The Weierstrass representation of closed surfaces in $\mathbf{R}^3$

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Abstract

In the present paper which a sequel to [19] and [20] a global Weierstrass representation of an arbitrary closed oriented surface of genus ≥ 1 in the the three-space is constructed. The Weierstrass spectrum of a torus immersed into $\mathbf{R}^3$ is introduced and finite-zone planes as well as finite-zone solutions to the modified Novikov-Veselov equations are constructed.

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§1. Introduction

The present article is a sequel to [19, 20]. The results presented here extend onto general surfaces the results obtained in [20] for surfaces of revolution and were exposed in a lot of talks of the author during the last year being at the end announced in [21]. These are

– a construction of a global Weierstrass representation for an arbitrary closed oriented surface, of genus $g \geq 1$, immersed into $\mathbf{R}^3$ (§2);

– a construction of a Weierstrass spectrum for a torus immersed into $\mathbf{R}^3$ and discussing its geometric properties (§3);

– a construction of finite-zone surfaces and finite-zone solutions to the modified Novikov–Veselov equations (§§4-5).

In §6 we discuss a relation of these constructions to the Willmore conjecture.

§2. The Weierstrass representation

2.1. The local Weierstrass representation.

The local Weierstrass representation of surfaces immersed into $\mathbf{R}^3$ is based on the following two lemmas.

Lemma 1 *Let W be a simply connected domain in $\mathbf{C}$ and let a vector function $\psi = (\psi_1, \psi_2) : W \to \mathbf{C}^2$ satisfy the equation*

$$\mathcal{D}\psi = 0 \tag{1}$$

where

$$\mathcal{D} = \begin{pmatrix} 0 & \partial \\ -\bar{\partial} & 0 \end{pmatrix} + \begin{pmatrix} U & 0 \\ 0 & U \end{pmatrix} \tag{2}$$

and a function $U(z, \bar{z})$ is real-valued. Then the following formulas

$$\begin{aligned} X^1(z,\bar{z}) &= \frac{i}{2}\int_\gamma \left((\bar{\psi}_2^2 + \psi_1^2)dz' - (\bar{\psi}_1^2 + \psi_2^2)d\bar{z}'\right), \\ X^2(z,\bar{z}) &= \frac{1}{2}\int_\gamma \left((\bar{\psi}_2^2 - \psi_1^2)dz' - (\bar{\psi}_1^2 - \psi_2^2)d\bar{z}'\right), \\ X^3(z,\bar{z}) &= \int_\gamma (\psi_1\bar{\psi}_2 dz' + \bar{\psi}_1\psi_2 d\bar{z}') \end{aligned} \tag{3}$$

define an immersion of W into $\mathbf{R}^3$. Moreover the induced metric takes the form

$$D(z,\bar{z})^2 dz d\bar{z}, \ \text{where } D(z,\bar{z}) = (|\psi_1(z,\bar{z})|^2 + |\psi_2(z,\bar{z})|^2),$$

the Gauss curvature is

$$K(z,\bar{z}) = -\frac{4}{D(z,\bar{z})^2}\partial\bar{\partial}\log D(z,\bar{z}),$$

and the mean curvature equals

$$H(z,\bar{z}) = 2\frac{U(z,\bar{z})}{D(z,\bar{z})}. \tag{4}$$

The immersion $W \to \mathbf{R}^3$ is constructed as follows. Take $z_0 \in W$ and map it into $0 \in \mathbf{R}^3$. After that define $X^i(z,\bar{z})$ by the value of the integral (3) taken over an arbitrary path, in W, connecting z_0 and z.

For $U = 0$, this is the classical Weierstrass representation of minimal surfaces.

After this lemma is formulated it is not difficult to prove it. The condition (1) together with the reality of U imply that the integrands in (3) are closed forms. Hence (3) defines an immersion W into $\mathbf{R}^3$ for which it is easy to compute all characteristics.

It seems that this lemma belongs to Eisenhart ([5]) who had written (1) as a condition only for ψ_1:

$$L\psi_1 = 0,$$

in terms of the second order scalar differential operator

$$L = \partial\bar{\partial} - \frac{\partial U}{U}\bar{\partial} + U^2. \tag{5}$$

In late 80's Abresch had derived the Weierstrass representation for constant mean curvature surfaces using the operator (2) and considered global representations for constructing explicit formulae for such surfaces of genus $g \geq 2$. Since this program is not realized until now, these results were only presented on the Luminy conference (1989).

