On the noncommutative residue for pseudodifferential operators with polyhomogeneous symbols

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Abstract. We study various aspects of the noncommutative residue for an algebra of pseudodifferential operators whose symbols have an expansion

$$a \sim \sum_{j=0}^{\infty} a_{m-j}, \quad a_{m-j}(x,\xi) = \sum_{l=0}^{k} a_{m-j,l}(x,\xi) \log^l |\xi|,$$

where $a_{m-j,l}$ is homogeneous in ξ of degree $m-j$. We call these symbols polyhomogeneous.

We study polyhomogeneous functions on symplectic cones and generalize the symplectic residue of GUILLEMIN to these functions. Similarly as for homogeneous functions, for a polyhomogeneous function this symplectic residue is an obstruction against being a sum of Poisson brackets.

For elliptic pseudodifferential operators with polyhomogeneous symbols we show that the ζ–function has a meromorphic continuation to the whole complex plane, however possibly with higher order poles.

This algebra of operators has a bigrading given by the order and the highest log–power occuring in the symbol expansion. We construct "higher" noncommutative residue functionals on the subspaces given by the log–grading. However, in contrast to the classical case we prove that the whole algebra does not admit any nontrivial traces.

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Contents

1. Introduction and summary of the results

Let M be a compact smooth Riemannian manifold without boundary. We denote by $\mathrm{CL}^*(M)$ the algebra of classical (1–step polyhomogeneous) pseudodifferential operators. $\mathrm{CL}^*(M)$ acts naturally as unbounded operators on the Hilbert space $\mathrm{L}^2(M)$ of square integrable functions. If $m < -\dim M$ then $\mathrm{CL}^m(M) \subset C_1(\mathrm{L}^2(M))$, the space of trace class operators.

The L^2–trace does not have a continuation as a trace functional on the whole algebra $\mathrm{CL}^*(M)$. For assume we had a trace τ on $\mathrm{CL}^*(M)$ that extends the L^2–trace. Then by the Atiyah–Singer index theorem for r large enough there exists an injective Fredholm operator $T \in \mathrm{CL}^1(M) \otimes M_r(\mathbb{C})$ of index -1. Let $T = U|T|$ be the polar decomposition where $|T| = |T^*T|^{1/2}$. $T^*T \in \mathrm{CL}^2(M) \otimes M_r(\mathbb{C})$ is an invertible elliptic operator and thus by the work of Seeley [16] we have $|T|^{-1} \in \mathrm{CL}^{-1}(M) \otimes M_r(\mathbb{C}), U = T|T|^{-1} \in \mathrm{CL}^0(M) \otimes M_r(\mathbb{C})$. Hence $U^*U = I$ and $I - UU^* \in \mathrm{CL}^{-\infty}(M) \otimes M_r(\mathbb{C})$ is a rank one projection. Since τ is a trace we get the contradiction

$$1 = \mathrm{Tr}_{\mathrm{L}^2}(I - UU^*) = (\tau \otimes \mathrm{tr}_{\mathbb{C}^r})(I - UU^*) = (\tau \otimes \mathrm{tr}_{\mathbb{C}^r})(I) - (\tau \otimes \mathrm{tr}_{\mathbb{C}^r})(U^*U) = 0. \quad (1.1)$$

However, in his seminal papers [18], [19] M. WODZICKI showed that, up to a constant, the algebra $\mathrm{CL}^*(M)$ has a unique trace which he called the noncommutative residue. A detailed account of the noncommutative residue was given by C. KASSEL [13]. B. FEDOSOV et al. [7] generalized the noncommutative residue to the BOUTET DE MONVEL algebra on a manifold with boundary.

There are several ways to define the noncommutative residue. The global definition which shows the intimate relation to ζ–functions and heat kernel expansions is as follows: given $A \in \mathrm{CL}^a(M)$. Then choose any elliptic operator $P \in \mathrm{CL}^m(M), m > 0$, whose leading symbol is positive. Then

$$\begin{aligned} \mathrm{Res}(A) &:= m\mathrm{Res}_{s=0}\mathrm{Tr}(AP^{-s}) \\ &= -m \times \text{ coefficient of } \log t \text{ in the asymptotic} \\ &\quad\ \text{expansion of } \mathrm{Tr}(Ae^{-tP}) \text{ as } t \to 0. \end{aligned}$$

$\mathrm{Res}(A)$ is in fact independent of the P chosen. This definition uses the fact that the generalized ζ–function $\mathrm{Tr}(AP^{-s})$ has a meromorphic continuation to $\mathbb{C}$ with simple poles at $\{\frac{\dim M + a - n}{m} \,|\, n \in \mathbb{Z}_+\} \cup \{-n \,|\, n \in \mathbb{Z}_+\}$ [8, Chap. 1]. Via the Mellin transform

$$\mathrm{Tr}(AP^{-s}) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1}\mathrm{Tr}(Ae^{-tP})dt$$

this is (almost) equivalent to the asymptotic expansion

$$\mathrm{Tr}(Ae^{-tP}) \sim_{t\to 0+} \sum_{j=0}^\infty c_j\, t^{\frac{j-a-\dim M}{m}} + \sum_{j=0}^\infty d_j\, t^j \log t. \quad (1.2)$$

Another approach to the noncommutative residue was discovered independently by M. WODZICKI and V. GUILLEMIN [11]. This approach works in the framework of symplectic cones. It shows that there is a local formula

$$\mathrm{Res}(A) = (2\pi)^{-\dim M} \int_{S^*M} a(x,\xi)_{-\dim M} |d\xi dx| \quad (1.3)$$

for the noncommuative residue in terms of the complete symbol of A. It is a remarkable fact that, although the complete symbol does not have an invariant meaning, the right hand side of (1.3) is well–defined.

Asymptotic expansions like (1.2) are an essential feature in the study of elliptic operators. They can also be achieved for certain operators with singularities, for example conical singularities [5], [4], [3] or boundary value problems [10], [9], [2]. In all these situations no higher $\log t$ powers occur. However, in [1] higher $\log t$ powers are postulated for equivariant heat trace expansions of the Laplacian. Unfortunately, so far it is unproved whether they really occur or not.

It was one of our motivations to write this paper to provide a natural algebra of pseudodifferential operators for which higher $\log t$–powers occur in the heat expansion (1.2). Although our class of operators is not really new, to the best of our knowledge it was never looked at with regard to noncommutative residue. Our point of view shows that the existence of the noncommutative residue as the unique trace depends heavily on the absence of higher $\log t$–powers.

The idea is quite simple: instead of looking at classical pseudodifferential operators we consider pseudodifferential operators whose symbol have an asymptotic expansion

$$a \sim \sum_{j=0}^{\infty} a_{m-j}, \tag{1.4}$$

where a_{m-j} is polyhomogeneous i.e.

$$a_{m-j}(x,\xi) = \sum_{l=0}^{k} a_{m-j,l}(x,\xi) \log^l |\xi| \tag{1.5}$$

and $a_{m-j,l}(x,\xi)$ is homogeneous of degree $m-j$ (Definition 3.1). We denote the class of pseudodifferential operators having this symbol expansion by $\mathrm{CL}^{m,k}(M)$, where m denotes the order and k the highest log–power occuring in the symbol expansion (Definition 3.3).

In fact, this class of operators is not new. It was considered before by SCHROHE in his thesis [15] where he constructed the complex powers for elliptic operators in this class.

In the present paper we show the heat expansion $\mathrm{Tr}(Ae^{-tP})$ for A with polyhomogeneous symbol and classical P. We use the method of GRUBB and SEELEY [9]. In contrast to (1.2) there occur higher $\log t$–powers (Theorem 3.7).

As a consequence of the heat expansion for $A \in \mathrm{CL}^{m,k}(M)$ we can define the "higher" noncommutative residue, $\mathrm{Res}_k(A)$, as the coefficient of the highest $\log t$–power in the expansion of $\mathrm{Tr}(Ae^{-tP})$ (Definition 4.1). It turns out that this residue has similar properties as the noncommutative residue of WODZICKI, in particular it is independent of the P chosen. It vanishes on appropriate commutators, i.e.

$$\mathrm{Res}_{k+l}([A,B]) = 0$$

if $A \in \mathrm{CL}^{m,k}(M)$ and $B \in \mathrm{CL}^{n,l}(M)$ (Theorem 4.4). There is also a local formula for Res_k (Corollary 4.8).

In the context of spectral triples higher noncommutative residues were also discovered and investigated by CONNES and MOSCOVICI [6, Chap. II].

As for the WODZICKI residue Res_k is an obstruction for being a commutator. Namely, if $A \in \mathrm{CL}^{m,k}(M)$ then there exist $P_1, \dots, P_N \in \mathrm{CL}^{1,0}(M)$ and $Q_1, \dots, Q_N \in \mathrm{CL}^{m,k}(M)$ such that $A - \sum_{j=1}^{N} [P_j, Q_j]$ is smoothing if and only if $\mathrm{Res}_k(A) = 0$ (Proposition 4.7). By a result of WODZICKI [18] (cf. also [12] for a generalization to Fourier integral operators) any smoothing operator is in fact a sum of commutators of classical pseudodifferential operators. Hence $A \in \mathrm{CL}^{m,k}(M)$ is a sum of commutators if and only if $\mathrm{Res}_k(A) = 0$ (Proposition 4.9). However, $\mathrm{Res}_{k+1}(A)$ is always zero for $A \in \mathrm{CL}^{m,k}(M)$, such that A can always be written as a sum of commutators if one increases the log–degree by one. As a consequence, there does not exist any trace functional on the algebra $\mathrm{CL}^{*,*}(M)$ (Corollary 4.11).

For proving the result about commutators we generalize a result due to GUILLEMIN [11] about homogeneous functions on symplectic cones. Namely, we generalize the notion of symplectic residue, which is closely related to the noncommuative residue, to functions on symplectic cones of the form

$$\sum_{j=0}^{k} f_j \log^j p, \tag{1.6}$$

where f_j is homogeneous and p is positive and homogeneous of degree 0. We call these functions polyhomogeneous.

