Surfaces in Lie sphere geometry and the stationary Davey-Stewartson hierarchy

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Abstract

We introduce two basic invariant forms which define generic surface in 3-space uniquely up to Lie sphere equivalence. Two particularly interesting classes of surfaces associated with these invariants are considered, namely, the Lie-minimal surfaces and the diagonally-cyclidic surfaces. For diagonally-cyclidic surfaces we derive the stationary modified Veselov-Novikov equation, whose role in the theory of these surfaces is similar to that of Calapso's equation in the theory of isothermic surfaces. Since Calapso's equation itself turns out to be related to the stationary Davey-Stewartson equation, these results shed some new light on differential geometry of the stationary Davey-Stewartson hierarchy. Diagonally-cyclidic surfaces are the natural Lie sphere analogs of the isothermally-asymptotic surfaces in projective differential geometry for which we also derive the stationary modified Veselov-Novikov equation with the different real reduction.

Parallels between invariants of surfaces in Lie sphere geometry and reciprocal invariants of hydrodynamic type systems are drawn in the conclusion.

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1 Introduction

Lie sphere geometry dates back to the dissertation of Lie in 1872 [1]. After that the subject was extensively developed by Blaschke and his coworkers and resulted in publication in 1929 of Blaschke's Vorlesungen über Differentialgeometrie [2], entirely devoted to the Lie sphere geometry of curves and surfaces. The modern multidimensional period of the theory was initiated by Pinkall's classification of Dupin hypersurfaces in E^4 [3], [4]. We refer also to Cecil's book [5] with the review of the last results in this direction. Since most of the recent research in Lie sphere geometry is concentrated around Dupin hypersurfaces and Dupin submanifolds, the general theory of Lie-geometric hypersurfaces seems not to be constructed so far. The aim of this paper is to shed some new light on Lie sphere geometry of (hyper)surfaces and to reveal its remarkable applications in the modern theory of integrable systems.

Let $M^2 \in E^3$ be a surface in the 3-dimensional space E^3 parametrized by the coordinates R^1, R^2 of the lines of curvature. Let k^1, k^2 and $g_{11}dR^{1^2} + g_{22}dR^{2^2}$ be the principal curvatures and the metric of M^2, respectively. In sect. 2 we introduce the basic Lie sphere invariants of the surface M^2, namely, the symmetric 2-form

$$\frac{\partial_1 k^1 \partial_2 k^2}{(k^1 - k^2)^2} \, dR^1 dR^2 \tag{1.1}$$

and the conformal class of the cubic form

$$\partial_1 k^1 g_{11} dR^{1^3} + \partial_2 k^2 g_{22} dR^{2^3}, \tag{1.2}$$

$\partial_i = \frac{\partial}{\partial R^i}$, which define "generic" surface M^2 uniquely up to Lie sphere equivalence. We recall that the group of Lie sphere transformations in E^{n+1} is a contact group, generated by conformal transformations and normal shifts, translating each point of the surface to a fixed distance $a = const$ along the normal direction. Conformal transformations and normal shifts generate in E^{n+1} a finite-dimensional Lie group isomorphic to $SO(n+2, 2)$. Lie sphere transformations can be equivalently characterized as the contact transformations, mapping spheres into spheres and preserving their oriented contact. In the implicit form objects (1.1) and (1.2) were contained already in [2]. Quadratic form (1.1) gives rise to the Lie-invariant functional

$$\int\int \frac{\partial_1 k^1 \partial_2 k^2}{(k^1 - k^2)^2} \, dR^1 dR^2 \tag{1.3}$$

the extremals of which are known as minimal surfaces in Lie sphere geometry ([2], § 94). Other possible representations of functional (1.3) and its relationship to the integrable hierarchy of Davey-Stewartson type are discussed in sect. 3. In sect. 4 we investigate the so-called diagonally-cyclidic surfaces ("diagonalzyklidische flächen" in the terminology of [2], p. 406), which can be characterized as surfaces M^2 possessing parametrization R^1, R^2 by the coordinates of lines of curvature such that the cubic form (1.2) becomes proportional to $dR^{1^3} + dR^{2^3}$. This class of surfaces is a straightforward generalization of isothermic surfaces in conformal geometry. It is demonstrated that for diagonally-cyclidic surfaces the Lie sphere density U defined by

$$U^2 = \frac{\partial_1 k^1 \partial_2 k^2}{(k^1 - k^2)^2}$$

satisfies the stationary modified Veselov-Novikov (mVN) equation

$$\begin{gathered}\partial_1^3 U + 3V\partial_1 U + \tfrac{3}{2}U\partial_1 V = \partial_2^3 U + 3W\partial_2 U + \tfrac{3}{2}U\partial_2 W\\ \partial_1 W = \partial_2(U^2)\\ \partial_2 V = \partial_1(U^2)\end{gathered}$$

which in the theory of diagonally-cyclidic surfaces plays a role similar to that of the Calapso equation [8]

$$\partial_1^2\left(\frac{\partial_1\partial_2 u}{u}\right) + \partial_2^2\left(\frac{\partial_1\partial_2 u}{u}\right) + \frac{1}{2}\partial_1\partial_2(u^2) = 0$$

in the theory of isothermic surfaces. Since Calapso's equation itself turns out to be related to the stationary Davey-Stewartson (DS) equation (see sect.4), this provides remarkable differential-geometric interpretation of the stationary flows of DS hierarchy and gives new insight into the recent results of [9] – [15] relating DS hierarchy with conformal geometry. The details of derivation of mVN equation for diagonally-cyclidic surfaces are included in the Appendix.

It is quite remarkable that in projective differential geometry there also exists a class of surfaces (the so-called isothermally-asymptotic surfaces, or Φ-surfaces) governed by the stationary mVN equation

$$\begin{gathered}p_{xxx} - 3Vp_x - \tfrac{3}{2}pV_x = p_{yyy} - 3Wp_y - \tfrac{3}{2}pW_y\\ W_x = (p^2)_y\\ V_y = (p^2)_x\end{gathered}$$

which can be reduced to that presented above by a complex change of variables $p \to iU, \quad W \to -W, \quad V \to -V$. These are surfaces, for which the Darboux cubic form is proportional to the sum of pure cubes $dx^3 + dy^3$ in the appropriate asymptotic parametrization x, y (particular examples are affine spheres related to the Tzitzeica equation). This observation reflects the duality between projective and Lie sphere geometries due to the Lie's famous line-sphere correspondence (see sect.4).

In sect. 5 we introduce Lie sphere invariants of higher dimensional hypersurfaces $M^n \in E^{n+1}$, namely the symmetric 2-form

$$\sum_{i\neq j} \frac{k_i^i k_j^j}{(k^i - k^j)^2}\, \omega^i \omega^j \tag{1.4}$$

and the conformal class of the cubic form

$$\sum_i k_i^i g_{ii}\, {\omega^i}^3 \tag{1.5}$$

defining "generic" hypersurface uniquely up to Lie sphere equivalence. Here k^i are principal curvatures, ω^i are principal covectors, $\sum g_{ii}{\omega^i}^2$ is the first fundamental form and coefficients k^i_i are defined by the expansions $dk^i = k^i_j\omega^j$ (we emphasize that hypersurface M^n of dimension $n \geq 3$ does not necessarily possess parametrization by the coordinates of lines of curvature). Objects (1.4) and (1.5) are the Lie-geometric analogs of the second fundamental form and the Darboux cubic form in projective differential geometry of hypersurfaces.

In sect. 6-9 interrelations between Lie sphere invariants and reciprocal invariants of hydrodynamic type systems

$$u^i_t = v^i_j(u)u^j_x, \quad i,j = 1,...,n \tag{1.6}$$

are discussed. We recall that reciprocal transformations are transformations from x, t to the new independent variables X, T defined by the formulae

$$\begin{aligned} dX &= B(u)dx + A(u)dt \\ dT &= N(u)dx + M(u)dt \end{aligned}$$

where $Bdx + Adt$ and $Ndx + Mdt$ are two integrals of system (1.6). Reciprocal transformations originate from gas dynamics and were extensively investigated in [16], [17]. In [18], [19] we introduced reciprocal invariants, defining a hydrodynamic type system uniquely up to reciprocal equivalence. The summary of these results in the 2-component case is given in sect. 6. In sect. 7-8 we recall the necessary information about Hamiltonian systems of hydrodynamic type and describe the general construction of [20], [21], relating Hamiltonian systems (1.6) and hypersurfaces in E^{n+1}. The main property of this correspondence is its "equivariance" in the sense that Lie sphere transformations of hypersurfaces correspond to "canonical" reciprocal transformations, that is, to those reciprocal transformations which preserve the Hamiltonian structure. In this approach Lie sphere invariants of hypersurfaces correspond to reciprocal invariants of hydrodynamic type systems, providing thus their differential-geometric interpretation.

In sect. 9 we write down reciprocal invariants of n-component systems for arbitrary $n \geq 3$ since they differ from those in case $n = 2$.

