A deformation theoretical approach to Weyl quantization on Riemannian manifolds

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28th July 1997

Abstract

Abstract. By using a sheaf theoretical language we introduce a notion of deformation quantization allowing not only for formal deformation parameters but for real or complex ones as well. As a model for this approach to deformation quantization we construct a quantization scheme for cotangent bundles of Riemannian manifolds. Hereby we essentially use a complete symbol calculus for pseudodifferential operators on a Riemannian manifold. Depending on a scaling parameter our quantization scheme corresponds to normally ordered, Weyl or antinormally ordered quantization. Finally it is shown that our quantization scheme induces a family of pairwise isomorphic strongly closed star-products on a cotangent bundle.

Mathematics Subject Classification (1991): 81S20 81S30 35S05

Key Words: deformations, quantization, pseudodifferential operators, star-products

Contents

Introduction

In the theory of star-products by Bayen, Flato, Fronsdal, Lichnerowicz and Sternheimer[2] one approaches quantization theory by deforming the product on the

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algebra of smooth functions on a symplectic manifold to a noncommutative product, namely the star-product, in a way such that Diracs quantization condition is fulfilled. The mathematical framework hereby used is the deformation theory of algebras from Gerstenhaber [8]. By the fundamental work of deWilde, Lecomte [27], Karasev,Maslov [12], Fedosov [6] and Omori, Yaedi, Yoshioka [14] one knows that every symplectic manifold has a star-product. The mathematical methods used by the different authors to prove the existence theorem for star-products vary considerably, from methods of Hochschild homology of algebras [27] over sheaf theory and Lagrangian geometry [12] to methods using symplectic connections [6] or differential geometry and Čhech cohomolgy [14]. Though the existence of star-products on symplectic manifolds is now well-known, it remains to single out - if possible - classes of canonical star-products fulfilling natural properties like closedness or functoriality with respect to an appropriate class of morphisms.

In this paper we construct a natural quantization scheme respectively star-product on a particular class of symplectic manifolds, namely cotangent bundles over Riemannian manifolds. This quantization comes from a generalization of the Weyl quantization on $\mathbb{R}^n$ - or in other words from the Moyal product on $\mathbb{R}^n$ - to arbitrary Riemannian manifolds. We show that Weyl quantization maps real classical observables to (formally) selfadjoint pseudodifferential operators. Moreover it is proved that the star-product corresponding to the Weyl quantization on Riemannian manifolds is strongly closed. Hence we give a positive answer to the question posed by Flato, Sternheimer [7], §2.6 on the strong closedness of star-products induced by different symbol calculi for pseudodifferential operators on manifolds.

The paper is organized as follows. In the first section we introduce our concept of a deformation quantization. By using a sheaf theoretical language we succeed in setting up a general language of deformation theory for (sheaves of) algebras. Thus we keep the language close to algebraic geometry (cf. Hartshorne [11]) and complex analysis (cf. Gindikin, Khenkin [9]), but unlike in algebraic geometry we allow sheaves of noncommutative algebras for deformation. The advantage of using sheaves lies in the fact that it allows to formulate not only deformations over a formal parameter but also deformations over an arbitrary commutative locally ringed space.

In the following section the theory of λ-symbol calculus for pseudodifferential operators on Riemannian manifolds is explained. On the one hand it comprises a generalization of the symbol calculus introduced by Widom [26] and on the other hand a generalization of the λ-ordered pseudodifferential calulus on Euclidean space $\mathbb{R}^n$ to manifolds. Furthermore in this section we provide some analytical properties of the λ-symbol calculus like a trace formula, adjointness relations and an asymptotic expansion for the λ-symbol of the product of two pseudodifferential operators.

In the third section the analytical tools from the previous one are used to construct a scale of quantizations of symbol sheaves resp. sheaves of observables polynomial in momentum. Finally we consider the scale of star-products induced by these quantizations and show that these star-products are all equivalent and strongly closed.

1 The concept of deformation quantization

First let us briefly recall the notion of a ringed space. A **ringed space** is a pair $(X, \mathcal{A})$ where X is a topological space and $\mathcal{A}$ a sheaf of rings on X. $\mathcal{A}$ is called the **structure sheaf** of the ringed space. It gives the topological space X an additional structure like that of a smooth manifold, complex space, scheme or supermanifold and can be regarded as a space of germs of *admissible generalized functions* on X. If the stalks $\mathcal{A}_x$ with x in X are all commutative (resp. noncommutative, local) we call $(X, \mathcal{A})$ a commutative (resp. noncommutative, locally) ringed space. If all the sections $\mathcal{A}(U)$ with U open in X are k-algebras, where k is a field, one says $(X, \mathcal{A})$ to be a k-ringed space or a ringed space over k.

A **morphism of ringed spaces** from $(X, \mathcal{A})$ to $(Y, \mathcal{B})$ is a pair (f, ϕ), where $f : X \to Y$ is a continuous map and $\phi : \mathcal{B} \to f_*(\mathcal{A})$ a morphism of sheaves on Y with values in the category of rings. Here $f_*(\mathcal{A})$ is the direct image of $\mathcal{A}$ via f, that means $f_*(\mathcal{A}(U)) = \mathcal{A}\left(f^{-1}(U)\right)$ for all open $U \subset Y$. If the ringed spaces $(X, \mathcal{A})$ and $(Y, \mathcal{B})$ are locally ringed, we require $\phi_x : \mathcal{B}_{f(x)} \to \mathcal{A}_x$ to be local for all x in X. In case $(X, \mathcal{A})$ and $(Y, \mathcal{B})$ are ringed spaces over k, it is assumed that $\phi(U) : \mathcal{B}(U) \to \mathcal{A}\left(f^{-1}(U)\right)$ is a homomorphism of k-algebras for all $U \subset X$ open.

The notion of a fibered morphism of ringed spaces is crucial for the definition of a deformation.

Definition 1.1 *A morphism $(f, \phi) : (X, \mathcal{A}) \to (P, \mathcal{S})$ of ringed spaces is called* **fibered**, *if the following conditions are fulfilled:*

(i) $(P, \mathcal{S})$ is a commutative locally ringed space.

(ii) $f : X \to P$ is surjective.

(iii) $\phi_x : \mathcal{S}_{f(x)} \to \mathcal{A}_x$ maps $\mathcal{S}_{f(x)}$ into the center of $\mathcal{A}_x$ for all $x \in X$.

The **fiber** *of F over a point p of P then is the ringed space $(X_p, \mathcal{A}_p)$ defined by*

$$X_p = f^{-1}(p), \quad \mathcal{A}_p = \mathcal{A}|_{f^{-1}(p)} / \mathfrak{m}_p \mathcal{A}|_{f^{-1}(p)},$$

where $\mathfrak{m}_p$ is the maximal ideal in $\mathcal{S}_p$ which acts on $\mathcal{A}|_{f^{-1}(p)}$ via ϕ.

A deformation now is a fibered morphism which in a certain sense is locally trivial.

Definition 1.2 *A* **deformation** *of a ringed space $(X, \mathcal{A})$ over the (commutative locally ringed)* **parameter space** *$(P, \mathcal{S})$ consists of a fibered morphism $D = (d, \Delta) : (Y, \mathcal{B}) \to (P, \mathcal{S})$ and a point $\bullet \in X$ such that $(d^{-1}(\bullet), \mathcal{B}_\bullet)$ is isomorphic to $(X, \mathcal{A})$ and $\mathcal{B}$ is* **locally trivial** *over $\mathcal{S}$ in the algebraic sense, i.e. for every $p \in P$, $y \in d^{-1}(P)$ and $m \in \mathbb{N}^*$ the homomorphism $\Delta_y : \mathcal{S}_p \to \mathcal{B}_y$ induces a flat morphism $\Delta_y^m : \mathcal{S}_p / \mathfrak{m}_p^m \to \mathcal{B}_y / \mathfrak{m}_p^m \mathcal{B}_y$, where $\mathfrak{m}_p$ is the maximal ideal of $\mathcal{S}_p$.*

If $(P, \mathcal{S}) = (\bullet, \mathbb{C}[[\hbar]])$, then the deformation is called a **formal** *one, if $(P, \mathcal{S}) = (\mathbb{R}, \mathcal{C}^\infty)$ the deformation is called* **smooth** *over $\mathbb{R}$.*

By Bourbaki [5] Chapter III, §5.2 the above flatness-condition guarantees in the formal case that for every $x \in X$ the stalk $\mathcal{B}_x$ is isomorphic to $\mathcal{A}_x[[\hbar]]$ as a vector space. In any case it guarantees local triviality (cf. Hartshorne [11]).

