{\mathcal{G}} \ltimes_\lambda \mathcal{M}$ *with inverse given by*

$$f^{-1}(\varphi \ltimes m) = (\mathbf{1}_{\mathcal{M}} \rtimes \varphi)(m \rtimes \hat{\mathbf{1}}) \tag{2.21}$$

Proof. Let $\triangleright : \hat{\mathcal{G}}^{cop} \otimes \mathcal{M} \to \mathcal{M}$ and $\triangleleft : \mathcal{M} \otimes \hat{\mathcal{G}} \to \mathcal{M}$ be the left and right actions dual to ρ and λ, respectively. Then (2.19) is equivalent to $\varphi \triangleright m = m \triangleleft \hat{S}^{-1}(\varphi)$ for all $\varphi \in \hat{\mathcal{G}}$ and $m \in \mathcal{M}$. Putting $\mathcal{A} = \hat{\mathcal{G}} \ltimes_\lambda \mathcal{M}$ in Lemma 2.1 we have from (2.13) $m\varphi = \varphi_{(1)}(\hat{S}(\varphi_{(2)}) \triangleright m)$ which is the relation (2.17) with $\hat{\mathcal{G}}$ replaced by $\hat{\mathcal{G}}^{cop}$. Hence, by Lemma 2.1 f is an algebra map whose inverse is easily verified to coincide with (2.21). □

We conclude this introductory part by describing crossed products in terms of the "generating matrix" formalism as advocated by the St. Petersburg school. Our presentation will closely follow the review of [N1]. First we note that since $\mathcal{G}$ is finite dimensional we may identify $\mathrm{Hom}_{\mathbb{C}}(\hat{\mathcal{G}}, V) \cong \mathcal{G} \otimes V$ for any $\mathbb{C}$-vector space V. In particular, the relation

$$T(\varphi) = (\varphi \otimes \mathrm{id})(\mathbf{T}), \quad \forall \varphi \in \hat{G}, \tag{2.22}$$

provides a one-to-one correspondence between algebra maps $T : \hat{\mathcal{G}} \to \mathcal{A}$ into some target algebra $\mathcal{A}$ and elements $\mathbf{T} \in \mathcal{G} \otimes \mathcal{A}$ satisfying

$$\mathbf{T}^{13}\mathbf{T}^{23} = (\Delta \otimes \mathrm{id})(\mathbf{T}) \tag{2.23}$$

where Eq. (2.23) is to be understood as an identity in $\mathcal{G} \otimes \mathcal{G} \otimes \mathcal{A}$, the upper indices indicating the canonical embedding of tensor factors (e.g. $\mathbf{T}^{23} = \mathbf{1}_{\mathcal{G}} \otimes \mathbf{T}$, etc.). Throughout, we will call elements $\mathbf{T} \in \mathcal{G} \otimes \mathcal{A}$ *normal*, if

$$(\epsilon \otimes \mathrm{id})(\mathbf{T}) = \mathbf{1}_{\mathcal{A}}$$

which in Eq. (2.22) is equivalent to $T : \hat{\mathcal{G}} \to \mathcal{A}$ being unit preserving. In what follows, the target algebra $\mathcal{A}$ may always be arbitrary. In the particular case $\mathcal{A} = \mathrm{End}\, V$ we would be talking of representations of $\hat{\mathcal{G}}$ on V, or more generally, as discussed in Lemma 2.4 below, of representations of $\mathcal{M} \rtimes \hat{\mathcal{G}}$ or $\hat{\mathcal{G}} \ltimes \mathcal{M}$, respectively, on V.

Definition 2.3. Let $\lambda : \mathcal{M} \to \mathcal{G} \otimes \mathcal{M}$ be a left coaction and let $\gamma : \mathcal{M} \to \mathcal{A}$ be an algebra map. An *implementer* of λ in $\mathcal{A}$ (with respect to γ) is an element $\mathbf{L} \in \mathcal{G} \otimes \mathcal{A}$ satisfying

$$[\mathbf{1}_{\mathcal{G}} \otimes \gamma(m)]\, \mathbf{L} = \mathbf{L}\, [(\mathrm{id}_{\mathcal{G}} \otimes \gamma)(\lambda(m))] \tag{2.24}$$

for all $m \in \mathcal{M}$. Similarly, an implementer in $\mathcal{A}$ of a right coaction $\rho : \mathcal{M} \to \mathcal{M} \otimes \mathcal{G}$ is an element $\mathbf{R} \in \mathcal{G} \otimes \mathcal{A}$ satisfying

$$\mathbf{R}\, [\mathbf{1}_{\mathcal{G}} \otimes \gamma(m)] = [(\mathrm{id} \otimes \gamma)(\rho^{op}(m))]\, \mathbf{R} \tag{2.25}$$

where $\rho^{op} = \tau_{\mathcal{M} \otimes \mathcal{G}} \circ \rho$.

We remark that Eq. (2.25) suggests that for right coactions one should rather use the convention $\mathbf{R} \in \mathcal{A} \otimes \mathcal{G}$ in place of introducing ρ^{op} as a left coaction of $\mathcal{G}^{cop}$. However, for later purposes it will be more convenient to place the "auxiliary copy" of $\mathcal{G}$ in the definition of implementers always to the left of $\mathcal{M}$. We now have

Lemma 2.4. *Let $\lambda : \mathcal{M} \to \mathcal{G} \otimes \mathcal{M}$ be a left coaction and let $\gamma : \mathcal{M} \to \mathcal{A}$ be an algebra map. Then the relation*

$$\gamma_L(\varphi \ltimes m) := (\varphi \otimes \mathrm{id})(\mathbf{L}) \gamma(m) \tag{2.26}$$

provides a one-to-one correspondence between algebra maps $\gamma_L : \hat{\mathcal{G}} \ltimes_\lambda \mathcal{M} \to \mathcal{A}$ extending γ and normal implementers $\mathbf{L} \in \mathcal{G} \otimes \mathcal{A}$ of λ satisfying

$$\mathbf{L}^{13} \mathbf{L}^{23} = (\Delta \otimes \mathrm{id})(\mathbf{L}) \tag{2.27}$$

Similarly, if $\rho : \mathcal{M} \to \mathcal{M} \otimes \mathcal{G}$ is a right coaction, then the relation

$$\gamma_R(m \rtimes \varphi) = \gamma(m)(\varphi \otimes \mathrm{id})(\mathbf{R}) \tag{2.28}$$

provides a one-to-one correspondence between algebra maps $\gamma_R : \mathcal{M} \rtimes \hat{\mathcal{G}} \to \mathcal{A}$ extending γ and normal implementers $\mathbf{R} \in \mathcal{G} \otimes \mathcal{A}$ of ρ satisfying

$$\mathbf{R}^{13} \mathbf{R}^{23} = (\Delta \otimes \mathrm{id})(\mathbf{R}) \tag{2.29}$$

Proof. Writing $R(\varphi) := (\varphi \otimes \mathrm{id})(\mathbf{R}) \equiv \gamma_R(\mathbf{1}_{\mathcal{M}} \rtimes \varphi) \in \mathcal{A}$ and using $\mathrm{Hom}_{\mathbb{C}}(\hat{\mathcal{G}}, \mathcal{A}) \cong \mathcal{G} \otimes \mathcal{A}$ the relation $\mathbf{R} \leftrightarrow \gamma_R$ is one-to-one. The implementer property (2.25) is then equivalent to $R(\varphi)\gamma(m) = \gamma(\varphi_{(1)} \triangleright m) R(\varphi_{(2)})$ and $\mathbf{R}$ is normal iff $\gamma_{\mathcal{R}}$ is unit preserving. Together with the remarks (2.22) - (2.23) this is further equivalent to γ_R defining an algebra map, similarly as in Lemma 2.1. The argument for γ_L is analogous. $\square$

Next, we note that the equivalence (2.16) $\Leftrightarrow$ (2.17) can be reformulated for implementers as follows

Lemma 2.5. *Under the conditions of Lemma 1.4 denote $\lambda(m) = m_{(-1)} \otimes m_{(0)}$ and $\rho(m) = m_{(0)} \otimes m_{(1)}$ and let $\mathbf{T} \in \mathcal{G} \otimes \mathcal{A}$ be normal. Dropping the symbol γ we then have*
i) $\mathbf{T}$ is an implementer of ρ if and only if

$$[\mathbf{1}_{\mathcal{G}} \otimes m]\, \mathbf{T} = [S^{-1}(m_{(1)}) \otimes \mathbf{1}_{\mathcal{A}}]\, \mathbf{T}\, [\mathbf{1}_{\mathcal{G}} \otimes m_{(0)}], \quad \forall m \in \mathcal{M} \tag{2.30}$$

ii) $\mathbf{T}$ is an implementer of λ if and only if

$$\mathbf{T}\, [\mathbf{1}_{\mathcal{G}} \otimes m] = [\mathbf{1}_{\mathcal{G}} \otimes m_{(0)}]\, \mathbf{T}\, [S^{-1}(m_{(-1)}) \otimes \mathbf{1}_{\mathcal{A}}], \quad \forall m \in \mathcal{M} \tag{2.31}$$

Proof. Suppose $\mathbf{T}$ is an implementer of ρ. Then by (2.25)

$$[S^{-1}(m_{(1)}) \otimes \mathbf{1}_{\mathcal{A}}]\,\mathbf{T}\,[\mathbf{1}_{\mathcal{G}} \otimes m_{(0)}] = [S^{-1}(m_{(2)})m_{(1)} \otimes m_{(0)}]\,\mathbf{T} = [\mathbf{1}_{\mathcal{G}} \otimes m]\,\mathbf{T}$$

by (2.2) and the antipode axioms. Conversely, if $\mathbf{T}$ satisfies (2.30) then

$$\mathbf{T}\,[\mathbf{1}_{\mathcal{G}} \otimes m] = [m_{(2)}S^{-1}(m_{(1)}) \otimes \mathbf{1}_{\mathcal{A}}]\,\mathbf{T}\,[\mathbf{1}_{\mathcal{G}} \otimes m_{(0)}] = [m_{(1)} \otimes m_{(0)}]\,\mathbf{T}$$

proving (2.25). Part ii) is proven analogously. □

3 Two-sided coactions and diagonal crossed products

In Part II we will give a straightforward generalization of the notion of coactions to quasi-Hopf algebras. However, in general an associated notion of a crossed product extension $\mathcal{M} \rtimes \hat{\mathcal{G}}$ will not be well defined as an associative algebra, basically because in the quasi-Hopf case the natural product in $\hat{\mathcal{G}}$ is not associative. We are now going to provide a new construction of what we call a *diagonal crossed product* which will allow to escape this obstruction when generalized to the quasi-Hopf case. Our diagonal crossed products are always based on *two-sided coactions* or, equivalently, on pairs of commuting left and right coactions. These structures are largely motivated by the specific example $\mathcal{M} = \mathcal{G}$ where our methods reproduce the quantum double $\mathcal{D}(\mathcal{G})$.

Definition 3.1. A two-sided coaction of $\mathcal{G}$ on an algebra $\mathcal{M}$ is an algebra map $\delta : \mathcal{M} \to \mathcal{G} \otimes \mathcal{M} \otimes \mathcal{G}$ satisfying

$$\begin{aligned}(\mathrm{id}_{\mathcal{G}} \otimes \delta \otimes \mathrm{id}_{\mathcal{G}}) \circ \delta &= (\Delta \otimes \mathrm{id}_{\mathcal{M}} \otimes \Delta) \circ \delta && (3.1)\\ (\epsilon \otimes \mathrm{id}_{\mathcal{M}} \otimes \epsilon) \circ \delta &= \mathrm{id}_{\mathcal{M}} && (3.2)\end{aligned}$$

An example of a two-sided coaction is given by $\mathcal{M} = \mathcal{G}$ and $\delta := D \equiv (\Delta \otimes \mathrm{id}) \circ \Delta$. More generally let $\lambda : \mathcal{M} \to \mathcal{G} \otimes \mathcal{M}$ and $\rho : \mathcal{M} \to \mathcal{M} \otimes \mathcal{G}$ be a left and a right coaction, respectively. We say that λ and ρ *commute*, if

$$(\lambda \otimes \mathrm{id}) \circ \rho = (\mathrm{id} \otimes \rho) \circ \lambda \tag{3.3}$$

It is straight forward to check that in this case

$$\delta := (\lambda \otimes \mathrm{id}) \circ \rho \equiv (\mathrm{id} \otimes \rho) \circ \lambda \tag{3.4}$$

provides a two-sided coaction. Conversely, we have

Lemma 3.2. *Let $\delta : \mathcal{M} \to \mathcal{G} \otimes \mathcal{M} \otimes \mathcal{G}$ be a two-sided coaction and define $\lambda := (\mathrm{id} \otimes \mathrm{id} \otimes \epsilon) \circ \delta$ and $\rho := (\epsilon \otimes \mathrm{id} \otimes \mathrm{id}) \circ \delta$. Then λ and ρ provide a pair of commuting left and right coactions, respectively, obeying Eq. (3.4)*

The proof Lemma 3.2 is again straightforward and therefore omitted. Similarly as in Eqs. (2.5) - (2.8) it will be useful to introduce the following notations

$$\begin{array}{rcccl} \delta(m) & = & m_{(-1)} \otimes m_{(0)} \otimes m_{(1)} & & \\ (\mathrm{id} \otimes \mathrm{id} \otimes \Delta)(\delta(m)) & \equiv & (\mathrm{id} \otimes \rho \otimes \mathrm{id})(\delta(m)) & = & m_{(-1)} \otimes m_{(0)} \otimes m_{(1)} \otimes m_{(2)} \\ (\Delta \otimes \mathrm{id} \otimes \mathrm{id})(\delta(m)) & \equiv & (\mathrm{id} \otimes \lambda \otimes \mathrm{id})(\delta(m)) & = & m_{(-2)} \otimes m_{(-1)} \otimes m_{(0)} \otimes m_{(1)} \\ (\Delta \otimes \mathrm{id} \otimes \Delta)(\delta(m)) & \equiv & (\mathrm{id} \otimes \delta \otimes \mathrm{id})(\delta(m)) & = & m_{(-2)} \otimes \dots \otimes m_{(2)} \end{array} \tag{3.5}$$

etc., implying again the usual summation conventions. We remark that in the quasi-coassociative setting of Part II the relation between two-sided coactions and pairs (λ, ρ) of left and right coactions becomes more involved, justifying the treatment of two-sided coactions as distinguished objects on their own right also in the present setting. Two-sided coactions may of course be considered as ordinary (right or left) coactions by rewriting them equivalently as

$$\rho_\delta := \tau_{\mathcal{G},(\mathcal{M}\otimes\mathcal{G})} \circ \delta : \mathcal{M} \to \mathcal{M} \otimes (\mathcal{G} \otimes \mathcal{G}^{cop}) \tag{3.6}$$

$$\lambda_\delta := \tau_{(\mathcal{G}\otimes\mathcal{M}),\mathcal{G}} \circ \delta : \mathcal{M} \to (\mathcal{G}^{cop} \otimes \mathcal{G}) \otimes \mathcal{M} \tag{3.7}$$

where the Hopf algebra structures on $\mathcal{G} \otimes \mathcal{G}^{cop}$ and $\mathcal{G}^{cop} \otimes \mathcal{G}$ are the usual tensor product operations induced from the structures on $\mathcal{G}$ and $\mathcal{G}^{cop}$, respectively. Conversely, any left coaction λ (right coaction ρ) may be considered as a two-sided coaction by putting $\delta(m) = \lambda(m) \otimes \mathbf{1}_\mathcal{G}$ or $\delta(m) = \mathbf{1}_\mathcal{G} \otimes \rho(m)$, respectively.

Next, in view of Lemma 3.2 we also have a one-to-one correspondence between two-sided coactions δ of $\mathcal{G}$ on $\mathcal{M}$ and pairs of mutually commuting left and right Hopf module actions, $\triangleright$ and $\triangleleft$, of $\hat{\mathcal{G}}$ on $\mathcal{M}$, the relation being given by

$$(\varphi \otimes \mathrm{id} \otimes \psi)(\delta(m)) = \psi \triangleright m \triangleleft \varphi, \tag{3.8}$$

where $\varphi, \psi \in \hat{\mathcal{G}}$ and $m \in \mathcal{M}$. This allows to construct as a new algebra the *right diagonal crossed product* $\mathcal{M} \bowtie \hat{\mathcal{G}}$ as follows.

Proposition 3.3. *Let $\delta = (\lambda \otimes \mathrm{id}_\mathcal{G}) \circ \rho = (\mathrm{id}_\mathcal{G} \otimes \rho) \circ \lambda$ be a two-sided coaction of $\mathcal{G}$ on $\mathcal{M}$ and let $\triangleright$ and $\triangleleft$ be the associated commuting pair of left and right actions of $\hat{\mathcal{G}}$ on $\mathcal{M}$. Define on $\mathcal{M} \otimes \hat{\mathcal{G}}$ the product*

$$(m \bowtie \varphi)(n \bowtie \psi) := (m(\varphi_{(1)} \triangleright n \triangleleft \hat{S}^{-1}(\varphi_{(3)})) \bowtie \varphi_{(2)}\psi) \tag{3.9}$$

where we write $(m \bowtie \varphi)$ in place of $(m \otimes \varphi)$ to distinguish the new algebraic structure. Then with this product $\mathcal{M} \otimes \hat{\mathcal{G}}$ becomes an associative algebra with unit $(\mathbf{1}_\mathcal{M} \bowtie \hat{\mathbf{1}})$ containing $\mathcal{M} \equiv \mathcal{M} \bowtie \hat{\mathbf{1}}$ and $\hat{\mathcal{G}} \equiv \mathbf{1}_\mathcal{M} \bowtie \hat{\mathcal{G}}$ as unital subalgebras.

Proof. For $m, m', n \in \mathcal{M}$ and $\varphi, \psi, \xi \in \hat{\mathcal{G}}$ we compute

$$\begin{aligned}
&\big[(m \bowtie \varphi)(m' \bowtie \psi)\big](n \bowtie \xi) = \big[m(\varphi_{(1)} \triangleright m' \triangleleft \hat{S}^{-1}(\varphi_{(3)})) \bowtie \varphi_{(2)}\psi\big](n \bowtie \xi)\\
&= \big[m(\varphi_{(1)} \triangleright m' \triangleleft \hat{S}^{-1}(\varphi_{(5)}))(\varphi_{(2)}\psi_{(1)} \triangleright n \triangleleft \hat{S}^{-1}(\psi_{(3)})\hat{S}^{-1}(\varphi_{(4)}))\big] \bowtie (\varphi_{(3)}\psi_{(2)}\xi)\\
&= m\big[\varphi_{(1)} \triangleright [m'(\psi_{(1)} \triangleright n \triangleleft \hat{S}^{-1}(\psi_{(3)}))] \triangleleft \hat{S}^{-1}(\varphi_{(3)})\big] \bowtie (\varphi_{(2)}\psi_{(2)}\xi)\\
&= (m \bowtie \varphi)\big[(m' \bowtie \psi)(n \bowtie \xi)\big].
\end{aligned}$$

which proves the associativity. The remaining statements follow trivially from $\varphi \triangleright \mathbf{1}_\mathcal{M} = \mathbf{1}_\mathcal{M} \triangleleft \varphi = \hat{\epsilon}(\varphi)\mathbf{1}_\mathcal{M}$ and the counit axioms. □

We emphasize that while Proposition 3.3 still is almost trivial as it stands, its true power only appears when generalized to quasi-Hopf algebras $\mathcal{G}$, which will be done in Part II.

Definition 3.4. Under the setting of Proposition 3.3 we define the *right diagonal crossed product* $\mathcal{M} \bowtie_\delta \hat{\mathcal{G}} \equiv \mathcal{M}\,{}_\lambda\!\bowtie_\rho \hat{\mathcal{G}}$ to be the vector space $\mathcal{M} \otimes \hat{\mathcal{G}}$ with associative multiplication structure (3.9).

Left diagonal crossed products will be constructed in Corollary 3.7 below. In cases where the two-sided coaction δ is unambiguously understood from the context we will also write $\mathcal{M} \bowtie \hat{\mathcal{G}}$. We emphasize already at this place that in Part II not every two-sided coaction will be given as $\delta = (\lambda \otimes \mathrm{id}_{\mathcal{G}}) \circ \rho$ (or $\delta = (\mathrm{id}_{\mathcal{G}} \otimes \rho) \circ \lambda$), in which case the notations $\mathcal{M} \bowtie_\delta \hat{\mathcal{G}}$ and $\mathcal{M}{}_\lambda\!\bowtie_\rho \hat{\mathcal{G}}$ will denote different (although still equivalent) extensions of $\mathcal{M}$. Here we freely use either one of them. If $\delta = \mathbf{1}_{\mathcal{G}} \otimes \rho$ or $\delta = \lambda \otimes \mathbf{1}_{\mathcal{G}}$ for some left coaction λ (right coaction ρ) then $\mathcal{M} \bowtie_\delta \hat{\mathcal{G}} = \mathcal{M} \rtimes_\rho \hat{\mathcal{G}}$ or $\mathcal{M} \bowtie_\delta \hat{\mathcal{G}} = \hat{\mathcal{G}} \ltimes_\lambda \mathcal{M}$, respectively, where in the later case we have to invoke the isomorphism (2.21). More generally, $\mathcal{M}{}_\lambda\!\bowtie_\rho \hat{\mathcal{G}}$ may be identified as a subalgebra of $\hat{\mathcal{G}} \ltimes_\lambda (\mathcal{M} \rtimes_\rho \hat{\mathcal{G}}) \equiv (\hat{\mathcal{G}} \ltimes_\lambda \mathcal{M}) \rtimes_\rho \hat{\mathcal{G}}$ using the injective algebra map

$$\begin{aligned} \mathcal{M}{}_\lambda\!\bowtie_\rho \hat{\mathcal{G}} \ni (m \bowtie \varphi) &\mapsto (\hat{\mathbf{1}} \ltimes m \rtimes \hat{\mathbf{1}})(\varphi_{(2)} \ltimes \mathbf{1}_{\mathcal{M}} \rtimes \varphi_{(1)}) \\ &\equiv \left[\varphi_{(2)} \ltimes (m \triangleleft \varphi_{(3)}) \rtimes \varphi_{(1)}\right] \in \hat{\mathcal{G}} \ltimes_\lambda \mathcal{M} \rtimes_\rho \hat{\mathcal{G}} \end{aligned} \tag{3.10}$$

which we leave to the reader to check. Eq. (3.10) also motivates our choice of calling the crossed product $\mathcal{M} \bowtie \hat{\mathcal{G}}$ "diagonal".

In the case $\mathcal{M} = \mathcal{G}$ and $\delta := D \equiv (\Delta \otimes \mathrm{id}) \circ \Delta$ the formula (3.9) coincides with the multiplication rule in the quantum double $\mathcal{D}(\mathcal{G})$ [Dr1,M3], i.e.

$$\mathcal{D}(\mathcal{G}) = \mathcal{G} \bowtie_D \hat{\mathcal{G}} \tag{3.11}$$

It is well known, that $\mathcal{D}(\mathcal{G})$ is itself again a Hopf algebra with coproduct Δ_D given by

$$\Delta_D(a \bowtie_D \varphi) = (a_{(1)} \bowtie_D \varphi_{(2)}) \otimes (a_{(2)} \bowtie_D \varphi_{(1)}) \tag{3.12}$$

where $a \in \mathcal{G}$ and $\varphi \in \hat{\mathcal{G}}$. It turns out that this result generalizes to diagonal crossed products as follows

Proposition 3.5. *Let $\delta : \mathcal{M} \to \mathcal{G} \otimes \mathcal{M} \otimes \mathcal{G}$ be a two-sided coaction. Then $\mathcal{M} \bowtie \hat{\mathcal{G}}$ admits a commuting pair of coactions $\lambda_D : \mathcal{M} \bowtie \hat{\mathcal{G}} \to \mathcal{D}(\mathcal{G}) \otimes (\mathcal{M} \bowtie \hat{\mathcal{G}})$ and $\rho_D : \mathcal{M} \bowtie \hat{\mathcal{G}} \to (\mathcal{M} \bowtie \hat{\mathcal{G}}) \otimes \mathcal{D}(\mathcal{G})$ given by*

$$\lambda_D(m \bowtie \varphi) = (m_{(-1)} \bowtie_D \varphi_{(2)}) \otimes (m_{(0)} \bowtie \varphi_{(1)}) \tag{3.13}$$

$$\rho_D(m \bowtie \varphi) = (m_{(0)} \bowtie_D \varphi_{(2)}) \otimes (m_{(1)} \bowtie \varphi_{(1)}) \tag{3.14}$$

where elements in $\mathcal{D}(\mathcal{G})$ are written as $(a \bowtie_D \varphi)$, $a \in \mathcal{G}$, $\varphi \in \hat{\mathcal{G}}$, and where we have used the notation (2.5)-(2.8).

Proof. In view of (3.12) the comodule axioms and the commutativity (3.4) are obvious. We are left to prove that λ_D and ρ_D provide algebra maps. To this end we use the following identities obviously holding for any two-sided coaction and all $m \in \mathcal{M}, \varphi, \psi \in \hat{\mathcal{G}}$

$$\rho(\varphi \triangleright m \triangleleft \psi) \;=\; m_{(0)} \triangleleft \psi \otimes (\varphi \rightharpoonup m_{(1)}) \tag{3.15}$$

$$m_{(0)} \otimes (m_{(1)} \leftharpoonup \psi) \;=\; (\psi \triangleright m_{(0)}) \otimes m_{(1)} \tag{3.16}$$

With this we now compute

$$\begin{aligned} &\rho_D(m \bowtie \varphi)\rho_D(n \bowtie \psi) = \\ &= \left[m_{(0)}(\varphi_{(4)} \triangleright n_{(0)} \triangleleft \hat{S}^{-1}(\varphi_{(6)})) \bowtie \varphi_{(5)}\psi_{(2)}\right] \otimes \left[m_{(1)}(\varphi_{(1)} \rightharpoonup n_{(1)} \leftharpoonup \hat{S}^{-1}(\varphi_{(3)})) \bowtie_D \varphi_{(2)}\psi_{(1)}\right] \\ &= \left[m_{(0)}(n_{(0)} \triangleleft \hat{S}^{-1}(\varphi_{(4)})) \bowtie \varphi_{(3)}\psi_{(2)}\right] \otimes \left[m_{(1)}(\varphi_{(1)} \rightharpoonup n_{(1)}) \bowtie_D \varphi_{(2)}\psi_{(1)}\right] \\ &= \left[m_{(0)}(\varphi_{(1)} \triangleright n \triangleleft \hat{S}^{-1}(\varphi_{(4)}))_{(0)} \bowtie \varphi_{(3)}\psi_{(2)}\right] \otimes \left[m_{(1)}(\varphi_{(1)} \triangleright n \triangleleft \hat{S}^{-1}(\varphi_{(4)}))_{(1)} \bowtie_D \varphi_{(2)}\psi_{(1)}\right] \\ &= \rho_D\left[m(\varphi_{(1)} \triangleright n \triangleleft \hat{S}^{-1}(\varphi_{(3)})) \bowtie \varphi_{(2)}\psi\right] \end{aligned}$$

where we have used (3.9) in the first equation, (3.16) and the antipode axioms in the second equation and (3.15) in the third equation. Hence ρ_D provides an algebra map. The argument for λ_D is analogous. $\square$

Before presenting further examples of diagonal crossed products let us recall the well known Hopf algebra identity $\mathcal{D}(\hat{\mathcal{G}}) = \mathcal{D}(\mathcal{G})^{cop}$ where the algebra isomorphism is given by

$$\mathcal{D}(\hat{\mathcal{G}}) \ni (\varphi \bowtie_{\hat{D}} a) \mapsto (\mathbf{1} \bowtie_D \varphi)(a \bowtie_D \hat{\mathbf{1}}) \in \mathcal{D}(\mathcal{G}) \tag{3.17}$$

Again this generalizes to diagonal crossed products in the sense that they may equivalently be modeled on the vector space $\hat{\mathcal{G}} \otimes \mathcal{M}$. To see this, we first note that there is an analogue of Lemma 2.1

Lemma 3.6. *Let $\mathcal{A}$ be an algebra containing $\mathcal{M}$ and $\hat{\mathcal{G}}$ as unital subalgebras and let $\triangleright$ and $\triangleleft$ be a commuting pair of left and right Hopf module actions of $\hat{\mathcal{G}}$ on $\mathcal{M}$. Then in $\mathcal{A}$ the relations*

$$\varphi m = [\varphi_{(1)} \triangleright m \triangleleft \hat{S}^{-1}(\varphi_{(3)})]\varphi_{(2)}, \quad \forall \varphi \in \hat{\mathcal{G}}, \forall m \in \mathcal{M} \tag{3.18}$$

are equivalent to

$$m\varphi = \varphi_{(2)}[\hat{S}^{-1}(\varphi_{(1)}) \triangleright m \triangleleft \varphi_{(3)}], \quad \forall \varphi \in \hat{\mathcal{G}}, \forall m \in \mathcal{M} \tag{3.19}$$

and if these hold then $\mathcal{M}\hat{\mathcal{G}} = \hat{\mathcal{G}}\mathcal{M} \subset \mathcal{A}$ is a subalgebra and $\mathcal{M} \bowtie \hat{\mathcal{G}} \ni (m \bowtie \varphi) \mapsto m\varphi \in \mathcal{M}\hat{\mathcal{G}}$ is an algebra epimorphism.

The proof of Lemma 3.6 is again straightforward from the antipode axioms and is left to the reader. Similarly as in Lemma 2.2 we now conclude

Corollary 3.7. *Under the setting of Proposition 3.3 define $\hat{\mathcal{G}} \bowtie \mathcal{M} \equiv \hat{\mathcal{G}} \bowtie_\delta \mathcal{M} \equiv \hat{\mathcal{G}}\,{}_\lambda\!\bowtie_\rho \mathcal{M}$ to be the vector space $\hat{\mathcal{G}} \otimes \mathcal{M}$ with multiplication rule*

$$(\varphi \bowtie m)(\psi \bowtie n) := \varphi\psi_{(2)} \bowtie (\hat{S}^{-1}(\psi_{(1)}) \triangleright m \triangleleft \psi_{(3)})n \tag{3.20}$$

Then $\hat{\mathcal{G}} \bowtie \mathcal{M}$ is an associative algebra and

$$\hat{\mathcal{G}} \bowtie \mathcal{M} \ni \varphi \bowtie m \mapsto (\mathbf{1}_{\mathcal{M}} \bowtie \varphi)(m \bowtie \hat{\mathbf{1}}) \equiv \varphi_{(1)} \triangleright m \triangleleft \hat{S}^{-1}(\varphi_{(3)}) \bowtie \varphi_{(2)} \in \mathcal{M} \bowtie \hat{\mathcal{G}} \tag{3.21}$$

provides an isomorphism restricting to the identity on $\mathcal{M}$ with inverse given by

$$\mathcal{M} \bowtie \hat{\mathcal{G}} \ni m \bowtie \varphi \mapsto (\hat{\mathbf{1}} \bowtie m)(\varphi \bowtie \mathbf{1}_{\mathcal{M}}) \equiv \varphi_{(2)} \bowtie \left(\hat{S}^{-1}(\varphi_{(1)}) \triangleright m \triangleleft \varphi_{(3)}\right) \in \hat{\mathcal{G}} \bowtie \mathcal{M} \tag{3.22}$$

Here we have used the same symbol $\bowtie$ on either side in order not to overload the notation. The reader is invited to check that in the case $\mathcal{M} = \mathcal{G}$ and $\delta = (\Delta \otimes \mathrm{id}) \circ \Delta$ we recover $\hat{\mathcal{G}} \bowtie_\delta \mathcal{M} = \mathcal{D}(\hat{\mathcal{G}})$. Also note that after a trivial permutation of tensor factors the multiplication rule (3.20) implies

$$(\hat{\mathcal{G}} \bowtie_\delta \mathcal{M})_{op} = \mathcal{M}_{op} \bowtie_{\delta_{op}} \hat{\mathcal{G}}^{cop}_{op}\ , \tag{3.23}$$

where "op" refers to the algebra with opposite multiplication and where $\delta_{op} := \delta^{321}$. We propose to call $\hat{\mathcal{G}} \bowtie \mathcal{M}$ the *left diagonal* and $\mathcal{M} \bowtie \hat{\mathcal{G}}$ the *right diagonal* crossed product.

4 Examples and applications

4.1 Double crossed products

In this Subsect. we relate our diagonal crossed product with the double crossed product construction of Majid [M3,4]. Here we adopt the version [M3, Thm.7.2.3], according to which a bialgebra $\mathcal{B}$ is a double crossed product, written as

$$\mathcal{B} = \mathcal{M} \bowtie_{double} \mathcal{H}, \tag{4.1}$$

iff $\mathcal{M}$ and $\mathcal{H}$ are sub–bialgebras of $\mathcal{B}$ such that the multiplication map

$$\mu : \mathcal{M} \otimes \mathcal{H} \ni m \otimes h \mapsto mh \in \mathcal{B}$$

provides an isomorphism of coalgebras. In this case the bialgebras $\mathcal{M}$ and $\mathcal{H}$ become a *matched pair* with mutual actions $->: \mathcal{H} \otimes \mathcal{M} \to \mathcal{M}$ and $<- \ : \mathcal{H} \otimes \mathcal{M} \to \mathcal{H}$ given by

$$\begin{aligned} h -> m &:= (\mathrm{id}_{\mathcal{M}} \otimes \epsilon_{\mathcal{H}})(\mu^{-1}(hm)) && (4.2)\\ h <- m &:= (\epsilon_{\mathcal{M}} \otimes \mathrm{id}_{\mathcal{H}})(\mu^{-1}(hm)), && (4.3) \end{aligned}$$

see [M3, Chap.7.2] for more details. The guiding example is again given by the quantum double satisfying

$$\mathcal{D}(\mathcal{G}) = \mathcal{G} \bowtie_{double} \hat{\mathcal{G}}^{cop}. \tag{4.4}$$

More generally, any diagonal crossed product $\mathcal{M}\,{}_{\lambda}\!\bowtie_{\rho} \hat{\mathcal{G}}$ becomes a double crossed product

$$\mathcal{M}\,{}_{\lambda}\!\bowtie_{\rho} \hat{\mathcal{G}} = \mathcal{M} \bowtie_{double} \hat{\mathcal{G}}^{cop} \tag{4.5}$$

provided $\mathcal{M}$ is equipped with a bialgebra structure $\Delta_{\mathcal{M}}$, $\epsilon_{\mathcal{M}}$ such that the tensor product coalgebra structures

$$\begin{aligned} \Delta_{\mathcal{B}}(m \bowtie \psi) &:= (m^{(1)} \bowtie \psi_{(2)}) \otimes (m^{(2)} \bowtie \psi_{(1)}) && (4.6)\\ \epsilon_{\mathcal{B}}(m \bowtie \psi) &:= \epsilon_{\mathcal{M}}(m)\hat{\epsilon}(\psi) && (4.7) \end{aligned}$$

give algebra maps $\Delta_{\mathcal{B}} : \mathcal{B} \to \mathcal{B} \otimes \mathcal{B}$ and $\epsilon_{\mathcal{B}} : \mathcal{B} \to \mathbb{C}$ (here we denote $\Delta(m) \equiv m^{(1)} \otimes m^{(2)}$). In this way all *generalized quantum doubles* $\mathcal{M} \bowtie_{double} \hat{\mathcal{G}}^{cop}$ in the sense of [M3, Ex.7.2.7] are as algebras actually diagonal crossed products in our sense. These may be described in terms of the bialgebra homomorphism $\kappa : \mathcal{M} \to \mathcal{G}$ induced by the Hopf skew-pairing $\mathcal{M} \otimes \hat{\mathcal{G}}^{cop} \to \mathbb{C}$ required in [M3, Chap. 7.2], which gives rise to the commuting pair of $\mathcal{G}$-coactions

$$\lambda := (\kappa \otimes \mathrm{id}_{\mathcal{M}}) \circ \Delta_{\mathcal{M}} \qquad , \qquad \rho := (\mathrm{id}_{\mathcal{M}} \otimes \kappa) \circ \Delta_{\mathcal{M}}. \tag{4.8}$$

For these models the "matched pair" actions (4.2) and (4.3) are given by the coadjoint actions

$$\begin{aligned} \psi -> m &= \psi_{(1)} \triangleright m \triangleleft \hat{S}^{-1}(\psi_{(2)}) && (4.9)\\ \psi <- m &= \epsilon_{\mathcal{M}}(\psi_{(1)} \triangleright m \triangleleft \hat{S}^{-1}(\psi_{(3)}))\, \psi_{(2)} \\ &= \hat{S}^{-1}(\kappa(m^{(1)})) \rightharpoonup \psi \leftharpoonup \kappa(m^{(2)}). && (4.10) \end{aligned}$$

Also note that the coactions (4.8) satisfy the compatibility condition

$$(\rho \otimes \mathrm{id}_{\mathcal{M}}) \circ \Delta_{\mathcal{M}} = (\mathrm{id}_{\mathcal{M}} \otimes \lambda) \circ \Delta_{\mathcal{M}} \tag{4.11}$$

and the homomorphism $\kappa : \mathcal{M} \to \mathcal{G}$ may be recovered as

$$\kappa = (\mathrm{id}_{\mathcal{G}} \otimes \epsilon_{\mathcal{M}}) \circ \lambda = (\epsilon_{\mathcal{M}} \otimes \mathrm{id}_{\mathcal{G}}) \circ \rho. \tag{4.12}$$

Conversely, we have

Proposition 4.1. *Let $(\mathcal{M}, \Delta_{\mathcal{M}}, \epsilon_{\mathcal{M}})$ be a bialgebra and let (λ, ρ) be a commuting pair of (left and right) $\mathcal{G}$-coactions on $\mathcal{M}$. Then the compatibility condition (4.11) is equivalent to*

$$(\mathrm{id}_{\mathcal{G}} \otimes \epsilon_{\mathcal{M}}) \circ \lambda = (\epsilon_{\mathcal{M}} \otimes \mathrm{id}_{\mathcal{G}}) \circ \rho =: \kappa \tag{4.13}$$

being a bialgebra homomorphism $\kappa : \mathcal{M} \to \mathcal{G}$ which satisfies Eq. (4.8). If these conditions are satisfied, then the diagonal crossed product $\mathcal{M}_{\lambda}\bowtie_{\rho} \hat{\mathcal{G}}$ is also a double crossed product with respect to the bialgebra structure (4.6), (4.7), i.e. in this case we have

$$\mathcal{M}_{\lambda}\bowtie_{\rho} \hat{\mathcal{G}} = \mathcal{M} \bowtie_{double} \hat{\mathcal{G}}^{cop}.$$

Proof. Applying $(\epsilon_{\mathcal{M}} \otimes \mathrm{id}_{\mathcal{G}} \otimes \epsilon_{\mathcal{M}})$ to (4.11) proves that the two expressions in (4.13) define the same algebra map $\kappa := \mathcal{M} \to \mathcal{G}$. Clearly, we have $\epsilon_{\mathcal{G}} \circ \kappa = \epsilon_{\mathcal{M}}$. To prove (4.8) we compute

$$\begin{aligned}
(\kappa \otimes \mathrm{id}_{\mathcal{M}}) \circ \Delta_{\mathcal{M}} &= (\mathrm{id}_{\mathcal{G}} \otimes \epsilon_{\mathcal{M}} \otimes \mathrm{id}_{\mathcal{M}}) \circ (\lambda \otimes \mathrm{id}_{\mathcal{M}}) \circ \Delta_{\mathcal{M}} \\
&= (\epsilon_{\mathcal{M}} \otimes \mathrm{id}_{\mathcal{G}} \otimes \epsilon_{\mathcal{M}} \otimes \mathrm{id}_{\mathcal{M}}) \circ (\mathrm{id}_{\mathcal{M}} \otimes \lambda \otimes \mathrm{id}_{\mathcal{M}}) \circ \Delta_{\mathcal{M}}^{(2)} \\
&= (\epsilon_{\mathcal{M}} \otimes \mathrm{id}_{\mathcal{G}} \otimes \epsilon_{\mathcal{M}} \otimes \mathrm{id}_{\mathcal{M}}) \circ (\rho \otimes \mathrm{id}_{\mathcal{M}} \otimes \mathrm{id}_{\mathcal{M}}) \circ \Delta_{\mathcal{M}}^{(2)} \\
&= (\epsilon_{\mathcal{M}} \otimes \mathrm{id}_{\mathcal{G}} \otimes \mathrm{id}_{\mathcal{M}}) \circ (\rho \otimes \mathrm{id}_{\mathcal{M}}) \circ \Delta_{\mathcal{M}} \\
&= (\epsilon_{\mathcal{M}} \otimes \lambda) \circ \Delta_{\mathcal{M}} = \lambda,
\end{aligned}$$

where $\Delta_{\mathcal{M}}^{(2)} = (\Delta_{\mathcal{M}} \otimes \mathrm{id}) \circ \Delta_{\mathcal{M}} = (\mathrm{id} \otimes \Delta_{\mathcal{M}}) \circ \Delta_{\mathcal{M}}$ and where we have used (4.11) in the third and the fifth line. The identity $\rho = (\mathrm{id} \otimes \kappa) \circ \Delta_{\mathcal{M}}$ is proven similarly. Finally, $\kappa : \mathcal{M} \to \mathcal{G}$ is a bialgebra map, since

$$\begin{aligned}
(\kappa \otimes \kappa) \circ \Delta_{\mathcal{M}} &= (\mathrm{id}_{\mathcal{G}} \otimes \kappa) \circ \lambda = (\mathrm{id}_{\mathcal{G}} \otimes \mathrm{id}_{\mathcal{G}} \otimes \epsilon_{\mathcal{M}}) \circ (\mathrm{id}_{\mathcal{G}} \otimes \lambda) \circ \lambda \\
&= (\Delta_{\mathcal{G}} \otimes \epsilon_{\mathcal{M}}) \circ \lambda = \Delta_{\mathcal{G}} \circ \kappa.
\end{aligned}$$

by the $\mathcal{G}$-coaction property for λ. According to the results of [M3, Sect. 7.2] this implies that the "generalized quantum double" $\mathcal{M} \bowtie_{double} \hat{\mathcal{G}}^{cop}$ as an algebra coincides with the diagonal crossed product $\mathcal{M}_{\lambda}\bowtie_{\rho} \hat{\mathcal{G}}$ in our sense. □

Using Theorem I we will give a short direct proof in Sect. 5 that under the compatibility conditions (4.11) the colgebra structures $\Delta_{\mathcal{B}}$ (4.6) and $\epsilon_{\mathcal{B}}$ (4.7) are indeed algebra maps.

We also remark that it seems natural to conjecture that the condition (4.11) is also necessary for $\mathcal{M}_{\lambda}\bowtie_{\rho} \hat{\mathcal{G}}$ being a double crossed product with respect to this coalgebra structure. However we have not succeeded in proving this.

4.2 Two–sided crossed products

We now provide further examples of diagonal crossed products. A simple recipe to produce two-sided $\mathcal{G}$-comodule algebras $(\mathcal{M}, \delta)$ is by taking a right $\mathcal{G}$-comodule algebra $(\mathcal{A}, \rho)$ and a left $\mathcal{G}$-comodule algebra $(\mathcal{B}, \lambda)$ and define $\mathcal{M} = \mathcal{A} \otimes \mathcal{B}$ and

$$\delta(A \otimes B) := B_{(-1)} \otimes (A_{(0)} \otimes B_{(0)}) \otimes A_{(1)} \tag{4.14}$$

where $A \in \mathcal{A},\ B \subset \mathcal{B},\ \rho(A) = A_{(0)} \otimes A_{(1)}$ and $\lambda(B) = B_{(-1)} \otimes B_{(0)}$. In terms of the $\hat{\mathcal{G}}$-actions $\triangleright$ on $\mathcal{A}$ and $\triangleleft$ on $\mathcal{B}$ dual to ρ and λ, respectively, the $\hat{\mathcal{G}}$-actions $\triangleright_{\mathcal{M}}$ and $\triangleleft_{\mathcal{M}}$ dual to (4.14) are given by

$$\varphi \triangleright_{\mathcal{M}} (A \otimes B) \triangleleft_{\mathcal{M}} \psi = (\varphi \triangleright A \otimes B \triangleleft \psi) \tag{4.15}$$

where $\varphi, \psi \in \hat{\mathcal{G}}$. Hence, we may construct the diagonal crossed product $\mathcal{M} \bowtie \mathcal{G}$ as before. It turns out that this example may be presented differently as a so-called *two-sided crossed product*.

Proposition 4.2. *Let $\triangleright : \hat{\mathcal{G}} \otimes \mathcal{A} \to \mathcal{A}$ and $\triangleleft : \mathcal{B} \otimes \hat{\mathcal{G}} \to \mathcal{B}$ be a left and a right Hopf module action, respectively. Define the "two-sided crossed product" $\mathcal{A} \rtimes \hat{\mathcal{G}} \ltimes \mathcal{B}$ to be the vector space $\mathcal{A} \otimes \hat{\mathcal{G}} \otimes \mathcal{B}$ with multiplication structure*

$$\big(A \rtimes \phi \ltimes B\big)\,\big(A' \rtimes \psi \ltimes B'\big) = A(\phi_{(1)} \triangleright A') \rtimes \phi_{(2)}\psi_{(1)} \ltimes (B \triangleleft \psi_{(2)})B' \ . \tag{4.16}$$

Then $\mathcal{A} \rtimes \hat{\mathcal{G}} \ltimes \mathcal{B}$ becomes an associative algebra with unit $\mathbf{1}_{\mathcal{A}} \rtimes \hat{\mathbf{1}} \ltimes \mathbf{1}_{\mathcal{B}}$ and

$$f : \mathcal{A} \rtimes \hat{\mathcal{G}} \ltimes \mathcal{B} \ni A \rtimes \phi \ltimes B \mapsto \big((A \otimes \mathbf{1}_{\mathcal{B}}) \bowtie \phi\big)\big((\mathbf{1}_{\mathcal{A}} \otimes B) \bowtie \hat{\mathbf{1}}\big) \in (\mathcal{A} \otimes \mathcal{B}) \bowtie \hat{\mathcal{G}} \tag{4.17}$$

provides an algebra isomorphism with inverse given by

$$f^{-1}\big((A \otimes B) \bowtie \phi\big) = (\mathbf{1}_{\mathcal{A}} \rtimes \hat{\mathbf{1}} \ltimes B)(A \rtimes \phi \ltimes \mathbf{1}_{B}) \tag{4.18}$$

Proof. To prove associativity of (4.16) we compute for $A, A', A'' \in \mathcal{A}$, $B, B', B'' \in \mathcal{B}$ and $\varphi, \psi, \xi \in \hat{\mathcal{G}}$

$$\begin{aligned}
&\big[(A \rtimes \varphi \ltimes B)(A' \rtimes \psi \ltimes B')\big](A'' \rtimes \xi \ltimes B'') = \\
&\quad = \big[A(\varphi_{(1)} \triangleright A') \rtimes \varphi_{(2)}\psi_{(1)} \ltimes (B \triangleleft \psi_{(2)})B'\big](A'' \rtimes \xi \ltimes B'') \\
&\quad = A(\varphi_{(1)} \triangleright A')(\varphi_{(2)}\psi_{(1)} \triangleright A'') \rtimes \varphi_{(3)}\psi_{(2)}\xi_{(1)} \ltimes (B \triangleleft \psi_{(3)}\xi_{(2)})(B' \triangleleft \xi_{(3)})B'' \\
&\quad = (A \rtimes \varphi \ltimes B)\big[A'(\psi_{(1)} \triangleright A'') \rtimes \psi_{(2)}\xi_{(1)} \ltimes (B' \triangleleft \xi_{(2)})B''\big] \\
&\quad = (A \rtimes \varphi \ltimes B)\big[(A' \rtimes \psi \ltimes B')(A'' \rtimes \xi \ltimes B'')\big]
\end{aligned}$$

Hence (4.16) is associative. Obviously $(\mathbf{1}_{\mathcal{A}} \rtimes \hat{\mathbf{1}} \ltimes \mathbf{1}_{B})$ is the unit in $\mathcal{A} \rtimes \hat{\mathcal{G}} \ltimes \mathcal{B}$. To prove that f is an algebra map we note $f(A \rtimes \varphi \ltimes B) = \big[A(\varphi_{(1)} \triangleright \mathbf{1}_{\mathcal{A}}) \otimes B \triangleleft S^{-1}(\varphi_{(3)})\big] \bowtie \varphi_{(2)} = \big[A \otimes B \triangleleft S^{-1}(\varphi_{(2)})\big] \bowtie \varphi_{(1)}$ implying

$$\begin{aligned}
&f(A(\varphi_{(1)} \triangleright A') \rtimes \varphi_{(2)}\psi_{(1)} \ltimes (B \triangleleft \psi_{(2)})B') = \\
&\quad = \big[A(\varphi_{(1)} \triangleright A') \otimes \big((B \triangleleft \varphi_{(3)})B'\big) \triangleleft S^{-1}(\varphi_{(3)}\psi_{(2)})\big] \bowtie \varphi_{(2)}\psi_{(1)} \\
&\quad = \big[A(\varphi_{(1)} \triangleright A') \otimes \big(B \triangleleft S^{-1}(\varphi_{(4)})\big)\big(B' \triangleleft S^{-1}(\psi_{(2)})S^{-1}(\varphi_{(3)})\big) \bowtie \varphi_{(2)}\psi_{(1)} \\
&\quad = \big[(A \otimes B \triangleleft S^{-1}(\varphi_{(2)})) \bowtie \varphi_{(1)}\big]\big[(A' \otimes B' \triangleleft S^{-1}(\psi_{(2)})) \bowtie \varphi_{(1)}\big] \\
&\quad = f(A \rtimes \varphi \ltimes B) f(A' \rtimes \psi \ltimes B')
\end{aligned}$$

Hence f is an algebra map. The proof that f^{-1} is given by (4.18) is left to the reader. □

As a particular example of the setting of Proposition 4.2 we may choose $\mathcal{A} = \mathcal{B} = \mathcal{G}$ with its canonical left and right $\hat{\mathcal{G}}$-action. It turns out that in this case the two-sided crossed product $\mathcal{G} \rtimes \hat{\mathcal{G}} \ltimes \mathcal{G} \equiv (\mathcal{G} \otimes \mathcal{G}) \bowtie \hat{\mathcal{G}}$ is isomorphic to the iterated crossed product $(\mathcal{G} \rtimes \hat{\mathcal{G}}) \rtimes \mathcal{G}$. More generally we have

Proposition 4.3. *Let $\mathcal{A}$ be a right $\mathcal{G}$-comodule algebra and consider the iterated crossed product $(\mathcal{A} \rtimes \hat{\mathcal{G}}) \rtimes \mathcal{G}$, where $\mathcal{G}$ acts on $\mathcal{A} \rtimes \hat{\mathcal{G}}$ in the usual way by $a \triangleright (A \rtimes \varphi) := A \rtimes (a \rightharpoonup \varphi)$, $A \in \mathcal{A}$, $a \in \mathcal{G}$, $\varphi \in \hat{\mathcal{G}}$. Then as an algebra $(\mathcal{A} \rtimes \hat{\mathcal{G}}) \rtimes \mathcal{G} = \mathcal{A} \rtimes \hat{\mathcal{G}} \ltimes \mathcal{G}$ with trivial identification.*

Proof. The claim follows from

$$A(\varphi_{(1)} \triangleright A') \rtimes \varphi_{(2)}\psi_{(1)} \ltimes (a \leftharpoonup \psi_{(2)})b = A(\varphi_{(1)} \triangleright A') \rtimes \varphi_{(2)}(a_{(1)} \rightharpoonup \psi)) \rtimes a_{(2)}b$$

as an identity in $\mathcal{A} \otimes \hat{\mathcal{G}} \otimes \mathcal{G}$, where we have used

$$\psi_{(1)} \otimes (a \leftharpoonup \psi_{(2)}) = \psi_{(1)}\langle a_{(1)}|\psi_{(2)}\rangle \otimes a_{(2)} = (a_{(1)} \rightharpoonup \psi) \otimes a_{(2)} \tag{4.19}$$

as an identity in $\hat{\mathcal{G}} \otimes \mathcal{G}$. □

It will be shown in Section 11.2 that being a particular example of a two-sided (and therefore of a diagonal) crossed product the analogue of $(\mathcal{A} \rtimes \hat{\mathcal{G}}) \rtimes \mathcal{G} \equiv \mathcal{A} \rtimes \hat{\mathcal{G}} \ltimes \mathcal{G}$ may also be constructed for quasi-Hopf algebras $\mathcal{G}$. However, in this case $\mathcal{A} \rtimes \hat{\mathcal{G}}$ (if defined to be the linear subspace $\mathcal{A} \otimes \hat{\mathcal{G}} \otimes \mathbf{1}_{\mathcal{G}}$) will no longer be a subalgebra of $\mathcal{A} \rtimes \hat{\mathcal{G}} \ltimes \mathcal{G}$. We will see in Section 11.4 that this fact is very much analogous to what happens in the field algebra constructions with quasi-Hopf symmetry as given by V. Schomerus [S].