GUILLEMIN's result says that a homogeneous function on a symplectic cone is a sum of Poisson brackets of homogeneous functions if and only if its symplectic residue vanishes. We define a residue for functions of the form (1.6) and prove an analogue of GUILLEMIN's result for these functions.

The paper is organized as follows: in Section 2 we present our generalization of the symplectic residue. It is divided into two parts. The first part deals with functions on $\mathbb{R}^n \setminus \{0\}$. Although this case is very elementary, it does not quite fit into the framework of symplectic cones and therefore we treat it separately. In this subsection we follow [7]. Loc. cit. deals with the noncommutative residue for manifolds with boundary. The second part of Section 2 deals with the symplectic residue on compact connected symplectic cones, and we proceed along the lines of [11].

In Section 3 we introduce the class $\mathrm{CL}^{*,*}(M)$ of pseudodifferential operators with polyhomogeneous symbol expansion.

Section 4 contains our main results.

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2. Polyhomogeneous functions

2.1 Polyhomogeneous functions on $\mathbb{R}^n \setminus \{0\}$

In this section we follow in part [7, Sec. 1].

Definition 2.1 *A function $f \in C^\infty(\mathbb{R}^n \setminus \{0\})$ is called* polyhomogeneous *of degree (a, k) if*

$$f(x) = \sum_{j=0}^{l} g_j(x) P_j(\log h_j(x)), \tag{2.1}$$

where the $P_j \in \mathbb{C}[t]$ are polynomials, $\deg P_j \le k$, and $g_j, h_j \in C^\infty(\mathbb{R}^n \setminus \{0\})$, $h_j > 0$, are homogeneous functions, g_j of degree a and h_j of degree b_j.

We denote the set of all polyhomogeneous functions of degree (a, k) by $\mathcal{P}^{a,k} = \mathcal{P}^{a,k}(\mathbb{R}^n)$ and put

$$\mathcal{P} := \bigoplus_{a \in \mathbb{R}, k \in \mathbb{Z}_+} \mathcal{P}^{a,k}. \tag{2.2}$$

More generally, if M is a manifold we denote by $\mathcal{P}^{a,k}(M, \mathbb{R}^n)$ the set of functions $f \in C^\infty(M \times (\mathbb{R}^n \setminus \{0\}))$ such that for each $x \in M$ we have $f(x, \cdot) \in \mathcal{P}^{a,k}$.

Lemma 2.2 *Each $f \in \mathcal{P}^{a,k}$ has a unique representation*

$$f(x) = \sum_{j=0}^{k} f_j(x) \log^j |x|$$

with $f_j \in \mathcal{P}^{a,0}$.

In the sequel for $f \in \mathcal{P}^{a,k}$ the coefficient of $\log^j |x|$ will always be denoted by f_j.

Proof The uniqueness is clear. It suffices to prove the existence for

$$f(x) = g(x) \log^l h(x)$$

with $g \in \mathcal{P}^{a,0}, h \in \mathcal{P}^{b,0}, l \le k$. Then we write

$$h(x) = |x|^b h(x/|x|) =: |x|^b h_1(x),$$

where $h_1 \in \mathcal{P}^{0,0}$. Thus $\log h_1 \in \mathcal{P}^{0,0}$ and hence

$$g(x) \log^l h(x) = \sum_{j=0}^{l} \binom{l}{j} g(x) (\log^{l-j} h_1(x)) b^j \log^j |x|. \qquad \square$$

An immediate consequence of Lemma 2.2 is the inclusion

$$\mathcal{P}^{a,k} \mathcal{P}^{b,l} \subset \mathcal{P}^{a+b,k+l}, \tag{2.3}$$

i.e. $\mathcal{P}$ is a bigraded algebra.

There is an analogue of Euler's theorem for polyhomogeneous functions. Namely, consider

$$f(x) = g(x)\log^l|x|, \quad g \in \mathcal{P}^{a,0}.$$

Then

$$\sum_{j=1}^n \frac{\partial f}{\partial x_j}x_j = \frac{d}{dt}|_{t=1}f(tx) = af(x) + lg(x)\log^{l-1}|x|, \tag{2.4}$$

from which one derives the identity

$$\sum_{j=1}^n \frac{\partial}{\partial x_j}(x_jF(x)) = f(x), \quad \text{if} \quad a \neq -n, \tag{2.5}$$

where

$$F(x) = g(x)\sum_{j=0}^l \frac{(-1)^{l-j}l!}{j!(n+a)^{l-j+1}}\log^j|x|, \quad a \neq -n. \tag{2.6}$$

Thus we have proved

Lemma 2.3 *If $a \neq -n$ then for $f \in \mathcal{P}^{a,k}$ there exists a $F \in \mathcal{P}^{a,k}$ such that*

$$f = \sum_{j=1}^n \frac{\partial}{\partial x_j}(x_jF).$$

We turn to the case $a = -n$ which is slightly more subtle. We denote by

$$\Delta := -\sum_{j=1}^n \frac{\partial^2}{\partial x_j^2} \tag{2.7}$$

the (positive) Laplacian in $\mathbb{R}^n$.

Definition 2.4 *Let $f \in \mathcal{P}^{-n,k}$,*

$$f(x) = \sum_{j=0}^k f_j(x)\log^j|x|.$$

Then we put

$$\operatorname{res}_j(f) := \int_{S^{n-1}} f_j(x)d\mathrm{vol}_S(x),$$

where vol_S denotes the volume form with respect to the standard metric on S^{n-1}.

Lemma 2.5 *Let*

$$f(x) = \sum_{j=0}^k f_j(x)\log^j|x| \in \mathcal{P}^{-n,k}. \tag{2.8}$$

If $n \neq 2$ then there exists a $F \in \mathcal{P}^{2-n,k}$ with $\Delta F = f$ if and only if $\operatorname{res}_k(f) = 0$.

If $n = 2$ then there exists a $F \in \mathcal{P}^{0,k}$ with $\Delta F = f$ if and only if $\operatorname{res}_k(f) = \operatorname{res}_{k-1}(f) = 0$.

Proof We identify $\mathbb{R}^n \setminus \{0\}$ with $\mathbb{R}_+ \times S^{n-1}$ via

$$\mathbb{R}_+ \times S^{n-1} \to \mathbb{R}^n \setminus \{0\}, \quad (r, \theta) \mapsto r\theta.$$

Then we have

$$f(r\theta) = \sum_{j=0}^{k} g_j(\theta) r^{-n} \log^j r.$$

The Laplacian is given by

$$\Delta = -r^{1-n}\frac{\partial}{\partial r}(r^{n-1}\frac{\partial}{\partial r}) + r^{-2}\Delta_S.$$

For F we make an Ansatz

$$F(r\theta) = \sum_{j=0}^{k} F_j(\theta) r^{2-n} \log^j r$$

and applying Δ to F we obtain the system of equations

$$\Delta_S F_k = g_k, \tag{2.9a}$$
$$\Delta_S F_{k-1} = g_{k-1} - k(n-2)F_k, \tag{2.9b}$$
$$\Delta_S F_j = g_j - (j+1)(n-2)F_{j+1} + (j+1)(j+2)F_{j+2}, \quad j \le k-2. \tag{2.9c}$$

The equation $\Delta_S F_k = g_k$ has a solution if and only if g_k is orthogonal to $\ker \Delta_S$ which consists of the constants. Hence the first equation has a solution if and only if $\mathrm{res}_k(f) = 0$. If $n = 2$ by the same reasoning the second equation has a solution if and only if $\mathrm{res}_{k-1}(f) = 0$. Hence we have proved the 'only if' part of the assertion.

Now let $n \neq 2$ and assume $\mathrm{res}_k(f) = 0$. Let F_k^0 be the unique solution of (2.9a) with $\int_{S^{n-1}} F_k^0 = 0$. Put

$$F_k := F_k^0 + \frac{1}{k(n-2)} \int_{S^{n-1}} g_{k-1}(x) d\mathrm{vol}_S(x).$$

Then $\Delta F_k = g_k$ and

$$k(n-2) \int_{S^{n-1}} F_k(x) d\mathrm{vol}_S(x) = \int_{S^{n-1}} g_{k-1}(x) d\mathrm{vol}_S(x).$$

Hence there exists a unique F_{k-1}^0 with $\Delta F_{k-1}^0 = g_{k-1} - k(n-2)F_k$. Proceeding in this way we obtain a solution of (2.9a)–(2.9c).

The case $n = 2$ is treated similar. $\square$

Lemma 2.6 *For $f \in \mathcal{P}^{1-n,k}$ we have* $\mathrm{res}_k(\frac{\partial}{\partial x_j} f) = 0$.

See [7, Lemma 1.2] for another proof of this lemma in the case $k = 0$.