In quantization theory one is interested in deformations of **Poisson spaces**, i.e. commutative ringed spaces $(X, \mathcal{A})$ which posses a Poisson bracket $\{\ ,\ \} : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$.

Definition 1.3 *Let $(X, \mathcal{A})$ be a Poisson space. A smooth (resp. formal) deformation of $(X, \mathcal{A})$ then consists of a smooth (resp. formal) deformation $D = (d, \Delta) : (X \times P, \mathcal{B}) \to (P, \mathcal{S})$ of $(X, \mathcal{A})$ and a quantization morphism $Q = (q, \mathfrak{q}) : (X \times P, \mathcal{B}) \to (X, \mathcal{A})$ with $q = \mathrm{pr}_1$ such that*

(i) In the formal case $\mathfrak{q}$ is the canonical embedding $\mathcal{A} \to \mathcal{B} \cong \mathcal{A}[[\hbar]]$.

(ii) The composition $\mathcal{A} \xrightarrow{\mathfrak{q}} \mathcal{B} \to \mathcal{B}_\bullet \cong \mathcal{A}$ is the identity.

(iii) $\mathfrak{q}(1) = 1$.

(iv) The canonical commutation relations

$$[\mathfrak{q}(a), \mathfrak{q}(b)] = -i\hbar\mathfrak{q}\left(\{a, b, \}\right) + o(\hbar^2) \tag{1}$$

are satisfied for every $a, b \in \mathcal{A}(U)$, $U \subset X$ open.

Let X be a symplectic manifold, and consider the sheaf $\mathcal{C}^\infty_M$ of smooth functions on X. Then $(X, \mathcal{C}^\infty_X)$ is a Poisson space. A formal quantization of $(X, \mathcal{C}^\infty_X)$ then gives rise to a **star-product** in the sense of Bayen et.al. [2] on $\mathcal{C}^\infty(X)[[\hbar]]$ and vice versa. In the case X is the cotangent bundle T^*M of a smooth manifold M we will consider the Poisson space $(M, \mathcal{D}_0)$ of **classical observables polynomial in momentum** or in other words of smooth functions on T^*M which are polynomial in the fibers. That means we have for $U \subset M$ open $\mathcal{D}_0(U) = \{a \in \mathcal{C}^\infty(T^*U) | \forall x \in U \; a \text{ is polynomial on } T^*_x M\}$, and the Poisson bracket on $\mathcal{D}_0$ comes from the canonical symplectic structure on T^*M.

In the following we will construct a scale of smooth and formal quantizations of the Poisson space $(M, \mathcal{D}_0)$ and the Poisson space of (asymptotic) symbols on M.

2 Weyl symbol calculus

Before setting up the Weyl symbol calculus for pseudodifferential operators on Riemannian manifolds let us fix some notation.

Let M always be a Riemannian manifold of dimension n, g its metric tensor and ω the canonical symplectic form on T^*M. Further let $W \subset TM$ be an open neighborhood of the zero section such that exp is injective on the fibers W. By $\psi : TM \to [0, 1]$ we denote a cut-off function having support in W and being identical to 1 on a neighborhood of the zero section of TM. The Riemannian structure on M induces a unique volume density μ on M, and for every $x \in M$ canonical volume densities μ_x on $T_x M$ and μ^*_x on $T^*_x M$. Denote for any $x \in M$ by $\rho_x : \tilde{W}_x \to \mathbb{R}^+$ with $\tilde{W}_x = \exp(W \cap (\{x\} \times M))$ the smooth function having the property

$$\rho_x \, \mu|_{\tilde{W}_x} = (\exp_x)_*(\mu_x). \tag{2}$$

Note that $\rho : \tilde{W} \to \mathbb{R}^+$, $(x, y) \mapsto \rho_x(y)$ with $\tilde{W} = \bigcup_{x \in M} \{x\} \times \tilde{W}_x$ is smooth as well. Now let $U \subset M$ be an open subset on which an orthonormal frame $e = (e_1, ..., e_n)$ exists. Then for any $x \in U$ there exist uniquely defined normal coordinates z_x of M at x such that $\partial_{z_x,k}|_x = e_k(x)$ for $k = 1, ..., n$. Finally denote by (z_x, ζ_x) the coordinates of the cotangent bundle T^*M induced by z_x.

By $\mathrm{S}^m(M)$, $m \in \mathbb{R}$ we understand the symbols on T^*M of Hörmander type, i. e. $\mathrm{S}^m(M)$ consists of all smooth functions $a \in \mathcal{C}^\infty(T^*M)$ such that uniformly on compact subsets $K \subset U$ of any coordinate patch $U \subset M$

$$\left|\partial_x^\alpha \, \partial_\xi^\beta \, a(x,\xi)\right| \leq C_{K,\alpha,\beta} < \xi >^{m-\beta}, \quad x \in K, \, \xi \in \mathbb{R}^n. \tag{3}$$

Hereby $(x_1, ..., x_n)$ are coordinates over U, $(x_1, ..., x_n, \xi_1, ...\xi_n)$ the induced coordinates on T^*U, $<\xi> = (1 + ||\xi||^2)^{1/2}$ and $C_{K,\alpha,\beta} > 0$. As usual we define $\mathrm{S}^\infty(M) = \cup_{m\in\mathbb{R}} \mathrm{S}^m(M)$ and $\mathrm{S}^{-\infty}(M) = \cap_{m\in\mathbb{R}} \mathrm{S}^m(M)$. Note that for any $m \in \mathbb{R} \cup \{\pm\infty\}$ we receive a sheaf S^m on M whose spaces of sections are given by $\mathrm{S}^m(U)$ for $U \subset M$ open. The space $\Psi^m(M)$ of pseudodifferential operators on M of order m consists of all pseudolocal continuous mappings $\mathcal{D}(M) \to \mathcal{D}'(M)$ which in a local coordinate system $\kappa : U \to \mathbb{R}^n$ have a representation of the form

$$\mathcal{D}(U) \ni u \mapsto \left(x \mapsto \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i<\xi,\kappa(x)-y>} \, a(\kappa(x), y, \xi) \, (u \circ \kappa^{-1})(y) \, dy \, d\xi \right) \in \mathcal{C}^\infty(U) \tag{4}$$

with $a \in \mathrm{S}^\infty(\kappa(U) \times \kappa(U), \mathbb{R}^n)$. Obviously the mapping $U \mapsto \Psi^m(U)$, where U runs through all open sets of M, defines a presheaf Ψ^m on M. By $\Psi^{-\infty}(M)$ we denote the space of all smoothing pseudodifferential operators on M, and by $\Psi^{-\infty}_{\mathrm{vtr}}(M)$ the smoothing pseudodifferential operators with vanishing trace.

Now let $\lambda \in [0, 1]$ be a parameter. We then associate to every $a \in \mathrm{S}^\infty(M)$ an operator $\mathrm{Op}_\lambda(a) \in \mathrm{Hom}\,(\mathcal{D}(M), \mathcal{D}'(M))$ by

$$< \mathrm{Op}_\lambda(a) \, u, v > = \frac{1}{(2\pi)^{n/2}} \int_{T^*M} a(\xi) \, \phi^\lambda_{u,v}(\xi) \, d\Omega(\xi), \tag{5}$$

where $\Omega = \omega^n$ is the Liouville volume form on T^*M, $u, v \in \mathcal{C}^\infty_{\mathrm{cpt}}(M)$ and $\phi^\lambda_{u,v}$ is the Wigner kernel

$$\phi^\lambda_{u,v}(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{T_{\pi(\xi)}M} e^{-i<\xi,X>} \, u(\exp(1-\lambda)X) \, \overline{v}(\exp(-\lambda X)) \, \psi(X) \, d\mu_x(X) \tag{6}$$

for the parameter value λ. In the case $\lambda = 0$ one can think of $\mathrm{Op}(a) = \mathrm{Op}_0(a)$ as the normally ordered operator associated to a, $\mathrm{Op}_{\mathrm{a}}(a) = \mathrm{Op}_1(a)$ is the antinormally ordered operator associated to a, and $\mathrm{Op}_{\mathrm{W}}(a) = \mathrm{Op}_{1/2}(a)$ gives the Weyl operator of a.