4.3 Hopf spin chains and lattice current algebras

Next, we point out that Proposition 4.2 and Proposition 4.3 also apply to the construction of Hopf algebraic quantum chains as considered in [NSz1, AFFS][3]. To see this let us shortly review the model of [NSz1], where one considers even (odd) integers to represent the sites (links) of a one-dimensional lattice and where one places a copy of $\mathcal{G} \cong \mathcal{A}_{2i}$ on each site and a copy of $\hat{\mathcal{G}} \cong \mathcal{A}_{2i+1}$ on each link. Non-vanishing commutation relations are then postulated only on neighboring site-link pairs, where one requires [NSz1]

$$\begin{aligned} A_{2i}(a)A_{2i-1}(\varphi) &= A_{2i-1}(a_{(1)} \rightharpoonup \varphi)A_{2i}(a_{(2)}) && (4.20)\\ A_{2i+1}(\varphi)A_{2i}(a) &= A_{2i}(\varphi_{(1)} \rightharpoonup a)A_{2i+1}(\varphi_{(2)}) && (4.21)\end{aligned}$$

Here $\mathcal{G} \ni a \mapsto A_{2i}(a) \in \mathcal{A}_{2i} \subset \mathcal{A}$ and $\hat{\mathcal{G}} \ni \varphi \mapsto A_{2i+1}(\varphi) \in \mathcal{A}_{2i+1} \subset \mathcal{A}$ denote the embedding of the single site (link) algebras into the global quantum chain $\mathcal{A}$. Denoting $\mathcal{A}_{i,j} \subset \mathcal{A}$ as the subalgebra generated by $\mathcal{A}_\nu$, $i \le \nu \le j$, we clearly have from (4.20) and (4.21)

$$\begin{aligned} \mathcal{A}_{i,j+1} &= \mathcal{A}_{i,j} \rtimes \mathcal{A}_{j+1} && (4.22)\\ \mathcal{A}_{i-1,j} &= \mathcal{A}_{i-1} \ltimes \mathcal{A}_{i,j} && (4.23)\end{aligned}$$

Hence, by Proposition 4.2, we recognize the two-sided crossed products

$$\mathcal{A}_{2i,2j+2} = \mathcal{A}_{2i,2j} \rtimes \hat{\mathcal{G}} \ltimes \mathcal{G} \tag{4.24}$$

More generally for all $i \le \nu \le j-1$ we have

$$\mathcal{A}_{2i,2j} = \mathcal{A}_{2i,2\nu} \rtimes \hat{\mathcal{G}} \ltimes \mathcal{A}_{2\nu+2,2j} \tag{4.25}$$

where $\hat{\mathcal{G}} \equiv \mathcal{A}_{2\nu+1}$. The advantage of looking at it in this way again comes from the fact that the constructions (4.24) and (4.25) generalize to quasi-Hopf algebras $\mathcal{G}$ whereas (4.22) and (4.23) do not. Using the identifications given in Eqs. (2.3)-(2.12) of [N1] a similar remark applies to the lattice current algebra of [AFFS]. This observation will be needed to formulate a theory of Hopf spin models and lattice current algebras at roots of unity, see Section 11.3.

[3] For earlier versions of lattice current algebras see also [AFSV, AFS, FG], the relation with the model of [NSz1] being clarified in [N1].

Also note that the identification (4.25) together with Proposition 4.2 and Proposition 3.5 immediately imply that quantum chains of the type (4.20), (4.21) admit localized commuting left and right coactions of the quantum double $\mathcal{D}(\mathcal{G})$, which is precisely the result of Theorem 4.1 of [NSz1]. In fact, applied to the example in Proposition 4.2 , Proposition 3.5 gives (for Eq. (4.27) use the identification (3.17))

Corollary 4.4. *Under the setting of Proposition 4.2 we have two commuting right coactions* $\rho_D : \mathcal{A} \rtimes \hat{\mathcal{G}} \ltimes \mathcal{B} \to (\mathcal{A} \rtimes \hat{\mathcal{G}} \ltimes \mathcal{B}) \otimes \mathcal{D}(\mathcal{G})$ *and* $\hat{\rho}_D : (\mathcal{A} \rtimes \hat{\mathcal{G}} \ltimes \mathcal{B}) \otimes \mathcal{D}(\hat{\mathcal{G}})$ *given by*

$$\rho_D(A \rtimes \varphi \ltimes B) = (A_{(1)} \rtimes \varphi_{(2)}) \otimes (A_{(2)} \bowtie_D \varphi_{(1)}) \tag{4.26}$$
$$\hat{\rho}_D(A \rtimes \varphi \ltimes B) = (A \rtimes \varphi_{(1)} \ltimes B_{(0)}) \otimes (\varphi_{(2)} \bowtie_{\hat{D}} B_{(-1)}) \tag{4.27}$$

Next, we remark that the identification (4.25) may be iterated in the obvious way. This observation also generalizes to the situation where in Proposition 4.2 $\mathcal{A}$ and $\mathcal{B}$ are both two-sided $\mathcal{G}$-comodules algebras with dual $\hat{\mathcal{G}}$ actions denoted $\triangleright_{\mathcal{A}}$, $\triangleleft_A$, $\triangleright_B$, $\triangleleft_B$, respectively. Then in the multiplication rule (4.16) only $\triangleright_{\mathcal{A}}$ and $\triangleleft_B$ appear and one easily checks, that for $\varphi, \psi \in \hat{\mathcal{G}}$ and $A \in \mathcal{A},\ B \in \mathcal{B}$ the definitions

$$\varphi \triangleright (A \rtimes \psi \ltimes B) := (A \rtimes \psi \ltimes (\varphi \triangleright_{\mathcal{B}} B)) \tag{4.28}$$
$$(A \rtimes \psi \ltimes B) \triangleleft \varphi := ((A \triangleleft_{\mathcal{A}} \varphi) \rtimes \psi \ltimes B) \tag{4.29}$$

again define a two-sided $\mathcal{G}$-comodule structure on $\mathcal{A} \ltimes \hat{\mathcal{G}} \rtimes \mathcal{B}$. Hence, we have a multiplication law on two-sided $\mathcal{G}$-comodule algebras which is in fact associative, i.e. as a two-sided $\mathcal{G}$-comodule algebra

$$(\mathcal{A} \rtimes \hat{\mathcal{G}} \ltimes \mathcal{B}) \rtimes \hat{\mathcal{G}} \ltimes C = \mathcal{A} \rtimes \hat{\mathcal{G}} \ltimes (B \rtimes \hat{\mathcal{G}} \ltimes C) \tag{4.30}$$

which the reader will easily check. Obviously, one may also consider mixed cases, e.g. where in (4.30) $\mathcal{A}$ is only a right $\mathcal{G}$-comodule algebra, but $\mathcal{B}$ and C are two-sided, in which case (4.30) would be an identity between right $\mathcal{G}$-comodule algebras.

We conclude this subsection by mentioning that diagonal crossed products also appear when formulating periodic boundary conditions for the quantum chain (4.20) - (4.21). In this case, starting with the open chain $\mathcal{A}_{0,2i}$ localized on $[0, 2i] \cap \mathbb{Z}$ one would like to add another copy of $\hat{\mathcal{G}}$ sitting on the link $2i+1 \equiv -1$ joining the sites $2i$ and 0 to form a periodic lattice. Algebraically this means that $\mathcal{A}_{2i+1} \cong \hat{\mathcal{G}}$ should have non-vanishing commutation relations with $\mathcal{A}_{2i} \cong \mathcal{G}$ and $\mathcal{A}_0 \cong \mathcal{G}$ in analogy with (4.20) and (4.21), i.e.

$$A_{2i+1}(\varphi) A_{2i}(a) = A_{2i}(\varphi_{(1)} \rightharpoonup a) A_{2i+1}(\varphi_{(2)}) \tag{4.31}$$
$$A_0(a) A_{2i+1}(\varphi) = A_{2i+1}(\varphi_{(1)}) A_0(a \leftharpoonup \varphi_{(2)}) \tag{4.32}$$

Written in this way Eqs.(4.31) and (4.32) are precisely the relations in

$$\mathcal{A}_{0,2i} \bowtie_\delta \hat{\mathcal{G}}$$

where $\delta : \mathcal{A}_{0,2i} \to \mathcal{G} \otimes \mathcal{A}_{0,2i} \otimes \mathcal{G}$ is the two–sided coaction given by $\delta|_{\mathcal{A}_0} = \Delta \otimes \mathbf{1}_{\mathcal{G}},\ \delta|_{\mathcal{A}_{2i}} = \mathbf{1}_{\mathcal{G}} \otimes \Delta$ and $\delta|_{\mathcal{A}_{1,2i-1}} = \mathbf{1}_{\mathcal{G}} \otimes \mathrm{id} \otimes \mathbf{1}_{\mathcal{G}}$. Hence, the periodic quantum chain appears as a diagonal crossed product of the open lattice chain by a copy of $\hat{\mathcal{G}}$ sitting on the link joining the end points.

5 Generating matrices

Similarly as in Lemma 2.4 we now describe the defining relations of diagonal crossed products in terms of a generating matrix. However, whereas in Lemma 2.4 the generating matrices $\mathbf{L}$ and $\mathbf{R}$ had to fulfill the *implementer* properties (2.24) or (2.25), respectively, the natural requirement here is that $\mathbf{T}$ *intertwines* the left and right coactions associated with δ.

Definition 5.1. Let (λ, ρ) be a commuting pair of left and right $\mathcal{G}$-coactions on $\mathcal{M}$ and let $\gamma : \mathcal{M} \to \mathcal{A}$ be an algebra map into some target algebra $\mathcal{A}$. Then a $\lambda\rho$-*intertwiner* in $\mathcal{A}$ (with respect to γ) is an element $\mathbf{T} \in \mathcal{G} \otimes \mathcal{A}$ satisfying

$$\mathbf{T}\lambda_{\mathcal{A}}(m) = \rho^{op}_{\mathcal{A}}(m)\mathbf{T}, \quad \forall m \in \mathcal{M} , \tag{5.1}$$

where $\lambda_{\mathcal{A}} \equiv (\gamma \otimes \mathrm{id}) \circ \lambda$ and $\rho_{\mathcal{A}} \equiv (\mathrm{id} \otimes \gamma) \circ \rho$. A $\lambda\rho$-intertwiner is called *coherent* if in $\mathcal{G} \otimes \mathcal{G} \otimes \mathcal{A}$ it satisfies

$$\mathbf{T}^{13}\mathbf{T}^{23} = (\Delta \otimes \mathrm{id})(\mathbf{T}) \tag{5.2}$$

Similarly as in Lemma 2.5 we then have

Lemma 5.2. *Let $(\mathcal{M}, \delta)$ be a two-sided $\mathcal{G}$-comodule algebra with associated commuting left and right $\mathcal{G}$-coactions (λ, ρ), and let $\gamma : \mathcal{M} \to \mathcal{A}$ be an algebra map. Then for $\mathbf{T} \in \mathcal{G} \otimes \mathcal{A}$ the following properties are equivalent:*
i) $\mathbf{T}$ *is a $\lambda\rho$-intertwiner*
ii) $\mathbf{T}\,[\mathbf{1}_G \otimes \gamma(m)] = [m_{(1)} \otimes \gamma(m_{(0)})]\,\mathbf{T}[S^{-1}(m_{(-1)}) \otimes \mathbf{1}_{\mathcal{A}}]$
iii) $[\mathbf{1}_G \otimes \gamma(m)]\,\mathbf{T} = [S^{-1}(m_{(1)}) \otimes \mathbf{1}_{\mathcal{A}}]\,\mathbf{T}[m_{(-1)} \otimes \gamma(m_{(0)})]$.

Proof. Suppose $\mathbf{T}$ is a $\lambda\rho$-intertwiner. Then

$$[m_{(1)} \otimes \gamma(m_{(0)})]\,\mathbf{T}\,[S^{-1}(m_{(-1)}) \otimes \mathbf{1}_{\mathcal{A}}] = \mathbf{T}\,[m_{(-1)}S^{-1}(m_{(-2)}) \otimes \gamma(m_{(0)})] = \mathbf{T}\,[\mathbf{1}_{\mathcal{G}} \otimes \gamma(m)]$$

by the antipode axiom. Conversely, if $\mathbf{T}$ satisfies (ii) then

$$\mathbf{T}\,[m_{(-1)} \otimes \gamma(m_{(0)})] = [m_{(1)} \otimes \gamma(m_{(0)})]\,\mathbf{T}\,[S^{-1}(m_{(-1)})m_{(-2)} \otimes \mathbf{1}_{\mathcal{A}}] = [m_{(1)} \otimes \gamma(m_{(0)})]\,\mathbf{T}$$

proving (i) $\Leftrightarrow$ (ii). The equivalence (i) $\Leftrightarrow$ (iii) follows similarly. $\square$

Note that by applying $(\varphi \otimes \mathrm{id})$ to both sides the equivalence of (ii) and (iii) in Lemma 5.2 precisely reflects the equivalence of (3.18) and (3.19) in Lemma 3.6.

We now arrive at a proof of Theorem I by concluding similarly as in Lemma 2.4.

Proposition 5.3. *Let $(\mathcal{M}, \delta)$ be a two-sided $\mathcal{G}$-comodule algebra with associated commuting pair of coactions (λ, ρ), and let $\gamma : \mathcal{M} \to \mathcal{A}$ be an algebra map. Then the relation*

$$\gamma_T(m \bowtie \varphi) = \gamma(m)(\varphi \otimes \mathrm{id})(\mathbf{T}) \tag{5.3}$$

provides a one-to-one correspondence between normal coherent $\lambda\rho$-intertwiners $\mathbf{T}$ and unital algebra maps $\gamma_T : \mathcal{M}\,{}_\lambda\!\bowtie_\rho\, \hat{\mathcal{G}} \to \mathcal{A}$ extending γ.

Proof. Let $T(\varphi) := (\varphi \otimes \mathrm{id})(\mathbf{T})$. Then Eq. (5.2) together with normality is equivalent to $\hat{\mathcal{G}} \ni \varphi \mapsto T(\varphi) \equiv \gamma_T(\mathbf{1}_{\mathcal{M}} \bowtie \varphi) \in \mathcal{A}$ being a unital algebra morphism and the correspondence $\mathbf{T} \leftrightarrow \gamma_T|_{\mathbf{1}_{\mathcal{M}} \bowtie \hat{\mathcal{G}}}$ is one-to-one. Clearly, γ_T extends γ and Lemma 5.2 ii) implies $\forall \varphi \in \hat{\mathcal{G}},\ m \in \mathcal{M}$

$$T(\varphi)\gamma(m) = \gamma(\varphi_{(1)} \triangleright m \triangleleft S^{-1}(\varphi_{(3)}))T(\varphi_{(2)})$$

and therefore γ_T is an algebra map. Conversely, since $(m \bowtie \varphi) = (m \bowtie \hat{\mathbf{1}})(\mathbf{1}_{\mathcal{M}} \bowtie \varphi)$, any algebra map $\gamma : \mathcal{M} \bowtie_\delta \hat{\mathcal{G}} \to \mathcal{A}$ is of the form (5.3). □

We remark that one could equivalently have chosen to work with

$$\gamma_T^{op}(\varphi \bowtie m) := (\varphi \otimes \mathrm{id})(\mathbf{T})\gamma(m) \tag{5.4}$$

to obtain algebra maps $\gamma_T^{op} : \hat{\mathcal{G}}_\lambda \bowtie_\rho \mathcal{M} \to \mathcal{A}$.

Note that Proposition 5.3 and Corollary 3.7 prove part 1.) and 3.) of Theorem I by putting $\mathcal{M}_1 := \mathcal{M}_\lambda \bowtie_\rho \hat{\mathcal{G}}$, $\mu_R := \mathrm{id}_{\mathcal{M} \otimes \hat{\mathcal{G}}}$ and μ_L as given in Eq. (3.21). The uniqueness of $\mathcal{M}_1$ up to equivalence follows by standard arguments.

Putting $\mathcal{A} = \mathcal{M}_1$ and $\gamma_T = \mathrm{id}$ in Proposition 5.3 we obtain $\mathbf{T} = \mathbf{\Gamma} \equiv e_\mu \otimes (\mathbf{1}_{\mathcal{M}} \bowtie e^\mu) \in \mathcal{G} \otimes \mathcal{M}_1$. Thus we call $\mathbf{\Gamma}$ the *universal $\lambda\rho$–intertwiner in $\mathcal{M}_1$.*

Applying the above formalism to the case $\mathcal{M} = \mathcal{G}$ and $\delta = D \equiv (\Delta \otimes \mathrm{id}) \circ \Delta$ we realize that (5.1) becomes (suppressing the symbol γ)

$$\mathbf{T}\Delta(a) = \Delta_{op}(a)\mathbf{T}, \quad \forall a \in \mathcal{A} \tag{5.5}$$

As already remarked, in this case $\mathcal{G} \bowtie_D \hat{\mathcal{G}} \equiv \mathcal{D}(\mathcal{G})$ is the quantum double of $\mathcal{G}$, in which case Proposition 5.3 coincides with [N1, Lem.5.2] describing $\mathcal{D}(\mathcal{G})$ as the unique algebra generated by $\mathcal{G}$ and the entries of a generating Matrix $\mathbf{D} \equiv \mathbf{\Gamma}_{\mathcal{D}(\mathcal{G})} \in \mathcal{G} \otimes \mathcal{D}(\mathcal{G})$ satisfying (5.2) and (5.5). More generally, Proposition 5.3 may be reformulated to say that in Lemma 3.6 the algebra embeddings $\hat{\mathcal{G}} \to \mathcal{A}$ satisfying (3.18) and (3.19) are in one-to-one correspondence with $\lambda\rho$-intertwiners $\mathbf{T} \in \mathcal{G} \otimes \mathcal{A}$ satisfying (5.2). We will see in Section 9 that this is the appropriate language to be generalized to the quasi-Hopf case.

We conclude this Section with demonstrating the good use of Proposition 5.3 by giving a short alternative proof of how the compatibility condition (4.11) in Proposition 4.1 makes diagonal crossed products into double crossed product bialgebras.

Proposition 5.4. *Under the conditions* (4.8), (4.11), *and* (4.12) *of Proposition 4.1 let $\mathcal{B} := \mathcal{M}_\lambda \bowtie_\rho \hat{\mathcal{G}}$. Then the coalgebra structures $\Delta_{\mathcal{B}}$ (4.6) and $\epsilon_{\mathcal{B}}$ (4.7) become algebra maps.*

Proof. Since $\Delta_{\mathcal{B}}$ extends $\Delta_{\mathcal{M}}$ we may use Proposition 5.3 by putting $\mathcal{A} := \mathcal{B} \otimes \mathcal{B}$, $\gamma := \Delta_{\mathcal{M}} : \mathcal{M} \to \mathcal{M} \otimes \mathcal{M} \subset \mathcal{A}$ and

$$\mathbf{T} := (\mathrm{id}_{\mathcal{G}} \otimes \Delta_{\mathcal{B}})(\mathbf{\Gamma}) \equiv \mathbf{\Gamma}^{13}\mathbf{\Gamma}^{12} \in \mathcal{G} \otimes \mathcal{A},$$

where $\mathbf{\Gamma} \in \mathcal{G} \otimes \mathcal{B}$ is the universal $\lambda\rho$-intertwiner. Thus, $\Delta_{\mathcal{B}} : \mathcal{B} \to \mathcal{A}$ provides an algebra map if and only if $\mathbf{T}$ is a normal coherent $\lambda\rho$-intertwiner with respect to $\Delta_{\mathcal{M}} : \mathcal{M} \to \mathcal{A}$. Now normality and coherence of $\mathbf{T}$ hold, since the restriction $\Delta_{\mathcal{B}}|_{\hat{\mathcal{G}}} \equiv \hat{\Delta}_{op}$ is a unital algebra map. To prove the $\lambda\rho$-intertwiner property for $\mathbf{T}$ we use the identities (4.8) to compute for all $m \in \mathcal{M}$

$$\begin{aligned}
\mathbf{T}(\mathrm{id}_{\mathcal{G}} \otimes \Delta_{\mathcal{M}})(\lambda(m)) &= \mathbf{\Gamma}^{13}\mathbf{\Gamma}^{12}[\kappa(m^{(1)}) \otimes m^{(2)} \otimes m^{(3)}] \\
&= \mathbf{\Gamma}^{13}\mathbf{\Gamma}^{12}[\lambda(m^{(1)}) \otimes m^{(2)}] \\
&= \mathbf{\Gamma}^{13}[\kappa(m^{(2)}) \otimes m^{(1)} \otimes m^{(3)}]\mathbf{\Gamma}^{12} \\
&= [\kappa(m^{(3)} \otimes m^{(1)}) \otimes m^{(2)}]\mathbf{\Gamma}^{13}\mathbf{\Gamma}^{12} \\
&= (\mathrm{id}_{\mathcal{G}} \otimes \Delta_{\mathcal{M}})(\rho^{op}(m))\mathbf{T},
\end{aligned}$$

where $m^{(1)} \otimes m^{(2)} \otimes m^{(3)} \equiv \Delta^{(2)}_{\mathcal{M}}(m)$. Hence, $\mathbf{T}$ is a $\lambda\rho$-intertwiner and therefore $\Delta_{\mathcal{B}}$ is an algebra map. Similarly, we prove that $\epsilon_{\mathcal{B}} : \mathcal{B} \to \mathbb{C}$ is multiplicative by putting $\mathcal{A} = \mathbb{C}$ and $\mathbf{T} = (\mathrm{id}_{\mathcal{G}} \otimes \epsilon_{\mathcal{B}})(\mathbf{\Gamma}) \equiv \mathbf{1}_{\mathcal{G}}$. In this case the $\lambda\rho$-intertwiner property reduces to

$$(\mathrm{id}_{\mathcal{G}} \otimes \epsilon_{\mathcal{M}})(\lambda(m)) = (\epsilon_{\mathcal{M}} \otimes \mathrm{id}_{\mathcal{G}})(\rho(m)),$$

which is precisely the condition (4.13) □

Part II

The Quasi–Coassociative Setting

In Part I we have reviewed the notions of left and right $\mathcal{G}$–coactions and crossed products and we have introduced as new concepts the notions of two–sided $\mathcal{G}$–coactions and diagonal crossed products, where throughout $\mathcal{G}$ had been supposed to be a standard coassociative Hopf algebra. As examples and applications we have mentioned the Drinfel'd double $\mathcal{D}(\mathcal{G})$ and the constructions of quantum chains based on a Hopf algebra $\mathcal{G}$.

We now proceed to generalize the above ideas to quasi–Hopf algebras $\mathcal{G}$. In Section 6 we give a short review of the definitions and properties of quasi–Hopf algebras as introduced by Drinfel'd [Dr2]. In Section 7 we propose an obvious generalization of the notion of right $\mathcal{G}$–coactions ρ on an algebra $\mathcal{M}$ to the case of quasi–Hopf algebras $\mathcal{G}$ (and similarly for left coactions λ). As for the coproduct on $\mathcal{G}$, the basic idea here is that $(\rho \otimes \mathrm{id}) \circ \rho$ and $(\mathrm{id} \otimes \Delta) \circ \rho$ are still related by an inner automorphism, implemented by a reassociator $\phi_\rho \in \mathcal{M} \otimes \mathcal{G} \otimes \mathcal{G}$. Similarly as for Drinfel'd's reassociator $\phi \in \mathcal{G} \otimes \mathcal{G} \otimes \mathcal{G}$, ϕ_ρ is required to obey a pentagon equation to guarantee McLane's coherence condition under iterated rebracketings. We also generalize Drinfel'd's notion of a twist transformation from coproducts to coactions.

It is important to realize that ϕ_ρ has to be non–trivial, if ϕ is non–trivial. On the other hand, ϕ_ρ might be non–trivial even if $\phi = \mathbf{1}_{\mathcal{G}} \otimes \mathbf{1}_{\mathcal{G}} \otimes \mathbf{1}_{\mathcal{G}}$, in which case the above mentioned pentagon equation reduces to a cocycle condition for ϕ_ρ as already considered by [DT,BCM,BM].

In Section 8 we pass to two–sided $\mathcal{G}$–coactions (δ, Ψ), which as in Section 3 could alternatively be considered as right $(\mathcal{G} \otimes \mathcal{G}^{cop})$–coactions in the above sense. Correspondingly, $\Psi \in \mathcal{G} \otimes \mathcal{G} \otimes \mathcal{M} \otimes \mathcal{G} \otimes \mathcal{G}$ is the reassociator for δ, which is again required to obey the appropriate pentagon equation. As in Section 3, associated with any two–sided $\mathcal{G}$-coaction (δ, Ψ) we have a pair (λ, ϕ_λ) and (ρ, ϕ_ρ) of left and right $\mathcal{G}$–coactions, respectively, which however in this case only *quasi–commute*. This means that there exists another reassociator $\phi_{\lambda\rho} \in \mathcal{G} \otimes \mathcal{M} \otimes \mathcal{G}$ such that

$$\phi_{\lambda\rho}\,(\lambda \otimes \mathrm{id}_{\mathcal{G}})(\rho(m)) = (\mathrm{id}_{\mathcal{G}} \otimes \rho)(\lambda(m))\,\phi_{\lambda\rho}, \quad \forall m \in \mathcal{M}.$$

Also, $\phi_{\lambda\rho}$ obeys in a natural way two pentagon identities involving (λ, ϕ_λ) and (ρ, ϕ_ρ), respectively. We show that twist equivalence classes of two–sided coactions are in one–to–one correspondence with twist equivalence classes of quasi–commuting pairs of coactions, i.e. any two–sided coaction δ is twist–equivalent to $(\lambda \otimes \mathrm{id}) \circ \rho$ (and also to $(\mathrm{id} \otimes \rho) \circ \lambda$) where $\lambda = (\mathrm{id}_{\mathcal{G}} \otimes \mathrm{id}_{\mathcal{M}} \otimes \epsilon) \circ \delta$ and $\rho = (\epsilon \otimes \mathrm{id}_{\mathcal{M}} \otimes \mathrm{id}_{\mathcal{G}}) \circ \delta$.

In Section 9 we give a representation theoretic interpretation of the notions of left, right and two–sided coactions by showing that they give rise to functors $\mathrm{Rep}\,_{\mathcal{G}} \times \mathrm{Rep}\,_{\mathcal{M}} \longrightarrow \mathrm{Rep}\,_{\mathcal{M}}$, $\mathrm{Rep}\,_{\mathcal{M}} \times \mathrm{Rep}\,_{\mathcal{G}} \longrightarrow \mathrm{Rep}\,_{\mathcal{M}}$ and $\mathrm{Rep}\,_{\mathcal{G}} \times \mathrm{Rep}\,_{\mathcal{M}} \times \mathrm{Rep}\,_{\mathcal{G}} \longrightarrow \mathrm{Rep}\,_{\mathcal{M}}$, respectively, furnished with natural associativity isomorphisms, obeying the analogue of McLane's coherence conditions for monoidal categories [ML].

In Section 10 we use our formalism to construct, for any two–sided $\mathcal{G}$–coaction (δ, Ψ) on $\mathcal{M}$, the left and right diagonal crossed products $\mathcal{M} \bowtie_\delta \hat{\mathcal{G}} \cong \hat{\mathcal{G}} \bowtie_\delta \mathcal{M}$ as equivalent associative algebra extensions of $\mathcal{M}$. Up to equivalence, these extensions only depend on the twist–equivalence class of δ's, and therefore on the twist–equivalence class of quasi–commuting pairs (λ, ρ). The basic strategy for defining the multiplication rules in these diagonal crossed products is to generalize the generating matrix formalism of Section 5 to the quasi–coassociative setting. In this way one is naturally lead to define $\lambda\rho$–intertwiners $\mathbf{T}$ as in Definition 5.1, where now the coherence condition (5.2) has to be replaced by appropriately injecting the reassociators $\phi_\lambda, \phi_{\lambda\rho}$ and ϕ_ρ into the l.h.s., similarly as in Drinfel'd's definition of a quasitriangular R–matrix for quasi–Hopf algebras. With these substitutions our main result is given by the following

Theorem II *Let $\mathcal{G}$ be a finite dimensional quasi–Hopf algebra and let $(\lambda, \phi_\lambda, \rho, \phi_\rho, \phi_{\lambda\rho})$ be a quasi–commuting pair of (left and right) $\mathcal{G}$–coactions on an associative algebra $\mathcal{M}$.*

1. *Then their exists a unital associative algebra extension $\mathcal{M}_1 \supset \mathcal{M}$ together with a linear map $\Gamma : \hat{\mathcal{G}} \longrightarrow \mathcal{M}_1$ satisfying the following universal property:*
   $\mathcal{M}_1$ is algebraically generated by $\mathcal{M}$ and $\Gamma(\hat{\mathcal{G}})$ and for any algebra map $\gamma : \mathcal{M} \longrightarrow \mathcal{A}$ into some target algebra $\mathcal{A}$ the relation
   $$\gamma_T(\Gamma(\varphi)) = (\varphi \otimes \mathrm{id})(\mathbf{T}) \tag{6.1}$$
   provides a one–to–one correspondence between algebra maps $\gamma_T : \mathcal{M}_1 \longrightarrow \mathcal{A}$ extending γ and normal elements $\mathbf{T} \in \mathcal{G} \otimes \mathcal{A}$ satisfying
   $$\mathbf{T}\, \lambda_{\mathcal{A}}(m) = \rho^{op}_{\mathcal{A}}(m)\, \mathbf{T}, \quad \forall m \in \mathcal{M} \tag{6.2}$$
   $$(\phi^{312}_\rho)_{\mathcal{A}}\, \mathbf{T}^{13}\, (\phi^{132}_{\rho\lambda})^{-1}_{\mathcal{A}}\, \mathbf{T}^{23}\, (\phi_\lambda)_{\mathcal{A}} = (\Delta \otimes \mathrm{id}_{\mathcal{A}})(\mathbf{T}), \tag{6.3}$$
   where $\lambda_{\mathcal{A}}(m) := (\mathrm{id} \otimes \gamma)(\lambda(m))$, $(\phi_\lambda)_{\mathcal{A}} := (\mathrm{id}_{\mathcal{G}} \otimes \mathrm{id}_{\mathcal{G}} \otimes \gamma)(\phi_\lambda)$, etc.

2. *If $\mathcal{M} \subset \tilde{M}_1$ and $\tilde{\Gamma} : \hat{\mathcal{G}} \longrightarrow \tilde{\mathcal{M}}_1$ satisfy the same universality property as in part 1.), then there exists a unique algebra isomorphism $f : \mathcal{M}_1 \longrightarrow \tilde{\mathcal{M}}_1$ restricting to the identity on $\mathcal{M}$, such that $\tilde{\Gamma} = f \circ \Gamma$*

3. *There exist elements $p_\lambda \in \mathcal{G} \otimes \mathcal{M}$ and $q_\rho \in \mathcal{M} \otimes \mathcal{G}$ such that the linear maps*
   $$\mu_L : \hat{\mathcal{G}} \otimes \mathcal{M} \ni (\varphi \otimes m) \mapsto (\mathrm{id} \otimes \varphi_{(1)})(q_\rho)\Gamma(\varphi_{(2)})m \;\in \mathcal{M}_1 \tag{6.4}$$
   $$\mu_R : \mathcal{M} \otimes \hat{\mathcal{G}} \ni (m \otimes \varphi) \mapsto m\Gamma(\varphi_{(1)})(\varphi_{(2)} \otimes \mathrm{id})(p_\lambda) \;\in \mathcal{M}_1 \tag{6.5}$$
   provide isomorphisms of vector spaces.

Putting $\mathbf{\Gamma} := e_\mu \otimes \Gamma(e^\mu) \in \mathcal{G} \otimes \mathcal{M}_1$ Theorem II implies that $\mathbf{\Gamma}$ itself satisfies the defining relations (6.2) and (6.3). As before, we call $\mathbf{\Gamma}$ the *universal $\lambda\rho$–intertwiner* in $\mathcal{M}_1$. We remark that it is more or less straightforward to check that the relations (6.2) and (6.3) satisfy all associativity constraints, such that the existence of $\mathcal{M}_1$ and its uniqueness up to isomorphy may not be too much of a surprise to the experts. In this way part 1. and 2. of Theorem II could also be proven without requiring an antipode on $\mathcal{G}$. The main non–trivial content of Theorem II is stated in part 3., saying that $\mathcal{M}_1$ may still be modeled on the underlying spaces $\hat{\mathcal{G}} \otimes \mathcal{M}$ or $\mathcal{M} \otimes \hat{\mathcal{G}}$, respectively [4]. However, as a warning against likely misunderstandings we

[4] To define the elements p_λ and q_ρ one needs an invertible antipode, see Eqs. (9.20), (9.23)

emphasize that in general (i.e. for $\phi_{\lambda\rho} \neq \mathbf{1}_{\mathcal{G}} \otimes \mathbf{1}_{\mathcal{M}} \otimes \mathbf{1}_{\mathcal{G}}$) neither of the maps

$$\mathcal{M} \otimes \hat{\mathcal{G}} \ni (m \otimes \varphi) \mapsto m\Gamma(\varphi) \in \mathcal{M}_1$$
$$\hat{\mathcal{G}} \otimes \mathcal{M} \ni (\varphi \otimes m) \mapsto \Gamma(\varphi)m \in \mathcal{M}_1$$

need to be injective (nor surjective)[5]. Also, in general neither of the linear subspaces $\Gamma(\hat{\mathcal{G}})$, $\mu_L(\hat{\mathcal{G}} \otimes \mathbf{1}_{\mathcal{M}})$ or $\mu_R(\mathbf{1}_{\mathcal{M}} \otimes \hat{\mathcal{G}})$ will be a subalgebra of $\mathcal{M}_1$. Still, the invertibility of the maps $\mu_{L/R}$ guarantees that there exist well defined associative algebra structures induced on $\hat{\mathcal{G}} \otimes \mathcal{M}$ and $M \otimes \hat{\mathcal{G}}$ via $\mu_{L/R}^{-1}$ from $\mathcal{M}_1$. As in Part I we denote these by

$$\hat{\mathcal{G}}\,{}_\lambda\!\bowtie_\rho \mathcal{M} \equiv \mu_L^{-1}(\mathcal{M}_1) \tag{6.6}$$
$$\mathcal{M}\,{}_\lambda\!\bowtie_\rho \hat{\mathcal{G}} \equiv \mu_R^{-1}(\mathcal{M}_1). \tag{6.7}$$

They are the analogues of the left and right diagonal crossed products, respectively, constructed in Proposition 3.3 and Corollary 3.7.

To actually prove Theorem II we go the opposite way, i.e. for any two–sided coaction (δ, Ψ) we will first explicitly construct left and right diagonal crossed products $\hat{\mathcal{G}} \bowtie_\delta \mathcal{M}$ and $\mathcal{M} \bowtie_\delta \hat{\mathcal{G}}$ as equivalent algebra extensions of $\mathcal{M}$ in Section 10.1. As in Part I these are defined on the underlying spaces $\hat{\mathcal{G}} \otimes \mathcal{M}$ and $\mathcal{M} \otimes \hat{\mathcal{G}}$, respectively. In Section 10.2 we describe these constructions in terms of so–called left and right *diagonal δ–implementers* $\mathbf{L}$ and $\mathbf{R}$ obeying the relations of Lemma 5.2(ii) and (iii), respectively, together with certain coherence conditions reflecting the multiplication rules in $\hat{\mathcal{G}} \bowtie \mathcal{M}$ and $\mathcal{M} \bowtie \hat{\mathcal{G}}$. In Section 10.3 we generalize Lemma 5.2 by showing that coherent (left or right) diagonal δ–implementers are always in one–to–one correspondence with (although not identical to) *coherent $\lambda\rho$–intertwiners* $\mathbf{T}$, i.e. generating matrices satisfying the relations (6.2) and (6.3) of Theorem II. This will finally lead to a proof of Theorem II in Section 10.4, where we show that for $\delta_l := (\lambda \otimes \mathrm{id}) \circ \rho$ and $\delta_r := (\mathrm{id} \otimes \rho) \circ \lambda$ any of the four choices $\hat{\mathcal{G}} \bowtie_{\delta_l} \mathcal{M}$, $\hat{\mathcal{G}} \bowtie_{\delta_r} \mathcal{M}$, $\mathcal{M} \bowtie_{\delta_l} \hat{\mathcal{G}}$, or $\mathcal{M} \bowtie_{\delta_r} \hat{\mathcal{G}}$ explicitly solve all properties claimed in Theorem II. Moreover, in terms of the notations (6.6), (6.7) we will have

$$\hat{\mathcal{G}}\,{}_\lambda\!\bowtie_\rho \mathcal{M} = \hat{\mathcal{G}} \bowtie_{\delta_l} \mathcal{M}$$
$$\mathcal{M}\,{}_\lambda\!\bowtie_\rho \hat{\mathcal{G}} = \mathcal{M} \bowtie_{\delta_r} \hat{\mathcal{G}}$$

with trivial identification.

In Section 11 we extend the examples and applications of Section 4 to the quasi–coassociative setting. In particular, taking $\mathcal{M} = \mathcal{G}$ and $(\lambda, \phi_\lambda) = (\rho, \phi_\rho) = (\Delta, \phi)$ we provide an explicit construction of the Drinfel'd double $\mathcal{D}(\mathcal{G}) := \mathcal{G} \bowtie \hat{\mathcal{G}} \cong \hat{\mathcal{G}} \bowtie \mathcal{G}$ for quasi–Hopf algebras $\mathcal{G}$, which recently has been defined in terms of a Tannaka–Krein like reconstruction in [M2]. We also give generalizations of the Hopf Spin models of [NSz] and the lattice current algebras of [AFFS] to the case of quasi–Hopf algebras.

6 Definitions and properties of quasi-Hopf algebras

In this Section we review the basic definitions and properties of quasi–Hopf algebras as introduced by Drinfel'd in [Dr2]. As before algebra morphisms are always supposed to be unital.

[5] In fact, we don't even know whether the map $\Gamma : \hat{\mathcal{G}} \longrightarrow \mathcal{M}_1$ necessarily has to be injective.

A *quasi-bialgebra* $(\mathcal{G}, \Delta, \epsilon, \phi)$ is an associative algebra $\mathcal{G}$ with unit together with algebra morphisms $\Delta : \mathcal{G} \longrightarrow \mathcal{G} \otimes \mathcal{G}$ (the coproduct) and $\epsilon : \mathcal{G} \longrightarrow \mathbb{C}$ (the counit), and an invertible element $\phi \in \mathcal{G} \otimes \mathcal{G} \otimes \mathcal{G}$, such that

$$(\mathrm{id} \otimes \Delta)(\Delta(a))\phi = \phi(\Delta \otimes \mathrm{id})(\Delta(a)), \quad a \in \mathcal{G} \tag{6.8}$$

$$(\mathrm{id} \otimes \mathrm{id} \otimes \Delta)(\phi)(\Delta \otimes \mathrm{id} \otimes \mathrm{id})(\phi) = (\mathbf{1} \otimes \phi)(\mathrm{id} \otimes \Delta \otimes \mathrm{id})(\phi)(\phi \otimes \mathbf{1}), \tag{6.9}$$

$$(\epsilon \otimes \mathrm{id}) \circ \Delta = \mathrm{id} = (\mathrm{id} \otimes \epsilon) \circ \Delta, \tag{6.10}$$

$$(\mathrm{id} \otimes \epsilon \otimes \mathrm{id})(\phi) = \mathbf{1} \otimes \mathbf{1} \tag{6.11}$$

It has been remarked by Drinfel'd that (6.8) - (6.11) also imply the identities $(\epsilon \otimes \mathrm{id} \otimes \mathrm{id})(\phi) = (\mathrm{id} \otimes \mathrm{id} \otimes \epsilon)(\phi) = \mathbf{1} \otimes \mathbf{1}$. A coproduct with the above properties is called *quasi-coassociative* and the element ϕ will be called the *reassociator*. As for Hopf algebras we will use the Sweedler notation $\Delta(a) = a_{(1)} \otimes a_{(2)}$, but since Δ is only quasi–coassociative we adopt the further convention

$$(\Delta \otimes \mathrm{id}) \circ \Delta(a) = a_{(1,1)} \otimes a_{(1,2)} \otimes a_{(2)} \quad \text{and} \quad (\mathrm{id} \otimes \Delta) \circ \Delta(a) = a_{(1)} \otimes a_{(2,1)} \otimes a_{(2,2)}, \text{ etc..}$$

Furthermore, here and throughout we use the notation

$$\phi = X^j \otimes Y^j \otimes Z^j; \quad \phi^{-1} = P^j \otimes Q^j \otimes R^j, \tag{6.12}$$

where we have suppressed the summation symbol. To give an example, Eq. (6.8) written with this notation looks like

$$a_{(1)}X^j \otimes a_{(2,1)}Y^j \otimes a_{(2,2)}Z^j = X^i a_{(1,1)} \otimes Y^i a_{(1,2)} \otimes Z^i a_{(2)}$$

As in the Hopf algebra case, one may define the dual $(\hat{\mathcal{G}}, \hat{\mu}, \hat{\Delta}, \hat{\epsilon})$, but one should note that $\hat{\mathcal{G}}$ is a different kind of object. Indeed, since Δ is not associative, the "multiplication" $\hat{\mu}$ on $\hat{\mathcal{G}}$ fails to be associative. On the other hand $(\hat{\mathcal{G}}, \hat{\Delta}, \hat{\epsilon})$ is a strictly coassociative coalgebra, in contrast to $\mathcal{G}$.

A quasi-bialgebra $\mathcal{G}$ is called a *quasi-Hopf algebra*, if there exists a linear antimorphism $S : \mathcal{G} \to \mathcal{G}$ and elements $\alpha, \beta \in \mathcal{G}$ satisfying for all $a \in \mathcal{G}$

$$S(a_{(1)})\alpha a_{(2)} = \alpha\epsilon(a); \quad a_{(1)}\beta S(a_{(2)}) = \beta\epsilon(a) \quad \text{and} \tag{6.13}$$

$$\sum_j X^j \beta S(Y^j)\alpha Z^j = 1 = \sum_j S(P^j)\alpha Q^j \beta S(R^j). \tag{6.14}$$

The map S is called an antipode. We will also always suppose that S is invertible. Note that as opposed to ordinary Hopf algebras, an antipode is not uniquely determined, provided it exists.

Together with a quasi–Hopf algebra $\mathcal{G} \equiv (\mathcal{G}, \Delta, \epsilon, \phi, S, \alpha, \beta)$ we also have $\mathcal{G}_{op}, \mathcal{G}^{cop}$ and $\mathcal{G}_{op}^{cop}$ as quasi–Hopf algebras, where "*op*" means opposite multiplication and "*cop*" means opposite comultiplication. The quasi–Hopf structures are obtained by putting $\phi_{op} := \phi^{-1}$, $\phi^{cop} := (\phi^{-1})^{321}$, $\phi_{op}^{cop} := \phi^{321}$, $S_{op} = S^{cop} = (S_{op}^{cop})^{-1} := S^{-1}$, $\alpha_{op} := S^{-1}(\beta)$, $\beta_{op} := S^{-1}(\alpha)$, $\alpha^{cop} := S^{-1}(\alpha)$, $\beta^{cop} := S^{-1}(\beta)$, $\alpha_{op}^{cop} := \beta$ and $\beta_{op}^{cop} := \alpha$.

Next we recall that the definition of a quasi-Hopf algebra is 'twist covariant' in the following sense: An element $F \in \mathcal{G} \otimes \mathcal{G}$ which is invertible and satisfies $(\epsilon \otimes \mathrm{id})(F) = (\mathrm{id} \otimes \epsilon)(F) = \mathbf{1}$, induces a so–called *twist transformation*

$$\Delta_F(a) := F\Delta(a)F^{-1}, \tag{6.15}$$

$$\phi_F := (\mathbf{1} \otimes F)\,(\mathrm{id} \otimes \Delta)(F)\,\phi\,(\Delta \otimes \mathrm{id})(F^{-1})\,(F^{-1} \otimes \mathbf{1}) \tag{6.16}$$

It has been noticed by Drinfel'd [Dr2] that $(\mathcal{G}, \Delta_F, \epsilon, \phi_F)$ is again a quasi bialgebra. Moreover, setting

$$\alpha_F := S(h^i)\alpha k^i, \quad \text{where} \sum_i h^i \otimes k^i = F^{-1}$$
$$\beta_F := f^i \beta S(g^i), \quad \text{where} \sum_i f^i \otimes g^i = F,$$

$(\mathcal{G}, \Delta_F, \epsilon, \phi_F, S, \alpha_F, \beta_F)$ is also a quasi-Hopf algebra. This means that a twist preserves the class of quasi-Hopf algebras.