Bobenko had shown that considering $\mathbf{R}^3$ as the space of imaginary quaternions some identities for the moving frame are written in terms of the Dirac equation for quaternion-valued functions, but constructing of general surfaces in terms of eigenfunctions of the Dirac operator and the globalization of this construction were not discussed by him ([1]).

In [12] Konopelchenko considered a "square root" of (5), i.e., the first order matrix differential operator (2), for definition of deformations of surfaces given by the formulas (3) via the modified Novikov–Veselov equations.

This replacement of the two-dimensional Schrödinger operator with a potential and a magnetic field (5) by the Dirac operator (2) with the potential U makes a representation more comfortable for applications.

Lemma 2 *Let W be a domain in $\mathbf{C}$ and let $X : W \to \mathbf{R}^3$ be a conformal immersion of W into $\mathbf{R}^3$: $z \to X(z,\bar{z}) = (X^1(z,\bar{z}), X^2(z,\bar{z}), X^3(z,\bar{z}))$. Assume that*

$$\frac{\partial X^3}{\partial z} \neq 0 \tag{6}$$

near $z_0 \in W$. Then near z_0 the functions

$$\psi_1(z,\bar{z}) = \sqrt{-\partial\Phi(z,\bar{z})}, \quad \psi_2(z,\bar{z}) = \sqrt{\bar{\partial}\Phi(z,\bar{z})}, \tag{7}$$

with

$$\Phi(z,\bar{z}) = X^2(z,\bar{z}) + iX^1(z,\bar{z}),$$

satisfy (1) with $U(z,\bar{z}) = H(z,\bar{z})D(z,\bar{z})/2$, where H is the mean curvature and $D^2 dz d\bar{z}$ is the metric of the surface $X(W) \subset \mathbf{R}^3$.

We explain the proof in brief following [19].

A condition that an immersion is conformal is written as

$$\left(\frac{\partial X^1}{\partial z}\right)^2 + \left(\frac{\partial X^2}{\partial z}\right)^2 + \left(\frac{\partial X^3}{\partial z}\right)^2 = 0. \tag{8}$$

The quadric $Q = \{x_1^2 + x_2^2 + x_3^2 = 0\} \subset \mathbf{C}P^2$ is diffeomorphic to the Grassmann manifold of two-dimensional oriented linear subspaces of $\mathbf{R}^3$ and for conformally immersed surface $X(W)$ the mapping $z \in W \to (X_z^1 : X_z^2 : X_z^3) \in Q$ is the Gauss map.

The Gauss map for the surface (3) is related to ψ by (7). Consider a conformal immersion satisfying (6) and by (7) construct near $z_0 \in W$ a vector function ψ. It follows from (6) and (8) that near z_0 the radicands in (7) do not vanish.

Take the following bases for the tangent planes to $X(W)$:

$$e_1 = \frac{1}{D}\frac{\partial X}{\partial x}, \quad e_2 = \frac{1}{D}\frac{\partial X}{\partial y},$$

where $D^2 dz d\bar{z}$ is the induced metric. Put $e_3 = e_1 \times e_2$. The derivation formulas take the form

$$\begin{aligned} \frac{\partial^2 X}{\partial x^2} &= \frac{\partial D}{\partial x} e_1 - \frac{\partial D}{\partial y} e_2 + D^2 h_{11} e_3, \\ \frac{\partial^2 X}{\partial x \, \partial y} &= \frac{\partial D}{\partial y} e_1 + \frac{\partial D}{\partial x} e_2 + D^2 h_{12} e_3, \\ \frac{\partial^2 X}{\partial y^2} &= -\frac{\partial D}{\partial x} e_1 + \frac{\partial D}{\partial y} e_2 + D^2 h_{22} e_3, \end{aligned} \tag{9}$$

where h_{ij} is the second fundamental form.

Now Lemma 2 is proved by substitution of (9) into the expressions for $\bar{\partial}\psi_1$ and $\partial\psi_2$ and by consequent straightforward computations.

Definition 1 *A representation of a surface Σ, immersed into $\mathbf{R}^3$, by the formulas (3) is called a Weierstrass representation.*

The function $U(z, \bar{z})$ of the form (4) is called the potential of the surface Σ with the distinguished conformal parameter z, or the potential of the Weierstrass representation of Σ.

