REMARKS ON PSEUDOCHARACTERS AND THE REAL CONTINUOUS BOUNDED COHOMOLOGY OF CONNECTED LOCALLY COMPACT GROUPS

Alexander I. Shtern

July 30, 1997

Abstract. Results on Borel pseudocharacters on locally compact groups are presented and their applications to the description of their second continuous bounded cohomology groups are indicated. In particular, it is proved that the second real continuous bounded cohomology group of a connected locally compact group is finite-dimensional.

Part I. Pseudocharacters on Almost Connected Locally Compact Groups

Recall the following definitions and facts [12] in the form of [14].

Definition 1. A real function f on a group G is called a *(real) quasi-character* on G if there exists a constant C_f such that

$$|f(st) - f(s) - f(t)| \leq C_f$$

for all $s, t \in G$, and a quasi-character φ is called a *(real) pseudocharacter* on G if

$$\varphi(x^n) = n\varphi(x)$$

for all $x \in G$ and all $n \in \mathbb{N}$.

Recall [12, 14] that any pseudocharacter φ satisfies the relation

$$\varphi(ab) = \varphi(ba), \qquad a, b \in G,$$

and, for any quasi-character f on G, there exists the limit

$$\varphi_f(g) = \lim n^{-1} f(g^n), \qquad g \in G, \tag{1}$$

the function φ_f is a pseudocharacter on G, and

$$|f(g) - \varphi_f(g)| \leq C_f, \qquad g \in G; \tag{2}$$

and it follows from (1) and (2) that φ_f is a unique pseudocharacter on G such that $f - \varphi_f$ is bounded on G.

If f is a continuous quasi-character, then it follows from (1) that the related pseudocharacter φ_f is bounded on a neighborhood of the identity element and belongs to the first Baire class on the entire group G. This pair of properties turns out to be characteristic.

This research was partially supported by the DFG Grant "Indextheorie", FA.-Nr. 04030206, and by the Russian Foundation for Basic Research under grant No. 96-01-00276.

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

Proposition 1. *A Borel pseudocharacter on a locally compact group that is bounded on a neighborhood of the identity element is necessarily related to a continuous quasi-character. In particular, any Borel pseudocharacter that is bounded on a neighborhood of the identity element belongs to the first Baire class.*

Proof. Let G be a locally compact group and let φ be a Borel pseudocharacter on G that is bounded on a neighborhood of the identity element of G. Let $C = C_\varphi$ be a constant related to φ as above. Apply an analog of the Blanc smoothing formula [4] in the form presented in [9, Chap. III, Proposition 1.7]). Namely, for a Borel pseudocharacter φ that is bounded on a compact neighborhood U of the identity element of G, consider a function χ supported by U, nonnegative, continuous on G, and with unit integral. As in [4, p. 146] and in [9, (1.16)] we set

$$\Phi_\varphi(g) = \int \chi(g^{-1}h)\varphi(h)\,dh = \int \chi(k)\varphi(gk)\,dk \tag{3}$$

for $g \in G$, where dh stands for some left Haar measure on G, and thus the integral with respect to k is taken here over U. Formula (3) works because the integrand is Haar measurable and locally bounded, and hence belongs to $L^\infty_{\mathrm{loc}}(G)$. It follows from (3) that

$$\begin{aligned} &\Phi_\varphi(g_1g_2) - \Phi_\varphi(g_1) - \Phi_\varphi(g_2) \\ &\quad = \int \chi((g_1g_2)^{-1}h)\varphi(h)\,dh - \int \chi(g_1^{-1}h)\varphi(h)\,dh - \int \chi(g_2^{-1}h)\varphi(h)\,dh \\ &\quad = \int_G \chi(k)(\varphi((g_1g_2)k) - \varphi(g_1k) - \varphi(g_2k))\,dk. \end{aligned} \tag{4}$$

Note that

$$|\varphi(ab) - \varphi(a) - \varphi(b)| \le C, \qquad a, b \in G,$$

and hence

$$|\varphi(gk) - \varphi(g) - \varphi(k)| \le C \qquad g \in G, \quad k \in U. \tag{5}$$

By assumption, there exists C_1 such that $|\varphi(k)| \le C_1$ for all $k \in U$. Then it follows from (5) that

$$|\varphi(gk) - \varphi(g)| \le C + C_1, \qquad g \in G, \qquad k \in U. \tag{6}$$

On substituting (6) into the natural estimate related to (4) we obtain

$$\begin{aligned} &|\Phi_\varphi(g_1g_2) - \Phi_\varphi(g_1) - \Phi_\varphi(g_2)| \\ &\quad \le \int_U \chi(k)|\varphi((g_1g_2)k) - \varphi(g_1k) - \varphi(g_2k)|\,dk \\ &\quad \le \int_U \chi(k)|\varphi(g_1g_2) - \varphi(g_1) - \varphi(g_2)|\,dk + 3C + 3C_1 \le 4C + 3C_1, \end{aligned}$$

which means that Φ_φ is a quasi-character on G. Moreover, by (6) we have the inequality

$$(7)\qquad \begin{aligned} |\Phi_\varphi(g)-\varphi(g)| &= |\int_U (\chi(k)\varphi(gk)-\chi(k)\varphi(g))\,dk| \\ &\le \int_U \chi(k)|\varphi(gk)-\varphi(g)|\,dk \le C+C_1, \qquad g\in G, \end{aligned}$$

and (7) immediately proves that the original pseudocharacter φ is related to the quasi-character Φ_φ, i.e., $\varphi=\varphi_{\Phi_\varphi}$.

As usual, formula (3) directly implies the continuity of Φ_φ on G, and this completes the proof of Proposition 1. □

Proposition 1 is an important tool in the study of first-Baire-class pseudocharacters on locally compact groups. We proceed with assertions related to such pseudocharacters. We first recall the following statement.