For Hopf algebras one knows that the antipode is an anti coalgebra morphism, i.e. $\Delta(a) = (S \otimes S)\big(\Delta^{op}(S^{-1}(a))\big)$. For quasi-Hopf algebras this is true only up to a twist: Following Drinfel'd we define the elemnts $\gamma, \delta \in \mathcal{G} \otimes \mathcal{G}$ by setting

$$\gamma := \sum_i (S(U^i) \otimes S(T^i)) \cdot (\alpha \otimes \alpha) \cdot (V^i \otimes W^i) \tag{6.17}$$
$$\delta := \sum_j (K^j \otimes L^j) \cdot (\beta \otimes \beta) \cdot (S(N^j) \otimes S(M^j)) \tag{6.18}$$

where

$$\sum_i T^i \otimes U^i \otimes V^i \otimes W^i = (\mathbf{1} \otimes \phi^{-1}) \cdot (\mathrm{id} \otimes \mathrm{id} \otimes \Delta)(\phi), \tag{6.19}$$
$$\sum_j K^j \otimes L^j \otimes M^j \otimes N^j = (\Delta \otimes \mathrm{id} \otimes \mathrm{id})(\phi) \cdot (\phi^{-1} \otimes \mathbf{1}). \tag{6.20}$$

With these definitions Drinfel'd has shown, that $f \in \mathcal{G} \otimes \mathcal{G}$ given by

$$f := (S \otimes S)(\Delta^{op}(P^i)) \cdot \gamma \cdot \Delta(Q^i \beta R^i). \tag{6.21}$$

defines a twist with inverse given by

$$f^{-1} = \Delta(S(P^j)\alpha Q^j) \cdot \delta \cdot (S \otimes S)(\Delta^{op}(R^j)), \tag{6.22}$$

such that for all $g \in \mathcal{G}$

$$f\Delta(g)f^{-1} = (S \otimes S)\big(\Delta^{op}(S^{-1}(g))\big). \tag{6.23}$$

The elements γ, δ and the twist f fulfill the relation

$$f\,\Delta(\alpha) = \gamma, \quad \Delta(\beta)\, f^{-1} = \delta. \tag{6.24}$$

Furthermore the corresponding twisted reassociator (s.(6.16)) is given by

$$\phi_f = (S \otimes S \otimes S)(\phi^{321}). \tag{6.25}$$

Taking $\mathcal{G}^{cop} \equiv (\mathcal{G}, \Delta^{op}, \epsilon, (\phi^{-1})^{321}, S^{-1}, S^{-1}(\alpha), S^{-1}(\beta))$ instead of $\mathcal{G}$, the above definition with the corresponding substitutions furnishes a twist $\hat{f}$ in place of (6.21) which is given by $\hat{f} = (S^{-1} \otimes S^{-1})(f)$. Hence,

$$h := \hat{f}^{21} \equiv (S^{-1} \otimes S^{-1})(f^{21}) \tag{6.26}$$

has the following properties

$$h\Delta(a)h^{-1} = (S^{-1}\otimes S^{-1})\big(\Delta^{op}(S(a))\big) \tag{6.27}$$

$$\phi_h = (S^{-1}\otimes S^{-1}\otimes S^{-1})(\phi^{321}) \tag{6.28}$$

$$h\,\Delta(S^{-1}(\alpha)) = (S^{-1}\otimes S^{-1})(\gamma^{21}). \tag{6.29}$$

These identities will be used frequently below as well as the following

Corollary 6.1. *For $a\in\mathcal{G}$ let $\Delta_L(a) := h\Delta(a)$ and $\Delta_R(a) := \Delta(a)h^{-1}$ where $h\in\mathcal{G}\otimes\mathcal{G}$ is the twist* (6.26). *Then*

$$(\mathrm{id}\otimes\Delta_L)(\Delta_L(a))\,\phi = (S^{-1}\otimes S^{-1}\otimes S^{-1})(\phi^{321})\,(\Delta_L\otimes\mathrm{id})(\Delta_L(a)) \tag{6.30}$$

$$\phi\,(\Delta_R\otimes\mathrm{id})(\Delta_R(a)) = (\mathrm{id}\otimes\Delta_R)(\Delta_R(a))\,(S^{-1}\otimes S^{-1}\otimes S^{-1})(\phi^{321}),\quad \forall a\in\mathcal{G} \tag{6.31}$$

Proof. Writing Eq. (6.28) as

$$(\mathbf{1}\otimes h)\,(\mathrm{id}\otimes\Delta)(h)\,\phi = (S^{-1}\otimes S^{-1}\otimes S^{-1})(\phi^{321})\,(h\otimes\mathbf{1})\,(\Delta\otimes\mathrm{id})(h),$$

multiplicating both sides from the right with $(\Delta\otimes\mathrm{id})(\Delta(a))$ and using (6.8) yields Eq. (6.30). Eq. (6.31) is proven analogously. □

7 Coactions of quasi–Hopf algebras

The generalization of the definition of coactions as given in (2.1 - 2.4) to the quasi–Hopf case is straightforward:

Definition 7.1. A *left coaction* of a quasi-bialgebra $(\mathcal{G},\mathbf{1}_\mathcal{G},\Delta,\epsilon,\phi)$ on a unital algebra $\mathcal{M}$ is an algebra morphism $\lambda:\mathcal{M}\longrightarrow\mathcal{G}\otimes\mathcal{M}$ together with an invertible element $\phi_\lambda\in\mathcal{G}\otimes\mathcal{G}\otimes\mathcal{M}$ satisfying

$$(\mathrm{id}\otimes\lambda)(\lambda(m))\,\phi_\lambda = \phi_\lambda\,(\Delta\otimes\mathrm{id})(\lambda(m)),\qquad \forall m\in\mathcal{M} \tag{7.1}$$

$$(\mathbf{1}_\mathcal{G}\otimes\phi_\lambda)(\mathrm{id}\otimes\Delta\otimes\mathrm{id})(\phi_\lambda)(\phi\otimes\mathbf{1}_\mathcal{M}) = (\mathrm{id}\otimes\mathrm{id}\otimes\lambda)(\phi_\lambda)(\Delta\otimes\mathrm{id}\otimes\mathrm{id})(\phi_\lambda), \tag{7.2}$$

$$(\epsilon\otimes\mathrm{id})\circ\lambda = \mathrm{id} \tag{7.3}$$

$$(\mathrm{id}\otimes\epsilon\otimes\mathrm{id})(\phi_\lambda) = (\epsilon\otimes\mathrm{id}\otimes\mathrm{id})(\phi_\lambda) = \mathbf{1}_\mathcal{G}\otimes\mathbf{1}_\mathcal{M} \tag{7.4}$$

Similarly a *right coaction* of $\mathcal{G}$ on $\mathcal{M}$ is an algebra morphism $\rho:\mathcal{M}\longrightarrow\mathcal{M}\otimes\mathcal{G}$ together with $\phi_\rho\in\mathcal{M}\otimes\mathcal{G}\otimes\mathcal{G}$ such that

$$\phi_\rho\,(\rho\otimes\mathrm{id})(\rho(m)) = (\mathrm{id}\otimes\Delta)(\rho(m))\,\phi_\rho,\qquad \forall m\in\mathcal{M} \tag{7.5}$$

$$(\mathbf{1}_\mathcal{M}\otimes\phi)(\mathrm{id}\otimes\Delta\otimes\mathrm{id})(\phi_\rho)(\phi_\rho\otimes\mathbf{1}_\mathcal{G}) = (\mathrm{id}\otimes\mathrm{id}\otimes\Delta)(\phi_\rho)(\rho\otimes\mathrm{id}\otimes\mathrm{id})(\phi_\rho), \tag{7.6}$$

$$(\mathrm{id}\otimes\epsilon)\circ\rho = \mathrm{id} \tag{7.7}$$

$$(\mathrm{id}\otimes\epsilon\otimes\mathrm{id})(\phi_\rho) = (\mathrm{id}\otimes\mathrm{id}\otimes\epsilon)(\phi_\rho) = \mathbf{1}_\mathcal{M}\otimes\mathbf{1}_\mathcal{G} \tag{7.8}$$

The triple $(\mathcal{M},\lambda,\phi_\lambda)$ $[(\mathcal{M},\rho,\phi_\rho)]$ is called a left [right] *comodule algebra* over $\mathcal{G}$.

We remark, that of the two counit conditions in Eqs. (7.4) and (7.8), respectively, actually either one of them already implies the other. Clearly, if $\mathcal{G}$ is a Hopf algebra, $\phi = \mathbf{1}_\mathcal{G}\otimes\mathbf{1}_\mathcal{G}\otimes\mathbf{1}_\mathcal{G}$ and $\phi_\lambda = \mathbf{1}_\mathcal{G}\otimes\mathbf{1}_\mathcal{G}\otimes\mathbf{1}_\mathcal{M}$, one recovers the definitions given in (2.1 - 2.4). Also, particular examples are given by $\mathcal{M}=\mathcal{G}$ and $\lambda=\rho=\Delta$, $\phi_\lambda=\phi_\rho=\phi$. In the general case equations

(7.2),(7.6) may be understood as a generalized pentagon equation, whereas (7.1),(7.5) mean, that λ, ρ respect the quasi-coalgebra structure of $\mathcal{G}$. One should notice, that because of the pentagon equations (7.2) and (7.6), ϕ_λ and ϕ_ρ have to be nontrivial if ϕ is nontrivial (i.e. if $\mathcal{G}$ is not a Hopf algebra). On the other hand ϕ_λ or ϕ_ρ may be nontrivial even if $\phi = \mathbf{1}_\mathcal{G} \otimes \mathbf{1}_\mathcal{G} \otimes \mathbf{1}_\mathcal{G}$, i.e. if $\mathcal{G}$ is a Hopf algebra. In fact, such a restricted setting has been investigated before, see [DT, Lemma 10], [BCM, Lemma 4.5] or [BM, Eqs. (1.2-3)]. In [BCM,BM] Eq. (7.5) is called a "twisted module condition" and Eq. (7.6) (for $\phi = \mathbf{1} \otimes \mathbf{1} \otimes \mathbf{1}$) a "cocycle condition". We will see in Section 10 that the twisted crossed products considered in [DT,BCM,BM] are in fact special types of our diagonal crossed products to be given in Definition 10.1 below.

As has been remarked for ordinary Hopf algebras in Section 2, after a permutation of tensor factors $\mathcal{M} \otimes \mathcal{G} \leftrightarrow \mathcal{G} \otimes \mathcal{M}$ a left coaction of $\mathcal{G}$ may always be considered as a right coaction of $\mathcal{G}^{cop}$ (and vice versa) where one would have to identify $\phi_\rho := (\phi_\lambda^{-1})^{321}$.

As with Hopf algebras, a left coaction $\lambda : \mathcal{M} \longrightarrow \mathcal{G} \otimes \mathcal{M}$ induces a map $\triangleleft : \mathcal{M} \otimes \hat{\mathcal{G}} \longrightarrow \mathcal{M}$ by

$$m \triangleleft \varphi := (\varphi \otimes \mathrm{id})(\lambda(m)), \quad \varphi \in \hat{\mathcal{G}},\, m \in \mathcal{M} \tag{7.9}$$

which by convenient abuse of notation and terminology we still call a "right action" of $\hat{\mathcal{G}}$ on $\mathcal{M}$, despite of the fact that $\hat{\mathcal{G}}$ may not be an associative algebra. Similarly, we put $\varphi \triangleright m := (\mathrm{id} \otimes \varphi)(\rho(m))$ and call this a left $\hat{\mathcal{G}}$–action on $\mathcal{M}$.

Similarly as for the coproduct Δ there is a natural notion of *twist equivalence* for coactions of quasi–Hopf algebras.

Lemma 7.2. *Let $\rho : \mathcal{M} \longrightarrow \mathcal{M} \otimes \mathcal{G}$ be a right coaction of a quasi–bialgebra $(\mathcal{G}, \Delta, \epsilon, \phi)$ and let $U \in \mathcal{M} \otimes \mathcal{G}$ be invertible such that $(\mathrm{id} \otimes \epsilon)(U) = \mathbf{1}_\mathcal{M}$. Then*

$$\rho'(m) := U\, \rho(m)\, U^{-1}$$

again defines a coaction $\rho' : \mathcal{M} \longrightarrow \mathcal{M} \otimes \mathcal{G}$ with respect to the same quasi–bialgebra structure on $\mathcal{G}$ with twisted reassociator

$$\phi'_\rho = (\mathrm{id}_\mathcal{M} \otimes \Delta)(U)\, \phi_\rho\, (\rho \otimes \mathrm{id}_\mathcal{G})(U^{-1})\, (U^{-1} \otimes \mathbf{1}_\mathcal{G}).$$

The proof of Lemma 7.2 is straightforward and therefore omitted. A similar statement holds for left coactions λ, where one would have to take $U \in \mathcal{G} \otimes \mathcal{M}$ and

$$\phi'_\lambda = (\mathbf{1}_\mathcal{G} \otimes U)\, (\mathrm{id}_\mathcal{G} \otimes \lambda)(U)\, \phi_\lambda\, (\Delta \otimes \mathrm{id}_\mathcal{M})(U^{-1}).$$

Note that twisting indeed defines an equivalence relation for coactions. Similarly, if Δ_F and ϕ_F are given by Eqs. (6.15) and (6.16), then any right (left) $\mathcal{G}$–coaction on $\mathcal{M}$ may also be considered as a coaction with respect to the F–twisted structures on $\mathcal{G}$ by putting $\rho_F = \rho$ ($\lambda_F = \lambda$) and

$$(\phi_\rho)_F := (\mathbf{1}_\mathcal{M} \otimes F)\, \phi_\rho, \quad (\phi_\lambda)_F := \phi_\lambda\, (F^{-1} \otimes \mathbf{1}_\mathcal{M}). \tag{7.10}$$

The reader is invited to check that with these definitions Eqs (7.2) and (7.6) are indeed also twist covariant.

8 Two–sided coactions

As already mentioned before, the fact that the dual $\hat{\mathcal{G}}$ fails to be an associative algebra is the reason why there is no generalization of the definitions of ordinary crossed products to the quasi–Hopf algebra case. Nevertheless this will be possible for our diagonal crossed product constructed from two–sided coactions. First we need

Definition 8.1. A *two–sided coaction* of a quasi–bialgebra $(\mathcal{G},\Delta,\epsilon,\phi)$ on an algebra $\mathcal{M}$ is an algebra map $\delta:\ \mathcal{M}\longrightarrow\mathcal{G}\otimes\mathcal{M}\otimes\mathcal{G}$ together with an invertible element $\Psi\in\mathcal{G}\otimes\mathcal{G}\otimes\mathcal{M}\otimes\mathcal{G}\otimes\mathcal{G}$ satisfying

$$(\mathrm{id}_{\mathcal{G}}\otimes\delta\otimes\mathrm{id}_{\mathcal{G}})(\delta(m))\,\Psi=\Psi\,(\Delta\otimes\mathrm{id}_{\mathcal{M}}\otimes\Delta)(\delta(m)),\quad\forall m\in\mathcal{M}\tag{8.1}$$

$$\begin{aligned}(\mathbf{1}_{\mathcal{G}}\otimes\Psi\otimes\mathbf{1}_{\mathcal{G}})\,&(\mathrm{id}_{\mathcal{G}}\otimes\Delta\otimes\mathrm{id}_{\mathcal{M}}\otimes\Delta\otimes\mathrm{id}_{\mathcal{G}})(\Psi)\,(\phi\otimes\mathbf{1}_{\mathcal{M}}\otimes\phi^{-1})\\&=(\mathrm{id}_{\mathcal{G}}\otimes\mathrm{id}_{\mathcal{G}}\otimes\delta\otimes\mathrm{id}_{\mathcal{G}}\otimes\mathrm{id}_{\mathcal{G}})(\Psi)\,(\Delta\otimes\mathrm{id}_{\mathcal{G}}\otimes\mathrm{id}_{\mathcal{M}}\otimes\mathrm{id}_{\mathcal{G}}\otimes\Delta)(\Psi)\end{aligned}\tag{8.2}$$

$$(\epsilon\otimes\mathrm{id}_{\mathcal{M}}\otimes\epsilon)\circ\delta=\mathrm{id}_{\mathcal{M}}\tag{8.3}$$

$$(\mathrm{id}_{\mathcal{G}}\otimes\epsilon\otimes\mathrm{id}_{\mathcal{M}}\otimes\epsilon\otimes\mathrm{id}_{\mathcal{G}})(\Psi)=(\epsilon\otimes\mathrm{id}_{\mathcal{G}}\otimes\mathrm{id}_{\mathcal{M}}\otimes\mathrm{id}_{\mathcal{G}}\otimes\epsilon)(\Psi)=\mathbf{1}_{\mathcal{G}}\otimes\mathbf{1}_{\mathcal{M}}\otimes\mathbf{1}_{\mathcal{G}}.\tag{8.4}$$

Again we remark, that either one of the two counit axioms in Eq. (8.4) already implies the other. Moreover, applying $\epsilon_i\otimes\mathrm{id}_{\mathcal{M}}\otimes\epsilon_j$, $1\le i,j\le 3$, to both sides of (8.2), where $\epsilon_i:\mathcal{G}^{\otimes^3}\to\mathcal{G}^{\otimes^2}$ is given by acting with ϵ on the i–th tensor factor, one gets additional identities, which are collected in the Appendix. We also note, that as in Eqs. (3.6) and (3.7) two–sided coactions could of course be considered as right coactions of $\mathcal{G}\otimes\mathcal{G}^{cop}$, or left coactions of $\mathcal{G}^{cop}\otimes\mathcal{G}$, respectively. Moreover, if (δ,Ψ) is a two–sided coaction of $\mathcal{G}$ on $\mathcal{M}$, then (δ,Ψ^{-1}) is a two–sided coaction of $\mathcal{G}_{op}$ on $\mathcal{M}_{op}$ and (δ_{op},Ψ_{op}) is a two–sided coaction of $\mathcal{G}^{cop}$ on $\mathcal{M}$, where

$$\delta_{op}:=\delta^{321},\quad\Psi_{op}:=\Psi^{54321}.\tag{8.5}$$

An example of a two–sided coaction is given by $\mathcal{M}=\mathcal{G}$, $\delta=(\Delta\otimes\mathrm{id})\circ\Delta$ and

$$\Psi:=[(\mathrm{id}\otimes\Delta\otimes\mathrm{id})(\phi)\otimes\mathbf{1}][\phi\otimes\mathbf{1}\otimes\mathbf{1}][(\delta\otimes\mathrm{id}\otimes\mathrm{id})(\phi^{-1})]$$

Similarly we could choose $\delta'=(\mathrm{id}\otimes\Delta)\circ\Delta$ and

$$\Psi':=[\mathbf{1}\otimes(\mathrm{id}\otimes\Delta\otimes\mathrm{id})(\phi^{-1})][\mathbf{1}\otimes\mathbf{1}\otimes\phi^{-1}][(\mathrm{id}\otimes\mathrm{id}\otimes\delta')(\phi)].$$

From this example one already realizes that in the present context the relation between two–sided coactions and pairs of commuting left and right coactions gets somewhat more involved as compared to Section 3, where we had $\delta=\delta'$. First, one easily checks that for any two–sided coaction (δ,Ψ) the definitions

$$\lambda:=(\mathrm{id}_{\mathcal{G}}\otimes\mathrm{id}_{\mathcal{M}}\otimes\epsilon)\circ\delta\qquad\qquad\phi_\lambda:=(\mathrm{id}_{\mathcal{G}}\otimes\mathrm{id}_{\mathcal{G}}\otimes\mathrm{id}_{\mathcal{M}}\otimes\epsilon\otimes\epsilon)(\Psi),\tag{8.6}$$

$$\rho:=(\epsilon\otimes\mathrm{id}_{\mathcal{M}}\otimes\mathrm{id}_{\mathcal{G}})\circ\delta\qquad\qquad\phi_\rho^{-1}:=(\epsilon\otimes\epsilon\otimes\mathrm{id}_{\mathcal{M}}\otimes\mathrm{id}_{\mathcal{G}}\otimes\mathrm{id}_{\mathcal{G}})(\Psi)\tag{8.7}$$

provide us again with a left coaction (λ,ϕ_λ) and a right coaction (ρ,ϕ_ρ). Moreover, putting $\delta^{(2)}:=(\mathrm{id}\otimes\delta\otimes\mathrm{id})\circ\delta$ we have

$$(\lambda\otimes\mathrm{id}_{\mathcal{G}})\circ\rho=(\epsilon\otimes\mathrm{id}_{\mathcal{G}}\otimes\mathrm{id}_{\mathcal{M}}\otimes\epsilon\otimes\mathrm{id}_{\mathcal{G}})\circ\delta^{(2)}\tag{8.8}$$

$$(\mathrm{id}_{\mathcal{G}}\otimes\rho)\circ\lambda=(\mathrm{id}_{\mathcal{G}}\otimes\epsilon\otimes\mathrm{id}_{\mathcal{M}}\otimes\mathrm{id}_{\mathcal{G}}\otimes\epsilon)\circ\delta^{(2)}\tag{8.9}$$

However, due to the appearance of the reassociator Ψ in the axiom (8.1), the two expressions (8.8) and (8.9) are in general unequal, and neither one needs to coincide with δ. Instead, as we will show now, λ and ρ *quasi–commute* and both expressions, $(\lambda\otimes\mathrm{id}_{\mathcal{G}})\circ\rho$ and $(\mathrm{id}_{\mathcal{G}}\otimes\rho)\circ\lambda$, define two–sided coactions which are *twist equivalent* to (δ,Ψ). First we give

Definition 8.2. Let $(\mathcal{G}, \Delta, \epsilon, \phi)$ be a quasi–bialgebra. By a *quasi–commuting pair* of $\mathcal{G}$–coactions on an algebra $\mathcal{M}$ we mean a quintuple $(\lambda, \rho, \phi_\lambda, \phi_\rho, \phi_{\lambda\rho})$, where (λ, ϕ_λ) and (ρ, ϕ_ρ) are left and right $\mathcal{G}$–coactions on $\mathcal{M}$, respectively, and where $\phi_{\lambda\rho} \in \mathcal{G} \otimes \mathcal{M} \otimes \mathcal{G}$ is invertible and satisfies

$$\phi_{\lambda\rho}\,(\lambda \otimes \mathrm{id})(\rho(m)) = (\mathrm{id} \otimes \rho)(\lambda(m))\,\phi_{\lambda\rho}, \qquad \forall m \in \mathcal{M} \tag{8.10}$$

$$(\mathbf{1}_\mathcal{G} \otimes \phi_{\lambda\rho})(\mathrm{id} \otimes \lambda \otimes \mathrm{id})(\phi_{\lambda\rho})(\phi_\lambda \otimes \mathbf{1}_\mathcal{G}) = (\mathrm{id} \otimes \mathrm{id} \otimes \rho)(\phi_\lambda)(\Delta \otimes \mathrm{id} \otimes \mathrm{id})(\phi_{\lambda\rho}) \tag{8.11}$$

$$(\mathbf{1}_\mathcal{G} \otimes \phi_\rho)(\mathrm{id} \otimes \rho \otimes \mathrm{id})(\phi_{\lambda\rho})(\phi_{\lambda\rho} \otimes \mathbf{1}_\mathcal{G}) = (\mathrm{id} \otimes \mathrm{id} \otimes \Delta)(\phi_{\lambda\rho})(\lambda \otimes \mathrm{id} \otimes \mathrm{id})(\phi_\rho) \tag{8.12}$$

Obviously, the conditions (8.10) - (8.12) apply to the case $\mathcal{M} = \mathcal{G}$, $\lambda = \rho = \Delta$ and $\phi_\lambda = \phi_\rho = \phi$. Also note, that acting with $(\epsilon \otimes \epsilon \otimes \mathrm{id}_\mathcal{M} \otimes \mathrm{id}_\mathcal{G})$ on Eq. (8.11) and with $(\mathrm{id}_\mathcal{G} \otimes \mathrm{id}_\mathcal{M} \otimes \epsilon \otimes \epsilon)$ on Eq. (8.12) and using the invertibility of $\phi_{\lambda\rho}$ one concludes the further identities

$$(\mathrm{id}_\mathcal{G} \otimes \mathrm{id}_\mathcal{M} \otimes \epsilon)(\phi_{\lambda\rho}) = \mathbf{1}_\mathcal{G} \otimes \mathbf{1}_\mathcal{M}, \quad (\epsilon \otimes \mathrm{id}_\mathcal{M} \otimes \mathrm{id}_\mathcal{G})(\phi_{\lambda\rho}) = \mathbf{1}_\mathcal{M} \otimes \mathbf{1}_\mathcal{G}. \tag{8.13}$$

Let us also observe that quasi–commutativity is stable under twisting. Indeed, if $U_\lambda \in \mathcal{G} \otimes \mathcal{M}$ is a twist from (λ, ϕ_λ) to $(\lambda', \phi'_\lambda)$ and $U_\rho \in \mathcal{M} \otimes \mathcal{G}$ is a twist from (ρ, ϕ_ρ) to (ρ', ϕ'_ρ), then $(\lambda, \rho, \phi_\lambda, \phi_\rho, \phi_{\lambda\rho})$ is a quasi–commuting pair if and only if $(\lambda', \rho', \phi'_\lambda, \phi'_\rho, \phi'_{\lambda\rho})$ is a quasi–commuting pair, where $\phi'_{\lambda\rho}$ is given by

$$\phi'_{\lambda\rho} = (\mathbf{1}_\mathcal{G} \otimes U_\rho)\,(\mathrm{id}_\mathcal{G} \otimes \rho)(U_\lambda)\,\phi_{\lambda\rho}\,(\lambda \otimes \mathrm{id}_\mathcal{G})(U_\rho^{-1})\,(U_\lambda \otimes \mathbf{1}_\mathcal{G})^{-1}. \tag{8.14}$$

In this case we say that $(\lambda, \rho, \phi_\lambda, \phi_\rho, \phi_{\lambda\rho})$ and $(\lambda', \rho', \phi'_\lambda, \phi'_\rho, \phi'_{\lambda\rho})$ are twist equivalent as quasi–commuting pairs of coactions.

Next we point out that two–sided coactions (δ, Ψ) may be twisted in the same fashion as one–sided ones.

Definition 8.3. Let (δ, Ψ) and (δ', Ψ') be two–sided coactions of $(\mathcal{G}, \Delta, \epsilon, \phi)$ on $\mathcal{M}$. Then (δ', Ψ') is called *twist equivalent* to (δ, Ψ), if there exists $U \in \mathcal{G} \otimes \mathcal{M} \otimes \mathcal{G}$ invertible such that

$$\delta'(m) = U\,\delta(m)\,U^{-1} \tag{8.15}$$

$$\Psi' = (\mathbf{1}_\mathcal{G} \otimes U \otimes \mathbf{1}_\mathcal{G})\,(\mathrm{id}_\mathcal{G} \otimes \delta \otimes \mathrm{id}_\mathcal{G})(U)\,\Psi\,(\Delta \otimes \mathrm{id}_\mathcal{M} \otimes \Delta)(U^{-1}) \tag{8.16}$$

$$(\epsilon \otimes \mathrm{id}_\mathcal{M} \otimes \epsilon)(U) = \mathbf{1}_\mathcal{M} \tag{8.17}$$

The reader is invited to check that for any two–sided coaction (δ, Ψ) and any invertible U satisfying (8.17) the definitions (8.15) and (8.16) indeed produce another two–sided coaction (δ', Ψ'). It is also easy to see that twisting does provide an equivalence relation between two–sided coactions. Moreover, similarly as for one–sided coactions one readily verifies that if (δ, Ψ) is a two–sided coaction of $(\mathcal{G}, \Delta, \epsilon, \phi)$ on $\mathcal{M}$, then for any twist $F \in \mathcal{G} \otimes \mathcal{G}$ the pair (δ, Ψ_F) is a two–sided coaction of $(\mathcal{G}, \Delta_F, \epsilon, \phi_F)$ on $\mathcal{M}$, where Δ_F and ϕ_F are the twisted structures on $\mathcal{G}$ given by Eqs. (6.15) and (6.16), and where

$$\Psi_F := \Psi\,(F^{-1} \otimes \mathbf{1}_\mathcal{M} \otimes F^{-1}) \tag{8.18}$$

We now show that quasi–commuting pairs of $\mathcal{G}$–coactions $(\lambda, \rho, \phi_\lambda, \phi_\rho, \phi_{\lambda\rho})$ always give rise to two–sided $\mathcal{G}$–coactions as follows

Proposition 8.4. *Let (λ, ϕ_λ) and (ρ, ϕ_ρ) be a pair of left and right $\mathcal{G}$–coactions, respectively, on $\mathcal{M}$, let $\phi_{\lambda\rho} \in \mathcal{G} \otimes \mathcal{M} \otimes \mathcal{G}$ be invertible and put*

$$\delta_l := (\lambda \otimes \mathrm{id}) \circ \rho \tag{8.19}$$

$$\Psi_l := (\mathrm{id}_\mathcal{G} \otimes \lambda \otimes \mathrm{id}_\mathcal{G}^{\otimes^2})\Big((\phi_{\lambda\rho} \otimes \mathbf{1}_\mathcal{G})(\lambda \otimes \mathrm{id}_\mathcal{G}^{\otimes^2})(\phi_\rho^{-1})\Big)\,[\phi_\lambda \otimes \mathbf{1}_\mathcal{G} \otimes \mathbf{1}_\mathcal{G}] \tag{8.20}$$

$$\delta_r := (\mathrm{id} \otimes \rho) \circ \lambda \tag{8.21}$$

$$\Psi_r := (\mathrm{id}_\mathcal{G}^{\otimes^2} \otimes \rho \otimes \mathrm{id}_\mathcal{G})\Big((\mathbf{1}_\mathcal{G} \otimes \phi_{\lambda\rho}^{-1})\,(\mathrm{id}_\mathcal{G}^{\otimes^2} \otimes \rho)(\phi_\lambda)\Big)\,[\mathbf{1}_\mathcal{G} \otimes \mathbf{1}_\mathcal{G} \otimes \phi_\rho^{-1}] \tag{8.22}$$

1. Consider the following conditions

(i) $(\lambda, \rho, \phi_\lambda, \phi_\rho, \phi_{\lambda\rho})$ *is a quasi–commuting pair of $\mathcal{G}$–coactions.*

(ii) (δ_l, Ψ_l) *is a two–sided $\mathcal{G}$–coaction*

(iii) (δ_r, Ψ_r) *is a two–sided $\mathcal{G}$–coaction*

Then (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii), and under these conditions $\phi_{\lambda\rho}$ is a twist equivalence from (δ_l, Ψ_l) to (δ_r, Ψ_r).

2. Under the conditions of 1.) let $(\lambda', \rho', \phi'_\lambda, \phi'_\rho, \phi'_{\lambda\rho})$ be a quasi–commuting pair of coactions obtained by twisting with $U_\lambda \in \mathcal{G} \otimes \mathcal{M}$ and $U_\rho \in \mathcal{M} \otimes \mathcal{G}$, and let $(\delta'_{l/r}, \Psi'_{l,r})$ be the associated two–sided coactions. Then $(U_\lambda \otimes \mathbf{1}_\mathcal{G})(\lambda \otimes \mathrm{id}_\mathcal{G})(U_\rho)$ is a twist from (δ_l, Ψ_l) to (δ'_l, Ψ'_l) and $(\mathbf{1}_\mathcal{G} \otimes U_\rho)(\mathrm{id}_\mathcal{G} \otimes \rho)(U_\lambda)$ is a twist from (δ_r, Ψ_r) to (δ'_r, Ψ'_r).

Proof. The proofs of the implications $(i) \Rightarrow (ii)$ and $(i) \Rightarrow (iii)$ as well as the proof of part 2.) are lengthy but straightforward and therefore omitted. The proof of the implication $(ii) \Rightarrow (i)$ is given in Appendix A, the implication $(iii) \Rightarrow (i)$ being analogous. □

Using this result we are now in the position to show that twist–equivalence classes of quasi–commuting pairs of coactions $(\lambda, \rho, \phi_\lambda, \phi_\rho, \phi_{\lambda\rho})$ are in one-to-one correspondence with twist equivalence classes of two–sided coactions (δ, Ψ), since up to twist equivalence any two–sided coaction is of the type $(\delta_{l/r}, \Psi_{l/r})$ given in (8.19) - (8.22).

Proposition 8.5. *Let (δ, Ψ) be a two–sided $\mathcal{G}$–coaction on $\mathcal{M}$ and let (λ, ϕ_λ) and (ρ, ϕ_ρ) be the pair of associated left and right $\mathcal{G}$-coactions, respectively, given in Eqs.* (8.6) *and* (8.7). *Define $U_{l/r}, \phi_{\lambda\rho} \in \mathcal{G} \otimes \mathcal{M} \otimes \mathcal{G}$ by*

$$U_l := (\epsilon \otimes \mathrm{id}_\mathcal{G} \otimes \mathrm{id}_\mathcal{M} \otimes \epsilon \otimes \mathrm{id}_\mathcal{G})(\Psi) \tag{8.23}$$

$$U_r := (\mathrm{id}_\mathcal{G} \otimes \epsilon \otimes \mathrm{id}_\mathcal{M} \otimes \mathrm{id}_\mathcal{G} \otimes \epsilon)(\Psi) \tag{8.24}$$

$$\phi_{\lambda\rho} := U_r U_l^{-1} \tag{8.25}$$

and let $(\delta_{l/r}, \Psi_{l/r})$ be given in terms of $(\lambda, \rho, \phi_\lambda, \phi_\rho, \phi_{\lambda\rho})$ by Eqs. (8.19) - (8.22). *Then*

(i) $(\lambda, \rho, \phi_\lambda, \phi_\rho, \phi_{\lambda\rho})$ *is a quasi–commuting pair of $\mathcal{G}$–coactions.*

(ii) $U_{l/r}$ *provides a twist equivalence from (δ, Ψ) to $(\delta_{l/r}, \Psi_{l/r})$.*

(iii) If (δ', Ψ') is twist equivalent to (δ, Ψ) then the associated quasi–commuting pair $(\lambda', \rho', \phi'_\lambda, \phi'_\rho, \phi'_{\lambda\rho})$ is twist equivalent to $(\lambda, \rho, \phi_\lambda, \phi_\rho, \phi_{\lambda\rho})$.

Proof. Part (iii) is straightforward and omitted. Part (i) follows from part (ii) by part 1. of Proposition 8.4, since the twist equivalence in part (ii) already guarantees that $(\delta_{l/r}, \Psi_{l/r})$ provide two–sided coactions. The proof of part (ii) is given in Appendix A. □

9 The representation theoretic interpretation

In this Section we provide a representation theoretic interpretation of our notions of left, right and two–sided coactions. From now on let $\mathcal{G}$ be a quasi–Hopf algebra with invertible antipode S. Let $\mathrm{Rep}\,\mathcal{M}$ and $\mathrm{Rep}\,\mathcal{G}$ be the category of unital representations of $\mathcal{M}$ and $\mathcal{G}$, respectively, where in $\mathrm{Rep}\,\mathcal{G}$ we only mean to speak of finite dimensional representations. We denote the objects in $\mathrm{Rep}\,\mathcal{G}$ by $(U,\pi_U),(V,\pi_V),(W,\pi_W),\ldots$, where $U,V,W\ldots$ denote the underlying representation spaces and $\pi_V:\mathcal{G}\longrightarrow \mathrm{End}_{\,\mathbb{C}}(V)$ the representation maps. Similarly, we denote the objects in $\mathrm{Rep}\,\mathcal{M}$ by $(\mathfrak{H},\gamma_{\mathfrak{H}}),(\mathfrak{K},\gamma_{\mathfrak{K}}),(\mathfrak{L},\gamma_{\mathfrak{L}}),\ldots$, where the Gothic symbols denote the representation spaces and where $\gamma_{\mathfrak{H}}:\mathcal{M}\longrightarrow \mathrm{End}_{\,\mathbb{C}}(\mathfrak{H})$, etc. We will also freely use the $\mathcal{G}$–module notation by writing $a\cdot v:=\pi_V(a)v$ and $V\equiv(V,\pi_V)$ (and analogously for $\mathcal{M}$–modules $\mathfrak{H}$). The set of morphisms $\mathrm{Hom}_{\mathcal{G}}(U,V)$ (also called intertwiners) is given by the linear maps $f:U\longrightarrow V$ satisfying $f\,\pi_U(a)=\pi_V(a)\,f,\ \forall a\in\mathcal{G}$.

It is well known (see e.g. [Dr2]) that for quasi–Hopf algebras $\mathcal{G}$ the category $\mathrm{Rep}\,\mathcal{G}$ becomes a rigid monoidal category, where the tensor product $(V\boxtimes W,\pi_V\boxtimes\pi_W)$ of two representations (V,π_V) and (W,π_W) is given by

$$V\boxtimes W:=V\otimes W \text{ and } \pi_V\boxtimes\pi_W:=(\pi_V\otimes\pi_W)\circ\Delta \tag{9.1}$$

whereas for morphisms ($\equiv\mathcal{G}$–module intertwiners) f,g one has $f\boxtimes g:=f\otimes g$. (The symbol $\otimes$ always denotes the usual tensor product in the category of vector spaces.) The associativity isomorphisms are given in terms of the reassociator ϕ by the natural family of $\mathcal{G}$–module isomorphisms

$$\phi_{UVW}:(U\boxtimes V)\boxtimes W\longrightarrow U\boxtimes(V\boxtimes W),\quad \phi_{UVW}:=(\pi_U\otimes\pi_V\otimes\pi_W)(\phi) \tag{9.2}$$

The unit object in $\mathrm{Rep}\,\mathcal{G}$ is given by $(\mathbb{C},\epsilon)$. Throughout, if $\mathbb{C}$ is viewed as a $\mathcal{G}$–module it is always meant to be equipped the module structure given by the one dimensional representation ϵ. The left and right dual of any representation (V,π_V) are defined by ${}^*V=V^*=\hat{V}:=\mathrm{Hom}_{\mathbb{C}}(V,\mathbb{C})$ and

$$\pi_{{}^*V}:=\pi_V^t\circ S,\quad \pi_{V^*}:=\pi_V^t\circ S^{-1}, \tag{9.3}$$

where ^{t} denotes the transposed map. The (left) rigidity structure is given by the family of morphisms ($\mathcal{G}$–module intertwiners)

$$a_V:{}^*V\boxtimes V\longrightarrow\mathbb{C},\qquad \hat{v}\otimes v\longmapsto\langle\hat{v}\mid\alpha_V\,v\rangle \tag{9.4}$$

$$b_V:\mathbb{C}\longrightarrow V\boxtimes{}^*V,\qquad 1\longmapsto\beta_V\,v_i\otimes v^i, \tag{9.5}$$

where $\alpha_V\equiv\pi_V(\alpha)$, $\beta_V\equiv\pi_V(\beta)$ and where $v_i\in V$ and $v^i\in\hat{V}$ are a choice of dual basses. Drinfel'd's antipode axioms for $\mathcal{G}$ precisely reflect the fact that a_V and b_V are morphisms in $\mathrm{Rep}\,\mathcal{G}$ fulfilling the *rigidity identities*

$$(\mathrm{id}_V\boxtimes a_V)\circ\phi_{V({}^*V)V}\circ(b_V\boxtimes\mathrm{id}_V)=\mathrm{id}_V \tag{9.6}$$

$$(a_V\boxtimes\mathrm{id}_{{}^*V})\circ\phi^{-1}_{({}^*V)V({}^*V)}\circ(\mathrm{id}_{{}^*V}\boxtimes b_V)=\mathrm{id}_{{}^*V}. \tag{9.7}$$

Also note that one has ${}^*(V^*)=({}^*V)^*=V$ with trivial identification.

Next, we recall that in any left–rigid monoidal category one has natural isomorphisms ${}^*(U\boxtimes V)\cong{}^*V\boxtimes{}^*U$. It has been mentioned by Drinfel'd [Dr2] that in our case these isomorphisms are given by

$$f_{UV}:U\boxtimes V\longrightarrow({}^*V\boxtimes{}^*U)^*,\quad u\otimes v\mapsto(\pi_V\otimes\pi_U)(f)\,(v\otimes u) \tag{9.8}$$

where we trivially identify the vector spaces $V \otimes W \equiv (\hat{V} \otimes \hat{W})^\wedge$ and where the twist $f \in \mathcal{G} \otimes \mathcal{G}$ is given by (6.21). The fact that f_{UV} is indeed a morphism in $\mathrm{Rep}\,\mathcal{G}$ follows from (6.23). Similarly, we have a natural family of isomorphisms

$$h_{UV} : U \boxtimes V \longrightarrow {}^*(V^* \boxtimes U^*), \quad u \otimes v \mapsto (\pi_V \otimes \pi_U)(h)\,(v \otimes u) \tag{9.9}$$

see Eqs. (6.26), (6.27).

Now a left $\mathcal{G}$–coaction (λ, ϕ_λ) on $\mathcal{M}$ naturally induces a *left action* of $\mathrm{Rep}\,\mathcal{G}$ on $\mathrm{Rep}\,\mathcal{M}$. By this we mean a functor

$$\odot : \mathrm{Rep}\,\mathcal{G} \times \mathrm{Rep}\,\mathcal{M} \longrightarrow \mathrm{Rep}\,\mathcal{M}, \tag{9.10}$$

where for $(V, \pi) \in \mathrm{Rep}\,\mathcal{G}$ and $(\mathfrak{H}, \gamma) \in \mathrm{Rep}\,\mathcal{M}$ we define $(V \odot \mathfrak{H}, \pi \odot \gamma) \in \mathrm{Rep}\,\mathcal{M}$ by

$$V \odot \mathfrak{H} = V \otimes \mathfrak{H}, \quad \pi \odot \gamma := (\pi \otimes \gamma) \circ \lambda, \tag{9.11}$$

whereas for morphisms we put $f \odot g := f \otimes g$. The counit axiom for λ implies $\epsilon \odot \gamma_{\mathfrak{H}} = \gamma_{\mathfrak{H}}$ for all $(\mathfrak{H}, \gamma_{\mathfrak{H}}) \in \mathrm{Rep}\,\mathcal{M}$ and the axioms for ϕ_λ imply the quasi–associativity relations

$$(\pi_V \boxtimes \pi_W) \odot \gamma_{\mathfrak{H}} \cong \pi_V \boxtimes (\pi_W \odot \gamma_{\mathfrak{H}})$$

where the isomorphism is given by

$$\phi_{VW\mathfrak{H}} := (\pi_V \otimes \pi_W \otimes \gamma_{\mathfrak{H}})(\phi_\lambda). \tag{9.12}$$

Finally, the pentagon axiom (7.2) provides us with the analogue of McLane's coherence conditions, i.e. the following commuting diagram

$$\begin{tikzcd}
 & (U \boxtimes V) \odot (W \odot \mathfrak{H}) \arrow[dr, "\phi_{UV(W\odot\mathfrak{H})}"] & \\
((U \boxtimes V) \boxtimes W) \odot \mathfrak{H} \arrow[ur, "\phi_{(U\boxtimes V)W\mathfrak{H}}"] \arrow[d, "\phi_{UVW} \odot \mathrm{id}_{\mathfrak{H}}"'] & & U \odot (V \odot (W \odot \mathfrak{H})) \\
(U \boxtimes (V \boxtimes W)) \odot \mathfrak{H} \arrow[r, "\phi_{U(V\boxtimes W)\mathfrak{H}}"] & U \odot ((V \boxtimes W) \odot \mathfrak{H}) \arrow[ur, "\mathrm{id}_U \odot \phi_{VW\mathfrak{H}}"'] &
\end{tikzcd} \tag{9.13}$$

With the obvious substitutions analogue statements hold for right coactions (ρ, ϕ_ρ), where now these induce a right action $\odot : \mathrm{Rep}\,\mathcal{M} \times \mathrm{Rep}\,\mathcal{G} \longrightarrow \mathrm{Rep}\,\mathcal{M}$ given for $(\gamma, \mathfrak{H}) \in \mathrm{Rep}\,\mathcal{M}$ and $(\pi, V) \in \mathrm{Rep}\,\mathcal{G}$ by

$$\mathfrak{H} \odot V := \mathfrak{H} \otimes V, \quad \gamma \odot \pi := (\gamma \otimes \pi) \circ \rho. \tag{9.14}$$

Finally, a quasi–commuting pair $(\lambda, \rho, \phi_\lambda, \phi_\rho, \phi_{\lambda\rho})$ provides us with both, a left and a right action of $\mathrm{Rep}\,\mathcal{G}$ on $\mathrm{Rep}\,\mathcal{M}$, together with a further family of associativity equivalences $(\pi_U \odot \gamma_{\mathfrak{H}}) \odot \pi_V \cong \pi_U \odot (\gamma_{\mathfrak{H}} \odot \pi_V)$, where now the isomorphisms are given by

$$\phi_{U\mathfrak{H}V} := (\pi_U \otimes \gamma_{\mathfrak{H}} \otimes \pi_V)(\phi_{\lambda\rho}). \tag{9.15}$$

Again, the conditions (8.11) and (8.12) imply further pentagon diagrams of the type (9.13) with objects of the type $UV\mathfrak{H}W$ or $U\mathfrak{H}VW$, respectively, in appropriate bracket positions.

In the obvious way the above may also be generalized to arbitrary two–sided coactions δ, in which case we would obtain a functor

$$\operatorname{Rep}\mathcal{G} \times \operatorname{Rep}\mathcal{M} \times \operatorname{Rep}\mathcal{G} \longrightarrow \operatorname{Rep}\mathcal{M}$$

denoted as

$$U \gtrdot \mathfrak{H} \lessdot V := U \otimes \mathfrak{H} \otimes V, \quad \pi_U \gtrdot \gamma_{\mathfrak{H}} \lessdot \pi_V := (\pi_U \otimes \gamma_{\mathfrak{H}} \otimes \pi_V) \circ \delta \tag{9.16}$$

together with associativity isomorphisms
$(\pi_U \boxtimes \pi_V) \gtrdot \gamma_{\mathfrak{H}} \lessdot (\pi_W \boxtimes \pi_Z) \cong \pi_U \gtrdot (\pi_V \gtrdot \gamma_{\mathfrak{H}} \lessdot \pi_W) \lessdot \pi_Z$ given by

$$\Psi_{UV\mathfrak{H}WZ} := (\pi_U \otimes \pi_V \otimes \gamma_{\mathfrak{H}} \otimes \pi_W \otimes \pi_Z)(\Psi) \tag{9.17}$$

and obeying analogue "two–sided" pentagon diagrams. Note that the operations $\gtrdot$ and $\lessdot$ are *not* defined individually, i.e. only the two–sided operation $\gtrdot \cdot \lessdot$ makes sense. According to Proposition 8.5 the relation between two–sided $\operatorname{Rep}\mathcal{G}$–actions and one–sided $\operatorname{Rep}\mathcal{G}$–actions is given by

$$\pi_V \odot \gamma_{\mathfrak{H}} := \pi_V \gtrdot \gamma_{\mathfrak{H}} \lessdot \epsilon, \quad \gamma_{\mathfrak{H}} \odot \pi_V := \epsilon \gtrdot \gamma_{\mathfrak{H}} \lessdot \pi_V \tag{9.18}$$

implying

$$(\pi_V \odot \gamma_{\mathfrak{H}}) \odot \pi_U \cong \pi_V \gtrdot \gamma_{\mathfrak{H}} \lessdot \pi_U \cong \pi_V \odot (\gamma_{\mathfrak{H}} \odot \pi_U) \tag{9.19}$$

where here the intertwiners are given by $(\pi_V \otimes \gamma_{\mathfrak{H}} \otimes \pi_U)(U_{l/r})$, respectively, see Proposition 8.5.

Motivated by this representation theoretic interpretation we now introduce various natural algebraic objects associated with left, right and two–sided coactions, respectively, which will play a central role later, when constructing and identifying different left and right versions of our diagonal crossed products. Associated with any left $\mathcal{G}$–coaction (λ, ϕ_λ) on $\mathcal{M}$ we define elements $p_\lambda, q_\lambda \in \mathcal{G} \otimes \mathcal{M}$ by

$$p_\lambda := \phi_\lambda^2 S^{-1}(\phi_\lambda^1 \beta) \otimes \phi_\lambda^3, \quad \text{where} \quad \phi_\lambda = \phi_\lambda^1 \otimes \phi_\lambda^2 \otimes \phi_\lambda^3, \tag{9.20}$$
$$q_\lambda := S(\bar{\phi}_\lambda^1)\alpha\bar{\phi}_\lambda^2 \otimes \bar{\phi}_\lambda^3, \quad \text{where} \quad \phi_\lambda^{-1} = \bar{\phi}_\lambda^1 \otimes \bar{\phi}_\lambda^2 \otimes \bar{\phi}_\lambda^3, \tag{9.21}$$

and where as before we have dropped summation indices and summation symbols. Similarly, associated with any right $\mathcal{G}$–coaction (ρ, ϕ_ρ) on $\mathcal{M}$ we define elements $p_\rho, q_\rho \in \mathcal{M} \otimes \mathcal{G}$ by

$$p_\rho := \bar{\phi}_\rho^1 \otimes \bar{\phi}_\rho^2 \beta S(\bar{\phi}_\rho^3), \quad \text{where} \quad \phi_\rho^{-1} = \bar{\phi}_\rho^1 \otimes \bar{\phi}_\rho^2 \otimes \bar{\phi}_\rho^3 \tag{9.22}$$
$$q_\rho := \phi_\rho^1 \otimes S^{-1}(\alpha\phi_\rho^3)\phi_\rho^2, \quad \text{where} \quad \phi_\rho = \phi_\rho^1 \otimes \phi_\rho^2 \otimes \phi_\rho^3 \tag{9.23}$$

Here α, β are the elements introduced in Eq. (6.13). In the case $\mathcal{M} = \mathcal{G}$ and $(\lambda, \phi_\lambda) = (\rho, \phi_\rho) = (\Delta, \phi)$ analogues of these elements have also been considered by [Dr2,S]. They allow for generalizations of formulas like $m_{(0)} \otimes S^{-1}(m_{(2)})m_{(1)} = m \otimes \mathbf{1}_{\mathcal{G}}$, etc. which will be needed to generalize Lemma 5.2.

Lemma 9.1.