2 Invariants of surfaces in Lie sphere geometry

In [2], p.392 Blaschke introduced the Lie-invariant 1-forms ω^1, ω^2 ($d\psi$, $d\bar{\psi}$ in Blaschke's notation) which assume the following form in the coordinates R^1, R^2 of the lines of curvature:

$$\begin{aligned} \omega^1 &= \frac{k^1_1}{k^1-k^2}\left(\frac{{k^2_2}^2 g_{11}}{{k^1_1}^2 g_{22}}\right)^{\frac{1}{6}} dR^1, \\ \omega^2 &= \frac{k^2_2}{k^2-k^1}\left(\frac{{k^1_1}^2 g_{22}}{{k^2_2}^2 g_{11}}\right)^{\frac{1}{6}} dR^2. \end{aligned} \tag{2.1}$$

Remark 1. In order to check Lie-sphere invariance of the 1-forms ω^1, ω^2 it is sufficient to check their invariance under the inversions and normal shifts, which can be verified by a direct calculation. Moreover, forms (2.1) do not change if the principal curvatures k^i are replaced by the radii of principal curvatures $w^i = \frac{1}{k^i}$ and the first fundamental form g_{ii} by the third fundamental form $G_{ii} = {k^i}^2 g_{ii}$.

Remark 2. Similar invariant 1-forms arise in the Möbius (conformal) and Laguerre geometries, which are subcases of the Lie sphere geometry. In the Möbius geometry we have the invariant 1-forms

$$\frac{\partial_1 k^1}{k^1 - k^2} dR^1, \qquad \frac{\partial_2 k^2}{k^2 - k^1} dR^2$$

and the invariant quadratic form

$$(k^1 - k^2)^2 (g_{11} {dR^1}^2 + g_{22} {dR^2}^2),$$

while in the Laguerre geometry they are

$$\frac{\partial_1 w^1}{w^1 - w^2} dR^1, \qquad \frac{\partial_2 w^2}{w^2 - w^1} dR^2$$

and

$$(w^1 - w^2)^2 (G_{11} {dR^1}^2 + G_{22} {dR^2}^2),$$

respectively (see [2], [28], [29]).

As far as ω^1 and ω^2 are invariant under Lie sphere transformations, so do the quadratic form

$$-\omega^1 \omega^2 = \frac{\partial_1 k^1 \partial_2 k^2}{(k^1 - k^2)^2} \, dR^1 dR^2$$

and the qubic form

$$ {\omega^1}^3 - {\omega^2}^3 = \frac{\partial_1 k^1 \partial_2 k^2}{(k^1 - k^2)^3 \sqrt{g_{11} g_{22}}} (\partial_1 k^1 g_{11} {dR^1}^3 + \partial_2 k^2 g_{22} {dR^2}^3), \qquad (2.2)$$

giving rise to (1.1) and (1.2), respectively. The reason for introducing these objects is their additional invariance under the interchange of indices 1 and 2 so that they play a role similar to that of "symmetric functions of the roots of polynomial" in the Viete theorem. Hence they define tensors which can be effectively computed in an arbitrary coordinate system (without solving algebraic equations).

In [2], §85 it is proved that up to certain exeptional cases a generic surface in E^3 is determined by the corresponding 1-forms ω^1, ω^2 uniquely up to Lie sphere transformations. Since we can reconstruct ω^1, ω^2 from the quadratic form (1.1) and the conformal class of the cubic form (1.2) (the multiple in (2.2) is not essential), we can formulate the following

Theorem 1. *A generic surface $M^2 \in R^3$ is defined by the quadratic form*

$$\frac{\partial_1 k^1 \partial_2 k^2}{(k^1-k^2)^2}\ dR^1 dR^2$$

and the conformal class of the cubic form

$$\partial_1 k^1 g_{11} dR^{1^3} + \partial_2 k^2 g_{22} dR^{2^3}$$

uniquely up to Lie sphere transformations.

The vanishing of the qubic form is equivalent to the conditions $\partial_1 k^1 = \partial_2 k^2 = 0$ which specify the so-called cyclids of Dupin. We recall that the vanishing of Darboux's cubic form in projective geometry specifies quadrics, which are thus projective "duals" of cyclids of Dupin.

Remark. The principal directions of the surface M^2 can be characterized as the zero directions of quadratic form (1.1). On the other hand, they are exactly those directions, where cubic form (1.2) reduces to the sum of pure cubes (without mixed terms). It should be pointed out, that any cubic form on the plane can be reduced to the sum of cubes, and the directions where it assumes the desired form are defined uniquely. In projective differential geometry of surfaces these are asymptotic directions of the Darboux cubic form.

3 Minimal surfaces in Lie sphere geometry

Lie-minimal surfaces are defined as the extremals of Lie-invariant functional (1.3)

$$\int\int \frac{\partial_1 k^1 \partial_2 k^2}{(k^1-k^2)^2}\ dR^1 dR^2$$

which is a natural analog of the Willmore functional

$$\int\int (k^1-k^2)^2 \sqrt{g_{11}g_{22}}\ dR^1 dR^2$$

in conformal geometry and the invariant functional

$$\int\int (w^1-w^2)^2 \sqrt{G_{11}G_{22}}\ dR^1 dR^2$$

in the Laguerre geometry. Due to the obvious identity

$$\frac{\partial_1 k^1 \partial_2 k^2}{(k^1-k^2)^2}\ dR^1 \wedge dR^2 = \frac{\partial_2 k^1 \partial_1 k^2}{(k^1-k^2)^2}\ dR^1 \wedge dR^2 - d\left(\frac{dk^2}{k^1-k^2}\right)$$

we see that for compact surfaces with $k^1 \neq k^2$ (for instance, immersed tori) functional (1.3) coincides with the functional

$$\int\int \frac{\partial_2 k^1 \partial_1 k^2}{(k^1-k^2)^2}\ dR^1 dR^2 = -\int\int ab\ dR^1 dR^2, \tag{3.1}$$

where we introduced the notation $a = \frac{\partial_2 k^1}{k^2 - k^1}, \quad b = \frac{\partial_1 k^2}{k^1 - k^2}$. We recall that in terms of the coefficients a and b the Peterson-Codazzi equations of the surface M^2 can be written as follows

$$\partial_2 \ln \sqrt{g_{11}} = a, \qquad \partial_1 \ln \sqrt{g_{22}} = b$$

while the equation for the radius-vector $\vec{r}$ assumes the form

$$\partial_1 \partial_2 \vec{r} = a \partial_1 \vec{r} + b \partial_2 \vec{r} \tag{3.2}$$

manifesting the fact that the net of the lines of curvature is conjugate.

Introducing the rotation coefficients β_{12}, β_{21} by the formulae

$$\beta_{12} = \frac{\partial_1 \sqrt{g_{22}}}{\sqrt{g_{11}}} = b \, \frac{\sqrt{g_{22}}}{\sqrt{g_{11}}}, \qquad \beta_{21} = \frac{\partial_2 \sqrt{g_{11}}}{\sqrt{g_{22}}} = a \, \frac{\sqrt{g_{11}}}{\sqrt{g_{22}}},$$

we can rewrite our functional in the form

$$\int \int \beta_{12} \beta_{21} \; dR^1 dR^2$$

which has the meaning of the "integral squared rotation" of the surface M^2.

Written in the form (3.1) our functional is closely related to the simplest quadratic conservation law of the (2+1)-dimensional hierarchy of the Davey-Stewartson (DS) type. To make it clear we recall the construction of [11], which defines the DS-type flow on conjugate nets in E^3. Let M^2 be a surface parametrized by conjugate coordinates R^1, R^2. The radius-vector $\vec{r}$ of such surface satisfies the equation (3.2) (at the moment we do not assume that our conjugate net is the net of lines of curvature). Let us define evolution of M^2 with respect to the "time" t by the formula

$$\vec{r}_t = \alpha {\partial_1}^2 \vec{r} + \beta {\partial_2}^2 \vec{r} + p \partial_1 \vec{r} + q \partial_2 \vec{r}, \tag{3.3}$$

where $\alpha, \beta = const$. The compatibility conditions of (3.2) and (3.3) give rise to the integrable system

$$\begin{gathered} a_t = \beta {\partial_2}^2 a - \alpha {\partial_1}^2 a + 2\alpha \partial_1 (ab) + \beta \partial_2 (a^2) + p \partial_1 a + \partial_2 (qa), \\ b_t = \alpha {\partial_1}^2 b - \beta {\partial_2}^2 b + 2\beta \partial_2 (ab) + \alpha \partial_1 (b^2) + q \partial_2 b + \partial_1 (pb), \\ \partial_2 p + 2\alpha \partial_1 a = 0, \\ \partial_1 q + 2\beta \partial_2 b = 0, \end{gathered} \tag{3.4}$$

whose relationship to the DS system was clarified in [11]. In fact system (3.4) is a linear combination of two simpler systems, corresponding to the choices ($\alpha = 1, \beta = 0, q = 0$) and ($\alpha = 0, \beta = 1, p = 0$), namely

$$\begin{gathered} a_t = -{\partial_1}^2 a + 2\partial_1 (ab) + p \partial_1 a, \\ b_t = {\partial_1}^2 b + \partial_1 (b^2) + \partial_1 (pb), \\ \partial_2 p + 2\partial_1 a = 0, \end{gathered} \tag{3.5}$$

and

$$a_t = {\partial_2}^2 a + \partial_2(a^2) + \partial_2(qa),$$

$$b_t = -{\partial_2}^2 b + 2\partial_2(ab) + q\partial_2 b, \tag{3.6}$$

$$\partial_1 q + 2\partial_2 b = 0,$$

respectively. Both systems (3.5) and (3.6) commute according to the general discussion in [31] and possess the quadratic integral

$$\int\!\!\int ab \; dR^1 dR^2$$

coinciding with (3.1). We emphasize, however, that the nets of lines of curvature are not preserved (in general) by these t-evolutions, although always remain conjugate.