Theorem 2.1 *Let $\lambda \in [0, 1]$. Then $\mathrm{Op}_\lambda : \mathrm{S}^\infty(M) \to \mathrm{Hom}\,(\mathcal{D}(M), \mathcal{D}'(M))$ is a linear mapping with the following properties:*

(i) For $a \in \mathrm{S}^m(M)$ the operator $\mathrm{Op}_\lambda(a)$ is pseudodifferential of order m and does not depend on the choice of the cut-off function ψ up to elements of $\Psi^{-\infty}_{\mathrm{vtr}}(M)$.

(ii) The formal adjoint of $\mathrm{Op}_\lambda(a)$ is given by $\mathrm{Op}_\lambda(a)^* = \mathrm{Op}_{1-\lambda}(\overline{a})$. In particular the Weyl operator of a symbol a is formally self-adjoint if and only if a is real-valued.

(*iii*) *Any operator* $\mathrm{Op}_\lambda(a)$ *has an integral representation of the form*

$$
\begin{aligned}
&[\mathrm{Op}_\lambda(a)\, u]\,(x) = \\
&= \frac{1}{(2\pi)^n} \int_{T_xM\times T_x^*M} e^{-i<\xi,X>}\, a\left(T^*_{\exp_x(\lambda X)} \exp_x^{-1}(\xi)\right)\, u(\exp(X))\, \tilde{\psi}(X)\, \rho_{\exp(\lambda X)}(x)\, dX\, d\xi,
\end{aligned} \tag{7}
$$

where $u \in \mathcal{D}(M)$, $x \in M$ *and* $\tilde{\psi}$ *is the cut-off function* $TX \ni X \mapsto \psi(T_X \exp X) \in [0,1]$.

(*iv*) *The Schwartz-kernel* $K_\lambda(a)$ *of* $\mathrm{Op}_\lambda(a)$ *is given as the oscillatory integral*

$$
\begin{aligned}
K_\lambda(a)(x,y) = &\rho_x^{-1}(y)\, \rho_{\exp(\lambda \exp_x^{-1}(y))}(x)\, \tilde{\psi}\left(\exp_x^{-1}(y)\right) \cdot \\
&\cdot \frac{1}{(2\pi)^n} \int_{T_x^*M} e^{-i<\xi,\exp_x^{-1}(y)>}\, a\left(T^*_{\exp_x(\lambda \exp_x^{-1}(y))} \exp_x^{-1}\right)\, d\xi.
\end{aligned} \tag{8}
$$

PROOF: (i) By definition $\mathrm{Op}_\lambda(a)$ has a kernel $K_\lambda(a) \in \mathcal{D}'(M\times M)$. Now the phase-function $T_{\pi(X)}M \ni \xi \mapsto <\xi, X> \in \mathbb{C}$ in Eq. (6) is singular if and only if $X = 0$. Hence the singular support of $K_\lambda(a)$ lies in the diagonal $\delta \subset M\times M$, and $\mathrm{Op}_\lambda(a)$ is a pseudodifferential operator of order m. The fact that $\mathrm{Op}_\lambda(a)$ does not depend on the choice of ψ up to elements of $\Psi^{-\infty}_{\mathrm{vtr}}(M)$ follows immediately from (iv) which will be proven later.
Let us now show (iii). Assuming $\lambda \neq 0$ and $a \in \mathrm{S}^{-\infty}(M)$ we get by using the Gauß Lemma

$$
\begin{aligned}
&< \mathrm{Op}_\lambda(a)\, u, v >= \\
&= \frac{1}{(2\pi)^n} \int_{T^*M} \int_{T_{\pi(\xi)}M} e^{-i<\xi,X>}\, a(\xi)\, u(\exp(1-\lambda)X)\, \overline{v}(\exp(-\lambda X))\, \psi(X)\, d\mu_{\pi(\xi)}(X)\, d\Omega(\xi) \\
&= \frac{1}{(2\pi\lambda)^n} \int_{T^*M} \int_{M} e^{i<\xi,\lambda^{-1}\exp^{-1}_{\pi(\xi)}(x)>}\, a(\xi)\, u\Big(\exp_{\pi(\xi)}\left(\lambda^{-1}(\lambda-1)\exp^{-1}_{\pi(\xi)}(x)\right)\Big)\, \overline{v}(x) \\
&\qquad\qquad \psi\left(-\lambda^{-1}\exp^{-1}_{\pi(\xi)}(x)\right)\, \rho_{\pi(\xi)}(x)\, d\mu(x)\, d\Omega(\xi) \\
&= \frac{1}{(2\pi\lambda)^n} \int_{M} \int_{T^*M} e^{-i<T^*\exp_x(\xi),\lambda^{-1}\exp_x^{-1}(\pi(\xi))>}\, a(\xi)\, u\Big(\exp_x\left(\lambda^{-1}\exp_x^{-1}(\pi(\xi))\right)\Big)\, \overline{v}(x) \\
&\qquad\qquad \psi\left(-\lambda^{-1}\exp^{-1}_{\pi(\xi)}(x)\right)\, \rho_{\pi(\xi)}(x)\, d\Omega(\xi)\, d\mu(x) \\
&= \frac{1}{(2\pi\lambda)^n} \int_{M} \overline{v}(x) \int_{T_xM\times T_x^*M} e^{-i<\xi,\lambda^{-1}X>}\, a\left(T^*_{\exp_x(X)}\exp_x^{-1}(\xi)\right)\, u\left(\exp_x\left(\lambda^{-1}X\right)\right) \\
&\qquad\qquad \psi(\lambda^{-1}T_X \exp X)\rho_{\exp_x X}(x)\, d\Omega_x(X,\xi)\, d\mu(x) \\
&= \frac{1}{(2\pi)^n} \int_{M} \overline{v}(x) \int_{T_xM\times T_x^*M} e^{-i<\xi,X>}\, a\left(T^*_{\exp_x(\lambda X)}\exp_x^{-1}(\xi)\right)\, u\left(\exp_x(X)\right) \\
&\qquad\qquad \tilde{\psi}(X)\, \rho_{\exp_x(\lambda X)}(x)\, dX\, d\xi\, d\mu(x).
\end{aligned} \tag{9}
$$

By continuity in λ this equation holds for $\lambda = 0$ as well. Now $\mathrm{S}^{-\infty}(M)$ is dense in $\mathrm{S}^m(M)$ in the topology of $\mathrm{S}^{\tilde{m}}(M)$ for any $\tilde{m} > m$. Continuity of both sides of Eq. (9) with respect to $a \in \mathrm{S}^{\tilde{m}}(M)$ entails that Eq. (9) is true for any $a \in \mathrm{S}^{\infty}(M)$. Hence (iii) follows.
(ii) Assume $u, v \in \mathcal{C}^{\infty}_{\mathrm{cpt}}(M, \mathbb{C})$. Then

$$\begin{aligned}
&< \mathrm{Op}_\lambda(a)^* u, v >= \overline{< \mathrm{Op}_\lambda(a) v, u >} \\
&= \overline{\frac{1}{(2\pi)^n} \int_{T^*M} a(\xi) \int_{T_{\pi(\xi)}M} e^{-i<\xi,X>} \, v\left(\exp(1-\lambda)X\right) \, \overline{u}\left(\exp(-\lambda X)\right) \, d\mu_{\pi(\xi)}(X) \, d\Omega} \\
&= \frac{1}{(2\pi)^n} \int_{T^*M} \overline{a}(\xi) \int_{T_{\pi(\xi)}M} e^{i<\xi,X>} \, \overline{v}\left(\exp(1-\lambda)X\right) \, u\left(\exp(-\lambda X)\right) \, d\mu_{\pi(\xi)}(X) \, d\Omega \\
&= \frac{1}{(2\pi)^n} \int_{T^*M} \overline{a}(\xi) \int_{T_{\pi(\xi)}M} e^{-i<\xi,X>} \, u\left(\exp(\lambda X)\right) \, \overline{v}\left(\exp(-(1-\lambda)X)\right) \, d\mu_{\pi(\xi)}(X) \, d\Omega \\
&=< \mathrm{Op}_{1-\lambda}(\overline{a}) u, v > .
\end{aligned} \tag{10}$$

Finally (iv) is an immediate consequence of (iii). □

Let us briefly consider as an example the case where M is Euclidean space $\mathbb{R}^n$. Then

$$\begin{aligned}
\left[\mathrm{Op}_\lambda(a) \, u\right](x) &= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{-i<\xi,X>} \, a\left(x - \lambda X, \xi\right) \, u(x+X) \, dX \, d\xi = \\
&= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i<\xi,x-y>} \, a\left((1-\lambda)x + \lambda y, \xi\right) \, u(y) \, dy \, d\xi,
\end{aligned} \tag{11}$$

which gives the well-known λ-ordered symbol calculus for pseudodifferential operators on $\mathbb{R}^n$.