1. *Let (λ,ϕ_λ) be a left $\mathcal{G}$–coaction on $\mathcal{M}$ and let p_λ, q_λ be given by* (9.20),(9.21). *Then the following identities hold for all $m\in\mathcal{M}$, where $\lambda(m)\equiv m_{(-1)}\otimes m_{(0)}$*

$$\lambda(m_{(0)})\, p_\lambda\, [S^{-1}(m_{(-1)})\otimes \mathbf{1}_{\mathcal{M}}] = p_\lambda\, [\mathbf{1}_{\mathcal{G}}\otimes m] \tag{9.24}$$
$$[S(m_{(-1)})\otimes \mathbf{1}_{\mathcal{M}}]\, q_\lambda\, \lambda(m_{(0)}) = [\mathbf{1}_{\mathcal{G}}\otimes m]\, q_\lambda \tag{9.25}$$
$$\lambda(q^2_\lambda)\, p_\lambda\, [S^{-1}(q^1_\lambda)\otimes \mathbf{1}_{\mathcal{M}}] = \mathbf{1}_{\mathcal{G}}\otimes \mathbf{1}_{\mathcal{M}} \tag{9.26}$$
$$[S(p^1_\lambda)\otimes \mathbf{1}_{\mathcal{M}}]\, q_\lambda\, \lambda(p^2_\lambda) = \mathbf{1}_{\mathcal{G}}\otimes \mathbf{1}_{\mathcal{M}} \tag{9.27}$$

Moreover, with $f,h\in\mathcal{G}\otimes\mathcal{G}$ being the twists given by (6.21),(6.26), *the following identities are valid*

$$\begin{aligned}&\phi_\lambda^{-1}\, (\mathrm{id}_{\mathcal{G}}\otimes\lambda)(p_\lambda)\, (\mathbf{1}_{\mathcal{G}}\otimes p_\lambda)\\ &\qquad = (\Delta\otimes \mathrm{id}_{\mathcal{M}})\big(\lambda(\phi^3_\lambda)p_\lambda\big)\, [h^{-1}\otimes \mathbf{1}_{\mathcal{M}}]\, [S^{-1}(\phi^2_\lambda)\otimes S^{-1}(\phi^1_\lambda)\otimes \mathbf{1}_{\mathcal{M}}]\end{aligned} \tag{9.28}$$
$$\begin{aligned}&(\mathbf{1}_{\mathcal{G}}\otimes q_\lambda)\, (\mathrm{id}_{\mathcal{G}}\otimes\lambda)(q_\lambda)\, \phi_\lambda\\ &\qquad = [S(\bar\phi^2_\lambda)\otimes S(\bar\phi^1_\lambda)\otimes \mathbf{1}_{\mathcal{M}}]\, [f\otimes \mathbf{1}_{\mathcal{M}}]\, (\Delta\otimes \mathrm{id}_{\mathcal{M}})\big(q_\lambda\lambda(\bar\phi^3_\lambda)\big)\end{aligned} \tag{9.29}$$

2. *Similarly, let (ρ,ϕ_ρ) be a right $\mathcal{G}$–coaction on $\mathcal{M}$ and let p_ρ, q_ρ be given by* (9.22) *and* (9.23). *Then the following identities hold for all $m\in\mathcal{M}$, where $\rho(m)\equiv m_{(0)}\otimes m_{(1)}$.*

$$\rho(m_{(0)})\, p_\rho\, [\mathbf{1}_{\mathcal{M}}\otimes S(m_{(1)})] = p_\rho\, [m\otimes \mathbf{1}_{\mathcal{G}}] \tag{9.30}$$
$$[\mathbf{1}_{\mathcal{M}}\otimes S^{-1}(m_{(1)})]\, q_\rho\, \rho(m_{(0)}) = [m\otimes \mathbf{1}_{\mathcal{G}}]\, q_\rho \tag{9.31}$$
$$\rho(q^1_\rho)\, p_\rho\, [\mathbf{1}_{\mathcal{M}}\otimes S(q^2_\rho)] = \mathbf{1}_{\mathcal{M}}\otimes \mathbf{1}_{\mathcal{G}} \tag{9.32}$$
$$[\mathbf{1}_{\mathcal{M}}\otimes S^{-1}(p^2_\rho)]\, q_\rho\, \rho(p^1_\rho) = \mathbf{1}_{\mathcal{M}}\otimes \mathbf{1}_{\mathcal{G}} \tag{9.33}$$

$$\begin{aligned}&\phi_\rho\, (\rho\otimes \mathrm{id}_{\mathcal{G}})(p_\rho)\, (p_\rho\otimes \mathbf{1}_{\mathcal{G}})\\ &\qquad = (\mathrm{id}_{\mathcal{M}}\otimes\Delta)\big(\rho(\bar\phi^1_\rho)p_\rho\big)\, [\mathbf{1}_{\mathcal{M}}\otimes f^{-1}]\, [\mathbf{1}_{\mathcal{M}}\otimes S(\bar\phi^3_\rho)\otimes S(\bar\phi^2_\rho)]\end{aligned} \tag{9.34}$$
$$\begin{aligned}&(q_\rho\otimes \mathbf{1}_{\mathcal{G}})\, (\rho\otimes \mathrm{id}_{\mathcal{G}})(q_\rho)\, \phi_\rho^{-1}\\ &\qquad = [\mathbf{1}_{\mathcal{M}}\otimes S^{-1}(\phi^3_\rho)\otimes S^{-1}(\phi^2_\rho)]\, [\mathbf{1}_{\mathcal{M}}\otimes h]\, (\mathrm{id}_{\mathcal{M}}\otimes\Delta)\big(q_\rho\rho(\phi^1_\rho)\big).\end{aligned} \tag{9.35}$$

Lemma 9.1 is proven in the Appendix. Note that part 2. of Lemma 9.1 is functorially equivalent to part 1., since (ρ,ϕ_ρ) is a right $\mathcal{G}$–coaction if and only if $(\rho^{op},(\phi_\rho^{-1})^{321})$ is a left $\mathcal{G}^{cop}$–coaction. Also, considering (ρ,ϕ_ρ^{-1}) as a right $\mathcal{G}_{op}$–coaction on $\mathcal{M}_{op}$, the roles of q_ρ and p_ρ interchange, which makes it enough to just prove Eqs. (9.31), (9.33) and (9.35) or the corresponding sets of equations in part 1.

We now give a representation theoretic interpretation of these elements by defining for $(V,\pi_V)\in \mathrm{Rep}\,\mathcal{G}$ and $(\mathfrak{H},\gamma_{\mathfrak{H}})\in \mathrm{Rep}\,\mathcal{M}$ the natural family of morphisms[6]

$$P_{V\mathfrak{H}}: \mathfrak{H}\longrightarrow V^*\odot(V\odot\mathfrak{H}),\quad \mathfrak{h}\mapsto v^i\otimes p_\lambda\cdot(v_i\otimes\mathfrak{h}) \tag{9.36}$$
$$Q_{V\mathfrak{H}}: {}^*V\odot(V\odot\mathfrak{H})\longrightarrow \mathfrak{H},\quad \hat v\otimes v\otimes\mathfrak{h}\mapsto (\hat v\otimes \mathrm{id})\Big(q_\lambda\cdot(v\otimes\mathfrak{h})\Big) \tag{9.37}$$

[6] Again a summation is understood, where $\{v_i\}$ is a basis of V with dual basis $\{v^i\}$

Eqs. (9.24) and (9.25) then say that these are indeed morphisms in Rep $\mathcal{M}$, and Eqs. (9.26) and (9.27) imply the "generalized left rigidity" identities

$$Q_{V^*(V\odot\mathfrak{H})} \circ (\mathrm{id}_V \odot P_{V\mathfrak{H}}) = \mathrm{id}_{V\odot\mathfrak{H}} \tag{9.38}$$
$$(\mathrm{id}_V \odot Q_{V\mathfrak{H}}) \circ P_{{}^*V(V\odot\mathfrak{H})} = \mathrm{id}_{V\odot\mathfrak{H}} \tag{9.39}$$

Finally, Eqs. (9.28) and (9.29) imply the coherence conditions given by the following commuting diagrams:

$$\begin{tikzcd}
\mathfrak{H} \arrow[r, "P_{V\mathfrak{H}}"] \arrow[d, "P_{(U\boxtimes V)\mathfrak{H}}"'] & V^*\odot(V\odot\mathfrak{H}) \arrow[r, "\mathrm{id}_{V^*}\odot P_{U(V\odot\mathfrak{H})}"] & V^*\odot\left[U^*\odot(U\odot(V\odot\mathfrak{H}))\right] \arrow[dd, "\mathrm{id}_{V^*}\odot(\mathrm{id}_{U^*}\odot\phi^{-1}_{UV\mathfrak{H}})"] \\
(U\boxtimes V)^*\odot[(U\boxtimes V)\odot\mathfrak{H}] \arrow[d, "f^{-1}_{V^*U^*}\odot\mathrm{id}_{(U\boxtimes V)\odot\mathfrak{H}}"'] & & \\
(V^*\boxtimes U^*)\odot[(U\boxtimes V)\odot\mathfrak{H}] \arrow[rr, "\phi_{V^*U^*[(U\boxtimes V)\odot\mathfrak{H}]}"] & & V^*\odot\left[U^*\odot((U\boxtimes V)\odot\mathfrak{H})\right]
\end{tikzcd} \tag{9.40}$$

$$\begin{tikzcd}
\mathfrak{H} & {}^*V\odot(V\odot\mathfrak{H}) \arrow[l, "Q_{V\mathfrak{H}}"'] & {}^*V\odot\left[{}^*U\odot(U\odot(V\odot\mathfrak{H}))\right] \arrow[l, "\mathrm{id}_{{}^*V}\odot Q_{U(V\odot\mathfrak{H})}"'] \\
{}^*(U\boxtimes V)\odot[(U\boxtimes V)\odot\mathfrak{H}] \arrow[u, "Q_{(U\boxtimes V)\mathfrak{H}}"] & & \\
({}^*V\boxtimes {}^*U)\odot[(U\boxtimes V)\odot\mathfrak{H}] \arrow[u, "h_{{}^*V^*U}\odot\mathrm{id}_{(U\boxtimes V)\odot\mathfrak{H}}"] & & {}^*V\odot\left[{}^*U\odot((U\boxtimes V)\odot\mathfrak{H})\right] \arrow[ll, "\phi^{-1}_{{}^*V^*U[(U\boxtimes V)\odot\mathfrak{H}]}"'] \arrow[uu, "\mathrm{id}_{{}^*V}\odot(\mathrm{id}_{{}^*U}\odot\phi_{UV\mathfrak{H}})"']
\end{tikzcd} \tag{9.41}$$

Similar statements hold of course for the natural family of morphisms

$$P_{\mathfrak{H}V} : \mathfrak{H} \longrightarrow (\mathfrak{H}\odot V)\odot {}^*V, \quad \mathfrak{h} \mapsto p_\rho \cdot (\mathfrak{h}\otimes v_i)\otimes v^i \tag{9.42}$$
$$Q_{\mathfrak{H}V} : (\mathfrak{H}\odot V)\odot V^* \longrightarrow \mathfrak{H}, \quad \mathfrak{h}\otimes v\otimes\hat{v} \mapsto (\mathrm{id}\otimes\hat{v})\Big(q_\rho\cdot(\mathfrak{h}\otimes v)\Big) \tag{9.43}$$

Later we will also need some additional identities in the case where $(\lambda, \phi_\lambda, \rho, \phi_\rho, \phi_{\lambda\rho})$ is a quasi–commuting pair of coactions.

Lemma 9.2. *Let $(\lambda, \phi_\lambda, \rho, \phi_\rho, \phi_{\lambda\rho})$ be a quasi–commuting pair of $\mathcal{G}$–coactions on $\mathcal{M}$ and let $p_{\lambda/\rho}$, $q_{\lambda/\rho}$ be given by Eqs.* (9.20) - (9.23). *Then putting $\bar\phi_{\lambda\rho} \equiv \phi^{-1}_{\lambda\rho}$*

$$\phi^{-1}_{\lambda\rho}\,(\mathrm{id}_{\mathcal G}\otimes\rho)(p_\lambda) = [\lambda(\phi^2_{\lambda\rho})p_\lambda\otimes\phi^3_{\lambda\rho}]\,[S^{-1}(\phi^1_{\lambda\rho})\otimes\mathbf{1}_{\mathcal M}\otimes\mathbf{1}_{\mathcal G}] \tag{9.44}$$

$$(\mathrm{id}_{\mathcal G}\otimes\rho)(q_\lambda)\,\phi_{\lambda\rho} = [S(\bar\phi^1_{\lambda\rho})\otimes\mathbf{1}_{\mathcal M}\otimes\mathbf{1}_{\mathcal G}]\,[q_\lambda\lambda(\bar\phi^2_{\lambda\rho})\otimes\bar\phi^3_{\lambda\rho}] \tag{9.45}$$

$$\phi_{\lambda\rho}\,(\lambda\otimes\mathrm{id}_{\mathcal G})(p_\rho) = [\bar\phi^1_{\lambda\rho}\otimes\rho(\phi^2_{\lambda\rho})p_\rho]\,[\mathbf{1}_{\mathcal G}\otimes\mathbf{1}_{\mathcal M}\otimes S(\bar\phi^3_{\lambda\rho})] \tag{9.46}$$

$$(\lambda\otimes\mathrm{id}_{\mathcal G})(q_\rho)\,\phi^{-1}_{\lambda\rho} = [\mathbf{1}_{\mathcal G}\otimes\mathbf{1}_{\mathcal M}\otimes S^{-1}(\phi^3_{\lambda\rho})]\,[\phi^1_{\lambda\rho}\otimes q_\rho\rho(\phi^2_{\lambda\rho})] \tag{9.47}$$

Lemma 9.2 is also proven in the appendix. Again we remark that for functorial reasons Eqs. (9.44) - (9.47) are all equivalent, see the arguments after Lemma 9.1. Note that for example Eq. (9.44) gives rise to the commuting diagram

$$\begin{tikzcd}
\mathfrak{H}\odot V \arrow[r, "P_{U(\mathfrak{H}\odot V)}"] \arrow[d, "P_{U\mathfrak{H}}\odot\mathrm{id}_V"'] & U^*\odot\big[U\odot(\mathfrak{H}\odot V)\big] \arrow[d, "\mathrm{id}_{U^*}\odot\phi^{-1}_{U\mathfrak{H}V}"] \\
\big[U^*\odot(U\odot\mathfrak{H})\big]\odot V \arrow[r, "\phi_{U^*(U\odot\mathfrak{H})V}"] & U^*\odot\big[(U\odot\mathfrak{H})\odot V\big]
\end{tikzcd} \tag{9.48}$$

Similar diagrams follow from (9.45) - (9.47).

We conclude this section with giving the analogue of the elements $p_{\lambda/\rho}$, $q_{\lambda/\rho}$ for two–sided coactions.

Lemma 9.3. *Let (δ, Ψ) be a two–sided $\mathcal{G}$–coactions on $\mathcal{M}$ and define p_δ, $q_\delta\in\mathcal{G}\otimes\mathcal{M}\otimes\mathcal{G}$ by*

$$p_\delta := \Psi^2 S^{-1}(\Psi^1\beta)\otimes\Psi^3\otimes\Psi^4\beta S(\Psi^5) \tag{9.49}$$

$$q_\delta := S(\bar\Psi^1)\alpha\bar\Psi^2\otimes\bar\Psi^3\otimes S^{-1}(\alpha\bar\Psi^5)\bar\Psi^4 \tag{9.50}$$

where $\bar\Psi\equiv\Psi^{-1}$. Then the following identities hold for all $m\in\mathcal{M}$, where $\delta(m)\equiv m_{(-1)}\otimes m_{(0)}\otimes m_{(1)}$ and where $f, h\in\mathcal{G}\otimes\mathcal{G}$ are the twists defined in (6.21) *and* (6.26).

$$p_\delta\,[\mathbf{1}_{\mathcal G}\otimes m\otimes\mathbf{1}_{\mathcal G}] = \delta(m_{(0)})\,p_\delta\,[S^{-1}(m_{(-1)})\otimes\mathbf{1}_{\mathcal M}\otimes S(m_{(1)})] \tag{9.51}$$

$$[\mathbf{1}_{\mathcal G}\otimes m\otimes\mathbf{1}_{\mathcal G}]\,q_\delta = [S(m_{(-1)})\otimes\mathbf{1}_{\mathcal M}\otimes S^{-1}(m_{(1)})]\,q_\delta\,\delta(m_{(0)}) \tag{9.52}$$

$$\delta(q^2_\delta)\,p_\delta\,[S^{-1}(q^1_\delta)\otimes\mathbf{1}_{\mathcal M}\otimes S(q^3_\delta)] = \mathbf{1}_{\mathcal G}\otimes\mathbf{1}_{\mathcal M}\otimes\mathbf{1}_{\mathcal G} \tag{9.53}$$

$$[S(p^1_\delta)\otimes\mathbf{1}_{\mathcal M}\otimes S^{-1}(p^3_\delta)]\,q_\delta\,\delta(p^2_\delta) = \mathbf{1}_{\mathcal G}\otimes\mathbf{1}_{\mathcal M}\otimes\mathbf{1}_{\mathcal G} \tag{9.54}$$

$$\begin{aligned}
&\Psi^{-1}\,(\mathrm{id}_{\mathcal G}\otimes\delta\otimes\mathrm{id}_{\mathcal G})(p_\delta)\,[\mathbf{1}_{\mathcal G}\otimes p_\delta\otimes\mathbf{1}_{\mathcal G}] && (9.55)\\
&\quad = (\Delta\otimes\mathrm{id}_{\mathcal M}\otimes\Delta)\big(\delta(\Psi^3)p_\delta\big)\,[h^{-1}\otimes\mathbf{1}_{\mathcal M}\otimes f^{-1}]\,[S^{-1}(\Psi^2)\otimes S^{-1}(\Psi^1)\otimes\mathbf{1}_{\mathcal M}\otimes S(\Psi^5)\otimes S(\Psi^4)] &&\\
&[\mathbf{1}_{\mathcal G}\otimes q_\delta\otimes\mathbf{1}_{\mathcal G}]\,(\mathrm{id}_{\mathcal G}\otimes\delta\otimes\mathrm{id}_{\mathcal G})(q_\delta)\,\Psi && (9.56)\\
&\quad = [S(\bar\Psi^2)\otimes S(\bar\Psi^1)\otimes\mathbf{1}_{\mathcal M}\otimes S^{-1}(\bar\Psi^5)\otimes S^{-1}(\bar\Psi^4)]\,[f\otimes\mathbf{1}_{\mathcal M}\otimes h]\,(\Delta\otimes\mathrm{id}_{\mathcal M}\otimes\Delta)\big(q_\delta\delta(\bar\Psi^3)\big) &&
\end{aligned}$$

Proof. This follows immediately from Lemma 9.1 by noting that after a permutation of tensor factors $\mathrm{id}_{\mathcal G}\otimes\tau_{\mathcal M,\mathcal G} : \mathcal{G}\otimes\mathcal{M}\otimes\mathcal{G}\longrightarrow\mathcal{G}\otimes\mathcal{G}\otimes\mathcal{M}$ any two–sided $\mathcal{G}$–coaction becomes a left $(\mathcal{G}\otimes\mathcal{G}^{cop})$–coaction. □

We remark that similarly as before the elements p_δ and q_δ give rise to natural families of morphisms

$$P_{U\mathfrak{H}V} : \mathfrak{H} \longrightarrow U^* \gtrdot (U \gtrdot \mathfrak{H} \lessdot V) \lessdot {}^*V$$
$$Q_{U\mathfrak{H}V} : {}^*U \gtrdot (U \gtrdot \mathfrak{H} \lessdot V) \lessdot V^* \longrightarrow \mathfrak{H}$$

obeying analogue "rigidity identities" and "coherence diagrams" as before. We leave the details to the reader.

We also remark without proof, that in the cases $\delta = \delta_l \equiv (\lambda \otimes \mathrm{id}) \circ \rho$ or $\delta = \delta_r \equiv (\mathrm{id} \otimes \rho) \circ \lambda$ for a quasi–commuting pair of $\mathcal{G}$–coactions $(\lambda, \rho, \phi_\lambda, \phi_\rho, \phi_{\lambda\rho})$ the elements q_δ and p_δ may be expressed in terms of the $q_{\lambda/\rho}$'s and $p_{\lambda/\rho}$'s by

$$q_{\delta_l} = (\mathbf{1}_\mathcal{G} \otimes q_\rho)\,(\mathrm{id}_\mathcal{G} \otimes \rho)(q_\lambda)\,\phi_{\lambda\rho} \tag{9.57}$$
$$p_{\delta_l} = \phi^{-1}_{\lambda\rho}\,(\mathrm{id}_\mathcal{G} \otimes \rho)(p_\lambda)\,(\mathbf{1}_\mathcal{G} \otimes p_\rho) \tag{9.58}$$
$$q_{\delta_r} = (q_\lambda \otimes \mathbf{1}_\mathcal{G})\,(\lambda \otimes \mathrm{id}_\mathcal{G})(q_\rho)\,\phi^{-1}_{\lambda\rho} \tag{9.59}$$
$$p_{\delta_r} = \phi_{\lambda\rho}\,(\lambda \otimes \mathrm{id}_\mathcal{G})(p_\rho)\,(p_\lambda \otimes \mathbf{1}_\mathcal{G}) \tag{9.60}$$

10 Diagonal crossed products

Having developed our theory of two–sided $\mathcal{G}$–coactions δ for quasi–bialgebras $\mathcal{G}$ we are now in the position to generalize the construction of the left and right diagonal crossed products $\hat{\mathcal{G}} \bowtie_\delta \mathcal{M}$ and $\mathcal{M} \bowtie_\delta \hat{\mathcal{G}}$ to the quasi–coassociative setting. Before writing down the concrete multiplication rules we would like to draw the reader's attention to some important conceptual differences in comparison with the results of Section 3, specifically those of Proposition 3.3, Lemma 3.6 and Corollary 3.7.

First, as already remarked, the natural "multiplication" $\hat{\mu} : \hat{\mathcal{G}} \otimes \hat{\mathcal{G}} \longrightarrow \hat{\mathcal{G}}$ given as the transpose of the coproduct $\Delta : \mathcal{G} \longrightarrow \mathcal{G} \otimes \mathcal{G}$ is *not* associative. Nevertheless, we will still write $\varphi\psi := \hat{\mu}(\varphi \otimes \psi)$, $\varphi, \psi \in \hat{\mathcal{G}}$, i.e.

$$\langle \varphi\psi \mid a \rangle := \langle \varphi \otimes \psi \mid \Delta(a) \rangle, \quad a \in \mathcal{G}. \tag{10.1}$$

Also note that the counit $\epsilon \equiv \mathbf{1}_{\hat{\mathcal{G}}} \in \hat{\mathcal{G}}$ still is a unit for $\hat{\mu}$. On the other hand, being the dual of an associative unital algebra, $\hat{\mathcal{G}}$ is a coassociative counital coalgebra with coproduct $\hat{\Delta}(\varphi) \equiv \varphi_{(1)} \otimes \varphi_{(2)}$ given by

$$\langle \hat{\Delta}(\varphi) \mid a \otimes b \rangle := \langle \varphi \mid ab \rangle, \quad a, b \in \mathcal{G} \tag{10.2}$$

We also note the identities

$$\hat{\Delta}(\mathbf{1}_{\hat{\mathcal{G}}}) = \mathbf{1}_{\hat{\mathcal{G}}} \otimes \mathbf{1}_{\hat{\mathcal{G}}} \tag{10.3}$$
$$\hat{\Delta}(\varphi\psi) = \hat{\Delta}(\varphi)\hat{\Delta}(\psi) \tag{10.4}$$

Denoting the natural left and right actions of $\mathcal{G}$ on $\hat{\mathcal{G}}$ by $a \rightharpoonup \varphi \equiv \varphi_{(1)}\langle \varphi_{(2)} \mid a \rangle$ and $\varphi \leftharpoonup a \equiv \langle \varphi_{(1)} \mid a \rangle \varphi_{(2)}$, $a \in \mathcal{G}$, $\varphi \in \hat{\mathcal{G}}$, this also implies

$$a \rightharpoonup (\varphi\psi) = (a_{(1)} \rightharpoonup \varphi)(a_{(2)} \rightharpoonup \psi) \tag{10.5}$$
$$(\varphi\psi) \leftharpoonup a = (\varphi \leftharpoonup a_{(1)})(\psi \leftharpoonup a_{(2)}). \tag{10.6}$$

The second important warning concerns the fact that although we will have $\hat{\mathcal{G}} \bowtie_\delta \mathcal{M} = \hat{\mathcal{G}} \otimes \mathcal{M}$ and $\mathcal{M} \bowtie_\delta \hat{\mathcal{G}} = \mathcal{M} \otimes \hat{\mathcal{G}}$ as linear spaces, the subspaces $\hat{\mathcal{G}} \otimes \mathbf{1}_{\mathcal{M}}$ and $\mathbf{1}_{\mathcal{M}} \otimes \hat{\mathcal{G}}$ will *not* be subalgebras in the diagonal crossed product. On the other hand, $\mathcal{M}$ will naturally be embedded as the unital subalgebra $\mathcal{M} \cong \mathbf{1}_{\hat{\mathcal{G}}} \otimes \mathcal{M} \cong \mathcal{M} \otimes \mathbf{1}_{\hat{\mathcal{G}}}$.

The third warning concerns the fact that $\hat{\mathcal{G}} \bowtie_\delta \mathcal{M} \cong \mathcal{M} \bowtie_\delta \hat{\mathcal{G}}$ will still be equivalent algebra extensions of $\mathcal{M}$, similarly as in Corollary 3.7. However the subspaces $\hat{\mathcal{G}} \bowtie \mathbf{1}_{\mathcal{M}}$ and $\mathbf{1}_{\mathcal{M}} \bowtie \hat{\mathcal{G}}$ will *not* be mapped onto each other under this isomorphism. (Recall that this was the case in Eqs. (3.21) and (3.22)).

10.1 The algebras $\hat{\mathcal{G}} \bowtie_\delta \mathcal{M}$ and $\mathcal{M} \bowtie_\delta \hat{\mathcal{G}}$

We now proceed to the details. Given a two–sided $\mathcal{G}$–coaction (δ, Ψ) on $\mathcal{M}$ we still write as before

$$\varphi \triangleright m \triangleleft \psi := (\psi \otimes \mathrm{id}_{\mathcal{M}} \otimes \varphi)(\delta(m)), \quad m \in \mathcal{M},\ \varphi, \psi \in \hat{\mathcal{G}}, \tag{10.7}$$

disregarding the fact that δ might be neither of the form (8.19) nor (8.21). We also introduce the elements $\Omega_L, \Omega_R \in \mathcal{G} \otimes \mathcal{G} \otimes \mathcal{M} \otimes \mathcal{G} \otimes \mathcal{G}$ given by

$$\Omega_L \equiv \Omega_L^1 \otimes \Omega_L^2 \otimes \Omega_L^3 \otimes \Omega_L^4 \otimes \Omega_L^5 := (\mathrm{id}_{\mathcal{G}} \otimes \mathrm{id}_{\mathcal{G}} \otimes \mathrm{id}_{\mathcal{M}} \otimes S^{-1} \otimes S^{-1})(\Psi^{-1}) \cdot h^{54} \tag{10.8}$$

$$\Omega_R \equiv \Omega_R^1 \otimes \Omega_R^2 \otimes \Omega_R^3 \otimes \Omega_R^4 \otimes \Omega_R^5 := (h^{-1})^{21} \cdot (S^{-1} \otimes S^{-1} \otimes \mathrm{id}_{\mathcal{M}} \otimes \mathrm{id}_{\mathcal{G}} \otimes \mathrm{id}_{\mathcal{G}})(\Psi), \tag{10.9}$$

where $h \equiv (S^{-1} \otimes S^{-1})(f^{21}) \in \mathcal{G} \otimes \mathcal{G}$ has been introduced in (6.26). As before, we drop all summation symbols and summation indices.

Definition 10.1. Let (δ, Ψ) be a two–sided coaction of a quasi–Hopf algebra $\mathcal{G}$ on an algebra $\mathcal{M}$. We define the *left diagonal crossed product* $\hat{\mathcal{G}} \bowtie_\delta \mathcal{M}$ to be the vector space $\hat{\mathcal{G}} \otimes \mathcal{M}$ equipped with the multiplication rule

$$(\varphi \bowtie m)(\psi \bowtie n) := \Big[(\Omega_L^1 \rightharpoonup \varphi \leftharpoonup \Omega_L^5)(\Omega_L^2 \rightharpoonup \psi_{(2)} \leftharpoonup \Omega_L^4)\Big] \bowtie \Big[\Omega_L^3 (\hat{S}^{-1}(\psi_{(1)}) \triangleright m \triangleleft \psi_{(3)})\, n\Big] \tag{10.10}$$

and we define the *right diagonal crossed product* $M \bowtie_\delta \hat{\mathcal{G}}$ to be the vector space $\mathcal{M} \otimes \hat{\mathcal{G}}$ with the multiplication rule

$$(m \bowtie \varphi)(n \bowtie \psi) := \Big[m\, (\varphi_{(1)} \triangleright n \triangleleft \hat{S}^{-1}(\varphi_{(3)}))\, \Omega_R^3\Big] \bowtie \Big[(\Omega_R^2 \rightharpoonup \varphi_{(2)} \leftharpoonup \Omega_R^4)(\Omega_R^1 \rightharpoonup \psi \leftharpoonup \Omega_R^5)\Big]. \tag{10.11}$$

A representation theoretic interpretation of these definitions will be given in Section 10.2, starting with Definition 10.7. In cases where the two–sided coaction is unambiguously understood from the context we also write $\hat{\mathcal{G}} \bowtie \mathcal{M}$ and $\mathcal{M} \bowtie \hat{\mathcal{G}}$. As in Section 3 the choice of placing $\hat{\mathcal{G}}$ either to the left or to the right of $\mathcal{M}$ stems from the fact that in $\hat{\mathcal{G}} \bowtie \mathcal{M}$ we have

$$(\varphi \bowtie m) = (\varphi \bowtie \mathbf{1}_{\mathcal{M}})(\mathbf{1}_{\hat{\mathcal{G}}} \bowtie m) \tag{10.12}$$

whereas in $\mathcal{M} \bowtie \hat{\mathcal{G}}$ we have

$$(m \bowtie \varphi) = (m \bowtie \mathbf{1}_{\hat{\mathcal{G}}})(\mathbf{1}_{\mathcal{M}} \bowtie \varphi), \tag{10.13}$$

as one easily checks from the definitions. Also note that for $\Omega_{L/R} = \mathbf{1}_{\mathcal{G}} \otimes \mathbf{1}_{\mathcal{G}} \otimes \mathbf{1}_{\mathcal{M}} \otimes \mathbf{1}_{\mathcal{G}} \otimes \mathbf{1}_{\mathcal{G}}$ we recover the definitions of Section 3. Furthermore, under the trivial permutation of tensor factors we have as in (3.23)

$$(\mathcal{M} \bowtie_{\delta} \hat{\mathcal{G}})_{op} = \hat{\mathcal{G}}_{op}^{cop} \bowtie_{\delta_{op}} \mathcal{M}_{op} \tag{10.14}$$

where $(\mathcal{M} \bowtie_{\delta} \hat{\mathcal{G}})_{op}$ denotes the diagonal crossed product with opposite multiplication, and where we recall our remark that with the definition (8.5) $(\delta_{op}, \Psi_{op}^{-1})$ defines a two–sided coaction of $\mathcal{G}_{op}^{cop}$ on $\mathcal{M}_{op}$. We now formulate our first main result.

Theorem 10.2.

(i) The diagonal crossed products $\hat{\mathcal{G}} \bowtie \mathcal{M}$ and $\mathcal{M} \bowtie \hat{\mathcal{G}}$ are associative algebras with unit $\mathbf{1}_{\hat{\mathcal{G}}} \bowtie \mathbf{1}_{\mathcal{M}}$ and $\mathbf{1}_{\mathcal{M}} \bowtie \mathbf{1}_{\hat{\mathcal{G}}}$, respectively.

(ii) $\mathcal{M} \equiv \mathbf{1}_{\hat{\mathcal{G}}} \bowtie \mathcal{M} \subset \hat{\mathcal{G}} \bowtie \mathcal{M}$ and $\mathcal{M} \equiv \mathcal{M} \bowtie \mathbf{1}_{\hat{\mathcal{G}}} \subset \mathcal{M} \bowtie \hat{\mathcal{G}}$ are unital algebra inclusions.

(iii) The algebras $\hat{\mathcal{G}} \bowtie \mathcal{M}$ and $\mathcal{M} \bowtie \hat{\mathcal{G}}$ provide equivalent extensions of $\mathcal{M}$, the isomorphism being given by

$$f : \hat{\mathcal{G}} \bowtie \mathcal{M} \ni (\varphi \bowtie m) \mapsto \big(q^2 \bowtie (S^{-1}(q^1) \rightharpoonup \varphi \leftharpoonup q^3)\big) \cdot (m \bowtie \hat{\mathbf{1}}) \in \mathcal{M} \bowtie \hat{\mathcal{G}} \tag{10.15}$$
$$f^{-1} : \mathcal{M} \bowtie \hat{\mathcal{G}} \ni (m \bowtie \varphi) \mapsto (\hat{\mathbf{1}} \bowtie m) \cdot \big((p^1 \rightharpoonup \varphi \leftharpoonup S^{-1}(p^3)) \bowtie p^2\big) \in \hat{\mathcal{G}} \bowtie \mathcal{M}, \tag{10.16}$$

where $p \equiv p_\delta$ and $q \equiv q_\delta$ are given in Eqs. (9.49) and (9.50).

Proof. One trivially checks the unit properties in part (i) and also the identities

$$(\varphi \bowtie m)(\mathbf{1}_{\hat{\mathcal{G}}} \bowtie n) = (\varphi \bowtie mn) \tag{10.17}$$
$$(n \bowtie \mathbf{1}_{\hat{\mathcal{G}}})(m \bowtie \varphi) = (nm \bowtie \varphi) \tag{10.18}$$

for all $m, n \in \mathcal{M}$ and all $\varphi \in \hat{\mathcal{G}}$, thereby proving part (ii). The proof of part (iii) is postponed to Section 10.2.

We now prove the associativity of the product in $(\hat{\mathcal{G}} \bowtie \mathcal{M})$, the case $(\mathcal{M} \bowtie \hat{\mathcal{G}})$ being analogous by the remark (10.14). First note that (10.10) and (10.17) immediately imply

$$[XY](\mathbf{1}_{\hat{\mathcal{G}}} \bowtie m) = X[Y(\mathbf{1}_{\hat{\mathcal{G}}} \bowtie m)] \tag{10.19}$$

for all $X, Y \in \hat{\mathcal{G}} \bowtie \mathcal{M}$ and all $m \in \mathcal{M}$. Next we show that

$$[X(\mathbf{1}_{\hat{\mathcal{G}}} \bowtie m)]Y = X[(\mathbf{1}_{\hat{\mathcal{G}}} \bowtie m)Y], \quad \forall X, Y \in \hat{\mathcal{G}} \bowtie \mathcal{M},\ m \in \mathcal{M} \tag{10.20}$$

To this end we use $(\mathrm{id} \otimes \epsilon)(h) = \mathbf{1}_{\mathcal{G}}$ and therefore $(\epsilon \otimes \mathrm{id}_{\mathcal{G}} \otimes \mathrm{id}_{\mathcal{M}} \otimes \mathrm{id}_{\mathcal{G}} \otimes \epsilon)(\Omega_L) = \mathbf{1}_{\mathcal{G}} \otimes \mathbf{1}_{\mathcal{M}} \otimes \mathbf{1}_{\mathcal{G}}$ to conclude for $m, m', n \in \mathcal{M}$ and $\psi \in \hat{\mathcal{G}}$

$$(\mathbf{1}_{\hat{\mathcal{G}}} \bowtie m)(\psi \bowtie n) = \psi_{(2)} \bowtie (S^{-1}(\psi_{(1)}) \triangleright m \triangleleft \psi_{(3)})n \tag{10.21}$$

and hence also

$$\begin{aligned}(\mathbf{1}_{\hat{\mathcal{G}}} \bowtie m')\big[(\mathbf{1}_{\hat{\mathcal{G}}} \bowtie m)(\psi \bowtie n)\big] &= \psi_{(3)} \bowtie \big[(S^{-1}(\psi_{(2)}) \triangleright m' \triangleleft \psi_{(4)})(S^{-1}(\psi_{(1)}) \triangleright m \triangleleft \psi_{(5)})n\big] \\ &= \psi_{(2)} \bowtie (S^{-1}(\psi_{(1)}) \triangleright m'm \triangleleft \psi_{(3)}) \\ &= (\mathbf{1}_{\hat{\mathcal{G}}} \bowtie m'm)(\psi \bowtie n)\end{aligned} \tag{10.22}$$

where we have used that δ is an algebra map. Moreover, (10.21) also implies for all $\varphi \in \hat{\mathcal{G}}$

$$
\begin{aligned}
&(\varphi \bowtie \mathbf{1}_{\mathcal{M}})\big[(\mathbf{1}_{\hat{\mathcal{G}}} \bowtie m)(\psi \bowtie n)\big] \\
&\qquad = \big[(\Omega_L^1 \rightharpoonup \varphi \leftharpoonup \Omega_L^5)(\Omega_L^2 \rightharpoonup \psi_{(2)} \leftharpoonup \Omega_L^4)\big] \bowtie \big[\Omega_L^3(S^{-1}(\psi_{(1)}) \triangleright m \triangleleft \psi_{(3)})n\big] \\
&\qquad = (\varphi \bowtie m)(\psi \bowtie n)
\end{aligned}
\tag{10.23}
$$

Putting (10.17), (10.19), (10.22) and (10.23) together, we have proven (10.20).

In view of (10.17), (10.19) and (10.20), to finish the proof of associativity we are now left with proving the following two identities

$$
(\varphi \bowtie \mathbf{1}_{\mathcal{M}})\big[(\psi \bowtie \mathbf{1}_{\mathcal{M}})(\chi \bowtie \mathbf{1}_{\mathcal{M}})\big] = \big[(\varphi \bowtie \mathbf{1}_{\mathcal{M}})(\psi \bowtie \mathbf{1}_{\mathcal{M}})\big](\chi \bowtie \mathbf{1}_{\mathcal{M}}) \tag{10.24}
$$

$$
(\mathbf{1}_{\hat{\mathcal{G}}} \bowtie m)\big[(\varphi \bowtie \mathbf{1}_{\mathcal{M}})(\psi \bowtie \mathbf{1}_{\mathcal{M}})\big] = \big[(\mathbf{1}_{\hat{\mathcal{G}}} \bowtie m)(\varphi \bowtie \mathbf{1}_{\mathcal{M}})\big](\psi \bowtie \mathbf{1}_{\mathcal{M}}) \tag{10.25}
$$

for all $\varphi, \psi, \chi \in \hat{\mathcal{G}}$ and all $m \in \mathcal{M}$. To prove these remaining identities we rewrite them using the generating matrix formalism. Let $\mathbf{L} \in \mathcal{G} \otimes (\hat{\mathcal{G}} \bowtie \mathcal{M})$ be given by $\mathbf{L} = e_\mu \otimes (e^\mu \bowtie \mathbf{1}_{\mathcal{M}})$, where $\{e_\mu\}$ is a basis in $\mathcal{G}$ with dual basis $\{e^\mu\}$ in $\hat{\mathcal{G}}$. We also abbreviate our notation by identifying $m \equiv (\mathbf{1}_{\hat{\mathcal{G}}} \bowtie m)$, $m \in \mathcal{M}$. Then Eqs. (10.24) and (10.25) are equivalent, respectively, to

$$
\mathbf{L}^{14}(\mathbf{L}^{24}\mathbf{L}^{34}) = (\mathbf{L}^{14}\mathbf{L}^{24})\mathbf{L}^{34} \tag{10.26}
$$

$$
[\mathbf{1}_{\mathcal{G}} \otimes \mathbf{1}_{\mathcal{G}} \otimes m](\mathbf{L}^{13}\mathbf{L}^{23}) = ([\mathbf{1}_{\mathcal{G}} \otimes \mathbf{1}_{\mathcal{G}} \otimes m]\mathbf{L}^{13})\mathbf{L}^{23}, \tag{10.27}
$$

where (10.26) is understood as an identity in $\mathcal{G}^{\otimes^3} \otimes (\hat{\mathcal{G}} \bowtie \mathcal{M})$ and (10.27) as an identity in $\mathcal{G}^{\otimes^2} \otimes (\hat{\mathcal{G}} \bowtie \mathcal{M})$. We now use that (10.10) and (10.12) imply

$$
[\mathbf{1}_{\mathcal{G}} \otimes \mathbf{1}_{\mathcal{M}}]\,\mathbf{L} = \mathbf{L} \tag{10.28}
$$

$$
[\mathbf{1}_{\mathcal{G}} \otimes m]\,\mathbf{L} = [S^{-1}(m_{(1)}) \otimes \mathbf{1}_{\mathcal{M}}]\,\mathbf{L}\,[m_{(-1)} \otimes m_{(0)}], \quad \forall m \in \mathcal{M} \tag{10.29}
$$

$$
\mathbf{L}^{13}\mathbf{L}^{23} = [S^{-1}(\bar{\Psi}^5) \otimes S^{-1}(\bar{\Psi}^4) \otimes \mathbf{1}_{\mathcal{M}}]\,[(\Delta_L \otimes \mathrm{id})(\mathbf{L})]\,[\bar{\Psi}^1 \otimes \bar{\Psi}^2 \otimes \bar{\Psi}^3] \tag{10.30}
$$

where we have introduced the notation $\delta(m) = m_{(-1)} \otimes m_{(0)} \otimes m_{(1)}$ and $\Psi^{-1} \equiv \bar{\Psi} = \bar{\Psi}^1 \otimes \bar{\Psi}^2 \otimes \bar{\Psi}^3 \otimes \bar{\Psi}^4 \otimes \bar{\Psi}^5$, and where $\Delta_L(a) := h\Delta(a)$, $a \in \mathcal{G}$, has been introduced in Corollary 6.1.

To prove (10.27) we use (10.29) twice together with (10.20) to get for the r.h.s. of (10.27)

$$
([\mathbf{1}_{\mathcal{G}} \otimes \mathbf{1}_{\mathcal{G}} \otimes m]\,\mathbf{L}^{13})\mathbf{L}^{23} = [S^{-1}(m_{(2)}) \otimes S^{-1}(m_{(1)}) \otimes \mathbf{1}_{\mathcal{M}}]\,\mathbf{L}^{13}\mathbf{L}^{23}\,[m_{(-2)} \otimes m_{(-1)} \otimes m_{(0)}],
$$

where we have used the notation $(\mathrm{id} \otimes \delta \otimes \mathrm{id}) \circ \delta(m) = m_{(-2)} \otimes m_{(-1)} \otimes m_{(0)} \otimes m_{(1)} \otimes m_{(2)}$. On the other hand, by the intertwiner property (6.27) together with (10.29),(10.30) the l.h.s. of (10.27) gives

$$
\begin{aligned}
&[\mathbf{1}_{\mathcal{G}} \otimes \mathbf{1}_{\mathcal{G}} \otimes m]\,(\mathbf{L}^{13}\mathbf{L}^{23}) \\
&\quad = [(S^{-1} \otimes S^{-1})\big(\Delta^{op}(m_{(1)})(\bar{\Psi}^5 \otimes \bar{\Psi}^4)\big) \otimes \mathbf{1}_{\mathcal{M}}]\,[(\Delta_L \otimes \mathrm{id})(\mathbf{L})]\,[\Delta(m_{(-1)}) \otimes m_{(0)}]\,[\bar{\Psi}^1 \otimes \bar{\Psi}^2 \otimes \bar{\Psi}^3].
\end{aligned}
$$

Using again (10.30) to rewrite the r.h.s. of this formula Eq. (10.27) follows from the defining property (8.1) of $\bar{\Psi} \equiv \Psi^{-1}$.

To prove (10.26), we use (10.30) to compute for the l.h.s (writing $\hat{\Psi}$ for another copy of Ψ)

$$
\begin{aligned}
\mathbf{L}^{14}(\mathbf{L}^{24}\mathbf{L}^{34}) &= [\mathbf{1}_{\mathcal{G}} \otimes S^{-1}(\bar{\Psi}^5) \otimes S^{-1}(\bar{\Psi}^4) \otimes \mathbf{1}_{\mathcal{M}}][(\mathrm{id} \otimes \Delta_L \otimes \mathrm{id})(\mathbf{L}^{13}\mathbf{L}^{23})][\mathbf{1}_{\mathcal{G}} \otimes \bar{\Psi}^1 \otimes \bar{\Psi}^2 \otimes \bar{\Psi}^3] \\
&= [(S^{-1} \otimes S^{-1} \otimes S^{-1})\Big((\bar{\hat{\Psi}}^5 \otimes \Delta^{op}(\bar{\hat{\Psi}}^4))(\mathbf{1}_{\mathcal{G}} \otimes \bar{\Psi}^5 \otimes \bar{\Psi}^4)\Big) \otimes \mathbf{1}_{\mathcal{M}}] \\
&\qquad \times [(\mathrm{id} \otimes \Delta_L \otimes \mathrm{id}) \circ (\Delta_L \otimes \mathrm{id})(\mathbf{L})]\,[\bar{\hat{\Psi}}^1 \otimes \Delta(\bar{\hat{\Psi}}^2) \otimes \bar{\hat{\Psi}}^3]\,[\mathbf{1}_{\mathcal{G}} \otimes \bar{\Psi}^1 \otimes \bar{\Psi}^2 \otimes \bar{\Psi}^3],
\end{aligned}
\tag{10.31}
$$

where for the second equality we have used the identity

$$\Delta_L(S^{-1}(a)bc) = (S^{-1}\otimes S^{-1})(\Delta^{op}(a))\Delta_L(b)\Delta(c)$$

following from (6.27). For the r.h.s. of (10.26) we get:

$$\begin{aligned}
&(\mathbf{L}^{14}\mathbf{L}^{24})\mathbf{L}^{34}\\
&\quad = [S^{-1}(\bar\Psi^5)\otimes S^{-1}(\bar\Psi^4)\otimes S^{-1}(\bar\Psi^3_{(1)})\otimes \mathbf{1}_{\mathcal{M}}][\Delta_L\otimes \mathrm{id}\otimes \mathrm{id})(\mathbf{L}^{13}\mathbf{L}^{23})][\bar\Psi^1\otimes\bar\Psi^2\otimes\bar\Psi^3_{(-1)}\otimes\bar\Psi^3_{(0)}]\\
&\quad = [(S^{-1}\otimes S^{-1}\otimes S^{-1})\Big((\Delta^{op}(\hat{\bar\Psi}^5)\otimes\hat{\bar\Psi}^4)(\bar\Psi^5\otimes\bar\Psi^4\otimes\bar\Psi^3_{(1)})\Big)\otimes\mathbf{1}_{\mathcal{M}}] \qquad (10.32)\\
&\qquad \times[(\Delta_L\otimes \mathrm{id}\otimes\mathrm{id})\circ(\Delta_L\otimes\mathrm{id})(\mathbf{L})]\,[\Delta(\hat{\bar\Psi}^1)\otimes\hat{\bar\Psi}^2\otimes\hat{\bar\Psi}^3][\bar\Psi^1\otimes\bar\Psi^2\otimes\bar\Psi^3_{(-1)}\otimes\bar\Psi^3_{(0)}],
\end{aligned}$$

where for the first equality we have used (10.29) to move $\bar\Psi^3$ to the right of $\mathbf{L}^{34}$ and in the second equality again (6.27). Now we use that by Corollary 2.1

$$(\mathrm{id}\otimes\Delta_L)(\Delta_L(a)) = (S^{-1}\otimes S^{-1}\otimes S^{-1})(\phi^{321})\,((\Delta_L\otimes\mathrm{id})(\Delta_L(a))\,\phi^{-1}, \quad \forall a\in\mathcal{G}.$$

Hence (10.31) and (10.32) are equal due to the pentagon identity (8.2) for Ψ, which proves (10.26). This concludes the proof of parts (i) and (ii) of Theorem 10.2. Part (iii) will be proven in Subsection 10.2 after Proposition 10.5. ☐

Before proceeding let us shortly discuss how in the present context one can see that ordinary crossed products $\mathcal{M}\rtimes_\rho\hat{\mathcal{G}}$ (or $\hat{\mathcal{G}}\ltimes_\lambda\mathcal{M}$) in general cannot be defined as associative algebras any more. In the strictly coassociative setting of Section 3 these could be considered as special types of diagonal crossed products, where $\delta = \mathbf{1}_{\mathcal{G}}\otimes\rho$ (or $\delta=\lambda\otimes\mathbf{1}_{\mathcal{G}}$). In the present setting it is not clear whether such δ's give well defined two–sided coactions, since in fact the maps

$$\lambda_0(m) := \mathbf{1}_{\mathcal{G}}\otimes m, \quad \rho_0(m) := m\otimes\mathbf{1}_{\mathcal{G}}$$

need not even be one–sided coactions. For this one would also need the existence of reassociators ϕ_{λ_0}, ϕ_{ρ_0} satisfying the axioms of Definition 2.1. On the other hand, suppose there exists ϕ_{λ_0} such that $(\lambda_0,\phi_{\lambda_0})$ is a well defined left coaction and let (ρ,ϕ_ρ) be a right coaction such that

$$(\mathrm{id}_{\mathcal{G}}\otimes\mathrm{id}_{\mathcal{G}}\otimes\rho)(\phi_{\lambda_0}) = \phi_{\lambda_0}\otimes\mathbf{1}_{\mathcal{G}}$$

Then one immediately checks that $(\lambda_0,\rho,\phi_{\lambda_0},\phi_\rho,\phi_{\lambda\rho} := \mathbf{1}_{\mathcal{G}}\otimes\mathbf{1}_{\mathcal{M}}\otimes\mathbf{1}_{\mathcal{G}})$ provides a quasi–commuting pair of coactions and therefore in this case $\mathcal{M}\rtimes_\rho\hat{\mathcal{G}} := \mathcal{M}\bowtie_\delta\hat{\mathcal{G}}$ would indeed be a well defined associative algebra, where $\delta = \mathbf{1}_{\mathcal{G}}\otimes\rho$ and $\Psi = (\mathbf{1}_{\mathcal{G}}\otimes\mathbf{1}_{\mathcal{G}}\otimes\phi_\rho^{-1})(\phi_{\lambda_0}\otimes\mathbf{1}_{\mathcal{G}}\otimes\mathbf{1}_{\mathcal{G}})$. An analogous statement holds for $\hat{\mathcal{G}}\ltimes_{\lambda_0}\mathcal{M}$. Such a scenario may of course be produced trivially by starting with $(\delta=\mathbf{1}_{\mathcal{G}}\otimes\rho, \Psi=\mathbf{1}_{\mathcal{G}}\otimes\mathbf{1}_{\mathcal{G}}\otimes\mathbf{1}_{\mathcal{M}}\otimes\mathbf{1}_{\mathcal{G}}\otimes\mathbf{1}_{\mathcal{G}})$ as a two–sided coaction on a strictly coassociative Hopf algebra $(\mathcal{G},\Delta)$ and subsequently passing to a twist equivalent quasi–Hopf algebra structure Δ_F on $\mathcal{G}$ with Ψ_F given by (8.18). As another example one may take a strictly coassociative Hopf algebra $(\mathcal{G},\Delta)$ with $\phi,\phi_{\lambda_0},\alpha,\beta,h$ being trivial, but (ρ,ϕ_ρ) being a $\mathcal{G}$–coaction on $\mathcal{M}$ in the sense of Definition 7.1, with a nontrivial cocycle ϕ_ρ as considered in [DT,BCM,BM]. Hence $\delta := \mathbf{1}_{\mathcal{G}}\otimes\rho, \Psi := \mathbf{1}_{\mathcal{G}}\otimes\mathbf{1}_{\mathcal{G}}\otimes\phi_\rho^{-1}$ would give a well defined two–sided $\mathcal{G}$–coaction in the sense of our Definition 8.1. In this case Ω_R in (10.9) would be given by

$$\Omega_R = \mathbf{1}_{\mathcal{G}}\otimes\mathbf{1}_{\mathcal{G}}\otimes\phi_\rho^{-1}$$

and defining $\sigma : \hat{\mathcal{G}} \otimes \hat{\mathcal{G}} \longrightarrow \mathcal{M}$ by

$$\sigma(\varphi \otimes \psi) := (\mathrm{id} \otimes \varphi \otimes \psi)(\phi_\rho^{-1})$$

one immediately verifies from Definition 10.1 that in this special case our right diagonal crossed product satisfies

$$\mathcal{M} \bowtie_\delta \hat{\mathcal{G}} = \mathcal{M} \#_\sigma \hat{\mathcal{G}}$$

where the right hand side is the twisted crossed product considered in [DT,BCM,BM].