4 Cyclidic curves and diagonally-cyclidic surfaces

With any surface M^2 we associate a 3-web of curves (that is, three 1-parameter families of curves) formed by the lines of curvature and cyclidic curves ("zyklidische kurven" in the terminology of Blaschke [2], §86) which are the zero directions of cubic form (1.2). In view of formula (2.2) the curves of this 3-web can be defined in terms of 1-forms (2.1) by the equations

$$\omega^1 = 0, \qquad \omega^2 = 0, \qquad \omega^1 - \omega^2 = 0, \tag{4.1}$$

respectively. Cyclidic curves naturally arise in the attempt to find those cyclids of Dupin, which are the "best" tangents to a given surface M^2 at a given point (see [2], §86 for the details). These curves are the natural analogs of the Darboux curves in projective differential geometry. Let us compute the connection 1-form ω of the 3-web (4.1), that is, the 1-form which is uniquely determined by the equations

$$d\omega^1 = \omega \wedge \omega^1, \quad d\omega^2 = \omega \wedge \omega^2,$$

(see [26], [27] for the introduction in web geometry). A direct computation results in

$$\begin{gathered}\omega = \tfrac{1}{3}\left(\tfrac{\partial_1\partial_2 k^2}{\partial_2 k^2} + \tfrac{\partial_1 k^1}{k^1-k^2}\right) dR^1 + \tfrac{1}{3}\left(\tfrac{\partial_1\partial_2 k^1}{\partial_1 k^1} + \tfrac{\partial_2 k^2}{k^2-k^1}\right) dR^2 + \\ \tfrac{1}{3} d \ln \tfrac{\partial_1 k^1 \partial_2 k^2}{(k^1-k^2)^5\sqrt{g_{11}g_{22}}}.\end{gathered} \tag{4.2}$$

Since both ω^1, ω^2 are invariant under Lie sphere transformations, so does the connection 1-form ω. From (4.2) it immediately follows that the curvature form $d\omega$ of 3-web (4.1) is given by

$$d\omega = \frac{1}{3} d\Omega,$$

where

$$\Omega=\left(\frac{\partial_1\partial_2 k^2}{\partial_2 k^2}+\frac{\partial_1 k^1}{k^1-k^2}\right)dR^1+\left(\frac{\partial_1\partial_2 k^1}{\partial_1 k^1}+\frac{\partial_2 k^2}{k^2-k^1}\right)dR^2. \tag{4.3}$$

As we will see in sect. 6 the object analogous to (4.3) arises in the theory of reciprocal invariants of hydrodynamic type systems.

An interesting class of diagonally-cyclidic surfaces (”diagonalzyklidische flächen” in the terminology of [2], p.406) is specified by the requirement, that 3-web (4.1) is hexagonal or, equivalently, has zero curvature:

$$d\omega=\frac{1}{3}d\Omega=0.$$

In this case there exist coordinates R^1, R^2 along the lines of curvature (note, that we have a reparametrization freedom $R^i\to\varphi^i(R^i)$), where ω^1,ω^2 assume the form

$$\omega^1=pdR^1, \quad \omega^2=-pdR^2$$

with nonzero common multiple p. Since in these coordinates the cubic form ${\omega^1}^3-{\omega^2}^3$ is proportional to ${dR^1}^3+{dR^2}^3$, these surfaces are the natural Lie-sphere analogs of isothermic surfaces in conformal geometry (we emphasize, that the class of isothermic surfaces is not invariant under the full group of Lie sphere transformations).

Equations governing diagonally-cyclidic surfaces can be easily written down as follows. Since the cubic form

$$\partial_1 k^1 g_{11}{dR^1}^3+\partial_2 k^2 g_{22}{dR^2}^3$$

is proportional to

$${dR^1}^3+{dR^2}^3,$$

we can put

$$g_{11}=\frac{e^{2\rho}}{\partial_1 k^1}, \qquad g_{22}=\frac{e^{2\rho}}{\partial_2 k^2}.$$

Inserting this representation in the Gauss-Peterson-Codazzi equations, we arrive at the following system for k^1, k^2, ρ:

$$\begin{gathered}\partial_1\rho=b+\tfrac{1}{2}\tfrac{\partial_1\partial_2 k^2}{\partial_2 k^2},\\ \partial_2\rho=a+\tfrac{1}{2}\tfrac{\partial_1\partial_2 k^1}{\partial_1 k^1},\\ \partial_1 k^1\left(\partial_1 b+\tfrac{b}{2}\partial_1\ln\tfrac{\partial_1 k^1}{\partial_2 k^2}\right)+\partial_2 k^2\left(\partial_2 a+\tfrac{a}{2}\partial_2\ln\tfrac{\partial_2 k^2}{\partial_1 k^1}\right)+k^1k^2e^{2\rho}=0,\end{gathered} \tag{4.4}$$

where $a=\frac{\partial_2 k^1}{k^2-k^1}, \quad b=\frac{\partial_1 k^2}{k^1-k^2}$.

Remark. System (4.4) is a Lie-sphere analog of the system

$$\begin{gathered}\partial_1\rho=b,\\ \partial_2\rho=a,\\ \partial_1^2\rho+\partial_2^2\rho+k^1k^2e^{2\rho}=0,\end{gathered} \tag{4.5}$$

describing isothermic surfaces in conformal geometry. The integrability and discretization (see [6], [7]) of system (4.5) are based on the $SO(4,1)$-linear problem, which comes from the following geometric fact: all isothermic surfaces possess Ribacour transformations preserving the metric up to a conformal factor. Moreover, the spectral parameter is due to the following scailing symmetry of system (4.5):

$$\tilde{R}^1 = \frac{1}{c}R^1, \quad \tilde{R}^2 = \frac{1}{c}R^2, \quad \tilde{k}^1 = ck^1, \quad \tilde{k}^2 = ck^2, \quad c = const.$$

We recall also that system (4.5) can be rewritten as the single fourth-order Calapso equation [8]

$$\partial_1^2\left(\frac{\partial_1\partial_2 u}{u}\right) + \partial_2^2\left(\frac{\partial_1\partial_2 u}{u}\right) + \frac{1}{2}\partial_1\partial_2(u^2) = 0 \tag{4.6}$$

for the conformal factor $u = e^{\rho}(k^1 - k^2)$. Another possible approach to the integrability of isothermic surfaces is based on the relationship of the Calapso equation to the DS-II equation

$$\begin{gathered} iu_t + u_{xx} - u_{yy} + uv = 0, \\ v_{xx} + v_{yy} = |u|^2_{xx} - |u|^2_{yy} \end{gathered}$$

which in the stationary case assumes the form

$$\begin{gathered} u_{xx} - u_{yy} + uv = 0, \\ v_{xx} + v_{yy} = |u|^2_{xx} - |u|^2_{yy}. \end{gathered}$$

Excluding v we arrive at the fourth-order equation with respect to u

$$\triangle\left(\frac{u_{xx} - u_{yy}}{u}\right) + |u|^2_{xx} - |u|^2_{yy} = 0$$

coinciding with (4.6) after the transformation $R^1 = x + y$, $R^2 = x - y$ and the reduction $u = \bar{u}$.

Similar approaches can be applied to system (4.4). Indeed, all diagonally-cyclidic surfaces possess Ribacour transformations preserving the cubic form up to a conformal factor ([2], p.420). Moreover, system (4.4) possesses a similar scaling symmetry

$$\tilde{R}^1 = \frac{1}{c}R^1, \quad \tilde{R}^2 = \frac{1}{c}R^2, \quad \tilde{k}^1 = c^3k^1, \quad \tilde{k}^2 = c^3k^2,$$

which is responsible for the spectral parameter. We hope to develop this geometric approach elsewhere. Another approach to the integrability of diagonally-cyclidic surfaces is based on their remarkable relationship to the mVN equation

$$\begin{gathered} U_t = U_{xxx} - U_{yyy} + 3U_xV - 3U_yW + \tfrac{3}{2}UV_x - \tfrac{3}{2}UW_y \\ W_x = (U^2)_y \\ V_y = (U^2)_x \end{gathered}$$

introduced in [34] which is the third-order flow in the DS-I hierarchy. In the Appendix we demonstrate, that the Lie sphere density U defined by

$$U^2 = \frac{\partial_1 k^1 \partial_2 k^2}{(k^1 - k^2)^2}$$

satisfies in case of diagonally-cyclidic surfaces the stationary mVN equation

$$\begin{gathered} U_{xxx} + 3U_x V + \tfrac{3}{2} U V_x = U_{yyy} + 3U_y W + \tfrac{3}{2} U W_y \\ W_x = (U^2)_y \\ V_y = (U^2)_x \end{gathered}$$

(here $\partial_1 = \partial_x, \ \partial_2 = \partial_y$). Note that U^2 is a conserved density of the mVN equation. A passage from equations (4.4) to he mVN equation requires quite complicated calculations which were performed with Mathematica (see the Appendix).