Lemma 1 [13, Lemma 2]**.** *Let f be a continuous pseudocharacter on a locally compact group G and let N be a closed normal subgroup of G such that the restriction of f to N vanishes. Then there exists a continuous pseudocharacter φ of the quotient group G/N such that $f=\varphi\circ\pi$, where π is the canonical epimorphism of G onto G/N.*

The following analog of this result holds for the first-Baire-class pseudocharacters that are bounded in a neighborhood of the identity element.

Let $\mathbb{R}$ be the additive group of reals.

Lemma 2. *Let f be a first-Baire-class pseudocharacter on a locally compact group G that is bounded in a neighborhood of the identity element and let N be a closed normal subgroup of G such that the restriction of f to N vanishes. Then there exists a first-Baire-class pseudocharacter φ of the quotient group G/N such that $f=\varphi\circ\pi$, where π is the canonical epimorphism of G onto G/N, and φ is bounded on a neighborhood of the identity element.*

Proof. Let $j\colon G/N\to G$ be a locally bounded section, for the canonical mapping $\pi\colon G\to G/N$, that is continuous at the identity element of G/N (see [16, Definition 2.2 and Theorem 2]). We set $\Phi=f\circ j$. The real function Φ on G/N is a quasi-character. Indeed, for any $a,b\in G/N$, the elements $j(a)j(b)$ and $j(ab)$ belong to the same N-coset in G, and hence

$$j(ab)=j(a)j(b)n$$

for some $n\in N$, where $f(n)=0$ by assumption. This implies the relation

$$|f(j(ab))-f(j(a)j(b))|=|f(j(ab))-f(j(a)j(b))-f(n)|\le C_f,$$

and therefore

$$\begin{aligned} |\Phi(ab)-\Phi(a)-\Phi(b)| &\le |f(j(ab))-f(j(a)j(b))| \\ &\quad + |f(j(a)j(b))-f(j(a))-f(j(b))| \le 2C_f. \end{aligned}$$

Let $\psi = \varphi_\Phi$ be the pseudocharacter on G/N related to Φ. Then $\psi \circ \pi$ and f are pseudocharacters on the group G. For any $g \in G$ we have

$$(\psi \circ \pi)(g) - f(g) = \psi(\pi(g)) - f(g) = \psi(\pi(g)) - f(j(\pi(g))n(g)),$$

with some $n(g) \in G$, and thus

$$\begin{aligned}
|(\psi \circ \pi)(g) - f(g)| &= |\psi(\pi(g)) - f(j(\pi(g))n(g))| \\
&\leq |\psi(\pi(g)) - f(j(\pi(g))) - f(n(g))| \\
&= |\psi(\pi(g)) - f(j(\pi(g)))| \\
&= |\varphi_\Phi(\pi(g)) - \Phi(\pi(g))| \leq C_\Phi.
\end{aligned}$$

The difference $\psi \circ \pi - f$ is therefore a bounded pseudocharacter, and hence vanishes. Thus, $\psi \circ \pi = f$. Let M be an open subset of $\mathbb{R}$. Then $K = \psi^{-1}(M)$ is a subset of G/N whose π-preimage is a Borel subset of G. Since π is continuous and open, we can readily see (for instance, by the transfinite induction) that ψ is Borel as well. Since the pseudocharacter f is bounded in a neighborhood U of the identity element and π is open, it follows that ψ is bounded on the image $\pi(U)$ of U, and $\pi(U)$ is a neighborhood of the identity element of G/N. Now the assertion of Lemma 2 is an immediate consequence of Proposition 1. $\square$

We now extend some results related to the continuous pseudocharacters on locally compact groups to the pseudocharacters that are defined on general locally compact groups, belong to the first Baire class, and are bounded on a neighborhood of the identity element. We begin with the following characterization (cf. [15]) of the connected locally compact groups on which any first-Baire-class pseudocharacter that is bounded on a neighborhood of the identity element is an ordinary (real) character.

Proposition 2. *Let G be a connected locally compact group. The following conditions are equivalent.*

1) *any first-Baire-class pseudocharacter on G that is bounded on a neighborhood of the identity element of G is an ordinary character;*
2) *the maximal semisimple quotient Lie group S of G has finite center, and G has no quotient Lie groups that are isomorphic to a one-dimensional real central extension of S by $\mathbb{R}$ defined by a nontrivial bounded continuous real 2-cocycle on S.*

For the proof of Proposition 2 we need the following auxiliary assertions.

Lemma 3. *Let G be a locally compact group. Any pseudocharacter on G that belongs to the first Baire class and is bounded on a neighborhood of the identity element of G vanishes on the maximal compact normal subgroup K of G. The corresponding pseudocharacter on the quotient group $H = G/K$ (see Lemma 2) vanishes on the commutator subgroup $[R, R]$ of the radical $R = \mathrm{rad}(H)$.*

Lemma 4. *Let G be a locally compact group that is either solvable or compact. Any pseudocharacter on G that belongs to the first Baire class and is bounded on a neighborhood of the identity element of G is a continuous real character on G if G is solvable and zero if G is compact.*

Lemma 5. *Let G be a locally compact group and let $R = \mathrm{G}$ be its radical. Each first-Baire-class pseudocharacter on G that is bounded on a neighborhood of the identity element can be factored through an extension of the group $H = G/R$ by $\mathbb{R}$ that is related to the trivial action of H on $\mathbb{R}$.*

Lemma 6. *Let G be an amenable locally compact group. Any pseudocharacter on G that belongs to the first Baire class and is bounded on a neighborhood of the identity element of G is a continuous real character on G.*

Proof of Lemma 4. Assume that A is a locally compact group that is amenable as a discrete group and that F is a first-Baire-class pseudocharacter on A that is bounded in a neighborhood of the identity element. It follows from the result of [12] cited above (see also [14]) that F is an ordinary real character of the group A regarded as a discrete group. However, any character F of a locally compact group A that is a Borel function and is bounded on a neighborhood of the identity element of A is automatically continuous (the proof of this fact is similar to that of the continuity of a unitary one-dimensional Borel character on a locally compact group).