Lemma 2 immediately implies ([19])

Lemma 3 *Every regular C^2-surface immersed into $\mathbf{R}^3$ near every its point admits a Weierstrass representation.*

2.2. The global Weierstrass representation.

For defining a Weierstrass representation globally for the whole surface Σ it needs to correctly define a bundle generated by ψ over Σ and an operator $\mathcal{D}$ acting in this bundle. This problem was solved in [19] and we recall this solution.

We consider closed oriented surfaces of genus $g \geq 1$.

By the Riemann uniformization theorem, every torus is conformally equivalent a flat torus $\mathbf{C}/\Lambda$ with Λ a lattice of rank 2.

Definition 2 *A torus Σ, immersed into $\mathbf{R}^3$, possesses a global Weierstrass representation if there exist a real potential U and functions ψ_1 and ψ_2 defined on the universal covering of Σ, i.e., on $\mathbf{C}$, such that*

1)

$$\begin{cases} U(z+\gamma) = U(z), \\ \psi_j(z+\gamma) = \varepsilon(\gamma)\psi_j(z), \\ \varepsilon(\gamma) = \pm 1 \end{cases} \tag{10}$$

for $z \in \mathbf{C}$ and $\gamma \in \Lambda$;

2) the vector function ψ satisfies (1) and for a suitable choice of coordinates in $\mathbf{R}^3$ defines by (3) an immersion of Σ.

For a sphere Σ with $g > 1$ handles the uniformization theorem tells that Σ is conformally equivalent to a quotient $\mathcal{H}/\Lambda$ with $\mathcal{H}$ the Lobachevskii upper-half plane and Λ a lattice in $PSL(2, \mathbf{R})$, the isometry group of $\mathcal{H}$.

Definition 3 *A sphere Σ with $g (> 1)$ handles, immersed into $\mathbf{R}^3$, possesses a global Weierstrass representation if there exist a real potential U and functions ψ_1 and ψ_2, defined on the universal covering of Σ, i.e., on $\mathcal{H}$, such that*

1)

$$\left\{ \begin{array}{l} U(\gamma(z)) = |cz+d|^2 U(z), \\ \psi_1(\gamma(z)) = (cz+d)\psi_1(z), \\ \psi_2(\gamma(z)) = (c\bar{z}+d)\psi_2(z) \end{array} \right. \tag{11}$$

for $z \in \mathcal{H}$ and $\gamma \in \Lambda$, represented by the matrix

$$\left(\begin{array}{cc} a & b \\ c & d \end{array} \right), \quad a, b, c, d \in \mathbf{R}, \quad ad - bc = 1;$$

2) the vector function ψ satisfies (1) and for a suitable choice of coordinates in $\mathbf{R}^3$ defines by (3) an immersion of Σ.

These definitions are quite natural and based on the following statement.

Lemma 4 *If a local Weierstrass representation of a closed oriented surface Σ of genus $g \geq 1$ is smoothly continued onto the whole universal covering $\tilde{\Sigma}$ then U and ψ satisfy (10) for $g = 1$ and (11) for $g \geq 2$.*

This lemma follows from the transformation rules for the mean curvature and the metric under changing of coordinates. Now we arrive at the following result([19]).

Theorem 1 *If a closed oriented surface Σ of genus ≥ 1 possesses a global Weierstrass representation then ψ is a global section of a spinor bundle over the constant curvature surface Σ_0 conformally equivalent to Σ and the Dirac operator (2) acts on this bundle.*

The following theorem demonstrates an importance of this representation.

Theorem 2 [1] *Every C^3-regular closed oriented surface Σ of genus $g \geq 1$, immersed into $\mathbf{R}^3$, possesses a global Weierstrass representation.*

A proof of Theorem 2.

By Lemma 3, the functions (7) define a local Weierstrass representation near every point with $\partial\Phi \neq 0$. For proving an existence of a global representation it suffices to correctly smoothly extend these functions onto neighborhoods of points with $\partial\Phi = 0$. By (8), the last condition is equivalent to $X^3_z = 0$.

For any compact immersed surface $\Sigma \subset \mathbf{R}^3$ there exist coordinates in $\mathbf{R}^3$ such that all the critical points of the "height function", a distance to the plane $X^3 = 0$, are nondegenerate in Morse's sense, i.e., at each of them the matrix of second

[1] This theorem also holds for spheres but we will consider this case separately.

derivatives of $X^3(z,\bar z)$ is nondegenerate ([16]). Take such coordinate system for the surface in study Σ.