In the following we determine operators $\mathrm{Op}_\lambda(a)$ of traceclass and calculate their trace in terms of a.

Proposition 2.2 *Let $a \in \mathrm{S}^m(M)$ with $m < -\dim(M)$ and $\pi(\operatorname{supp} a) \subset M$ compact. Then $\mathrm{Op}_\lambda(a)$ is traceclass, and a is integrable with respect to the Liouville measure on T^*M. Furthermore*

$$\operatorname{tr} \mathrm{Op}_\lambda(a) = \frac{1}{(2\pi)^n} \int_{T^*M} a \, d\Omega. \tag{12}$$

Proof: The claim follows immediately from Theorem 2.1 (iv) by the following equality:

$$\begin{aligned}
\operatorname{tr} \mathrm{Op}_\lambda(a) &= \int_M K_\lambda(a)(x,x) \, d\mu(x) = \frac{1}{(2\pi)^n} \int_M \int_{T^*_x M} a(\xi) \, d\xi \, d\mu(x) \\
&= \frac{1}{(2\pi)^n} \int_{T^*M} a \, d\Omega.
\end{aligned} \tag{13}$$

□

Next we will study quasiinverses of the maps Op_λ, the so-called symbol maps.

Theorem 2.3 *(i) Let $\varphi : \tilde{W} \to \mathbb{R}$ be the phase-function $(x,\xi) \mapsto <\xi, \exp^{-1}_{\pi(\xi)}(x)>$, where $\tilde{W} = \left\{(x,\xi) \in X \times T^*X \,|\, x \in \exp \tilde{W}_{\pi(\xi)}\right\}$. Then for every $A \in \Psi^\infty(M)$ the* **normal symbol** *$\sigma(A) = \sigma_0(A) \in \mathrm{S}^\infty(M)$ is defined by*

$$T^*M \ni \xi \mapsto \sigma(A)(\xi) = A\left(\psi \circ \exp^{-1}_{\pi(\xi)} \, e^{i\varphi(\cdot,\xi)}\right)\,(\pi(\xi)) \in \mathbb{C}. \tag{14}$$

The normal symbol provides a map $\sigma : \Psi^\infty(M) \to \mathrm{S}^\infty(M)$ which is quasiinvers to Op*, i.e. which is invers to* Op *modulo $\mathrm{S}^{-\infty}(M)$ resp. $\Psi^{-\infty}(M)$.*

*(ii) For every $\lambda \in [0,1]$ there exists a map $\sigma_\lambda : \Psi^\infty(M) \to \mathrm{S}^\infty(M)$ which is quasiinvers to Op_λ. The symbol $\sigma_\lambda(A)$ is called the λ-***symbol** *of the operator $A \in \Psi^\infty(M)$.*

(iii) The following asymptotic expansions hold with $x = \pi(\xi)$:

$$\sigma_0(A)(\xi) \sim \sum_{\alpha,\beta\in\mathbb{N}^n} \frac{(-i\lambda)^{|\alpha+\beta|}}{\alpha!\,\beta!} \left[\partial^\alpha_{z_x}\,\partial^{\alpha+\beta}_{\zeta_x}\sigma_\lambda(A)\right](\xi)\;\left[\partial^\beta_{z_x}\rho_{(\cdot)}(x)\right](x) \tag{15}$$

$$\begin{aligned}\sigma_\lambda(A)(\xi) \sim \sigma_0(A)(\xi) + \sum_{k\geq 1}(-1)^k \sum_{\alpha_1,\dots,\alpha_k\in\mathbb{N}^*} \frac{(-i\lambda)^{|\alpha_1+\dots+\alpha_k|}}{\alpha_1!\cdot\ldots\cdot\alpha_k!} \Bigg[\partial^{\alpha_1}_{z_x}\partial^{\alpha_1}_{\zeta_x}\Big|_{\substack{x_1=x\\ \zeta_1=\xi}} \\ \left(\partial^{\alpha_2}_{z_{x_1}}\partial^{\alpha_2}_{\zeta_{x_1}}\Big|_{\substack{x_2=x_1\\ \zeta_2=\zeta_1}}\left(\dots\,\partial^{\alpha_k}_{z_{x_{k-1}}}\partial^{\alpha_k}_{\zeta_{x_{k-1}}}\Big|_{\substack{x_k=x_{k-1}\\ \zeta_k=\zeta_{k-1}}}\left(\sigma_0(A)(\zeta_k)\,\rho_{x_k}(x)\right)\dots\right)\rho_{x_1}(x)\right)\Bigg].\end{aligned} \tag{16}$$

In the last asymptotic expansion the partial derivatives $\partial^{\alpha_k}_{z_{x_k}}\partial^{\alpha_k}_{\zeta_{x_k}}\Big|_{\substack{x_{k+1}=y\\ \zeta_{k+1}=\xi}}$ act on the variables x_{k+1} resp. ζ_{k+1} and are evaluted at $x_{k+1} = y$ resp. $\zeta_{k+1} = \xi$.

PROOF: The proof of (i) is an immediate consequence of WIDOM [26], Proposition 3.5 or PFLAUM [16], Theorem 3.2. Consider the orthonormal basis $V = (V_1, \dots, V_n)$ of T_xM with $V_k = \partial_{z_x,k}|_x$ for $k = 1, \dots, n$, and let $\zeta = (\zeta_1, \dots, \zeta_n)$ be the dual basis of V. Then by Theorem 2.1 (iii) and the asymptotic expansion of the symbol of a pseudodifferential operator on $\mathbb{R}^n \cong T_xM$ (cf. GRIGIS, SJÖSTRAND [10], Theorem 3.4) we have the following asymptotic expansion of the normal symbol of $A = \mathrm{Op}_\lambda(a)$ with $a \in \mathrm{S}^m(M)$

$$\begin{aligned}\sigma_0(A)(\xi) &\sim \sum_{\alpha\in\mathbb{N}^n} \frac{i^{-|\alpha|}}{\alpha!}\,\partial^\alpha_V\,\partial^\alpha_\zeta\,\left[a\left(T^*_{\exp\lambda V}\exp^{-1}_x(\zeta)\right)\rho_{\exp\lambda V}(x)\right]\Big|_{V=0,\zeta=\xi} \\ &= \sum_{\alpha\in\mathbb{N}^n} \frac{i^{-|\alpha|}}{\alpha!}\lambda^{|\alpha|}\,\partial^\alpha_{z_x}\,\partial^\alpha_{\zeta_x}\left[a\,\rho_{(\cdot)}(x)\right](\xi) \\ &= \sum_{\alpha,\beta\in\mathbb{N}^n} \frac{(-i\lambda)^{|\alpha+\beta|}}{\alpha!\,\beta!}\left[\partial^\alpha_{z_x}\,\partial^{\alpha+\beta}_{\zeta_x}a\right](\xi)\;\left[\partial^\beta_{z_x}\rho_{(\cdot)}(x)\right](x).\end{aligned} \tag{17}$$

If we can yet show that this asymptotic expansion determines the symbol a up to smoothing symbols, we could set $\sigma_\lambda(A) = a$ and thus prove Eq. (15) and (ii).