Let us prove that any Borel pseudocharacter F on a compact group Q that is bounded on a neighborhood of the identity element vanishes on Q. Indeed, for any element $a \in Q$, the subgroup A (of Q) generated by a is Abelian and compact; hence, it follows from the first part of the lemma that the restriction of f to A is a continuous real character of A, and thus this restriction is trivial. In particular, $F(a) = 0$. Since a is an arbitrary element of Q, this means that F is equal to zero on Q. □

The proof of Lemma 3 readily follows from Lemma 4. □

Proof of Lemma 5. By Lemma 4, the restriction $\chi = f|_R$ of f to R is a continuous character of R. Note that the character χ is trivial not only on the commutator subgroup $[R, R]$ of R, but also on the subset $M = \operatorname{Ker}\chi$, which is a nontrivial subgroup of R, and hence of G. We recall that f is constant on any conjugacy class. Hence, M is an intersection of two sets that are invariant under the inner automorphisms, namely, of the set $\operatorname{Ker} f$ and the set R. Therefore, M is a normal subgroup of G. Applying Lemma 2 we can see that f is defined by a pseudocharacter F on the quotient group $H = G/M$, and F is a first-Baire-class pseudocharacter that is bounded on a neighborhood of the identity element of H.

The quotient group $R_H = R/M$ is clearly topologically isomorphic to $\mathbb{R}$. Therefore, if the restriction of f to R is nontrivial, then the restriction of F to the subgroup R_H is a nonzero linear function on R_H. Since F is constant on any conjugacy class of H, it follows that each point of R_H is a conjugacy class of the group H. Hence, the action of the quotient group $Q = (G/M)/(R/M)$ on R_H is trivial. This completes the proof of Lemma 5. □

Proof of Lemma 6. The assertion is immediate because f is an ordinary real character, and we can readily prove that the assumptions of the lemma imply its continuity (cf. the proof of Lemma 3). □

Proof of Proposition 2. 1) $\Longrightarrow$ 2). Recall that G is a connected locally compact group. Assume that any first-Baire-class pseudocharacter on G that is bounded on a neighborhood of the identity element of G is an ordinary character. This means that any first-Baire-class pseudocharacter on the maximal semisimple quotient Lie group S of G that is bounded on a neighborhood of the identity element of S is an ordinary character as well. Let us prove that S has finite center.

Let $S = KAN$ be an Iwasawa decomposition of S, under usual notation (in particular, K is the analytic subgroup of S related to a maximal compact Lie subalgebra of the Lie algebra of S, A is Abelian, N is nilpotent, and AN is a simply connected solvable Lie group). Assume that the center of S is infinite. Any continuous real character χ of K (which can be nonzero if and only if the center of S is infinite) defines a quasi-character f on S by the formula

$$f(kan) = \chi(k), \qquad k \in K, \quad a \in A, \quad n \in N,$$

and this quasi-character f is continuous on G. The corresponding pseudocharacter φ on G is then clearly nontrivial.

Therefore, the center of S is finite. Now we assume that G has a quotient Lie group H that is isomorphic to a one-dimensional real central extension of S by $\mathbb{R}$ defined by a nontrivial bounded continuous real 2-cocycle ψ on S. Let us construct a nontrivial first-Baire-class pseudocharacter on H that is bounded on a neighborhood of the identity element of H. This pseudocharacter can then be lifted to a nontrivial pseudocharacter of G.

Consider the group

$$H = \{(s,r), s \in S, r \in \mathbb{R}\}$$

endowed with the product

$$(s_1,r_1)(s_2,r_2) = (s_1 s_2, r_1 + r_2 + \psi(s_1,s_2)), \qquad s_1, s_2 \in S, \quad r_1, r_2 \in \mathbb{R}.$$

The mapping $f\colon H \to \mathbb{R}$ given by the rule

$$(s,r) \mapsto r, \qquad (s,r) \in H,$$

is a (continuous) quasi-character on H, which is immediate by the conditions imposed on ψ. If the corresponding pseudocharacter φ is a character, then it is Borel, and therefore it is continuous. However, in this case H is isomorphic to the direct product of S and $\mathbb{R}$, and hence the cocycle ψ is trivial, which contradicts the assumption. Hence, no such group H can be isomorphic to a quotient group of G, and this completes the proof of the implication 1) $\Longrightarrow$ 2).

Note that the corresponding pseudocharacter φ is given by the formula

$$\varphi(s,r) = r + \lim_{n\to\infty} n^{-1} \sum_{k=1}^{n-1} \psi(s,s^k), \qquad (s,r) \in G.$$

In general, *the limit*

$$\Delta_\psi(s) = \lim_{n\to\infty} n^{-1} \sum_{k=1}^{n-1} \psi(s,s^k) \tag{8}$$

exists for any bounded real Borel 2-cocycle ψ on a group S and for any $s \in S$.

2) $\Longrightarrow$ 1). Let f be a pseudocharacter on a connected locally compact group G that belongs to the first Baire class and is bouunded on a neighborhood of the identity element of G. Let N be a normal subgroup of G that belongs to a neighborhood of the identity element of G on which f is bounded and for which the quotient group G/N is a Lie group (which is also connected). The restriction of f to N is bounded, and hence vanishes. By Lemma 2, the pseudocharacter f can be expressed in the form $f = \varphi \circ \pi$, where φ is a first-Baire-class pseudocharacter of the quotient group G/N and π is the canonical epimorphism of G onto G/N, and φ is bounded on a neighborhood of the identity element. Therefore, it suffices to prove the assertion under the additional assumption that G is a connected Lie group.