10.2 Left and right diagonal δ–implementers

From the associativity proof of Theorem 10.2 in terms of the "generating matrix" $\mathbf{L}$ we immediately read off an analogue of Proposition 5.3 describing the conditions under which an algebra map $\gamma : \mathcal{M} \longrightarrow \mathcal{A}$ into some target algebra $\mathcal{A}$ extends to an algebra map from the diagonal crossed products into $\mathcal{A}$. First, in view of Eq. (10.29), given $\gamma : \mathcal{M} \longrightarrow \mathcal{A}$, we consider a left action $\succ$ and a right action $\prec$ of $\mathcal{G} \otimes \mathcal{M} \otimes \mathcal{G}$ on $\mathcal{G} \otimes \mathcal{A}$ given for $\mathbf{X} \in \mathcal{G} \otimes \mathcal{A}$, and $a, b \in \mathcal{G}$, $m \in \mathcal{M}$ by

$$(a \otimes m \otimes b) \succ \mathbf{X} := [b \otimes \gamma(m)]\, \mathbf{X}\, [S^{-1}(a) \otimes \mathbf{1}_\mathcal{A}] \tag{10.33}$$

$$\mathbf{X} \prec (a \otimes m \otimes b) := [S^{-1}(b) \otimes \mathbf{1}_\mathcal{A}]\, \mathbf{X}\, [a \otimes \gamma(m)] \tag{10.34}$$

Note that for $\mathcal{A} = \hat{\mathcal{G}} \bowtie \mathcal{M}$ and $\gamma : \mathcal{M} \longrightarrow \mathcal{A}$ the canonical inclusion, Eq. (10.29) now reads

$$[\mathbf{1}_\mathcal{G} \otimes m]\, \mathbf{L} = \mathbf{L} \prec \delta(m), \quad \forall m \in \mathcal{M}$$

Also note that the action $\succ$ commutes with right multiplication by $\mathbf{1}_\mathcal{G} \otimes \gamma(m)$ and the action $\prec$ commutes with left multiplication by $\mathbf{1}_\mathcal{G} \otimes \gamma(m)$. Thus we are lead to the following

Definition 10.3. Let $\gamma : \mathcal{M} \longrightarrow \mathcal{A}$ be an algebra map into some target algebra $\mathcal{A}$ and let (δ, Ψ) be a two–sided $\mathcal{G}$–coaction on $\mathcal{M}$. A *left diagonal δ– implementer* in $\mathcal{A}$ (with respect to γ) is an element $\mathbf{L} \in \mathcal{G} \otimes \mathcal{A}$ satisfying for all $m \in \mathcal{M}$

$$[\mathbf{1}_\mathcal{G} \otimes \gamma(m)]\, \mathbf{L} = \mathbf{L} \prec \delta(m) \tag{10.35}$$

Similarly, a *right diagonal δ–implementer* in $\mathcal{A}$ (with respect to γ) is an element $\mathbf{R} \in \mathcal{G} \otimes \mathcal{A}$ satisfying for all $m \in \mathcal{M}$

$$\mathbf{R}\, [\mathbf{1}_\mathcal{G} \otimes \gamma(m)] = \delta(m) \succ \mathbf{R} \tag{10.36}$$

A left δ–implementer $\mathbf{L}$ (right δ–implementer $\mathbf{R}$) is called *coherent* if, respectively,

$$\mathbf{L}^{13}\, \mathbf{L}^{23} = [\Omega_L^5 \otimes \Omega_L^4 \otimes \mathbf{1}_\mathcal{A}]\, (\Delta \otimes \mathrm{id})(\mathbf{L})\, [\Omega_L^1 \otimes \Omega_L^2 \otimes \gamma(\Omega_L^3)] \tag{10.37}$$

$$\mathbf{R}^{13}\, \mathbf{R}^{23} = [\Omega_R^4 \otimes \Omega_R^5 \otimes \gamma(\Omega_R^3)]\, (\Delta \otimes \mathrm{id})(\mathbf{R})\, [\Omega_R^2 \otimes \Omega_R^1 \otimes \mathbf{1}_\mathcal{A})], \tag{10.38}$$

where $\Omega_{L/R}$ have been defined in Eqs. (10.8) and (10.9).

To unburden our terminology from now by a left (right) δ–implementer we will always mean a left (right) diagonal δ–implementer in the sense of the above definition. We trust that the reader will not be confused by this slight inconsistence of terminology (which arises in comparison with Definition 2.3, since two–sided coactions might also be looked upon as one–sided ones).

As before, we also call $\mathbf{L}/\mathbf{R}$ *normal*, if $(\epsilon \otimes \mathrm{id})(\mathbf{L}/\mathbf{R}) = \mathbf{1}_{\mathcal{A}}$. Note that in the coassociative setting of Lemma 5.2 left and right δ–implementers always coincide. In the present context we will still have one–to–one correspondences between left and right δ–implementers, however the identifications will not be the trivial ones. This observation will eventually lead to a proof of Theorem 10.2(iii). Before approaching this goal let us first note the immediate

Corollary 10.4. *Let $(\mathcal{M}, \delta, \Psi)$ be a two–sided $\mathcal{G}$–comodule algebra and let $\gamma : \mathcal{M} \longrightarrow \mathcal{A}$ be an algebra map into some target algebra $\mathcal{A}$. Then the relations*

$$\gamma_L(\varphi \bowtie m) = (\varphi \otimes \mathrm{id})(\mathbf{L})\, \gamma(m) \tag{10.39}$$

$$\gamma_R(m \bowtie \varphi) = \gamma(m)\, (\varphi \otimes \mathrm{id})(\mathbf{R}) \tag{10.40}$$

provide one–to–one correspondences between algebra maps $\gamma_L : \hat{\mathcal{G}} \bowtie \mathcal{M} \longrightarrow \mathcal{A}$ ($\gamma_R : M \bowtie \hat{\mathcal{G}} \longrightarrow \mathcal{A}$) extending γ and coherent left δ–implementers $\mathbf{L}$ (coherent right δ–implementers $\mathbf{R}$), respectively, where γ_L/γ_R is unital if and only if $\mathbf{L}/\mathbf{R}$ is normal.

Proof. This follows immediately from Eqs. (10.29) and (10.30) and the analogue relations in $\mathcal{M} \bowtie_\delta \hat{\mathcal{G}}$. □

To approach the proof of part (iii) of Theorem 10.2 we now show that there is a coherence preserving isomorphism between the spaces of left and right δ–implementers. At this point the reader may find it helpful to also consult the representation theoretic interpretation of left and right δ–implementers given at the end of this subsection (starting with Definition 10.7), where Proposition 10.5 is expressed by the commuting diagram (10.53).

Proposition 10.5. *Under the setting of Corollary 10.4 let $p \equiv p_\delta$ and $q \equiv q_\delta$ be given by Eqs.* (9.49) *and* (9.50)*. Then the map*

$$\mathbf{L} \mapsto \mathbf{R} := \mathbf{L} \prec p \tag{10.41}$$

provides a coherence and normality preserving isomorphism from the space of left δ–implementers onto the space of right δ–implementers with inverse given by

$$\mathbf{R} \mapsto \mathbf{L} := q \succ \mathbf{R} \tag{10.42}$$

Proof. Throughout we are going to drop the symbol γ, in particular $\mathbf{1}_{\mathcal{M}} \equiv \gamma(\mathbf{1}_{\mathcal{M}}) \equiv \mathbf{1}_{\mathcal{A}}$. By (8.4) we have $(\epsilon \otimes \mathrm{id}_{\mathcal{M}} \otimes \epsilon)(p) = (\epsilon \otimes \mathrm{id}_{\mathcal{M}} \otimes \epsilon)(q) = \mathbf{1}_{\mathcal{M}}$ and therefore normality of $\mathbf{L}/\mathbf{R}$ implies normality of $\mathbf{L} \prec p$ and $q \succ \mathbf{R}$, respectively. If $\mathbf{L}$ is a left δ–implementer then, using (10.33) - (10.35)

$$\begin{aligned}
\delta(m) \succ (\mathbf{L} \prec p) &\equiv [(\mathbf{1}_{\mathcal{G}} \otimes m_{(0)})\, \mathbf{L}] \prec [p\, (S^{-1}(m_{(-1)}) \otimes \mathbf{1}_{\mathcal{M}} \otimes S(m_{(1)}))] \\
&= \mathbf{L} \prec [\delta(m_{(0)})\, p\, (S^{-1}(m_{(-1)}) \otimes \mathbf{1}_{\mathcal{M}} \otimes S(m_{(1)}))] \\
&= \mathbf{L} \prec [p\, (\mathbf{1}_{\mathcal{G}} \otimes m \otimes \mathbf{1}_{\mathcal{G}})] \\
&\equiv (\mathbf{L} \prec p)\, [\mathbf{1}_{\mathcal{G}} \otimes m]
\end{aligned}$$

for all $m \in \mathcal{M}$, where in the third line we have used (9.51). Hence $\mathbf{L} \prec p$ is a right δ–implementer. Analogously, if $\mathbf{R}$ is a right δ–implementer, then (9.52) implies that $q \succ \mathbf{R}$ is a left δ–implementer. Next, if $\mathbf{L}$ is a left δ–implementer, then

$$\begin{aligned} q \succ (\mathbf{L} \prec p) &= [(\mathbf{1}_{\mathcal{G}} \otimes q^2)\,\mathbf{L}] \prec [p\,(S^{-1}(q^1) \otimes \mathbf{1}_{\mathcal{M}} \otimes S(q^3))] \\ &= \mathbf{L} \prec [\delta(q^2)\,p\,(S^{-1}(q^1) \otimes \mathbf{1}_{\mathcal{M}} \otimes S(q^3))] \\ &= \mathbf{L} \end{aligned}$$

by (9.53). Similarly, using (9.54), one shows for a right δ–implementer $\mathbf{R}$

$$(q \succ \mathbf{R}) \prec p = [(S(p^1) \otimes \mathbf{1}_{\mathcal{M}} \otimes S^{-1}(p^3))\,q\,\delta(p^2)] \succ \mathbf{R} = \mathbf{R}.$$

We are left to show that $\mathbf{L}$ is coherent if and only if $\mathbf{R} := \mathbf{L} \prec p$ is coherent. So if $\mathbf{L}$ is a coherent left δ–implementer we compute for $\mathbf{R} = \mathbf{L} \prec p$ from (10.37) and (10.8)

$$\begin{aligned} \mathbf{R}^{13}\,\mathbf{R}^{23} &= \big[S^{-1}(p^3)\Omega_L^5 \otimes S^{-1}(p^2_{(1)}\hat{p}^3)\Omega_L^4 \otimes \mathbf{1}_{\mathcal{M}}\big]\,(\Delta \otimes \mathrm{id})(\mathbf{L})\,\big[\Omega_L^1\,p^1 \otimes \Omega_L^2\,p^2_{(-1)}\hat{p}^1 \otimes \Omega_L^3\,p^2_{(0)}\hat{p}^2\big] \\ &= (S^{-1} \otimes S^{-1} \otimes \mathrm{id}_{\mathcal{M}})\Big(f^{21}(\bar{\Psi}^5 \otimes \bar{\Psi}^4)(p^3 \otimes p^2_{(1)}\hat{p}^3) \otimes \mathbf{1}_{\mathcal{M}}\Big) \\ &\qquad (\Delta \otimes \mathrm{id})(\mathbf{L})\,\big[\bar{\Psi}^1 p^1 \otimes \bar{\Psi}^2 p^2_{(-1)}\hat{p}^1 \otimes \bar{\Psi}^3 p^2_{(0)}\hat{p}^2\big] \end{aligned} \tag{10.43}$$

where $\hat{p}$ is another copy of p. On the other hand, using

$$(\Delta \otimes \mathrm{id})(\mathbf{R}) = [\Delta(S^{-1}(p^3)) \otimes \mathbf{1}_{\mathcal{M}}]\,(\Delta \otimes \mathrm{id})(\mathbf{L})\,[\Delta(p^1) \otimes p^2] \tag{10.44}$$

one computes

$$\begin{aligned} &[\Omega_R^4 \otimes \Omega_R^5 \otimes \Omega_R^3]\,(\Delta \otimes \mathrm{id})(\mathbf{R})\,[\Omega_R^2 \otimes \Omega_R^1 \otimes \mathbf{1}_{\mathcal{M}})] \\ &\quad = \big[(\Omega_R^4 \otimes \Omega_R^5)\Delta(S^{-1}(\Omega^3_{R(1)}p^3)) \otimes \mathbf{1}_{\mathcal{M}}\big]\,(\Delta \otimes \mathrm{id})(\mathbf{L})\,\big[\Delta(\Omega^3_{R(-1)}p^1) \otimes \Omega^3_{R(0)}p^2\big]\,\big[\Omega_R^2 \otimes \Omega_R^1 \otimes \mathbf{1}_{\mathcal{A}}\big] \\ &\quad = (S^{-1} \otimes S^{-1} \otimes \mathrm{id}_{\mathcal{M}})\Big(f^{21}\Delta^{op}(\Psi^3_{(1)}p^3)(f^{-1})^{21}\big(S(\Psi^4) \otimes S(\Psi^5)\big) \otimes \mathbf{1}_{\mathcal{M}}\Big) \\ &\qquad\qquad \times (\Delta \otimes \mathrm{id})(\mathbf{L})\,\Big[\Delta(\Psi^3_{(-1)}p^1)h^{-1}\big(S^{-1}(\Psi^2) \otimes S^{-1}(\Psi^1)\big) \otimes \Psi^3_{(0)}p^2\Big] \end{aligned} \tag{10.45}$$

where in the first equation we have used (10.44) and (10.35), in the second equation (10.9) and (6.23). Comparing the r.h.s. of (10.43) and (10.45) we conclude that they are equal due to Eq. (9.55). Hence the coherence condition (10.38) is satisfied. This proves that $\mathbf{R} := \mathbf{L} \prec p$ is coherent provided $\mathbf{L}$ is coherent. By functoriality, the inverse conclusion follows analogously from $\mathbf{L} := q \succ \mathbf{R}$ and the remark (10.14). This concludes the proof of Proposition 10.5. $\square$

Proof of Theorem 10.2 (iii): We are now in the position to give the proof of Theorem 10.2 (iii). Using the notation (10.33) the map $f : \hat{\mathcal{G}} \bowtie \mathcal{M} \longrightarrow \mathcal{M} \bowtie \hat{\mathcal{G}}$ in (10.15) is of the form

$$f(\varphi \bowtie m) = (\varphi \otimes \mathrm{id})\big(q \succ \mathbf{R}\big)\,(m \bowtie \mathbf{1}_{\hat{\mathcal{G}}})$$

where $\mathbf{R} = e_i \otimes (\mathbf{1}_{\mathcal{M}} \bowtie e^i)$ is the generating matrix in $\mathcal{M} \bowtie \hat{\mathcal{G}}$. By Proposition 10.5 $\mathbf{L} := q \succ \mathbf{R}$ is a coherent left δ–implementer in $\mathcal{M} \bowtie \hat{\mathcal{G}}$ and therefore, by Corollary 10.4, f is an algebra map. Also by Proposition 10.5, f is bijective with inverse given by

$$f^{-1}(m \bowtie \varphi) = (\mathbf{1}_{\hat{\mathcal{G}}} \bowtie m)(\varphi \otimes \mathrm{id})\big(\mathbf{L} \prec p\big),$$

where $\mathbf{L} = e_i \otimes (e^i \bowtie \mathbf{1}_{\mathcal{M}})$ is the generating matrix in $\hat{\mathcal{G}} \bowtie \mathcal{M}$. This proves (10.16) and therefore concludes the proof of Theorem 10.2 (iii). $\square$

Next, we show that the diagonal crossed products associated with twist equivalent two–sided coactions are equivalent algebra extensions.

Proposition 10.6.

1. *Let (δ, Ψ) and (δ', Ψ') be twist equivalent two–sided $\mathcal{G}$–coactions on $\mathcal{M}$. Then the diagonal crossed products $\mathcal{M} \bowtie_\delta \hat{\mathcal{G}}$ and $\mathcal{M} \bowtie_{\delta'} \hat{\mathcal{G}}$ are equivalent extensions of $\mathcal{M}$.*
2. *Let (δ, Ψ) be a two–sided $\mathcal{G}$–coaction on $\mathcal{M}$ with respect to the coproduct $\Delta : \mathcal{G} \longrightarrow \mathcal{G} \otimes \mathcal{G}$, and let (δ, Ψ_F) be the two–sided coaction with respect to a twist equivalent coproduct Δ_F on $\mathcal{G}$, see Eq. (8.18). Denote the associated diagonal crossed products by $\mathcal{M} \bowtie \hat{\mathcal{G}}$ and $\mathcal{M} \bowtie \hat{\mathcal{G}}_F$, respectively. Then $\mathcal{M} \bowtie \hat{\mathcal{G}} = \mathcal{M} \bowtie \hat{\mathcal{G}}_F$ with trivial identification.*

Proof. 1. Let $U \in \mathcal{G} \otimes \mathcal{M} \otimes \mathcal{G}$ be a normal twist transformation from (δ, Ψ) to (δ', Ψ') and let $\mathbf{R} \in \mathcal{G} \otimes (\mathcal{M} \bowtie_\delta \hat{\mathcal{G}})$ and $\mathbf{R}' \in \mathcal{G} \otimes (\mathcal{M} \bowtie_{\delta'} \hat{\mathcal{G}})$ be the generating matrices. By Corollary 10.4, to provide a homomorphism

$$f : \mathcal{M} \bowtie_\delta \hat{\mathcal{G}} \longrightarrow \mathcal{M} \bowtie_{\delta'} \hat{\mathcal{G}}$$

restricting to the identity on $\mathcal{M}$ we have to find a coherent normal right δ–implementer $\tilde{\mathbf{R}} \in \mathcal{G} \otimes (\mathcal{M} \bowtie_{\delta'} \hat{\mathcal{G}})$. We claim that the canonical choice

$$\tilde{\mathbf{R}} := U^{-1} \succ \mathbf{R}' \tag{10.46}$$

will do the job. Indeed, $\tilde{\mathbf{R}}$ obviously is a normal right δ–implementer and one is left with checking the coherence condition with respect to (δ, Ψ). Using (8.16) this is straight forward and is left to the reader.

To prove part 2. we note that $\Psi_F = \Psi\,(F^{-1} \otimes \mathbf{1}_{\mathcal{M}} \otimes F^{-1})$ implies by (10.9) $(\Omega_R)_F = F^{21}\,\Omega_R\,(F^{-1})^{45}$ since the element $h \in \mathcal{G} \otimes \mathcal{G}$ transforms under a twist according to $h_F = (S^{-1} \otimes S^{-1})(F_{op}^{-1})\,h\,F^{-1}$. Hence, by Definition 10.3 $\mathbf{R}$ is coherent with respect to (δ, Ψ, Δ) if and only if it is coherent with respect to $(\delta, \Psi_F, \Delta_F)$. ◻

Of course, analogous statements hold for the left diagonal crossed products. We emphasize that Proposition 10.6 implies that - in the semisimple case - the diagonal crossed product $\mathcal{M} \bowtie_\delta \hat{\mathcal{G}}$ is actually completely determined (up to equivalence) by the "two–sided fusion rules" $\mathrm{Rep}\,\mathcal{G} \times \mathrm{Rep}\,\mathcal{M} \times \mathrm{Rep}\,\mathcal{G} \longrightarrow \mathrm{Rep}\,\mathcal{M}$ induced by (δ, Ψ). A more detailed exploration of this observation will be given elsewhere.

Moreover, in view of Proposition 8.5 we may from now on restrict ourselves to two–sided coactions of the form $(\delta, \Psi) = (\delta_{l/r}, \Psi_{l/r})$ for a quasi–commuting pair $(\lambda, \phi_\lambda, \rho, \phi_\rho, \phi_{\lambda\rho})$, where $\delta_l = (\lambda \otimes \mathrm{id}) \circ \rho$ and $\delta_r = (\mathrm{id} \otimes \rho) \circ \lambda$, see Eqs. (8.19) - (8.22).

In this light it will also be appropriate to introduce as an alternative notation consistent with (6.6),(6.7)

$$\hat{\mathcal{G}}\,{}_\lambda\!\bowtie_\rho \mathcal{M} := \hat{\mathcal{G}} \bowtie_{\delta_l} \mathcal{M} \tag{10.47}$$

$$\mathcal{M}\,{}_\lambda\!\bowtie_\rho \hat{\mathcal{G}} := \mathcal{M} \bowtie_{\delta_r} \hat{\mathcal{G}} \tag{10.48}$$

By Theorem 10.2(iii) these are equivalent to $\mathcal{M} \bowtie_{\delta_l} \hat{\mathcal{G}}$ and $\hat{\mathcal{G}} \bowtie_{\delta_r} \mathcal{M}$, respectively, and by Proposition 10.6(1.) we have $\mathcal{M} \bowtie_{\delta_r} \hat{\mathcal{G}} \cong \mathcal{M} \bowtie_{\delta_l} \hat{\mathcal{G}}$ and $\hat{\mathcal{G}} \bowtie_{\delta_l} \mathcal{M} \cong \hat{\mathcal{G}} \bowtie_{\delta_r} \mathcal{M}$, where according to (10.46) and Prop. 8.4(1.)

$$\mathbf{L}_{\delta_l} = \mathbf{L}_{\delta_r} \prec \phi_{\lambda\rho} \tag{10.49}$$

$$\mathbf{R}_{\delta_r} = \phi_{\lambda\rho} \succ \mathbf{R}_{\delta_l} \tag{10.50}$$

Thus we get four equivalent versions of diagonal crossed products associated with any quasi–commuting pair $(\lambda, \rho, \phi_\lambda, \phi_\rho, \phi_{\lambda\rho})$ of $\mathcal{G}$–coactions on $\mathcal{M}$, all of which will be shown in Section 10.4 to be a realization of the abstract algebra $\mathcal{M}_1$ in Theorem II.

We conclude this subsection by giving a representation theoretic interpretation of our notion of left and right δ–implementers. To this end let $(\mathfrak{H}, \gamma_{\mathfrak{H}})$ be a fixed representation of $\mathcal{M}$. Putting $\mathcal{A} := \mathrm{End}_{\mathbb{C}}(\mathfrak{H})$, we consider $\gamma \equiv \gamma_{\mathfrak{H}} : \mathcal{M} \longrightarrow \mathcal{A}$ as an algebra map. This leads to

Definition 10.7. Let (δ, Ψ) be a two–sided $\mathcal{G}$–coaction on $\mathcal{M}$. A representation $(\mathfrak{H}, \gamma)$ of $\mathcal{M}$ is called *δ–coherent* if there exists a normal coherent left δ–implementer $\mathbf{L} \in \mathcal{G} \otimes \mathrm{End}_{\mathbb{C}}(\mathfrak{H})$ (equivalently right δ–implementer $\mathbf{R} \in \mathcal{G} \otimes \mathrm{End}_{\mathbb{C}}(\mathfrak{H})$).

Corollary 10.4 then says that a representation of $\mathcal{M}$ is δ–coherent if and only if it extends to a representation of the diagonal crossed products $\hat{\mathcal{G}} \bowtie_\delta \mathcal{M} \cong \mathcal{M} \bowtie_\delta \hat{\mathcal{G}}$.

We now provide a category theoretic description of δ–coherent representations $(\mathfrak{H}, \gamma)$. Associated with a left δ–implementer $\mathbf{L} \in \mathcal{G} \otimes \mathcal{A}$, $\mathcal{A} := \mathrm{End}_{\mathbb{C}}(\mathfrak{H})$, we define a natural family of $\mathcal{M}$–linear morphisms

$$l_V : V \gtrdot \mathfrak{H} \lessdot V^* \longrightarrow \mathfrak{H}, \quad v \otimes \mathfrak{h} \otimes \hat{v} \mapsto \Big((\hat{v} \otimes \mathrm{id}) \circ \mathbf{L}_V\Big)(v \otimes \mathfrak{h}) \tag{10.51}$$

where $(V, \pi_V) \in \mathrm{Rep}\,\mathcal{G}$ and $\mathbf{L}_V := (\pi_V \otimes \mathrm{id})(\mathbf{L})$ and where we have used the notation (9.16). Eq. (10.35) guarantees that l_V is in fact a morphism in $\mathrm{Rep}\,\mathcal{M}$, and naturality means[7]

$$l_V \circ (g \otimes \mathrm{id}_{\mathfrak{H}} \otimes \mathrm{id}_{W^*}) = l_V \circ (\mathrm{id}_V \otimes \mathrm{id}_{\mathfrak{H}} \otimes g^t)$$

for any morphism $g : V \longrightarrow W$ in $\mathrm{Rep}\,\mathcal{G}$. The normality condition for $\mathbf{L}$ implies $l_{\mathbb{C}} = \mathrm{id}_{\mathfrak{H}}$ and the coherence condition (10.37) for $\mathbf{L}$ translates into the following coherence condition for the l_V's

$$l_{V \boxtimes W} = l_V \circ (\mathrm{id}_V \otimes l_W \otimes \mathrm{id}_{V^*}) \circ \Omega^L_{VW\mathfrak{H}W^*V^*}$$

where $\Omega^L_{VW\mathfrak{H}W^*V^*} : (V \boxtimes W) \gtrdot \mathfrak{H} \lessdot (V \boxtimes W)^* \longrightarrow V \gtrdot (W \gtrdot \mathfrak{H} \lessdot W^*) \lessdot V^*$ is the natural $\mathcal{M}$–linear isomorphism given by

$$\Omega^L_{VW\mathfrak{H}W^*V^*} = \Psi_{VW\mathfrak{H}W^*V^*} \circ (\mathrm{id}_V \otimes \mathrm{id}_W \otimes \mathrm{id}_{\mathfrak{H}} \otimes f^{-1}_{W^*V^*})$$

see (9.8), (9.17) and (10.8). Similarly, a right δ–implementer $\mathbf{R} \in \mathcal{G} \otimes \mathcal{A}$ gives rise to a natural coherent family of $\mathcal{M}$–linear morphisms

$$r_V : \mathfrak{H} \longrightarrow V^* \gtrdot \mathfrak{H} \lessdot V, \quad \mathfrak{h} \mapsto v^i \otimes \mathbf{R}^{21}_V(\mathfrak{h} \otimes v_i) \tag{10.52}$$

where $\mathbf{R}_V := (\pi_V \otimes \mathrm{id})(\mathbf{R})$. As above, naturality means[7]

$$(\mathrm{id}^*_V \otimes \mathrm{id}_{\mathfrak{H}} \otimes g) \circ r_V = (g^t \otimes \mathrm{id}_{\mathfrak{H}} \otimes \mathrm{id}_W) \circ r_W$$

for all ($\mathcal{G}$–linear) morphisms $g : V \longrightarrow W$, and coherence means

$$r_{U \boxtimes V} = \Omega^R_{V^*U^*\mathfrak{H}UV} \circ (\mathrm{id}_{V^*} \otimes r_U \otimes \mathrm{id}_V) \circ r_V$$

[7] Actually, in the terminology of [ML,Chap.IX.4] l_V and r_V are *dinatural* ($\equiv$ diagonal–natural) transformations. We thank T. Kerler for pointing this out to us.

where $\Omega^R_{V^*U^*\mathfrak{H}UV} : V^* \gtrdot (U^* \gtrdot \mathfrak{H} \lessdot U) \lessdot V \longrightarrow (U \boxtimes V)^* \gtrdot \mathfrak{H} \lessdot (U \boxtimes V)$ is given by

$$\Omega^R_{V^*U^*\mathfrak{H}UV} := (f_{V^*U^*} \otimes \mathrm{id}_{\mathfrak{H}} \otimes \mathrm{id}_U \otimes \mathrm{id}_V) \circ \Psi^{-1}_{V^*U^*\mathfrak{H}UV},$$

see Eqs. (10.9) and (10.38). The transformation from left δ–implementers to right δ–implementers given in Proposition 10.5 may now be described as follows. In analogy to Eqs. (9.36) and (9.37) we define $\mathcal{M}$–linear morphisms

$$\begin{aligned} P_{V\mathfrak{H}W} :\ & \mathfrak{H} \longrightarrow V^* \gtrdot (V \gtrdot \mathfrak{H} \lessdot W) \lessdot {}^*W \\ & \mathfrak{h} \longmapsto v^i \otimes p_\delta \cdot (v_i \otimes \mathfrak{h} \otimes w_j) \otimes w^j \\ Q_{V\mathfrak{H}W} :\ & {}^*V \gtrdot (V \gtrdot \mathfrak{H} \lessdot W) \lessdot W^* \longrightarrow \mathfrak{H} \\ & \hat{v} \otimes v \otimes \mathfrak{h} \otimes w \otimes \hat{w} \longmapsto (\hat{v} \otimes \mathrm{id}_{\mathfrak{H}} \otimes \hat{w})\Big(q_\delta \cdot (v \otimes \mathfrak{h} \otimes w)\Big) \end{aligned}$$

obeying

$$\begin{aligned} Q_{V^*(V \gtrdot \mathfrak{H} \lessdot W)^*W} \circ (\mathrm{id}_V \otimes P_{V\mathfrak{H}W} \otimes \mathrm{id}_W) &= \mathrm{id}_{V \gtrdot \mathfrak{H} \lessdot W} \\ (\mathrm{id}_V \otimes Q_{V\mathfrak{H}W} \otimes \mathrm{id}_W) \circ P_{{}^*V(V \gtrdot \mathfrak{H} \lessdot W)W^*} &= \mathrm{id}_{V \gtrdot \mathfrak{H} \lessdot W} \end{aligned}$$

where we have used Eqs. (9.51) - (9.54). Proposition 10.5 then implies that the transition from l_V's to r_V's and back is given by the following commuting diagram

$$\begin{tikzcd} V \gtrdot \mathfrak{H} \lessdot V^* \arrow[r, "l_V"] \arrow[d, "\mathrm{id}_V \gtrdot r_V \lessdot \mathrm{id}_{V^*}"'] & \mathfrak{H} \arrow[r, "r_V"] \arrow[dr, "P_{V\mathfrak{H}V^*}"'] & V^* \gtrdot \mathfrak{H} \lessdot V \\ V \gtrdot (V^* \gtrdot \mathfrak{H} \lessdot V) \lessdot V^* \arrow[ur, "Q_{V^*\mathfrak{H}V}"'] & & V^* \gtrdot (V \gtrdot \mathfrak{H} \lessdot V^*) \lessdot V \arrow[u, "\mathrm{id}_{V^*} \gtrdot l_V \lessdot \mathrm{id}_V"'] \end{tikzcd} \tag{10.53}$$

10.3 Coherent $\lambda\rho$–intertwiners

In this subsection we are going to generalize Lemma 5.2 by providing a normality and coherence preserving one–to–one correspondence between right δ_r–implementers $\mathbf{R}$ or left δ_l–implementers $\mathbf{L}$, respectively, and $\lambda\rho$–intertwiners $\mathbf{T}$, where $\delta_r := (\mathrm{id} \otimes \rho) \circ \lambda$ and $\delta_l := (\lambda \otimes \mathrm{id}) \circ \rho$. This will finally lead to a proof of Theorem II. We start with a generalization of Definition 5.1

Definition 10.8. Let $(\lambda, \phi_\lambda, \rho, \phi_\rho, \phi_{\lambda\rho})$ be a quasi–commuting pair of $\mathcal{G}$–coactions on $\mathcal{M}$ and let $\gamma : \mathcal{M} \otimes \mathcal{A}$ be a unital algebra map into some target algebra $\mathcal{A}$. A *$\lambda\rho$–intertwiner* in $\mathcal{A}$ (with respect to γ) is an element $\mathbf{T} \in \mathcal{G} \otimes \mathcal{A}$ satisfying

$$\mathbf{T}\, \lambda_{\mathcal{A}}(m) = \rho^{op}_{\mathcal{A}}(m)\, \mathbf{T}, \quad \forall m \in \mathcal{A} \tag{10.54}$$

A $\lambda\rho$–intertwiner is called *normal* if $(\epsilon \otimes \mathrm{id})(\mathbf{T}) = \mathbf{1}_{\mathcal{A}}$ and it is called *coherent*, if

$$(\phi^{312}_\rho)_{\mathcal{A}}\, \mathbf{T}^{13}\, (\phi^{-1}_{\lambda\rho})^{132}_{\mathcal{A}}\, \mathbf{T}^{23}\, (\phi_\lambda)_{\mathcal{A}} = (\Delta \otimes \mathrm{id})(\mathbf{T}) \tag{10.55}$$

where the index $\mathcal{A}$ refers to the image $\gamma(\mathcal{M}) \subset \mathcal{A}$, see also Eqs. (6.2) and (6.3).

We first point out that Eq. (10.55) is consistent with (10.54) in the following sense

Lemma 10.9. *Under the conditions of Definition 10.8 let* $\mathbf{T}$ *be a* $\lambda\rho$*–intertwiner in* $\mathcal{A}$ *and define* $B \in \mathcal{G} \otimes \mathcal{G} \otimes \mathcal{A}$ *by*

$$B := (\phi_\rho^{312})_{\mathcal{A}}\, \mathbf{T}^{13}\, (\phi_{\lambda\rho}^{-1})_{\mathcal{A}}^{132}\, \mathbf{T}^{23}\, (\phi_\lambda)_{\mathcal{A}} \tag{10.56}$$

Then we have for all $m \in \mathcal{M}$

$$B\, (\Delta \otimes \mathrm{id}_{\mathcal{A}})\big(\lambda(m)_{\mathcal{A}}\big) = (\Delta \otimes \mathrm{id}_{\mathcal{A}})\big(\rho^{op}(m)_{\mathcal{A}}\big)\, B \tag{10.57}$$

Proof. This is straightforward from the intertwiner properties of $\mathbf{T}$ and the reassociators $\phi_\lambda, \phi_{\lambda\rho}$ and ϕ_ρ, see (10.54), (7.5), (7.1) and (8.10). $\square$

We now state the generalization of Lemma 5.2, where we recall our notation for the left and right actions given in Eqs. (10.33) and (10.34).

Proposition 10.10.

1. *Under the conditions of Definition 10.8 let* $\delta_r := (\mathrm{id} \otimes \rho) \circ \lambda$ *and* $\Psi_r \in \mathcal{G} \otimes \mathcal{G} \otimes \mathcal{M} \otimes \mathcal{G} \otimes \mathcal{G}$ *as in* (8.22) *and let* $p_\lambda, q_\lambda \in \mathcal{G} \otimes \mathcal{M}$ *be given by Eqs.* (9.20),(9.21). *Then the assignment*

$$\mathbf{T} \longmapsto \mathbf{R} := \mathbf{T}\, (\mathrm{id} \otimes \gamma)(p_\lambda) \tag{10.58}$$

provides a normality and coherence preserving isomorphism from the space of $\lambda\rho$*–intertwiners onto the space of right* δ_r*–implementers with inverse given by*

$$\mathbf{R} \longmapsto \mathbf{T} := (\mathrm{id}_{\mathcal{G}} \otimes \rho)(q_\lambda) \succ \mathbf{R} \tag{10.59}$$

2. *Similarly let* $\delta_l := (\lambda \otimes \mathrm{id}) \circ \rho$ *and* $\Psi_l \in \mathcal{G} \otimes \mathcal{G} \otimes \mathcal{M} \otimes \mathcal{G} \otimes \mathcal{G}$ *as in* (8.20), *and let* $p_\rho, q_\rho \in \mathcal{M} \otimes \mathcal{G}$ *be given by Eqs.* (9.22),(9.23). *Then the assignment*

$$\mathbf{T} \longmapsto \mathbf{L} := (\mathrm{id} \otimes \gamma)(q_\rho^{op})\, \mathbf{T} \tag{10.60}$$

provides a normality and coherence preserving isomorphism from the space of $\lambda\rho$*–intertwiners onto the space of left* δ_l*–implementers with inverse given by*

$$\mathbf{L} \longmapsto \mathbf{T} := \mathbf{L} \prec (\lambda \otimes \mathrm{id}_{\mathcal{G}})(p_\rho) \tag{10.61}$$

Again the reader may find it helpful to consult the representation theoretic interpretation given at the end of this subsection, where the statements of Proposition 10.10 are expressed in terms of the commuting diagrams (10.69) and (10.70). We also recall our previous results regarding the transformations between left and right δ_r–implementers or left and right δ_l–implementers (Proposition 10.5), as well as the twist transformations given in Eqs. (10.49) and (10.50). Using the identities (9.57)–(9.60) together with Proposition 10.10 one may check that all these transformations also give rise to commuting diagrams.

Proof. Throughout, by a convenient abuse of notation, we are going to omit the symbol γ. Again we only need to prove part 1, since part 2 is functorially equivalent. If $\mathbf{T}$ is a $\lambda\rho$–intertwiner then

$$\begin{aligned}\delta_r(m) \succ \mathbf{R} &\equiv \rho^{op}(m_{(0)})\, \mathbf{T}\, p_\lambda\, [S^{-1}(m_{(-1)}) \otimes \mathbf{1}_{\mathcal{A}}] \\ &= \mathbf{R}\, [\mathbf{1}_{\mathcal{G}} \otimes m]\end{aligned}$$

by (10.54) and (9.24), and therefore $\mathbf{R}$ is a right δ_r–implementer. Moreover, (9.26) implies

$$\begin{aligned}(\mathrm{id}_{\mathcal{G}} \otimes \rho)(q_\lambda) \succ (\mathbf{T}\, p_\lambda) &\equiv \rho^{op}(q_\lambda^2)\, \mathbf{T}\, p_\lambda\, [S^{-1}(q_\lambda^1) \otimes \mathbf{1}_{\mathcal{M}}] \\ &= \mathbf{T}\, \lambda(q_\lambda^2)\, p_\lambda\, [S^{-1}(q_\lambda^1) \otimes \mathbf{1}_{\mathcal{M}}] \\ &= \mathbf{T}\end{aligned}$$

Conversely if $\mathbf{R}$ is a right δ_r–implementer, then

$$\begin{aligned}\rho^{op}(m)\, \mathbf{T} &\equiv (\mathrm{id}_{\mathcal{G}} \otimes \rho)\big((\mathbf{1}_{\mathcal{G}} \otimes m) q_\lambda\big) \succ \mathbf{R} \\ &= \Big[[(\mathrm{id}_{\mathcal{G}} \otimes \rho)(q_\lambda)\, \delta_r(m_{(0)})] \succ \mathbf{R}\Big]\, [m_{(-1)} \otimes \mathbf{1}_{\mathcal{M}}] \\ &= \mathbf{T}\, \lambda(m)\end{aligned}$$

where in the second equation we have used (9.25) and in the last line the right δ_r–implementer property (10.36) of $\mathbf{R}$. Hence $\mathbf{T}$ is a $\lambda\rho$–intertwiner. Moreover

$$\begin{aligned}\big[(\mathrm{id}_{\mathcal{G}} \otimes \rho)(q_\lambda) \succ \mathbf{R}\big]\, p_\lambda \equiv\ & \big[(\mathrm{id}_{\mathcal{G}} \otimes \rho)\big((S(p_\lambda^1) \otimes \mathbf{1}_{\mathcal{A}})\, q_\lambda\big)\big] \succ \big(\mathbf{R}\, [\mathbf{1}_{\mathcal{G}} \otimes p_\lambda^2]\big) \\ =\ & (\mathrm{id}_{\mathcal{G}} \otimes \rho)\Big((S(p_\lambda^1) \otimes \mathbf{1}_{\mathcal{A}})\, q_\lambda\, \lambda(p_\lambda^2)\Big) \succ \mathbf{R} \\ =\ & \mathbf{R}\end{aligned}$$

Thus the correspondence $\mathbf{T} \leftrightarrow \mathbf{R}$ is one–to–one, and since q_λ and p_λ are normal it is clearly normality preserving.

To prove that it is also coherence preserving assume now that the $\lambda\rho$–intertwiner $\mathbf{T}$ satisfies the coherence condition (10.55) and let $\mathbf{R} = \mathbf{T}\, p_\lambda$. Then

$$(\Delta \otimes \mathrm{id})(\mathbf{R})\, [h^{-1} \otimes \mathbf{1}_{\mathcal{M}}] = \phi_\rho^{312}\, \mathbf{T}^{13}\, (\phi_{\lambda\rho}^{-1})^{132}\, A \tag{10.62}$$

where $A \in \mathcal{G} \otimes \mathcal{G} \otimes \mathcal{M}$ is given by

$$\begin{aligned}A &= \mathbf{T}^{23}\, \phi_\lambda\, (\Delta \otimes \mathrm{id}_{\mathcal{M}})(p_\lambda)\, [h^{-1} \otimes \mathbf{1}_{\mathcal{M}}] \\ &= \mathbf{T}^{23}\, (\mathrm{id}_{\mathcal{G}} \otimes \lambda)\big(\lambda(\bar\phi_\lambda^3)\, p_\lambda\big)\, [\mathbf{1}_{\mathcal{G}} \otimes p_\lambda]\, [S^{-1}(\bar\phi_\lambda^2) \otimes S^{-1}(\bar\phi_\lambda^1) \otimes \mathbf{1}_{\mathcal{M}}] \\ &= (\mathrm{id}_{\mathcal{G}} \otimes \rho^{op})\big(\lambda(\bar\phi_\lambda^3)\, p_\lambda\big)\, \mathbf{R}^{23}\, [S^{-1}(\bar\phi_\lambda^2) \otimes S^{-1}(\bar\phi_\lambda^1) \otimes \mathbf{1}_{\mathcal{M}}]\end{aligned} \tag{10.63}$$

Here we have used (9.28) in the second line and the intertwining property (10.54) of $\mathbf{T}$ in the third line. Using the intertwiner property (8.10) of $\phi_{\lambda\rho}$ and (9.44) we further compute

$$\begin{aligned}&\mathbf{T}^{13}\, (\phi_{\lambda\rho}^{-1})^{132}\, (\mathrm{id}_{\mathcal{G}} \otimes \rho^{op})\big(\lambda(\bar\phi_\lambda^3)\, p_\lambda\big) \\ &\quad = \Big[(\rho^{op} \otimes \mathrm{id})\big(\rho(\bar\phi_\lambda^3)\, (\phi_{\lambda\rho}^2 \otimes \phi_{\lambda\rho}^3)\big)\, (\mathbf{R} \otimes \mathbf{1}_{\mathcal{M}})\Big]^{132}\, [S^{-1}(\phi_{\lambda\rho}^1) \otimes \mathbf{1}_{\mathcal{G}} \otimes \mathbf{1}_{\mathcal{M}}]\end{aligned} \tag{10.64}$$

Putting (10.62) - (10.64) together we finally conclude

$$(\Delta \otimes \mathrm{id})(\mathbf{R})\, [h^{-1} \otimes \mathbf{1}_{\mathcal{M}}] = [\bar\Psi_r^4 \otimes \bar\Psi_r^5 \otimes \bar\Psi_r^3]\, \mathbf{R}^{13}\, \mathbf{R}^{23}\, [S^{-1}(\bar\Psi_r^2) \otimes S^{-1}(\bar\Psi_r^1) \otimes \mathbf{1}_{\mathcal{M}}] \tag{10.65}$$

where $\bar\Psi_r \in \mathcal{G} \otimes \mathcal{G} \otimes \mathcal{M} \otimes \mathcal{G} \otimes \mathcal{G}$ is given by

$$\bar\Psi_r := (\mathbf{1}_{\mathcal{G}} \otimes \mathbf{1}_{\mathcal{G}} \otimes \phi_\rho)(\mathrm{id}_{\mathcal{G}} \otimes \mathrm{id}_{\mathcal{G}} \otimes \rho \otimes \mathrm{id}_{\mathcal{G}})\Big((\mathrm{id}_{\mathcal{G}} \otimes \mathrm{id}_{\mathcal{G}} \otimes \rho)(\Phi_\lambda^{-1})\, (\mathbf{1}_{\mathcal{G}} \otimes \phi_{\lambda\rho})\Big)$$

By (8.22) we have $\bar\Psi_r = \Psi_r^{-1}$ and therefore (10.65) is equivalent to the coherence condition (10.38) for $\mathbf{R}$ as a right (δ_r, Ψ_r)–implementer.

Conversely, assume now that $\mathbf{R}$ is a coherent right (δ_r, Ψ_r)–implementer and let $\mathbf{T} := (\mathrm{id}_{\mathcal{G}} \otimes \rho)(q_\lambda) \succ \mathbf{R}$. To prove that $\mathbf{T}$ is coherent we have to show that

$$(\Delta \otimes \mathrm{id})(\mathbf{T}) = B$$

where B is given by Eq. (10.56). Now writing $\mathbf{R} = \mathbf{T}\, p_\lambda$ and going backwards through the derivation (10.62)←(10.65) we conclude

$$(\Delta \otimes \mathrm{id})(\mathbf{T}\, p_\lambda) = B\, (\Delta \otimes \mathrm{id})(p_\lambda) \tag{10.66}$$

Thus, if p_λ were invertible we could immediately conclude that $\mathbf{T}$ is coherent. It turns out that we may use Eq. (9.26) as a substitute for the invertibility of p_λ, since it implies

$$\begin{aligned}
(\Delta \otimes \mathrm{id})(\mathbf{T}) &\equiv (\Delta \otimes \mathrm{id})\Big((\mathrm{id}_{\mathcal{G}} \otimes \rho)(q_\lambda) \succ (\mathbf{T}\, p_\lambda)\Big) \\
&= (\Delta \otimes \mathrm{id})\big(\rho^{op}(q_\lambda^2)\big)\, (\Delta \otimes \mathrm{id})(\mathbf{T} p_\lambda)\, \big[\Delta(S^{-1}(q_\lambda^1)) \otimes \mathbf{1}_{\mathcal{M}}\big] \\
&= (\Delta \otimes \mathrm{id})\big(\rho^{op}(q_\lambda^2)\big)\, B\, (\Delta \otimes \mathrm{id})\Big(p_\lambda\, (S^{-1}(q_\lambda^1) \otimes \mathbf{1}_{\mathcal{M}})\Big) \\
&= B\, (\Delta \otimes \mathrm{id})\Big(\lambda(q_\lambda^2)\, p_\lambda\, (S^{-1}(q_\lambda^1) \otimes \mathbf{1}_{\mathcal{M}})\Big) \\
&= B
\end{aligned}$$

Here we have used the definition (10.33) in the second line, (10.66) in the third line, (10.57) in the fourth line and again (9.26) in the last line. Thus $\mathbf{T}$ is a coherent $\lambda\rho$–intertwiner, which concludes the proof of Proposition 10.10. □

We conclude this subsection with a representation categorical interpretation of the notion of $\lambda\rho$–intertwiners. As before, given a fixed representation $(\mathfrak{H}, \gamma)$ of $\mathcal{M}$ we consider $\gamma : \mathcal{M} \longrightarrow \mathcal{A} \equiv \mathrm{End}_{\mathbb{C}}(\mathfrak{H})$ as an algebra map. Similarly as in Definition 10.7 we say

Definition 10.11. Let $(\lambda, \phi_\lambda, \rho, \phi_\rho, \phi_{\lambda\rho})$ be a quasi–commuting pair of $\mathcal{G}$–coactions on $\mathcal{M}$. A representation $(\mathfrak{H}, \gamma_{\mathfrak{H}})$ of $\mathcal{M}$ is called *$\lambda\rho$–coherent*, if there exists a normal coherent $\lambda\rho$–intertwiner $\mathbf{T} \in \mathcal{G} \otimes \mathrm{End}_{\mathbb{C}}(\mathfrak{H})$.