Proof Let

$$f(x) = \sum_{i=0}^{k} f_i(x) \log^i |x|, \quad f_i \in \mathcal{P}^{1-n,0}.$$

Then

$$\frac{\partial f}{\partial x_j}(x) = \sum_{i=0}^{k} \frac{\partial f_i}{\partial x_j}(x) \log^i |x| + \sum_{i=0}^{k-1} \frac{x_j f_{i+1}(x)}{|x|^2}(i+1) \log^i |x|,$$

hence

$$\mathrm{res}_k\left(\frac{\partial}{\partial x_j} f\right) = \int_{S^{n-1}} \frac{\partial f_k}{\partial x_j} \mathrm{dvol}_S.$$

We consider the vector field

$$X_j(x) := (1 - x_j^2)\frac{\partial}{\partial x_j} - \sum_{i \neq j} x_j x_i \frac{\partial}{\partial x_i}.$$

Obviously, $X_j | S^{n-1}$ is tangential to S^{n-1}. One checks by direct calculation

$$\mathrm{div}_S X_j = (1-n)x_j. \tag{2.10}$$

On the other hand, by Euler's identity,

$$\begin{aligned} X_j f_k &= \frac{\partial f_k}{\partial x_j} - x_j \sum_{i=1}^{n} x_i \frac{\partial f_k}{\partial x_i} \\ &= \frac{\partial f_k}{\partial x_j} + (n-1) x_j f_k, \end{aligned}$$

hence in view of (2.10)

$$\begin{aligned} \int_{S^{n-1}} \frac{\partial f_k}{\partial x_j} \mathrm{dvol}_S &= \int_{S^{n-1}} X_j f_k + (1-n) x_j f_k \mathrm{dvol}_S \\ &= \int_{S^{n-1}} (-\mathrm{div}_S(X_j) + (1-n)x_j) f_k \mathrm{dvol}_S = 0. \end{aligned}$$

□

Proposition 2.7 *Let $f \in \mathcal{P}^{a,k}$. Then there exist $f_j \in \mathcal{P}^{a,k}$ such that $f = \sum_{j=1}^{n} \frac{\partial}{\partial x_j} f_j$ if and only if $a \neq -n$ or $\mathrm{res}_k(f) = 0$.*

Proof The case $a \neq -n$ follows from Lemma 2.3. Let $a = -n$. If $\mathrm{res}_k(f) = \mathrm{res}_{k-1}(f) = 0$ then by Lemma 2.4 there exists $F \in \mathcal{P}^{2-n,k}$ such that $f = \Delta F$ and we reach the conclusion in this case.

Thus it suffices to show that the function

$$|x|^{-n} \log^{k-1} |x|$$

is a sum of derivatives. But

$$\sum_{j=1}^{n} \frac{\partial}{\partial x_j}(x_j |x|^{-n} \log^k |x|) = k|x|^{-n} \log^{k-1} |x|$$

does the job. □

We will also need the results of [7, (1.1)–(1.5), Lemma 1.1 and 1.3] in our context. We briefly summarize the facts. Let

$$\sigma := \sum_{j=0}^{n} (-1)^{j+1} \xi_j d\xi_1 \wedge \ldots \wedge \widehat{d\xi_j} \wedge \ldots \wedge d\xi_n \in \Omega^{n-1}(\mathbb{R}^n). \tag{2.11}$$

$\sigma|S^{n-1}$ is the volume form. Moreover, for $f \in \mathcal{P}^{-n,0}$ the form $f\sigma$ is closed by Euler's theorem (2.4). Thus for any bounded domain $D \subset \mathbb{R}^n$, $0 \in D$, with smooth boundary we have for $f \in \mathcal{P}^{-n,0}$

$$\mathrm{res}_k(f) = \int_{S^{n-1}} f_k(x) d\mathrm{vol}_{S^{n-1}}(x) = \int_{S^{n-1}} f_k \sigma = \int_{\partial D} f_k \sigma. \tag{2.12}$$

Moreover, if $T \in \mathrm{GL}(n, \mathbb{R})$ then

$$\begin{aligned} \mathrm{res}_k(T^* f) &= \int_{S^{n-1}} f_k \circ T \sigma = \mathrm{sgn}\,(\det T) \int_{T(S^{n-1})} f_k T^{-1\,*} \sigma \\ &= \frac{1}{|\det T|} \int_{T(S^{n-1})} f_k \sigma = \frac{1}{|\det T|} \mathrm{res}_k(f). \end{aligned} \tag{2.13}$$

Finally we note

Lemma 2.8 [7, Lemma 1.3] *Let* $f \in \mathcal{P}^{a,k}$. *Then* $\mathrm{res}_k(\xi^\alpha \partial^\beta f) = 0$ *if* $|\beta| > |\alpha|$.

Proof This follows from Lemma 2.6 by induction since if $\partial^\beta = \frac{\partial}{\partial \xi_j} \partial^\gamma$ then

$$\xi^\alpha \partial^\beta f = \frac{\partial}{\partial \xi_j} (\xi^\alpha \partial^\gamma f) - \frac{\partial \xi^\alpha}{\partial \xi_j} \partial^\gamma f. \qquad \square$$

2.2 Polyhomogeneous functions on symplectic cones

In this section we study polyhomogeneous functions on an arbitrary symplectic cone. Our exposition parallels [11, Sec. 6].

Let Y be a symplectic cone. This is a principal bundle

$$\pi : Y \to X$$

with structure group $\mathbb{R}_+$. Denote by $\varrho_a : Y \to Y$ the action of $a \in \mathbb{R}_+$. That Y is a symplectic cone means that Y is symplectic, with symplectic form ω, and

$$\varrho_a^* \omega = a\omega.$$

We assume furthermore that Y is *connected* and X is *compact.*

The main example of course is the cotangent bundle with the zero section removed, $T^*M \setminus 0$, over a compact connected manifold M of dimension $\dim M > 1$.

Definition 2.9 *A function* $f \in C^\infty(Y)$ *is called* polyhomogeneous *of degree* (a, k) *if*

$$f = \sum_{j=0}^{l} g_j P_j(\log h_j)$$

with $g_j, h_j \in C^\infty(Y)$, $P_j \in \mathbb{C}[t]$, *where* g_j *is homogeneous of degree* a, h_j *is homogeneous of degree* b_j, *and* $h_j > 0$ *everywhere. Furthermore,* $\deg P_j \leq k$.

Again, we denote the set of all polyhomogeneous functions of degree (a,k) by $\mathcal{P}^{a,k}$. Then (2.2), (2.3) hold similarly.

We fix, for once and for all, $p \in \mathcal{P}^{1,0}$ such that p is everywhere positive. Then by Euler's identity we have $dp \neq 0$ everywhere. We put

$$Z := \{y \in Y \,|\, p(y) = 1\}.$$

p plays the role of $|\cdot|$ and Z plays the role of S^{n-1} in the preceding section.

Lemma 2.10 *Each $f \in \mathcal{P}^{a,k}$ has a representation*

$$f = \sum_{j=0}^{k} f_j \log^j p.$$

Furthermore, f_k is independent of the choice of p.

Proof Consider

$$g \log^l h$$

with $g \in \mathcal{P}^{a,0}, h \in \mathcal{P}^{b,0}$. Then $h_1 := p^{-b}h$ is 0–homogeneous and positive, hence $\log h_1 \in \mathcal{P}^{0,0}$. Thus

$$g \log^l h = \sum_{j=0}^{l} \tbinom{l}{j} g(x)(\log^{l-j} h_1) b^j \log^j p,$$

from which we see that the coefficient of $\log^l p$ is independent of p. □

Definition 2.11 *We put*

$$\mathrm{res}_k(f) := \mathrm{Res}\, f_k,$$

where $\mathrm{Res}\, f_k$ *is the symplectic residue of* GUILLEMIN. *By the preceding Lemma* res_k *is well defined.*

For the convenience of the reader and since we have to introduce some notation anyway we briefly recall the definition of the symplectic residue (cf. [11, Sec. 6]).

Via $\Phi_t := \varrho_{e^t}$ we obtain a one parameter group of diffeomorphisms of Y. Let $\Xi \in C^\infty(TY)$ be the infinitesimal generator of this group. Put $\alpha := i_\Xi \omega \in \Omega^1(Y)$, and let

$$\mu := \alpha \wedge \omega^{n-1}.$$

If $f \in \mathcal{P}^{-n,0}$ then the form $f\mu$ is horizontal and invariant, hence there is a unique $(2n-1)$–form μ_f such that $f\mu = \pi^*\mu_f$. Then

$$\mathrm{Res}(f) := \int_X \mu_f.$$

One can show (cf. [11, Proof of Lemma 6.3]) that also

$$\mathrm{Res}(f) = \int_Z f\mu.$$

We denote by $\{\cdot,\cdot\}$ the Poisson bracket associated with the symplectic structure.

Lemma 2.12 *If* $f \in \mathcal{P}^{a,k}, g \in \mathcal{P}^{1,l}$ *then* $\{f, g\} \in \mathcal{P}^{a,k+l}$ *and* $\mathrm{res}_{k+l}\{f, g\} = 0$.

Proof W.l.o.g. we may assume

$$f = \phi \log^k p, \quad g = \psi \log^l p,$$

with $\phi \in \mathcal{P}^{a,0}, \psi \in \mathcal{P}^{1,0}$. Then

$$\begin{aligned} \{f, g\} &= \{\phi, \psi\} \log^{k+l} p + k\{p, \psi\}\phi p^{-1} \log^{k+l-1} p \\ &\quad + \{\phi, p\} l \psi p^{-1} \log^{k+l-1} p, \end{aligned}$$

hence $\{f, g\} \in \mathcal{P}^{a,k+l}$ and $\mathrm{res}_{k+l}\{f, g\} = \mathrm{Res}\{\phi, \psi\} = 0$ by [11, Prop. 6.1]. □

Now we can state the generalization of [11, Thm. 6.2] to polyhomogeneous functions.

Theorem 2.13 1. *If* $a \neq -n$ *then* $\{\mathcal{P}^{1,0}, \mathcal{P}^{a,k}\} = \mathcal{P}^{a,k}$.
2. *If* $a = -n$ *then* $\{\mathcal{P}^{1,0}, \mathcal{P}^{a,k}\} = \ker \mathrm{res}_k \subset \mathcal{P}^{a,k}$.

Proof We follow the proof of loc. cit. and choose functions $g_1, \dots, g_N \in \mathcal{P}^{1,0}$ such that their differentials span the cotangent space of Y at every point. Let

$$D_i : \mathcal{P}^{a,k} \to \mathcal{P}^{a,k}, \quad D_i f := \{g_i, f\}.$$

We introduce a pre–Hilbert space structure on $\mathcal{P}^{a,k}$ as follows: we identify $\mathcal{P}^{a,k}$ with $C^\infty(Z, \mathbb{C}^{k+1})$ via

$$\sum_{j=0}^{k} f_j \log^j p \mapsto (f_0, \dots, f_k)|Z. \tag{2.14}$$

Next let ν be the restriction of μ to Z. This is a volume form and hence it defines a L^2–structure on $C^\infty(Z, \mathbb{C}^{k+1}) \simeq \mathcal{P}^{a,k}$.

