Investigation and application of the dressing action on surfaces of constant mean curvature

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1 Introduction

The dressing method for the generating of new solutions of integrable systems was introduced in 1979 by Zakharov and Shabat [16]. It was soon traced back to a natural loop group action on the solution space of integrable systems [1, 11]. With the introduction of the theory of integrable systems into geometry, most notably the theory of surfaces of constant mean curvature (CMC surfaces) [10, 2], the dressing method also entered the realm of differential geometry.

Let it serve as an illustration of the power of the dressing action that, due to [8] (see also [3]), all CMC immersions of finite type, in particular all CMC tori, are included in the dressing orbit of the standard cylinder. This result was used in [4] to reproduce the classification of CMC tori given in [10] in terms of the dressing action.

Moreover, while the integrable systems methods in general only apply to a certain class of CMC surfaces, those without umbilics, the dressing action can easily be applied also to CMC surfaces with umbilics. The latter was made possible by a general loop group theoretic approach to CMC surfaces, the so called DPW method [7].

Similar to the Weierstraß representation of minimal surfaces, the DPW method starts with a holomorphic function E and a meromorphic function f, both defined on some open, simply connected subset $\mathcal{D}$ of the complex plane, and constructs an S^1-family of isometric conformal CMC immersions $\Psi_{\lambda\in S^1} : \mathcal{D} \to \mathbb{R}^3$, the associated family, from these data. As a first application of the dressing action, in [6] and [15] it was shown that the dressing action can be used to describe the set of admissible input data (E, f) for the DPW method.

While $E\mathrm{d}z^2$ is simply the Hopf differential of the resulting CMC immersion, the function f has no such simple geometric interpretation. The problem shows especially if one wants to construct CMC immersions Ψ which are invariant under a symmetry group $\Gamma \subset \mathrm{Aut}\mathcal{D}$ of biholomorphic automorphisms of $\mathcal{D}$, i.e. $\Psi \circ \gamma = T\Psi$, where $\gamma \in \Gamma$ and T is a (proper) Euclidean motion in space. It is clear, that $E\mathrm{d}z^2$ has to be automorphic w.r.t. Γ, i.e. $\gamma^*(E\mathrm{d}z^2) = E\mathrm{d}z^2$ or $E \circ \gamma = (\gamma')^{-2}E$. But the meromorphic function f satisfies no such automorphicity conditions. Instead, as was shown by the authors in [5], f transforms by complicated dressing transformations under Γ. For further reference see [5]. Thus, in the DPW formulation the understanding of compact and symmetric CMC immersions is intimately related to the understanding of the dressing action. This served as a strong incentive to further investigate the dressing action, in particular on the meromorphic data of the DPW method.

Clearly, the Hopf differential is invariant under dressing. Wu [14] constructed a large set of algebraic invariants under dressing and was able to find normalized representatives in each dressing orbit [13]. These results are particularly useful, since the dressing orbits, as the finite type results indicate, are very large.

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In this paper we want to describe one of the major properties of dressing as a group action: the isotropy group of a CMC immersion under the dressing action.

As the main result of this paper we will show in Theorem 2.5 that for immersions whose Hopf differential has zeroes, the isotropy group is trivial. In other words, the dressing action is simple on surfaces with umbilics. This is essentially different from the situation for CMC surfaces of finite type. Since these are all contained in the dressing orbit of the standard cylinder, it is easy to see, that for these surfaces there is an infinite dimensional isotropy group. In fact, as was shown in [4], the existence of this isotropy group can be viewed as the major prerequisite for Krichever's method.

Theorem 2.5 leads to several applications for CMC immersions with umbilics. In this paper we present two 'no-go'-theorems for CMC surfaces with umbilics.

As a first candidate for a non-simply connected CMC surface with umbilics we investigate the case $\mathcal{D} = D_1$, the open unit disk, and $f \equiv C$, where C is a complex constant. In this case f is clearly invariant under any group $\Gamma \subset \mathrm{Aut}(D_1)$. We will show in Section 3.1, that as long as the Hopf differential has umbilics, such an f cannot produce a CMC immersion which is invariant under a group Γ of biholomorphic automorphisms of D_1.

As a second candidate we look for symmetric surfaces with umbilics in the dressing orbits of arbitrary automorphic DPW data, a very large class of surfaces. In Section 3.2, it will be shown that for such CMC immersions all members of the associated family Ψ_λ, $\lambda \in S^1$, constructed from the same DPW data, share the same symmetry in $\mathbb{R}^3$. In particular, if $\Psi = \Psi_{\lambda=1}$ is e.g. compact in $\mathbb{R}^3$ of genus $g \geq 2$ then all surfaces Ψ_λ are compact with the same Fuchsian group Γ.

In view of the case of CMC tori, in which at most a countable number of associated surfaces is compact, this may indicate that there are no symmetric examples in the dressing orbits of automorphic DPW data. However, as we want to emphasize, the latter is not a theorem, just a conjecture. In any case, the results of Section 3.2 show that even if there exists an algebro-geometric method for CMC surfaces with umbilics, it will have to look essentialy different from Krichever's method for the finite type case.

At the end of this introduction we also would like to add, that in spite of the discouraging results above, we were able to construct a large family of non-simply connected CMC surfaces with umbilics using the dressing action in the DPW method. These results will be published elsewhere.

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2 The dressing action

In this chapter we will review the definition and basic properties of the dressing action on CMC surfaces in the framework of the DPW construction. We will also investigate the isotropy group of a CMC surface, represented by its meromorphic DPW data, under the dressing action. For further reference see [4, Section 2].

2.2 For each real constant r, $0 < r < 1$, let $\Lambda_r \mathbf{SL}(2,\mathbb{C})_\sigma$ denote the group of smooth maps $g(\lambda)$ from C_r, the circle of radius r, to $\mathbf{SL}(2,\mathbb{C})$, which satisfy the twisting condition

$$g(-\lambda) = \sigma(g(\lambda)), \tag{2.2.1}$$

where $\sigma : \mathbf{SL}(2,\mathbb{C}) \to \mathbf{SL}(2,\mathbb{C})$ is defined by conjugation with the Pauli matrix $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. The Lie algebras of these groups, which we denote by $\Lambda_r \mathbf{sl}(2,\mathbb{C})_\sigma$, consist of maps $x : C_r \to \mathbf{sl}(2,\mathbb{C})$, which satisfy a similar twisting condition as the group elements

$$x(-\lambda) = \sigma_3 x(\lambda) \sigma_3. \tag{2.2.2}$$

In order to make these loop groups complex Banach Lie groups, we equip them, as in [7], with some H^s-topology for $s > \frac{1}{2}$.