Proposition 10.10 then says, that $\lambda\rho$–coherence is equivalent to δ_l-coherence for $\delta_l = (\lambda \otimes \mathrm{id}) \circ \rho$ (or to δ_r–coherence for $\delta_r = (\mathrm{id} \otimes \rho) \circ \lambda$). Associated with a $\lambda\rho$–intertwiner $\mathbf{T} \in \mathcal{G} \otimes \mathrm{End}_{\mathbb{C}}(\mathfrak{H})$ we now define a natural family of $\mathcal{M}$–linear morphisms

$$t_V : V \odot \mathfrak{H} \longrightarrow \mathfrak{H} \odot V, \quad v \otimes \mathfrak{h} \mapsto \mathbf{T}^{21}(\mathfrak{h} \otimes v), \tag{10.67}$$

where $(V, \pi_V) \in \mathrm{Rep}\, \mathcal{G}$ and $\mathbf{T}_V^{21} := (\mathrm{id} \otimes \pi_V)(\mathbf{T}^{21})$. Eq. (10.54) guarantees that t_V is a morphism in $\mathrm{Rep}\, \mathcal{M}$ and naturality means that

$$t_W \circ (g \otimes \mathrm{id}_{\mathfrak{H}}) = (\mathrm{id}_{\mathfrak{H}} \otimes g) \circ t_V$$

for any morphism $g : V \longrightarrow W$ in $\mathrm{Rep}\, \mathcal{G}$. The normality condition for $\mathbf{T}$ implies $t_{\mathbb{C}} = \mathrm{id}_{\mathfrak{H}}$ and the coherence condition for $\mathbf{T}$ translates into the following coherence condition for t_V

$$t_{V \boxtimes W} = \phi_{VW\mathfrak{H}} \circ (\mathrm{id}_V \otimes t_W) \circ \phi^{-1}_{V\mathfrak{H}W} \circ (t_V \otimes \mathrm{id}_W) \circ \phi_{\mathfrak{H}VW} \tag{10.68}$$

Note that (10.68) looks precisely like one of the coherence conditions for the braiding in a braided quasi–tensor category with nontrivial associativity isomorphisms. Indeed, as will be shown in [HN1], for the case $\mathcal{M} = \mathcal{G}$, the family of t_V's may be used to define a braiding in the representation category of the quantum double $\mathcal{D}(\mathcal{G}) \equiv \mathcal{G} \bowtie \hat{\mathcal{G}}$.

Using the morphisms $P_{V\mathfrak{H}}, P_{\mathfrak{H}V}$ and $Q_{V\mathfrak{H}}, Q_{\mathfrak{H}V}$ given in (9.36),(9.37) and (9.42),(9.43), the relation between $\lambda\rho$–intertwiners $\mathbf{T}$, right δ_r–implementers $\mathbf{R}$ and left δ_l–implementers $\mathbf{L}$ may now be described by the following commuting diagrams connecting the intertwiner morphisms t_V with the maps r_V (associated with $\mathbf{R}$) and l_V (associated with $\mathbf{L}$) as given in (10.51) and (10.52).

$$\begin{tikzcd}
(V \odot \mathfrak{H}) \odot V^* \arrow[r, "l_V"] \arrow[d, "t_V \odot \mathrm{id}_{V^*}"'] & \mathfrak{H} \arrow[r, "r_V"] \arrow[dr, "P_{V\mathfrak{H}}"] & V^* \odot (\mathfrak{H} \odot V) \\
(\mathfrak{H} \odot V) \odot V^* \arrow[ur, "Q_{\mathfrak{H}V}"'] & & V^* \odot (V \odot \mathfrak{H}) \arrow[u, "\mathrm{id}_{V^*} \odot t_V"']
\end{tikzcd} \tag{10.69}$$

$$\begin{tikzcd}
\big((V \odot \mathfrak{H}) \odot V^*\big) \odot V \arrow[dr, "l_V \odot \mathrm{id}_V"'] & V \odot \mathfrak{H} \arrow[l, "P_{(V\odot\mathfrak{H})V^*}"'] \arrow[r, "\mathrm{id}_V \odot r_V"] \arrow[d, "t_V"] & V \odot \big(V^* \odot (\mathfrak{H} \odot V)\big) \arrow[dl, "Q_{V^*(\mathfrak{H}\odot V)}"] \\
& \mathfrak{H} \odot V &
\end{tikzcd} \tag{10.70}$$

10.4 Proof of the main theorem

We are finally in the position to prove Theorem II as stated in the introduction to Part II. The uniqueness of $\mathcal{M}_1$ (up to equivalence) in part 2 of Theorem II follows by standard arguments. Indeed, suppose $\mathcal{M} \subset \tilde{\mathcal{M}}_1$ is another extension satisfying the same properties for $\tilde{\Gamma} : \hat{\mathcal{G}} \longrightarrow \tilde{\mathcal{M}}_1$. Choosing $\mathcal{A} = \tilde{\mathcal{M}}_1$ and $\gamma : \mathcal{M} \longrightarrow \mathcal{A}$ the inclusion map we have $\mathrm{id} : \tilde{\mathcal{M}}_1 \longrightarrow \tilde{\mathcal{M}}_1 \equiv \mathcal{A}$ as an extension of γ. Hence, putting

$$\tilde{\mathbf{\Gamma}} := e_\mu \otimes \tilde{\Gamma}(e^\mu) \in \mathcal{G} \otimes \tilde{\mathcal{M}}_1$$

we conclude from the universality property of $\tilde{\mathcal{M}}_1$ that $\tilde{\mathbf{\Gamma}}$ is a normal, coherent $\lambda\rho$–intertwiner in $\tilde{\mathcal{M}}_1$. Since $\mathcal{M}_1$ also solves the universality property, there exists an algebra map

$$f \equiv \gamma_{\tilde{\mathbf{\Gamma}}} : \mathcal{M}_1 \longrightarrow \tilde{\mathcal{M}}_1$$

restricting to the identity on $\mathcal{M}$ and satisfying $f \circ \Gamma = \tilde{\Gamma}$. Since $\mathcal{M}_1$ is algebraically generated by $\mathcal{M}$ and $\Gamma(\hat{\mathcal{G}})$, f is uniquely fixed by these conditions. Interchanging the role of $\mathcal{M}_1$ and $\tilde{\mathcal{M}}_1$ we also have a map $\tilde{f} : \tilde{\mathcal{M}}_1 \longrightarrow \mathcal{M}_1$ and clearly $\tilde{f} = f^{-1}$.

We now prove the existence of $\mathcal{M}_1$ by choosing $\mathcal{M}_1 := \mathcal{M}\,{}_\lambda\!\bowtie_\rho \hat{\mathcal{G}} \equiv \mathcal{M} \bowtie_{\delta_r} \hat{\mathcal{G}}$ and $\Gamma : \hat{\mathcal{G}} \longrightarrow \mathcal{M}_1$,

$$\Gamma(\varphi) := \Big((q_\lambda^2)_{(0)} \bowtie_{\delta_r} \big(S^{-1}(q_\lambda^1) \rightharpoonup \varphi \leftharpoonup (q_\lambda^2)_{(1)} \big) \Big) \tag{10.71}$$

where $q_\lambda \in \mathcal{G}\otimes\mathcal{M}$ is given by (9.21) and where for $m \in \mathcal{M}$ we write $\rho(m) \equiv m_{(0)}\otimes m_{(1)}$. Here (δ_r, Ψ_r) is the two–sided $\mathcal{G}$–coaction constructed in Proposition 8.4. Putting $\mathbf{R}_{\delta_r} := e_\mu \otimes (\mathbf{1}_\mathcal{M} \bowtie e^\mu) \in \mathcal{G}\otimes\mathcal{M}_1$ and comparing with the notation (10.33) we conclude

$$\Gamma(\varphi) = (\varphi\otimes \mathrm{id}_{\mathcal{M}_1})(\mathbf{\Gamma}) \tag{10.72}$$

where $\mathbf{\Gamma} \in \mathcal{G}\otimes\mathcal{M}_1$ is given by

$$\mathbf{\Gamma} = (\mathrm{id}_\mathcal{G}\otimes\rho)(q_\lambda) \succ \mathbf{R}_{\delta_r} \tag{10.73}$$

Hence, by part 1 of Proposition 10.10 $\mathbf{\Gamma}$ is a normal coherent $\lambda\rho$–intertwiner in $\mathcal{M}_1$. Moreover, putting $p \equiv p_\lambda \in \mathcal{G}\otimes\mathcal{M}$ given in Eq. (9.20) we get for $m \in \mathcal{M}$ and $\varphi \in \hat{\mathcal{G}}$

$$\begin{aligned}\mu_R(m\otimes\varphi) :&= (m \bowtie \mathbf{1}_{\hat{\mathcal{G}}})\,\Gamma(\varphi_{(1)})\,(\varphi_{(2)}\otimes\mathrm{id})(p)\\ &= (m \bowtie \mathbf{1}_{\hat{\mathcal{G}}})\,(\varphi\otimes\mathrm{id})(\mathbf{\Gamma}\,p)\\ &= (m \bowtie \mathbf{1}_{\hat{\mathcal{G}}})\,(\mathbf{1}_\mathcal{M}\bowtie\varphi)\\ &\equiv (m\bowtie\varphi)\end{aligned} \tag{10.74}$$

by Proposition 10.10 1.), and therefore $\mu_R : \mathcal{M}\otimes\hat{\mathcal{G}} \longrightarrow \mathcal{M}_1$ becomes the identity map. This also shows that $\mathcal{M}_1$ is algebraically generated by $\mathcal{M} \equiv (\mathcal{M}\bowtie\mathbf{1}_{\hat{\mathcal{G}}})$ and $\Gamma(\hat{\mathcal{G}})$.

To prove that $\mathcal{M}_1$ solves the universality property we recall from Corollary 10.4 that there is a one–to–one correspondence between extensions $\hat{\gamma} : \mathcal{M}_1 \longrightarrow \mathcal{A}$ of algebra maps $\gamma : \mathcal{M} \longrightarrow \mathcal{A}$ and normal, coherent right δ_r–implementers $\mathbf{R}_\mathcal{A} \in \mathcal{G}\otimes\mathcal{A}$, such that

$$(\mathrm{id}_\mathcal{G}\otimes\hat{\gamma})(\mathbf{R}_{\delta_r}) = \mathbf{R}_\mathcal{A}.$$

Thus any such extension satisfies by (10.73)

$$(\mathrm{id}_\mathcal{G}\otimes\hat{\gamma})(\mathbf{\Gamma}) = \mathbf{T}_\mathcal{A} := (\mathrm{id}_\mathcal{G}\otimes\rho)(q_\lambda) \succ \mathbf{R}_\mathcal{A}$$

By Proposition 10.10 $\mathbf{T}_\mathcal{A} \in \mathcal{G}\otimes\mathcal{A}$ is a normal, coherent $\lambda\rho$–intertwiner and the correspondence $\mathbf{T}_\mathcal{A} \leftrightarrow \mathbf{R}_\mathcal{A}$ is one–to–one. Thus $\mathcal{M}_1$ solves the universal property specified in part 1. of Theorem II.

We are left with showing that with $q_\rho \in \mathcal{M}\otimes\mathcal{G}$ given by (9.23) also $\mu_L : \hat{\mathcal{G}}\otimes\mathcal{M} \longrightarrow \mathcal{M}_1$ given in (6.4) provides a linear isomorphism, which under the identification $\hat{\mathcal{G}}\otimes\mathcal{M} = \hat{\mathcal{G}} \bowtie_{\delta_l} \mathcal{M}$ in fact becomes an algebra map. To this end we use the twist equivalence $f : \mathcal{M}_1 \equiv \mathcal{M} \bowtie_{\delta_r} \hat{\mathcal{G}} \to \mathcal{M} \bowtie_{\delta_l} \hat{\mathcal{G}}$ of Prop. 10.6 given by

$$(\mathrm{id}_\mathcal{G}\otimes f)(\mathbf{R}_{\delta_r}) = \phi_{\lambda\rho} \succ \mathbf{R}_{\delta_l},$$

see Eq. (10.50). Putting $\mathbf{L}_{\delta_l} := e_\mu \otimes (e^\mu \bowtie \mathbf{1}_\mathcal{M}) \in \mathcal{G}\otimes(\hat{\mathcal{G}}\bowtie_{\delta_l}\mathcal{M})$ this gives

$$\begin{aligned}(\mathrm{id}_\mathcal{G}\otimes f\circ\mu_L)(\mathbf{L}_{\delta_l}) &\equiv (\mathrm{id}_\mathcal{G}\otimes f)(q_\rho^{op}\,\mathbf{\Gamma}) \equiv (\mathrm{id}_\mathcal{G}\otimes f)\big((\mathbf{1}_\mathcal{G}\otimes q_\rho)\succ\mathbf{\Gamma}\big)\\ &= \big[(\mathbf{1}_\mathcal{G}\otimes q_\rho)\,(\mathrm{id}_\mathcal{G}\otimes\rho)(q_\lambda)\,\phi_{\lambda\rho}\big] \succ \mathbf{R}_{\delta_l}\\ &= q_{\delta_l} \succ \mathbf{R}_{\delta_l}\end{aligned}$$

where we have used (10.73) and (9.57). Hence, by Theorem 10.2(iii) $f\circ\mu_L : \hat{\mathcal{G}} \bowtie_{\delta_l} \mathcal{M} \to \mathcal{M} \bowtie_{\delta_l} \hat{\mathcal{G}}$ provides an isomorphism. Since f is invertible this concludes the proof of part 3 of Theorem II. $\square$

We think it to be gratifying that in view of this proof and the preceding results all the different appearances of left and right diagonal crossed products described by associated left/right $\delta_{l/r}$–implementers may be replaced by the one universal $\lambda\rho$–intertwiner $\mathbf{\Gamma}$, which is independent of any left–right conventions. This is also our main motivation for formulating Theorem II in this way. In particular, from now on we may dispense with all the nasty calculations involving $\delta_{l/r}$–implementers and just work with the much more convenient generating relations (6.2), (6.3).

11 Examples and applications

11.1 The quantum double $\mathcal{D}(\mathcal{G})$

In view of the identification of the quantum double $\mathcal{D}(\mathcal{G})$ of an ordinary Hopf algebra $\mathcal{G}$ with the diagonal crossed product $\mathcal{G} \bowtie \hat{\mathcal{G}}$ in (3.11) we propose the following

Definition 11.1. Let $(\mathcal{G}, \Delta, \epsilon, \phi)$ be a quasi–Hopf algebra. The diagonal crossed product $\mathcal{D}(\mathcal{G}) := \hat{\mathcal{G}}\,{}_\lambda\!\bowtie_\rho \mathcal{G} \cong \mathcal{G}\,{}_\lambda\!\bowtie_\rho \hat{\mathcal{G}}$ associated with the quasi–commuting pair $(\lambda = \rho = \Delta,\ \phi_\lambda = \phi_\rho = \phi_{\lambda\rho} = \phi)$ of $\mathcal{G}$–coactions on $\mathcal{M} \equiv \mathcal{G}$ is called the *quantum double of* $\mathcal{G}$.

Following the notations of [N1], the universal $\lambda\rho$–intertwiner of the quantum double will be denoted by $\mathbf{D} \equiv \mathbf{\Gamma}_{\mathcal{D}(\mathcal{G})} \in \mathcal{G} \otimes \mathcal{D}(\mathcal{G})$. Hence it obeys the relations $(\epsilon \otimes \mathrm{id})(\mathbf{D}) = \mathbf{1}_{\mathcal{D}(\mathcal{G})}$ and

$$\mathbf{D}\,\Delta(a) = \Delta^{op}(a)\,\mathbf{D}, \quad \forall a \in \mathcal{G} \tag{11.1}$$

$$\phi^{312}\,\mathbf{D}^{13}\,(\phi^{-1})^{132}\,\mathbf{D}^{23}\,\phi = (\Delta \otimes \mathrm{id})(\mathbf{D}) \tag{11.2}$$

where we have suppressed the embedding $\mathcal{G} \hookrightarrow \mathcal{D}(\mathcal{G})$. Eq. (11.1) motivates to call $\mathbf{D}$ the *universal flip operator* for Δ. Theorem II implies that the category of representations of our quantum double $\mathcal{D}(\mathcal{G})$ is nothing else but the so–called "double category of $\mathcal{G}$–modules", which was used by S. Majid [M2] to define the quantum double of a quasi–Hopf algebra using a Tannaka–Krein like reconstruction procedure. Indeed, in our terminology this is precisely the subcategory of *$\lambda\rho$–coherent representations* $(\mathfrak{H}, \gamma_{\mathfrak{H}}, \mathbf{T}_{\mathfrak{H}})$ in $\mathrm{Rep}\,\mathcal{G}$ (where $\lambda = \rho = \Delta$), see Definition 10.11, with morphisms given by the $\mathcal{G}$–linear maps $\tau : \mathfrak{H} \to \mathfrak{H}'$ satisfying $(\mathrm{id} \otimes \tau)(\mathbf{T}_{\mathfrak{H}}) = \mathbf{T}_{\mathfrak{H}'}$, where $\mathbf{T}_{\mathfrak{H}}$ and $\mathbf{T}_{\mathfrak{H}'}$ are the normal coherent $\lambda\rho$–intertwiners in $\mathcal{G} \otimes \mathrm{End}_{\mathbb{C}}(\mathfrak{H})$ and $\mathcal{G} \otimes \mathrm{End}_{\mathbb{C}}(\mathfrak{H}')$, respectively. A more detailed account of this will be given in [HN1], where we will also show, that $\mathcal{D}(\mathcal{G})$ is in fact a quasitriangular quasi–Hopf algebra.

Here we restrict ourselves with proving the quasi–bialgebra structure on $\mathcal{D}(\mathcal{G})$. Analogously as in Proposition 3.5 this will also guarantee that every diagonal crossed product $\mathcal{M}_1 = \mathcal{M}\,{}_\lambda\!\bowtie_\rho \hat{\mathcal{G}}$ naturally admits a quasi–commuting pair $(\lambda_D, \rho_D, \phi_{\lambda_D}, \phi_{\rho_D}, \phi_{\lambda_D\rho_D})$ of coactions of $\mathcal{D}(\mathcal{G})$ on $\mathcal{M}_1$. We begin with constructing $\lambda_D : \mathcal{M}_1 \to \mathcal{D}(\mathcal{G}) \otimes \mathcal{M}_1$ and $\rho_D : \mathcal{M}_1 \to \mathcal{M}_1 \otimes \mathcal{D}(\mathcal{G})$ as algebra maps extending the left and right coactions $\lambda : \mathcal{M}_1 \supset \mathcal{M} \to \mathcal{G} \otimes \mathcal{M} \subset \mathcal{D}(\mathcal{G}) \otimes \mathcal{M}_1$ and $\rho : \mathcal{M}_1 \supset \mathcal{M} \to \mathcal{M} \otimes \mathcal{G} \subset \mathcal{M}_1 \otimes \mathcal{D}(\mathcal{G})$, respectively (see (3.13),(3.14)).

Lemma 11.2. *Let $(\lambda, \rho, \phi_\lambda, \phi_\rho, \phi_{\lambda\rho})$ be a quasi–commuting pair of $\mathcal{G}$–coactions on $\mathcal{M}$ and let $\mathcal{M}_1 \equiv \hat{\mathcal{G}}\,{}_\lambda\!\bowtie_\rho \mathcal{M}$ be the associated diagonal crossed product with universal $\lambda\rho$–intertwiner $\mathbf{\Gamma} \in \mathcal{G} \otimes \mathcal{M}_1$. Then there exist uniquely determined algebra maps $\lambda_D : \mathcal{M}_1 \to \mathcal{D}(\mathcal{G}) \otimes \mathcal{M}_1$ and*

$\rho_D : \mathcal{M}_1 \to \mathcal{M}_1 \otimes \mathcal{D}(\mathcal{G})$ *satisfying (we suppress all embeddings* $\mathcal{M} \hookrightarrow \mathcal{M}_1$ *and* $\mathcal{G} \hookrightarrow \mathcal{D}(\mathcal{G})$*)*

$$\lambda_D(m) = \lambda(m), \quad \forall m \in \mathcal{M} \subset \mathcal{M}_1 \tag{11.3}$$
$$(\mathrm{id} \otimes \lambda_D)(\mathbf{\Gamma}) = (\phi_{\lambda\rho}^{-1})^{231}\, \mathbf{\Gamma}^{13}\, \phi_\lambda^{213}\, \mathbf{D}^{12}\, \phi_\lambda^{-1} \ \in \mathcal{G} \otimes \mathcal{D}(\mathcal{G}) \otimes \mathcal{M}_1 \tag{11.4}$$
$$\rho_D(m) = \rho(m), \quad \forall m \in \mathcal{M} \subset \mathcal{M}_1 \tag{11.5}$$
$$(\mathrm{id} \otimes \rho_D)(\mathbf{\Gamma}) = (\phi_\rho^{-1})^{231}\, \mathbf{D}^{13}\, \phi_\rho^{213}\, \mathbf{\Gamma}^{12}\, \phi_{\lambda\rho}^{-1} \ \in \mathcal{G} \otimes \mathcal{D}(\mathcal{G}) \otimes \mathcal{M}_1 \tag{11.6}$$

Proof. Viewing the left $\mathcal{G}$–coaction $\lambda : \mathcal{M} \longrightarrow \mathcal{G} \otimes \mathcal{M}$ as a map $\lambda : \mathcal{M} \longrightarrow \mathcal{D}(\mathcal{G}) \otimes \mathcal{M}_1$, Theorem II states that λ_D is a unital algebra map extending λ if and only if $\mathbf{T}_D := (\mathrm{id} \otimes \lambda_D)(\mathbf{\Gamma}) \in \mathcal{G} \otimes (\mathcal{D}(\mathcal{G}) \otimes \mathcal{M}_1)$ is a normal coherent $\lambda\rho$–intertwiner. Now normality of $\mathbf{T}_D$ follows from the normality of $\mathbf{\Gamma}$. To prove that $\mathbf{T}_D$ is a $\lambda\rho$–intertwiner we compute for all $m \in \mathcal{M}$

$$\begin{aligned}
\mathbf{T}_D\, (\mathrm{id}_\mathcal{G} \otimes \lambda_D)(\lambda(m)) &= (\phi_{\lambda\rho}^{-1})^{231}\, \mathbf{\Gamma}^{13}\, \phi_\lambda^{213}\, \mathbf{D}^{12}\, \phi_\lambda^{-1}\, (\mathrm{id}_\mathcal{G} \otimes \lambda_D)(\lambda(m)) \\
&= \big[(\lambda_D \otimes \mathrm{id}_\mathcal{G})(\rho(m))\big]^{231}\, (\phi_{\lambda\rho}^{-1})^{231}\, \mathbf{\Gamma}^{13}\, \phi_\lambda^{213}\, \mathbf{D}^{12}\, \phi_\lambda^{-1} \\
&= (\mathrm{id}_\mathcal{G} \otimes \lambda_D)(\rho^{op}(m))\, \mathbf{T}_D
\end{aligned}$$

where both sides are viewed as elements in $\mathcal{G} \otimes \mathcal{D}(\mathcal{G}) \otimes \mathcal{M}_1$. Here we have used the intertwining properties of $\mathbf{\Gamma}$ and $\mathbf{D}$ and of the three reassociators.

To show that $\mathbf{T}_D$ also satisfies the coherence condition, i.e. Eq. (6.3), we compute in $\mathcal{G} \otimes \mathcal{G} \otimes \mathcal{D}(\mathcal{G}) \otimes \mathcal{M}_1$ - again suppressing all embeddings

$$\begin{aligned}
(\Delta \otimes \mathrm{id})(\mathbf{T}_D) &= \big[(\mathrm{id} \otimes \mathrm{id} \otimes \Delta)(\phi_{\lambda\rho}^{-1})\, [\mathbf{1}_\mathcal{G} \otimes \phi_\rho]\big]^{3412} \\
&\quad \mathbf{\Gamma}^{14}\, (\phi_{\lambda\rho}^{-1})^{142}\, \mathbf{\Gamma}^{24} \big[[\mathbf{1}_\mathcal{G} \otimes \phi_\lambda]\, (\mathrm{id} \otimes \Delta \otimes \mathrm{id})(\phi_\lambda)[\phi \otimes \mathbf{1}_\mathcal{M}]\big]^{3124} \mathbf{D}^{13}\, (\phi^{-1})^{132}\, \mathbf{D}^{23} \\
&\quad [\phi \otimes \mathbf{1}_\mathcal{M}](\Delta \otimes \mathrm{id} \otimes \mathrm{id})(\phi_\lambda^{-1}) \\
&= \big[(\lambda \otimes \mathrm{id} \otimes \mathrm{id})(\phi_\rho)[\phi_{\lambda\rho}^{-1} \otimes \mathbf{1}_\mathcal{G}](\mathrm{id} \otimes \rho \otimes \mathrm{id})(\phi_{\lambda\rho}^{-1})\big]^{3412} \\
&\quad \mathbf{\Gamma}^{14}\, (\phi_{\lambda\rho}^{-1})^{142}\, \mathbf{\Gamma}^{24} \big[(\mathrm{id} \otimes \mathrm{id} \otimes \lambda)(\phi_\lambda)(\Delta \otimes \mathrm{id} \otimes \mathrm{id})(\phi_\lambda)\big]^{3124} \mathbf{D}^{13}\, (\phi^{-1})^{132}\, \mathbf{D}^{23} \\
&\quad (\mathrm{id} \otimes \Delta \otimes \mathrm{id})(\phi_\lambda^{-1})[\mathbf{1}_\mathcal{G} \otimes \phi_\lambda^{-1}](\mathrm{id} \otimes \mathrm{id} \otimes \lambda)(\phi_\lambda) \\
&= \big[(\lambda \otimes \mathrm{id} \otimes \mathrm{id})(\phi_\rho)[\phi_{\lambda\rho}^{-1} \otimes \mathbf{1}_\mathcal{G}]\big]^{3412}\, \mathbf{\Gamma}^{14} \\
&\quad \big[(\mathrm{id} \otimes \lambda \otimes \mathrm{id})(\phi_{\lambda\rho}^{-1})\, [\mathbf{1}_\mathcal{G} \otimes \phi_{\lambda\rho}^{-1}]\, (\mathrm{id} \otimes \mathrm{id} \otimes \rho)(\phi_\lambda)\big]^{3142}\, \mathbf{\Gamma}^{24}\, \mathbf{D}^{13} \\
&\quad \big[(\Delta \otimes \mathrm{id} \otimes \mathrm{id})(\phi_\lambda)[\phi^{-1} \otimes \mathbf{1}_\mathcal{M}]\, (\mathrm{id} \otimes \Delta \otimes \mathrm{id})(\phi_\lambda^{-1})\big]^{1324} \\
&\quad \mathbf{D}^{23}\, [\mathbf{1}_\mathcal{G} \otimes \phi_\lambda^{-1}](\mathrm{id} \otimes \mathrm{id} \otimes \lambda)(\phi_\lambda) \\
&= \big[(\lambda \otimes \mathrm{id} \otimes \mathrm{id})(\phi_\rho)[\phi_{\lambda\rho}^{-1} \otimes \mathbf{1}_\mathcal{G}]\big]^{3412}\, \mathbf{\Gamma}^{14}\, \phi_\lambda^{314}\, \mathbf{D}^{13} \\
&\quad \big[(\Delta \otimes \mathrm{id} \otimes \mathrm{id})(\phi_{\lambda\rho}^{-1})\, (\mathrm{id} \otimes \mathrm{id} \otimes \rho)(\phi_\lambda^{-1})\big]^{1324}\, \mathbf{\Gamma}^{24} \\
&\quad \phi_\lambda^{324}\, \mathbf{D}^{23}\, [\mathbf{1}_\mathcal{G} \otimes \phi_\lambda^{-1}](\mathrm{id} \otimes \mathrm{id} \otimes \lambda)(\phi_\lambda) \\
&= (\mathrm{id} \otimes \mathrm{id} \otimes \lambda_D)\Big(\phi_\rho^{312}\, \mathbf{\Gamma}^{13}\, (\phi_{\lambda\rho}^{-1})^{132}\, \mathbf{\Gamma}^{23}\, \phi_\lambda\Big)
\end{aligned}$$

Here we have used several pentagon identities for the reassociators involved and the intertwining and coherence properties of $\mathbf{\Gamma}$ and $\mathbf{D}$. In the first equality we used (6.3) for $\mathbf{\Gamma}$ and $\mathbf{D}$, and in the second the pentagons (8.12) and (7.2). For the third equality we used the intertwining properties of $\mathbf{D}$ and $\mathbf{\Gamma}$ to move two more reassociators between $\mathbf{D}^{13}$ and $\mathbf{D}^{23}$ and to more between $\mathbf{\Gamma}^{14}$ and $\mathbf{\Gamma}^{24}$. To arrive at the fourth equality we commuted $\mathbf{D}^{13}$ and $\mathbf{\Gamma}^{24}$ and used the pentagons (8.11) and (7.2) and then again the intertwining properties of $\mathbf{D}$ and $\mathbf{\Gamma}$ to bring two reassociators back between $\mathbf{D}^{13}$ and $\mathbf{\Gamma}^{24}$. The last equality holds by (11.3), (11.4). Thus we have shown that $\mathbf{T}_D$ is coherent and therefore the definitions (11.3), (11.4) uniquely define a unital algebra map λ_D extending λ. Similarly one shows that ρ_D defines a unital algebra map $\rho_D : \mathcal{M}_1 \longrightarrow \mathcal{M}_1 \otimes \mathcal{D}(\mathcal{G})$ extending ρ. □

Choosing in Lemma 11.2 also $\mathcal{M} = \mathcal{G}$ (i.e. $\mathcal{M}_1 = \mathcal{D}(\mathcal{G})$) we arrive at the following

Theorem 11.3. *Let $(\mathcal{G}, \Delta, \epsilon, \phi)$ be a quasi–Hopf algebra, denote $i_D : \mathcal{G} \hookrightarrow \mathcal{D}(\mathcal{G})$ the canonical embedding and let $\mathbf{D} \in \mathcal{G} \otimes \mathcal{D}(\mathcal{G})$ be the universal flip operator.*

(i) Then $(\mathcal{D}(\mathcal{G}), \Delta_D, \epsilon_D, \phi_D)$ is a quasi–bialgebra, where

$$\phi_D := (i_D \otimes i_D \otimes i_D)(\phi) \tag{11.7}$$

$$\epsilon_D(i_D(a)) := \epsilon(a), \quad (\mathrm{id} \otimes \epsilon_D)(\mathbf{D}) := \mathbf{1}_{\mathcal{D}(\mathcal{G})} \tag{11.8}$$

$$\Delta_D(i_D(a)) := (i_D \otimes i_D)(\Delta(a)), \quad \forall a \in \mathcal{G} \tag{11.9}$$

$$(i_D \otimes \Delta_D)(\mathbf{D}) := (\phi_D^{-1})^{231}\, \mathbf{D}^{13}\, \phi_D^{213}\, \mathbf{D}^{12}\, \phi_D^{-1} \tag{11.10}$$

(ii) Under the setting of Lemma 11.2 denote $i_{\mathcal{M}_1} : \mathcal{M} \hookrightarrow \mathcal{M}_1$ the embedding and define

$$\begin{aligned}
\phi_{\lambda_D} &:= (i_D \otimes i_D \otimes i_{\mathcal{M}_1})(\phi_\lambda) \in \mathcal{D}(\mathcal{G}) \otimes \mathcal{D}(\mathcal{G}) \otimes \mathcal{M}_1 \\
\phi_{\rho_D} &:= (i_{\mathcal{M}_1} \otimes i_D \otimes i_D)(\phi_\rho) \in \mathcal{M}_1 \otimes \mathcal{D}(\mathcal{G}) \otimes \mathcal{D}(\mathcal{G}) \\
\phi_{\lambda_D \rho_D} &:= (i_D \otimes i_{\mathcal{M}_1} \otimes i_D)(\phi_{\lambda\rho}) \in \mathcal{D}(\mathcal{G}) \otimes \mathcal{M}_1 \otimes \mathcal{D}(\mathcal{G})
\end{aligned}$$

Then $(\lambda_D, \rho_D, \phi_{\lambda_D}, \phi_{\rho_D}, \phi_{\lambda_D\rho_D})$ provides a quasi–commuting pair of $\mathcal{D}(\mathcal{G})$–coactions on $\mathcal{M}_1 \equiv \hat{\mathcal{G}}\,{}_\lambda\!\bowtie_\rho \mathcal{M}$.

Proof. Setting $\mathcal{M} := \mathcal{G}$ and $\lambda = \Delta$ in Lemma 11.2 implies that Δ_D is a unital algebra morphism. The property of ϵ_D being a counit for Δ_D follows directly from the fact that $(\mathrm{id} \otimes \epsilon \otimes \mathrm{id})(\phi) = \mathbf{1}_\mathcal{G} \otimes \mathbf{1}_\mathcal{G}$. To show that Δ_D is quasi–coassociative one computes that

$$[\mathbf{1}_\mathcal{G} \otimes \phi_D] \cdot (\mathrm{id} \otimes \Delta_D \otimes \mathrm{id})\Big((\mathrm{id} \otimes \Delta_D)(\mathbf{D})\Big) = (\mathrm{id} \otimes \mathrm{id} \otimes \Delta_D)\Big((\mathrm{id} \otimes \Delta_D)(\mathbf{D})\Big) \cdot [\mathbf{1}_\mathcal{G} \otimes \phi_D],$$

where one has to use (11.10), the pentagon equation for ϕ and the intertwiner property (11.1) of $\mathbf{D}$ similarly as in the proof of Lemma 11.2. Thus Δ_D is quasi–coassociative and this concludes the proof of part (i).

Part (ii) is shown by direct calculation using the intertwiner properties of $\mathbf{\Gamma}$ and $\mathbf{D}$ and several pentagon identities for the reassociators involved. The details are left to the reader. □

Note that viewed in $\mathcal{D}(\mathcal{G}) \otimes \mathcal{D}(\mathcal{G})$ and $\mathcal{D}(\mathcal{G})^{\otimes^3}$, respectively, the relations (11.1), (11.2) and (11.10) are precisely the defining properties of a quasitriangular R–matrix [Dr2]. Thus $R_D := (i_D \otimes \mathrm{id})(\mathbf{D})$ is an R–matrix for $\mathcal{D}(\mathcal{G})$. It is shown in [HN1] that there also exists an antipode S_D for $\mathcal{D}(\mathcal{G})$ extending the antipode of $\mathcal{G}$, thus making the quantum double $\mathcal{D}(\mathcal{G})$ into a quasitriangular quasi–Hopf algebra.

11.2 Two–sided crossed products

As in the associative case, a simple recipee to produce two–sided $\mathcal{G}$–comodule algebras $(\mathcal{M},\delta)$ is by tensoring a right $\mathcal{G}$–module algebra $(\mathcal{A},\rho_{\mathcal{A}})$ and a left $\mathcal{G}$–comodule algebra $(\mathcal{B},\lambda_{\mathcal{B}})$, i.e. by setting $\mathcal{M}=\mathcal{A}\otimes\mathcal{B}$ and

$$\delta(A\otimes B):=B_{(-1)}\otimes(A_{(0)}\otimes B_{(0)})\otimes A_{(1)}$$

as in Eq. (4.14). Obviously $\delta=(\lambda\otimes\mathrm{id})\circ\rho=(\mathrm{id}\otimes\rho)\circ\lambda$, where (λ,ϕ_λ) and (ρ,ϕ_ρ) are the trivially extended left and right coactions $(\lambda_{\mathcal{B}},\phi_{\lambda_{\mathcal{B}}})$ and $(\rho_{\mathcal{A}},\phi_{\rho_{\mathcal{A}}})$, respectively. Hence $(\lambda,\rho,\phi_\lambda,\phi_\rho,\phi_{\lambda\rho}=\mathbf{1}_{\mathcal{G}}\otimes\mathbf{1}_{\mathcal{M}}\otimes\mathbf{1}_{\mathcal{G}})$ is a *strictly commuting* pair of coactions. In the terminology of Proposition 8.4 we have $\delta=\delta_r=\delta_l$, whereas $\Psi=\Psi_r=\Psi_l$ is given by

$$\Psi=[\mathbf{1}_{\mathcal{G}}\otimes\mathbf{1}_{\mathcal{G}}\otimes\phi_\rho^{-1}]\,[\phi_\lambda\otimes\mathbf{1}_{\mathcal{G}}\otimes\mathbf{1}_{\mathcal{G}}].$$

According to Theorem II the diagonal crossed product $\mathcal{M}_1=(\mathcal{A}\otimes\mathcal{B})_\lambda\bowtie_\rho\hat{\mathcal{G}}$ is generated by $\{A,B,\Gamma(\varphi)\mid A\in\mathcal{A},\ B\in\mathcal{B},\ \varphi\in\hat{\mathcal{G}}\}$ satisfying the defining relations

$$A\,B=B\,A \tag{11.11}$$
$$[\mathbf{1}_{\mathcal{G}}\otimes B]\,\mathbf{\Gamma}=\mathbf{\Gamma}\,\lambda(B) \tag{11.12}$$
$$\rho^{op}(A)\,\mathbf{\Gamma}=\mathbf{\Gamma}\,[\mathbf{1}_{\mathcal{G}}\otimes A] \tag{11.13}$$
$$(\Delta\otimes\mathrm{id})(\mathbf{\Gamma})=\phi_\rho^{312}\,\mathbf{\Gamma}^{13}\,\mathbf{\Gamma}^{23}\,\phi_\lambda \tag{11.14}$$

where $\mathbf{\Gamma}=e_\mu\otimes\Gamma(e^\mu)$ is the universal $\lambda\rho$–intertwiner.

The next Proposition is an analogue of Proposition 4.2 saying that the diagonal crossed product $(\mathcal{A}\otimes\mathcal{B})_\lambda\bowtie_\rho\hat{\mathcal{G}}$ may be realized as a *two–sided crossed product* $\mathcal{A}\rtimes_\rho\hat{\mathcal{G}}\ltimes_\lambda\mathcal{B}$. Note that the isomorphism μ in Eq. (11.15) below is different from the isomorphisms μ_R and μ_L constructed in Theorem II.

Proposition 11.4. *Let $\triangleright:\hat{\mathcal{G}}\otimes\mathcal{A}\to\mathcal{A}$ and $\triangleleft:\mathcal{B}\otimes\hat{\mathcal{G}}\to\mathcal{B}$ be the left and the right action corresponding to a right $\mathcal{G}$–coaction (ρ,ϕ_ρ) on $\mathcal{A}$ and a left $\mathcal{G}$–coaction (λ,ϕ_λ) on $\mathcal{B}$, respectively. Extend λ and ρ trivially to $\mathcal{A}\otimes\mathcal{B}$ and let $\mathcal{M}_1:=(\mathcal{A}\otimes\mathcal{B})_\lambda\bowtie_\rho\hat{\mathcal{G}}\equiv(\mathcal{A}\otimes\mathcal{B})\bowtie_{\delta_r}\hat{\mathcal{G}}$, $\delta_r:=(\mathrm{id}\otimes\rho)\circ\lambda$, be the diagonal crossed product with universal $\lambda\rho$–intertwiner $\mathbf{\Gamma}\in\mathcal{G}\otimes\mathcal{M}_1$.*

(i) There is a linear bijection $\mu:\mathcal{A}\otimes\hat{\mathcal{G}}\otimes\mathcal{B}\longrightarrow\mathcal{M}_1$ given by

$$\mu(A\otimes\varphi\otimes B)=A\,\Gamma(\varphi)\,B \tag{11.15}$$

where we have suppressed the embeddings $\mathcal{A}\hookrightarrow\mathcal{M}_1$ and $\mathcal{B}\hookrightarrow\mathcal{M}_1$.

(ii) Denote the induced algebra structure on $\mathcal{A}\otimes\hat{\mathcal{G}}\otimes\mathcal{B}$ by $\mathcal{A}\rtimes_\rho\hat{\mathcal{G}}\ltimes_\lambda\mathcal{B}\equiv\mu^{-1}(\mathcal{M}_1)$. Then we get the following multiplication structure with unit $\mathbf{1}_{\mathcal{A}}\rtimes\hat{\mathbf{1}}\ltimes\mathbf{1}_{\mathcal{B}}$ on $\mathcal{A}\rtimes_\rho\hat{\mathcal{G}}\ltimes_\lambda\mathcal{B}$

$$\begin{aligned}&\big(A\rtimes\varphi\ltimes B\big)\,\big(A'\rtimes\psi\ltimes B'\big)\\&=A(\varphi_{(1)}\triangleright A')\bar{\phi}_\rho^1\rtimes[\bar{\phi}_\lambda^1\rightharpoonup\varphi_{(2)}\leftharpoonup\bar{\phi}_\rho^2]\,[\bar{\phi}_\lambda^2\rightharpoonup\psi_{(1)}\leftharpoonup\bar{\phi}_\rho^3]\ltimes\bar{\phi}_\lambda^3(B\triangleleft\psi_{(2)})B'\end{aligned} \tag{11.16}$$

Proof. Let $p_\lambda\in\mathcal{G}\otimes\mathcal{B}\equiv\mathcal{G}\otimes(\mathbf{1}_{\mathcal{A}}\otimes\mathcal{B})$ be given by Eq. (9.20). Then, using (11.12)

$$\begin{aligned}\mu\Big(A\otimes\varphi_{(1)}\otimes(\varphi_{(2)}\otimes\mathrm{id}_{\mathcal{B}})\big(\lambda(B)p_\lambda\big)\Big)&=A\,(\varphi\otimes\mathrm{id}_{\mathcal{M}_1})(\mathbf{\Gamma}\,\lambda(B)\,p_\lambda)\\&=A\,B\,(\varphi\otimes\mathrm{id}_{\mathcal{M}_1})(\mathbf{\Gamma}\,p_\lambda)\\&=\mu_R(A\otimes B\otimes\varphi)\end{aligned}$$

where $\mu_R : (\mathcal{A} \otimes \mathcal{B}) \otimes \hat{\mathcal{G}} \to \mathcal{M}_1$ is the linear bijection constructed in part 3. of Theorem II, see also (10.74). Hence μ is surjective. Conversely, let $\mathbf{R} := \mathbf{\Gamma}\, p_\lambda \in \mathcal{G} \otimes \mathcal{M}_1$ then by Proposition 10.10 $\mathbf{R}$ is a right δ_r–implementer and

$$\mathbf{\Gamma} = (\mathrm{id}_{\mathcal{G}} \otimes \rho)(q_\lambda) \succ \mathbf{R} \equiv [\mathbf{1}_{\mathcal{G}} \otimes q_\lambda^2]\, \mathbf{R}\, [S^{-1}(q_\lambda^1) \otimes \mathbf{1}_{\mathcal{M}_1}]$$

where $q_\lambda \in \mathcal{G} \otimes \mathcal{B}$ if given by (9.21) and where we have used that ρ is trivial on $\mathcal{B}$. Hence we get for all $B \in \mathcal{B}$

$$\begin{aligned}\mathbf{\Gamma}\, [\mathbf{1}_{\mathcal{G}} \otimes B] &= [\mathbf{1}_{\mathcal{G}} \otimes q_\lambda^2]\, [\delta_r(B) \succ \mathbf{R}]\, [S^{-1}(q_\lambda^1) \otimes \mathbf{1}_{\mathcal{M}_1}] \\ &= [\mathbf{1}_{\mathcal{G}} \otimes q_\lambda^2 B_{(0)}]\, \mathbf{R}\, [S^{-1}(q_\lambda^1 B_{(-1)}) \otimes \mathbf{1}_{\mathcal{M}_1}] \end{aligned} \tag{11.17}$$

where $B_{(-1)} \otimes B_{(0)} = \lambda(B)$ and where we have used $\delta_r(B) = B_{(-1)} \otimes (\mathbf{1}_{\mathcal{A}} \otimes B_{(0)}) \otimes \mathbf{1}_{\mathcal{G}}$. Eq. (11.17) implies for all $A \in \mathcal{A}, \varphi \in \hat{\mathcal{G}}$ and $B \in \mathcal{B}$

$$A\, \Gamma(\varphi)\, B = \mu_R\Big(A \otimes \big(\hat{S}^{-1}(\varphi_{(2)}) \otimes \mathrm{id}_{\mathcal{B}}\big)\big(q_\lambda \lambda(B)\big) \otimes \varphi_{(1)}\Big)$$

and therefore the injectivity of μ_R implies the injectivity of μ.

This proves part (i). Part (ii) follows since one straightforward checks that the multiplication rule (11.16) is equivalent to the defining relations (11.12)-(11.14). $\square$

11.3 Hopf spin chains and lattice current algebras

In this subsection we describe how the Hopf algebraic quantum chains considered in [NSz] generalize to quasi–Hopf algebras $\mathcal{G}$ (the case of weak quasi–Hopf algebras like semisimple quotients of quantum groups at roots of unity requires only minor changes as sketched in Part III). To this end we use the two–sided crossed product theory of Section 11.2 to generalize the constructions (4.24),(4.25) and (4.30).

First we show, that analogously as in (4.30) the two–sided crossed product construction given in Proposition 11.4 may be iterated if one of the two algebras $\mathcal{A}$ and $\mathcal{B}$ admits a quasi–commuting pair of coactions.

Proposition 11.5. *Let $(\mathcal{A}, \rho_{\mathcal{A}}, \phi_{\rho_{\mathcal{A}}})$, $(\mathcal{C}, \lambda_{\mathcal{C}}, \phi_{\lambda_{\mathcal{C}}})$ and $(\mathcal{B}, \rho_{\mathcal{B}}, \lambda_{\mathcal{B}}, \phi_{\lambda_{\mathcal{B}}}, \phi_{\rho_{\mathcal{B}}}, \phi_{\lambda_{\mathcal{B}},\rho_{\mathcal{B}}})$ be a right, a left and a two–sided comodule algebra, respectively, and denote the universal $\lambda\rho$–intertwiners*

$$\mathbf{\Gamma}_{\mathcal{AB}} := e_\mu \otimes (\mathbf{1}_{\mathcal{A}} \rtimes e^\mu \ltimes \mathbf{1}_{\mathcal{B}}) \in \mathcal{A} \rtimes_{\rho_{\mathcal{A}}} \hat{\mathcal{G}} \ltimes_{\lambda_{\mathcal{B}}} \mathcal{B} \tag{11.18}$$

$$\mathbf{\Gamma}_{\mathcal{BC}} := e_\mu \otimes (\mathbf{1}_{\mathcal{B}} \rtimes e^\mu \ltimes \mathbf{1}_{\mathcal{C}}) \in \mathcal{B} \rtimes_{\rho_{\mathcal{B}}} \hat{\mathcal{G}} \ltimes_{\lambda_{\mathcal{C}}} \mathcal{C} \tag{11.19}$$

Then

(i) $\mathcal{A} \rtimes_{\rho_{\mathcal{A}}} \hat{\mathcal{G}} \ltimes_{\lambda_{\mathcal{B}}} \mathcal{B}$ admits a right $\mathcal{G}$–coaction $(\tilde{\rho}, \phi_{\tilde{\rho}})$ given by $\phi_{\tilde{\rho}} := \phi_{\rho_{\mathcal{B}}}$, $\tilde{\rho}|_{(\mathcal{A} \otimes \mathcal{B})} := \mathrm{id}_{\mathcal{A}} \otimes \rho_{\mathcal{B}}$ and

$$(\mathrm{id}_{\mathcal{G}} \otimes \tilde{\rho})(\mathbf{\Gamma}_{\mathcal{AB}}) := (\mathbf{\Gamma}_{\mathcal{AB}} \otimes \mathbf{1}_{\mathcal{G}})\, \phi^{-1}_{\lambda_{\mathcal{B}}\rho_{\mathcal{B}}} \tag{11.20}$$

(ii) $\mathcal{B} \rtimes_{\rho_{\mathcal{B}}} \hat{\mathcal{G}} \ltimes_{\lambda_{\mathcal{C}}} \mathcal{C}$ admits a left $\mathcal{G}$–coaction $(\tilde{\lambda}, \phi_{\tilde{\lambda}})$ given by $\phi_{\tilde{\lambda}} := \phi_{\lambda_{\mathcal{B}}}$, $\tilde{\lambda}|_{(\mathcal{B} \otimes \mathcal{C})} := \lambda_{\mathcal{B}} \otimes \mathrm{id}_{\mathcal{C}}$ and

$$(\mathrm{id}_{\mathcal{G}} \otimes \tilde{\lambda})(\mathbf{\Gamma}_{\mathcal{BC}}) := (\phi^{-1}_{\lambda_{\mathcal{B}}\rho_{\mathcal{B}}})^{231}\, \mathbf{\Gamma}^{13}_{\mathcal{BC}} \tag{11.21}$$

(iii) We have an algebra isomorphism

$$(\mathcal{A} \rtimes_{\rho_\mathcal{A}} \hat{\mathcal{G}} \ltimes_{\lambda_\mathcal{B}} \mathcal{B}) \rtimes_{\tilde{\rho}} \hat{\mathcal{G}} \ltimes_{\lambda_\mathcal{C}} \mathcal{C} \equiv \mathcal{A} \rtimes_{\rho_\mathcal{A}} \hat{\mathcal{G}} \ltimes_{\tilde{\lambda}} (\mathcal{B} \rtimes_{\rho_\mathcal{B}} \hat{\mathcal{G}} \ltimes_{\lambda_\mathcal{C}} \mathcal{C}) \tag{11.22}$$

given by the trivial identification.

