On the Determinant of One-Dimensional Elliptic Boundary Value Problems

Matthias Lesch and Jürgen Tolksdorf

Institut für Mathematik, Humboldt–Universität zu Berlin, Unter den Linden 6, D–10099 Berlin, Germany,
e-mail: lesch@mathematik.hu-berlin.de, tolkdorf@mathematik.hu-berlin.de,
Internet: http://spectrum.mathematik.hu-berlin.de/~lesch

August 6, 1997

Abstract

We discuss the ζ−regularized determinant of elliptic boundary value problems on a line segment. Our framework is applicable for separated and non-separated boundary conditions.

Contents

1. Introduction

In [BFK1, BFK2, BFK3], Burghelea, Friedlander and Kappeler calculated the $\zeta-$regularized determinant of elliptic differential operators on a line segment with periodic and separated boundary conditions. In [BFK2] they also discussed pseudodifferential operators over S^1, e.g. on $[0,1]$ with periodic boundary conditions. In [L2], the first named author of this paper considered the $\zeta-$regularized determinant of second order Sturm-Liouville operators with regular singularities at the boundary. The common phenomenon of [BFK1, BFK2, BFK3] and of [L2] is that the $\zeta-$regularized determinant is expressed in terms of a determinant (in the sense of linear algebra) of an endomorphism of a finite-dimensional vector space of solutions of the corresponding homogeneous differential equation.

In this paper we want to show that this phenomenon remains valid for arbitrary (e.g. non-separated, non-periodic) boundary conditions and that there exists a simple proof which works for all types of boundary conditions simultaneously. However, our result is less explicit than the results of [BFK1], [BFK3] for periodic and separated boundary conditions, respectively. The reason is that for arbitrary boundary conditions we could not prove a general deformation result for the variation of the leading coefficient. On the other hand while [BFK3] is limited to even order operators we deal with operators of arbitrary order (see also the discussion at the end of Section 3.1).

The main feature of our approach is the new proof of the variation formula, Proposition 3.1 below, which uses the explicit formulas for the resolvent kernel. This together with some general considerations about $\zeta-$regularized determinants and regularized limits (Section 2.3) easily give the main results, Theorem 3.2 and Theorem 3.3.

Since this paper may be viewed as the second part of [L2], we refer the reader to the end of Section 1 of loc. cit. for a more detailed historical discussion of $\zeta-$regularized determinants for one-dimensional operators. Nevertheless, we try to keep this paper notationally as self-contained as possible.

The paper is organized as follows: In Section 2 we introduce some notation and review the basic facts about the $\zeta-$regularized determinant of an operator. In Section 3 we state and prove our main results.

This work was supported by Deutsche Forschungsgemeinschaft.

2. Generalities

2.1 Regularized integrals

First we briefly recall regularized limits and integrals (c.f. [L1, (1.8)-(1.13c)]). Let $f:\mathbb{R}\to\mathbb{C}$ be a function having an asymptotic expansion

$$f(x) \sim_{x\to 0+} \sum_{\operatorname{Re}\alpha\le 0} x^\alpha P_\alpha(\log x) + o(1), \tag{2.1}$$

where $P_\alpha\in\mathbb{C}[t]$ are polynomials and $P_\alpha=0$ for all but finitely many α. Then we put

$$\operatorname*{LIM}_{x\to 0} f(x) := P_0(0). \tag{2.2}$$

If f has an expansion like (2.1) as $x\to\infty$ then $\mathrm{LIM}_{x\to\infty}$ is defined likewise.

Next let $f : \mathbb{R} \to \mathbb{C}$ such that

$$\begin{aligned} f(x) &= \sum_\alpha x^\alpha \, P_\alpha(\log x) + f_1(x), \\ &= \sum_\beta x^\beta \, Q_\beta(\log x) + f_2(x), \end{aligned} \tag{2.3}$$

with $P_\alpha, Q_\beta \in \mathbb{C}[t], f_1 \in \mathrm{L}^1_{loc}([0,\infty)), f_2 \in \mathrm{L}^1([1,\infty))$. Then, we define

$$⨍_0^\infty f(x)\,dx := \operatorname*{LIM}_{a\to 0} \int_a^1 f(x)\,dx + \operatorname*{LIM}_{b\to\infty} \int_1^b f(x)\,dx. \tag{2.4}$$

We note that for a slightly more restricted class of functions, this regularized integral can also be defined by the Mellin transform ([BS], [L1, Sec. 2.1], [L2, (1.12)]). Note that

$$⨍_0^\infty x^\alpha \log^k x \, dx = 0 \tag{2.5}$$

for $\alpha \in \mathbb{C}, k \in \mathbb{Z}_+$ (cf. [L2, (1.13 a-c)]).

2.2 Boundary value problems on a line segment

We consider a linear differential operator

$$l := \sum_{k=0}^n a_k(x)\, D^k, \qquad D := -i\frac{d}{dx} \tag{2.6}$$

defined on the bounded interval $I := [a,b]$ with matrix coefficients $a_k \in C^\infty(I, M(m,\mathbb{C}))$. We assume (2.6) to be elliptic, i.e. $\det a_n(x) \neq 0, x \in I$. A priori, the differential operator l acts on $C^\infty(I,\mathbb{C}^m)$. We consider the following boundary condition:

$$\mathcal{B}(f) := R_a \begin{pmatrix} f(a) \\ f'(a) \\ \vdots \\ f^{(n-1)}(a) \end{pmatrix} + R_b \begin{pmatrix} f(b) \\ f'(b) \\ \vdots \\ f^{(n-1)}(b) \end{pmatrix} = 0, \tag{2.7}$$

where $R_a, R_b \in M(nm,\mathbb{C})$ are matrices of size $nm \times nm$.

We denote by $L := l_\mathcal{B}$ the differential operator (2.6), restricted to the domain

$$\mathcal{D}(L) := \{f \in H^n(I,\mathbb{C}^m) \mid \mathcal{B}(f) = 0\}. \tag{2.8}$$

Let $\phi : I \to M(nm,\mathbb{C})$ be the fundamental matrix of l, which means that ϕ is the solution of the initial value problem

$$\begin{aligned} \phi'(x) &= A\,\phi(x), \\ \phi(a) &= \mathbf{1}_{nm}. \end{aligned} \tag{2.9}$$

Here, $A \in C^\infty(I, M(nm,\mathbb{C}))$ is the matrix

$$A := \begin{pmatrix} 0 & \mathbf{1}_m & 0 & \dots & 0 \\ 0 & 0 & \mathbf{1}_m & \dots & 0 \\ \vdots & \vdots & \ddots & & \vdots \\ 0 & 0 & 0 & \dots & \mathbf{1}_m \\ \beta_0 & \beta_1 & \beta_2 & \dots & \beta_{n-1} \end{pmatrix}, \tag{2.10}$$

where, respectively, $\beta_k \equiv -\alpha_n^{-1}\,\alpha_k := -(-i)^{-n}\,a_n^{-1}\,(-i)^k\,a_k,\ \ k = 0,\cdots,n-1$, i.e. $\alpha_k := (-i)^k\,a_k,\ k = 0,\cdots,n$ and $\mathbf{1}_m \in M(m,\mathbb{C})$ denotes the $m\times m$ unit-matrix.