Furthermore, we will use the following subgroups of $\Lambda_r\mathbf{SL}(2,\mathbb{C})_\sigma$: Let $\mathbf{B}$ be a subgroup of $\mathbf{SL}(2,\mathbb{C})$ and $\Lambda^+_{r,B}\mathbf{SL}(2,\mathbb{C})_\sigma$ be the group of maps in $\Lambda_r\mathbf{SL}(2,\mathbb{C})_\sigma$, which can be extended to holomorphic maps on

$$I^{(r)} = \{\lambda \in \mathbb{C}; |\lambda| < r\}, \tag{2.2.3}$$

the interior of the circle C_r, and take values in $\mathbf{B}$ at $\lambda = 0$. Analogously, let $\Lambda^-_{r,B}\mathbf{SL}(2,\mathbb{C})_\sigma$ be the group of maps in $\Lambda_r\mathbf{SL}(2,\mathbb{C})_\sigma$, which can be extended to the exterior

$$E^{(r)} = \{\lambda \in \mathbb{C}\mathrm{P}_1; |\lambda| > r\} \tag{2.2.4}$$

of C_r and take values in $\mathbf{B}$ at $\lambda = \infty$. If $\mathbf{B} = \{I\}$ (based loops) we write the subscript $*$ instead of $\mathbf{B}$, if $\mathbf{B} = \mathbf{SL}(2,\mathbb{C})$ we omit the subscript for Λ entirely.

Also, by an abuse of notation, we will denote by $\Lambda_r\mathbf{SU}(2)_\sigma$ the subgroup of maps in $\Lambda_r\mathbf{SL}(2,\mathbb{C})_\sigma$, which can be extended holomorphically to the open annulus

$$A^{(r)} = \{\lambda \in \mathbb{C}; r < |\lambda| < \frac{1}{r}\} \tag{2.2.5}$$

and take values in $\mathbf{SU}(2)$ on the unit circle.

Corresponding to these subgroups, we analogously define Lie subalgebras of $\Lambda_r\mathbf{sl}(2,\mathbb{C})_\sigma$.

We quote the following results from [9] and [7]:

(i) For each solvable subgroup $\mathbf{B}$ of $\mathbf{SL}(2,\mathbb{C})$, which satisfies $\mathbf{SU}(2)\cdot\mathbf{B} = \mathbf{SL}(2,\mathbb{C})$ and $\mathbf{SU}(2)\cap\mathbf{B} = \{I\}$, multiplication

$$\Lambda_r\mathbf{SU}(2)_\sigma \times \Lambda^+_{r,B}\mathbf{SL}(2,\mathbb{C})_\sigma \longrightarrow \Lambda_r\mathbf{SL}(2,\mathbb{C})_\sigma$$

is a diffeomorphism onto. The associated splitting

$$g = Fg_+ \tag{2.2.6}$$

of an element g of $\Lambda_r\mathbf{SL}(2,\mathbb{C})_\sigma$, s.t. $F \in \Lambda_r\mathbf{SU}(2)_\sigma$ and $g_+ \in \Lambda^+_{r,B}\mathbf{SL}(2,\mathbb{C})_\sigma$ will be called Iwasawa decomposition.

(ii) Multiplication

$$\Lambda^-_{r,*}\mathbf{SL}(2,\mathbb{C})_\sigma \times \Lambda^+_r\mathbf{SL}(2,\mathbb{C})_\sigma \longrightarrow \Lambda_r\mathbf{SL}(2,\mathbb{C})_\sigma \tag{2.2.7}$$

is a diffeomorphism onto the open and dense subset $\Lambda^-_{r,*}\mathbf{SL}(2,\mathbb{C})_\sigma \cdot \Lambda^+_r\mathbf{SL}(2,\mathbb{C})_\sigma$ of $\Lambda_r\mathbf{SL}(2,\mathbb{C})_\sigma$, called the "big cell" [12]. The associated splitting

$$g = g_-g_+ \tag{2.2.8}$$

of an element g of the big cell, where $g_- \in \Lambda^-_{r,*}\mathbf{SL}(2,\mathbb{C})_\sigma$ and $g_+ \in \Lambda^+_r\mathbf{SL}(2,\mathbb{C})_\sigma$, will be called Birkhoff factorization.

2.3 Let $\Psi : \mathcal{D} \to \mathbb{R}^3$ be a conformal CMC-immersion. Define the extended frame $F(z,\lambda) : \mathcal{D} \to \Lambda\mathbf{SU}(2)_\sigma$ as in [7] (see also the appendix of [6]). Furthermore, define $g_- : \mathcal{D} \to \Lambda^-_{r,*}\mathbf{SL}(2,\mathbb{C})_\sigma$ by the Birkhoff splitting

$$F(z,\lambda) = g_-(z,\lambda)g_+(z,\lambda). \tag{2.3.1}$$

Then g_- is a meromorphic function on $\mathcal{D}$ with poles in the set $\mathcal{S} \subset \mathcal{D}$ of points, where $F(z,\lambda)$ is not in the "big cell", i.e. where the Birkhoff splitting (2.3.1) of $F(z,\lambda)$ is not defined. It should also be noted that, by [4, Lemma 2.2], the maximal analytic continuation of g_- does not depend on the chosen radius r. I.e. the meromorphic potential of a CMC immersion does not depend on r.

For given meromorphic g_-, we can recover the extended frame F by the Iwasawa decomposition

$$g_- = F g_+^{-1}. \tag{2.3.2}$$

For smoothness questions, see [6].