Outside critical points define ψ by (7) and show that these functions do not ramify at critical points.

Let q be a critical point of X^3 and let z be a conformal parameter near q such that $z(q)=0$. Since this point is nondegenerate, we have

$$X^3(z,\bar z) = \alpha z^2 + \bar\alpha \bar z^2 + \beta z \bar z + O(|z|^3) \tag{12}$$

with $|\alpha|+|\beta| \neq 0$. Show that (7) correctly defines ψ_1 near q and notice that these conversations work also for ψ_2.

Since $\partial\Phi(q)=0$, we have

$$\frac{\partial X^1(q)}{\partial z} = i\frac{\partial X^2(q)}{\partial z}. \tag{13}$$

Differentiating the left-hand side of (8) by z, we infer from (12) and (13) that

$$(\partial\Phi)_z(q) = \frac{\partial^2 X^2(q)}{\partial z^2} + i\frac{\partial^2 X^1(q)}{\partial z^2} = 0. \tag{14}$$

Analogously differentiating it by $\bar z$, we derive

$$(\partial\Phi)_{\bar z}(q) = \frac{\partial^2 X^2(q)}{\partial z\,\partial \bar z} + i\frac{\partial^2 X^1(q)}{\partial z\,\partial \bar z} = 0.$$

Assume that $\alpha \neq 0$. Differentiating the left-hand side of (8) twice by z and taking (12), (13) and (14) into account, we obtain

$$\left(\frac{\partial^2 X^3(q)}{\partial z^2}\right)^2 + \frac{\partial X^2(q)}{\partial z}\left(\frac{\partial^3 X^2(q)}{\partial z^3} + i\frac{\partial^3 X^1(q)}{\partial z^3}\right) = 0$$

and, since the surface is regular, $X^2_z(q) \neq 0$. Hence

$$(\partial\Phi)_{zz}(q) \neq 0. \tag{15}$$

If $\alpha = 0$, then $\beta \neq 0$ and, differentiating the left-hand side of (8) twice by $\bar z$, we obtain

$$(\partial\Phi)_{\bar z\bar z}(q) \neq 0. \tag{16}$$

From (15) and (16) we conclude that $\partial\Phi$ has a double zero at q and if, for instance, the inequality (15) holds then $\partial\Phi = z^2\cdot f(z,\bar z)$ where the branches of $\sqrt{f}$ do not ramify at q. Hence the branches of $\sqrt{-\partial\Phi}$ do not ramify at q.

Since we consider an arbitrary critical point of the function X^3, each branch of $\sqrt{-\partial\Phi}$ is correctly defined and has no ramifications. Therefore on the universal covering of Σ the vector function ψ is correctly defined by (7).

This proves Theorem 2.

We arrive at the following conclusion

Zero-eigenfunctions of Dirac operators (2), with real potentials, on spinor bundles (10) and (11) over closed oriented constant curvature surfaces Σ of genus ≥ 1 are in one-to-one correspondence with immersions of their universal coverings into $\mathbf{R}^3$ with Gauss maps descending through Σ.

§3. The spectrum of Weierstrass representation

Let Σ be a torus immersed into $\mathbf{R}^3$ and conformally equivalent to a flat torus $\mathbf{C}/\Lambda$. Assume that it possesses a global Weierstrass representation which is, by Theorem 2, valid if it is C^3-regular :

1) There exists a function $U(z) : \mathbf{C} \to \mathbf{R}$, the potential of Σ, such that $U(z+\gamma) = U(z)$ for $z \in \mathbf{C}, \gamma \in \Lambda$;

2) There exist ε_1 and ε_2 such that $\varepsilon_j \in \{0,1\}$ and they determine a spinor bundle over $\mathbf{C}/\Lambda$;

3) There exists a function ψ such that $\mathcal{D}\psi = 0$ and it satisfies the periodicity conditions (10).

By Lemma 1, every solution to (1) defines an immersion of a surface into $\mathbf{R}^3$. We consider a "linear basis" for the family of such surfaces, i.e., the set of Floquet functions of the operator (2).

Definition 4 *A function $\psi : \mathbf{C} \to \mathbf{C}$ is called a Floquet function (with zero eigenvalue) of the operator $\mathcal{D}$ (2) with the quasimomenta (k_1, k_2) if $\mathcal{D}\psi = 0$ and*

$$\psi(z+\gamma) = \exp\left(2\pi i (Re\,\gamma \cdot k_1 + Im\,\gamma \cdot k_2)\right)\psi(z) \tag{17}$$

for $\gamma \in \Lambda$.