Now consider $f \log^j p, f \in \mathcal{P}^{a,0}$. Then

$$D_i(f \log^j p) = (D_i f) \log^j p + j f p^{-1}(D_i p) \log^{j-1} p.$$

Thus putting

$$q_i := p^{-1} D_i p \in \mathcal{P}^{0,0}$$

and $d_i := D_i|\mathcal{P}^{a,0}$ the identification (2.14) transforms D_i into

$$D_i \cong \begin{pmatrix} d_i & q_i & 0 & \dots & 0 \\ 0 & d_i & 2q_i & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ & & & d_i & kq_i \\ 0 & \dots & & 0 & d_i \end{pmatrix} : C^\infty(Z, \mathbb{C}^{k+1}) \to C^\infty(Z, \mathbb{C}^{k+1}). \tag{2.15}$$

Now we consider

$$\Delta := \sum_{j=1}^{N} D_i D_i^t. \tag{2.16}$$

Δ is a self–adjoint elliptic differential operator on $C^\infty(Z, \mathbb{C}^{k+1})$. Hence

$$\mathrm{Im}\, \Delta^\perp = \ker \Delta.$$

Therefore, consider

$$f = \sum_{j=0}^{k} f_j \log^j p \in \ker \Delta.$$

In view of (2.15)

$$d_i^t f_j + j q_i f_{j-1} = 0, \quad i = 1, \ldots, N;\ j = 0, \ldots, k,$$

where we have put $f_{-1} := 0$.

By [11, (6.15)] we have

$$d_i^t = -d_i + (2a+n) q_i.$$

Abbreviating $r := -(n+2a)$ we obtain

$$-\{g_i, f_j\} - r p^{-1} f_j \{g_i, p\} + j f_{j-1} p^{-1} \{g_i, p\} = 0,$$

or

$$\{g_i, p^r f_j\} = j f_{j-1} p^{r-1} \{g_i, p\}. \tag{2.17}$$

Since the differentials of the g_i span the cotangent space at every point we conclude that $p^r f_0 = c_0$ is constant.

By induction we assume that $p^r f_{j'} = 0$ for $j' < j-1$ and $p^r f_{j-1} = c_{j-1}$ is constant. Then, in view of (2.17)

$$\{g_i, p^r f_j\} = j c_{j-1} \{g_i, \log p\},$$

thus

$$\{g_i, p^r f_j - j c_{j-1} \log p\} = 0,$$

(cf. [11, Lemma 6.6]). Again since the differentials of the g_i span the cotangent space this implies

$$f_j = c_j p^{-r} + j c_{j-1} p^{-r} \log p.$$

But since $f_j \in \mathcal{P}^{a,0}$ the constant c_{j-1} must be 0.

Summing up we have proved

$$f_j = 0, j < k, \quad f_k = c p^{-r}.$$

This implies $c = 0$ or $r = -a$. Since $r = -(n+2a)$ this is equivalent to $c = 0$ or $r = n$. Since $\int_Z p^{-r} \nu \neq 0$ we reach the conclusion. □

3. Pseudodifferential operators with polyhomogeneous symbols

Let M be a smooth manifold of dimension n. We denote by $\mathrm{L}^*(M)$ the algebra of pseudodifferential operators with complete symbols of HÖRMANDER type (1,0). I. e. if U is an open chart then $A \in \mathrm{L}^m(U)$ can be written

$$Au(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} \int_U a(x,y,\xi) u(y) e^{i<x-y,\xi>} dy d\xi, \tag{3.1}$$

where $a \in C^\infty(U \times U \times \mathbb{R}^n)$ satisfies

$$|\partial_x^\alpha \partial_y^\beta \partial_\xi^\gamma a(x,y,\xi)| \le C_{\alpha,\beta,\gamma,K}(1+|\xi|)^{m-|\gamma|}, \tag{3.2}$$

for $\xi \in \mathbb{R}^n, (x,y) \in K$, and $K \subset U$ compact. The class of these symbols is as usual denoted by $\mathrm{S}^m(U \times U, \mathbb{R}^n)$.

We denote by $\mathrm{L}^*(M,E)$ the algebra of pseudodifferential operators acting on sections of the C^∞ vector bundle E.

We are now going to introduce a subclass of $\mathrm{L}^*(M)$ which generalizes the classical pseudodifferential operators.

We proceed locally and introduce polyhomogeneous symbols:

Definition 3.1 *For an open set $U \subset \mathbb{R}^n$ we denote by $\mathrm{CS}^{m,k}(U,\mathbb{R}^n)$ the set of symbols $a \in \cap_{\varepsilon>0}\mathrm{S}^{m+\varepsilon}(U,\mathbb{R}^n)$ having an asymptotic expansion*

$$a \sim \sum_{j=0}^\infty \psi(\xi) a_{m-j}(x,\xi)$$

where $a_{m-j} \in \mathcal{P}^{m-j,k}(U,\mathbb{R}^n)$ and $\psi \in C^\infty(\mathbb{R}^n)$ with $\psi(\xi)=0$ for $|\xi| \le 1/4$ and $\psi(\xi)=1$ for $|\xi| \ge 1/2$.

Remark 3.2 1. $\mathrm{CS}^{m,0}(U,\mathbb{R}^n)$ are just the classical (1-step polyhomogeneous) symbols.

2. One could more generally consider symbols having expansions

$$a \sim \sum_{j=0}^\infty a_j, \tag{3.3}$$

where $a_j \in \mathcal{P}^{m_j,k}(U,\mathbb{R}^n), \lim_{j\to\infty} m_j = -\infty$. These symbols were already considered by SCHROHE [15] who constructed the complex powers for the corresponding class of elliptic pseudodifferential operators.

Our class of operators, which is more closely to classical operators, is large enough for our purposes. However, our results (with suitable modifications) remain true for the slightly more general symbols (3.3).

Definition 3.3 *We denote by $\mathrm{CL}^{m,k}(U)$ the class of pseudodifferential operators which can be written in the form (3.1) with $a \in \mathrm{CS}^{m,k}(U \times U, \mathbb{R}^n)$.*

It is fairly straightforward to check that this class of pseudodifferential operators satisfies the usual rules of calculus. We summarize the results for the convenience of the reader.

Proposition 3.4 1. *If $A \in \mathrm{CL}^{m,k}(U)$ is properly supported then the complete symbol σ_A is in $\mathrm{CS}^{m,k}(U,\mathbb{R}^n)$ and*

$$\begin{aligned}\sigma_A &\sim \sum_{\alpha\in\mathbb{Z}_+^n} \frac{i^{-\alpha}}{\alpha!}\partial_\xi^\alpha \partial_y^\alpha a(x,y,\xi)|_{y=x} \\ &\sim \sum_{j=0}^\infty \sum_{|\alpha|+l=j} \frac{i^{-\alpha}}{\alpha!}\partial_\xi^\alpha \partial_y^\alpha a_{m-l}(x,y,\xi)|_{y=x}.\end{aligned} \tag{3.4}$$

2. *If* $A \in \mathrm{CL}^{m,k}(U), B \in \mathrm{CL}^{m',k'}(U)$ *are properly supported then* $AB \in \mathrm{CL}^{m+m',k+k'}(U)$ *and*

$$\begin{aligned}\sigma_{AB} &\sim \sum_{\alpha \in \mathbb{Z}_+^n} \frac{i^{-\alpha}}{\alpha!}(\partial_\xi^\alpha \sigma_A(x,\xi))\partial_x^\alpha \sigma_B(x,\xi) \\ &\sim \sum_{j=0}^{\infty} \sum_{|\alpha|+l+l'=j} \frac{i^{-\alpha}}{\alpha!}(\partial_\xi^\alpha a_{m-l}(x,\xi))\partial_x^\alpha b_{m'-l'}(x,\xi),\end{aligned} \tag{3.5}$$

if $\sigma_A \sim \sum a_{m-j}, \sigma_B \sim \sum b_{m'-j}$.

3. *If* $A \in \mathrm{CL}^{m,k}(U)$ *then* A^t *in* $\mathrm{CL}^{m,k}(U)$ *and*

$$\begin{aligned}\sigma_{A^t} &\sim \sum_{\alpha \in \mathbb{Z}_+^n} \frac{i^{-\alpha}}{\alpha!}\partial_\xi^\alpha \partial_x^\alpha \sigma_A(x,\xi)^* \\ &\sim \sum_{j=0}^{\infty} \sum_{|\alpha|+l=j} \frac{i^{-\alpha}}{\alpha!}\partial_\xi^\alpha \partial_x^\alpha a_{m-l}(x,\xi)^*.\end{aligned} \tag{3.6}$$

Proof This follows from [17, Theorems 3.1–3.4] and the obvious inclusions

$$\partial_\xi^\alpha \partial_x^\beta \mathrm{CS}^{m,k}(U) \subset \mathrm{CS}^{m-|\alpha|,k}(U), \mathrm{CS}^{m,k}(U) \cdot \mathrm{CS}^{m',k'}(U) \subset \mathrm{CS}^{m+m',k+k'}(U). \qquad \square$$

Proposition 3.5 *Let* $\kappa : U \to V$ *be a diffeomorphism and* $A \in \mathrm{CL}^{m,k}(U)$ *properly supported. Then* $\kappa_* A$ *given by* $(\kappa_* A)u := (A(u \circ \kappa)) \circ \kappa^{-1}$ *is in* $\mathrm{CL}^{m,k}(V)$ *and*

$$\sigma_{\kappa_* A}(y,\eta)|_{y=\kappa(x)} \sim \sum_{\alpha \in \mathbb{Z}_+^n} \frac{1}{\alpha!}(\partial_\xi^\alpha \sigma_A)(x, D\kappa^t(x)\eta) D_z^\alpha e^{i<\kappa_x''(z),\eta>}|_{z=x}, \tag{3.7}$$

where

$$\kappa_x''(z) = \kappa(z) - \kappa(x) - D\kappa(x)(z-x).$$

Furthermore,

$$\Phi_\alpha(x,\eta) := D_z^\alpha e^{i<\kappa_x''(z),\eta>}|_{z=x}$$

is a polynomial in η *of degree not higher than* $|\alpha|/2$.