Particular solutions of the stationary mVN equation can be obtained by the ansatz

$$W = -\frac{2}{3}\frac{U_{yy}}{U} + \frac{1}{3}\left(\frac{U_y}{U}\right)^2$$

$$V = -\frac{2}{3}\frac{U_{xx}}{U} + \frac{1}{3}\left(\frac{U_x}{U}\right)^2$$

where U satisfies the Tzitzeica equation

$$(\ln U)_{xy} = -U^2 + \frac{c}{U}, \quad c = const.$$

In this case the first equation is satisfied identically, while the last two become just two conservation laws of the Tzitzeica equation.

We have demonstrated that the stationary flows of the DS hierarchy have a natural interpretation within the contexts of conformal and Lie sphere geometries. This observation agrees with the results of [9], [10], [12], [13], [14], [15] where dynamics of surfaces, induced by the odd flows of DS-II hierarchy, was investigated. In particular, it was argued that the integrals of DS-II hierarchy define conformally invariant functionals and the corresponding stationary flows define certain conformally invariant classes of surfaces. We hope that results presented above contribute to these investigations. We emphasize also, that stationary points of some of the DS-integrals are probably invariant under the full group of Lie sphere transformations, rather then just conformal group (for example, the class of diagonally-cyclidic surfaces, corresponding to the stationary mVN equation, is invariant under the full Lie sphere group).

Remark. Diagonally-cyclidic surfaces have a natural projective counterpart, namely, the so-called isothermally-asymptotic surfaces for which the 3-web, formed by the asymptotic lines and Darboux's curves is hexagonal (Darboux's curves are the zero curves of Darboux's cubic form). This class of surfaces can be equivalently characterized by the existence of asymptotic coordinates where Darboux's cubic form becomes isothermic. We refer to [36] for further discussion and references conserning isothermally-asymptotic surfaces (Φ-surfaces in the terminology of [36]). Isothermally-asymptotic surfaces are related to a different real reduction of the mVN equation. Here we present the details of it's derivation. Let M^2 be a surface in projective space parametrized by asymptotic coordinates x, y with the radius-vector $\vec{r}$ satisfying the equations

$$\begin{aligned} \vec{r}_{xx} &= a\vec{r}_x + p\vec{r}_y, \\ \vec{r}_{yy} &= q\vec{r}_x + b\vec{r}_y. \end{aligned} \tag{4.7}$$

With any surface (4.7) we associate the symmetric 2-form

$$pq\ dxdy \tag{4.8}$$

and the conformal class of the Darboux cubic form

$$pdx^3 + qdy^3 \tag{4.9}$$

which define "generic" surface M^2 uniquely up to projective equivalence and play a role similar to that of (1.1) and (1.2) in the Lie sphere geometry. Darboux's curves are the zero curves of the Darboux cubic form. The hexagonality conditions of the 3-web formed by the asymptotic lines and the Darboux curves is equivalent to the existence of asymptotic parametrization x, y such that cubic form (4.9) becomes proportional to

$$dx^3 + dy^3,$$

that is, to the condition $p = q$. An important subclass of isothermally-asymptotic surfaces are the proper affine spheres, for which the radius-vector $\vec{r}$ satisfies the equations

$$\vec{r}_{xx} = -\tfrac{p_x}{p}\vec{r}_x + p\vec{r}_y$$

$$\vec{r}_{yy} = p\vec{r}_x - \tfrac{p_y}{p}\vec{r}_y$$

$$\vec{r}_{xy} = \tfrac{1}{p}\vec{r}$$

with p satisfying the Tzitzeica equation

$$(\ln p)_{xy} = p^2 - \frac{1}{p}.$$

In the case $p = q$ the compatibility conditions of (4.7) reduce to

$$\begin{gathered} (p_x + ap + \tfrac{1}{2}b^2 - b_y)_x = \tfrac{3}{2}(p^2)_y \\ (p_y + bp + \tfrac{1}{2}a^2 - a_x)_y = \tfrac{3}{2}(p^2)_x \\ a_y = b_x. \end{gathered} \tag{4.10}$$

Equations (4.10) can be rewritten in the form

$$b_y = p_x + ap + \tfrac{1}{2}b^2 - \tfrac{3}{2}W, \qquad W_x = (p^2)_y$$

$$a_x = p_y + bp + \tfrac{1}{2}a^2 - \tfrac{3}{2}V, \qquad V_y = (p^2)_x \tag{4.11}$$

$$a_y = f, \qquad b_x = f.$$

Crossdifferentiation of (4.11) gives the expressions for f_x, f_y

$$\begin{aligned} f_x &= p_{yy} + ap^2 + \tfrac{1}{2}pb^2 - \tfrac{3}{2}pW + bp_y + af - (p^2)_x \\ f_y &= p_{xx} + bp^2 + \tfrac{1}{2}pa^2 - \tfrac{3}{2}pV + ap_x + bf - (p^2)_y \end{aligned} \tag{4.12}$$

the compatibility conditions of which result in the stationary mVN equation

$$\begin{gathered} p_{xxx} - 3Vp_x - \tfrac{3}{2}pV_x = p_{yyy} - 3Wp_y - \tfrac{3}{2}pW_y \\ W_x = (p^2)_y \\ V_y = (p^2)_x \end{gathered} \tag{4.13}$$

which can be reduced to that presented above by a complex change of variables $p \to iU, \;\; W \to -W, \;\; V \to -V$. Integrating the compatible systems (4.12), (4.11) and (4.7) for a given solution p, W, V of (4.13) we arrive at the explicit formula for the radius-vector $\vec{r}$.

Particular solutions of the stationary mVN equation (4.13) can be obtained by the ansatz

$$W = \frac{2}{3}\frac{p_{yy}}{p} - \frac{1}{3}\left(\frac{p_y}{p}\right)^2$$

$$V = \frac{2}{3}\frac{p_{xx}}{p} - \frac{1}{3}\left(\frac{p_x}{p}\right)^2$$

where p satisfies the Tzitzeica equation

$$(\ln p)_{xy} = p^2 + \frac{c}{p}, \qquad c = const.$$

In this case the first equation is satisfied identically, while the last two become just two conservation laws of the Tzitzeica equation. If $c \neq 0$, then it can be normalized to -1; the corresponding surfaces are the proper affine spheres. The case $c = 0$ corresponds to the improper affine spheres whose affine normals are parallel and the radius-vector $\vec{r}$ satisfies the equations

$$\vec{r}_{xx} = -\tfrac{p_x}{p}\vec{r}_x + p\vec{r}_y$$

$$\vec{r}_{yy} = p\vec{r}_x - \tfrac{p_y}{p}\vec{r}_y$$

$$\vec{r}_{xy} = \tfrac{1}{p}\vec{l}$$

where $\vec{l}$ is a constant vector (direction of the affine normal) and p satisfies the Liouville equation

$$(\ln p)_{xy} = p^2.$$

The fact that Tzitzeica's equation defines a subclass of solutions of the stationary VN equation was observed recently in [37] (I would like to thank W. Schief for providing me with this reference). Our results give differential-geometric interpretation of this formal observation.

Isothermally-asymptotic surfaces are known to possess Bäcklund transformations such that the initial and the transformed surfaces are two focal surfaces of a W-congruence, which preserves asymptotic lines and the Darboux curves. This can be the starting point for the modern approach to isothermally-asymptotic surfaces, their discretization in the spirit of [35], etc.

5 Invariants of higher dimensional hypersurfaces in Lie sphere geometry

In this section we announce several results on Lie sphere geometry on higher dimensional hypersurfaces, postponing the detailed proofs to a separate publication.

Let M^n be hypersurface with principal curvatures k^i and principal covectors ω^i, so that the i-th principal direction of M^n is defined by the equations $\omega^j = 0, \quad j \neq i$. It should be pointed out that generic hypersurface of dimension ≥ 3 does not possess parametrization R^i by the lines of curvature as in the 2-dimensional case. Differentiating covectors ω^i and principal curvatures k^i we arrive at the structure equations

$$d\omega^i = c^i_{jk}\omega^j \wedge \omega^k, \qquad (5.1)$$
$$dk^i = k^i_j \omega^j.$$

Let also

$$ds^2 = \sum_1^n g_{ii}{\omega^i}^2$$

be the first fundamental form of hypersurface M^n.