Notice that a function ψ satisfying (17) has the form

$$\psi(z) = \exp\left(2\pi i (x k_1 + y k_2)\right)\varphi(z) \tag{18}$$

where $\varphi(z)$ is periodic with respect to Λ.

The following lemma is clear.

Lemma 5 *A function ψ of the form (18) satisfies the equation*

$$\left[\begin{pmatrix} 0 & \partial \\ -\bar\partial & 0 \end{pmatrix} + \begin{pmatrix} U & 0 \\ 0 & U \end{pmatrix}\right]\psi = \lambda\psi \tag{19}$$

if and only if

$$\mathcal{D}_k\varphi = \lambda\varphi \tag{20}$$

where

$$\mathcal{D}_k = \begin{pmatrix} 0 & \partial \\ -\bar\partial & 0 \end{pmatrix} + \begin{pmatrix} U & \pi(k_2 + ik_1) \\ \pi(k_2 - ik_1) & U \end{pmatrix}.$$

Take a constant C such that the operator

$$\mathcal{A} = \left[\begin{pmatrix} 0 & \partial \\ -\bar\partial & 0 \end{pmatrix} + \begin{pmatrix} C & 0 \\ 0 & C \end{pmatrix}\right]$$

is invertible on $L_2(\mathbf{C}/\Lambda)$. An existence of such constant is easily verified by using the Fourier transform. Then the equation (20) possesses a solution from $L_2(\mathbf{C}/\Lambda)$ if and only if the equation

$$\left[1 + \begin{pmatrix} U - (C+\lambda) & \pi(k_2 + ik_1) \\ \pi(k_2 - ik_1) & U - (C+\lambda) \end{pmatrix}\begin{pmatrix} C & \partial \\ -\bar\partial & C \end{pmatrix}^{-1}\right]\xi = 0 \tag{21}$$

is solvable in $L_2(\mathbf{C}/\Lambda)$.

Since $\mathcal{A}^{-1}$ increases a smoothness, by the Sobolev embedding theorem, $\mathcal{A}^{-1}$ is compact. Since U is continuous, the operator of multiplication by

$$\begin{pmatrix} U-(C+\lambda) & \pi(k_2+ik_1) \\ \pi(k_2-ik_1) & U-(C+\lambda) \end{pmatrix}.$$

is bounded. Hence, we have

Lemma 6 *A pencil of operators*

$$\mathcal{D}_k \circ \begin{pmatrix} C & \partial \\ -\bar\partial & C \end{pmatrix}^{-1} - 1 =$$

$$\begin{pmatrix} U-(C+\lambda) & \pi(k_2+ik_1) \\ \pi(k_2-ik_1) & U-(C+\lambda) \end{pmatrix} \begin{pmatrix} C & \partial \\ -\bar\partial & C \end{pmatrix}^{-1} \tag{22}$$

is polynomial in k_1, k_2 and λ and consists of compact operators from $L_2(\mathbf{C}/\Lambda)$ to $L_2(\mathbf{C}/\Lambda)$.

Now by using the polynomial Fredholm alternative ([11]) we obtain

Theorem 3 *There exist analytic subsets $\hat\Gamma_U \subset \mathbf{C}^3$ and $\Gamma_U \subset \mathbf{C}^2$ of positive codimensions such that*

1) the equation (21) is solvable in $L_2(\mathbf{C}/\Lambda)$ if and only if $(k_1,k_2,\lambda) \in \hat\Gamma_U$;

2) $\Gamma_U = \hat\Gamma_U \cap \{\lambda = 0\}$.

These subsets are called the Floquet spectrum and the zero Floquet spectrum of $\mathcal{D}$, respectively.

A proof of Theorem 3.

By the Keldysh theorem ([11]), if A_μ is a polynomial, in $\mu \in \mathbf{C}^n$, pencil of compact operators, then the set of μ, for which the equation $(1+A_\mu)\xi = 0$ is solvable, forms an analytic subvariety in $\mathbf{C}^n$.

We are left to prove that these subsets have positive codimensions.