Next, we define the dressing action of $\Lambda_r^+\mathbf{SL}(2,\mathbb{C})_\sigma$, $0 < r \leq 1$, on $\mathcal{F}$, the set of extended frames of CMC-immersions. For $F(z,\lambda) \in \mathcal{F}$ and $h_+ \in \Lambda_r^+\mathbf{SL}(2,\mathbb{C})_\sigma$ we set

$$h_+(\lambda)F(z,\lambda) = (h_+.F)(z,\lambda)q_+(z,\lambda), \tag{2.3.3}$$

where the r.h.s. of (2.3.3) is defined by the Iwasawa decomposition in $\Lambda_r\mathbf{SL}(2,\mathbb{C})_\sigma$ of h_+F, i.e. $q_+ : \mathcal{D} \to \Lambda_r^+\mathbf{SL}(2,\mathbb{C})_\sigma$. In addition at $\lambda = 0$ the matrix $q_+(z,\lambda)$ takes values in the solvable subgroup B of $\mathbf{SL}(2,\mathbb{C})$, s.t.

$$B \cap \mathbf{SU}(2) = \{I\}. \tag{2.3.4}$$

It is easily proved (see e.g. [3]) that $h_+.F$ is again in $\mathcal{F}$. Therefore, Eq. (2.3.3) really defines an action on $\mathcal{F}$. On the matrices g_- defined by (2.3.1) the dressing is defined by

$$h_+(\lambda)g_-(z,\lambda) = \hat{g}_-(z,\lambda)p_+(z,\lambda). \tag{2.3.5}$$

Here, $\hat{g}_- = h_+.g_-$ and $p_+ : \mathcal{D} \to \Lambda_r^+\mathbf{SL}(2,\mathbb{C})_\sigma$ are defined by the Birkhoff splitting (2.3.1) of h_+g_-. Note that $h_+.F = \hat{g}_-\hat{g}_+$ for some $\hat{g} : \mathcal{D} \to \Lambda_r^+\mathbf{SL}(2,\mathbb{C})_\sigma$. Since g_- and $\hat{g}_-$ are both meromorphic in z, also $p_+ = \hat{g}_+q_+g_+^{-1}$ is meromorphic in z.

The extended frames are normalized by

$$F(0,\lambda) = I, \qquad \lambda \in S^1, \tag{2.3.6}$$

which implies

$$g_-(0,\lambda) = I, \qquad \lambda \in S^1. \tag{2.3.7}$$

Let now the meromorphic potential be defined by

$$\xi(z,\lambda) = g_-^{-1}\mathrm{d}g_-, \tag{2.3.8}$$

then it is of the form

$$\xi(z,\lambda) = \lambda^{-1}\begin{pmatrix} 0 & f \\ \frac{E}{f} & 0 \end{pmatrix}\mathrm{d}z, \tag{2.3.9}$$

where f is a nonvanishing meromorphic function. We will always assume $E \not\equiv 0$, i.e. we will exclude the case that the surface is part of a round sphere. To construct a CMC-immersion from a given meromorphic potential of the form (2.3.9), the functions f and E cannot be chosen arbitrarily. They have to satisfy additional conditions, given in [6].

The matrix g_- and therefore also the frame F are uniquely determined by the meromorphic potential and the initial condition (2.3.6).

From Eq. (2.3.5) it follows, that ξ transforms under dressing with $h_+ \in \Lambda_r^+\mathbf{SL}(2,\mathbb{C})_\sigma$ as

$$h_+.\xi = p_+^{-1}\xi p_+ + p_+^{-1}\mathrm{d}p_+ = \lambda^{-1}\begin{pmatrix} 0 & h_+.f \\ \frac{E}{h_+.f} & 0 \end{pmatrix}\mathrm{d}z. \tag{2.3.10}$$

Note, that $E\mathrm{d}z^2$, the Hopf differential of the CMC-immersion Ψ, is invariant under dressing.

We now set

$$p_+ = \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \tag{2.3.11}$$

Then, for the matrix entries of p_+ we get with $\hat{f} = h_+.f$:

$$\begin{aligned}
\lambda a' &= b\frac{E}{\hat{f}} - fc, && (2.3.12)\\
\lambda b' &= a\hat{f} - fd, && (2.3.13)\\
\lambda c' &= d\frac{E}{\hat{f}} - \frac{E}{f}a, && (2.3.14)\\
\lambda d' &= c\hat{f} - \frac{E}{f}b, && (2.3.15)
\end{aligned}$$

where $(\cdot)'$ denotes differentiation w.r.t. z.

2.4 Let us investigate the isotropy group $I(F)$ of an extended frame F under dressing. For an extended frame F, $I(F)$ is defined as the group of all $h_+ \in \Lambda_r^+ \mathbf{SL}(2,\mathbb{C})_\sigma$, s.t. $h_+.F = F$.

Assume, that $h_+ \in I(F)$. Then Eq. (2.3.5) defines a meromorphic function p_+ on $\mathcal{D}$, s.t.

$$\xi = h_+.\xi = p_+^{-1}\xi p_+ + p_+^{-1}\mathrm{d}p_+, \tag{2.4.1}$$

or, using (2.3.12)–(2.3.15),

$$\begin{aligned}
\lambda a' &= b\frac{E}{f} - fc = -\lambda d', && (2.4.2)\\
\lambda b' &= (a-d)f, && (2.4.3)\\
\lambda c' &= (d-a)\frac{E}{f}. && (2.4.4)
\end{aligned}$$

Lemma: *Let F be an extended frame and let ξ be the associated meromorphic potential. Let $h_+ \in I(F)$ and let $p_+ : \mathcal{D} \to \Lambda_r^+ \mathbf{SL}(2,\mathbb{C})_\sigma$ be the associated solution of Eq. (2.4.1). Define b as the upper right entry of p_+. If $b \equiv 0$, then $h_+ = I$.*

Proof: If $b \equiv 0$, then Eq. (2.4.3) gives $a = d$. This together with $\det p_+ = ad = 1$ shows, that $a = d \equiv \pm 1$. By (2.4.2) we get $c \equiv 0$. This implies, that $p_+ = \pm I$. Therefore, by Eq. (2.3.5),

$$h_+ g_- = \pm g_-. \tag{2.4.5}$$

This shows, that $h_+ = \pm I$. The case $h_+ = -I$ is excluded, since in this case, by Eq. (2.3.3), $q_+ = -I \in B \cap \mathbf{SU}(2)$, which contradicts (2.3.4). □