Sometimes we also write $\phi(x;l)$ to make the dependence on the operator (2.6) explicit. We introduce the matrices

$$\mathcal{R} := \mathcal{R}(l, R_a, R_b) := R_a + R_b\,\phi(b;l), \tag{2.11}$$

$$\mathcal{R}(z) := \mathcal{R}(l+z, R_a, R_b) := R_a + R_b\,\phi(b;l+z). \tag{2.12}$$

It is a well-known fact that the operator L is invertible if and only if the matrix $\mathcal{R}$ is invertible. In this case the inverse operator L^{-1} is a trace class operator with kernel

$$K(x,y) = \begin{cases} -\left[\phi(x)\,\mathcal{R}^{-1}\,R_b\,\phi(b)\,\phi^{-1}(y)\right]_{1n}\,\alpha_n(y)^{-1} & \text{if} \quad y > x, \\ -\left[\phi(x)\,(\mathcal{R}^{-1}\,R_b\,\phi(b) - \mathbf{1})\,\phi^{-1}(y)\right]_{1n}\,\alpha_n(y)^{-1} & \text{if} \quad y < x. \end{cases} \tag{2.13}$$

Here, $[\quad]_{1n}$ means the upper right entry of a $n\times n$ block matrix. Note that $K(x,y) \in \mathrm{M}(m,\mathbb{C})$.

2.3 The $\zeta-$regularized determinant

We briefly discuss $\zeta-$regularized determinants in an abstract setting. Let $\mathcal{H}$ be a Hilbert space and let L be an (unbounded) operator in $\mathcal{H}$.

For $\alpha < \beta$ we denote by

$$C_{\alpha,\beta} := \{z \in \mathbb{C}\setminus\{0\} \mid \alpha \le \arg z \le \beta\} \tag{2.14}$$

a sector in the complex plane. We assume that the operator L has θ as a principal angle. By this we mean that there exists an $\epsilon > 0$ such that

$$\operatorname{spec} L \cap C_{\theta-\epsilon,\theta+\epsilon} = \emptyset. \tag{2.15}$$

Furthermore, we assume that

$$\|(L-z)^{-1}\|_{\mathcal{L}(\mathcal{H})} \le c\,|z|^{-1}, \quad z \in C_{\theta-\epsilon,\theta+\epsilon}, \quad |z| \ge \mathrm{R}, \tag{2.16}$$

where $(L-z)^{-1}$ is trace class and there is an asymptotic expansion in $C_{\theta-\epsilon,\theta+\epsilon}$ as $z\to\infty$

$$\operatorname{Tr}(L-z)^{-1} \sim_{z\to\infty} \sum_{\operatorname{Re}\alpha \ge -1-\delta} z^\alpha\,P_\alpha(\log z) + o(|z|^{-1-\delta}), \tag{2.17}$$

where, again, $P_\alpha \in \mathbb{C}[t]$ are polynomials and $P_\alpha \ne 0$ for at most finitely many α.

Moreover, we assume that

$$\deg P_{-1} = 0, \tag{2.18}$$

i.e., there are no terms like $z^{-1}\log^k(z), k \ge 1$.

The trace class property of $(L-z)^{-1}$ implies that

$$\lim_{\substack{z\to\infty \\ z\in C_{\theta-\delta,\theta+\delta}}} \operatorname{Tr}(L-z)^{-1} = 0 \tag{2.19}$$

for any $\delta < \epsilon$. Thus, $P_\alpha = 0$ if $\operatorname{Re}\alpha \ge 0$.

In view of (2.16) we can construct the complex powers of the operator L as follows (cf. [Se2, Sec. 1], [Sh, Sec. 10.1]): let $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$ be the contour in $\mathbb{C}$ with

$$\begin{aligned} \Gamma_1 &:= \{r\, e^{i(\theta+2\pi)} \mid \rho < r < \infty\}, \\ \Gamma_2 &:= \{\rho\, e^{i(\theta+\varphi)} \mid 0 < \varphi < 2\pi\}, \\ \Gamma_3 &:= \{r\, e^{i\theta} \mid \rho < r < \infty\}. \end{aligned} \tag{2.20}$$

Here, the contour Γ is traversed such that the set $\mathbb{C}\backslash\{r\, e^{i\theta} \mid r > \rho\}$ lies "inside" Γ. Moreover, ρ is chosen so small that $\operatorname{spec} L \cap \{z \in \mathbb{C} \mid |z| \le \rho\} \subset \{0\}$.

Then, put for $\operatorname{Re} z < 0$

$$L_z := \frac{i}{2\pi} \int_\Gamma \lambda^z\, (L-\lambda)^{-1}\, d\lambda. \tag{2.21}$$

Here, the complex powers λ^z are defined by $(r\, e^{i(\theta+\varphi)})^z := r^z\, e^{iz(\theta+\varphi)}, 0 \le \varphi \le 2\pi$. The same proof as in [Se1, Thm.1] (cf. also [Sh, Prop. 10.1]) shows that $z \mapsto L_z$ is a holomorphic semigroup of bounded operators in the Hilbert space $\mathcal{H}$.

For $k \in \mathbb{Z}, k < 0$ we have $L_k = (L_{-1})^k$. Moreover, if $0 \notin \operatorname{spec} L$ then $L_{-1} = L^{-1}$. If $0 \in \operatorname{spec} L$, then LL_{-1} is a projection onto a complementary subspace of $\ker L$. Therefore, we shall write L^z instead of L_z.

By assumption $(L-z)^{-1}$ is trace class and in view of (2.16) we can estimate the trace norm

$$\begin{aligned} \|(L-z)^{-1}\|_{\mathrm{tr}} &\le \|(L-z_0)^{-1}\|_{\mathrm{tr}}\, \|(L-z_0)(L-z)^{-1}\| \\ &\le \|(L-z_0)^{-1}\|_{\mathrm{tr}}\, (1 + |z-z_0|\, \|(L-z)^{-1}\|) \\ &\le C, \quad |z| \ge \mathrm{R}. \end{aligned} \tag{2.22}$$

Therefore, if $\operatorname{Re} z < -1$ the integral (2.21) converges in the trace norm and the ζ−function of L with respect to the principal angle θ

$$\begin{aligned} \zeta_{L,\theta}(s) &:= \operatorname{Tr}(L^{-s}) = \sum_{\lambda \in \operatorname{spec}(L)\backslash\{0\}} \lambda^{-s} \\ &= \frac{i}{2\pi} \int_\Gamma z^{-s} \operatorname{Tr}(L-z)^{-1}\, dz \end{aligned} \tag{2.23}$$

is a holomorphic function for $\operatorname{Re} s > 1$.