Theorem 2. *A generic hypersurface M^n $(n \geq 3)$ is defined by quadratic form (1.4)*

$$\sum_{i \neq j} \frac{k^i_i k^j_j}{(k^i - k^j)^2}\ \omega^i \omega^j$$

and the conformal class of cubic form (1.5)

$$\sum_i k^i_i g_{ii}\ {\omega^i}^3$$

uniquely up to Lie sphere equivalence.

As "generic" it is sufficient to understand a surface with $k^i_i \neq 0$. This genericity assumption is essential since, for instance, there exist examples of Dupin hypersurfaces (that is, hypersurfaces with $k^i_i = 0$) which are not Lie-equivalent. Theorem 2 is an analog of the corresponding theorem in projective differential geometry stating that hypersurface in projective space P^n of dimension $n \geq 4$ is uniquely determined by the conformal classes of of its second fundamental form and Darboux's cubic form (see [30] for the exact statements and further references).

Remark 1. In case $k^i_i \neq 0$ cubic form (1.5) encodes all the information about the lines of curvature of hypersurface M^n. Indeed, principal directions are uniquely defined as those directions where cubic form (1.5) reduces to the sum of pure cubes (without mixed terms). Moreover, principal directions are zero directions of quadratic form (1.4). However, this last condition does not define them uniquely as in the 2-dimensional situation.

Remark 2. The invariant quadratic form (1.4) defines the invariant volume form, giving rise in the case $n = 3$ to the invariant functional

$$\int\int\int \frac{k^1_1 k^2_2 k^3_3}{(k^1 - k^2)(k^1 - k^3)(k^2 - k^3)}\ \omega^1\omega^2\omega^3, \tag{5.2}$$

the extremals of which should be called minimal hypersurfaces in Lie sphere geometry in analogy with the 2-dimensional case. It does not look likely that this functional was investigated so far.

Remark 3. In principle for $n \geq 3$ there exist additional Lie-sphere invariants besides those mentioned in Theorem 2, namely:

1. The cross-ratios

$$\frac{(k^i - k^j)(k^n - k^l)}{(k^n - k^j)(k^i - k^l)} \tag{5.3}$$

of any four principal curvatures.

2. The covectors

$$\frac{k^i_i(k^j - k^l)}{(k^i - k^j)(k^i - k^l)}\ \omega^i \qquad (i \neq j \neq l). \tag{5.4}$$

For instance, in case $n = 3$ we have three invariant covectors

$$\Omega^1 = \frac{k^1_1(k^2 - k^3)}{(k^1 - k^2)(k^1 - k^3)}\ \omega^1, \quad \Omega^2 = \frac{k^2_2(k^3 - k^1)}{(k^2 - k^1)(k^2 - k^3)}\ \omega^2,$$

$$\Omega^3 = \frac{k^3_3(k^1 - k^2)}{(k^3 - k^1)(k^3 - k^2)}\ \omega^3$$

giving rise to the invariant quadratic form

$$\Omega^{1^2} + \Omega^{2^2} + \Omega^{3^2}$$

whose volume functional

$$\int\int\int \Omega^1\Omega^2\Omega^3$$

coincides with (5.2).

3. Conformal class of the quadratic form

$$g_{11}\left(\prod_{l\neq 1}(k^1-k^l)\right)^{\frac{2}{n-2}}\omega^{1\,2}+\ldots+g_{nn}\left(\prod_{l\neq n}(k^n-k^l)\right)^{\frac{2}{n-2}}\omega^{n\,2}. \tag{5.5}$$

Lie-invariant class of hypersurfaces with conformally flat quadratic form (5.5) deserves a special investigation.

4. Differential $d\Omega$ of the 1-form

$$\Omega=\left(\sum_{l\neq 1}\frac{k^l_1-\frac{k^1_1}{n-1}}{k^1-k^l}\right)\omega^1+\ldots+\left(\sum_{l\neq n}\frac{k^l_n-\frac{k^n_n}{n-1}}{k^n-k^l}\right)\omega^n. \tag{5.6}$$

In generic case $k^i_i\neq 0$ objects (5.3)-(5.6) can be expressed through the forms (1.4) and (1.5). However they are important in the nongeneric situations, when some (or all) of k^i_i vanish so that (1.4) and (1.5) become zero. In particular, cross-ratios of principal curvatures play essential role in the study of Dupin hypersurfaces – see [32], [33]. In this respect it seems interesting to understand the role of conformal class (5.5) and the 2-form $d\Omega$ in the modern Lie-geometric approach to Dupin hypersurfaces.

For hypersurfaces with nonholonomic nets of lines of curvature Theorem 2 leads to a nice geometric corollary, which we will discuss in the simplest nontrivial 3-dimensional case. Let us consider the structure equations (5.1) of the 3-dimensional hypersurface M^3. Then there are only two possibilities:

1. Holonomic case: all three coefficients $c^1_{23}, c^2_{31}, c^3_{12}$ are equal to zero. Such hypersurfaces possess parametrization by the lines of curvature.

2. Nonholonomic case: all three coefficients $c^1_{23}, c^2_{31}, c^3_{12}$ are nonzero.

It immediately follows from the Peterson-Codazzi equations that for $n=3$ intermediate cases are forbidden. In the nonholonomic case we can normalize covectors $\omega^1,\omega^2,\omega^3$ in such a way that the structure equations assume the form

$$\begin{aligned}
d\omega^1&=a\omega^1\wedge\omega^2+b\omega^1\wedge\omega^3+\omega^2\wedge\omega^3,\\
d\omega^2&=p\omega^2\wedge\omega^1+q\omega^2\wedge\omega^3+\omega^3\wedge\omega^1,\\
d\omega^3&=r\omega^3\wedge\omega^1+s\omega^3\wedge\omega^2+\omega^1\wedge\omega^2.
\end{aligned} \tag{5.7}$$

This normalization fixes ω^i uniquely. As follows from the results of [24], in the 3-dimensional nonholonomic situation Peterson-Codazzi equations completely determine quadratic form (1.4) and qubic form (1.5) through the coefficients a,b,p,q,r,s in the structure equations (5.7). Hence we can formulate the following result.

Theorem 3. *Nonholonomic 3-dimensional hypersurface M^3 is defined by its structure equations (5.7) uniquely up to Lie sphere equivalence.*

We can reformulate this result as follows: two 3-dimensional nonholonomic hypersurfaces are Lie-equivalent if and only if there exists a point correspondence between them, mapping the lines of curvature of one of them onto the lines of curvature of the other. Hence 3-dimensional nonholonomic hypersurface is uniquely determined by geometry of it's lines of curvature.

This Theorem should remain valid for higher-dimensional hypersurfaces if we generalize the notion of "nonholonomicity" in a proper way (e.g. $c^i_{jk} \neq 0$ for all $i \neq j \neq k$ which probably can be weakend).

6 Reciprocal transformations of hydrodynamic type systems. Reciprocal invariants

In this section we consider 2-component systems of hydrodynamic type

$$u^i_t = v^i_j(u)u^j_x, \qquad i,j = 1,2 \tag{6.1}$$

which naturally arise in polytropic gas dynamics, chromatography, plasticity , etc. and describe wide variety of models of continuous media. The main advantage of the 2-component case is the existence of the so-called Riemann invariants: coordinates, where equations (6.1) assume the diagonal form

$$\begin{aligned} R^1_t &= \lambda^1(R)R^1_x, \\ R^2_t &= \lambda^2(R)R^2_x, \end{aligned} \tag{6.2}$$

considerably simplifying their investigation. Any system (6.2) possesses infinitely many conservation laws

$$h(R)dx + g(R)dt \tag{6.3}$$

with the densities $h(R)$ and the fluxes $g(R)$ governed by the equations

$$\partial_i g = \lambda^i \partial_i h, \qquad i = 1,2 \tag{6.4}$$

($\partial_i = \partial/\partial R^i$) which are completely equivalent to the condition $h_t = g_x$, manifesting closedness of 1-form (6.3). Crossdifferentiation of (6.4) results in the second-order equation

$$\partial_1\partial_2 h = \frac{\partial_2\lambda^1}{\lambda^2 - \lambda^1}\partial_1 h + \frac{\partial_1\lambda^2}{\lambda^1 - \lambda^2}\partial_2 h, \tag{6.5}$$

for the conserved densities of system (6.2). Thus conservation laws of system (6.2) depend on two arbitrary functions of one variable. Let us choose two particular conservation laws $B(R)dx + A(R)dt$ and $N(R)dx + M(R)dt$ and introduce new independent variables X, T by the formulae

$$\begin{aligned} dX &= Bdx + Adt \\ dT &= Ndx + Mdt \end{aligned} \tag{6.6}$$

which are correct since the right hand sides are closed. Changing from x, t to X, T in (6.2) we arrive at the transformed system

$$\begin{aligned} R^1_t &= \Lambda^1(R) R^1_x \\ R^2_t &= \Lambda^2(R) R^2_x \end{aligned} \tag{6.7}$$

where the new characteristic velocities Λ^i are given by the formulae

$$\Lambda^i = \frac{\lambda^i B - A}{M - \lambda^i N}, \qquad i = 1, 2. \tag{6.8}$$