We want to investigate the set of scalar meromorphic functions (a,b,c,d) on $\mathcal{D}$, s.t. the set of equations (2.4.2)–(2.4.4) is satisfied. To this end, we first rewrite (2.4.2)–(2.4.4) as a single third order differential equation in b:

First, we differentiate Eq. (2.4.3) and use Eq. (2.4.2) to get

$$\lambda b'' = \frac{f'}{f}\lambda b' + 2fa' \tag{2.4.6}$$

or

$$a' = \frac{\lambda}{2f}\left(b'' - \frac{f'}{f}b'\right). \tag{2.4.7}$$

From Eq. (2.4.2) we also get, using (2.4.7),

$$c = -\frac{\lambda}{f}a' + \frac{E}{f^2}b = -\frac{\lambda^2}{2f^2}b'' + \frac{\lambda^2 f'}{2f^3}b' + \frac{E}{f^2}b. \tag{2.4.8}$$

From (2.4.3) and (2.4.4) it follows, that

$$b'\frac{E}{f} = -c' f. \tag{2.4.9}$$

By differentiating Eq. (2.4.2) we furthermore get, using Eq. (2.4.9),

$$\lambda a'' = 2b'\frac{E}{f} - f'c + b\left(\frac{E}{f}\right)'. \tag{2.4.10}$$

Differentiating (2.4.6) and using (2.4.7), (2.4.10) and (2.4.8), we get the following ordinary differential equation in b:

$$\lambda^2\left(b''' - 3\frac{f'}{f}b'' - \left(\left(\frac{f'}{f}\right)' - 2\left(\frac{f'}{f}\right)^2\right)b'\right) = 4Eb' + 2\left(E' - 2\frac{f'}{f}E\right)b. \tag{2.4.11}$$

Since p_+ takes values in the twisted loop group $\Lambda_r\mathbf{SL}(2,\mathbb{C})_\sigma$, the function $b(z,\lambda)$ is odd in λ. If we write

$$b(z,\lambda) = \sum_{n=0}^{\infty} b_n(z)\lambda^n, \tag{2.4.12}$$

then all coefficients b_n for which n is even are identically zero.
From Eq. (2.4.11) we get

$$2Eb_1' + \left(E' - 2\frac{f'}{f}E\right)b_1 = 0 \tag{2.4.13}$$

and the recursion relation

$$b_{n-2}''' - 3\frac{f'}{f}b_{n-2}'' - \left(\left(\frac{f'}{f}\right)' - 2\left(\frac{f'}{f}\right)^2\right)b_{n-2}' = 4Eb_n' + 2\left(E' - 2\frac{f'}{f}E\right)b_n,\ n \geq 3. \tag{2.4.14}$$

We can solve Eq. (2.4.13) for b_1 and we get

$$b_1 = C_b\sqrt{\frac{f^2}{E}}, \tag{2.4.15}$$

where C_b is a constant.
The conjugation of Eq. (2.3.10) with the matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ leads to a transformation of the system of equations (2.4.2)–(2.4.4). In terms of matrix entries this transformation reads

$$f \longrightarrow \frac{E}{f}, \qquad a \longleftrightarrow d, \qquad b \longleftrightarrow c. \tag{2.4.16}$$

Using this transformation, we can immediately write down the ODE for the coefficient c by replacing f by $\frac{E}{f}$ in Eq. (2.4.11):

$$\lambda^2\left(c''' - 3\left(\frac{E'}{E} - \frac{f'}{f}\right)c'' - \left(\left(\frac{E'}{E} - \frac{f'}{f}\right)' - 2\left(\frac{E'}{E} - \frac{f'}{f}\right)^2\right)c'\right) = 4Ec' - 2\left(E' - 2\frac{f'}{f}E\right)c. \tag{2.4.17}$$

We define the λ-coefficients c_n of c by

$$c(z,\lambda) = \sum_{n=0}^{\infty} c_n(z)\lambda^n. \tag{2.4.18}$$

Like b also c is odd in λ. Therefore, for n even, the coefficients c_n vanish. From (2.4.17) we get a recursion relation for the c_n:

$$2Ec_1' - \left(E' - 2\frac{f'}{f}\right) c_1 = 0 \tag{2.4.19}$$

and

$$c_{n-2}''' - 3\left(\frac{E'}{E} - \frac{f'}{f}\right) c_{n-2}'' - \left(\left(\frac{E'}{E} - \frac{f'}{f}\right)' - 2\left(\frac{E'}{E} - \frac{f'}{f}\right)^2\right) c_{n-2}' = 4Ec_n' - 2\left(E' - 2\frac{f'}{f}E\right) c_n, \tag{2.4.20}$$

$n \geq 3$. We can solve Eq. (2.4.19) for c_1 and we get

$$c_1 = C_c \sqrt{\frac{E}{f^2}}, \tag{2.4.21}$$

where C_c is a constant.

Using only Eq. (2.4.14) we get

Theorem: *Let $\Psi : \mathcal{D} \to \mathbb{R}^3$ be a conformal CMC-immersion with extended frame $F(z,\lambda)$ and define the dressing action as above. If the isotropy group $I(F)$ of F under dressing is nontrivial, i.e. $I(F) \neq \{I\}$, then the Hopf differential E of Ψ is the square of a meromorphic function.*

Proof: We assume that there exists $I \neq h_+ \in I(F)$. Let p_+ be the associated solution of (2.4.1). We define a, b, c, and d as in Eq. (2.3.11). Define b_n, $n \in \mathbb{N}$, by (2.4.12). By Lemma 2.4, $b \not\equiv 0$. Thus there exists a smallest index $N \in \mathbb{N}$, N odd, for which $b_n \not\equiv 0$, i.e. $b_N \not\equiv 0$ and $b_n \equiv 0$ for $n < N$. The λ^N-coefficient b_N is meromorphic and satisfies

$$2Eb_N' + \left(E' - 2\frac{f'}{f}E\right) b_N = 0, \tag{2.4.22}$$

since the l.h.s. of Eq. (2.4.14) vanishes. Eq. (2.4.22) has the solution

$$b_N = C\sqrt{\frac{f^2}{E}}, \tag{2.4.23}$$

where C is a complex constant, $C \neq 0$. Since b_N is meromorphic, we get that the Hopf differential

$$E = \left(C\frac{f}{b_N}\right)^2 \tag{2.4.24}$$

is the square of a meromorphic function. □

Corollary: *Let Ψ and F be as in Theorem 2.4. If the Hopf differential of Ψ has a zero of odd order, then $I(F) = \{I\}$.*