Furthermore, the asymptotic expansion (2.17) implies that $\zeta_{L,\theta}(s)$ has a meromorphic continuation to $\operatorname{Re} s > -\delta$ with poles in the set $\{\alpha + 1 \mid P_\alpha \ne 0\}$. The order of the pole $\alpha + 1$ is either $\deg P_\alpha$ if $\alpha + 1 \in \mathbb{Z}$ or $\deg P_\alpha + 1$ if $\alpha + 1 \notin \mathbb{Z}$ (see for instance [BL, Lemma 2.1]). Because of the assumption (2.18) $\zeta_{L,\theta}(s)$ is regular at 0.

Following Ray and Singer, [RS], we put $\det_\theta L = 0$ if $0 \in \operatorname{spec} L$, and otherwise

$$\det{}_\theta L := \exp(-\zeta'_{L,\theta}(0)). \tag{2.24}$$

It is convenient to deal with the principal angle $\theta = \pi$. We therefore consider the operator

$$\widetilde{L} := e^{i(\pi-\theta)}\, L. \tag{2.25}$$

Obviously, this operator has $\theta = \pi$ as a principal angle and it satisfies (2.16)-(2.18), too. Furthermore,

$$\widetilde{L}^{-s} = e^{is(\theta-\pi)}\, L^{-s}, \tag{2.26}$$

and thus

$$\zeta_{\widetilde{L},\pi}(s) = e^{is(\theta-\pi)}\, \zeta_{L,\theta}(s). \tag{2.27}$$

Consequently,

$$\begin{aligned} \zeta_{L,\theta}(0) &= \zeta_{\widetilde{L},\pi}(0), \\ \zeta'_{L,\theta}(0) &= \zeta'_{\widetilde{L},\pi}(0) + i(\pi-\theta)\,\zeta_{\widetilde{L},\pi}(0) \end{aligned} \tag{2.28}$$

and therefore

$$\det{}_\theta L = e^{i(\theta-\pi)\,\zeta_{\widetilde{L},\pi}(0)}\, \det{}_\pi \widetilde{L}. \tag{2.29}$$

In the sequel we thus assume $\theta = \pi$. We then write the expansion (2.17) in the form

$$\operatorname{Tr}(L+x)^{-1} \sim_{x\to\infty} \sum_{\operatorname{Re}\alpha \geq -1-\delta} x^{\alpha}\, P_\alpha(\log x) + o(x^{-1-\delta}), \quad x \geq 0. \tag{2.17'}$$

Of course, there exist formulas relating the P_α in (2.17) and the corresponding P_α in (2.17').

Lemma 2.1 *Let the operator L be given as above with principal angle $\theta = \pi$. Then,*

$$\zeta_{L,\pi}(s) = \frac{\sin \pi s}{\pi} \,⨍_0^\infty x^{-s}\operatorname{Tr}(L+x)^{-1}\,dx, \tag{2.30}$$

$$\zeta'_{L,\pi}(0) = ⨍_0^\infty \operatorname{Tr}(L+x)^{-1}\,dx\,. \tag{2.31}$$

Proof With respect to the decomposition $\mathcal{H} = (\ker L) \oplus (\ker L)^\perp$, the operator L reads

$$L = \begin{pmatrix} 0 & T \\ 0 & L_1 \end{pmatrix}, \tag{2.32}$$

where L_1 is invertible. In view of (2.5) we have

$$⨍_0^\infty x^{-s}\operatorname{Tr}(L+x)^{-1}\,dx = ⨍_0^\infty x^{-s}\operatorname{Tr}(L_1+x)^{-1}\,dx \tag{2.33}$$

and thus we may assume L to be invertible.

From the estimate (2.22) we conclude that the following integral is absolutely convergent for $1 < \operatorname{Re} s < 2$:

$$\begin{aligned} \int_0^\infty x^{-s}\,[\operatorname{Tr}(L+x)^{-1} - \operatorname{Tr}(L^{-1})]\,dx &= \sum_{\lambda\in\operatorname{spec}(L)\setminus\{0\}} \int_0^\infty x^{-s}\,[(\lambda+x)^{-1} - \lambda^{-1}]\,dx \\ &= \sum_{\lambda\in\operatorname{spec}(L)\setminus\{0\}} ⨍_0^\infty x^{-s}\,(\lambda+x)^{-1}\,dx \\ &= \frac{\pi}{\sin \pi s} \sum_{\lambda\in\operatorname{spec}(L)\setminus\{0\}} \lambda^{-s}. \end{aligned} \tag{2.34}$$

Here, we have used (2.5) again. Hence, the first formula is proved.

Since (2.17'), (2.18) and [L1, (1.12)] imply

$$\int_0^\infty x^{-s}\,\mathrm{Tr}(L+x)^{-1}\,dx = \frac{P_{-1}(0)}{s} + \int_0^\infty \mathrm{Tr}(L+x)^{-1}\,dx + O(s), \quad s\to 0 \tag{2.35}$$

we reach the conclusion by noting that $\frac{\sin \pi s}{\pi} = s + O(s^3), s \to 0$. □

Lemma 2.2 *Let L be as before, $\theta = \pi$. Then, we have the asymptotic expansion*

$$\log \det_\pi(L+x) \sim_{x\to\infty} \sum_{\mathrm{Re}\,\alpha \geq -1-\delta} x^{\alpha+1}\,Q_\alpha(\log x) + O(x^{-\delta}), \tag{2.36}$$

where $P_\alpha = (\alpha+1)\,Q_\alpha + Q'_\alpha$. Furthermore, $Q_{-1}(\log x) = P_{-1}(0)\log x$. In particular

$$\underset{x\to\infty}{\mathrm{LIM}} \log \det_\pi(L+x) = 0. \tag{2.37}$$

Proof Since L^{-1} is trace class, it follows that $\log \det_\pi(L+x)$ is differentiable and

$$\frac{d}{dx}\log \det_\pi(L+x) = \mathrm{Tr}(L+x)^{-1}. \tag{2.38}$$

Hence,

$$\underset{y\to\infty}{\mathrm{LIM}} \log \det_\pi(L+y) - \log \det_\pi(L+x) = \int_x^\infty \mathrm{Tr}(L+y)^{-1}\,dy. \tag{2.39}$$

Comparing this equation for $x = 0$ with the preceding lemma yields (2.37). Hence,

$$\begin{aligned} \log \det_\pi(L+x) \quad &= \quad -\int_x^\infty \mathrm{Tr}(L+y)^{-1}\,dy \\ &\sim_{x\to\infty} \sum_{\mathrm{Re}\,\alpha \geq -1-\delta} -\int_x^\infty y^\alpha\,P_\alpha(\log y)\,dy + O(x^{-\delta}) \end{aligned} \tag{2.40}$$

and we reach the conclusion. □

The reader might ask why we argued so complicated in order to get the first equality of (2.40). It appears to be a direct consequence of (2.31) via the apparently "trivial" calculation

$$\begin{aligned} \log \det_\pi(L+x) &= -\int_0^\infty \mathrm{Tr}(L+x+y)^{-1}\,dy \\ &= -\int_x^\infty \mathrm{Tr}(L+y)^{-1}\,dy. \end{aligned} \tag{2.41}$$