Remark. In principle one can apply transformation (6.6) directly to system (6.1) without rewriting it in Riemann invariants. In this case the transformed equations assume the form

$$u^i_T = V^i_j(u) u^j_X,$$

with the new matrix V given by

$$V = (Bv - AE)(ME - Nv)^{-1}, \quad E = id.$$

Transformations of type (6.6) are known as "reciprocal" and have been extensively investigated in [16], [17] (see also [22] and [18] - [21] for further discussion). Following [18], [19] we introduce the reciprocal invariants:

the symmetric 2-form

$$\frac{\partial_1 \lambda^1 \partial_2 \lambda^2}{(\lambda^1 - \lambda^2)^2}\, dR^1 dR^2 \tag{6.9}$$

and the differential

$$d\Omega \tag{6.10}$$

of the 1-form

$$\Omega = \left(\frac{\partial_1 \partial_2 \lambda^2}{\partial_2 \lambda^2} + \frac{\partial_1 \lambda^1}{\lambda^1 - \lambda^2} \right) dR^1 + \left(\frac{\partial_1 \partial_2 \lambda^1}{\partial_1 \lambda^1} + \frac{\partial_2 \lambda^2}{\lambda^2 - \lambda^1} \right) dR^2 \tag{6.11}$$

(Ω itself is not reciprocally invariant). Note that both objects (6.9) and (6.10) do not change under the reparametrization of Riemann invariants $R^1 \to \varphi^1(R^1), \ \ R^2 \to \varphi^2(R^2)$.

Remark. In order to check the invariance of (6.9) and (6.10) under arbitrary reciprocal transformations it is sufficient to check their invariance under the following elementary ones:

$$\begin{aligned} dX &= B dx + A dt, \\ dT &= dt, \end{aligned}$$

which changes only x and preserves t (under this transformation λ^i goes to $\Lambda^i = \lambda^i B - A$) and

$$\begin{aligned} dX &= dt, \\ dT &= dx, \end{aligned}$$

which transforms λ^i into $\Lambda^i = \frac{1}{\lambda^i}$. The invariance of (6.9) and (6.10) under these elementary transformations can be checked by a direct calculation. Since any reciprocal transformation is a composition of elementary ones, we arrive at the required invariance.

It is quite remarkable that invariants (6.9) and (6.10) form a complete set in the following sense: if the invariants of one system can be mapped onto the invariants of the other one by the appropriate change of coordinates R^i, than both these systems are reciprocally related and the corresponding reciprocal transformation (6.6) can be constructed explicitely (see [18], [19] for the discussion).

7 Hamiltonian systems

System (6.1) is called Hamiltonian, if it can be represented in the form

$$u^i_t = \epsilon^i \delta^{ij} \frac{d}{dx}\left(\frac{\delta H}{\delta u^j}\right), \quad \epsilon^i = \pm 1,$$

with the Hamiltonian operator $\epsilon^i \delta^{ij} \frac{d}{dx}$ and the Hamiltonian $H = \int h(u)dx$. In this case the matrix v^i_j is just the Hessian of the density h (for definiteness we choose $\epsilon^i = 1$), so that system (6.1) assumes the form

$$\begin{pmatrix} u^1 \\ u^2 \end{pmatrix}_t = \begin{pmatrix} h_{11} & h_{12} \\ h_{12} & h_{22} \end{pmatrix} \begin{pmatrix} u^1 \\ u^2 \end{pmatrix}_x \tag{7.1}$$

(here h_{ij} means $\frac{\partial^2 h}{\partial_{u^i} \partial_{u^j}}$). For systems (6.2) in Riemann invariants the necessary and sufficient condition for the existence of the Hamiltonian representation (7.1) is given by the following

Lemma [25]. System (6.2) is Hamiltonian if and only if there exists flat diagonal metric $ds^2 = g_{11}(R)dR^{1^2} + g_{22}(R)dR^{2^2}$ such that

$$\begin{aligned} \partial_2 \ln \sqrt{g_{11}} &= \tfrac{\partial_2 \lambda^1}{\lambda^2 - \lambda^1}, \\ \partial_1 \ln \sqrt{g_{22}} &= \tfrac{\partial_1 \lambda^2}{\lambda^1 - \lambda^2}. \end{aligned} \tag{7.2}$$

Introducing the Lame coefficients $H_1 = \sqrt{g}_{11}, \quad H_2 = \sqrt{g}_{22}$ and the rotation coefficients β_{12}, β_{21} by the formulae

$$\partial_1 H_2 = \beta_{12} H_1, \quad \partial_2 H_1 = \beta_{21} H_2, \tag{7.3}$$

we can rewrite the flatness condition of the metric ds^2 in a simple form

$$\partial_1 \beta_{12} + \partial_2 \beta_{21} = 0. \tag{7.4}$$

The coordinates u^1, u^2 in (7.1) are just flat coordinates of the metric ds^2, where it assumes the standard Euclidean form $du^{1^2} + du^{2^2}$.

Under reciprocal transformations (6.6) the metric coefficients g_{ii} transform according to the formulae

$$G_{ii} = g_{ii} \frac{(M - \lambda^i N)^2}{(BM - AN)^2}, \qquad i = 1, 2 \tag{7.5}$$

(see [22], [23]), so that the transformed metric coefficients G_{ii} and the transformed characteristic velocities Λ^i satisfy the same equations (7.2). It is important to emphasize that reciprocal transformations do not preserve in general the flatness condition of the metric ds^2 and hence destroy the Hamiltonian structure. However, for any Hamiltonian system there always exist sufficiently many "canonical" reciprocal transformations preserving the flatness condition [21], [23].

Let us introduce the cubic form

$$\partial_1 \lambda^1 g_{11} dR^{1^3} + \partial_2 \lambda^2 g_{22} dR^{2^3}. \tag{7.6}$$

Using formulae (6.8) and (7.5) one can immediately check, that this qubic form is conformally invariant under reciprocal transformations: it aquires the multiple $\frac{1}{BM-AN}$, so that the zero curves of (7.6) are reciprocally invariant. Hence with any Hamiltonian system we can associate besides the invariants (6.9) and (6.10) the reciprocally invariant 3-web of curves formed by coordinate lines $R^1 = const$, $R^2 = const$ and the zero curves of qubic form (7.6) which are defined by the equation

$$(\partial_1 \lambda^1 g_{11})^{\frac{1}{3}} dR^1 + (\partial_2 \lambda^2 g_{22})^{\frac{1}{3}} dR^2 = 0.$$

A calculation similar to that in sect.4 shows that invariant (6.10) is just the curvature form of this 3-web.

Remark. It will be interesting to obtain explicit formulae for reciprocal invariants (6.9), (6.10) and (7.6) in the flat coordinates u^i in terms of the Hamiltonian density h.

As we already know the objects similar to (6.9), (6.10) and (7.6) arise in the Lie sphere geometry of surfaces. To clarify this point we recall the construction of [20], [21] relating Hamiltonian systems and surfaces in the Euclidean space.

8 Hamiltonian systems and surfaces in E^3

Let us consider a 2-component Hamiltonian system (7.1)

$$\begin{pmatrix} u^1 \\ u^2 \end{pmatrix}_t = \begin{pmatrix} h_{11} & h_{22} \\ h_{12} & h_{22} \end{pmatrix} \begin{pmatrix} u^1 \\ u^2 \end{pmatrix}_x$$

and apply the reciprocal transformation

$$dX = Bdx + Adt,$$
$$dT = dt,$$

where

$$B = \frac{{u^1}^2 + {u^2}^2 + 1}{2}, \qquad A = h_1 u^1 + h_2 u^2 - h$$

(this is indeed an integral of system (7.1)). The transformed system assumes the form

$$\begin{pmatrix} u^1 \\ u^2 \end{pmatrix}_t = \begin{pmatrix} h_{11}B - A & h_{12}B \\ h_{12}B & h_{22}B - A \end{pmatrix} \begin{pmatrix} u^1 \\ u^2 \end{pmatrix}_x \tag{8.1}$$

To reveal geometric meaning of system (8.1) we introduce a surface M^2 in the Euclidean space $E^3(x^1, x^2, x^3)$ with the radius-vector

$$\vec{r} = \begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix} = \begin{pmatrix} h_1 - u^1 \frac{A}{B} \\ \\ h_2 - u^2 \frac{A}{B} \\ \\ -\frac{A}{B} \end{pmatrix} \tag{8.2}$$

As one can verify by a straightforward calculation, the unit normal of the surface M^2 is given by

$$\vec{n} = \begin{pmatrix} \frac{u^1}{B} \\ \\ \frac{u^2}{B} \\ \\ \frac{1}{B} - 1 \end{pmatrix}$$

Let us define the matrix w^i_j by the formula

$$\frac{\partial \vec{r}}{\partial u^j} = \sum_{i=1}^{2} w^i_j \frac{\partial \vec{n}}{\partial u^i}.$$