2.5 We can actually extend Corollary 2.4 to all surfaces with umbilics. Let us state the main result of this paper:

Theorem: *Let $\Psi : \mathcal{D} \to \mathbb{R}^3$ be a conformal CMC-immersion with extended frame $F(z,\lambda)$ and define the dressing action as above. If the surface defined by Ψ has an umbilic, i.e. if its Hopf differential E has a zero, then the isotropy group $I(F)$ of F under dressing is trivial, i.e. $I(F) = \{I\}$.*

Before we prove Theorem 2.5, we will draw a simple conclusion from Eq. (2.4.14) and Eq. (2.4.20):

Lemma: *Let $\xi = \lambda^{-1}\begin{pmatrix} 0 & f \\ \frac{E}{f} & 0 \end{pmatrix}\mathrm{d}z$ be a meromorphic potential and let p_+ be a meromorphic matrix function which satisfies Eq. (2.4.1). Define $b(z,\lambda)$ and $c(z,\lambda)$ by Eq. (2.3.11). Then the following holds:*

1. If f is defined at $z_0 \in \mathcal{D}$ then b is defined at z_0.

2. If f has a pole of order j at z_0, then b has at most a pole of order $2(j-1)$ at z_0.

3. If $\frac{E}{f}$ is defined at z_0, then c is defined at z_0.

4. If $\frac{E}{f}$ has a pole of order j at z_0, then c has at most a pole of order $2(j-1)$ at z_0.

Proof: We know, that the coefficients $b_n(z)$ and $c_n(z)$ are all meromorphic in $\mathcal{D}$. Let us denote by k_n the order of the pole of b_n at z_0 for $n \in \mathbb{N}$, $k_n = 0$ if b_n is defined at z_0. The function $\frac{f'}{f}$ has at most a simple pole at z_0. Let $n_f \in \mathbb{Z}$ be the residue of $\frac{f'}{f}$ at z_0.

If $k_{n-2} > 0$ then the l.h.s. of Eq. (2.4.14) has at most a pole of order $k_{n-2}+3$ at z_0. The function b_{n-2} is of the form

$$b_{n-2}(z) = \beta z^{-k_{n-2}} + v(z) \tag{2.5.1}$$

with $\beta \neq 0$, and $z^{k_{n-2}-1}v(z)$ locally holomorphic at z_0. The coefficient of $z^{-(k_{n-2}+3)}$ on the l.h.s. of Eq. (2.4.14) is given by

$$\beta(-(k_{n-2}+2)(k_{n-2}+1)k_{n-2} - 3n_f(k_{n-2}+1)k_{n-2} + (-n_f - 2n_f^2)k_{n-2}). \tag{2.5.2}$$

Therefore, the l.h.s. of Eq. (2.4.14) has a pole of order $k_{n-2}+3 > 3$ at z_0 iff

$$(k_{n-2}+2)(k_{n-2}+1) + 3n_f(k_{n-2}+1) + n_f + 2n_f^2 \neq 0. \tag{2.5.3}$$

This in turn is equivalent to

$$k_{n-2} \neq -2(n_f+1) \qquad \text{and} \qquad k_{n-2} \neq -(n_f+1). \tag{2.5.4}$$

1. If $n_f \geq 0$, i.e. f has no pole at z_0, then the condition (2.5.4) is always satisfied if $k_{n-2} > 0$. Therefore, if b_{n-2} has a pole of order $k_{n-2} > 0$ at z_0, then the l.h.s. of Eq. (2.4.14) has a pole of order $k_{n-2}+3 > 3$ at z_0. Thus, also the r.h.s. of Eq. (2.4.14) has a pole of order ≥ 4 at z_0. This is only possible, if b_n has a pole at z_0. If $k_n > 0$ then the r.h.s. of Eq. (2.4.14) has a pole of order at most $k_n - m + 1$ at z_0. Here, $m \geq 0$ is the zero order of E at z_0, $m = 0$ if $E(z_0) \neq 0$. By comparing the pole orders, we get

$$k_n \geq k_{n-2} + m + 2 > k_{n-2}. \tag{2.5.5}$$

Let us assume, that there exists $N \in \mathbb{N}$, s.t. $k_N > 0$. Using (2.5.5) we get that b_{N+2l}, $l > 0$, has a pole of order

$$k_{N+2l} \geq k_N + l(m+2),\ l > 0, \tag{2.5.6}$$

at z_0. It follows, that b has an essential singularity at $z = z_0$ for all $\lambda \in S^1$. This contradicts the meromorphicity of p_+. Therefore, all b_n are holomorphic in $\mathcal{D}$.

2. If $n_f < 0$ then $j = -n_f$, and the condition (2.5.3) is certainly satisfied if

$$k_{n-2} > -2(n_f + 1). \tag{2.5.7}$$

In this case, we can argue as in the proof of 1. that Eq. (2.5.5) is satisfied for k_n. Since therefore $k_n > k_{n-2}$, we get that also for k_n Eq. (2.5.7) is satisfied. Assume, that there exists $N \in \mathbb{N}$, s.t. $k_N > -2(n_f + 1)$. Then, by the argument above, Eq. (2.5.7) is satisfied for all $n \geq N$ and we get, as in the first part:

$$k_{N+2l} \geq k_N + l(m+2),\ l > 0. \tag{2.5.8}$$

This shows that b has an essential singularity at z_0, contradicting the meromorphicity of p_+. Therefore, b can have at most a pole of order $-2(n_f + 1) = 2(j-1)$.