Consider the restriction of f to G. Let us prove that if any semisimple quotient Lie group of G has finite center and if f is continuous, then f is a character of G. On applying Lemma 2 we see that the pseudocharacter f on G can be factored through a one-dimensional central extension G_1 of a semisimple Lie group with finite center (for instance, the maximal semisimple quotient Lie group S of G) by a normal subgroup isomorphic to $\mathbb{R}$. Therefore, we can restrict ourselves to the case in which the group G itself has this structure. Let $\mathcal{G}$ be the universal covering group of the group G. Let F be the natural lifting of f to $\mathcal{G}$ (constructed by taking the composition $F = f \circ \pi$, where π is the natural surjection $\pi \colon \mathcal{G} \to G$).

By construction, the group $\mathcal{G}$ is simply connected, and on applying the Levy–Mal'tsev theorem we can present the group $\mathcal{G}$ as a semidirect product $\mathcal{S}\mathcal{R}$, where $\mathcal{S}$ is a simply connected semisimple Lie subgroup of $\mathcal{G}$ and the normal subgroup $\mathcal{R}$ is the (solvable) radical of $\mathcal{G}$, which is clearly isomorphic to $\mathbb{R}$. In this case, the pseudocharacter F on $\mathcal{G}$ is determined by the restrictions $F_{\mathcal{S}}$ and $F_{\mathcal{R}}$ of F to $\mathcal{S}$ and $\mathcal{F}$, respectively. Both there restrictions are pseudocharacters, on the corresponding groups, that belong to the first Baire class and are bounded on neighborhoods of the identity elements of $\mathcal{S}$ and $\mathcal{F}$, respectively.

If $F_{\mathcal{R}}$ vanishes, then F is determined by the pseudocharacter $F_{\mathcal{S}}$ of its semisimple quotient group. However, it follows from the construction that F vanishes on the discrete kernel of π. Therefore, $F_{\mathcal{S}}$ must vanish on a subgroup of the center of $\mathcal{S}$ whose index in this center is finite. As was shown above in the proof of Proposition 2, in this case the pseudocharacter $F_{\mathcal{S}}$ is zero, and following the line of reasoning of [13, Theorem 2] we can see that $F = 0$, and hence $f = 0$. Hence, each nontrivial pseudocharacter on G that belongs to the first Baire class and is bounded on a neighborhood of the identity element of G must be nontrivial on the radical of G, and we now assume that, for a given f, the restriction of f to R is nonzero. It suffices to prove that if f is not an ordinary character, then G has a quotient Lie group that is isomorphic to a one-dimensional real central extension of S by $\mathbb{R}$ defined by a nontrivial bounded continuous real 2-cocycle on S. To this end we apply Lemma 5 and pass to the quotient group of G for which the radical is isomorphic to $\mathbb{R}$. Therefore, we may replace the group G by this quotient group and assume that the group G itself is a central extension of S by $\mathbb{R}$ (see Lemma 5) and f is a nontrivial pseudocharacter on G. Let us prove that G is a nontrivial (nonsplittable) extension.

Let s be a Borel section of the natural mapping $\pi \colon G \to S$ (as is known, such a section exists in our setting, see, e.g., [17, 5.1.1]), and let φ be the real (Borel) 2-cocycle on S given by

$$\varphi(x,y)=f(s(x)s(y))-f(s(x))-f(s(y)), x,y\in S.$$

Since f is a pseudocharacter, the function φ is bounded. Let us first show that φ is a 2-cocycle on the quotient group S. We must verify the relation

$$\begin{aligned} &f(s(g)s(h))-f(s(g))-f(s(h))+(f(s(gh)s(k))-f(s(gh))-f(s(k)) \\ &=f(s(h)s(k))-f(s(h))-f(s(k))+(f(s(g)s(hk))-f(s(g))-f(s(hk)). \end{aligned} \tag{9}$$

The elements $s(g)s(h)$ and $s(gh)$ certainly belong to the same coset by the radical R, which is a normal subgroup isomorphic to $\mathbb{R}$. Moreover, in this case R is central because

$$f(g^{-1}xg)=f(x), \qquad x\in R, \quad g\in S.$$

Furthermore, since the restriction of f to R is a continuous (nontrivial) real character of R, we may introduce a multiplicative coordinate on R and assume that f defines an additive coordinate on $\mathbb{R}$, and thus identify f on R with the logarithmic function on $\mathbb{R}$. Now we can write

$$s(gh)=s(g)s(h)x_{g,h}, \tag{10}$$

where the mapping $G\times G\to R$ given by

$$\{g,h\}\mapsto x_{g,h}, \qquad g,h\in G, \quad x_{g,h}\in R,$$

is a bounded Borel mapping. For any element $k\in G$, the closed subgroup L_k of G generated by k and R is certainly solvable, and hence amenable; therefore, by Lemma 6, the restriction of f to this subgroup is a continuous (ordinary) real character of L_k. In particular, this means that

$$f(kr)=f(k)+f(r), \qquad k\in G, \quad r\in\mathbb{R},$$

and thus it follows from (10) that

$$f(s(gh))=f(s(g)s(h)x_{g,h})=f(s(g)s(h))+\log(x_{g,h})$$

for all $g,h\in S$. Now the direct computation of the form

$$\begin{aligned} s(ghk)x_{gh,k}^{-1}&=s(gh)s(k)s(k)=s(g)s(h)s(k)x_{g,h} \\ &=s(g)s(hk)x_{h,k}^{-1}x_{g,h}=s(ghk)x_{g,hk}^{-1}x_{h,k}^{-1}x_{g,h}, \end{aligned}$$