Proof Except the assertion $\kappa_* A \in \mathrm{CL}^{m,k}(V)$ this is [17, Thm. 4.2]. $\kappa_* A \in \mathrm{CL}^{m,k}(V)$ now follows from (3.7). $\square$

As usual, this result allows to define operators of the class $\mathrm{CL}^{m,k}$ on manifolds. For a smooth manifold M we denote by $\mathrm{CL}^{m,k}(M)$ the corresponding space of pseudodifferential operators. We note that (3.7) shows that the leading symbol of $A \in \mathrm{CL}^{m,k}(M)$ can be considered as an element of $\mathcal{P}^{m,k}(T^*M)$. Because of its importance we single out this observation:

Proposition 3.6 *The leading symbol which is locally defined by* $\sigma_{\mathrm{L}}^m(A) := a_m$ *induces a surjective linear map*

$$\sigma_{\mathrm{L}} : \mathrm{CL}^{m,k}(M) \longrightarrow \mathcal{P}^{m,k}(T^*M) \cong C^\infty(\mathrm{S}^*M, \mathbb{C}^{k+1})$$

with $\ker \sigma_{\mathrm{L}} = \mathrm{CL}^{m-1,k}(M)$. *Furthermore, for* $A \in \mathrm{CL}^{a,k}(M), B \in \mathrm{CL}^{b,l}(M)$ *we have* $\sigma_{\mathrm{L}}^{a+b}(AB) = \sigma_{\mathrm{L}}^a(A)\sigma_{\mathrm{L}}^b(B)$.

Proof That σ_{L} is well defined follows immediately from Proposition 3.5. Similar to 2.14 the isomorphism $\mathcal{P}^{m,k}(T^*M) \cong C^\infty(\mathrm{S}^*M, \mathbb{C}^{k+1})$ is given by

$$\sum_{j=0}^{k} f_j \log^j |.| \mapsto (f_0, \ldots, f_k)|\mathrm{S}^*M,$$

where S^*M denotes the cosphere bundle of M. Surjectivity of σ_{L} is proved by the standard construction of gluing together local pseudodifferential data. □

For defining the polyhomogeneous residue of an operator $A \in \mathrm{CL}^{a,k}(M)$ we need the meromorphic continuation of the function $\mathrm{Tr}(AP^{-s})$ for a classical elliptic pseudodifferential operator P. In principle, this could be done for any elliptic $P \in \mathrm{CL}^{m,k}(M)$. The meromorphic continuation of $\mathrm{Tr}(P^{-s})$, $P \in \mathrm{CL}^{m,k}(M)$, was already proved in [15]. However, for treating $\mathrm{Tr}(AP^{-s})$ one has to modify Sections 1 and 2 of [9]. Therefore, we decided to content ourselves to classical elliptic $P \in \mathrm{CL}^{m,0}(M)$, which is enough for our purposes. Then the method of loc. cit. directly applies. Nevertheless, as mentioned on p. 488 of loc. cit. the results there could be generalized to operators of class $\mathrm{CL}^{m,k}$.

First we state the expansion result for the resolvent (cf. [9, Thm. 2.7]). For the definition of ellipticity with parameter see loc. cit. Def. 2.7.

Theorem 3.7 *Let M be a compact manifold of dimension n, E a C^∞ vector bundle over M, and $A \in \mathrm{CL}^{a,k}(M,E)$. Assume furthermore that $P \in \mathrm{CL}^{m,0}(M,E)$ is elliptic with parameter $\mu \in \Gamma$. Then for $\lambda \in -\Gamma^m := \{-\mu^m \,|\, \mu \in \Gamma\}$ and $N \in \mathbb{N}$ with $-Nm + a < -n$ the kernel $(A(P-\lambda)^{-N})(x,y)$ of $A(P-\lambda)^{-N}$ satisfies on the diagonal*

$$(A(P-\lambda)^{-N})(x,x) \sim \sum_{j=0}^{\infty}\sum_{l=0}^{k} c_{jl}(x)\lambda^{\frac{n+a-j}{m}-N}\log^l \lambda \quad + \sum_{j=0}^{\infty}\sum_{l=0}^{k+1} d_{jl}(x)\lambda^{-j-N}\log^l \lambda,$$

as $|\lambda| \to \infty$, uniformly in closed subsectors of Γ.

Proof This is proved similar to [9, Thm 2.7]. For the convenience of the reader we indicate the steps of the proof of loc. cit. that have to be modified. During this proof we will use freely the notation of [9].

Let $\mathcal{P}$ be a pseudodifferential parametrix of $(P-\lambda)^{-N}$ as constructed in [9, p. 503]. Then

$$A(P-\lambda)^{-N} = A\mathcal{P}^N + \sum_{l\geq 1} A\mathcal{P}R_{(l)} \tag{3.8}$$

where $\mathcal{P}R_{(l)} \in \mathrm{OP}(S^{-\infty,-(l+1)m})$ and thus also $A\mathcal{P}R_{(l)} \in \mathrm{OP}(S^{-\infty,-(l+1)m})$. Since the symbol expansion of $A\mathcal{P}R_{(l)}$ depends on μ only as a rational function of $\lambda = -\mu^m$ the kernel of $A\mathcal{P}R_{(l)}$ has the asymptotic expansion

$$K_{A\mathcal{P}R_{(l)}}(x,x,\lambda) \sim_{\lambda\to\infty} \sum_{\sigma=0}^{\infty} c_{l,\sigma}(x)\lambda^{-l-N-\sigma} \tag{3.9}$$

([9, p. 504]).

To expand the kernel of $A\mathcal{P}^N$ we note that $A\mathcal{P}^N = \mathrm{OP}(q)$ where $q \in S^{a+\varepsilon-Nm,0} \cap S^{a+\varepsilon,-Nm}$ for every $\varepsilon > 0$. Note that although A is not weakly polyhomogeneous in the sense of [9] we have $A \in \cap_{\varepsilon>0} L^{a+\varepsilon}(M)$.

Now q has an expansion $q \sim \sum_{j\geq 0} q_j$, where

$$q_j = \sum_{|\alpha|+l+l'=j} \frac{i^{-|\alpha|}}{\alpha!} (\partial_\xi^\alpha a_{a-l}) \partial_x^\alpha p_{-mN-l'}, \tag{3.10}$$

$a_{a-l} \in \mathcal{P}^{a-l,k}(T^*M)$, $p_{-mN-l'}$ is $-mN-l'$–homogeneous in (ξ,μ) for $|\xi|\geq 1$.

The kernels of the remainders $r_J = q - \sum_{0\leq j<J} q_j$ are expanded as in [9, (2.3)] which gives again integer powers λ^{-N-l} in the expansion of the kernel. Here we have to note again that $p_{-mN-l'}(x,\xi,\mu)$ is a rational function in $-\mu^m$.

Picking one of the summands of q_j we are finally facing the problem of expanding the integral

$$\int_{\mathbb{R}^n} a(x,\xi)\,\log^t|\xi|\, b(x,\xi,\mu)\, d\xi, \quad t\leq k, \tag{3.11}$$

where $a(x,\xi)$ is $(a-l-|\alpha|)$–homogeneous for $|\xi|\geq 1$ and $b(x,\xi,\mu)$ is $(-mN-l')$–homogeneous in (ξ,μ) for $|\xi|\geq 1$. Furthermore, $b(x,\xi,\mu)$ is a rational function of $-\mu^m$ with coefficients homogeneous in ξ.

Now we split the integral (3.11) into the three pieces $|\xi|\geq|\mu|, 1\leq|\xi|\leq|\mu|, |\xi|\leq 1$ as in [9, (2.6)].

By homogeneity for $|\xi|\geq 1$ we find

$$\begin{aligned} &\int_{|\xi|\geq|\mu|} a(x,\xi)\,\log^t|\xi|\, b(x,\xi,\mu)d\xi \\ = \;& |\mu|^{a+n+j-mN} \int_{|\xi|\geq 1} a(x,\xi)\,(\log|\xi\mu|)^t\, b(x,\xi,\frac{\mu}{|\mu|})d\xi \\ = \;& \sum_{\sigma=0}^{t} \binom{t}{\sigma} \int_{|\xi|\geq 1} a(x,\xi)\,\log^{t-\sigma}|\xi|\, b(x,\xi,\frac{\mu}{|\mu|})\, d\xi\, |\mu|^{a+n-j-mN}\,\log^\sigma|\mu|. \end{aligned} \tag{3.12}$$

Since b is holomorphic in μ by [9, Lemma 2.3] this is actually an expansion in terms of the functions $\mu^{a+n-j-mN}\log^\sigma\mu$.

Now we expand using [9, Theorem 1.12]

$$b(x,\xi,\mu) = \sum_{0\leq\nu<M} b_\nu(x,\xi)\mu^{-m\nu-Nm} + R_M(x,\xi,\mu) \tag{3.13}$$

where b_ν is homogeneous in $|\xi|$ for $|\xi|\geq 1$ and

$$R_M(x,\xi,\mu) = \mathrm{O}(<\xi>^{mM-l'}\,\mu^{-m(M+N)}), \tag{3.14}$$

thus

$$\begin{aligned} &\int_{|\xi|\leq 1} a(x,\xi)\,\log^t|\xi|\, b(x,\xi,\mu)d\xi \\ = \;& \sum_{0\leq\sigma<M} \mu^{-m\nu-Nm} \int_{|\xi|\leq 1} a(x,\xi)\,\log^t|\xi|\, b_\nu(x,\xi)d\xi + O(\mu^{-m(M+N)}). \end{aligned} \tag{3.15}$$

Furthermore,

$$\begin{aligned}
&\int_{1\le|\xi|\le|\mu|} a(x,\xi)\,\log^t|\xi|\,b_\nu(x,\xi)d\xi\,\mu^{-m\nu-Nm}\\
=\;& \mu^{-m\nu-Nm}c_\nu(x)\int_1^{|\mu|} r^{a-l-|\alpha|+m\nu-l'+n-1}\log^t r\,dr\\
=\;& \begin{cases} \displaystyle\sum_{\sigma=0}^{t} c_{\nu,\sigma}(x)(\log^\sigma|\mu|)\mu^{-m\nu-Nm}|\mu|^{a-j+m\nu+n}, & a-j+m\nu+n\neq 0,\\ \displaystyle\frac{c_\nu(x)}{t+1}(\log^{t+1}|\mu|)\mu^{-m\nu-Nm}, & a+m\nu+n-j=0.\end{cases}
\end{aligned}\tag{3.16}$$

Again invoking [9, Lemma 2.3] we end up with the desired expansion. The remainder term coming from R_M is estimated exactly as in [9, p. 498]. The proof is complete. □

Remark 3.8 (3.16) shows that $d_{k+1,l}=0$ if $a\notin\mathbb{Z}_+$.