Proof. (i) To show that (11.20) provides a well defined algebra map $\tilde{\rho} : \mathcal{A} \rtimes_{\rho_\mathcal{A}} \hat{\mathcal{G}} \ltimes_{\lambda_\mathcal{B}} \mathcal{B} \to (\mathcal{A} \rtimes_{\rho_\mathcal{A}} \hat{\mathcal{G}} \ltimes_{\lambda_\mathcal{B}} \mathcal{B}) \otimes \mathcal{G}$ extending $\mathrm{id}_\mathcal{A} \otimes \rho_\mathcal{B}$ we have to check that the relations (11.12)-(11.14) are respected. To this end we put $\mathbf{T}_{\mathcal{AB}} := (\mathbf{\Gamma}_{\mathcal{AB}} \otimes \mathbf{1}_\mathcal{G})\, \phi^{-1}_{\lambda_\mathcal{B}\rho_\mathcal{B}}$ and compute for $B \in \mathcal{B}$

$$\begin{aligned}
[\mathbf{1}_\mathcal{G} \otimes \tilde{\rho}(B)]\, \mathbf{T}_{\mathcal{AB}} &= (\mathbf{\Gamma}_{\mathcal{AB}} \otimes \mathbf{1}_\mathcal{G})\, (\lambda_\mathcal{B} \otimes \mathrm{id})(\rho_\mathcal{B}(B))\, \phi^{-1}_{\lambda_\mathcal{B}\rho_\mathcal{B}} \\
&= \mathbf{T}_{\mathcal{AB}}\, (\mathrm{id}_G \otimes \tilde{\rho})(\lambda_\mathcal{B}(B))
\end{aligned}$$

which is the relation (11.12). Trivially one also has (since $\phi_{\lambda_\mathcal{B}\rho_\mathcal{B}} \in \mathcal{G} \otimes (\mathbf{1}_\mathcal{A} \otimes \mathcal{B}) \otimes \mathcal{G}$)

$$\mathbf{T}_{\mathcal{AB}}\, [\mathbf{1}_\mathcal{G} \otimes A \otimes \mathbf{1}_\mathcal{G}] = [\rho^{op}(A) \otimes \mathbf{1}_\mathcal{G}]\, \mathbf{T}_{\mathcal{AB}}$$

verifying (11.13). Finally, the coherence condition (11.14) is respected, since in $\mathcal{G} \otimes \mathcal{G} \otimes (\mathcal{A} \rtimes \hat{\mathcal{G}} \ltimes \mathcal{B}) \otimes \mathcal{G}$ we have

$$\begin{aligned}
(\Delta \otimes \mathrm{id} \otimes \mathrm{id})(\mathbf{T}_{\mathcal{AB}}) &= (\Delta \otimes \mathrm{id})(\mathbf{\Gamma}_{\mathcal{AB}})^{123}\, (\Delta \otimes \mathrm{id} \otimes \mathrm{id})(\phi^{-1}_{\lambda_\mathcal{B}\rho_\mathcal{B}}) \\
&= (\Delta \otimes \mathrm{id})(\mathbf{\Gamma}_{\mathcal{AB}})^{123}\, [\phi^{-1}_{\lambda_\mathcal{B}} \otimes \mathbf{1}_\mathcal{G}]\, (\mathrm{id} \otimes \lambda_\mathcal{B} \otimes \mathrm{id})(\phi^{-1}_{\lambda_\mathcal{B}\rho_\mathcal{B}})\, [\mathbf{1}_\mathcal{G} \otimes \phi^{-1}_{\lambda_\mathcal{B}\rho_\mathcal{B}}]\, (\mathrm{id} \otimes \mathrm{id} \otimes \rho_\mathcal{B})(\phi_{\lambda_\mathcal{B}}) \\
&= \phi^{312}_{\rho_\mathcal{A}}\, \mathbf{\Gamma}^{13}_{\mathcal{AB}}\, \mathbf{\Gamma}^{23}_{\mathcal{AB}}\, (\mathrm{id} \otimes \lambda_\mathcal{B} \otimes \mathrm{id})(\phi^{-1}_{\lambda_\mathcal{B}\rho_\mathcal{B}})\, [\mathbf{1}_\mathcal{G} \otimes \phi^{-1}_{\lambda_\mathcal{B}\rho_\mathcal{B}}]\, (\mathrm{id} \otimes \mathrm{id} \otimes \rho_\mathcal{B})(\phi_{\lambda_\mathcal{B}}) \\
&= \phi^{312}_{\rho_\mathcal{A}}\, \mathbf{\Gamma}^{13}_{\mathcal{AB}}\, (\phi^{-1}_{\lambda_\mathcal{B}\rho_\mathcal{B}})^{134}\, \mathbf{\Gamma}^{23}_{\mathcal{AB}} (\phi^{-1}_{\lambda_\rho})^{234}\, (\mathrm{id} \otimes \mathrm{id} \otimes \rho_\mathcal{B})(\phi_{\lambda_\mathcal{B}}) \\
&= (\mathrm{id} \otimes \mathrm{id} \otimes \tilde{\rho})(\phi^{312}_{\rho_\mathcal{A}})\, \mathbf{T}^{13}_{\mathcal{AB}}\, \mathbf{T}^{23}_{\mathcal{AB}}\, (\mathrm{id} \otimes \mathrm{id} \otimes \tilde{\rho})(\phi_{\lambda_\mathcal{B}})
\end{aligned}$$

Here we have used the pentagon identity (8.11) in the second line, the coherence property (11.14) of $\mathbf{\Gamma}_{\mathcal{AB}}$ in the third line and finally the intertwining property (11.12). Thus $\tilde{\rho}$ provides a well defined algebra map, which is also unit preserving since $(\epsilon \otimes \mathrm{id} \otimes \mathrm{id})(\mathbf{T}_{\mathcal{AB}}) = (\mathbf{1}_\mathcal{A} \rtimes \mathbf{1}_{\hat{\mathcal{G}}} \ltimes \mathbf{1}_\mathcal{B}) \otimes \mathbf{1}_\mathcal{G}$. Similarly one shows by a straight forward calculation that the pair $(\tilde{\rho}, \phi_{\tilde{\rho}})$ satisfies (7.5). Since $\phi_{\tilde{\rho}} = \phi_{\rho_\mathcal{B}}$, the pentagon equation (7.6) and the counit equations (7.7) and (7.8) are clearly satisfied. This proves part (i). Part (ii) follows analogously. To prove part (iii) we have to check that we may consistently identify

$$\mathcal{G} \otimes \Big[(\mathcal{A} \rtimes \hat{\mathcal{G}} \ltimes \mathcal{B}) \rtimes \hat{\mathcal{G}} \ltimes \mathcal{C}\Big] \ni \mathbf{\Gamma}_{\mathcal{AB}} \stackrel{!}{\equiv} \mathbf{\Gamma}_{\mathcal{A}(\mathcal{B} \rtimes \hat{\mathcal{G}} \ltimes \mathcal{C})} \in \mathcal{G} \otimes \Big[\mathcal{A} \rtimes \hat{\mathcal{G}} \ltimes (\mathcal{B} \rtimes \hat{\mathcal{G}} \ltimes \mathcal{C})\Big] \tag{11.23}$$

$$\mathcal{G} \otimes \Big[(\mathcal{A} \rtimes \hat{\mathcal{G}} \ltimes \mathcal{B}) \rtimes \hat{\mathcal{G}} \ltimes \mathcal{C}\Big] \ni \mathbf{\Gamma}_{(\mathcal{A} \rtimes \hat{\mathcal{G}} \ltimes \mathcal{B})\mathcal{C}} \stackrel{!}{\equiv} \mathbf{\Gamma}_{\mathcal{BC}} \in \mathcal{G} \otimes \Big[\mathcal{A} \rtimes \hat{\mathcal{G}} \ltimes (\mathcal{B} \rtimes \hat{\mathcal{G}} \ltimes \mathcal{C})\Big] \tag{11.24}$$

For this the nontrivial commutation relations to be looked at are according to (11.12),(11.13)

$$\begin{aligned}
\mathbf{\Gamma}_{(\mathcal{A} \rtimes \hat{\mathcal{G}} \ltimes \mathcal{B})\mathcal{C}}\, [\mathbf{1}_\mathcal{G} \otimes X] &= \tilde{\rho}^{op}(X)\, \mathbf{\Gamma}_{(\mathcal{A} \rtimes \hat{\mathcal{G}} \ltimes \mathcal{B})\mathcal{C}}, \quad X \in \mathcal{A} \rtimes \hat{\mathcal{G}} \ltimes \mathcal{B} \\
[\mathbf{1}_\mathcal{G} \otimes Y]\, \mathbf{\Gamma}_{\mathcal{A}(\mathcal{B} \rtimes \hat{\mathcal{G}} \ltimes \mathcal{C})} &= \mathbf{\Gamma}_{\mathcal{A}(\mathcal{B} \rtimes \hat{\mathcal{G}} \ltimes \mathcal{C})}\, \tilde{\lambda}(Y), \quad Y \in \mathcal{B} \rtimes \hat{\mathcal{G}} \ltimes \mathcal{C}
\end{aligned}$$

The remaining cases being trivial, it is enough to consider $X \in \Gamma_{\mathcal{AB}}(\hat{\mathcal{G}})$ and $Y \in \Gamma_{\mathcal{BC}}(\hat{\mathcal{G}})$ for which we get

$$\begin{aligned}
\mathbf{\Gamma}^{23}_{(\mathcal{A} \rtimes \hat{\mathcal{G}} \ltimes \mathcal{B})\mathcal{C}}\, \mathbf{\Gamma}^{13}_{\mathcal{AB}} &= (\mathrm{id}_\mathcal{G} \otimes \tilde{\rho})(\mathbf{\Gamma}_{\mathcal{AB}})^{132}\, \mathbf{\Gamma}^{23}_{\mathcal{A}(\mathcal{B} \rtimes \hat{\mathcal{G}} \ltimes \mathcal{C})} = \mathbf{\Gamma}^{13}_{\mathcal{AB}}\, (\phi^{-1}_{\lambda_\mathcal{B}\rho_\mathcal{B}})^{132}\, \mathbf{\Gamma}^{23}_{(\mathcal{A} \rtimes \hat{\mathcal{G}} \ltimes \mathcal{B})\mathcal{C}} \\
\mathbf{\Gamma}^{23}_{\mathcal{BC}}\, \mathbf{\Gamma}^{13}_{\mathcal{A}(\mathcal{B} \rtimes \hat{\mathcal{G}} \ltimes \mathcal{C})} &= \mathbf{\Gamma}^{13}_{\mathcal{A}(\mathcal{B} \rtimes \hat{\mathcal{G}} \ltimes \mathcal{C})}\, (\mathrm{id}_\mathcal{G} \otimes \tilde{\lambda})(\mathbf{\Gamma}_{\mathcal{BC}})^{213} = \mathbf{\Gamma}^{13}_{\mathcal{A}(\mathcal{B} \rtimes \hat{\mathcal{G}} \ltimes \mathcal{C})}\, (\phi^{-1}_{\lambda_\mathcal{B}\rho_\mathcal{B}})^{132}\, \mathbf{\Gamma}^{23}_{\mathcal{BC}}
\end{aligned}$$

by the definitions (11.20) and (11.21). This shows that the identifications (11.23) and (11.24) are indeed consistent and therefore proves part (iii) $\square$

Note that due to part (iii) of the above proposition the notations $\mathbf{\Gamma}_{\mathcal{AB}}$ and $\mathbf{\Gamma}_{\mathcal{BC}}$ as in (11.18),(11.19) are still well defined in iterated two–sided crossed products and the commutation relations of "neighboring" $\lambda\rho$–intertwiners are given by

$$\mathbf{\Gamma}_{\mathcal{AB}}^{13}\,\phi_{\lambda_{\mathcal{B}}\rho_{\mathcal{B}}}^{-1}\,\mathbf{\Gamma}_{\mathcal{BC}}^{23} = \mathbf{\Gamma}_{\mathcal{BC}}^{23}\,\mathbf{\Gamma}_{\mathcal{AB}}^{13} \tag{11.25}$$

Due to Proposition 11.5 the definition of Hopf spin chains as reviewed in Section 4.3 immediately generalizes to the quasi–coassociative case. As in Section 4.3 we interpret even integers as sites and odd integers as links of a one dimensional lattice and we set $\mathcal{A}_{2i} \cong \mathcal{G}$, $\mathcal{A}_{2i+1} \cong \hat{\mathcal{G}}$, the latter just being a linear space. A local net of associative algebras $\mathcal{A}_{n,m}$ is then constructed inductively for all $n, m \in 2\mathbb{Z}, n \leq m$, by first putting

$$\mathcal{A}_{2i,2i+2} := \mathcal{A}_{2i} \rtimes \mathcal{A}_{2i+1} \ltimes \mathcal{A}_{2i+2} \cong \mathcal{G} \rtimes \hat{\mathcal{G}} \ltimes \mathcal{G}$$

where $\mathcal{G}$ is equipped with its canonical two–sided comodule structure ($\lambda = \rho = \Delta$, $\phi_\lambda = \phi_\rho = \phi_{\lambda\rho} = \phi$). Due to Proposition 11.5 this procedure may be iterated as in (4.24), by setting

$$\mathcal{A}_{2i,2j+2} := \mathcal{A}_{2i,2j} \rtimes \hat{\mathcal{G}} \ltimes \mathcal{G} \equiv \mathcal{G} \rtimes \hat{\mathcal{G}} \ltimes \mathcal{A}_{2i-2,2j+2}$$

where the last equality follows from (11.22) by iteration. More generally one has as in (4.25) for all $i \leq \mu \leq j-1$

$$\mathcal{A}_{2i,2j} = \mathcal{A}_{2i,2\mu} \rtimes \hat{\mathcal{G}} \ltimes \mathcal{A}_{2\mu+2,2j} \tag{11.26}$$

Defining the generating "link operators" $\mathbf{L}_{2i+1} := \mathbf{\Gamma}_{\mathcal{A}_{2i}\mathcal{A}_{2i+2}}$ as in (11.18), the commutation relation (11.25) implies

$$\mathbf{L}_{2i-1}^{13}\,\phi_{2i}^{-1}\,\mathbf{L}_{2i+1}^{23} = \mathbf{L}_{2i+1}^{23}\,\mathbf{L}_{2i-1}^{13}, \quad \phi_{2i} := (\mathrm{id} \otimes \mathrm{id} \otimes A_{2i})(\phi).$$

Writing $A_{2i+1}(\varphi) = (\varphi \otimes \mathrm{id})(\mathbf{L}_{2i+1})$ this is equivalent to

$$A_{2i-1}(\varphi)\,A_{2i+1}(\psi) = A_{2i+1}(\psi \leftharpoonup \bar{\phi}^3)\,A_{2i}(\bar{\phi}^2)\,A_{2i-1}(\bar{\phi}^1 \rightharpoonup \varphi) \tag{11.27}$$

Thus link operators on neighboring links do not commute any more in contrast to the coassociative setting! But the algebras $\mathcal{A}_{2i-4,2i-2}$ and $\mathcal{A}_{2i+2,2i+4}$ still commute, which means that the above construction still yields a local net of algebras.

Next we remark that Lemma 11.2 applied to the special case of two–sided crossed products (Prop. 11.4) provides us with localized left and right coactions of the quantum double $\mathcal{D}(\mathcal{G})$ on the above quantum chain. This generalizes the $\mathcal{D}(\mathcal{G})$–cosymmetry discovered by [NSz].

Also the construction of the periodic chain by closing a finite open chain $\mathcal{A}_{0,2i}$ may again be described by a diagonal crossed product $\mathcal{A}_{0,2i}\,{}_\lambda\!\bowtie_\rho\,\hat{\mathcal{G}}$, where λ and ρ are nontrivial only on $\mathcal{A}_0 \rtimes \mathcal{A}_1 \ltimes \mathcal{A}_2$ and $\mathcal{A}_{2i-2} \rtimes \mathcal{A}_{2i-1} \ltimes \mathcal{A}_{2i}$, respectively, where they are defined as in Proposition 11.5.

Let us conclude this section by indicating, how the equivalence of the Hopf spin chains of [NSz] and the lattice current algebras of [AFFS] as shown in [N1] generalizes to the quasi–Hopf setting. Following [N1] we define the generating current operators by

$$\mathbf{J}_{2i+1} := (\mathrm{id} \otimes A_{2i})(R^{op})\,\mathbf{L}_{2i+1}, \tag{11.28}$$

where $R \in \mathcal{G} \otimes \mathcal{G}$ is supposed to be quasi–triangular. This yields the following commutation relations

$$\begin{aligned}
[\mathbf{1}_{\mathcal{G}} \otimes A_{2i}(a)]\, \mathbf{J}_{2i-1} &= \mathbf{J}_{2i-1}\, [a_{(1)} \otimes A_{2i}(a_{(2)})], \quad \forall a \in \mathcal{G} \\
[a_{(1)} \otimes A_{2i}(a_{(2)})]\, \mathbf{J}_{2i+1} &= \mathbf{J}_{2i+1}\, [\mathbf{1}_{\mathcal{G}} \otimes A_{2i}(a)] \\
\mathbf{J}^{13}_{2i+1}\, \mathbf{J}^{23}_{2i+1} &= \hat{R}_{2i}\, \phi_{2i}\, (\Delta \otimes \mathrm{id})(\mathbf{J})\, \phi^{-1}_{2i+2} \\
\mathbf{J}^{13}_{2i-1}\, \hat{R}_{2i} \mathbf{J}^{23}_{2i+1} &= \mathbf{J}^{23}_{2i+1}\, \mathbf{J}^{13}_{2i-1}
\end{aligned}$$

where $\hat{R}_{2i} := (\mathrm{id} \otimes \mathrm{id} \otimes A_{2i})(\phi^{213} R^{12} \phi^{-1})$ and $\phi_{2i} := (\mathrm{id} \otimes \mathrm{id} \otimes A_{2i})(\phi)$. These relations generalize the defining relations of lattice current algebras as given in [AFFS] to quasi–Hopf algebras. They have also appeared in [AGS] as lattice Chern–Simons algebras (restricted to the boundary of a disk) in the weak quasi–Hopf algebra setting, where the copies of $\mathcal{G}$ sitting at the sites are interpreted as gauge transformations.

A more detailed account of these constructions will be given elsewhere [H].

11.4 Field algebra construction with quasi–Hopf symmetry

In this section we sketch how the field algebra construction of Mack and Schomerus [MS,S] may be described by two–sided crossed products. Here we anticipate the result of Part III, where it is shown how to generalize our constructions to *weak* quasi–Hopf algebras $\mathcal{G}$ (i.e. satisfying $\Delta(\mathbf{1}) \neq \mathbf{1} \otimes \mathbf{1}$). Choosing $\mathcal{B} \equiv \mathcal{G}$ and $\lambda_{\mathcal{B}} \equiv \Delta$, one gets a unital algebra $\mathcal{M}_1 := (\mathcal{A} \otimes \mathcal{G}) \bowtie \hat{\mathcal{G}} \cong \mathcal{A} \rtimes \hat{\mathcal{G}} \ltimes \mathcal{G}$ associated with every right $\mathcal{G}$–comodule algebra $(\mathcal{A}, \rho, \phi_\rho)$. Here $\mathcal{A} \subset \mathcal{M}_1$ is to be interpreted as the algebra of observables, the universal $\lambda\rho$–intertwiner $\mathbf{\Gamma}$ is a "master" field operator, and $\mathcal{G} \subset \mathcal{A} \rtimes \hat{\mathcal{G}} \ltimes \mathcal{G}$ represents the global "quantum symmetry". The fields are said to *transform covariantly*, which means that

$$[\mathbf{1}_{\mathcal{G}} \otimes a]\, \mathbf{\Gamma} = \mathbf{\Gamma}\, \lambda(a) \equiv \mathbf{\Gamma}\, \Delta(a), \quad a \in \mathcal{G}, \tag{11.29}$$

whereas the observables $A \in \mathcal{A}$ are *$\mathcal{G}$–invariant*

$$a\, A = A\, a, \quad \forall a \in \mathcal{G}, A \in \mathcal{A}$$

The linear subspace $\mathcal{F} \equiv \mathcal{A} \rtimes \hat{\mathcal{G}} := \mathcal{A} \rtimes \hat{\mathcal{G}} \ltimes \mathbf{1}_{\mathcal{G}}$ is called the *field algebra*. Note that in the quasi–coassociative setting $\mathcal{F}$ is *not* a subalgebra of $\mathcal{A} \rtimes \hat{\mathcal{G}} \ltimes \mathcal{B}$. But similarly as in [MS] one may define a new non–associative "product" $\times$ on $\mathcal{F}$ by setting $A \times A' = AA'$ for $A, A' \in \mathcal{A}$ and

$$\mathbf{\Gamma}^{13} \times \mathbf{\Gamma}^{23} := (\Delta \otimes \mathrm{id})(\mathbf{\Gamma}) \tag{11.30}$$

This product is quasi–coassociative in the sense that

$$[\phi \otimes \mathbf{1}]\, \left(\mathbf{\Gamma}^{14} \times \mathbf{\Gamma}^{24}\right) \times \mathbf{\Gamma}^{34} = \mathbf{\Gamma}^{14} \times \left(\mathbf{\Gamma}^{24} \times \mathbf{\Gamma}^{34}\right) [\phi \otimes \mathbf{1}]$$

and it satisfies

$$[\mathbf{1}_{\mathcal{G}} \otimes \mathbf{1}_{\mathcal{G}} \otimes a]\, (\mathbf{\Gamma}^{13} \times \mathbf{\Gamma}^{23}) = (\mathbf{\Gamma}^{13} \times \mathbf{\Gamma}^{23})\, (\Delta \otimes \mathrm{id})(\Delta(a))$$

which is the reason why it is called *covariant product* in [MS]. Moreover, if $\mathcal{G}$ is quasitriangular, the field operators satisfy the braiding relations

$$\mathbf{\Gamma}^{13} \times \mathbf{\Gamma}^{23} = R^{12}\, (\mathbf{\Gamma}^{23} \times \mathbf{\Gamma}^{13})\, (R^{-1})^{12}$$

The difference of this approach with the setting of [MS,S] lies in the fact that here the field operators appear in the form of irreducible matrix–multiplets

$$F_I^{ij} := (\pi_I^{ij} \otimes \mathrm{id})(\mathbf{\Gamma}), \quad \pi_I \in \mathrm{Irrep}\, \mathcal{G}$$

Correspondingly, the superselection sectors of the observable algebra $\mathcal{A}$ are given by the *amplimorphisms*

$$\rho_I^{ij}(A) \equiv (\pi^{ij} \otimes \mathrm{id})(\rho(A)) = F_I^{ik}\, A\, (F_I^{kj})^*$$

which is equivalent to the defining relation (11.13). In this sense the above construction fits to the formulation of DHR–sector theory as proposed for lattice theories in [SzV,NSz]. In the terminology of [NSz] the $\mathcal{G}$–coactions $\rho : \mathcal{A} \to \mathcal{A} \otimes \mathcal{G}$ would then be a "universal cosymmetry".

We will show elsewhere [HN2], that such a ρ may indeed be constructed also for continuum theories (more precisely for "rational" theories, where the number of sectors is finite). This will be done by providing suitable multiplets $W_I^i \in \mathcal{A}$ satisfying

$$\sum_i W_I^i W_I^{i*} = \mathbf{1}_{\mathcal{A}}, \quad \forall I$$

and putting

$$\rho_I^{ij}(A) := W_I^{i*}\, \sigma_I(A)\, W_I^j$$

where σ_I is a DHR–endomorphism representing the irreducible sector I. The field operators of [MS,S] are then recovered as

$$\Psi_I^i = W_I^j\, F_I^{ji}$$

In general the W_I^i's will not generate a Cuntz algebra (i.e. they will not be orthogonal isometries) and therefore the amplimorphisms ρ_I will be non–unital, implying $\mathcal{G}$ to be a *weak* quasi–Hopf algebra. The reassociator $\phi_\rho \in \mathcal{A} \otimes \mathcal{G} \otimes \mathcal{G}$ will be obtained by suitably "blowing up" the quantum field theoretic "$6j$–symbols" with the help of the W_I^i's. In this way we will finally be able to show [HN2], that the inclusion $\mathcal{A} \subset \mathcal{M}_1 \equiv \mathcal{A} \rtimes \hat{\mathcal{G}} \ltimes \mathcal{G}$ provides a finite index and depth–2 inclusion of von–Neumann factors satisfying $\mathcal{A}' \cap \mathcal{M}_1 = \mathcal{G}$.

Part III
Weak quasi–Hopf algebras

In this last part we will generalize the definitions and constructions of Part II to *weak quasi–Hopf algebras* as introduced in [MS]. This contains the physical relevant examples such as truncated quantum groups at roots of unity. As will be shown, it is nearly straightforward to extend all results obtained so far to the case of weak quasi–Hopf algebras. The new feature of *weak* quasi–Hopf algebras is to allow the coproduct to be non unital, i.e. $\Delta(\mathbf{1}_\mathcal{G}) \neq \mathbf{1}_\mathcal{G} \otimes \mathbf{1}_\mathcal{G}$. This results in a *truncation* of tensor products of representations, i.e. the representation $(\pi_V \otimes \pi_W) \circ \Delta$ operates only on the subspace $V \boxtimes W := (\pi_V \otimes \pi_W)(\Delta(\mathbf{1}_\mathcal{G}))(V \otimes W)$. Also the invertibility requirement on certain universal elements - such as the reassociator or the R–matrix - is weakened by only postulating the existence of so–called *quasi–inverses*. Correspondingly coactions and two–sided coactions of a weak quasi-Hopf algebra are no longer supposed to be unital and the associated reassociators are only required to possess quasi–inverses. The diagonal crossed products $\mathcal{M}_1 \equiv \hat{\mathcal{G}} \bowtie \mathcal{M}$ may then again be defined by the same relations as before, with the additional requirement that the universal $\lambda\rho$–intertwiner $\mathbf{\Gamma} \in \mathcal{G} \otimes \mathcal{M}_1$ has to satisfy $\mathbf{\Gamma}\, \lambda(\mathbf{1}_{\mathcal{M}}) \equiv \rho^{op}(\mathbf{1}_{\mathcal{M}})\, \mathbf{\Gamma} = \mathbf{\Gamma}$. This will also imply that now as a linear space $\mathcal{M}_1$ is only isomorphic to a certain subspace $\hat{\mathcal{G}}\,{}_\lambda\!\bowtie_\rho \mathcal{M} \subset \hat{\mathcal{G}} \otimes \mathcal{M}$ (or $\mathcal{M}\,{}_\lambda\!\bowtie_\rho \hat{\mathcal{G}} \subset \mathcal{M} \otimes \hat{\mathcal{G}}$). More specifically Theorem II now reads

Theorem III *Let $\mathcal{G}$ be a finite dimensional weak quasi–Hopf algebra and let $(\lambda, \phi_\lambda, \rho, \phi_\rho, \phi_{\lambda\rho})$ be a quasi–commuting pair of (left and right) $\mathcal{G}$–coactions on an associative algebra $\mathcal{M}$.*

1. *Then their exists a unital associative algebra extension $\mathcal{M}_1 \supset \mathcal{M}$ together with a linear map $\Gamma : \hat{\mathcal{G}} \longrightarrow \mathcal{M}_1$ satisfying the following universal property:*
   $\mathcal{M}_1$ is algebraically generated by $\mathcal{M}$ and $\Gamma(\hat{\mathcal{G}})$ and for any algebra map $\gamma : \mathcal{M} \longrightarrow \mathcal{A}$ into some target algebra $\mathcal{A}$ the relation
   $$\gamma_T(\Gamma(\varphi)) = (\varphi \otimes \mathrm{id})(\mathbf{T})$$
   provides a one–to–one correspondence between algebra maps $\gamma_T : \mathcal{M}_1 \longrightarrow \mathcal{A}$ extending γ and elements $\mathbf{T} \in \mathcal{G} \otimes \mathcal{A}$ satisfying $(\epsilon \otimes \mathrm{id})(\mathbf{T}) = \gamma(\mathbf{1}_{\mathcal{M}})$ and
   $$\begin{aligned} \mathbf{T}\, \lambda_{\mathcal{A}}(m) &= \rho^{op}_{\mathcal{A}}(m)\, \mathbf{T}, \quad \forall m \in \mathcal{M} \\ \mathbf{T}\, \lambda_{\mathcal{A}}(\mathbf{1}_{\mathcal{M}}) &\equiv \rho^{op}_{\mathcal{A}}(\mathbf{1}_{\mathcal{M}})\, \mathbf{T} = \mathbf{T} \\ (\phi^{312}_\rho)_{\mathcal{A}}\, \mathbf{T}^{13}\, (\phi^{132}_{\lambda\rho})^{-1}_{\mathcal{A}}\, \mathbf{T}^{23}\, (\phi_\lambda)_{\mathcal{A}} &= (\Delta \otimes \mathrm{id}_{\mathcal{A}})(\mathbf{T}), \end{aligned}$$
   where $\lambda_{\mathcal{A}}(m) := (\mathrm{id} \otimes \gamma)(\lambda(m))$, $(\phi_\lambda)_{\mathcal{A}} := (\mathrm{id}_{\mathcal{G}} \otimes \mathrm{id}_{\mathcal{G}} \otimes \gamma)(\phi_\lambda)$, etc., and where $\phi^{-1}_{\lambda\rho}$ is the quasi–inverse of $\phi_{\lambda\rho}$.

2. *If $\mathcal{M} \subset \tilde{M}_1$ and $\tilde{\Gamma} : \hat{\mathcal{G}} \longrightarrow \tilde{\mathcal{M}}_1$ satisfy the same universality property as in part 1.), then there exists a unique algebra isomorphism $f : \mathcal{M}_1 \longrightarrow \tilde{\mathcal{M}}_1$ restricting to the identity on $\mathcal{M}$, such that $\tilde{\Gamma} = f \circ \Gamma$*

3. *There exist elements $p_\lambda \in \mathcal{G} \otimes \mathcal{M}$ and $q_\rho \in \mathcal{M} \otimes \mathcal{G}$ such that the linear maps*
   $$\begin{aligned} \mu_L &: \hat{\mathcal{G}} \otimes \mathcal{M} \ni (\varphi \otimes m) \mapsto (\mathrm{id} \otimes \varphi_{(1)})(q_\rho)\Gamma(\varphi_{(2)})m \in \mathcal{M}_1 \\ \mu_R &: \mathcal{M} \otimes \hat{\mathcal{G}} \ni (m \otimes \varphi) \mapsto m\Gamma(\varphi_{(1)})(\varphi_{(2)} \otimes \mathrm{id})(p_\lambda) \in \mathcal{M}_1 \end{aligned}$$
   are surjective.

4. *Let $P_L : \hat{\mathcal{G}} \otimes \mathcal{M} \to \hat{\mathcal{G}} \otimes \mathcal{M}$ and $P_R : \mathcal{M} \otimes \hat{\mathcal{G}} \to \mathcal{M} \otimes \hat{\mathcal{G}}$ be the linear maps given by*
   $$\begin{aligned} P_L(\varphi \otimes m) \equiv \varphi \bowtie m &:= \varphi_{(2)} \otimes (\varphi_{(3)} \otimes \mathrm{id} \otimes \hat{S}^{-1}(\varphi_{(1)}))\big(\delta_l(\mathbf{1}_{\mathcal{M}})\big)m \\ P_R(m \otimes \varphi) \equiv m \bowtie \varphi &:= m(\hat{S}^{-1}(\varphi_{(3)}) \otimes \mathrm{id}_{\mathcal{M}} \otimes \varphi_{(1)})\big(\delta_r(\mathbf{1}_{\mathcal{M}})\big) \otimes \varphi_{(2)} \end{aligned}$$
   where $\delta_l = (\lambda \otimes \mathrm{id}) \circ \rho$ and $\delta_r = (\rho \otimes \mathrm{id}) \circ \lambda$. Then P_L and P_R are projections and $Ker\, \mu_{L/R} = Ker\, P_{L/R}$.

Part 3. and 4. of Theorem III imply that we may put $\hat{\mathcal{G}}_\lambda \bowtie_\rho M := P_L(\hat{\mathcal{G}} \otimes \mathcal{M})$ and $\mathcal{M}\,{}_\lambda\!\bowtie_\rho \hat{\mathcal{G}} := P_R(\mathcal{M} \otimes \hat{\mathcal{G}})$ to conclude that analogously as in Eqs. (6.6) and (6.7) the restrictions

$$\begin{aligned} \mu_L &: \hat{\mathcal{G}}\,{}_\lambda\!\bowtie_\rho \mathcal{M} \longrightarrow \mathcal{M}_1 \\ \mu_R &: \mathcal{M}\,{}_\lambda\!\bowtie_\rho \hat{\mathcal{G}} \longrightarrow \mathcal{M}_1 \end{aligned}$$

are linear isomorphisms inducing a concrete realization of the abstract algebra $\mathcal{M}_1$ on the subspaces $\hat{\mathcal{G}}_\lambda\bowtie_\rho \mathcal{M} \subset \hat{\mathcal{G}}\otimes\mathcal{M}$ and $\mathcal{M}_\lambda\bowtie_\rho \hat{\mathcal{G}} \subset \mathcal{M}\otimes\hat{\mathcal{G}}$, respectively. As before, we call these concrete realizations the *left and right diagonal crossed products*, respectively, associated with the quasi–commuting pair of $\mathcal{G}$–coactions $(\lambda, \rho, \phi_\lambda, \phi_\rho, \phi_{\lambda\rho})$ on $\mathcal{M}$[9]. To actually prove Theorem III we follow the same strategy as in Part II., i.e. we first construct these diagonal crossed products explicitly and then show that they solve the universal properties defining $\mathcal{M}_1$.

12 Weak quasi–bialgebras

We start with a little digression on the notion of *quasi–inverses*. Let $\mathcal{A}$ be an associative algebra and let $x, p, q \in \mathcal{A}$ satisfy

$$px = x = xq \tag{12.1}$$

$$p^2 = p, \qquad q^2 = q \tag{12.2}$$

Then we say that $y \in \mathcal{A}$ is a *quasi–inverse* of x with respect to (p, q), if

$$yx = q, \qquad xy = p, \qquad yxy = y \tag{12.3}$$

Clearly, given (p, q), a quasi–inverse of x is uniquely determined, provided it exists. This is why we also write $y = x^{-1}$, if the idempotents (p, q) are understood. Also note that we have $qy = y = yp$ and $xyx = x$ and therefore x is the quasi–inverse of y with respect to (q, p). All this generalizes in the obvious way to $\mathcal{A}$–module morphisms $x \in \mathrm{Hom}_\mathcal{A}(V, W)$, $p \in \mathrm{End}_{\,\mathcal{A}}(W)$ and $q \in \mathrm{End}_{\,\mathcal{A}}(V)$, in which case the quasi–inverse would be an element $x^{-1} \in \mathrm{Hom}_\mathcal{A}(W, V)$.

Note that in place of (12.1) we could equivalently add to (12.3) the requirement

$$xyx = x \tag{12.4}$$

In our setting of weak quasi–Hopf algebras the idempotents p, q always appear as images of $\mathbf{1}_\mathcal{G}$ of non–unital algebra maps defined on $\mathcal{G}$, like $\Delta(\mathbf{1})$, $\Delta^{op}(\mathbf{1})$, $(\Delta\otimes\mathrm{id})(\Delta(\mathbf{1})), \ldots$ etc., whereas the element x will be an intertwiner between two such maps, like a reassociator ϕ etc. Hence, throughout we will adopt the convention that if $\alpha : \mathcal{G} \to \mathcal{A}$ and $\beta : \mathcal{G} \to \mathcal{A}$ are two algebra maps and $x \in \mathcal{A}$ satisfies

$$x\,\alpha(g) = \beta(g)\,x, \quad \forall g \in \mathcal{G}$$

then the quasi–inverse $y = x^{-1} \in \mathcal{A}$ is defined to be the unique (if existing) element satisfying

$$yx = \alpha(\mathbf{1}), \quad xy = \beta(\mathbf{1})$$
$$xyx = x, \quad yxy = y$$

Clearly this implies conversely

$$\alpha(g)\,y = y\,\beta(g), \quad \forall g \in \mathcal{G}$$

and therefore $x = y^{-1}$. We also note the obvious identities

$$\beta(g) = x\,\alpha(g)\,x^{-1}, \quad \alpha(g) = x^{-1}\,\beta(g)\,x$$
$$x\,\alpha(\mathbf{1}) = \beta(\mathbf{1})\,x = x, \quad \alpha(\mathbf{1})\,x^{-1} = x^{-1}\,\beta(\mathbf{1}) = x^{-1}$$

[9] Actually we have again four versions of diagonal crossed products as in Part II, given by $\hat{\mathcal{G}} \bowtie_{\delta_l} \mathcal{M} \equiv \hat{\mathcal{G}}_\lambda\bowtie_\rho \mathcal{M}$, $\hat{\mathcal{G}} \bowtie_{\delta_r} \mathcal{M}$, $\mathcal{M} \bowtie_{\delta_l} \hat{\mathcal{G}}$ and $\mathcal{M} \bowtie_{\delta_r} \hat{\mathcal{G}} \equiv \mathcal{M}_\lambda\bowtie_\rho \hat{\mathcal{G}}$.

After this digression we now define, following [MS] a *weak quasi–bialgebra* $(\mathcal{G}, \mathbf{1}, \Delta, \epsilon, \phi)$ to be an associative algebra $\mathcal{G}$ with unit $\mathbf{1}$, a non–unital algebra map $\Delta : \mathcal{G} \to \mathcal{G} \otimes \mathcal{G}$, an algebra map $\epsilon : \mathcal{G} \to \mathbb{C}$ and an element $\phi \in \mathcal{G} \otimes \mathcal{G} \otimes \mathcal{G}$ satisfying (6.8)-(6.10), whereas (6.11) is replaced by

$$(\mathrm{id} \otimes \epsilon \otimes \mathrm{id})(\phi) = \Delta(\mathbf{1}) \tag{12.5}$$

and where in place of invertibility ϕ is supposed to have a quasi–inverse $\bar{\phi} \equiv \phi^{-1}$ with respect to the intertwining property (6.8). Hence we have $\phi\bar{\phi}\phi = \phi$, $\bar{\phi}\phi\bar{\phi} = \bar{\phi}$ as well as

$$\phi\,\bar{\phi} = (\mathrm{id} \otimes \Delta)(\Delta(\mathbf{1})), \quad \bar{\phi}\,\phi = (\Delta \otimes \mathrm{id})(\Delta(\mathbf{1})) \tag{12.6}$$

implying the further identities

$$(\mathrm{id} \otimes \Delta)(\Delta(a)) = \phi\,(\Delta \otimes \mathrm{id})(\Delta(a))\,\bar{\phi}, \quad \forall a \in \mathcal{G} \tag{12.7}$$

$$\phi = \phi\,(\Delta \otimes \mathrm{id})(\Delta(\mathbf{1})), \quad \bar{\phi} = \bar{\phi}\,(\mathrm{id} \otimes \Delta)(\Delta(\mathbf{1})) \tag{12.8}$$

$$(\mathrm{id} \otimes \epsilon \otimes \mathrm{id})(\bar{\phi}) = \Delta(\mathbf{1}) \tag{12.9}$$

A weak quasi–bialgebra is called *weak quasi–Hopf algebra*, if there exists a unital algebra anti-morphism S and elements $\alpha, \beta \in \mathcal{G}$ satisfying (6.13) and (6.14). We will also always suppose that S is invertible. The remarks about the quasi–Hopf algebras $\mathcal{G}_{op}$, $\mathcal{G}^{cop}$ and $\mathcal{G}_{op}^{cop}$ remain valid as in Section 6.

A quasi–invertible element $F \in \mathcal{G} \otimes \mathcal{G}$ satisfying $(\epsilon \otimes \mathrm{id})(F) = (\mathrm{id} \otimes \epsilon)(F) = \mathbf{1}$ induces a *twist transformation* from $(\mathcal{G}, \Delta, \epsilon, \phi)$ to $(\mathcal{G}, \Delta_F, \epsilon, \phi_F)$ as in (6.15) and (6.16), where $\mathcal{G}_F := (\mathcal{G}, \mathbf{1}, \Delta_F, \epsilon, \phi_F)$ is again a weak quasi–bialgebra. The two bialgebras $\mathcal{G}_F$ and $\mathcal{G}$ are called *twist–equivalent*.

Finally the properties of the twists f, h defined as in (6.21) and (6.26) are still valid. In particular (6.22) defines the quasi–inverse of f with respect to the intertwining property (6.23).

13 Weak coactions

The notion of coactions may easily be generalized as well. By a left $\mathcal{G}$–coaction (λ, ϕ_λ) of a weak quasi–bialgebra $\mathcal{G}$ on a unital algebra $\mathcal{M}$ we mean a (not necessarily unital) algebra map $\lambda : \mathcal{M} \to \mathcal{G} \otimes \mathcal{M}$ and a quasi–invertible element $\phi_\lambda \in \mathcal{G} \otimes \mathcal{G} \otimes \mathcal{M}$ satisfying (7.1)-(7.3) as in Definition 7.1 and

$$(\mathrm{id} \otimes \epsilon \otimes \mathrm{id})(\phi_\lambda) = (\epsilon \otimes \mathrm{id} \otimes \mathrm{id})(\phi_\lambda) = \lambda(\mathbf{1}_\mathcal{M}) \tag{13.1}$$

The definition of right coactions is generalized analogously. Lemma 7.2 about twist equivalences of coactions stays valid, where one has to make the adjustments that a twist $U \in \mathcal{M} \otimes \mathcal{G}$ only is supposed to be quasi–invertible.

By now it should become clear how one has to proceed: Definition 8.1 of two–sided coactions is generalized by allowing δ to be non–unital and Ψ to be non–invertible but with quasi–inverse $\bar{\Psi} \equiv \Psi^{-1}$ and by replacing (8.4) by

$$(\mathrm{id}_\mathcal{G} \otimes \epsilon \otimes \mathrm{id}_\mathcal{M} \otimes \epsilon \otimes \mathrm{id}_\mathcal{G})(\Psi) = (\epsilon \otimes \mathrm{id}_\mathcal{G} \otimes \mathrm{id}_\mathcal{M} \otimes \mathrm{id}_\mathcal{G} \otimes \epsilon)(\Psi) = \delta(\mathbf{1}_\mathcal{M}) \tag{13.2}$$

The definitions of quasi–commuting pairs of coactions, twist equivalences of two–sided coactions etc. are generalized similarly. With these adjustments all results of Section 8 stay valid for weak quasi–Hopf algebras and are proven analogously.

Let us now shortly review the representation theoretical interpretation as given in Section 9 in the light of the present setting. Due to the coproduct being non–unital the definition of the tensor product functor in $\mathrm{Rep}\,\mathcal{G}$ has to be slightly modified. First note that the element $\Delta(\mathbf{1})$ (as well as higher coproducts of $\mathbf{1}$) is idempotent and commutes with all elements in $\Delta(\mathcal{G})$. Thus, given two representations $(V,\pi_V),(W,\pi_W)$, the operator $(\pi_V\otimes\pi_W)(\Delta(\mathbf{1}))$ is a projector, whose image is precisely the $\mathcal{G}$–invariant subspace of $V\otimes W$ on which the tensor product representation operates non trivial. Thus one is led to define the tensor product $\boxtimes$ of two representations of $\mathcal{G}$ by setting

$$V\boxtimes W := (\pi_V\otimes\pi_W)(\Delta(\mathbf{1}))\,V\otimes W,\quad \pi_V\boxtimes\pi_W := (\pi_V\otimes\pi_W)\circ\Delta|_{V\boxtimes W}$$

One readily verifies that with these definition ϕ_{UVW} - restricted to the subspace $(U\boxtimes V)\boxtimes W$ - furnish a natural family of isomorphisms defining an associativity constraint for the tensor product functor $\boxtimes$. Moreover, $\mathrm{Rep}\,\mathcal{G}$ becomes a rigid monoidal category with rigidity structure defined as before by (9.3)-(9.7).

The *left action* of $\mathrm{Rep}\,\mathcal{G}$ on $\mathrm{Rep}\,\mathcal{M}$ induced by a left $\mathcal{G}$–coaction (λ,ϕ_λ) on $\mathcal{M}$ has to be modified analogously by defining

$$V\odot\mathfrak{H} := (\pi\otimes\gamma)(\lambda(\mathbf{1}_{\mathcal{M}}))\,V\otimes\mathfrak{H},\quad \pi\odot\gamma := (\pi\otimes\gamma)\circ\Delta|_{V\odot\mathfrak{H}}$$

The modifications of right actions and of two–sided actions of $\mathrm{Rep}\,\mathcal{G}$ on $\mathrm{Rep}\,\mathcal{M}$ (induced by (ρ,ϕ_ρ) and by (δ,Ψ), respectively) should by now be obvious and are left to the reader.

With these adjustments all categorical identities such as the definition of natural families and commuting diagrams given in Section 9 stay valid. Translating these into algebraic identities one has to take some care with identities in higher tensor products of $\mathcal{G}$. The only equations which have to be modified are (9.26),(9.27), where the r.h.s. becomes $\lambda(\mathbf{1}_{\mathcal{M}})$ instead of $\mathbf{1}_{\mathcal{G}}\otimes\mathbf{1}_{\mathcal{M}}$ and similarly (9.32/9.33) and (9.53/9.54) where the r.h.s. has to be replaced by $\rho(\mathbf{1}_{\mathcal{M}})$ and by $\delta(\mathbf{1}_{\mathcal{M}})$, respectively. This is rather obvious from the categorical point of view, since for example (9.26) is directly connected with (9.38) where the r.h.s. is given by $\mathrm{id}_{V\odot\mathfrak{H}}\equiv(\pi_V\otimes\pi_{\mathfrak{H}})(\lambda(\mathbf{1}_{\mathcal{M}}))$.

14 Diagonal crossed products

The definition of diagonal crossed products $\hat{\mathcal{G}}\bowtie_\delta\mathcal{M}$ and $\mathcal{M}\bowtie_\delta\hat{\mathcal{G}}$ as equivalent algebra extensions of $\mathcal{M}$, given in Definition 10.1 and Theorem 10.2 need some more care in the present context. We will proceed in two steps. First we define an associative algebra structure on $\hat{\mathcal{G}}\otimes\mathcal{M}$ (or $\mathcal{M}\otimes\hat{\mathcal{G}}$) exactly as in Definition 10.1. Unfortunately in general this algebra is not unital unless the two–sided coaction δ is unital. But the element $\mathbf{1}_{\hat{\mathcal{G}}}\otimes\mathbf{1}_{\mathcal{M}}$ is still a right unit ($\mathbf{1}_{\mathcal{M}}\otimes\mathbf{1}_{\hat{\mathcal{G}}}$ is still a left unit) and in particular idempotent. The second step then consists in defining the subalgebra $\hat{\mathcal{G}}\bowtie_\delta\mathcal{M}\subset\hat{\mathcal{G}}\otimes\mathcal{M}$ as the right ideal generated by $\mathbf{1}_{\hat{\mathcal{G}}}\otimes\mathbf{1}_{\mathcal{M}}$, i.e. $\hat{\mathcal{G}}\bowtie_\delta\mathcal{M} := (\mathbf{1}_{\hat{\mathcal{G}}}\otimes\mathbf{1}_{\mathcal{M}})\cdot(\hat{\mathcal{G}}\otimes\mathcal{M})$ (the left ideal generated by $\mathbf{1}_{\mathcal{M}}\otimes\mathbf{1}_{\hat{\mathcal{G}}}$, i.e. $\mathcal{M}\bowtie_\delta\hat{\mathcal{G}} := (\mathcal{M}\otimes\hat{\mathcal{G}})\cdot(\mathbf{1}_{\mathcal{M}}\otimes\mathbf{1}_{\hat{\mathcal{G}}})$). These algebras are then unital algebra extensions of $\mathcal{M}\equiv\mathbf{1}_{\hat{\mathcal{G}}}\bowtie\mathcal{M}$ and of $\mathcal{M}\equiv\mathcal{M}\bowtie\mathbf{1}_{\hat{\mathcal{G}}}$, respectively. As in Section 10 one may proceed to a description by left and right δ–implementers and equivalently by $\lambda\rho$–intertwiners, thus getting a proof of Theorem III.