By Eq. (15) a and $\sigma_0(A)$ coincide modulo $\mathrm{S}^{m-1}(M)$. Assume now that modulo $\mathrm{S}^{m-k}(M)$, $k \geq 0$ there is only one $a \in \mathrm{S}^m(M)$ fulfilling Eq. (15). Denote by $t_l(a)$,

$l \in \mathbb{N}$ the sum of all terms in the right hand side of Eq. (17) such that $|\alpha+\beta| = l$, and let $\tilde{a} \in \mathrm{S}^m(M)$ be a symbol fulfilling $\sigma_0(A) \sim \sum_{l\in\mathbb{N}} t_l(\tilde{a})$. By assumption we have $a - \tilde{a} \in \mathrm{S}^{m-k}(M)$. Hence $t_l(a-\tilde{a}) \in \mathrm{S}^{m-k-1}(M)$ for $l \geq 1$ and

$$a - \tilde{a} = \sum_{l \leq k+1} t_l(a - \tilde{a}) = 0 \mod \mathrm{S}^{m-k-1}(M). \tag{18}$$

Inductively this shows $a - \tilde{a} \in \mathrm{S}^{-\infty}(M)$.

It remains to prove Eq.(16). Define inductively symbols $a_k \in \mathrm{S}^m(M)$ by

$$a_0 = \sigma_0(A), \quad a_{k+1} = \sigma_0(A) - \sum_{\alpha\in\mathbb{N}^*} \frac{(-i\lambda)^{|\alpha|}}{\alpha!} \partial^\alpha_{z_x}\partial^\alpha_{\zeta_x}\left[a_k\, \rho_{(\cdot)}(x)\right](\xi). \tag{19}$$

Then $a_1 - a_0 \in \mathrm{S}^{m-1}(M)$, and by induction on k $a_{k+1} - a_k \in \mathrm{S}^{m-k-1}(M)$. Hence (a_k) is a Cauchy-sequence with respect to the topology of asymptotic convergence. Its limit $a = \lim_{k\to\infty} a_k$ satisfies Eq. (17), so $a = \sigma_\lambda(A)$. But by definition the symbol a is equal to the right hand side of Eq. (16). This proves the claim. □

The above theorem enables us to calculate the λ-symbol of a pseudodifferential operator out of its normal symbol and vice versa. In the following proposition this method is used to determine the various symbols of the Laplacian on a Riemannian manifold.

Proposition 2.4 *Assume $D \in \mathcal{D}\mathrm{iff}(M) \subset \mathrm{Hom}(\mathcal{C}^\infty(M,\mathbb{C}), \mathcal{C}^\infty(M,\mathbb{C}))$ to be a second order differential operator on the Riemannian manifold M, and express the normal symbol $a = \sigma_0(D)$ of D in the form*

$$a(\xi) = T^{kl}\xi_k\xi_l + X^k\xi_k + r, \quad \xi \in T^*M, \tag{20}$$

where T is a symmetric contravariant 2-tensor field, X a vector field and r a smooth function on M. Then the λ-symbol of D is given in terms of T, X and r by

$$\begin{aligned}\sigma_\lambda(D)(\xi) = & T^{kl}\xi_k\xi_l + i\,2\lambda\left(\nabla_k T^{kl}\right)\xi_l - \lambda^2\,\nabla_k\nabla_l T^{kl} + \\ & + X^k\xi_k - i\lambda\nabla_k X^k + r, \quad \xi \in T^*M.\end{aligned} \tag{21}$$

Vice versa, if $a \in \mathrm{S}^2(M)$ is a symbol of the form (20), then

$$\begin{aligned}\mathrm{Op}_\lambda(a) = & -T^{kl}\nabla_k\nabla_l - 2\lambda\left(\nabla_k T^{kl}\right)\nabla_l - \lambda^2\,\nabla_k\nabla_l T^{kl} - \\ & - iX^k\nabla_k - i\lambda\nabla_k X^k + r.\end{aligned} \tag{22}$$

In the particular case, where $a(\xi) = ||\xi||^2$, the equality

$$\mathrm{Op}_\lambda(a) = -\Delta_g \tag{23}$$

holds, where Δ_g is the Laplacian with respect to g.

Proof: Let us assume for simplicity $X = 0$ and $r = 0$. The calculations for nonvanishing X and r are similar and will be omitted. Recall that the Christoffel symbols of ∇ and the inverse Jacobian $\rho_{(\cdot)}(x)$ have the following expansions with respect to the normal coordinates z_x (cf. BERLINE, GETZLER, VERGNE [3], Chapter 1.3):

$$\Gamma^k{}_{ij}(y) = -\frac{1}{3}\mathrm{R}^k_{ijl}(x)\, z_{x,l}(y) + o(z_x(y)^3), \tag{24}$$

$$\rho_y(x) = 1 + \frac{1}{6}\mathrm{Ric}_{kl}(x)\, z_{x,k}(y) z_{x,l}(y) + o(z_x(y)^3), \tag{25}$$

where R is the curvature and Ric the Ricci-tensor of g. Then we have for all $x, y \in M$ close enough with respect to the normal coordinates z_x:

$$\begin{aligned} \nabla_k \nabla_l T^{kl}(x) &= \partial_{z_x,k}\partial_{z_x,l} T^{kl}(x) + \left(\partial_{z_x,k}\Gamma^k_{lr}\right) T^{rl} + \left(\partial_{z_x,k}\Gamma^l_{lr}\right) T^{kr} \\ &= \partial_{z_x,k}\partial_{z_x,l} T^{kl}(x) - \frac{1}{3}\left(\mathrm{R}^k{}_{lrk}(x)\, T^{rl}(x) + \mathrm{R}^l{}_{lrk}(x)\, T^{kr}(x)\right) \\ &= \partial_{z_x,k}\partial_{z_x,l} T^{kl}(x) + \frac{1}{3}\mathrm{Ric}_{kl}(x)\, T^{kl}(x). \end{aligned} \tag{26}$$

Now express the symbol $a = \sigma_\lambda(D)$ in the form

$$\sigma_\lambda(D)(\xi) = T^{kl}\xi_k\xi_l + Y^k\xi_k + s, \tag{27}$$

where Y is a smooth vector field and s a smooth function on M. Hence we receive by Eq. (15)

$$\begin{aligned} a(\xi) =& \sigma_\lambda(D)(\xi) - i\,2\lambda \left[\partial_{z_{\pi(\xi)},k} T^{kl}\right](\pi(\xi))\, \zeta_{\pi(\xi),l}(\xi) - \\ &- \lambda^2 \left[\partial_{z_{\pi(\xi)},k}\partial_{z_{\pi(\xi)},l} T^{kl} + T^{kl}\,\partial_{z_{\pi(\xi)},k}\partial_{z_{\pi(\xi)},l}\, \rho_{(\cdot)}(\pi(\xi))\right](\pi(\xi)) - i\lambda\partial_{z_{\pi(\xi)},k} Y^k \\ =& \sigma_\lambda(D)(\xi) - i\,2\lambda \left(\nabla_k T^{kl}\right)\xi_l - \lambda^2\left(\nabla_k\nabla_l T^{kl}\right)(\pi(\xi)) - i\lambda\nabla_k Y^k. \end{aligned} \tag{28}$$

This gives

$$\sigma_\lambda(D)(\xi) = T^{kl}\xi_k\xi_l + i\,2\lambda\left(\nabla_k T^{kl}\right)\xi_l + \lambda^2\left(\nabla_k\nabla_l T^{kl}\right)(\pi(\xi)) + i\lambda\nabla_k Y^k \tag{29}$$

for all $\xi \in T^*M$, and implies $Y^l = i\,2\lambda\nabla_k T^{kl}$. Now (21) follows immediately. In the case $a(\xi) = ||\xi^2||$ we have $D := \mathrm{Op}_0(||\xi||^2) = -\Delta_g$. As $\nabla g = 0$ Eq. (21) entails

$$\sigma_\lambda(D)(\xi) = ||\xi^2||, \tag{30}$$

hence Eq. (23) follows. By analogous arguments one proves Eq. (22). □

Finally in this section we want to provide an asymptotic expansion for the λ-symbol of the product of two pseudodifferential operators in terms of the symbols of its components.