As a consequence of Theorem 3.7 we have an asymptotic expansion

$$\operatorname{Tr}(A(P-\lambda)^{-N}))\sim\sum_{j=0}^{\infty}\sum_{l=0}^{k}c_{jl}\lambda^{\frac{n+a-j}{m}-N}\log^l\lambda\quad+\sum_{j=0}^{\infty}\sum_{l=0}^{k+1}d_{jl}\lambda^{-j-N}\log^l\lambda. \tag{3.17}$$

If the eigenvalues of the leading symbol of P lie in $\operatorname{Re}\lambda>0$ then via the appropriate Cauchy integral we obtain the heat expansion

$$\operatorname{Tr}(Ae^{-tP})\sim_{t\to 0+}\sum_{j=0}^{\infty}t^{\frac{j-n-a}{m}}c_j(\log t)+\sum_{j=0}^{\infty}t^j d_j(\log t) \tag{3.18}$$

with polynomials $c_j,d_j\in\mathbb{C}[x],\deg c_j\le k,\deg d_j\le k+1$.

Furthermore, the generalized ζ–function

$$\operatorname{Tr}(AP^{-s})=\frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}\operatorname{Tr}(Ae^{-tP})dt \tag{3.19}$$

has a meromorphic continuation to $\mathbb{C}$ with poles in $\{\frac{a+n-j}{m}\,|\,j\in\mathbb{Z}_+\}\cup\{-j\,|\,j\in\mathbb{Z}_+\}$. If $\frac{j-n-a}{m}\notin\mathbb{Z}_+$ then the pole has order $\deg c_j+1$, while the pole $-j, j\in\mathbb{Z}_+$, has order $\deg(c_{mj+n+a}+d_j)$. Here we have put $c_\mu:=0$ if $\mu\notin\mathbb{Z}_+$. (cf. e.g. [3, Lemma 2.1], [14, Sec 2.1], [10]).

Using the notation $f(s)=:\sum_k \operatorname{Res}_k f(s_0)(s-s_0)^{-k}$ for Laurent expansions we note explicitly

$$\operatorname{Res}_{k+1}\operatorname{Tr}(AP^{-s})|_{s=\frac{a+n-j}{m}} = \frac{(-1)^k}{\Gamma(\frac{a+n-j}{m})}\frac{d^k}{dt^k}c_j(t)|_{t=0},\quad \frac{j-n-a}{m}\notin\mathbb{Z}_+, \tag{3.20}$$

$$\operatorname{Res}_{k+1}\operatorname{Tr}(AP^{-s})|_{s=-j} = (-1)^{k+j+1}j!\frac{d^{k+1}}{dt^{k+1}}d_j(t)|_{t=0}. \tag{3.21}$$

4. The polyhomogeneous noncommutative residue

In this section we consider a compact closed manifold M of dimension n. Let E be a C^∞ vector bundle over M and $A \in \mathrm{CL}^{a,k}(M, E)$ a pseudodifferential operator with polyhomogeneous symbol.

We pick an elliptic classical pseudodifferential operator $P \in \mathrm{CL}^{m,0}(M, E)$ of order $m > 0$ whose leading symbol is *scalar* and *positive*. For instance we could take a generalized Laplace operator for P. The assumption that the leading symbol of P is scalar guarantees that the commutator $[P, A]$ lies in $\mathrm{CL}^{a+m-1,k}(M, E)$, which makes life easier in the sequel. We emphasize however that this assumption is not really important.

We put

$$\nabla_P^0 A := A, \quad \nabla_P^{j+1} A := [P, \nabla^j A], \tag{4.1}$$

and by induction we have

$$\nabla_P^j A \in \mathrm{CL}^{a+j(m-1),k}(M, E). \tag{4.2}$$

Definition 4.1 *We put*

$$\begin{aligned}
\mathrm{Res}_k(A, P) &:= m^{k+1}\mathrm{Res}_{k+1}\mathrm{Tr}(AP^{-s})|_{s=0} \\
&= m^{k+1}(-1)^{k+1} d_0^{(k+1)}(0) \\
&= m^{k+1}(-1)^{k+1}(k+1)! \times \text{ \textit{coefficient of} } \log^{k+1} t \text{ \textit{in the asymptotic}} \\
&\quad\ \text{\textit{expansion of}} \,\mathrm{Tr}(Ae^{-tP}) \text{ \textit{as} } t \to 0. \\
&= m^{k+1}(-1)^{N}(k+1)! \times \text{ \textit{coefficient of} } \lambda^{-N} \log^{k+1} \lambda \text{ \textit{in the asymptotic}} \\
&\quad\ \text{\textit{expansion of}} \,\mathrm{Tr}(A(P-\lambda)^{-N}) \text{ \textit{as} } \lambda \to \infty.
\end{aligned}$$

We abbreviate

$$\Delta_N := \{(t_1, \ldots, t_N) \in \mathbb{R}^N \mid 0 \le t_1 \le \ldots \le t_N \le 1\}. \tag{4.3}$$

Then we have the well–known formula[1]

$$e^{-tP} A = \sum_{j=0}^{N-1} \frac{(-t)^j}{j!} (\nabla_P^j A) e^{-tP} + R_N(A, P, t), \tag{4.4}$$

where

$$R_N(A, P, t) = (-1)^N \int_{t\Delta_N} e^{-t_1 P} (\nabla_P^N A) e^{-(t-t_1)P} dt_1 \ldots dt_N. \tag{4.5}$$

Lemma 4.2 *If $p, \varepsilon, N > 0$ such that $(N-a)/m - p - \varepsilon > 0$ then we have the estimate*

$$\|R_N(A, P, t)(P + c)^p\| \le C t^{\frac{N-a}{m} - p - \varepsilon}, \quad 0 < t \le 1.$$

Here $c > 0$ is any constant such that $P + c$ is invertible.

[1] Actually it was not well–known to the author; he has learned the formula and the following lemma from Henri Moscovici

Proof Since the leading symbol of P is positive the operator $P+c$ is invertible for some $c>0$. $\nabla_P^N A \in \mathrm{CL}^{a+N(m-1),k}(M,E) \subset \mathrm{L}^{a+N(m-1)+m\varepsilon}(M,E)$, hence $(P+c)^{\frac{N-a}{m}-N-\varepsilon}\nabla_P^N A$ is bounded and thus we obtain the estimate

$$\begin{aligned}\|e^{-t_1P}(\nabla_P^N A)e^{-(t-t_1)P}(P+c)^p\| &\le C\|(P+c)^{\frac{a-N}{m}+N+p+\varepsilon}e^{-(t-t_1)P}\| \\ &\le C(t-t_1)^{\frac{N-a}{m}-N-p-\varepsilon}, \quad 0\le t_1<t\le 1.\end{aligned}$$

Integrating this inequality gives the desired estimate. □

Proposition 4.3 *Let $A\in \mathrm{CL}^{a,k}(M,E)$ and let $P\in \mathrm{CL}^{m,0}(M,E)$ be a classical elliptic pseudodifferential operator of order $m>0$ whose leading symbol is scalar and positive. Then*

1. $\mathrm{Res}_k(A,P^\alpha)=\mathrm{Res}_k(A,P)$ *for any* $\alpha>0$.
2. *If P_u is a smooth 1–parameter family of such operators then $\mathrm{Res}_k(A,P_u)$ is independent of the parameter u.*
3. *If $B\in \mathrm{CL}^{b,l}(M,E)$ then $\mathrm{Res}_{k+l}([A,B],P)=0$.*

Proof 1. is straightforward.

2. In view of Lemma 4.2 we have for N large enough

$$\begin{aligned}\frac{d}{du}Ae^{-tP_u} &= -\int_0^t Ae^{-(t-t_1)P_u}(\frac{d}{du}P_u)e^{-t_1P_u}dt_1 \\ &= \sum_{j=0}^{N-1}\frac{(-t)^{j+1}}{(j+1)!}(A\nabla_P^j\frac{d}{du}P_u)e^{-tP}+R_N(t),\end{aligned}$$

where

$$\|R_N(t)\|_{\mathrm{tr}}=O(t), \quad t\to 0.$$

Here $\|\cdot\|_{\mathrm{tr}}$ denotes the trace norm and this trace norm estimate follows from Lemma 4.2 since $(P+c)^{-p}$ is trace class if $p>\dim M/m$.

Thus we conclude

$$\frac{d}{du}\mathrm{Tr}(Ae^{-tP_u})=\sum_{j=0}^{N-1}\frac{(-1)^{j+1}}{(j+1)!}t^{j+1}\mathrm{Tr}(A\nabla_P^j\frac{d}{du}P_u)e^{-tP})+O(t), \quad t\to 0.$$

Since $\mathrm{Tr}(A\nabla_P^j(\frac{d}{du}P_u)e^{-tP})$ has no $t^{-j-1}\log^{k+1}t$ term in its asymptotic expansion (cf. (3.18)) we reach the conclusion.