which clearly implies relation (9), proves that the function φ is a (Borel) real bounded 2-cocycle on H indeed. Moreover, note that we can define the special section $\sigma\colon H \to G$ by setting $\sigma(h) \in G$ to be the (unique) element in G such that $\pi(\sigma(h)) = h$ and $f(\sigma(h)) = 0$ for any $h \in H$. For this section we clearly have

$$\log(x_{g,h}) = -f(\sigma(g)\sigma(h)), \qquad g, h \in G,$$

and therefore the pseudocharacter f defines the cocycle x uniquely up to a coboundary. The corresponding extension G is splittable if and only if x is a coboundary. However, G is a semidirect product if and only if there exists a section s that is a homomorphism. Since s is Borel, it is an analytic homomorphism into G, and the composition $f \circ s$ is a continuous pseudocharacter on H. This pseudocharacter clearly determines a pseudocharacter $F = f \circ s \circ \pi$ of G, and $f - F$ is bounded on the image of s. Hence, $f = F$ on the image of s, and thus the image of s is a closed set (because it clearly coincides with the zero set of the continuous pseudocharacter $f - F$). However, this means that G is the direct product of the subgroups $s(H)$ and $\mathbb{R}$, and the difference $\Phi = f - F$ is a (continuous) character, which is trivial on $s(H)$ and a nontrivial linear function on $\mathbb{R}$. Therefore, $f = F + \Phi$, where F is a first-Baire-class pseudocharacter that vanishes on $\mathbb{R}$, and hence on the radical, and thus is a pseudocharacter on S (that is bounded on a neighborhood of the identity element). Therefore, F is equal to zero (by the first part of the proof of the implication 2) $\Longrightarrow$ 1)). This shows that $f = \Phi$, which contradicts the assumption that f is not a character. Therefore, G is not splittable indeed, and this completes the proof of the implication 2) $\Longrightarrow$ 1), and hence of Proposition 2. □

Definition 2. A first-Baire-class pseudocharacter f on a connected locally compact group G that is bounded on a neighborhood of the identity element and nontrivial on the radical of G is said to be *pure*.

Remark 1. On a connected semisimple Lie group S, any first-Baire-class pseudocharacter is continuous. Indeed, let $S = KAN$ be an Iwasawa decomposition of G as above. The restriction of any first-Baire-class real pseudocharacter on G defines a real character χ of the Lie group K (which can be nonzero if and only if the center of S is infinite), and, in turn, χ defines a quasi-character f on S by the formula

$$f(kan) = \chi(k), \qquad k \in K, \quad a \in A, \quad n \in N.$$

The quasi-character f is clearly continuous on G, and the corresponding pseudocharacter φ on G turns out to be continuous as well, as readily follows from direct calculations related to convenient one-parameter subgroups and from the fact that any pseudocharacter is constant on each of the conjugacy classes (cf. Example 1 below). Thus, the space of (continuous or first-Baire-class) pseudocharacters on S is naturally isomorphic to the dual space of the (maximal) vector subspace of K.

Remark 2. The pure first-Baire-class pseudocharacters on one-dimensional central extensions of semisimple Lie groups are closely related to the pseudocharacters mentioned in Remark 1, see Example 2 below.

Proposition 3 [15, Theorem 2]**.** *Let G be a locally compact group and let G_0 be the connected component of G. Any pseudocharacter φ on G that belongs to the* *first Baire class and is bounded on a neighborhood of the identity element of G, can be factored as follows*:

1) *through the disconnected quotient group of G/G_0 provided that φ vanishes on G_0, in which case φ is continuous*;
2) *through the quotient group of G by the radical* $\mathrm{rad}(G)$ *of G (here and henceforth, the radical of a locally compact group is its maximal connected normal solvable subgroup, see* [11, Proposition 3.7]) *provided that φ does not vanish on G_0 but vanishes on* $\mathrm{rad}(G)$ *(in this case, the restriction of the corresponding factorization of φ to $G_0/\mathrm{rad}(G)$ is continuous by Remark* 1), *and*
3) *through a quotient group of G that is isomorphic to an extension, of $\mathbb{R}$ by the group $H = G/\mathrm{rad}(G)$, defined by a bounded continuous system of factors on $H \times H$ (a bounded continuous* 2*-cocycle on H) related to the trivial action of H on $\mathbb{R}$ (if φ does not vanish on* $\mathrm{rad}(G)$*)*.

   Conversely, if the action of $G_0/\mathrm{rad}(G)$ on the (Abelian) quotient group $\mathrm{rad}(G)/[\mathrm{rad}(G), \mathrm{rad}(G)]$ *has a one-dimensional trivial quotient action, say, on $T(\simeq \mathbb{R})$, then to any bounded continuous* 2*-cocycle on the quotient group $G_0/\mathrm{rad}(G)$ with coefficients in T, a first-Baire-class pseudocharacter on G_0 corresponds, which is bounded on a neighborhood of zero, and this pseudocharacter is nontrivial if and only if the cocycle is nontrivial.*

Remark 3. If G is connected, then either any pseudocharacter on G of the type 3) is the sum of a continuous (real) character and a pseudocharacter of the type 2) or the maximal semisimple quotient of G that has no compact normal subgroups is not simply connected, as follows from the Levy–Mal'tsev theorem.

Proof of Proposition 3. It follows from the proof of Lemma 4 that each first-Baire-class pseudocharacter on G that is bounded on a neighborhood of the identity element can be factored through an extension of the group $H = G/\mathrm{rad}(G)$ by $\mathbb{R}$. The corresponding extension is related to the trivial action of H on $\mathbb{R}$. The remaining part of the proof of Proposition 3 readily follows from Proposition 2 and from the continuity of any first-Baire-class pseudocharacter, on a totally disconnected locally compact group G/G_0, that is bounded on a neighborhood of the identity element (see [15, Theorem 2]; the proof is based on the results of [18]. $\square$

Remark 4. The possibilities listed in Proposition 3 can occur indeed, as can be shown by Examples 1 and 2 below.