Definition 14.1. Let (δ,Ψ) be a two–sided coaction of a weak quasi–Hopf algebra $\mathcal{G}$ on an algebra $\mathcal{M}$. We define $\hat{\mathcal{G}}\otimes_\delta\mathcal{M}$ to be the vector space $\hat{\mathcal{G}}\otimes\mathcal{M}$ with multiplication structure given as in (10.10) and the left *diagonal crossed product* $\hat{\mathcal{G}} \bowtie_\delta \mathcal{M}$ to be the subspace

$$\hat{\mathcal{G}} \bowtie_\delta \mathcal{M} := (\mathbf{1}_{\hat{\mathcal{G}}} \otimes \mathbf{1}_{\mathcal{M}}) \cdot (\hat{\mathcal{G}} \otimes_\delta \mathcal{M}) \tag{14.1}$$

Analogously $\mathcal{M} \otimes_\delta \hat{\mathcal{G}}$ is defined to be the vector space $\mathcal{M} \otimes \hat{\mathcal{G}}$ with multiplication structure (10.11) and the right *diagonal crossed product* $\mathcal{M} \bowtie_\delta \hat{\mathcal{G}}$ to be the subspace

$$\mathcal{M} \bowtie_\delta \hat{\mathcal{G}} := (\mathcal{M} \otimes_\delta \hat{\mathcal{G}}) \cdot (\mathbf{1}_{\mathcal{M}} \otimes \mathbf{1}_{\hat{\mathcal{G}}}) \tag{14.2}$$

The elements spanning $\hat{\mathcal{G}} \bowtie_\delta \mathcal{M}$ and $\mathcal{M} \bowtie_\delta \hat{\mathcal{G}}$ will be denoted by, respectively

$$\varphi \bowtie m := (\mathbf{1}_{\hat{\mathcal{G}}} \otimes \mathbf{1}_{\mathcal{M}})(\varphi \otimes m) \equiv (\varphi \otimes \mathbf{1}_{\mathcal{M}})(\mathbf{1}_{\mathcal{M}} \otimes m) \equiv \varphi_{(2)} \otimes \big(\hat{S}^{-1}(\varphi_{(1)}) \triangleright \mathbf{1}_{\mathcal{M}} \triangleleft \varphi_{(3)}\big) m \tag{14.3}$$

$$m \bowtie \varphi := (m \otimes \varphi)(\mathbf{1}_{\mathcal{M}} \otimes \mathbf{1}_{\hat{\mathcal{G}}}) \equiv (m \otimes \mathbf{1}_{\hat{\mathcal{G}}})(\mathbf{1}_{\mathcal{M}} \otimes \varphi) \equiv m\big(\varphi_{(1)} \triangleright \mathbf{1}_{\mathcal{M}} \triangleleft \hat{S}^{-1}(\varphi_{(3)})\big) \otimes \varphi_{(2)} \tag{14.4}$$

Note that $\hat{\mathcal{G}} \otimes_\delta \mathcal{M} = \hat{\mathcal{G}} \bowtie_\delta \mathcal{M}$, if $\delta(\mathbf{1}_{\mathcal{M}}) = \mathbf{1}_{\mathcal{G}} \otimes \mathbf{1}_{\mathcal{M}} \otimes \mathbf{1}_{\mathcal{G}}$, which means that the above definition generalizes Definition 10.1. We now state the analogue of Theorem 10.2.

Theorem 14.2.

(i) *$\hat{\mathcal{G}} \otimes_\delta \mathcal{M}$ and $\mathcal{M} \otimes_\delta \hat{\mathcal{G}}$ are associative algebras with left unit $\mathbf{1}_{\hat{\mathcal{G}}} \otimes \mathbf{1}_{\mathcal{M}}$ and right unit $\mathbf{1}_{\mathcal{M}} \otimes \mathbf{1}_{\hat{\mathcal{G}}}$, respectively. Consequently, the diagonal crossed products $\hat{\mathcal{G}} \bowtie_\delta \mathcal{M}$ and $\mathcal{M} \bowtie_\delta \hat{\mathcal{G}}$ are subalgebras of $\hat{\mathcal{G}} \otimes_\delta \mathcal{M}$ and $\mathcal{M} \otimes_\delta \hat{\mathcal{G}}$, respectively, with unit given by $\mathbf{1}_{\hat{\mathcal{G}}} \bowtie \mathbf{1}_{\mathcal{M}} \equiv \mathbf{1}_{\hat{\mathcal{G}}} \otimes \mathbf{1}_{\mathcal{M}}$ and $\mathbf{1}_{\mathcal{M}} \bowtie \mathbf{1}_{\hat{\mathcal{G}}} \equiv \mathbf{1}_{\mathcal{M}} \otimes \mathbf{1}_{\hat{\mathcal{G}}}$, respectively.*

(ii) *$\mathcal{M} \equiv \mathbf{1}_{\hat{\mathcal{G}}} \otimes \mathcal{M} = \mathbf{1}_{\hat{\mathcal{G}}} \bowtie \mathcal{M} \subset \hat{\mathcal{G}} \bowtie_\delta \mathcal{M}$ and $\mathcal{M} \equiv \mathcal{M} \otimes \mathbf{1}_{\hat{\mathcal{G}}} = \mathcal{M} \bowtie \mathbf{1}_{\hat{\mathcal{G}}} \subset \mathcal{M} \bowtie_\delta \hat{\mathcal{G}}$ are unital algebra inclusions.*

(iii) *The algebras $\hat{\mathcal{G}} \bowtie \mathcal{M}$ and $\mathcal{M} \bowtie \hat{\mathcal{G}}$ provide equivalent extensions of $\mathcal{M}$ with isomorphism given as in* (10.15),(10.16).

Proof. We will sketch the proof of part (i) for $\hat{\mathcal{G}} \otimes \mathcal{M}$, the case $\mathcal{M} \otimes \hat{\mathcal{G}}$ being analogous. From (10.10) one computes that

$$\begin{aligned}
(\varphi \otimes m)(\mathbf{1}_{\hat{\mathcal{G}}} \otimes \mathbf{1}_{\mathcal{M}}) &= (\varphi \otimes m) \\
(\mathbf{1}_{\hat{\mathcal{G}}} \otimes \mathbf{1}_{\mathcal{M}})(\varphi \otimes m) &= \varphi_{(2)} \otimes \big(\hat{S}^{-1}(\varphi_{(1)}) \triangleright \mathbf{1}_{\mathcal{M}} \triangleleft \varphi_{(3)}\big)\, m \\
&= \big(\mathbf{1}_{(-1)} \rightharpoonup \varphi \leftharpoonup S^{-1}(\mathbf{1}_{(1)})\big) \otimes \mathbf{1}_{(0)}\, m
\end{aligned}$$

where $\delta(\mathbf{1}_{\mathcal{M}}) = \mathbf{1}_{(-1)} \otimes \mathbf{1}_{(0)} \otimes \mathbf{1}_{(1)}$. This shows that $\mathbf{1}_{\hat{\mathcal{G}}} \otimes \mathbf{1}_{\mathcal{M}}$ is a right unit in $\hat{\mathcal{G}} \otimes_\delta \mathcal{M}$ but in general not a left unit.

To proof the associativity of the product one proceeds as in the proof of Theorem 10.2. Here one has to take some notational care when translating (10.24/10.25) into relations of a generating matrix $\mathbf{L}$. First, it is necessary to distinguish $\tilde{\mathbf{L}} \equiv \tilde{L}^1 \otimes \tilde{L}^2 := e_\mu \otimes (e^\mu \otimes \mathbf{1}_{\mathcal{M}}) \in \mathcal{G} \otimes (\hat{\mathcal{G}} \otimes \mathcal{M})$ and $\mathbf{L} := e_\mu \otimes (e^\mu \bowtie \mathbf{1}_{\mathcal{M}}) \in \mathcal{G} \otimes (\hat{\mathcal{G}} \bowtie \mathcal{M})$. Eq. (10.28) must then be replaced by

$$[\mathbf{1}_{\mathcal{G}} \otimes \mathbf{1}_{\mathcal{M}}]\, \tilde{\mathbf{L}} = \mathbf{L} \tag{14.5}$$

and Eqs. (10.29), (10.30) are at first only valid for $\mathbf{L}$ but not for $\tilde{\mathbf{L}}$, since $\mathbf{1}_{\mathcal{M}} \equiv \mathbf{1}_{\hat{\mathcal{G}}} \otimes \mathbf{1}_{\mathcal{M}}$ is not a left unit in $\hat{\mathcal{G}} \otimes_\delta \mathcal{M}$. This can be cured by rewriting for example (10.29) for $\tilde{\mathbf{L}}$ more carefully as

$$(\tilde{L}^1 \otimes m\tilde{L}^2) = S^{-1}(m_{(1)})\, \tilde{L}^1\, m_{(-1)} \otimes \tilde{L}^2\, m_{(0)}, \quad \forall m \in \mathcal{M}$$

in which form it would still be valid. Taking this into account and using that $\bar{\Psi}\,(\mathbf{1}_{\mathcal{G}} \otimes \delta(\mathbf{1}_{\mathcal{M}}) \otimes \mathbf{1}_{\mathcal{G}}) = \bar{\Psi}$, the proof proceeds as the one of Theorem 10.2 (i). The proof of the remaining parts of Theorem 10.2 is straightforwardly adjusted in the same spirit. $\square$

From now on we will disregard the "unphysical" non–unital algebras $\mathcal{M} \otimes_\delta \hat{\mathcal{G}}$ and $\hat{\mathcal{G}} \otimes_\delta \mathcal{M}$, and stay with $\hat{\mathcal{G}} \bowtie \mathcal{M}$ and $\mathcal{M} \bowtie \hat{\mathcal{G}}$ as our objects of interest. With the appropriate notations (14.3),(14.4) all relations of Section 10.1 remain valid for these algebras. We also remark that the left multiplication by $\mathbf{1}_{\hat{\mathcal{G}}} \otimes \mathbf{1}_{\mathcal{M}}$ (right multiplication by $\mathbf{1}_{\mathcal{M}} \otimes \mathbf{1}_{\hat{\mathcal{G}}}$) precisely gives the projections $P_{L/R}$ mentioned in part 4. of Theorem III.

Defining left and right δ–implementers as in Definition 10.3 all results in Section 10.2 also carry over to the present setting. A $\lambda\rho$–intertwiner is then supposed to have the additional property that

$$\mathbf{T}\,\lambda_{\mathcal{A}}(\mathbf{1}_{\mathcal{M}}) \equiv \rho^{op}_{\mathcal{A}}(\mathbf{1}_{\mathcal{M}})\,\mathbf{T} = \mathbf{T} \tag{14.6}$$

With this the one–to–one correspondence $\mathbf{T} \leftrightarrow \mathbf{R}$ and $\mathbf{T} \leftrightarrow \mathbf{L}$ of Proposition 10.10 is still valid, where (14.6) becomes equivalent to $\mathbf{L} \prec \delta(\mathbf{1}_{\mathcal{M}}) = \mathbf{L}$ and $\delta(\mathbf{1}_{\mathcal{M}}) \succ \mathbf{R} = \mathbf{R}$, respectively, which follow from (10.35) and (10.36). One may now prove Theorem III analogously as Theorem II where the modifications in part 3. and 4. have their origin in (14.6).

15 Examples and applications

Finally, all examples given in Section 11 generalize to weak quasi–Hopf algebras. The definition of the quantum double $\mathcal{D}(\mathcal{G})$ for weak quasi–Hopf algebras as given in Theorem 11.3 yields a weak quasi–bialgebra.

The definition of *two–sided crossed products* as in Proposition 11.4 (i.e. an alternative description of the algebra $\mathcal{M}_{\lambda}\!\bowtie_{\rho} \hat{\mathcal{G}}$ for the case that $\mathcal{M} = \mathcal{A} \otimes \mathcal{B}$), has to be slightly modified. The unital algebra $\mathcal{A} \rtimes_\rho \hat{\mathcal{G}} \ltimes_\lambda \mathcal{B}$ is now defined on the subspace of $\mathcal{A} \otimes \hat{\mathcal{G}} \otimes \mathcal{B}$ given by

$$\mathcal{A} \rtimes \hat{\mathcal{G}} \ltimes \mathcal{B} := \operatorname{span}\big\{A \rtimes \varphi \ltimes B \equiv A\mathbf{1}_{\mathcal{A}}^{(0)} \otimes \mathbf{1}_{\mathcal{A}}^{(1)} \triangleright \varphi \triangleleft \mathbf{1}_{\mathcal{B}}^{(-1)} \otimes \mathbf{1}_{\mathcal{B}}^{(0)} B \mid A \in \mathcal{A}, B \in \mathcal{B}, \varphi \in \hat{\mathcal{G}}\big\}, \tag{15.1}$$

where $\mathbf{1}_{\mathcal{A}}^{(0)} \otimes \mathbf{1}_{\mathcal{A}}^{(1)} \equiv \rho(\mathbf{1}_{\mathcal{A}})$ and $\mathbf{1}_{\mathcal{B}}^{(-1)} \otimes \mathbf{1}_{\mathcal{B}}^{(0)} \equiv \lambda(\mathbf{1}_{\mathcal{B}})$ Again we have a linear bijection $\mu : \mathcal{A} \rtimes \hat{\mathcal{G}} \ltimes \mathcal{B} \to \mathcal{M}_1 = (\mathcal{A} \otimes \mathcal{B})_{\lambda}\!\bowtie_{\rho} \hat{\mathcal{G}}$

$$\mu(A \rtimes \hat{\mathcal{G}} \ltimes B) = A\,\Gamma(\varphi)\,B$$

inducing the multiplication rule described by Eq. (11.16). With these notations all results of Section 11.3 are still valid for weak quasi–Hopf algebras. As for part (iii) of Proposition 11.5 we note that (15.1) still allows the identification

$$(A \rtimes \varphi \ltimes B) \rtimes \psi \ltimes C = A \rtimes \varphi \ltimes (B \rtimes \psi \ltimes C)$$

Moreover, putting $(\mathcal{B}, \lambda, \phi_\lambda) = (\mathcal{G}, \Delta, \phi)$ this is also the setting underlying the field algebra constructions proposed in Section 11.4.

Appendix

In this appendix we will give the proofs which have been omitted in the previous sections. Let us first collect additional identities of the reassociator $\Psi \in \mathcal{G} \otimes \mathcal{G} \otimes \mathcal{M} \otimes \mathcal{G} \otimes \mathcal{G}$ given in

Definition 8.1, which follow by applying $\epsilon_i \otimes \mathrm{id}_{\mathcal{M}} \otimes \epsilon_j$, $1 \leq i, j \leq 3$, to both sides of (8.2), where $\epsilon_i : \mathcal{G}^{\otimes^3} \to \mathcal{G}^{\otimes^2}$ is given by acting with ϵ on the i–th tensor factor.

$$\begin{aligned}
&[(\mathrm{id}_{\mathcal{G}}^{\otimes^2} \otimes \mathrm{id}_{\mathcal{M}} \otimes \epsilon \otimes \mathrm{id}_{\mathcal{G}})(\Psi) \otimes \mathbf{1}_{\mathcal{G}}]\,[(\epsilon \otimes \Delta \otimes \mathrm{id}_{\mathcal{M}} \otimes \mathrm{id}_{\mathcal{G}}^{\otimes^2})(\Psi)] \\
&\qquad = [(\epsilon \otimes \mathrm{id}_{\mathcal{G}} \otimes \lambda \otimes \mathrm{id}_{\mathcal{G}}^{\otimes^2})(\Psi)]\,[(\mathrm{id}_{\mathcal{G}}^{\otimes^2} \otimes \mathrm{id}_{\mathcal{M}} \otimes \epsilon \otimes \Delta)(\Psi)] && \text{(A.1)} \\
&[(\mathrm{id}_{\mathcal{G}}^{\otimes^2} \otimes \mathrm{id}_{\mathcal{M}} \otimes \mathrm{id}_{\mathcal{G}} \otimes \epsilon)(\Psi) \otimes \mathbf{1}_{\mathcal{G}}]\,[(\epsilon \otimes \Delta \otimes \mathrm{id}_{\mathcal{M}} \otimes \mathrm{id}_{\mathcal{G}}^{\otimes^2})(\Psi)] \\
&\qquad = [(\epsilon \otimes \mathrm{id}_{\mathcal{G}} \otimes \delta \otimes \epsilon \otimes \mathrm{id}_{\mathcal{G}})(\Psi)]\,\Psi && \text{(A.2)} \\
&[\mathbf{1}_{\mathcal{G}} \otimes (\epsilon \otimes \mathrm{id}_{\mathcal{G}} \otimes \mathrm{id}_{\mathcal{M}} \otimes \epsilon \otimes \mathrm{id}_{\mathcal{G}})(\Psi) \otimes \mathbf{1}_{\mathcal{G}}]\,\Psi \\
&\qquad = [(\mathrm{id}_{\mathcal{G}} \otimes \epsilon \otimes \lambda \otimes \mathrm{id}_{\mathcal{G}}^{\otimes^2})(\Psi)]\,[(\mathrm{id}_{\mathcal{G}}^{\otimes^2} \otimes \mathrm{id}_{\mathcal{M}} \otimes \epsilon \otimes \Delta)(\Psi)] && \text{(A.3)} \\
&[\mathbf{1}_{\mathcal{G}} \otimes (\mathrm{id}_{\mathcal{G}} \otimes \epsilon \otimes \mathrm{id}_{\mathcal{M}} \otimes \mathrm{id}_{\mathcal{G}}^{\otimes^2})(\Psi)]\,[(\mathrm{id}_{\mathcal{G}}^{\otimes^2} \otimes \mathrm{id}_{\mathcal{M}} \otimes \Delta \otimes \epsilon)(\Psi)] \\
&\qquad = [(\mathrm{id}_{\mathcal{G}}^{\otimes^2} \otimes \rho \otimes \mathrm{id}_{\mathcal{G}} \otimes \epsilon)(\Psi)]\,[(\Delta \otimes \epsilon \otimes \mathrm{id}_{\mathcal{M}} \otimes \mathrm{id}_{\mathcal{G}}^{\otimes^2})(\Psi)] && \text{(A.4)} \\
&[\mathbf{1}_{\mathcal{G}} \otimes (\epsilon \otimes \mathrm{id}_{\mathcal{G}} \otimes \mathrm{id}_{\mathcal{M}} \otimes \mathrm{id}_{\mathcal{G}}^{\otimes^2})(\Psi)]\,[(\mathrm{id}_{\mathcal{G}}^{\otimes^2} \otimes \mathrm{id}_{\mathcal{M}} \otimes \Delta \otimes \epsilon)(\Psi)] \\
&\qquad = [(\mathrm{id}_{\mathcal{G}} \otimes \epsilon \otimes \delta \otimes \mathrm{id}_{\mathcal{G}} \otimes \epsilon)(\Psi)]\,\Psi && \text{(A.5)} \\
&[\mathbf{1}_{\mathcal{G}} \otimes (\mathrm{id}_{\mathcal{G}} \otimes \epsilon \otimes \mathrm{id}_{\mathcal{M}} \otimes \mathrm{id}_{\mathcal{G}} \otimes \epsilon)(\Psi) \otimes \mathbf{1}_{\mathcal{G}}]\,\Psi \\
&\qquad = [(\mathrm{id}_{\mathcal{G}}^{\otimes^2} \otimes \rho \otimes \epsilon \otimes \mathrm{id}_{\mathcal{G}})(\Psi)]\,[(\Delta \otimes \epsilon \otimes \mathrm{id}_{\mathcal{M}} \otimes \mathrm{id}_{\mathcal{G}}^{\otimes^2})(\Psi)]. && \text{(A.6)}
\end{aligned}$$

Here we have used $\epsilon_i(\phi) = \mathbf{1}_{\mathcal{G}} \otimes \mathbf{1}_{\mathcal{G}}$ and we have introduced the notation $\lambda := (\mathrm{id}_{\mathcal{G}} \otimes \mathrm{id}_{\mathcal{M}} \otimes \epsilon) \circ \delta$, $\rho := (\epsilon \otimes \mathrm{id}_{\mathcal{M}} \otimes \mathrm{id}_{\mathcal{G}}) \circ \delta$. Note that Eqs. (A.4) - (A.6) are the left–right mirror images of (A.1) - (A.3), and that the axioms (8.1) - (8.4) are symmetric under this reflection.

Proof of Proposition 8.4: We prove the implication $(ii) \Rightarrow (i)$: So given that (δ_l, Ψ_l) is a two–sided coaction we use the counit axioms for ϕ_λ and ϕ_ρ to conclude

$$(\mathrm{id}_{\mathcal{G}}^{\otimes^2} \otimes \mathrm{id}_{\mathcal{M}} \otimes \mathrm{id}_{\mathcal{G}} \otimes \epsilon)(\Psi_l) = (\mathrm{id}_{\mathcal{G}} \otimes \lambda \otimes \mathrm{id}_{\mathcal{G}})(\phi_{\lambda\rho})\,(\phi_\lambda \otimes \mathbf{1}_{\mathcal{G}}) \tag{A.7}$$

$$(\mathrm{id}_{\mathcal{G}} \otimes \epsilon \otimes \mathrm{id}_{\mathcal{M}} \otimes \mathrm{id}_{\mathcal{G}} \otimes \epsilon)(\Psi_l) = \phi_{\lambda\rho} \tag{A.8}$$

In particular this implies $(\epsilon \otimes \mathrm{id}_{\mathcal{M}} \otimes \mathrm{id}_{\mathcal{G}})(\phi_{\lambda\rho}) = \mathbf{1}_{\mathcal{M}} \otimes \mathbf{1}_{\mathcal{G}}$ and $(\mathrm{id}_{\mathcal{G}} \otimes \mathrm{id}_{\mathcal{M}} \otimes \epsilon)(\phi_{\lambda\rho}) = \mathbf{1}_{\mathcal{G}} \otimes \mathbf{1}_{\mathcal{M}}$ and therefore also

$$(\epsilon \otimes \mathrm{id}_{\mathcal{G}} \otimes \mathrm{id}_{\mathcal{M}} \otimes \epsilon \otimes \mathrm{id}_{\mathcal{G}})(\Psi_l) = \mathbf{1}_{\mathcal{G}} \otimes \mathbf{1}_{\mathcal{M}} \otimes \mathbf{1}_{\mathcal{G}} \tag{A.9}$$

$$(\mathrm{id}^{\otimes^2} \otimes \mathrm{id}_{\mathcal{M}} \otimes \epsilon \otimes \epsilon)(\Psi_l) = \phi_\lambda \tag{A.10}$$

$$(\epsilon \otimes \epsilon \otimes \mathrm{id}_{\mathcal{M}} \otimes \mathrm{id}^{\otimes^2})(\Psi_l) = \phi_\rho^{-1} \tag{A.11}$$

Now we use $\lambda = (\mathrm{id}_{\mathcal{G}} \otimes \mathrm{id}_{\mathcal{M}} \otimes \epsilon) \circ \delta_l$ and $\rho = (\epsilon \otimes \mathrm{id}_{\mathcal{M}} \otimes \mathrm{id}_{\mathcal{G}}) \circ \delta_l$ implying

$$\delta_r = (\mathrm{id}_{\mathcal{G}} \otimes \epsilon \otimes \mathrm{id}_{\mathcal{M}} \otimes \epsilon) \circ \delta_l^{(2)} = Ad\,\phi_{\lambda\rho} \circ \delta_l$$

by (A.8) and the assumption (8.2) for (δ_l, Ψ_l). This proves (8.10). Eq. (8.11) now follows by acting with $(\mathrm{id}_{\mathcal{G}}^{\otimes^2} \otimes \mathrm{id}_{\mathcal{M}} \otimes \mathrm{id}_{\mathcal{G}} \otimes \epsilon)$ on (A.4).

To prove Eq. (8.12) we first use that by (A.9), (A.11) and (A.3)

$$(\lambda \otimes \mathrm{id}_{\mathcal{G}}^{\otimes^2})(\phi_\rho^{-1}) = (\epsilon \otimes \mathrm{id}_{\mathcal{G}} \otimes \mathrm{id}_{\mathcal{M}} \otimes \mathrm{id}_{\mathcal{G}}^{\otimes^2})(\Psi_l).$$

Hence Eq. (8.12) follows from

$$
\begin{aligned}
&(\mathrm{id}\otimes\rho\otimes\mathrm{id})(\phi_{\lambda\rho})\,(\phi_{\lambda\rho}\otimes\mathbf{1}_{\mathcal G})(\lambda\otimes\mathrm{id}\otimes\mathrm{id})(\phi_\rho^{-1})\\
&\quad=[(\mathrm{id}_{\mathcal G}\otimes\epsilon\otimes\rho\otimes\mathrm{id}_{\mathcal G}\otimes\epsilon)(\Psi_l)][(\mathrm{id}_{\mathcal G}\otimes\epsilon\otimes\mathrm{id}_{\mathcal M}\otimes\mathrm{id}_{\mathcal G}\otimes\epsilon)(\Psi_l)\otimes\mathbf{1}_{\mathcal G}][(\epsilon\otimes\mathrm{id}_{\mathcal G}\otimes\mathrm{id}_{\mathcal M}\otimes\mathrm{id}_{\mathcal G}^{\otimes^2})(\Psi_l)]\\
&\quad=[(\mathrm{id}_{\mathcal G}\otimes\epsilon\otimes\rho\otimes\mathrm{id}_{\mathcal G}\otimes\epsilon)(\Psi_l)][(\mathrm{id}_{\mathcal G}\otimes\epsilon\otimes\mathrm{id}_{\mathcal M}\otimes\mathrm{id}_{\mathcal G}^{\otimes^2})(\Psi_l)]\\
&\quad=[\mathbf{1}_{\mathcal G}\otimes(\epsilon\otimes\epsilon\otimes\mathrm{id}_{\mathcal M}\otimes\mathrm{id}_{\mathcal G}^{\otimes^2})(\Psi_l)][(\mathrm{id}_{\mathcal G}\otimes\epsilon\otimes\mathrm{id}_{\mathcal M}\otimes\Delta\otimes\epsilon)(\Psi_l)]\\
&\quad=(\mathbf{1}_{\mathcal G}\otimes\phi_\rho^{-1})\,(\mathrm{id}_{\mathcal G}\otimes\mathrm{id}_{\mathcal M}\otimes\Delta)(\phi_{\lambda\rho})
\end{aligned}
$$

Here we have acted with $(\mathrm{id}_{\mathcal G}\otimes\epsilon\otimes\mathrm{id}_{\mathcal M}\otimes\mathrm{id}_{\mathcal G}^{\otimes^2})$ on (A.2) in the second equation and on (A.4) in the third equation. Hence we have shown $(ii)\Rightarrow(i)$. The implication $(iii)\Rightarrow(i)$ is proven analogously.

We are left with showing that under the conditions $(i)-(iii)$ $U\equiv\phi_{\lambda\rho}$ provides a twist from (δ_l,Ψ_l) to (δ_r,Ψ_r). Now the intertwiner property (8.15) holds by definition and (8.17) follows from (8.19). To prove (8.16) for $\Psi'=\Psi_r$ and $\Psi=\Psi_l$ we note that (8.12) and (8.20) imply

$$
\Psi_l=(\mathrm{id}_{\mathcal G}\otimes\delta\otimes\mathrm{id})(\phi_{\lambda\rho}^{-1})[\mathbf{1}_{\mathcal G}\otimes(\lambda\otimes\mathrm{id}_{\mathcal G}\otimes\mathrm{id}_{\mathcal M})(\phi_\rho^{-1})]\,(\mathrm{id}_{\mathcal G}\otimes\lambda\otimes\Delta)(\phi_{\lambda\rho})\,[\phi_\lambda\otimes\mathbf{1}_{\mathcal G}\otimes\mathbf{1}_{\mathcal G}]
$$

and therefore, putting $U=\phi_{\lambda\rho}$ in (8.16)

$$
\begin{aligned}
&(\mathbf{1}_{\mathcal G}\otimes U\otimes\mathbf{1}_{\mathcal G})\,(\mathrm{id}_{\mathcal G}\otimes\delta\otimes\mathrm{id}_{\mathcal G})(U)\,\Psi_l\\
&\quad=[\mathbf{1}_{\mathcal G}\otimes(\phi_{\lambda\rho}\otimes\mathbf{1}_{\mathcal G})(\lambda\otimes\mathrm{id}_{\mathcal G}\otimes\mathrm{id}_{\mathcal G})(\phi_\rho^{-1})](\mathrm{id}_{\mathcal G}^{\otimes^2}\otimes\mathrm{id}_{\mathcal M}\otimes\Delta)\Big((\mathrm{id}_{\mathcal G}\otimes\lambda\otimes\mathrm{id}_{\mathcal G})(\phi_{\lambda\rho})(\phi_\lambda\otimes\mathbf{1}_{\mathcal G})\Big)\\
&\quad=[\mathbf{1}_{\mathcal G}\otimes(\mathrm{id}_{\mathcal G}\otimes\rho\otimes\mathrm{id}_{\mathcal G})(\phi_{\lambda\rho}^{-1})][\mathbf{1}_{\mathcal G}\otimes\mathbf{1}_{\mathcal G}\otimes\phi_\rho^{-1}]\\
&\qquad(\mathrm{id}_{\mathcal G}^{\otimes^2}\otimes\mathrm{id}_{\mathcal M}\otimes\Delta)\Big((\mathbf{1}_{\mathcal G}\otimes\phi_{\lambda\rho})(\mathrm{id}_{\mathcal G}\otimes\lambda\otimes\mathrm{id}_{\mathcal G})(\phi_{\lambda\rho})(\phi_\lambda\otimes\mathbf{1}_{\mathcal G})\Big)\\
&\quad=[\mathbf{1}_{\mathcal G}\otimes(\mathrm{id}_{\mathcal G}\otimes\rho\otimes\mathrm{id}_{\mathcal G})(\phi_{\lambda\rho}^{-1})][\mathbf{1}_{\mathcal G}\otimes\mathbf{1}_{\mathcal G}\otimes\phi_\rho^{-1}]\,(\mathrm{id}_{\mathcal G}^{\otimes^2}\otimes(\mathrm{id}_{\mathcal M}\otimes\Delta)\circ\rho)(\phi_\lambda)\,(\Delta\otimes\mathrm{id}_{\mathcal M}\otimes\Delta)(\phi_{\lambda\rho})\\
&\quad=\Psi_r\,(\Delta\otimes\mathrm{id}_{\mathcal M}\otimes\Delta)(U)
\end{aligned}
$$

Here we have used (8.12) in the second equality, (8.11) in the third equality and (7.5) in the last equality. This proves Eq. (8.16) for $\Psi=\Psi_l$ and $\Psi'=\Psi_r$. Hence $\phi_{\lambda\rho}$ provides a twist from (δ_l,Ψ_l) to (δ_r,Ψ_r). This concludes the proof of Proposition 8.4. □

Proof of Proposition 8.5 (ii): The identities $\delta_{l/r}=AdU_{l/r}\circ\delta$ follow immediately from (8.1), (8.8) and (8.9). We are left to show that

$$
(\mathbf{1}_{\mathcal G}\otimes U_l\otimes\mathbf{1}_{\mathcal G})\,(\mathrm{id}_{\mathcal G}\otimes\delta\otimes\mathrm{id}_{\mathcal G})(U_l)\,\Psi=\Psi_l\,(\Delta\otimes\mathrm{id}_{\mathcal M}\otimes\Delta)(U_l) \tag{A.12}
$$

thus proving that (δ_l,Ψ_l) is a two–sided coaction twist equivalent to (δ,Ψ) via U_l[10]. Using (8.23) and (A.2) the l.h.s. of (A.12) gives

$$
\begin{aligned}
&(\mathbf{1}_{\mathcal G}\otimes U_l\otimes\mathbf{1}_{\mathcal G})\,(\mathrm{id}_{\mathcal G}\otimes\delta\otimes\mathrm{id}_{\mathcal G})(U_l)\,\Psi\\
&\quad=\big[\mathbf{1}_{\mathcal G}\otimes(\epsilon\otimes\mathrm{id}_{\mathcal G}\otimes\mathrm{id}_{\mathcal M}\otimes\epsilon\otimes\mathrm{id}_{\mathcal G})(\Psi)\otimes\mathbf{1}_{\mathcal G}\big]\big[(\mathrm{id}_{\mathcal G}^{\otimes^2}\otimes\mathrm{id}_{\mathcal M}\otimes\mathrm{id}_{\mathcal G}\otimes\epsilon)(\Psi)\otimes\mathbf{1}_{\mathcal G}\big]\\
&\qquad\big[(\epsilon\otimes\Delta\otimes\mathrm{id}_{\mathcal M}\otimes\mathrm{id}_{\mathcal G}^{\otimes^2})(\Psi)\big]\\
&\quad=\big[(\mathrm{id}_{\mathcal G}\otimes\epsilon\otimes\lambda\otimes\mathrm{id}_{\mathcal G}\otimes\epsilon)(\Psi)\otimes\mathbf{1}_{\mathcal G}\big]\big[(\mathrm{id}_{\mathcal G}^{\otimes^2}\otimes\mathrm{id}_{\mathcal M}\otimes\epsilon\otimes\mathrm{id}_{\mathcal G})(\Psi)\otimes\mathbf{1}_{\mathcal G}\big]\\
&\qquad\big[(\epsilon\otimes\Delta\otimes\mathrm{id}_{\mathcal M}\otimes\mathrm{id}_{\mathcal G}^{\otimes^2})(\Psi)\big],
\end{aligned} \tag{A.13}
$$

[10] Recall that (A.12) together with $\delta_{l/r}=AdU_{l/r}\circ\delta$ already guarantees that (δ_l,Ψ_l) satisfies all axioms of Definition 8.1.

where in the second equation we have used Eq. (A.3) acted upon by $(\mathrm{id}_{\mathcal{G}}^{\otimes^2} \otimes \mathrm{id}_{\mathcal{M}} \otimes \mathrm{id}_{\mathcal{G}} \otimes \epsilon)$. On the other hand, the r.h.s. of (A.12) gives, using (8.20), (8.23)-(8.25), (8.6), (8.7) and (7.1)

$$
\begin{aligned}
&\Psi_l\,(\Delta \otimes \mathrm{id}_{\mathcal{M}} \otimes \Delta)(U_l) \\
&\quad = \big[(\mathrm{id}_{\mathcal{G}} \otimes \epsilon \otimes \lambda \otimes \mathrm{id}_{\mathcal{G}} \otimes \epsilon)(\Psi) \otimes \mathbf{1}_{\mathcal{G}}\big]\big[(\epsilon \otimes \mathrm{id}_{\mathcal{G}} \otimes \lambda \otimes \epsilon \otimes \mathrm{id}_{\mathcal{G}})(\Psi^{-1}) \otimes \mathbf{1}_{\mathcal{G}}\big] \\
&\qquad \big[(\mathrm{id}_{\mathcal{G}}^{\otimes^2} \otimes \mathrm{id}_{\mathcal{M}} \otimes \epsilon \otimes \epsilon)(\Psi) \otimes \mathbf{1}_{\mathcal{G}} \otimes \mathbf{1}_{\mathcal{G}}\big]\big[(\epsilon \otimes \epsilon \otimes (\Delta \otimes \mathrm{id}_{\mathcal{M}}) \circ \lambda \otimes \mathrm{id}_{\mathcal{G}}^{\otimes^2})(\Psi)\big] \\
&\qquad \big[(\epsilon \otimes \Delta \otimes \mathrm{id}_{\mathcal{M}} \otimes \epsilon \otimes \Delta)(\Psi)\big] \\
&\quad = \big[(\mathrm{id}_{\mathcal{G}} \otimes \epsilon \otimes \lambda \otimes \mathrm{id}_{\mathcal{G}} \otimes \epsilon)(\Psi) \otimes \mathbf{1}_{\mathcal{G}}\big]\big[(\epsilon \otimes \mathrm{id}_{\mathcal{G}} \otimes \lambda \otimes \epsilon \otimes \mathrm{id}_{\mathcal{G}})(\Psi^{-1}) \otimes \mathbf{1}_{\mathcal{G}}\big] \\
&\qquad \big[(\mathrm{id}_{\mathcal{G}}^{\otimes^2} \otimes \mathrm{id}_{\mathcal{M}} \otimes \epsilon \otimes \epsilon)(\Psi) \otimes \mathbf{1}_{\mathcal{G}} \otimes \mathbf{1}_{\mathcal{G}}\big]\big[(\epsilon \otimes \Delta \otimes \mathrm{id}_{\mathcal{M}} \otimes \epsilon \otimes \mathrm{id}_{\mathcal{G}})(\Psi) \otimes \mathbf{1}_{\mathcal{G}}\big] \\
&\qquad \big[(\epsilon \otimes \Delta \otimes \mathrm{id}_{\mathcal{M}} \otimes \mathrm{id}_{\mathcal{G}}^{\otimes^2})(\Psi)\big] \\
&\quad = \big[(\mathrm{id}_{\mathcal{G}} \otimes \epsilon \otimes \lambda \otimes \mathrm{id}_{\mathcal{G}} \otimes \epsilon)(\Psi) \otimes \mathbf{1}_{\mathcal{G}}\big]\big[(\mathrm{id}_{\mathcal{G}}^{\otimes^2} \otimes \mathrm{id}_{\mathcal{M}} \otimes \epsilon \otimes \mathrm{id}_{\mathcal{G}})(\Psi) \otimes \mathbf{1}_{\mathcal{G}}\big] \\
&\qquad \big[(\epsilon \otimes \Delta \otimes \mathrm{id}_{\mathcal{M}} \otimes \mathrm{id}_{\mathcal{G}}^{\otimes^2})(\Psi)\big]
\end{aligned}
\tag{A.14}
$$

Here we have used Eq. (A.1) acted upon by $(\epsilon \otimes \Delta \otimes \mathrm{id}_{\mathcal{M}} \otimes \mathrm{id}_{\mathcal{G}}^{\otimes^2})$ in the second equation and Eq. (A.1) acted upon by $(\mathrm{id}_{\mathcal{G}}^{\otimes^2} \otimes \mathrm{id}_{\mathcal{M}} \otimes \epsilon \otimes \mathrm{id}_{\mathcal{G}})$ in the last equation. Comparing (A.14) and (A.13) we have proven (A.12). Hence U_l provides a twist equivalence from (δ, Ψ) to (δ_l, Ψ_l). Similarly, taking a left–right mirror image of the above proof, U_r provides a twist equivalence from (δ, Ψ) to (δ_r, Ψ_r). $\square$

Proof of Lemma 9.1: As has been remarked after Lemma 9.1 it suffices to prove Eqs. (9.31),(9.33) and (9.35). Let us begin with (9.31). Denoting the multiplication in $\mathcal{G}^{op}$ by μ^{op} one computes

$$
\begin{aligned}
&[\mathbf{1}_{\mathcal{M}} \otimes S^{-1}(m_{(1)})]\, q_\rho\, \rho(m_{(0)}) = [\phi_\rho^1 \otimes S^{-1}(\alpha \phi_\rho^3 m_{(1)}) \phi_\rho^2]\, \rho(m_{(0)}) \\
&\quad = (\mathrm{id}_{\mathcal{M}} \otimes \mu^{op}) \circ (\mathrm{id}_{\mathcal{M}} \otimes \mathrm{id}_{\mathcal{G}} \otimes S^{-1})\Big([\mathbf{1}_{\mathcal{M}} \otimes \mathbf{1}_{\mathcal{G}} \otimes \alpha]\, \phi_\rho\, (\rho \otimes \mathrm{id}_{\mathcal{G}})(\rho(m))\Big) \\
&\quad = (\mathrm{id}_{\mathcal{M}} \otimes \mu^{op}) \circ (\mathrm{id}_{\mathcal{M}} \otimes \mathrm{id}_{\mathcal{G}} \otimes S^{-1})\Big([\mathbf{1}_{\mathcal{M}} \otimes \mathbf{1}_{\mathcal{G}} \otimes \alpha]\, (\mathrm{id}_{\mathcal{M}} \otimes \Delta)(\rho(m))\, \phi_\rho\Big) \\
&\quad = [m \otimes \mathbf{1}_{\mathcal{G}}]\, q_\rho,
\end{aligned}
$$

where we have plugged in the definition (9.23) of q_ρ and used the intertwiner property (7.5) of ϕ_ρ and the antipode property (6.13). This proves (9.31).

To prove (9.33) we introduce for $a, b, c \in \mathcal{G}$ the notation $\sigma(a \otimes b \otimes c) := c\, S^{-1}(\alpha b \beta)\, a$, to compute for the l.h.s.

$$
\begin{aligned}
[\mathbf{1}_{\mathcal{M}} \otimes S^{-1}(p_\rho^2)]\, q_\rho\, \rho(p_\rho^1) &\equiv \big[\phi_\rho^1 \otimes \bar{\phi}_\rho^3\, S^{-1}(\alpha \phi_\rho^3 \bar{\phi}_\rho^2 \beta) \phi_\rho^2\big]\, \rho(\bar{\phi}_\rho^1) \\
&= (\mathrm{id}_{\mathcal{M}} \otimes \sigma)\Big([\phi_\rho \otimes \mathbf{1}_{\mathcal{G}}]\, (\rho \otimes \mathrm{id}_{\mathcal{G}} \otimes \mathrm{id}_{\mathcal{G}})(\phi_\rho^{-1})\Big) \\
&= (\mathrm{id}_{\mathcal{M}} \otimes \sigma)\Big((\mathrm{id}_{\mathcal{M}} \otimes \Delta \otimes \mathrm{id}_{\mathcal{G}})(\phi_\rho^{-1})\, [\mathbf{1}_{\mathcal{M}} \otimes \phi^{-1}]\, (\mathrm{id}_{\mathcal{M}} \otimes \mathrm{id}_{\mathcal{G}} \otimes \Delta)(\phi_\rho)\Big) \\
&= \mathbf{1}_{\mathcal{M}} \otimes \bar{\phi}^3 S^{-1}(\alpha \bar{\phi}^2 \beta) \bar{\phi}^1 = \mathbf{1}_{\mathcal{M}} \otimes \mathbf{1}_{\mathcal{G}},
\end{aligned}
$$

where we have used the pentagon identity (7.6), then the two antipode properties (6.13) together with $(\mathrm{id} \otimes \mathrm{id} \otimes \epsilon)(\phi_\rho) = \mathbf{1}_{\mathcal{M}} \otimes \mathbf{1}_{\mathcal{G}}$ to drop the reassociators ϕ_ρ and ϕ_ρ^{-1} and finally (6.14).

The proof of (9.35) is more complicated. First we rewrite

$$
\text{l.h.s. } (9.35) = \omega(X),
$$

where

$$X = [\phi_\rho^1 \otimes \phi_\rho^2 \otimes \mathbf{1}_\mathcal{G} \otimes \mathbf{1}_\mathcal{G} \otimes \phi_\rho^3]\,[(\rho \otimes \mathrm{id}_\mathcal{G} \otimes \mathrm{id}_\mathcal{G})(\phi_\rho) \otimes \mathbf{1}_\mathcal{G}]\,[\phi_\rho^{-1} \otimes \mathbf{1}_\mathcal{G} \otimes \mathbf{1}_\mathcal{G}] \tag{A.15}$$

and where the map $\omega : \mathcal{M} \otimes \mathcal{G}^{\otimes^4} \to \mathcal{M} \otimes \mathcal{G}^{\otimes^2}$ is given by

$$\omega(m \otimes a \otimes b \otimes c \otimes d) := m \otimes S^{-1}(\alpha d)\, a \otimes S^{-1}(\alpha c)\, b$$

To rewrite the r.h.s. of (9.35) in the same fashion we first use the identities (6.27) and (6.29) and the definition (9.23) of q_ρ to compute

$$\begin{aligned}
[\mathbf{1}_\mathcal{M} \otimes h]\,(\mathrm{id}_\mathcal{M} \otimes \Delta)(q_\rho) &= [\mathbf{1}_\mathcal{M} \otimes h]\,[\phi_\rho^1 \otimes \Delta\Big(S^{-1}(\alpha\phi_\rho^3)\phi_\rho^2\Big)] \\
&= [\phi_\rho^1 \otimes (S^{-1} \otimes S^{-1})(\Delta^{op}(\phi_\rho^3))]\,[\mathbf{1}_\mathcal{M} \otimes h]\,[\mathbf{1}_\mathcal{M} \otimes \Delta(S^{-1}(\alpha)\phi_\rho^2)] \\
&= \big[\mathbf{1}_\mathcal{M} \otimes (S^{-1} \otimes S^{-1})\big(\gamma^{op}\Delta^{op}(\phi_\rho^3)\,\big)\big]\,[\phi_\rho^1 \otimes \Delta(\phi_\rho^2)].
\end{aligned}$$

Now we use the formula (6.17) for γ implying

$$(S^{-1} \otimes S^{-1})(\gamma^{op}) = S^{-1}(\alpha\bar\phi^3\phi^3_{(2)})\phi^1 \otimes S^{-1}(\alpha\bar\phi^2\phi^3_{(1)})\bar\phi^1\phi^2$$

to obtain

$$\text{r.h.s. (9.35)} = \omega(Y),$$

where

$$\begin{aligned}
Y = [\mathbf{1}_\mathcal{M} \otimes \mathbf{1}_\mathcal{G} \otimes \phi^{-1}]\,(\mathrm{id}_\mathcal{M} \otimes \mathrm{id}_\mathcal{G} \otimes \mathrm{id}_\mathcal{G} \otimes \Delta)\big([\mathbf{1}_\mathcal{M} \otimes \phi]\,(\mathrm{id}_\mathcal{M} \otimes \Delta \otimes \mathrm{id}_\mathcal{G})(\phi_\rho)\big)& \\
((\mathrm{id} \otimes \Delta) \circ \rho \otimes \mathrm{id}_\mathcal{G} \otimes \mathrm{id}_\mathcal{G})(\phi_\rho)&
\end{aligned} \tag{A.16}$$

Using the pentagon eq. (7.6) to replace the second and third reassociator in (A.16) yields

$$\begin{aligned}
Y &= [\mathbf{1}_\mathcal{M} \otimes \mathbf{1}_\mathcal{G} \otimes \phi^{-1}]\,(\mathrm{id}_\mathcal{M} \otimes \mathrm{id}_\mathcal{G}^{\otimes^2} \otimes \Delta)\Big((\mathrm{id}^{\otimes^2} \otimes \Delta)(\phi_\rho)\,(\rho \otimes \mathrm{id}^{\otimes^2})(\phi_\rho)\,[\phi_\rho^{-1} \otimes \mathbf{1}_\mathcal{G}]\Big) \\
&\qquad ((\mathrm{id} \otimes \Delta) \circ \rho \otimes \mathrm{id}^{\otimes^2})(\phi_\rho) \\
&= (\mathrm{id}^{\otimes^2} \otimes (\Delta \otimes \mathrm{id}) \circ \Delta)(\phi_\rho)\,(\rho \otimes \mathrm{id} \otimes \mathrm{id})\Big([\mathbf{1}_\mathcal{M} \otimes \phi^{-1}]\,(\mathrm{id}^{\otimes^2} \otimes \Delta)(\phi_\rho)\,(\rho \otimes \mathrm{id}^{\otimes^2})(\phi_\rho)\Big) \\
&\qquad [\phi_\rho^{-1} \otimes \mathbf{1}_\mathcal{G} \otimes \mathbf{1}_\mathcal{G}] \\
&= (\mathrm{id}^{\otimes^2} \otimes (\Delta \otimes \mathrm{id}) \circ \Delta)(\phi_\rho)\,(\rho \otimes \Delta \otimes \mathrm{id})(\phi_\rho)\,[(\rho \otimes \mathrm{id}^{\otimes^2})(\phi_\rho) \otimes \mathbf{1}_\mathcal{G}]\,[\phi_\rho^{-1} \otimes \mathbf{1}_\mathcal{G} \otimes \mathbf{1}_\mathcal{G}]
\end{aligned} \tag{A.17}$$

where in the second equation we have used (6.8) and (7.5) to shift the reassociators ϕ^{-1} and ϕ_ρ^{-1} by one step to the right, and in the third equation again the pentagon identity (7.6). Hence, when computing $\omega(Y)$ the second factor in (A.17) may be dropped due to the antipode property (6.13) and the two coproducts in the first factor disappear by the same reason. Comparing with (A.15) proves, that $\omega(X) = \omega(Y)$ and therefore both sides of (9.35) are equal. $\square$

Proof of Lemma 9.2. We prove the identity (9.47). Introducing for $a, b \in \mathcal{G}$ the map $\nu(a \otimes b) := S^{-1}(\alpha b)\, a$ and using the formula (9.23) for q_ρ we compute

$$\begin{aligned}
(\lambda \otimes \mathrm{id})(q_\rho)\,\phi_{\lambda\rho}^{-1} &\equiv [\lambda(\phi_\rho^1) \otimes S^{-1}(\alpha\phi_\rho^3)\phi_\rho^2]\,\phi_{\lambda\rho}^{-1} \\
&= (\mathrm{id}_\mathcal{G} \otimes \mathrm{id}_\mathcal{M} \otimes \nu)\Big((\lambda \otimes \mathrm{id} \otimes \mathrm{id})(\phi_\rho)\,[\phi_{\lambda\rho}^{-1} \otimes \mathbf{1}_\mathcal{G}]\Big) \\
&= (\mathrm{id}_\mathcal{G} \otimes \mathrm{id}_\mathcal{M} \otimes \nu)\Big((\mathrm{id}^{\otimes^2} \otimes \Delta)(\phi_{\lambda\rho}^{-1})[\mathbf{1}_\mathcal{G} \otimes \phi_\rho]\,(\mathrm{id} \otimes \rho \otimes \mathrm{id})(\phi_{\lambda\rho})\Big) \\
&= [\mathbf{1}_\mathcal{G} \otimes \mathbf{1}_\mathcal{G} \otimes S^{-1}(\phi_{\lambda\rho}^3)]\,[\mathbf{1}_\mathcal{G} \otimes q_\rho]\,[\phi_{\lambda\rho}^1 \otimes \rho(\phi_{\lambda\rho}^2)].
\end{aligned}$$

Here we have plugged in the pentagon equation (8.12) and used the fact that ϕ_{λ_ρ} may be dropped due to (8.13) and the antipode property (6.13). This proves eq. (9.47) and therefore Lemma 9.2 (see remark after Lemma 9.2). $\square$

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