3. and 4. follow from the proof of 1. and 2. by replacing f by $\frac{E}{f}$ and Eq. (2.4.14) by Eq. (2.4.20). □

2.6 The fact that dressing is a group action allows us to prove the following

Lemma: *Let F be the extended frame of a CMC immersion and let $\hat{h}_+ \in \Lambda_r^+\mathbf{SL}(2,\mathbb{C})_\sigma$. Define $\hat{F} = \hat{h}_+.F$. Then the isotropy groups $I(F)$ and $I(\hat{F})$ are isomorphic. The isomorphism is given by conjugation with $\hat{h}_+$ in $\Lambda_r^+\mathbf{SL}(2,\mathbb{C})_\sigma$.*

Proof: Assume, that $h_+ \in I(F)$. Then we have

$$h_+.F = F. \tag{2.6.1}$$

Since dressing is a group action, we can rewrite this as

$$h_+.(\hat{h}_+^{-1}.\hat{F}) = \hat{h}_+^{-1}.\hat{F}. \tag{2.6.2}$$

This shows that

$$(\hat{h}_+ h_+ \hat{h}_+^{-1}).\hat{F} = \hat{F}, \tag{2.6.3}$$

i.e. $\hat{h}_+ h_+ \hat{h}_+^{-1} \in I(\hat{F})$. Conversely, for each $h_+ \in I(\hat{F})$ we see in the same way, that $\hat{h}_+^{-1} h_+ \hat{h}_+ \in I(F)$. □

Using Lemma 2.4, Lemma 2.5, and Lemma 2.6, we can now give the

Proof of Theorem 2.5: Let f be defined by Eq. (2.3.9). We assume, that E has a zero of order $m > 0$ at some $z_0 \in \mathcal{D}$. Let $h_+ \in I(F)$ and let p_+ be the corresponding solution of (2.4.1). Let b be the upper right entry of p_+ and let b_n, $n \in \mathbb{N}$, be defined by Eq. (2.4.12).

Assume $b \not\equiv 0$. If N is the smallest index for which $b_n \not\equiv 0$ then by the same argument as in the proof of Theorem 2.4 we get

$$b_N = C\sqrt{\frac{f^2}{E}}. \tag{2.6.4}$$

Case I: f is locally holomorphic without zero at z_0. Then b is, by Lemma 2.5, defined and locally holomorphic at z_0. But b_N has by (2.6.4) a pole of order m at z_0, a contradiction. Therefore, we get $b \equiv 0$ and, by Lemma 2.4, $h_+ = I$. This shows that in this case $I(F) = \{I\}$.

Case II: f has a pole or zero at z_0. In [6, Section 3.12] it was shown, that in the $r = 1$-dressing orbit of each extended frame F there is a frame $\hat{F}$, s.t. the function $\hat{f}$, defined by the associated meromorphic potential, has neither a pole nor a zero at z_0. Since $\Lambda^+\mathbf{SL}(2,\mathbb{C})_\sigma$ is a subgroup of $\Lambda_r^+\mathbf{SL}(2,\mathbb{C})_\sigma$ for each $0 < r \leq 1$, the same statement holds for each r-dressing orbit of an extended frame. The isotropy group of $\hat{F}$ is therefore, by the proof of Case I, trivial, i.e. $I(\hat{F}) = \{I\}$. By Lemma 2.6, for two elements F and $\hat{F}$ in the same dressing orbit the isotropy groups $I(F)$ and $I(\hat{F})$ are isomorphic. This shows that $I(F) = I(\hat{F}) = \{I\}$, which finishes the proof. $\square$

Let us emphasize the main result again by reformulating it in the following form:

Corollary: *If the isotropy group of a CMC immersion under dressing is nontrivial, then the surface has no umbilics.*

3 Applications

In this section we will apply the results of the last section, i.e. Theorem 2.5 to symmetric surfaces of constant mean curvature.

Let now ξ be a meromorphic potential given by (2.3.9) with $f \equiv C = \text{const}$ and E a holomorphic function *with zeroes* on the open unit disk $\mathcal{D} = D_1$. I.e. the CMC immersion $\Psi : \mathcal{D} \to \mathbb{R}^3$ associated to ξ has umbilics.

We will show, that for such a surface the symmetry group $\mathrm{Sym}(\Psi)$ of biholomorphic automorphisms which leave the surface invariant up to a proper Euclidean motion in space, will never contain a fixed point free automorphism. In other words, it is not possible using a constant function f to construct a not simply connected CMC surface over $\mathcal{D} = D_1$ with umbilics.

3.1

The following is well known:

Lemma: *The group of biholomorphic automorphisms* $\mathrm{Aut}(D_1)$ *of the unit disk is the following group of Moebius transformations:*

$$\mathrm{Aut}(D_1) = \{\gamma : z \longmapsto \frac{az+b}{\bar{b}z+\bar{a}},\ a, b \in \mathbb{C}, |a|^2 - |b|^2 = 1\}.$$

An element $\gamma \in \mathrm{Aut}(D_1)$ *has a fixed point inside* D_1 *iff it describes a rotation around the origin* $z = 0$*, i.e.* $|a| = 1$ *and* $b = 0$.

We will use this result together with Theorem 2.5 to prove

Theorem: *Let* $\Psi : D_1 \to \mathbb{R}^3$ *be a CMC immersion with umbilics whose meromorphic potential is of the form*

$$\xi = \lambda^{-1} \begin{pmatrix} 0 & f \\ \frac{E}{f} & 0 \end{pmatrix} \mathrm{d}z,\ f \equiv C \in \mathbb{C}. \tag{3.1.1}$$

If the extended frame $F : D_1 \to \Lambda\mathbf{SU}(2)_\sigma$ *of* Ψ *satisfies*

$$F(\gamma(z), \lambda) = \chi(\lambda) F(z, \lambda) k(z) \tag{3.1.2}$$

for some $\gamma \in \mathrm{Aut}(D_1)$, $\chi \in \Lambda\mathbf{SU}(2)_\sigma$, *and* $k : D_1 \to \mathbf{U}(1)$ *then* γ *is a rotation around the origin* $z = 0 \in D_1$.