Geometrically w^i_j is just the inverse of the Weingarten operator (shape operator) of the surface M^2. Using the formulae for $\vec{r}$ and $\vec{n}$ we arrive at the following expression for the matrix w^i_j:

$$\begin{pmatrix} h_{11}B - A & h_{12}B \\ h_{12}B & h_{22}B - A \end{pmatrix}$$

which coincides with that in (8.1). Hence the matrix of system (8.1) is just the inverse of the Weingarten operator of the associated surface M^2. The characteristic velocities w^i of system (8.1) are related to that of system (7.1) by the formula

$$w^i = \lambda^i B - A, \tag{8.3}$$

and have geometric meaning of radii of principal curvatures of the surface M^2. Moreover, the Riemann invariants of both systems (7.1) and (8.1) coincide and play the role of parameters of the lines of curvature. Equations (8.1) can be equivalently represented in the conservative form

$$\vec{n}_t = \vec{r}_x.$$

Some further properties of the correspondence (8.2) (in the general n-component case) were discussed in [20], [21], in particular:

— commuting Hamiltonian systems correspond via formula (8.2) to surfaces with the same spherical image of the lines of curvature;

— multi-Hamiltonian systems correspond to surfaces, possessing nontrivial deformations preserving the Weingarten operator;

— "canonical" reciprocal transformations, preserving the Hamiltonian structure, correspond to Lie sphere transformations of the associated surfaces;

— the flat metric ds^2 defining Hamiltonian structure (see lemma in sect.7) corresponds to the third fundamental form of the associated surface.

Since the correspondence between systems (7.1) and (8.1) is reciprocal, invariants (6.9), (6.10) and (7.6) coincide respectively with the symmetric 2-form

$$\frac{\partial_1 w^1 \partial_2 w^2}{(w^1 - w^2)^2}\, dR^1 dR^2,$$

the skew-symmetric 2-form $d\Omega$, where

$$\Omega = \left(\frac{\partial_1\partial_2 w^2}{\partial_2 w^2} + \frac{\partial_1 w^1}{w^1 - w^2}\right) dR^1 + \left(\frac{\partial_1\partial_2 w^1}{\partial_1 w^1} + \frac{\partial_2 w^2}{w^2 - w^1}\right) dR^2$$

and the conformal class of the qubic form

$$\partial_1 w^1 G_{11} dR^{1^3} + \partial_2 w^2 G_{22} dR^{2^3}$$

where now R^1, R^2 are the parameters of lines of curvature, w^1, w^2 are the radii of principal curvatures and G_{11}, G_{22} are the components of the third fundamental form of the associated surface M^2. Since these objects preserve their form if we rewrite them in terms of principal curvatures k^i and the components of the metric g_{ii}, they concide with the Lie sphere invariants of the surface M^2. This provides remarkable differential-geometric interpretation of reciprocal invariants of hydrodynamic type systems.

9 Reciprocal transformations and reciprocal invariants of n-component systems

Let us consider an n-component system of hydrodynamic type

$$u^i_t = v^i_j(u) u^j_x, \qquad i, j = 1, ..., n \tag{9.1}$$

with the characteristic velocities λ^i and the corresponding left eigenvectors $\vec{l^i} = (l^i_j)$ which satisfy the formulae
$$\sum_k l^i_k v^k_j = \lambda^i l^i_j.$$
Introducing the 1-forms $\omega^i = l^i_j du^j$ (note that $\vec{l^i}$ and ω^i are defined up to rescailing $\vec{l^i} \to p^i \vec{l^i}, \;\; \omega^i \to p^i \omega^i$), we can rewrite equations (9.1) in the equivalent exterior form

$$\omega^i \wedge (dx + \lambda^i dt) = 0, \qquad i = 1, ..., n. \tag{9.2}$$

Differentiation of ω^i and λ^i results in the "structure equations" of system (9.1):

$$d\omega^i = c^i_{jk} \omega^j \wedge \omega^k, \tag{9.3}$$

$$d\lambda^i = \lambda^i_j \omega^j. \tag{9.4}$$

Systems in Riemann invariants are specified by the conditions $c^i_{jk} = 0$ for any triple of indices $i \neq j \neq k$. Indeed, in this case the forms ω^i satisfy the equations $d\omega^i \wedge \omega^i = 0$ for any $i = 1, ..., n$ and hence can be normalized so as to become just $\omega^i = dR^i$. In the coordinates R^i equations (9.2) assume the familiar Riemann-invariant form

$$R^i_t = \lambda^i R^i_x, \qquad i = 1, ..., n. \tag{9.5}$$

The exterior representation (9.2) is a natural analog of representation (9.5) which is applicable in the nondiagonalizable case as well. We emphasize that for $n \geq 3$ Riemann invariants do not exist in general.

Applying to (9.1) the reciprocal transformation

$$\begin{aligned} dX &= B dx + A dt, \\ dT &= N dx + M dt, \end{aligned}$$

we arrive at the transformed equations

$$u^i_T = V^i_j(u) u^j_X$$

with the new matrix V given by

$$V = (Bv - AE)(ME - Nv)^{-1}, \qquad E = id$$

or, in the exterior form,
$$\omega^i \wedge (dX + \Lambda^i dT) = 0$$
where
$$\Lambda^i = \frac{\lambda^i B - A}{M - \lambda^i N}.$$
Hence the forms ω^i as well as the structure equations (9.3) do not change, while λ^i transform as in the 2-component case — see formula (6.8).

Remark. In the n-component case equations (6.4) for the densities and fluxes of conservation laws $hdx + gdt$ assume the form

$$g_i = \lambda^i h_i, \qquad i = 1, ..., n$$

where g_i and h_i are defined by the expansions

$$dg = g_i \omega^i, \quad dh = h_i \omega^i.$$

In [18], [19] we introduced the following reciprocally invariant objects:

1. The symmetric 2-form

$$\sum_{i \neq j} \frac{\lambda^i_i \lambda^j_j}{(\lambda^i - \lambda^j)^2} \, \omega^i \omega^j \tag{9.6}$$

2. The skew-symmetric 2-form

$$d\Omega \tag{9.7}$$

where

$$\Omega = \left(\sum_{k \neq 1} \frac{\lambda^k_1 - \frac{\lambda^1_1}{n-1}}{\lambda^1 - \lambda^k} \right) \omega^1 + \ldots + \left(\sum_{k \neq n} \frac{\lambda^k_n - \frac{\lambda^n_n}{n-1}}{\lambda^n - \lambda^k} \right) \omega^n \tag{9.8}$$

(Ω itself is not reciprocally invariant). Note that both objects (9.6) and (9.7) do not change if we reparametrize the 1-forms in the structure equations: $\omega^i \to p^i \omega^i$. Objects (9.6) and (9.7) are the natural analogs of the corresponding invariants (6.9) and (6.10) in the 2-component case. However, for $n \geq 3$ the form Ω depends only on the first derivatives of the characteristic velocities λ^i rather that on the second derivatives as in the 2-component case.

In principle for $n \geq 3$ there exist additional reciprocal invariants, namely the 1-forms

$$\frac{\lambda^i_i (\lambda^j - \lambda^l)}{(\lambda^i - \lambda^j)(\lambda^i - \lambda^l)} \omega^i \qquad (i \neq j \neq l)$$

as well as the cross-ratios

$$\frac{(\lambda^i - \lambda^j)(\lambda^k - \lambda^l)}{(\lambda^k - \lambda^j)(\lambda^i - \lambda^l)}$$

of any four characteristic velocities.

However, as follows from [18], [19], the structure equations (9.3) and the invariants (9.6), (9.7) in fact define generic system of hydrodynamic type uniquely up to reciprocal equivalence (under "generic" it is sufficient to understand genuinely nonlinear system, that is, a system with $\lambda^i_i \neq 0$ for any i).

10 Appendix. Derivation of the mVN equation for diagonally-cyclidic surfaces

Our aim is to show, that system (4.4) on ρ, k^1, k^2 can be rewritten in terms of the Lie sphere density U defined as

$$U^2 = \frac{\partial_1 k^1 \partial_2 k^2}{(k^1 - k^2)^2}.$$

In view of the correspondence between surfaces in E^3 and Hamiltonian systems discussed in sect. 8 the classification of diagonally-cyclidic surfaces can be equivalently reformulated as the classification of those Hamiltonian systems

$$\begin{aligned} R^1_t &= \lambda^1 R^1_x \\ R^2_t &= \lambda^2 R^2_x \end{aligned}$$

for which the flat metric $ds^2 = g_{11} d{R^1}^2 + g_{22} d{R^2}^2$ defining the Hamiltonian structure can be represented in the form

$$g_{11} = \frac{e^{2\rho}}{\partial_1 \lambda^1}, \qquad g_{22} = \frac{e^{2\rho}}{\partial_2 \lambda^2}.$$