However, in general for functions f like (2.3) we have

$$\int_0^\infty f(x+y)\,dy \neq \int_x^\infty f(y)\,dy. \tag{2.42}$$

Consequently, some care must be in order. Since the operator L^{-1} is trace class, the phenomenon (2.42) does not occur for $\operatorname{Tr}(L+x)^{-1}$. More precisely, if $f \in L^1_{loc}([0,\infty))$ and

$$f(x) \sim_{x\to\infty} \sum_\alpha x^\alpha P_\alpha(\log x), \tag{2.43}$$

then

$$⨍_0^\infty f(x+y)\,dy - ⨍_x^\infty f(y)\,dy = \operatorname*{LIM}_{b\to\infty} \int_b^{b+x} f(y)\,dy \tag{2.44}$$

and in general this vanishes only if $P_\alpha = 0$ for $\alpha \in \mathbb{Z}_+$. As an illustrative example we consider $f(x) := x^\alpha, \alpha \in \mathbb{Z}$. Then, we get

$$⨍_0^\infty (x+y)^\alpha\,dy = \begin{cases} -\frac{x^{\alpha+1}}{\alpha+1} & \text{if} \quad \alpha \in \mathbb{Z}_-\setminus\{-1\}, \\ -\ln x & \text{if} \quad \alpha = -1, \\ 0 & \text{if} \quad \alpha \in \mathbb{Z}_+; \end{cases} \tag{2.45}$$

$$⨍_x^\infty f(y)\,dy = \begin{cases} -\frac{x^{\alpha+1}}{\alpha+1} & \text{if} \quad \alpha \neq -1, \\ -\ln x & \text{if} \quad \alpha = -1. \end{cases} \tag{2.46}$$

Hence,

$$⨍_0^\infty f(x+y)\,dy - ⨍_x^\infty f(y)\,dy = \begin{cases} \frac{x^{\alpha+1}}{\alpha+1} & \text{if} \quad \alpha \in \mathbb{Z}_+, \\ 0 & \text{if} \quad \alpha \notin \mathbb{Z}_+. \end{cases} \tag{2.47}$$

3. Main results

From now on we restrict ourselves to boundary value problems on a line segment as introduced in Sec. 2.2. Let $(l,\mathcal{B})$ be an elliptic boundary value problem, $L := l_{\mathcal{B}}$. More precisely, we assume that $(l,\mathcal{B})$ is elliptic in the sense of [Se1, Def.1] and that it satisfies Agmon's condition [Se1, Def.2]. Agmon's condition assures that the coefficient $a_n(x)$ has a certain principal angle, θ. Then we can find an angle θ', arbitrary close to θ, such that θ' is a principal angle for L and $a_n(x)$. Henceforth we shall write θ for a common principal angle of $a_n(x)$ and L. In short: we will refer to an operator $L = l_{\mathcal{B}}$, defining an elliptic boundary value problem $(l,\mathcal{B})$, as an *admissible* operator.

3.1 Operators of order ≥ 2

If in addition $n \geq 2$ then the conditions (2.16)-(2.18) are fulfilled by the work of Seeley [Se1, Se2]. Namely, (2.16) follows from [Se1, Lemma 15] and by [Se2, Thm.2] we have an asymptotic expansion as $z \to \infty$ in $C_{\theta-\epsilon,\theta+\epsilon}$

$$\operatorname{Tr}(L-z)^{-1} \sim_{z\to\infty} \sum_{k=0}^\infty a_k\, z^{\frac{1-k}{n}-1}. \tag{3.1}$$

(2.18) is automatically fulfilled since there are no log-terms in (3.1).

Summing up, we see that $\det_\theta L$ is well-defined for $n \geq 2$. First order operators are slightly more complicated since in this case $(L-z)^{-1}$ is not of trace class. This problem will be treated separately in subsection 3.2.

First, we study the behavior of $\det_\theta L$ under deformations of the coefficients of l.

Proposition 3.1 *Assume that the coefficients $a_0, \ldots, a_{n-2}$ depend smoothly on a parameter t. Let L_t be the corresponding family of operators. If L_t is invertible then we have*

$$\partial_t \log\det_\theta L_t = \partial_t \log\det \mathcal{R}_t \tag{3.2}$$

where $\mathcal{R}_t := \mathcal{R}(l_t, R_a, R_b)$, cf. (2.11).

Proof The inclusion $H^2([a,b],\mathbb{C}^m) \hookrightarrow L^2([a,b],\mathbb{C}^m)$ is trace class and $\mathcal{D}(L) \subset H^n([a,b],\mathbb{C}^m)$. Hence the operators $D^k L^{-1}$ are trace class, as well, for $k = 0, \ldots, n-2$. Hence,

$$\begin{aligned}
\partial_t \log\det_\theta L_t &= \operatorname{Tr}((\partial_t L_t) L_t^{-1}) \\
&= \sum_{j=0}^{n-2} \int_a^b \operatorname{tr}_{\mathbb{C}^m} \Big(\partial_t a_j(t;x)\,(D^j K_t)(x,x)\Big)\, dx \\
&= \sum_{j=0}^{n-2} \int_a^b \operatorname{tr}_{\mathbb{C}^m} \Big(\partial_t \alpha_j(t;x)\, K_t^{(j)}(x,x)\Big)\, dx \\
&= \sum_{j=0}^{n-2} \int_a^b \operatorname{tr}_{\mathbb{C}^m} \Big(-\alpha_n^{-1}(x)\partial_t\alpha_j(t;x)\,[\widetilde{K}_t^{(j)}(x,x)]_{1n}\Big)\, dx \\
&= \sum_{j=0}^{n-2} \int_a^b \operatorname{tr}_{\mathbb{C}^m} \Big(\partial_t \beta_j(t;x)\,[\widetilde{K}_t^{(j)}(x,x)]_{1n}\Big)\, dx,
\end{aligned} \tag{3.3}$$

where

$$\widetilde{K}_t(x,y) := \begin{cases} \phi_t(x)\,\mathcal{R}_t^{-1}\,R_b\,\phi_t(b)\,\phi_t^{-1}(y) & \text{if} \quad y > x, \\ \phi_t(x)\,(\mathcal{R}_t^{-1}\,R_b\,\phi_t(b) - \mathbf{1})\,\phi_t^{-1}(y) & \text{if} \quad y < x. \end{cases} \tag{3.4}$$