Proposition 2.5 *Let $a \in \mathrm{S}^m(M)$, $b \in \mathrm{S}^{m'}(M)$ be two symbols on M. Then the λ-symbol $c = \sigma_\lambda(\mathrm{Op}_\lambda(a)\cdot\mathrm{Op}_\lambda(b)) \in \mathrm{S}^{m+m'}(M)$ of the product of the two pseudodifferential operators $\mathrm{Op}_\lambda(a)$ and $\mathrm{Op}_\lambda(b)$ has an asymptotic expansion of the form*

$$\begin{aligned} c(\xi) \sim& ab - i\sum_{l=1}^{n}\left[(1-\lambda)\left(\partial_{\zeta_{\pi(\xi),l}} a\right)\left(\partial_{z_{\pi(\xi),l}} b\right) - \lambda\left(\partial_{z_{\pi(\xi),l}} a\right)\left(\partial_{\zeta_{\pi(\xi),l}} b\right)\right](\xi) + \\ &+ \sum_{\substack{\alpha,\tilde\alpha,\beta,\tilde\beta\in\mathbb{N}^n \\ \left|\beta+\frac{1}{2}\tilde\beta\right|\geq 2}} f^\lambda_{\alpha,\beta,\tilde\alpha,\tilde\beta}(\xi)\left(\partial^\alpha_{z_{\pi(\xi)}}\partial^\beta_{\zeta_{\pi(\xi)}} a\right)\left(\partial^{\tilde\alpha}_{z_{\pi(\xi)}}\partial^{\tilde\beta}_{\zeta_{\pi(\xi)}} b\right)(\xi), \end{aligned} \tag{31}$$

where the $f^{\lambda}_{\alpha,\beta,\tilde{\alpha},\tilde{\beta}}$ are polynomials in the fibers of T^*M of degree $\leq \left|\beta + \frac{1}{2}\tilde{\beta}\right|$ and depend only on the derivatives of the curvature of g.

Proof: For the case of the normal symbol σ_0 Eq. (31) holds true by Widom [26], Proposition 3.6 or Pflaum [16], Theorem 5.4. Let $A = \mathrm{Op}_\lambda(a)$ and $B = \mathrm{Op}_\lambda(b)$. Then by Theorem 2.3 (iii) one has expansions

$$\sigma_0(A) = a - i\lambda \sum_{l=1}^{n} \partial_{z_{\pi(\xi),l}}\, \partial_{\zeta_{\pi(\xi),l}} a + r_a, \tag{32}$$

$$\sigma_0(B) = a - i\lambda \sum_{l=1}^{n} \partial_{z_{\pi(\xi),l}}\, \partial_{\zeta_{\pi(\xi),l}} b + r_b, \tag{33}$$

where $r_a \in \mathrm{S}^{m-2}(M)$ and $r_b \in \mathrm{S}^{m'-2}(M)$. Using Eq. (31) for the case of $\lambda = 0$ we receive

$$\sigma_0(AB) = ab - i \sum_{l=1}^{n} \left[\left(\partial_{\zeta_{\pi(\xi),l}} a\right) \left(\partial_{z_{\pi(\xi),l}} b\right) + \lambda \left(a\, \partial_{z_{\pi(\xi),l}} \partial_{\zeta_{\pi(\xi),l}} b + b\, \partial_{z_{\pi(\xi),l}} \partial_{\zeta_{\pi(\xi),l}} a \right)\right] + r_0, \tag{34}$$

where $r_0 \in \mathrm{S}^{m+m'-2}(M)$. Applying Theorem 2.3 (iii) again finally yields

$$\sigma_\lambda(AB) = ab - i \sum_{l=1}^{n} \left[(1-\lambda) \left(\partial_{\zeta_{\pi(\xi),l}} a\right) \left(\partial_{z_{\pi(\xi),l}} b\right) - \lambda \left(\partial_{z_{\pi(\xi),l}} a\right) \left(\partial_{\zeta_{\pi(\xi),l}} b\right)\right] + r_\lambda, \tag{35}$$

where $r_\lambda \in \mathrm{S}^{m+m'-2}(M)$. It remains to show that the remainder terms r_λ have the required form. For $\lambda = 0$ it is already known that r_0 has the claimed asymptotic expansion (see Pflaum [16], Theorem 5.4), hence Theorem 2.3 (iii) entails that r_λ has it as well. □

3 Quantization of symbols

With the help of the analytical tools from the previous section we now want to to construct a scale of concrete deformations of the symbol sheaf over a Riemannian manifold.

Let us define for every $m \in \mathbb{R}$ a sheaf $\mathcal{B}^m$ on $X = M \times \mathbb{R}$ by

$$\mathcal{B}^m(U \times I) = \{a \in \mathcal{C}^\infty(T^*U \times I) | I \ni \hbar \mapsto a(\cdot, \hbar) \in \mathrm{S}^m(U) \text{ is continuous}\}, \tag{36}$$

where $U \subset M$ and $I \subset \mathbb{R}$ are open. The sheaves $\mathcal{B}^m$ give rise to further sheaves $\mathcal{B}^\infty = \lim_{n\to\infty} \mathcal{A}^m$ and $\mathcal{B}^{-\infty} = \lim_{n\to-\infty} \mathcal{A}^m$. We denote their quotient $\mathcal{B}^\infty/\mathcal{B}^{-\infty}$ by $\mathcal{B}$. Now let $\hbar \in \mathbb{R} \setminus \{0\}$, and denote by $\iota_\hbar : \mathrm{S}^\infty(M) \to \mathrm{S}^\infty(M)$ the map $a \mapsto (T^*M \ni \xi \mapsto a(\hbar\xi) \in \mathbb{C})$. Obviously $\iota_\hbar$ is invertible with inverse $\iota_{\hbar^{-1}}$. With the help of the morphisms $\iota_\hbar$ we are now able to define for every $\lambda \in [0,1]$ a bilinear map $*_\lambda$ on the sheaf $\mathcal{B}^\infty$ by

$$(a,b) \mapsto (a *_\lambda b)(\cdot, \hbar) = \begin{cases} \iota_{\hbar^{-1}} \sigma_\lambda \left[\mathrm{Op}_\lambda\left(\iota_\hbar(a(\cdot,\hbar))\right) \cdot \mathrm{Op}_\lambda\left(\iota_\hbar(b(\cdot,\hbar))\right)\right] & \text{for } \hbar \in I \setminus \{0\}, \\ a(\cdot, 0) \cdot (b(\cdot, 0) & \text{for } \hbar = 0, \end{cases} \tag{37}$$

where $a, b \in \mathcal{B}^\infty(U \times I)$ and $U \subset M$, $I \subset \mathbb{R}$ open. By Proposition 2.5 $a *_\lambda b$ is smooth even at $\hbar = 0$ and lies in $\mathcal{B}^\infty(U \times I)$ indeed. Moreover it induces an associative product $*_\lambda$ on $\mathcal{B}$. We denote the resulting sheaf of algebras by $\mathcal{B}_\lambda$ and call it the sheaf of λ-**ordered quantized symbols**. In case $\lambda = 0$ (resp. $\lambda = 1$ or $\lambda = \frac{1}{2}$) $\mathcal{B}_\lambda$ will also be called the sheaf of **normally** (resp. **antinormally** or **Weyl**) **ordered quantized symbols** on M.

Let us now show that $(X, \mathcal{B}_\lambda)$ gives rise to a smooth deformation of the ringed space $(M, \mathrm{S}^\infty/\mathrm{S}^{-\infty})$ with distinguished point $\hbar = 0$. Denote by $D = (\bar{d}, \Delta) : (X, \mathcal{B}_\lambda) \to (\mathbb{R}, \mathcal{C}^\infty_\mathbb{R})$ the fibered morphism

$$\begin{array}{rccc} \bar{d} = pr_2 : & X & \to & \mathbb{R}, \\ & (x, \hbar) & \to & \hbar, \end{array} \tag{38}$$

$$\begin{array}{rccl} \Delta_I : & \mathcal{C}^\infty(I) & \to & \mathcal{B}_\lambda(M \times I) \\ & f & \mapsto & f \circ pr_2 + \mathcal{B}^{-\infty}(M \times I). \end{array} \tag{39}$$