For the connected locally compact groups we thus have only two nontrivial possibilities.

Corollary 1. *Any first-Baire-class pseudocharacter on a connected locally compact group G that is bounded, on a neighborhood of the identity element, and nontrivial (i.e., is not a continuous character of G) defines a continuous character on any amenable subgroup of G. This pseudocharacter can be factored through a quotient group of G that is isomorphic to an extension H', of the maximal semisimple quotient group H of the group G by the additive group $\mathbb{R}$ of reals, where H' is defined by a bounded continuous system of factors on $H \times H$ (a bounded continuous* 2*-cocycle on H) related to the trivial action of H on $\mathbb{R}$. This pseudocharacter is continuous if and only if it is pure.*

Conversely, to any bounded continuous real 2-cocycle on H, a first-Baire-class pseudocharacter on G corresponds (that is bounded on a neighborhood of the identity element). This pseudocharacter is nontrivial if and only if the cocycle is nontrivial.

The proof immediately follows from Proposition 3.

Propositions 2 and 3, together with Corollary 1, give a correct version of the assertions in [13, Theorem 2 and Corollary 3]. A gap in Corollary 3 was indicated by M. Grosser in [8].

Example 1. Here we show that the natural mapping of the universal covering group G of the group $H = \mathrm{SL}(2,\mathbb{R})$ onto $\mathbb{R}$ that sends $g \in G$ to the K-component of the Iwasawa decomposition of G is a quasi-character f whose pseudocharacter φ (which is a unique nontrivial pseudocharacter on G up to a nonzero multiple) is continuous.

According to [1], the group G can be parametrized as follows. Let

$$G = \{\,(\gamma,\omega),\ \gamma \in \mathbb{C},\ |\gamma| < 1,\ \omega \in \mathbb{R}\,\},$$

and let the covering $\pi\colon G \to \mathrm{SL}(2,\mathbb{R})$ be given by the formula

$$\pi(g) = \begin{pmatrix} e^{i\omega}(1-|\gamma|^2)^{-1/2} & e^{i\omega}\gamma(1-|\gamma|^2)^{-1/2} \\ e^{-i\omega}\overline{\gamma}(1-|\gamma|^2)^{-1/2} & e^{-i\omega}(1-|\gamma|^2)^{-1/2} \end{pmatrix} \in \mathrm{SL}(2,\mathbb{R}). \tag{11}$$

In this case, the multiplication rule on G is given by the formulas

$$g = g'g'', \qquad g' = (\gamma',\omega'), \quad g'' = (\gamma'',\omega''),$$

where

$$\gamma = (\gamma' + \gamma'' e^{-2i\omega'})(1 + \overline{\gamma'}\gamma'' e^{-2i\omega'})^{-1}, \tag{12}$$

$$\omega = \omega' + \omega'' + \frac{1}{2i}\log\left\{\frac{1+\overline{\gamma'}\gamma'' e^{-2i\omega'}}{1+\gamma'\overline{\gamma''}e^{2i\omega'}}\right\}; \tag{13}$$

here and henceforth, the logarithm is understood in the sense of its principal value (belonging to the interval $(-\pi,\pi)$).

Relations (12) and (13) immediately show that the formula

$$f\colon g = (\gamma,\omega) \mapsto \omega, \qquad g \in G,$$

defines a continuous real quasi-character f on G.

To calculate the corresponding pseudocharacter, we use the fact that any pseudocharacter is constant of any conjugacy class and find good representatives of the conjugacy classes.

Let

$$g = (\gamma,\omega), \quad h = (\delta,\sigma) \in G.$$

In this case we have

$$h^{-1} = (-\delta e^{2i\sigma}, -\sigma)$$

and

(14)

$$hgh^{-1} = \Big(\frac{\delta+\gamma e^{-2i\sigma}-\delta e^{-2i\omega}(1+\delta\overline{\gamma}e^{2i\sigma})}{1+\overline{\delta}\gamma e^{-2i\sigma}-(\overline{\delta}+\overline{\gamma}e^{2i\sigma})\delta e^{-2i\omega}},$$
$$\omega+\frac{1}{2i}\log\left\{\frac{1+\overline{\delta}\gamma e^{-2i\sigma}}{1+\delta\overline{\gamma}e^{2i\sigma}}\right\}$$
$$+\frac{1}{2i}\log\left\{\frac{(1+\overline{\delta}\gamma e^{-2i\sigma}-(\overline{\delta}+\overline{\gamma}e^{2i\sigma})\delta e^{-2i\omega})(1+\delta\overline{\gamma}e^{2i\sigma})}{(1+\delta\overline{\gamma}e^{2i\sigma}-(\delta+\gamma e^{-2i\sigma})\overline{\delta}e^{2i\omega})(1+\overline{\delta}\gamma e^{-2i\sigma})}\right\}\Big).$$

Relation (14) makes it possible to write out formulas for elements of G that are conjugate to elements with clear value of the pseudocharacter φ corresponding to the quasi-character f.