Note that $\widetilde{K}_t^{(j)}$ is continuous on the diagonal for $j = 0, \ldots, n-2$, but $\widetilde{K}_t^{(n-1)}$ has a jump. This is one of the reasons that this proposition is limited to the case of constant a_{n-1}. Thus we have

$$\begin{aligned}
\partial_t \log\det_\theta L_t &= \operatorname{Tr}((\partial_t L_t)\, L_t^{-1}) \\
&= \int_a^b \operatorname{tr}_{\mathbb{C}^{nm}} \Big((\partial_t A_t)(x)\,\widetilde{K}_t(x,x)\Big)\, dx.
\end{aligned} \tag{3.5}$$

By use of (2.9) we then calculate:

$$\begin{aligned}
\operatorname{tr}_{\mathbb{C}^{nm}} \Big((\partial_t A_t)(x)\,\widetilde{K}_t(x,x)\Big) &= \operatorname{tr}_{\mathbb{C}^{nm}} \Big[(\partial_t A_t)(x)\,\phi_t(x)\,\mathcal{R}_t^{-1}\,R_b\,\phi_t(b)\,\phi_t^{-1}(x)\Big] \\
&= \operatorname{tr}_{\mathbb{C}^{nm}} \Big[(\partial_x\partial_t\phi_t)(x)\,\mathcal{R}_t^{-1}\,R_b\,\phi_t(b)\,\phi_t^{-1}(x)\Big] \\
&\quad - \Big[(\partial_x\phi_t)(x)\,\phi_t^{-1}(x)\,(\partial_t\phi_t)(x)\,\mathcal{R}_t^{-1}\,R_b\,\phi_t(b)\,\phi_t^{-1}(x)\Big] \\
&= \partial_x \operatorname{tr}_{\mathbb{C}^{nm}} \Big[(\partial_t\phi_t)(x)\,\mathcal{R}_t^{-1}\,R_b\,\phi_t(b)\,\phi_t^{-1}(x)\Big]
\end{aligned} \tag{3.6}$$

to obtain

$$\begin{aligned}\partial_t \log\det{}_\theta L_t &= \operatorname{tr}_{\mathbb{C}^{nm}}\left[(\partial_t\phi_t)(b)\,\mathcal{R}_t^{-1}\,R_b\right]\\ &= \partial_t \log\det\mathcal{R}_t, \end{aligned}\tag{3.7}$$

which proves the statement. □

Theorem 3.2 *Let $(l,\mathcal{B})$ be an admissible operator of order $n\geq 2$, $L:=l_{\mathcal{B}}$. Assume the principal angle θ equals π. For $\mathcal{R}(z):=\mathcal{R}(l+z,\mathcal{B})$ we obtain an asymptotic expansion*

$$\log\det\mathcal{R}(z)\sim_{z\to\infty}\sum_{\substack{k=0\\k\neq 1}}^{\infty} b_k\,z^{\frac{1-k}{n}}+b_1+\zeta_{L,\pi}(0)\log z \tag{3.8}$$

in a conic neighborhood of $\mathbb{R}_+$. Furthermore,

$$\log\det{}_\pi\,(L+z)=\log\det\mathcal{R}(z)-\operatorname*{LIM}_{w\to\infty}\log\det\mathcal{R}(w). \tag{3.9}$$

Proof In view of (3.1) we have an asymptotic expansion

$$\operatorname{Tr}(L+z)^{-1}\sim_{z\to\infty}\sum_{k=0}^{\infty}a_k\,z^{\frac{1-k}{n}-1}. \tag{3.10}$$

We apply the preceding proposition with $a_0(z;x)=a_0(x)+z$ and $a_k(z;x)=a_k(x)$, $k\geq 1$. Then

$$\begin{aligned}\partial_z\log\det\mathcal{R}(z) &= \partial_z\log\det{}_\pi\,(L+z)\\ &= \operatorname{Tr}(L+z)^{-1}\\ &\sim_{z\to\infty} \sum_{k=0}^{\infty}a_k\,z^{\frac{1-k}{n}-1}. \end{aligned}\tag{3.11}$$

This proves the first assertion. Note that from Lemma 2.1 one easily concludes $a_1=\zeta_{L,\pi}(0)$. By (3.11) $\log\det\mathcal{R}(z)-\log\det_\pi(L+z)$ is a constant. Then the second assertion follows from (2.37). □

Theorem 3.3 *Let again $(l,\mathcal{B})$ be an admissible operator of order $n\geq 2$, $L:=l_{\mathcal{B}}$, with principal angle $\theta=\pi$. We put*

$$\log\mathcal{C}(l,\mathcal{B}):=-\operatorname*{LIM}_{z\to\infty}\log\det\mathcal{R}(z). \tag{3.12}$$

Then,

$$\det{}_\pi L=\mathcal{C}(l,\mathcal{B})\det\mathcal{R},\quad \mathcal{R}:=\mathcal{R}(0). \tag{3.13}$$

Furthermore, $\mathcal{C}(l,\mathcal{B})$ depends only on a_n, a_{n-1} and the boundary operator $\mathcal{B}$, i.e. $\mathcal{C}(l,\mathcal{B})=\mathcal{C}_1(a_n,a_{n-1},R_a,R_b)$.

Proof (3.12) and (3.13) are immediate consequences of the preceding Theorem. To prove the last statement we consider two admissible operators

$$l_j := \sum_{k=0}^{n} a_{k,j}(x)\, D^k, \quad j = 0, 1\,, \tag{3.14}$$

where $a_{n,0} = a_{n,1}$, $a_{n-1,0} = a_{n-1,1}$ and the boundary condition $\mathcal{B}$ is fixed.

We put

$$l_t := t\, l_1 + (1-t)\, l_0, \quad 0 \le t \le 1.$$

We would like to apply Proposition 3.1. However, it may happen that $\operatorname{spec} l_t \cap \{z \in \mathbb{C} \mid z \le 0\} \ne \emptyset$ for some t. But since π is a principal angle for the leading symbol of l_t there exists a $z_0 > 0$ such that $L_t + z$ is invertible for all $z \ge z_0$.

By Proposition 3.1 we then have $\mathcal{C}(l_0 + z, \mathcal{B}) = \mathcal{C}(l_1 + z, \mathcal{B})$ for $z > z_0$. Since both functions are holomorphic we are done. □

Note that formulas (3.12) and (3.13) express the $\zeta-$regularized determinant of L completely in terms of the solutions of the homogeneous differential equation $(L+z)u = 0$. It seems impossible, however, to find an explicit formula for the coefficient $\mathcal{C}(l, \mathcal{B})$ in full generality. But in cases where the fundamental matrix $\mathcal{R}(z)$ can be calculated explicitly one can also find an expression for $\mathcal{C}(l, \mathcal{B})$.

Now we are going to discuss in detail non-separated boundary conditions for second order operators. We therefore consider the following

Example: Let $A, B, C, D \in M(m, \mathbb{C})$ and consider the operator

$$l := -\frac{d^2}{dx^2} + q(x)$$

with boundary operator $\mathcal{B} = (R_a, R_b)$, where

$$R_a := \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \quad R_b := \mathbf{1}_{2m}.$$

It turns out that the operator $L = l_{\mathcal{B}}$ is admissible iff the meromorphic function

$$M(z) := \det\left(-z\frac{B}{2} + \frac{A+D}{2} - \frac{1}{z}\frac{C}{2}\right) \tag{3.15}$$

does not vanish identically. Hence, let us assume $M(z) \ne 0$. Note that M is a Laurent polynomial.