Furthermore let $\mathcal{C}^\infty_\hbar$ be the stalk of $\mathcal{C}^\infty_M$ at $\hbar \in \mathbb{R}$ and $\mathfrak{m}_\hbar$ the maximal ideal of $\mathcal{C}^\infty_\hbar$. Then it follows by Proposition 2.5 that $\Delta\left(\mathcal{C}^\infty_\mathbb{R}\right)$ lies in the center of $\mathcal{B}_\lambda$. Moreover by the definition of $\mathcal{B}$ the sheaves $\mathcal{B}_{*_\lambda,0} := \mathcal{B}_\lambda|_{M\times\{0\}} / \mathfrak{m}_0 \, \mathcal{B}_\lambda|_{M\times\{0\}}$ and $\mathrm{S}^\infty_M/\mathrm{S}^{-\infty}$ are isomorphic. It remains to show the flatness-condition for $\mathcal{B}$. We have for $m \in \mathbb{N}^*$, $\hbar \in \mathbb{R}$ and $x \in M$ the following isomorphies:

$$\mathcal{C}^\infty_\hbar / \mathfrak{m}^m_\hbar \cong \mathbb{C}^m, \tag{40}$$

$$\mathcal{B}_{(x,\hbar)}/\mathfrak{m}^m_\hbar \mathcal{B}_{(x,\hbar)} \cong \left(\mathrm{S}^\infty_M/\mathrm{S}^{-\infty}\right)^m_x \cong \left(\mathcal{C}^{\infty(\mathbb{N})}_x\right)^m. \tag{41}$$

But $\left(\mathcal{C}^{\infty(\mathbb{N})}_x\right)^m$ is definitely flat over $\mathbb{C}^m$, hence $\mathcal{B}$ is locally trivial over $\mathcal{C}^\infty_\mathbb{R}$ and D a deformation of the sheaf $\mathrm{S}^\infty_M/\mathrm{S}^{-\infty}$ of **asymptotic symbols** on M. As the Poisson bracket on $\mathcal{C}^\infty(T * M)$ leaves the sheaf $\mathrm{S}^{-\infty}_M$ invariant, it induces a Poisson structure on the ringed space $(M, \mathrm{S}^\infty_M/\mathrm{S}^{-\infty})$. The abovely defined deformation D will give rise to a quantization of $(M, \mathrm{S}^\infty_M/\mathrm{S}^{-\infty})$.

Theorem 3.1 *Let M be a Riemannian manifold with metric g, $\lambda \in [0,1]$ and $\mathcal{B}_\lambda$ the sheaf of λ-ordered quantized symbols. Then the morphism $D = (\bar{d}, \Delta) : (X, \mathcal{B}_\lambda) \to (\mathbb{R}, \mathcal{C}^\infty_\mathbb{R})$ together with the morphism*

$$Q = (q, \mathfrak{q}) : (X, \mathcal{B}_\lambda) \to (M, \mathrm{S}^\infty/\mathrm{S}^{-\infty}) \tag{42}$$

$$M \times \mathbb{R} \ni (x, \hbar) \mapsto q(x, \hbar) = x \in M \tag{43}$$

$$\mathrm{S}^\infty_M/\mathrm{S}^{-\infty} \ni a \mapsto \mathfrak{q}(a) = a \circ \mathrm{pr}_1 \in \mathcal{B}_\lambda \tag{44}$$

*CoDeTe comprises a smooth quantization of the Poisson space $(M, \mathrm{S}^\infty_M/\mathrm{S}^{-\infty})$ of asymptotic symbols on M, the so-called λ-***ordered quantization*** of $\mathrm{S}^\infty_M/\mathrm{S}^{-\infty}$. For every $\hbar \neq 0$ the map*

$$\mathcal{B}^\infty(M) \ni a \mapsto \mathrm{Op}_\lambda\left(\iota_\hbar(a(\cdot, \hbar))\right) \in \Psi^\infty(M) \tag{45}$$

induces an isomorphism from the sheaf $\mathcal{B}_{*_\lambda,\hbar} := \mathcal{B}_\lambda|_{M\times\{0\}} / \mathfrak{m}_\hbar \, \mathcal{B}_\lambda|_{M\times\{0\}}$ to the sheaf $\Psi^\infty_M/\Psi^{-\infty}$.

Proof: The canonical commutation relations for the quantization map $\mathfrak{q}$ are fulfilled by Eq. (35). The other conditions for a smooth quantization have been shown above respectively are obvious. Hence the theorem is shown. □

Corollary 3.2 *The λ-ordered quantization of the sheaf of asymptotic symbols induces a smooth quantization of the sheaf $\mathcal{D}_0$ of fiberwise polynomial smooth functions on T^*M to the sheaf $\mathcal{D}\mathrm{iff}_M$ of differential operators on M. Its quantization map $\mathfrak{q}_{\lambda,\hbar} : \mathcal{D}_0 \to \mathcal{D}\mathrm{iff}_M$ for a particular value $\hbar \in \mathbb{R}^*$ is given by*

$$\mathfrak{q}_{\lambda,\hbar}(a) = \mathrm{Op}_\lambda\left(\iota_\hbar(a)\right). \tag{46}$$

In the case of Weyl quantization, i.e. $\lambda = \frac{1}{2}$, $\mathfrak{q}_{\lambda,\hbar}$ maps selfadjoint classical observables to (formally) selfadjoint operators on the Hilbert space $L^2(M)$.

Proof: As $\mathcal{D}_0$ is a subsheaf of $\mathrm{S}^\infty_M/\mathrm{S}^{-\infty}$ and $\mathcal{D}\mathrm{iff}_M$ a subsheaf of $\Psi^\infty_M/\Psi^{-\infty}$, the claim follows immediately from the preceding result and Theorem 2.1 (ii). □

Let us now give some examples of quantizations of classical observables polynomial in momentum by applying the corollary. First consider the harmonic oscillator on $M = \mathbb{R}^n$ with Hamiltonian $H(\xi) = \frac{1}{2m}||\xi||^2 + \frac{1}{2}m\omega^2||x||^2$, where m is the mass of the oscillator and ω its angular frequency. Then H has λ-ordered quantization $\mathfrak{q}_{\lambda,\hbar}(H) = -\frac{\hbar^2}{2m}\Delta + \frac{1}{2}m\omega^2||x||^2$. More generally, if M is a Riemannian manifold and $H(\xi) = \frac{1}{2m}||\xi||^2 + V(\pi(\xi))$ the Hamiltonian of a particle moving on M in an external field with potential V, the quantized Hamiltonian is the differential operator $\mathfrak{q}_{\lambda,\hbar}(H) = -\frac{\hbar^2}{2m}\Delta + V$. In the special case of the Hydrogen atom with classical Hamiltonian $H = \frac{1}{2m}||\xi^2|| - \frac{e^2}{||x||}$ we receive $\mathfrak{q}_{\lambda,\hbar}(H) = -\frac{\hbar^2}{2m}\Delta - \frac{e^2}{||x||}$. The quantization of the angular momentum $L_j = x_k\xi_l - x_l\xi_k$ with (j,k,l) is a cyclic permutation of $(1,2,3)$, is given by $\mathfrak{q}_{\lambda,\hbar}(L_j) = -i\hbar\left(x_k\partial_l - x_l\partial_k\right)$. More interesting than the quantization of the angular momentum is the quantization of the Lenz-Runge vector $A_j = \frac{1}{m}\left(L_k\xi_l - L_l\xi_k\right) + e^2\frac{x_j}{||x||}$. Using Proposition 2.4 we get

$$\mathfrak{q}_{\lambda,\hbar}(A_j) = -\frac{\hbar^2}{m}\left(x_k\partial_k\partial_j + x_l\partial_l\partial_j - x_j\left(\partial_k^2 + \partial_l^2\right) + 2\lambda\partial_j\right) + e^2\frac{x_j}{||x||}. \tag{47}$$

Now $\mathfrak{q}_{\lambda,\hbar}(A_j)$ is formally selfadjoint if and only if $\lambda = \frac{1}{2}$, i.e. in the case of Weyl quantization. By Theorem 2.1 (ii) it is a general feature of Weyl quantization that it maps selfadjoint classical obsevables to (formally) selfadjoint operators on the Hilbert space $L^2(M)$ of square integrable functions on M. By construction Weyl quantization is functorial on the category of Riemannian manifolds with isometric embeddings as morphisms. Hence Weyl quantization seems to be the correct quantization scheme for cotangent bundles on Riemannian manifolds.

Finally in this article we consider the star-products induced by λ-ordered quantizations on cotangent bundles.