Corollary: *It is impossible to obtain non-simply connected CMC immersions* $\Psi : D_1 \to \mathbb{R}^3$ *with umbilics, in particular compact CMC surfaces of genus* $g \geq 2$*, from meromorphic potentials of the form (3.1.1).*

<u>Proof of Theorem 3.1:</u> We first write (3.1.2) as

$$(g_- g_+) \circ \gamma = \chi_- \chi_+ g_- g_+ k \tag{3.1.3}$$

where $F = g_- g_+$ and $\chi = \chi_- \chi_+$ denote the Birkhoff splitting of F and χ, respectively. Of course, $g_-^{-1} \mathrm{d} g_- = \xi$ is the meromorphic potential. Since $\chi(\lambda) = F(\gamma(0), \lambda)$, we get

$$\chi_- = g_-(\gamma(0)), \qquad \chi_+ k = g_+(\gamma(0)).$$

If we set

$$\hat{g}_- = \chi_-^{-1} g_- \circ \gamma) = \chi_+ g_- g_+ k (g_+ \circ \gamma)^{-1} \tag{3.1.4}$$

then

$$\hat{g}_-^{-1} \mathrm{d} \hat{g}_- = (g_- \circ \gamma)^{-1} \mathrm{d}(g_- \circ \gamma) = \xi \circ \gamma. \tag{3.1.5}$$

Since also $\hat{g}_-(0) = \chi_-^{-1}(g_-(\gamma(0))) = I$ we get

$$\xi \circ \gamma = \chi_+ . \xi, \tag{3.1.6}$$

where '.' denotes the dressing action.

Moreover, we already know that

$$(g_- \circ \gamma)(0) = \chi_- \tag{3.1.7}$$

and

$$\chi = \chi_- \chi_+ \in \Lambda \mathbf{SU}(2)_\sigma. \tag{3.1.8}$$

Now let us assume, that ξ is of the form (3.1.1). Let $C = c e^{i\phi}$. Dressing with the matrix $\mathrm{diag}(\exp(-i\frac{\phi}{2}), \exp(i\frac{\phi}{2})) \in \Lambda_r^+ \mathbf{SL}(2, \mathbb{C})_\sigma$ transforms f into a positive real constant. On the other hand, by [5, Corollary 4.2], dressing with a λ-independent unitary matrix just amounts to a rigid rotation of the CMC immersion in space. Thus w.l.o.g. we can assume that C is a positive real constant.

Equation (3.1.6) gives

$$\begin{aligned} \chi_+ . f &= (f \circ \gamma)\gamma' = \frac{C}{(\overline{b} z + \overline{a})^2} \\ &= \frac{C \overline{a}^{-2}}{(1 + \frac{\overline{b}}{C \overline{a}}(Cz))^2} = \frac{C \overline{a}^{-2}}{(1 + \frac{\overline{b}}{C \overline{a}} \int_0^z f \mathrm{d}z)^2} \\ &= T_D(\sqrt{C} \overline{a}^{-1}) T_U(\frac{\overline{b}}{C \overline{a}})(f) \end{aligned} \tag{3.1.9}$$

where $f \equiv C$ and T_U, T_D denote the basic dressing transformations investigated in [6, Section 3]. I.e. $T_D(t)$ denotes dressing with $\mathrm{diag}(t, t^{-1})$ and $T_U(t)$ denotes dressing with $\begin{pmatrix} 1 & 0 \\ t\lambda & 1 \end{pmatrix}$.

Now Theorem 2.5 implies that on surfaces with umbilics the dressing action is free. Thus, (3.1.9) determines the matrix χ_+ uniquely:

$$\chi_+ = \begin{pmatrix} \frac{s}{\overline{a}} & 0 \\ 0 & \frac{\overline{a}}{s} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{\overline{b}\lambda}{C\overline{a}} & 1 \end{pmatrix} = \begin{pmatrix} \frac{s}{\overline{a}} & 0 \\ \frac{\overline{b}\lambda}{Cs} & \frac{\overline{a}}{s} \end{pmatrix}, \tag{3.1.10}$$

where we have chosen $s = \sqrt{C}$ to be the positive root. Now we have to find χ_- s.t. (3.1.8) holds. We make the ansatz

$$\chi_- = \begin{pmatrix} 1 & q\lambda^{-1} \\ 0 & 1 \end{pmatrix}.$$

Then

$$\chi = \chi_-\chi_+ = \begin{pmatrix} \frac{s}{\bar{a}} + \frac{\bar{b}q}{Cs} & \frac{q\bar{a}}{s}\lambda^{-1} \\ \frac{\bar{b}}{Cs}\lambda & \frac{\bar{a}}{s} \end{pmatrix}$$

has to be unitary for all $\lambda \in S^1$. This is equivalent to

$$q = -\frac{b}{\bar{a}C}, \tag{3.1.11}$$

$$\frac{s}{\bar{a}} + \frac{\bar{b}q}{Cs} = \frac{a}{s}. \tag{3.1.12}$$

From this it follows, using $s^2 = C$, that

$$C - \frac{|b|^2}{C^2} = |a|^2. \tag{3.1.13}$$

The λ^{-1}-coefficient of Eq. (3.1.7) gives

$$q = \int_0^{\gamma(0)} f \mathrm{d}z = C\gamma(0) = C\frac{b}{\bar{a}},$$

which together with (3.1.11) gives $C = 1$. Finally, (3.1.13) implies for $C = 1$, that $|a|^2 + |b^2| = 1$, which together with $|a|^2 - |b|^2 = 1$ forces $b = 0$ and $|a| = 1$, i.e. γ is a rotation around the origin $z = 0$ in D_1. $\square$

3.2 The next best candidates for symmetric surfaces are expected to lie in the dressing orbit of automorphic meromorphic potentials. I.e. if $\gamma \in \Gamma$ is an automorphism in the symmetry group $\mathrm{Aut}_\Psi \mathcal{D}$, $\mathcal{D} = \mathbb{C}$ or $\mathcal{D} = D_1$, then we look at a meromorphic potential

$$\xi_0 = \lambda^{-1} \begin{pmatrix} 0 & f \\ \frac{E}{f} & 0 \end{pmatrix} \mathrm{d}z,$$

s.t. $E \circ \gamma = {\gamma'}^{-2} E$, $f \circ \gamma = {\gamma'}^{-1} f$. Since ξ_0 is an automorphic one form w.r.t. γ, we conclude for the integral g_-^0:

$$g_-^0(\gamma(z), \lambda) = \rho_-^0(\lambda) g_-^0(z, \lambda), \tag{3.2.1}$$

where $\rho_-^0(\lambda) \in \Lambda_{r,*}^- \mathbf{SL}(2, \mathbb{C})_\sigma$.