By the preceding proposition $\mathcal{C}(l, \mathcal{B}) =: \mathcal{C}_2(A, B, C, D)$ is independent of q. More precisely,

Proposition 3.4 *$\mathcal{C}(l, \mathcal{B})^{-1}$ is equal to the leading coefficient of the Laurent polynomial $M(z)$; $\zeta_{L,\pi}(0)$ equals $\frac{1}{2}$ the degree of $M(z)$.*

Proof As remarked before it suffices to consider the case $q = 0$. Then the fundamental solution $\phi(x, z) := \phi(x, l + z)$ reads

$$\phi(x, z) = \begin{pmatrix} \cosh[(x-a)\sqrt{z}]\, \mathbf{1}_m & \dfrac{\sinh[(x-a)\sqrt{z}]}{\sqrt{z}}\, \mathbf{1}_m \\ \sqrt{z}\, \sinh[(x-a)\sqrt{z}]\, \mathbf{1}_m & \cosh[(x-a)\sqrt{z}]\, \mathbf{1}_m \end{pmatrix}, \tag{3.16}$$

where, again, $\mathbf{1}_m$ denotes the $m \times m$ unit matrix.

For the rest of the proof all matrices will be 2×2 block matrices with $m \times m$ block entries and for simplicity we will omit $\mathbf{1}_m$. Abbreviating $c := b - a, w := \sqrt{z}$ we find

$$\begin{aligned} \phi(b,z) &= \begin{pmatrix} \cosh cw & \frac{\sinh cw}{w} \\ w \sinh cw & \cosh cw \end{pmatrix}, \\ &= W \begin{pmatrix} e^{cw} & 0 \\ 0 & e^{-cw} \end{pmatrix} W^{-1}, \end{aligned}$$

where

$$W = \begin{pmatrix} 1 & 1 \\ w & -w \end{pmatrix}.$$

Thus,

$$\begin{aligned} \det \mathcal{R}(z) &= \det(R_a + R_b \phi(b,z)) && (3.17) \\ &= \det(R_a + W \begin{pmatrix} e^{cw} & 0 \\ 0 & e^{-cw} \end{pmatrix} W^{-1}) && (3.18) \\ &= \det(W^{-1} R_a W + \begin{pmatrix} e^{cw} & 0 \\ 0 & e^{-cw} \end{pmatrix}) && (3.19) \\ &= e^{mcw} \det\nolimits_{\mathbb{C}^m}[W^{-1} R_a W]_{22} + O(w^{m+1} e^{(m-1)cw}), \end{aligned}$$

since

$$W^{-1} R_a W = w X_1 + X_0 + w^{-1} X_{-1}, \quad X_i \in M(2m, \mathbb{C}),\ i = -1, 0, 1.$$

Here,

$$[W^{-1} R_a W]_{22} = -\frac{w}{2} B + \frac{A+D}{2} - \frac{1}{2w} C$$

denotes the lower right entry of the 2×2 block matrix $W^{-1} R_a W$.

This implies

$$\begin{aligned} \log \det \mathcal{R}(z) &= mc\sqrt{z} + \log\Big[\det\nolimits_{\mathbb{C}^m}[W^{-1} R_a W]_{22} + O(z^{\frac{m+1}{2}} e^{-c\sqrt{z}})\Big] \\ &= mc\sqrt{z} + \log M(\sqrt{z}) + O\left(\frac{z^{\frac{m+1}{2}} e^{-c\sqrt{z}}}{M(\sqrt{z})}\right). \end{aligned}$$

Since $M(w)$ is a Laurent polynomial we may write

$$M(\sqrt{z}) = \lambda z^{k/2} + O(z^{\frac{k-1}{2}}), \quad z \to \infty$$

and thus

$$\log M(\sqrt{z}) = \frac{k}{2} \log z + \log \lambda + O(z^{-\frac{1}{2}}) \quad z \to \infty$$

and we reach the conclusion. □

The leading coefficient of $M(z)$ is in general difficult to describe. Thus, it seems hard to find a more explicit formula for $\mathcal{C}(l,\mathcal{B})$ than given in the preceding Proposition.

We discuss some special cases:

1. B invertible:

$$\begin{aligned} M(z) &= (-1)^m (\det \frac{B}{2})\, z^m \det(\mathbf{1}_m + O(z^{-1})) \\ &= (-1)^m (\det \frac{B}{2})\, z^m + O(z^{m-1})); \end{aligned}$$

2. $B = 0,\ A + D$ invertible:

$$M(z) = \det(\frac{A+D}{2}) + O(z^{-1});$$

3. $B = 0, A + D = 0$:

$$M(z) = \frac{(-1)^m}{2^m} \det C\, z^{-m}.$$

Hence,

$$\mathcal{C}_2(A,B,C,D) = \begin{cases} \dfrac{(-1)^m\, 2^m}{\det B} & \text{if} \quad \det B \neq 0, \\ \dfrac{2^m}{\det(A+D)} & \text{if} \quad B = 0,\ \det(A+D) \neq 0, \\ \dfrac{(-1)^m\, 2^m}{\det C} & \text{if} \quad B = 0,\ A + D = 0. \end{cases} \tag{3.20}$$

Of course, this does not cover all possible cases. The periodic boundary conditions are given by $R_a = -\mathbf{1}_{2m}$ and thus

$$\mathcal{C}_2(A,B,C,D) = (-1)^m,$$

which is consistent with [BFK1, Thm.1].

Next, we discuss how $\mathcal{C}(a_n, a_{n-1}, \mathcal{B})$ depends on the coefficients a_n and a_{n-1}. We start with the dependence on the subleading coefficient a_{n-1} and use the standard trick to eliminate it (cf. also [BFK3, Prop.2.2]).