3. We use again Lemma 4.2 and we find for N large enough

$$Ae^{-tP}B=\sum_{j=0}^{N-1}\frac{(-t)^j}{j!}A(\nabla_P^jB)e^{-tP}+R_N(t),$$

where $\|R_N(t)\|_{\mathrm{tr}}=O(t), \quad t\to 0$. Thus

$$\mathrm{Tr}([A,B]e^{-tP})=-\sum_{j=1}^{N-1}\frac{(-1)^j}{j!}t^j\mathrm{Tr}(A(\nabla_P^jB)e^{-tP})+O(t), \quad t\to 0.$$

Since $A\nabla_P^jB\in \mathrm{CL}^{a+b+j(m-1),k+l}(M,E)$ the highest $\log t$ power occuring as coefficient of t^{-j} in the asymptotic expansion of $\mathrm{Tr}(A(\nabla_P^jB)e^{-tP})$ as $t\to 0$ is $\le k+l$ and hence $\mathrm{Res}_{k+l}([A,B],P)=0$. □

Summing up we have proved:

Theorem 4.4 *For* $A \in \mathrm{CL}^{a,k}(M,E)$

$$\mathrm{Res}_k(A) := \mathrm{Res}_k(A,P)$$

is well–defined independent of the particular P *chosen.* Res_k *is a linear functional on* $\mathrm{CL}^{*,k}(M,E)$ *which vanishes on appropriate commutators. More precisely,*

$$\mathrm{Res}_{k+l}([A,B]) = 0 \tag{4.6}$$

for $A \in \mathrm{CL}^{a,k}(M,E), B \in \mathrm{CL}^{b,l}(M,E)$.

We note that Res_0 coincides with the noncommutative residue of WODZICKI. This follows directly from the definition and [13, Thm. 1.4].

Although (4.6) holds Res_k is not a trace on the full algebra $\mathrm{CL}^{*,*}(M,E)$. We will see in Theorem 4.11 below that this algebra does not have any nontrivial traces.

Note furthermore that $\mathrm{Res}_k(A) = 0$ if $\mathrm{ord}(A) < -n$ because then A is of trace class and hence $\mathrm{Tr}(AP^{-s})$ is regular at $s = 0$.

4.1 The residue density

Proposition 4.5 (cf. [19, Prop. 3.2]) *Let* M *be a (not necessarily compact) smooth manifold. For* $A \in \mathrm{CL}^{m,k}(M,E,F)$ *there exists a density*

$$\omega_k(A) \in C^\infty(M, \mathrm{Hom}(E,F) \otimes |\Omega|)$$

with the following properties:

1. $A \mapsto \omega_k(A)$ *is* $\mathbb{C}$*–linear.*

2. *If* $\varphi \in C_0^\infty(M)$ *then* $\omega_k(\varphi A) = \varphi \omega_k(A)$.

3. *If* $\kappa : U \to V \subset \mathbb{R}^n$ *is a local chart then*

$$\kappa^*(\omega_k(\kappa_* A)) = \omega_k(A).$$

4. *In a local coordinate chart we have*

$$\begin{aligned} \omega_k(A)(x) &= \frac{(k+1)!}{(2\pi)^n} \mathrm{res}_k(a_{-n}(x,\cdot))|dx| \\ &= \frac{(k+1)!}{(2\pi)^n} \Big(\int_{|\xi|=1} a_{-n,k}(x,\xi)|d\xi| \Big) |dx|. \end{aligned} \tag{4.7}$$

5. *If the complete symbol of* $A \in \mathrm{CL}^{m,k}(M,E,F)$ *vanishes at* $p \in M$ *then* $\omega_k(A)(p) = 0$. *If* $m < -\dim M$ *then* $\omega_k(A) = 0$.

6. *Let* $E = F$, $A \in \mathrm{CL}^{a,k}(M,E), B \in \mathrm{CL}^{b,l}(M,E)$. *If* A *or* B *have compact support then*

$$\int_M \mathrm{tr}_{E_x} \omega_{k+l}([A,B])(x) = 0.$$

Proof We take (4.7) as the definition of $\omega_k(A)$. To show that $\omega_k(A)$ is well defined we proceed along the lines of [7, Thm. 1.4].

Let $U \subset \mathbb{R}^n$, $A \in \mathrm{CL}^{m,k}(U,E,F)$ and $\kappa : U \to V$ a diffeomorphism. Since this consideration is local we may assume E, F to be trivial bundles. By Proposition 3.5 we have

$$\sigma_{\kappa_* A}(\kappa(x),\eta) \sim \sum_{\alpha\in\mathbb{Z}_+^n} (\partial_\xi^\alpha \sigma_A)(x, D\kappa^t(x)\eta)\Phi_\alpha(x,\eta),$$

where $\Phi_\alpha(x,\eta)$ is a polynomial in η of degree not higher than $|\alpha|/2$ and $\Phi_0 = 1$. Hence in view of (2.12), (2.13), and Lemma 2.8

$$\begin{aligned}
&\mathrm{res}_k[\sigma_{\kappa_* A}(\kappa(x),\eta)_{-n}]|d\kappa(x)| \\
= \;& |\det D\kappa(x)| \sum_{\alpha\in\mathbb{Z}_+^n} \mathrm{res}_k[((\partial_\xi^\alpha \sigma_A)(x, D\kappa^t(x)\eta)\Phi_\alpha(x,\eta))_{-n}]|dx| \\
= \;& |\det D\kappa(x)| \mathrm{res}_k[\sigma_A(x, D\kappa^t(x)\eta)_{-n}]|dx| \\
= \;& \mathrm{res}_k[\sigma_A(x,\eta)_{-n}]|dx|.
\end{aligned}$$

This proves 3. and 4. Furthermore, 1., 2., and 5. are simple consequences of 4.

Now it suffices to prove 6. locally. This is done exactly as in the proof of [7, Thm. 1.4] and we omit the details. □

Lemma 4.6 *Let M be a compact Riemannian manifold,* $\dim M = n$. *Furthermore, let $Q \in \mathrm{CL}^{-n,k}(M,E)$ with leading symbol $q_n(x,\xi) = |\xi|^{-n}\log^k|\xi|$. Then*

$$\mathrm{Res}_k(Q) = \int_M \omega_k(Q) \neq 0.$$

Proof This lemma could be derived from the proof of [9, Thm 2.1]. But since our situation is much simpler we will give an ad hoc proof.

Note first that in a coordinate system

$$\omega_k(Q)(x) = \frac{(k+1)!}{(2\pi)^n}\mathrm{vol}(S^{n-1})d\mathrm{vol}_M(x)$$

and thus

$$\int_M \omega_k(Q) = \frac{(k+1)!}{(2\pi)^n}\mathrm{vol}(S^{n-1})\mathrm{vol}(M) \neq 0.$$

We pick an elliptic operator $P \in \mathrm{CL}^{m,0}(M,E), m > 0$, whose leading symbol $p(x,\xi)$ is scalar and positive.

Then since Q has order $-n$ the coefficient of $\lambda^{-N}\log^{k+1}\lambda$ in the asymptotic expansion of $\mathrm{Tr}(A(P-\lambda)^{-N})$ equals the corresponding coefficient of

$$(2\pi)^{-n}\int_{T^*M} \varphi(|\xi|)|\xi|^{-n}\log^k|\xi|(p(x,\xi)-\lambda)^{-N}d\xi dx.$$

Here, $\varphi \in C^\infty(\mathbb{R})$ is a function with $\varphi(t) = 0$, if $t \le 1/2$ and $\varphi(t) = 1$, if $t \ge 1$.

Now

$$\begin{aligned}
&\int_{\mathbb{R}^n} \varphi(|\xi|)|\xi|^{-n}\log^k|\xi|(p(x,\xi)-\lambda)^{-N}d\xi\\
=\ &\lambda^{-N}\int_{\mathbb{R}^n} \varphi(|\lambda^{1/m}\xi|)|\xi|^{-n}\log^k|\lambda^{1/m}\xi|(p(x,\xi)-1)^{-N}d\xi\\
=\ &\lambda^{-N}\int_{|\xi|\le 1} \varphi(|\lambda^{1/m}\xi|)|\xi|^{-n}\log^k|\lambda^{1/m}\xi|(p(x,\xi)-1)^{-N}d\xi + O(\lambda^{-N}\log^k\lambda)\\
=\ &\lambda^{-N}\int_0^1\int_{|\xi|=1} \varphi(|\lambda^{1/m}\varrho|)\varrho^{-1}\log^k|\lambda^{1/m}\varrho|(p(x,\varrho\xi)-1)^{-N}d\xi d\varrho + O(\lambda^{-N}\log^k\lambda)\\
=\ &(-1)^N\lambda^{-N}(\log^k\lambda^{1/m})\mathrm{vol}(S^{n-1})\int_0^1 \varphi(|\lambda^{1/m}\varrho|)\varrho^{-1}d\varrho + O(\lambda^{-N}\log^k\lambda)\\
=\ &\frac{(-1)^N}{m^{k+1}}\mathrm{vol}(S^{n-1})\lambda^{-N}\log^{k+1}\lambda + O(\lambda^{-N}\log^k\lambda)
\end{aligned}$$

and hence

$$\begin{aligned}
&(2\pi)^{-n}\int_{T^*M} \varphi(|\xi|)|\xi|^{-n}\log^k|\xi|(p(x,\xi)-\lambda)^{-N}d\xi dx\\
=\ &\Big(\frac{(-1)^N}{m^{k+1}(k+1)!}\int_M \omega_k(Q)\Big)\lambda^{-N}\log^{k+1}\lambda + O(\lambda^{-N}\log^k\lambda)
\end{aligned}$$

which proves the assertion in view of Definition 4.1. □

Proposition 4.7 *Let M be a connected compact manifold of dimension $n>1$. Choose a $Q\in \mathrm{CL}^{-n,k}(M)$ with $\mathrm{Res}_k(Q)=\int_M\omega_k(Q)\neq 0$. Then there exist $P_1,\dots,P_N\in \mathrm{CL}^{1,0}(M)$ such that for any $A\in\mathrm{CL}^{a,k}(M)$ there exist $Q_1,\dots,Q_N\in\mathrm{CL}^{a,k}(M)$ such that*

$$A-\sum_{j=1}^N[P_j,Q_j]-\frac{\int_M\omega_k(A)}{\mathrm{Res}_k(Q)}Q\in \mathrm{L}^{-\infty}(M).$$

Proof We choose $P_1,\dots,P_N\in\mathrm{CL}^{1,0}(M)$ such that the differentials of the leading symbols span $T^*M\setminus 0$ at every point. We consider the leading symbol $\sigma^a_{\mathrm{L}}(A)\in\mathcal{P}^{a,k}(T^*M)$ of A. If $a\neq -n$ then by Theorem 2.13 there exist $Q_1^{(1)},\dots,Q_N^{(1)}\in\mathrm{CL}^{a,k}(M)$ such that

$$A-\sum_{j=1}^N[P_j,Q_j^{(1)}]\in\mathrm{CL}^{a-1,k}(M).$$

(Note that the leading symbol of a commutator is the Poisson bracket of the leading symbols).