Let now $h_+ \in \Lambda_r^+ \mathbf{SL}(2, \mathbb{C})_\sigma$ and define g_- by the dressing action of h_+ on g_-^0,

$$h_+(\lambda) g_-^0(z, \lambda) = g_-(z, \lambda) p_+(z, \lambda) \tag{3.2.2}$$

with $p_+ : \mathcal{D} \to \Lambda_r^+ \mathbf{SL}(2, \mathbb{C})_\sigma$. We denote by ξ the corresponding meromorphic potential $\xi = g_-^{-1} \mathrm{d} g_-$. We then get the following result:

Proposition: *Let γ be an automorphism of $\mathcal{D}$, $\mathcal{D} = \mathbb{C}$ or $\mathcal{D} = D_1$, and let ξ_0 be a meromorphic potential with umbilics on $\mathcal{D}$ which satisfies $\gamma^* \xi_0 = \xi_0$. Let $h_+ \in \Lambda_r^+ \mathbf{SL}(2, \mathbb{C})_\sigma$ and define ξ, g_-^0, ρ_-^0, and g_- as above. Then there is a unique $\rho \in \Lambda_r \mathbf{SL}(2, \mathbb{C})_\sigma$ and a unique map $w_+ : \mathcal{D} \to \Lambda_r^+ \mathbf{SL}(2, \mathbb{C})_\sigma$, such that*

$$g_- \circ \gamma = \rho g_- w_+. \tag{3.2.3}$$

Moreover,

$$\rho = h_+ \rho_-^0 h_+^{-1}. \tag{3.2.4}$$

Proof: First we note that

$$g_- \circ \gamma = h_+(g_-^0 \circ \gamma)(p_+^{-1} \circ \gamma) = h_+ \rho_-^0 h_+^{-1} g_- p_+ (p_+^{-1} \circ \gamma). \tag{3.2.5}$$

Thus, (3.2.3) holds, with $w_+ = p_+(p_+^{-1} \circ \gamma)$ and $\rho = h_+ \rho_-^0 h_+^{-1} \in \Lambda_r \mathbf{SL}(2, \mathbb{C})_\sigma$.

Now let $\rho \in \Lambda_r \mathbf{SL}(2, \mathbb{C})_\sigma$ be an arbitrary matrix, such that (3.2.3) holds for some $w_+ : \mathcal{D} \to \Lambda_r^+ \mathbf{SL}(2, \mathbb{C})_\sigma$. Using the definition (3.2.2) of g_- we get from (3.2.3)

$$\rho^{-1} h_+ \rho_-^0 h_+^{-1} g_- = g_- w_+ (p_+ \circ \gamma) p_+^{-1}. \tag{3.2.6}$$

Since $g_-(0, \lambda) = I$, we get $\rho^{-1} h_+ \rho_-^0 h_+^{-1} \in \Lambda_r^+ \mathbf{SL}(2, \mathbb{C})_\sigma$. Thus, $\rho^{-1} h_+ \rho_-^0 h_+^{-1}$ is in the isotropy group of ξ under dressing. Since we assumed that ξ_0 and therefore also ξ has umbilics, Theorem 2.5 gives

$$\rho = h_+ \rho_-^0 h_+^{-1}, \tag{3.2.7}$$

from which the uniqueness of w_+ also follows. □

Theorem: *Under the same assumptions as in the proposition above the following two statements are equivalent:*

1. *The automorphism γ is in the symmetry group of the CMC immersion Ψ corresponding to ξ, i.e. $\gamma \in \mathrm{Aut}_\Psi \mathcal{D}$,*
2. *The integral g_- of ξ is invariant under γ, i.e. $g_- \circ \gamma = g_-$.*

Proof: 2. $\Rightarrow$ 1. follows immediately from [5, Corollary 4.2]. For the converse statement assume that $\gamma \in \mathrm{Aut}_\Psi \mathcal{D}$. Then g_- transforms by [5, Theorem 4.2] and [4, Lemma 2.2] like (3.2.3), where $\rho \in \Lambda_r \mathbf{SU}(2)_\sigma \subset \Lambda_r \mathbf{SL}(2, \mathbb{C})_\sigma$ and $w_+ : \mathcal{D} \to \Lambda_r^+ \mathbf{SL}(2, \mathbb{C})_\sigma$. By Proposition 3.2 we have $\rho = h_+ \rho_-^0 h_+^{-1}$. Hence, on the circle S_r the eigenvalues of ρ and ρ_-^0 coincide. The matrix ρ is by [5, Section 3] also the monodromy matrix of the extended frame F of Ψ,

$$F(\gamma(z), \lambda) = \rho(\lambda) F(z, \lambda) k(z) \tag{3.2.8}$$

for some $k : \mathcal{D} \to \mathbf{U}(1)$. Therefore, by [4, Lemma 2.2], the unitary matrix ρ can be extended holomorphically to $\mathbb{C}^*$. And since ρ_-^0 can be extended holomorphically to the exterior of the circle S_r, we get that the eigenvalues of ρ and ρ_-^0 are holomorphic functions on $\mathbb{CP}_1$. Thus, they are constant and equal to the eigenvalue of ρ_-^0 at $\lambda = \infty$. By $\rho_-^0(\lambda \to \infty) = I$ we get $\rho = I$, which together with (3.2.3) gives the desired result. □

Remark: It should be noted here, that due to Theorem 3.2, the situation for CMC immersions with umbilics is very different from the situation without umbilics: As was shown in [8], *all* CMC immersions of finite type, among which are all CMC tori, can be obtained by dressing the translationally invariant meromorphic potential $\xi = \lambda^{-1} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} dz$ of the standard cylinder. However, as follows from [4, Eq. (3.4.5)], *none* of these surfaces (not even the standard cylinder itself) has a constant monodromy matrix $\rho = \chi \equiv I$. I.e. Theorem 3.2 can obviously not be extended to potentials without umbilics. Therefore, there is no immediate way to generalize the application of the dressing group to finite type surfaces, as it was done in [4], to surfaces with umbilics. In fact the triviality of the dressing isotropy groups of CMC immersions with umbilics seems to indicate, that for surfaces with umbilics, in particular compact surfaces of genus $g \geq 2$, there is no algebraic geometric method corresponding to Krichever's method.

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