For this let again $L = l_{\mathcal{B}}$ be an admissible operator, $\mathcal{B} = (R_a, R_b)$. Let also $U : I \to M(m, \mathbb{C})$ be the unique solution of the initial value problem

$$\begin{aligned} U'(x) &= -\frac{i}{n}\,(a_n^{-1} a_{n-1})(x)\, U(x) \\ U(a) &= \mathbf{1}_m\,. \end{aligned} \tag{3.21}$$

The determinant of $U(x)$ is given by

$$\det U(x) = \exp\Big(-\frac{i}{n}\int_a^x \operatorname{tr}(a_n^{-1}\, a_{n-1})(y)\, dy\Big) \neq 0. \tag{3.22}$$

By conjungation of l with U we find

$$l^u := U^{-1}\, l\, U = \sum_{j=0}^{n} \tilde{a}_j(x)\, D^j \tag{3.23}$$

with $\tilde{a}_n(x) = (U^{-1}a_n U)(x)$ and $\tilde{a}_{n-1} = 0$. Since $\operatorname{spec}\tilde{a}_n = \operatorname{spec} a_n$, $\tilde{a}_n$ has the same principal angle as a_n. Furthermore, for $L^u := U^{-1}LU$ we have

$$\operatorname{spec} L^u = \operatorname{spec} L \tag{3.24}$$

and hence $\det_\theta L^u = \det_\theta L$. Next, we determine the transformed boundary operator $\mathcal{B}^u := (R_a^u, R_b^u)$. If

$$\phi = \begin{pmatrix} \varphi_1 & \cdots & \varphi_n \\ \vdots & & \vdots \\ \varphi_1^{(n-1)} & \cdots & \varphi_n^{(n-1)} \end{pmatrix} \tag{3.25}$$

denotes a fundamental matrix of l, then the corresponding fundamental matrix of l^u reads

$$\begin{aligned} \tilde{\phi}^u &= \begin{pmatrix} U^{-1}\varphi_1 & \cdots & U^{-1}\varphi_n \\ \vdots & & \vdots \\ (U^{-1}\varphi_1)^{(n-1)} & \cdots & (U^{-1}\varphi_n)^{(n-1)} \end{pmatrix} \\ &= T(U)\,\phi, \end{aligned} \tag{3.26}$$

where

$$T(U)_{ij}(x) := \begin{cases} \binom{i}{j}\,(\partial_x^{(i-j)}\, U^{-1})(x) & \text{if} \quad 0 \le j \le i \le n-1, \\ 0 & \text{if} \quad j > i\,. \end{cases} \tag{3.27}$$

Note that $\det T(U) = (\det U)^{-n}$.

However, the fundamental matrix $\tilde{\phi}^u$ is not normalized. We therefore put

$$\begin{aligned} \phi^u(x) &:= \tilde{\phi}^u(x)\,(\tilde{\phi}^u)^{-1}(a) \\ &= T(U)(x)\,\phi(x)\,T(U)^{-1}(a). \end{aligned} \tag{3.28}$$

We now determine the boundary conditions for L^u. Let $g \in \mathcal{D}(L^u)$. Then, $g = U^{-1}f$ with $f \in \mathcal{D}(L)$. With $F := (f, \dots f^{(n-1)})^t, G := (g, \dots g^{(n-1)})^t$ we have

$$G = T(U)\,F \tag{3.29}$$

and thus

$$\begin{aligned} 0 &= R_a\, F(a) + R_b\, F(b) \\ &= R_a\, T(U)^{-1}(a)\, G(a) + R_b\, T(U)^{-1}(b)\, G(b) \\ &=: \tilde{R}_a^u\, G(a) + \tilde{R}_b^u\, G(b). \end{aligned} \tag{3.30}$$

We put

$$\begin{aligned} R_a^u &:= T(U)(b)\,\tilde{R}_a^u \\ R_b^u &:= T(U)(b)\,\tilde{R}_b^u. \end{aligned} \tag{3.31}$$

Note that R_a^u, R_b^u define the same boundary conditions as $\tilde{R}_a^u, \tilde{R}_b^u$.

Then,

$$\begin{aligned} \mathcal{R}(l^u + z, R_a^u, R_b^u) &= R_a^u + R_b^u\, \phi^u(b, l^u + z) \\ &= T(U)(b)\, \mathcal{R}(l + z, R_a, R_b)\, T(U)^{-1}(a) \end{aligned} \tag{3.32}$$

and thus

$$\det \mathcal{R}(l^u + z, R_a^u, R_b^u) = (\det U(b))^{-n} \det \mathcal{R}(l + z, R_a, R_b). \tag{3.33}$$

Consequently,

$$\det{}_\theta L^u = \det{}_\theta L \tag{3.34}$$

implies

$$\mathcal{C}(a_n, a_{n-1}, R_a, R_b) = (\det U(b))^{-n}\, \mathcal{C}(U^{-1} a_n U, 0, R_a^u, R_b^u). \tag{3.35}$$

We thus have proved the

Proposition 3.5 *Let $L = l_{\mathcal{B}}$ be an admissible operator and let $U(x)$ be the unique solution of the initial value problem* (3.21)*. Then, the operator $L^u = l^u_{\mathcal{B}^u}$ is also admissible. It has the same principal angle θ as L and*

$$\mathcal{C}(a_n, a_{n-1}, R_a, R_b) = \exp\left(i \int_a^b \operatorname{tr}(a_n^{-1}\, a_{n-1})(y)\, dy \right) \mathcal{C}(U^{-1} a_n U, 0, R_a^u, R_b^u), \tag{3.36}$$

with

$$R_a^u := T(U)(b)\, R_a\, T(U)^{-1}(a), \tag{3.37}$$

$$R_b^u := T(U)(b)\, R_b\, T(U)^{-1}(b) \tag{3.38}$$

and $T(U)$ is defined by formula (3.27)*.*

As an application, we consider the operator

$$l := -\frac{d^2}{dx^2} + p(x)\, \frac{d}{dx} + q(x)$$

with the same boundary operator $\mathcal{B} = (R_a, R_b)$ as given in the preceding example. Again, by Proposition 3.3 it is sufficent to consider $q = 0$. Notice that the leading coefficient a_2 of the operator l is invariant with respect to conjungation with U. Hence, in the two specific cases where, respectively, B is invertible, or $B = 0$ and $A + D$ is invertible, we can simply make use of (3.20) to obtain

$$\mathcal{C}_2(p, A, B, C, D) = e^{\frac{1}{2}\int_a^b \operatorname{tr} p(x)\, dx} \begin{cases} \frac{(-1)^m\, 2^m}{\det B} & \text{if } \det B \neq 0, \\ \frac{2^m}{\det(A+D)} & \text{if } B = 0,\ \det(A + D) \neq 0. \end{cases} \tag{3.39}$$

We now turn to the dependence of $\mathcal{C}(l, \mathcal{B})$ on the leading symbol of the differential operator l. The aim is to get an explicit formula analogous to (3.35) in the case of a_{n-1}. Unfortunately, this is much more involved than in the case of separated boundary conditions. Following [BFK3] one considers the family of operators $l_t := \alpha_t\,(D^n + l')$, where l' denotes a differential operator of order $n-1$ and $\alpha_t(x)$, $t \in [0,1]$ is a smooth variation of $a_n(x)$ such that $\alpha_0 = \mathrm{Id}$ and $\alpha_1 = a_n$. Then, the question arises whether the corresponding operators $L_t := (l_t, \mathcal{B})$ are admissible for all $t \in [0,1]$. To answer this question seems to be hopeless for the general situation discussed in this paper. Note, however, for a given admissible operator $L = (l, \mathcal{B})$ the constant $C(l, \mathcal{B})$ can be calculated if the fundamental solution of the corresponding homogeneous equation is known.