If $a-1\neq -n$ we iterate the procedure. Thus if $a\notin\{l\in\mathbb{Z}\,|\,l\ge -n\}$ then by induction we find operators $Q_j^{(l)}\in\mathrm{CL}^{a,k}(M)$ such that

$$A^{(l)}:=A-\sum_{j=1}^N[P_j,Q_j^{(l)}]\in\mathrm{CL}^{a-l,k}(M). \tag{4.8}$$

If $a \in \{l \in \mathbb{Z} \,|\, l \geq -n\}$ then (4.8) holds for $l \leq a+n$. In view of Proposition 4.5 we have

$$\int_M \omega_k(A^{(a+n)} - \frac{\int_M \omega_k(A)}{\mathrm{Res}_k(Q)} Q) = 0.$$

Using again Theorem 2.13 we find by induction operators $Q_j^{(l)} \in \mathrm{CL}^{a,k}(M)$ such that

$$A^{(l)} := A - \sum_{j=1}^{N} [P_j, Q_j^{(l)}] - \frac{\int_M \omega_k(A)}{\mathrm{Res}_k(Q)} Q \in \mathrm{CL}^{a-l,k}(M). \tag{4.9}$$

Now choose $Q_j \in \mathrm{CL}^{a,k}(M)$ with $Q_j - Q_j^{(l)} \in \mathrm{CL}^{a-l,k}(M)$. Then we reach the conclusion. □

Corollary 4.8 *For $A \in \mathrm{CL}^{a,k}(M, E)$ we have*

$$\mathrm{Res}_k(A) = \int_M \mathrm{tr}_{E_x} \omega_k(A)(x) = \frac{(k+1)!}{(2\pi)^n} \int_{S^*M} a_{-n,k}(x,\xi) |d\xi dx|.$$

Proof For $E = \mathbb{C}$ this is an immediate consequence of Theorem 4.4, Proposition 4.5 and the preceding proposition. For arbitrary E it follows from the facts that there exists a bundle F making $E \oplus F$ trivial and $\mathrm{CL}^{a,k}(M, \mathbb{C}^r) \cong \mathrm{CL}^{a,k}(M) \otimes M_r(\mathbb{C})$. □

We note explicitly as a consequence of Proposition 4.7 that if $A \in \mathrm{CL}^{a,k}(M)$ there are always $Q_j \in \mathrm{CL}^{a,k+1}(M)$ such that

$$A - \sum_{j=1}^{N} [P_j, Q_j] \in \mathrm{L}^{-\infty}(M).$$

This can actually be improved, using the fact that every classical pseudodifferential operator with vanishing residue is a sum of commutators. This result is due to Wodzicki [18], a generalization to algebras of Fourier integral operators is due to Guillemin [12]. In particular every smoothing operator is a sum of commutators of classical pseudodifferential operators. Since [18] is written in Russian we briefly sketch the argument: one actually has to show that $\mathrm{CL}^*(M)/[\mathrm{CL}^*(M), \mathrm{CL}^*(M)]$ is one–dimensional. Given any trace τ on $\mathrm{CL}^*(M)$ consider first $\tau | \mathrm{L}^{-\infty}(M)$. By [12, Appendix] $\mathrm{L}^{-\infty}(M)/[\mathrm{L}^{-\infty}(M), \mathrm{L}^{-\infty}(M)] \cong \mathbb{C}$ and it is spanned by the L^2–trace. Thus $\tau|\mathrm{L}^{-\infty}(M) = c\,\mathrm{Tr}_{\mathrm{L}^2}$. The same argument as in the introduction (1.1) then shows $c = 0$. Hence τ induces a trace on $\mathrm{CL}^*(M)/\mathrm{L}^{-\infty}(M)$ and thus in view of Proposition 4.7 it is a constant multiple of the Wodzicki residue.

Summing up we can improve Proposition 4.7 as follows:

Proposition 4.9 *Let M be a connected compact manifold of dimension $n > 1$. Choose a $Q \in \mathrm{CL}^{-n,k}(M)$ with $\mathrm{Res}_k(Q) = \int_M \omega_k(Q) \neq 0$. Then there exist $P_1, \ldots, P_N \in \mathrm{CL}^{1,0}(M)$ such that for any $A \in \mathrm{CL}^{a,k}(M)$ there exist $Q_1, \ldots, Q_N \in \mathrm{CL}^{a,k}(M)$ and classical pseudodifferential operators $R_1, \ldots, R_N, S_1, \ldots, S_N \in \mathrm{CL}^{*,0}(M)$ such that*

$$A = \sum_{j=1}^{N} [P_j, Q_j] + \sum_{j=1}^{N} [R_j, S_j] + \frac{\int_M \omega_k(A)}{\mathrm{Res}_k(Q)} Q.$$

In particular, since $\mathrm{Res}_{k+1}(A) = 0$ *there exist* $Q_1, \ldots, Q_N \in \mathrm{CL}^{a,k+1}(M)$ *and* $R_1, \ldots, R_N$, $S_1, \ldots, S_N \in \mathrm{CL}^{*,0}(M)$ *such that*

$$A = \sum_{j=1}^{N} [P_j, Q_j] + \sum_{j=1}^{N} [R_j, S_j].$$

Remark 4.10 This result generalizes the corresponding result for classical pseudodifferential operators due to WODZICKI [18] [13, Prop. 5.4]. In [12], WODZICKI's result was generalized to certain algebras of Fourier integral operators.

An immediate consequence is the

Corollary 4.11 *There are no nontrivial traces on the algebra* $\mathrm{CL}^{*,*}(M)$.

References

[1] J. BRÜNING and E. HEINTZE: *The asymptotic expansion of Minakshisundaram–Pleijel in the equivariant case.* Duke Math. J. **51** (1984), 959–980.

[2] J. BRÜNING and M. LESCH: *On the eta–invariant of certain non–local boundary value problems.* Preprint, Humboldt–Universität zu Berlin, 1996; to appear in Duke Math. J.

[3] J. BRÜNING and M. LESCH: *On the spectral geometry of algebraic curves.* J. reine angew. Math. **474** (1996), 25–66.

[4] J. BRÜNING and R. SEELEY: *The resolvent expansion for second order regular singular operators.* J. Funct. Anal. **73** (1987), 369–429.

[5] J. CHEEGER: *Spectral geometry of singular Riemannian spaces.* J. Diff. Geom. **18** (1983), 575–657.

[6] A. CONNES and H. MOSCOVICI: *The local index theorem in noncommutative Geometry.* GAFA **5** (1995), 174–243.

[7] B. FEDOSOV, F. GOLSE, E. LEICHTNAM, and E. SCHROHE: *The noncommutative residue for manifolds with boundary.* J. Funct. Anal. **142** (1996), 1–31.

[8] P. GILKEY: *Invariance theory, the heat equation, and the Atiyah–Singer index theorem.* 2. edition, CRC Press, Boca Raton, 1995.

[9] G. GRUBB and R. SEELEY: *Weakly parametric pseudodifferential operators and Atiyah–Patodi–Singer boundary problems.* Invent. Math. **121** (1995), 481–529.

[10] G. GRUBB and R. T. SEELEY: *Zeta and eta functions for Atiyah–Patodi–Singer operators.* J. Geom. Anal. **6** (1996), 31–77.

[11] V. GUILLEMIN: *A new proof of Weyl's formula on the asymptotic distribution of eigenvalues.* Adv. Math. **55** (1985), 131–160.

[12] V. GUILLEMIN: *Residue traces for certain algebras of Fourier integral operators.* J. Funct. Anal. **115** (1993), 391–417.

[13] C. KASSEL: *Le résidue non commutatif (d'après M. Wodzicki).* Séminaire Bourbaki, 41ème année, 1988–89, no 708, 199–229.

[14] M. LESCH: *Operators of Fuchs type, conical singularities, and asymptotic methods.* Teubner Texte zur Mathematik Vol. 136, Teubner–Verlag, Leipzig, 1997.

[15] E. SCHROHE: *Complex powers of elliptic pseudodifferential operators.* Integral Equations Operator Theory **9** (1986), 337–354.

[16] R. T. Seeley: *Complex powers of an elliptic operator.* Proc. Symp. Pure Math. **10** (1967), 288–307.

[17] M. A. Shubin: *Pseudodifferential operators and spectral theory.* Springer–Verlag, Berlin–Heidelberg–New York, 1980.

[18] M. Wodzicki: *Spectral asymmetry and local invariants (in Russian).* Habilitation thesis, Moscow: Steklov Math. Inst., 1984.

[19] M. Wodzicki: *Noncommutative residue: Chapter I. Fundamentals.* In Y. Manin (ed.), *K–theory, arithmetic and geometry*, Lect. Notes Math. **1289**. Springer–Verlag, Berlin–Heidelberg–New York, 1987.

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