Indeed, this is an immediate consequence of isotermicity of the cubic form. Equations (7.2) and the flatness condition for the metric ds^2 imply the following system for $\rho, \lambda^1, \lambda^2$:

$$\begin{aligned} \partial_1 \rho &= B + \tfrac{1}{2} \tfrac{\partial_1 \partial_2 \lambda^2}{\partial_2 \lambda^2}, \\ \partial_2 \rho &= A + \tfrac{1}{2} \tfrac{\partial_1 \partial_2 \lambda^1}{\partial_1 \lambda^1}, \end{aligned} \tag{10.1}$$

$$\partial_1 \lambda^1 \left(\partial_1 B + \frac{B}{2} \partial_1 \ln \frac{\partial_1 \lambda^1}{\partial_2 \lambda^2} \right) + \partial_2 \lambda^2 \left(\partial_2 A + \frac{A}{2} \partial_2 \ln \frac{\partial_2 \lambda^2}{\partial_1 \lambda^1} \right) = 0 \tag{10.2}$$

where $A = \frac{\partial_2 \lambda^1}{\lambda^2 - \lambda^1}, \quad B = \frac{\partial_1 \lambda^2}{\lambda^1 - \lambda^2}$ (compare with (4.4)). Since

$$U^2 = \frac{\partial_1 k^1 \partial_2 k^2}{(k^1 - k^2)^2} = \frac{\partial_1 \lambda^1 \partial_2 \lambda^2}{(\lambda^1 - \lambda^2)^2}$$

the desired equation for U will be obtained after we rewrite (10.1) – (10.2) in terms of U. Crossdifferentiating (10.1) and introducing $k = \frac{1}{2} \ln \frac{\partial_2 \lambda^2}{\partial_1 \lambda^1}$ we obtain

$$\partial_1 \partial_2 k = \partial_1 A - \partial_2 B,$$

while (10.2) assumes the form

$$\partial_2 (e^k A) + \partial_1 (e^{-k} B) = 0.$$

Moreover, the equations for the characteristic velocities

$$\partial_1\lambda_1 = Ue^{-k}(\lambda^1-\lambda^2), \qquad \partial_2\lambda_1 = A(\lambda^2-\lambda^1)$$

$$\partial_1\lambda_2 = B(\lambda^1-\lambda^2), \qquad \partial_2\lambda_2 = Ue^{k}(\lambda^1-\lambda^2)$$

give as the compatibility conditions two additional relations

$$\partial_1 A = AB + U^2 - \partial_2(Ue^{-k}),$$

$$\partial_2 B = AB + U^2 + \partial_1(Ue^{k}).$$

Thus the problem is reduced to that of excluding variables A, B, k and deriving the equation for U from the following much simpler looking system

$$\partial_1\partial_2 k = \partial_1 A - \partial_2 B, \tag{10.3}$$

$$\partial_2(e^k A) + \partial_1(e^{-k}B) = 0, \tag{10.4}$$

$$\partial_1 A = AB + U^2 - \partial_2(Ue^{-k}), \tag{10.5}$$

$$\partial_2 B = AB + U^2 + \partial_1(Ue^{k}). \tag{10.6}$$

To proceed further we introduce new variables m, n by the formulae

$$-\partial_2(Ue^{-k}) = Um, \qquad \partial_1(Ue^{k}) = Un,$$

so that $\partial_1 k, \partial_2 k$ can be expressed as follows:

$$\partial_1 k = -\frac{\partial_1 U}{U} + e^{-k}n, \qquad \partial_2 k = \frac{\partial_2 U}{U} + e^{k}m. \tag{10.7}$$

The compatibility conditions of (10.7) with (10.3) (which now assumes the form $\partial_1\partial_2 k = U(m-n)$) give the following equations for m, n:

$$\begin{aligned} \partial_2 n &= \tfrac{\partial_2 U}{U}n + e^k(\partial_1\partial_2 \ln U + U(m-n) + mn) \\ \partial_1 m &= \tfrac{\partial_1 U}{U}m - e^{-k}(\partial_1\partial_2 \ln U - U(m-n) + mn). \end{aligned} \tag{10.8}$$

Rewriting (10.4) in the form

$$e^k\partial_2 A + e^k A\left(\frac{\partial_2 U}{U} + e^k m\right) + e^{-k}\partial_1 B - e^{-k}B\left(-\frac{\partial_1 U}{U} + e^{-k}n\right) = 0$$

and introducing the new variable F by the formulae

$$e^k\partial_2 A + e^k A\left(\frac{\partial_2 U}{U} + e^k m\right) = F + \frac{1}{2}e^k A^2 - \frac{1}{2}e^{-k}B^2,$$

$$e^{-k}\partial_1 B - e^{-k}B\left(-\frac{\partial_1 U}{U} + e^{-k}n\right) = -F - \frac{1}{2}e^k A^2 + \frac{1}{2}e^{-k}B^2,$$

we can express the derivatives of A and B as follows

$$
\begin{aligned}
\partial_1 A &= AB + U^2 + Um, \\
\partial_2 B &= AB + U^2 + Un, \\
\partial_2 A &= -A(\tfrac{\partial_2 U}{U} + e^k m) + \tfrac{1}{2}A^2 - \tfrac{1}{2}e^{-2k}B^2 + e^{-k}F, \\
\partial_1 B &= -B(\tfrac{\partial_1 U}{U} - e^{-k} n) - \tfrac{1}{2}e^{2k}A^2 + \tfrac{1}{2}B^2 - e^k F.
\end{aligned}
\tag{10.9}
$$

Let us introduce also the new functions G and H by the formulae

$$
\begin{aligned}
\partial_1 n &= e^{2k}F + \tfrac{1}{2}e^{-k}(3n^2 + 4nU + U^2) - 2n\tfrac{\partial_1 U}{U} - 3\partial_1 U + \tfrac{1}{2}e^k H, \\
\partial_2 m &= e^{-2k}F - \tfrac{1}{2}e^{k}(3m^2 + 4mU + U^2) - 2m\tfrac{\partial_2 U}{U} - 3\partial_2 U + \tfrac{1}{2}e^{-k} G.
\end{aligned}
\tag{10.10}
$$

The compatibility conditions of eqns. (10.9) give the following expressions for $\partial_1 F, \partial_2 F$:

$$
\begin{aligned}
&\partial_1 F = -F(\tfrac{\partial_1 U}{U} - e^{-k} n) + e^{2k}Um(U+m) \\
&+\tfrac{U}{2}(G + 2e^{-k}F - e^{2k}(3m^2 + 4mU + U^2)), \\
&\partial_2 F = -F(\tfrac{\partial_2 U}{U} + e^{k} m) + e^{-2k}Un(U+n) \\
&-\tfrac{U}{2}(H + 2e^{k}F + e^{-2k}(3n^2 + 4nU + U^2)).
\end{aligned}
\tag{10.11}
$$

The compatibility conditions of (10.8) and (10.10) give the expressions for $\partial_1 G, \partial_2 H$ which can be represented as follows:

$$
\begin{aligned}
\partial_1(G + 2\tfrac{\partial_2^2 U}{U} - (\tfrac{\partial_2 U}{U})^2) &= -3\partial_2(U^2), \\
\partial_2(H - 2\tfrac{\partial_1^2 U}{U} + (\tfrac{\partial_1 U}{U})^2) &= 3\partial_1(U^2).
\end{aligned}
\tag{10.12}
$$

Finally, the compatibility conditions of (10.11) imply

$$
\partial_2(GU^2) + \partial_1(HU^2) = 0,
\tag{10.13}
$$

so that equations (10.3)-(10.6) are reduced to (10.12)-(10.13). The desired stationary mVN equation results now after introducing V, W by the formulae

$$
G + 2\frac{\partial_2^2 U}{U} - \left(\frac{\partial_2 U}{U}\right)^2 = -3W,
$$

$$
H - 2\frac{\partial_1^2 U}{U} + \left(\frac{\partial_1 U}{U}\right)^2 = 3V.
$$

All these calculations were performed with Mathematica and can be easily verified.

11 Concluding remarks

Here we just list some of the unsolved problems.

1. In our discussion of Lie-sphere invariants of surfaces (reciprocal invariants of hydrodynamic-type systems) the choice of coordinates R^i plays a crucial role. In case of surfaces these are coordinates of the lines of curvature (Riemann invariants in case of hydrodynamic type systems). This choice is not accidental, since the lines of curvature are preserved by the Lie sphere group while Riemann invariants are preserved under reciprocal transformations. In fact only in these special coordinates do our invariants assume particularly symmetric and simple form. However, from the point of view of applications it is desirable to have a kind of invariant tensor formula, which will allow computation of these objects in an arbitrary coordinate system, for instance, in conformal parametrization in case of surfaces or in flat coordinates in case of Hamiltonian systems.

2. In [9], [10] Konopelchenko introduced dynamics of surfaces, governed by the mVN equation. In this approach the integrals of mVN correspond to certain functionals on surfaces, which were conjectured in [12], [13], [14] to be conformally invariant. In particular, the simplest quadratic integral of mVN corresponds to the Willmore functional. This conjecture was proved recently in [15]. Since functional (1.3) is conformally invariant (indeed, it is invariant under the full group of Lie sphere transformations which contains conformal group), it would be interesting to understand its relationship to the mVN hierarchy.

3. It seems to be an interesting problem to describe the class of surfaces, for which evolutions (3.5), (3.6) preserve the lines of curvature.

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