3.2 Operators of order 1

In this subsection we briefly indicate how Theorem 3.2 and 3.3 generalize to operators of order one. Let $L = l_{\mathcal{B}}$ be an admissible operator of order one with principal angle π. A priori $(L+x)^{-1}$ is not of trace class. However, the trace of $(L+x)^{-1}$ can be regularized.

In the sequel we use the notation $\mathrm{Res}_k\, f(z_0)$ for the coefficient of $(z-z_0)^k$ in the Laurent expansion of the meromorphic function f.

The function $\mathrm{Tr}(L+z)^{-s}$ is meromorphic with simple poles in $1, 0, -1, \ldots$, which follows from (3.43) below, and we put

$$\overline{\mathrm{Tr}}(L+z)^{-1} := \mathrm{Res}_0\,(L+z)^{-s}|_{s=1}. \tag{3.40}$$

Then,

$$\begin{aligned}
\frac{d}{dz}\mathrm{logdet}_\pi\,(L+z) &= -\frac{d}{ds}|_{s=0}\,\frac{d}{dz}\,\mathrm{Tr}(L+z)^{-s} \\
&= \frac{d}{ds}|_{s=0}\, s\mathrm{Tr}(L+z)^{-s-1} \\
&= \overline{\mathrm{Tr}}(L+z)^{-1} \\
&= \mathrm{Tr}(\,(L+z)^{-1} - L^{-1}\,) + \overline{\mathrm{Tr}}(L^{-1}),
\end{aligned} \tag{3.41}$$

since $(L+z)^{-1} - L^{-1} = -z(L+z)^{-1}L^{-1}$ is of trace class.

The same calculation as (2.34) shows that

$$\begin{aligned}
\zeta_{L,\pi}(s) &= \frac{\sin \pi s}{\pi} \int_0^\infty x^{-s}\,\mathrm{Tr}[(L+x)^{-1} - L^{-1}]\,dx \\
&= \frac{\sin \pi(s-1)}{\pi(s-1)} \int_0^\infty x^{1-s}\,\mathrm{Tr}(L+x)^{-2}\,dx.
\end{aligned} \tag{3.42}$$

By the work of Seeley we have an asymptotic expansion

$$\mathrm{Tr}(L+x)^{-2} \sim_{x\to\infty} \sum_{k=0}^{\infty} a_k\, x^{-k-1}, \tag{3.43}$$

which implies in view of $\sin \pi(s-1) = \pi(s-1)^2 + O((s-1)^3)$ that

$$\overline{\mathrm{Tr}}(L^{-1}) = \mathrm{Res}_0\,\zeta_{L,\pi}(1) = \int_0^\infty \mathrm{Tr}(L+x)^{-2}\,dx. \tag{3.44}$$

On the other hand, we have

$$\overline{\mathrm{Tr}}(L^{-1}) = \int_0^\infty \mathrm{Tr}(L+x)^{-2}\,dx = \operatorname*{LIM}_{R\to\infty}\left\{-\int_0^R \frac{d}{dx}\mathrm{Tr}[(L+x)^{-1} - L^{-1}]\,dx\right\}$$

$$= -\operatorname*{LIM}_{R\to\infty}\mathrm{Tr}[(L+R)^{-1} - L^{-1}]. \tag{3.45}$$

Comparing this with (3.41) gives

$$\operatorname*{LIM}_{z\to\infty}\frac{d}{dz}\log\det{}_\pi(L+z) = \operatorname*{LIM}_{z\to\infty}\overline{\mathrm{Tr}}(L+z)^{-1} = 0. \tag{3.46}$$

Summing up we can state the anolog of Lemma 2.1 and 2.2 for operators of first order:

Lemma 3.6 *Let L be as before. Then we have*

$$\zeta_{L,\pi}(s) = \frac{\sin\pi s}{\pi}\int_0^\infty x^{-s}\,\overline{\mathrm{Tr}}(L+x)^{-1}\,dx, \tag{3.47}$$

$$\zeta'_{L,\pi}(0) = \int_0^\infty \overline{\mathrm{Tr}}(L+x)^{-1}\,dx\,. \tag{3.48}$$

and the asymptotic expansions in a conic neighborhood of $\mathbb{R}_+$

$$\overline{\mathrm{Tr}}(L+x)^{-1} \sim_{x\to\infty} \sum_{k=1}^\infty \frac{a_k}{k}\,x^{-k} + a_0\log x, \tag{3.49}$$

$$\log\det{}_\pi(L+x) \sim_{x\to\infty} \sum_{k=2}^\infty \frac{a_k}{k(k-1)}\,x^{1-k} + \zeta_{L,\pi}(0)\log x + a_0\,x\log x + a_0\,x, \tag{3.50}$$

in particular

$$\operatorname*{LIM}_{x\to\infty}\overline{\mathrm{Tr}}(L+x)^{-1} = 0, \tag{3.51}$$

$$\operatorname*{LIM}_{x\to\infty}\log\det{}_\pi(L+x) = 0. \tag{3.52}$$

Proof The equation (3.47) follows from (3.42) and (3.48) follows from (3.47), similar to (2.35); (3.49), (3.50) follow from integrating the expansion (3.43); (3.51) follows from (3.46) and finally, (3.52) is proved exactly as (2.37). □

Theorem 3.7 *Let L be as before. For $\mathcal{R}(x) = \mathcal{R}(l+x,\mathcal{B})$ we have an asymptotic expansion*

$$\log\det\mathcal{R}(x) \sim_{x\to\infty} \sum_{k=2}^\infty \frac{a_k}{k(k-1)}\,x^{1-k} + \zeta_{L,\pi}(0)\log x + b + a_0\,x\log x + c\,x. \tag{3.53}$$

Furthermore,

$$\log\det{}_\pi(L+x) - \log\det\mathcal{R}(x) = -b - (c-a_0)x. \tag{3.54}$$

Proof We cannot apply Proposition 3.1 to $L+z$ since the operator L is of order one. But the same computation as in the proof of Proposition 3.1 shows that

$$\begin{aligned}
\frac{d}{dx}\overline{\mathrm{Tr}}(L+x)^{-1} &= -\mathrm{Tr}(L+x)^{-2} \\
&= -\int_a^b \mathrm{tr}(L+x)^{-2}(t,t)\,dt \\
&= \frac{d}{dx}\int_a^b \mathrm{tr}[(L+x)^{-1}(t,t+0)]\,dt \\
&= \frac{d^2}{dx^2}\log\,\det\mathcal{R}(x), \qquad (3.55)
\end{aligned}$$

hence,

$$\log\det_\pi(L+x) - \log\,\det\mathcal{R}(x) \qquad (3.56)$$

is a polynomial of degree one and we are done. □

As a consequence, we end up with the formula

$$\det_\pi L = e^{-b}\,\det\mathcal{R}, \qquad (3.57)$$

where $b := \mathrm{LIM}_{x\to\infty}\log\,\det\mathcal{R}(x)$.

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