It follows from (14) that a given element $g=(\gamma,\omega)\in G$ is conjugate to an element h of the form $k=(\varepsilon,n\pi)\in G$, for some ε with $|\varepsilon|<1$ and $n\in\mathbb{Z}$, if and only if

$$e^{2i\omega}\frac{1+\overline{\delta}\gamma e^{-2i\sigma}-(\overline{\delta}+\overline{\gamma}e^{2i\sigma})\delta e^{-2i\omega}}{1+\delta\overline{\gamma}e^{2i\sigma}-(\delta+\gamma e^{-2i\sigma})\overline{\delta}e^{2i\omega}}=1,$$

or

$$e^{2i\omega}+\overline{\delta}\gamma e^{2i(\omega-\sigma)}-|\delta|^2-\delta\overline{\gamma}e^{2i\sigma}=1+\delta\overline{\gamma}e^{2i\sigma}-\delta e^{2i\omega}-\overline{\delta}\gamma e^{2i(\omega-\sigma)},$$

for some $h=(\delta,\sigma)\in G$. We set $|\gamma|=r<1$, $|\delta|=\rho<1$, and define $e^{i\theta}$, for $r\rho>0$, by

$$\overline{\gamma}\delta=2r\rho e^{i\theta}.$$

In this case we must have $\sin(\omega)(1+\rho^2)=2r\rho x$ for some $x=\sin(\omega-2\sigma+\theta)$. Therefore, a required ρ exists if and only if

$$r>|\sin\omega|,$$

and this condition implies the relation

$$\varphi_f(g)=n\pi,\qquad n=[\omega/\pi+1/2].$$

We can similarly see from (14) that a given element $g=(\gamma,\omega)\in G$ is conjugate to an element h of the form $k=(0,\tau)\in G$, for some $\tau\in\mathbb{R}$, if and only if

$$\delta+\gamma e^{-2i\sigma}-\delta e^{-2i\omega}(1+\delta\overline{\gamma}e^{2i\sigma})=0$$

for some $h=(\delta,\sigma)\in G$, which means that the quadratic equation

$$\rho^2-2\rho\frac{\sin\omega}{r}+1=0,$$

where $r=|\gamma|$ again, must have a root ρ with $|\rho|<1$, which is possible if and only if

$$r<|\sin\omega|.$$

This makes it possible to find the corresponding values of τ and immediately verify that

$$\lim_{r \nearrow |\sin\omega|} \tau = [\omega/\pi + 1/2]\pi.$$

Certainly, for g with $r < |\sin\omega|$ we have

$$\varphi_f(g) = \tau.$$

Finally, let

$$r = \sin\omega,$$

where $r = |\gamma|$ as above. The direct calculation (after applying (14) with $\delta = 0$) shows that we can choose a conjugate element with the same ω and

$$\gamma = -i\sin(\omega)e^{i\omega}.$$

This gives

$$\omega = [\omega/\pi + 1/2]\pi + \Delta\omega,$$

where

$$|\Delta\omega| < \pi/2,$$

and hence on such an element g, the pseudocharacter φ_f takes the obvious value

$$\varphi_f(g) = [\omega/\pi + 1/2]\pi,$$

and this result completes the proof of the continuity of the nontrivial pseudocharacter f.

Example 2. Here we show that a pure pseudocharacter of the (essentially unique) nontrivial extension of the adjoint group $\mathrm{SL}(2,\mathbb{R})$ by $\mathbb{R}$ is continuous, which is in fact shown by the direct calculations above. Indeed, let us construct the pseudocharacter f related to the Guichardet–Wigner 2-cocycle x on the group $G = \mathrm{PSL}(2,\mathbb{R})$ ([11], see also [7]). Along with x, we consider the restriction y of the cocycle x to the discrete subgroup $H = \mathrm{SL}(2,\mathbb{Z})$.

Let x be the mapping $x\colon G \times G \to \mathbb{R}$ such that, for any $g, h \in G$, $x(g,h)$ is the oriented Lobachevski area of the Lobachevski triangle, on the upper half-plane $\mathbb{C}^+$, with the vertices z, gz, hgz, where $z \in \mathbb{C}^+$ [7] (the value of this area does not depend on the choice of z). This cocycle x of G and its restriction y to H are continuous, bounded, and nontrivial 2-cocycles of G and H, respectively. Another formula for the above cocycle (up to a nonzero factor) is presented in [11] and shows that the Guichardet–Wigner cocycle x can be related to the above cocycle in Example 1 as follows. Adopt the notation of Example 1. We set

$$j(g) = \omega, \qquad g = (\gamma, \omega) \in G$$

and write

$$y(g,h) = j(gh) - j(g) - j(h), \qquad g, h \in G.$$

It follows from (11) and (13) that the function y on $G \times G$ depends on $\pi(g)$ and $\pi(h)$ only and thus determines a continuous function x on $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$, which is a

2-cocycle on $\mathrm{SL}(2,\mathbb{R})$. Therefore, the 2-cocycles on $\mathrm{SL}(2,\mathbb{R})$ in these two examples are the same, and hence the related difference Δ_x (see (8)) between the pseudocharacter and the quasi-character in these examples is the same. In particular, this implies the continuity of the pseudocharacter related to the Guichardet-Wigner cocycle on $\mathrm{SL}(2,\mathbb{R})$.

Certainly, the similar relationship between the nontrivial pseudocharacters on a simply connected simple Lie group G and the nontrivial pseudocharacters, on a one-dimensional central extension H of a non-simply-connected quotient group G' of G, related to nontrivial real continuous 2-cocycles on G' follows directly from the explicit formulas for the Guichardet–Wigner cocycles [11]. Since the calculations of the asymptotic behavior (see (8)) related to the cocycles on the simple Lie factors of a given semisimple Lie group S can be performed directly by applying formulas from [11, Theorems 1 and 2], we can state the following conjecture.

Conjecture. An arbitrary first-Baire-class pseudocharacter bounded on a neighborhood of the identity element of a connected locally compact group is continuous.