Theorem 3.3 *Let M be a Riemannian manifold. Then for every $\lambda \in [0,1]$ the λ-ordered quantization induces a strongly closed formal star-product $\star_\lambda$ on the space $\mathcal{C}^\infty(T^*M)$ of smooth functions on the symplectic manifold T^*M. Any two of these star-products are equivalent.*

PROOF: By Proposition 2.5 the quantization $\mathfrak{q}_{\lambda,\hbar}$ induces the star-product

$$
\begin{aligned}
a \star_\lambda b = ab - i\hbar \sum_{l=1}^{n} \Big[(1-\lambda) \left(\partial_{\zeta_{\pi(\xi),l}} a\right) \left(\partial_{z_{\pi(\xi),l}} b\right) - \lambda \left(\partial_{z_{\pi(\xi),l}} a\right) \left(\partial_{\zeta_{\pi(\xi),l}} b\right) \Big] + \\
+ \sum_{\substack{\alpha,\tilde{\alpha},\beta,\tilde{\beta} \in \mathbb{N}^n \\ \left|\beta + \frac{1}{2}\tilde{\beta}\right| \geq 2}} \hbar^{|\beta+\tilde{\beta}|} f^{\lambda}_{\alpha,\beta,\tilde{\alpha},\tilde{\beta}} \left(\partial^{\alpha}_{z_{\pi(\xi)}} \partial^{\beta}_{\zeta_{\pi(\xi)}} a\right) \left(\partial^{\tilde{\alpha}}_{z_{\pi(\xi)}} \partial^{\tilde{\beta}}_{\zeta_{\pi(\xi)}} b\right),
\end{aligned}
\tag{48}
$$

where $a, b \in \mathcal{C}^\infty(T^*M)$. By definition of $\star_\lambda$ and Proposition 2.2

$$
\mathcal{C}^\infty(T^*M)[[\hbar]] \ni a = \sum_{k\in\mathbb{N}} a_k \hbar^k \mapsto \frac{1}{(2\pi\hbar)^n} \int_{T^*M} a \, d\Omega \in \mathbb{C}[\hbar^{-1}, \hbar]] \tag{49}
$$

defines a $\mathbb{C}[\hbar^{-1}, \hbar]]$-valued trace on $(\mathcal{C}^\infty(T^*M)[[\hbar]], \star_\lambda)$, hence $\star_\lambda$ is strongly closed. To prove equivalence of two starproducts $\star_{\lambda_1}$ and $\star_{\lambda_2}$ with $\lambda_1, \lambda_2 \in [0,1]$, it suffices to show that $\star_\lambda$ and $\star_0$ are equivalent for any $\lambda \in [0,1]$. By Theorem 2.3 (iii) the map

$$
\mathcal{C}^\infty \ni a \mapsto \sum_{\alpha,\beta\in\mathbb{N}^n} \frac{(-i\lambda\hbar)^{|\alpha+\beta|}}{\alpha!\,\beta!} \left[\partial^{\alpha}_{z_x} \partial^{\alpha+\beta}_{\zeta_x} a\right] \left[\partial^{\beta}_{z_x} \rho_{(\cdot)}(x)\right] \in \mathcal{C}^\infty(T^*M)[[\hbar]] \tag{50}
$$

defines an isomorphism from $(\mathcal{C}^\infty(T^*M)[[\hbar]], \star_\lambda)$ to $(\mathcal{C}^\infty(T^*M)[[\hbar]], \star_0)$. This proves the theorem. □

References

[1] Sean Bates and Alan Weinstein, *Lectures on Geometric Quantization*, Dept. Math., University of California, Berkeley, California, 1993, Lecture Notes.

[2] F. Bayen, M. Flato, C. Fronsdal, A. Lichnerowicz, and D. Sternheimer, *Deformation theory and quantization*, Ann. Physics **111** (1978), 61–151.

[3] N. Berline, E. Getzler, and M. Vergne, *Heat kernels and Dirac operators*, Grundlehren der mathematischen Wissenschaften, vol. 298, Springer–Verlag, Berlin, Heidelberg, New York, 1992.

[4] Robert J. Blattner, *Some remarks on quantization*, Symplectic Geometry and Mathematical Physics, Progr. in Math., vol. 99, Birkhäuser, 1991, pp. 37–47.

[5] Nicolas Bourbaki, *Commutative Algebra*, Hermann, Paris, 1972.

[6] Boris Fedosov, *Deformation Quantization and Index Theory*, Akademie Verlag, 1996.

[7] Moshe Flato and Daniel Sternheimer, *Star products, quantum groups, cyclic cohomolgy and pseudodifferential calculus*, Contemp. Math. **175** (1994), 53–72.

[8] Murray Gerstenhaber, *On the deformations of rings and algebras*, Ann. of Math. (2) **79** (1964), 59–103.

[9] S. G. Gindikin and G. M. Khenkin (Eds.), *Several Complex Variables IV*, Encyclopaedia of Mathematical Sciences, vol. 10, Springer-Verlag, Berlin, Heidelberg, New York, 1990.

[10] Alain Grigis and Johannes Sjøstrand, *Microlocal Analysis for Differential Operators*, London Mathematical Society Lecture note series, vol. 196, Cambridge University Press, 1994.

[11] Robin Hartshorne, *Algebraic Geometry*, vol. 52, Springer-Verlag, Berlin, Heidelberg, New York, 1977.

[12] M. Karasev and V. Maslov, *Nonlinear Poisson Brackets*, Translations of Mathematical Monographs, vol. 119, American Mathematical Society, 1993.

[13] B. Kostant, *Quantization and unitary representations*, Lecture Notes in Mathematics, vol. 170, Springer-Verlag, 1970, pp. 87–208.

[14] Hideki Omori, Yoshiaki Maeda, and Akira Yoshioka, *Weyl manifolds and deformation quantization*, Adv. in Math. **85** (1991), 224–255.

[15] Markus J. Pflaum, *Local Analysis of Deformation Quantization*, Ph.D. thesis, Fakultät für Mathematik der Ludwig-Maximilians-Universität, München, November 1995.

[16] ______, *The normal symbol on Riemannian manifolds*, dg-ga/9612011 (1995).

[17] J. Rawnsley, M. Cahen, and S. Gutt, *Quantization of Kähler manifolds I: geometric interpretation of Berezin's quantization*, J. Geom. Phys. **7** (1990), no. 1, 45–62.

[18] John Rawnsley, *Deformation quantization of Kähler manifolds*, Symplectic Geometry and Mathematical Physics, Progr. in Math., vol. 99, Birkhäuser, 1991, pp. 366–373.

[19] ______, *Quantization on Kähler manifolds*, Differential Geometric Methods in Theoretical Physics, Proceedings, Rapallo, Italy 1990 (C. Bartocci, U. Bruzo, and R. Cianci, eds.), Lecture Notes in Physics, vol. 375, Springer-Verlag, 1991, pp. 155–161.

[20] Marc A. Rieffel, *Quantization and C^*-algebras*, Contemp. Math. **167** (1994), 67–97.

[21] Yuri Safarov, *Pseudodifferential operators and linear connections*, Proc. London Math. Soc. (3) (to appear).

[22] M. A. Shubin, *Pseudodifferential Operators and Spectral Theory*, Springer-Verlag, Berlin, Heidelberg, New York, 1987.

[23] Jedrezej Śniatycki, *Geometric Quantization and Quantum Mechanics*, Applied Mathematical Sciences, vol. 30, Springer-Verlag, New-York, Heidelberg, Berlin, 1980.

[24] J.-M. Souriau, *Structures des systemes dynamique*, Dunod, Paris, 1970.

[25] J. Underhill, *Quantization on a manifold with connection*, J. Math. Phys. **19** (1978), no. 9, 1932–1935.

[26] Harold Widom, *A complete symbol calculus for pseudodifferential operators*, Bull. Sci. Math. (2) **104** (1980), 19–63.

[27] M. De Wilde and P. Lecomte, *Existance of star-products and of formal deformatioons of the Poisson Lie algebra of arbitrary symplectic manifolds*, Lett. Math. Phys. **7** (1983), 487–496.

[28] Liu Zhang-Ju and Quian Min, *Gauge invariant quantization on Riemannian manifolds*, Trans. Amer. Math. Soc. **331** (1992), no. 1, 321–333.