Part II. Applications to Continuous Bounded Cohomology

We recall that $H_b^*(G,\mathbb{R})$ is the homology of the complex

$$0 \to \mathbb{R} \xrightarrow{d=0} CB^1(G) \xrightarrow{d} CB^2(G) \to \cdots,$$

where $CB^n(G)$ stands for the space of continuous bounded functions from G^n to $\mathbb{R}$ and

$$\begin{aligned} df(x_1,\dots,x_n) = f(x_2,\dots,x_n) \\ + \sum_{i=1}^{n}(-1)^i f(x_1,\dots,x_i x_{i+1},\dots,x_{n+1}) \\ + (-1)^{n+1} f(x_1,\dots,x_n) \end{aligned}$$

for $f \in CB^n(G)$ $(n \geq 1)$. As was noted in [3, 2] (see also [5]) for the case of discrete groups (and can be immediately repeated in the topological case), the kernel of the natural mapping

$$H_b^2(G,\mathbb{R}) \to H^2(G,\mathbb{R}) \tag{15}$$

(which arises on regarding a bounded cocycle as a cocycle) is naturally isomorphic to the quotient space of the continuous quasi-characters on G by the bounded continuous functions or, which gives an isomorphic space (as we saw above), to the quotient space of the Borel pseudocharacters on G (in the papers [2, 3, 5], the *pseudocharacters* arise under the title *quasi-morphismes homogènes*) by the subspace of (continuous real additive) characters of G. Clearly, the image of the mapping (15) is isomorphic to a quotient space of $H_b^2(G,\mathbb{R})$ (namely, by the kernel of the mapping (15)). Hence, the space $H_b^2(G,\mathbb{R})$ itself is isomorphic to the direct sum of the kernel of the mapping (15) (which can be described in terms of the pseudocharacters of G) and of the subspace of $H^2(G,\mathbb{R})$ related to bounded continuous 2-cocycles.

Remark 5. Note that, according to Corollary 1, any pseudocharacter on G that is related to continuous cohomology can be factored through a quotient group of G that is isomorphic to a one-dimensional central extension of the semisimple quotient of G. Hence, any pseudocharacter on G that is related to continuous cohomology can be factored through the quotient group by the commutator subgroup of the radical of G.

Proposition 4. *The second real continuous bounded cohomology group $H^2_b(G,\mathbb{R})$ of a connected locally compact group G is finite-dimensional.*

Proof. Let us prove first that the kernel of the mapping (15) is finite-dimensional. It follows from the above reasoning that all first-Baire-class pseudocharacters on a connected locally compact group G vanish both on the commutator subgroup of the radical and on any compact normal subgroup. Hence, by Lemma 2, any such pseudocharacter can be factored through some quotient group G' of G that is a central extension of a semisimple Lie group S by a finite-dimensional vector space V (this extension is central because any pseudocharacter is constant on any conjugacy class, and the conjugation defines a finite-dimensional representation of the semisimple factor, and hence all orbits are either singletons or unbounded sets, and this extension is nonsplittable because it admits nontrivial pseudocharacters), and this quotient group G' is the same for all nontrivial pseudocharacters under consideration.

Since the extension G' is central, it follows that the 2-cocycle determined by a given pseudocharacter under consideration is constant on the cosets by the vector normal subgroup V and thus it defines a 2-cocycle on the semisimple quotient group S. We can readily see that the corresponding 2-cohomology class is defined uniquely (this follows from the fact that the restriction of a Borel pseudocharacter to any amenable subgroup is a continuous real character of this subgroup), and hence we obtain a mapping (which is clearly linear) from the linear space of the pseudocharacters under consideration on G' (or of the corresponding 2-cohomology classes) into the linear space of real continuous 2-cocycles on S (these are also bounded, but this is unessential here), or, after applying the natural epimorphism, into the corresponding cohomology space. Therefore, the group (or the vector space) of our pseudocharacters is isomorphic to a subgroup of the second continuous cohomology group of S, which is well known. Indeed, $H^2_c(S)$ is isomorphic to the space of real linear mappings of the quotient Lie algebra $\mathfrak{k}/[\mathfrak{k},\mathfrak{k}]$ into $\mathbb{R}$, where $\mathfrak{k}$ is the compact part of the Lie algebra of the group S.

This identification immediately proves that the linear space of these pseudocharacters is finite-dimensional. Moreover, since 1) a pseudocharacter on a simple Lie group or its one-dimensional central extension can exist only if the center of the maximal compact Lie subalgebra of the Lie algebra of this Lie group is nontrivial, 2) the maximal compact Lie subalgebra of a simple Lie algebra can have at most on-dimensional center, and 3) a nontrivial pseudocharacter on the corresponding Lie group is unique up to a multiple, it follows that the dimension of the space of locally bounded Borel pseudocharacters on a connected locally compact group G does not exceed that of the center of the maximal compact Lie subalgebra of the

Lie algebra of the maximal semisimple quotient Lie group of G that has no compact simple factors.

The remaining part of the second real bounded continuous cohomology group of G is also finite-dimensional, which follows from the above remarks and from the assertions [9, Chap. III, Corollary 7.3] and [9, Chap. III, Corollary 5.4]. This completes the proof of Proposition 4. $\square$

Note that we have proved the following assertion as a by-product.

Corollary 2. *If any semisimple quotient group of a given locally compact group has trivial second continuous cohomology group* (*or, which is the same, if the center of the maximal compact Lie subalgebra of the Lie algebra of G is trivial*), *then the corresponding mapping* (15) *is injective.*

Remark 6. It follows from the explicit formula [9, Chap. 2, Subsec. 7.5] for cocycles realizing the cohomology classes in $H^2_c(S)$, where S is a semisimple Lie group with nontrivial $H^2_c(S)$, that all these cocycles have bounded representatives.

The author expresses his deep gratitude to the Humboldt Universität (Berlin), the Westfälische Wilhelms-Universität (Münster), and GMD (Sankt Augustin) and especially to Professor Eberhard Kirchberg, Professor Thomas Friedrich, Professor Wolfgang Lück, Professor Fritz Schwarz, and Professor Eckehart Hotzel for warm hospitality in July 1997.

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Dept. of Mechanics and Mathematics, Moscow State University, Vorob′evy gory, 119899 Moscow, Russia