Let $k_1 = \lambda = 0$. Then (21) is equivalent to

$$\left(\begin{pmatrix} C & \partial \\ -\bar\partial & C \end{pmatrix} + \begin{pmatrix} 0 & \pi k_2 \\ \pi k_2 & 0 \end{pmatrix} \begin{pmatrix} 1 & \frac{U-C}{\pi k_2} \\ \frac{U-C}{\pi k_2} & 1 \end{pmatrix} \right) \xi = 0 \tag{23}$$

and as $|k_2| \to \infty$ it degenerates into

$$\left(\begin{pmatrix} 0 & \partial \\ -\bar\partial & 0 \end{pmatrix} + \begin{pmatrix} 0 & \pi k_2 \\ \pi k_2 & 0 \end{pmatrix} \right) \xi = 0.$$

From the last equation by methods of perturbation theory it is inferred that there exists k_2 (with sufficiently large $|k_2|$) for which the equation (23) is not solvable in $L_2(\mathbf{C}/\Lambda)$. This proves Theorem 3.

Kuchment had strengthened the Keldysh theorem for some operators:

There exists an entire function $Z : \mathbf{C}^3 \to \mathbf{C}$ such that the equation (21) is solvable in $L_2(\mathbf{C}/\Lambda)$ if and only if $Z(k_1,k_2,\lambda) = 0$.

Roughly speaking, Z is a regularized determinant of the pencil (22). In [15] this is proved for scalar even order elliptic operators but his reasonings work also for the operator (2).

It follows from (18) that $\hat{\Gamma}_U$ and Γ_U are invariant under the action of the dual lattice Λ^*:

$$k_1 \to k_1 + \mathrm{Re}\,\gamma^*, k_2 \to k_2 + \mathrm{Im}\,\gamma^*, \gamma^* \in \Lambda^*. \tag{24}$$

Remind that Λ^* consists of $\gamma^* \in \mathbf{C}$ such that $(\gamma, \gamma^*) = \mathrm{Re}\,\gamma \cdot \mathrm{Re}\,\gamma^* + \mathrm{Im}\,\gamma \cdot \mathrm{Im}\,\gamma^* \in \mathbf{Z}$ for all $\gamma \in \Lambda$. It is also clear that the definitions of $\hat{\Gamma}_U$ and Γ_U are independent on a choice of a conformal parameter z on the torus.

Definition 5 *A complex surface W_Σ defined as the quotient space of Γ_U for the action (24) is called the Weierstrass spectrum of the torus Σ.*

The genus of the normalization of W_Σ is called the spectral genus of Σ.

This spectrum was introduced by the author who had pointed out its relation to the Willmore functional, i.e., an integral of the squared mean curvature, as the simplest Kruskal integral. For tori of revolution this was analyzed in details in [20].

More effective definition of the Floquet spectrum for multidimensional operators is given by perturbation theory. For the Schrödinger operator and for $\partial_y - \partial_x^2$ this was done by Krichever (see [14], where an approach to define spectral curves by using the Keldysh theorem is also mentioned with a reference to our unpublished paper).

The Willmore functional $\mathcal{W}$ is conformally invariant in the following sense. Let $g : \mathbf{R}^3 \to \mathbf{R}^3$ be a conformal transformation saving a closed surface Σ in a compact domain, then $\mathcal{W}(\Sigma) = \mathcal{W}(g(\Sigma))$. The author had conjectured that the whole Weierstrass spectrum is conformally invariant in this sense. Quite soon after its formulation two different proofs of this conjecture had been obtained.

Pinkall using methods of ([10]) had written the spectral problem $\mathcal{D}\psi = 0$ in conformally invariant terms which implies the conformal invariance of W_Σ.

Grinevich and Schmidt had shown that, since the transformation formulas, for the potential, corresponding to infinitesimal conformal transformations of a surface are quadratic in ψ (it was noticed for tori in revolution in [20]), by an analog of the Melnikov theorem for periodic operators, Γ_U is conformally invariant ([8]).

A relation of the Weierstrass spectrum to the spectral curves of special soliton tori (constant mean curvature, Willmore, see [7, 9, 17]) will be discussed elsewhere.

§4. Finite-zone planes and tori

In the next two paragraphs we somewhere only sketch proofs which are usual for the finite-zone theory ([4, 13]). Moreover for the one-dimensional limit of (2) a derivation of the theta formulas is exposed in [18] and symmetries of Γ are discussed in [3].

Consider more general operator

$$L = \begin{pmatrix} 0 & \partial \\ -\bar{\partial} & 0 \end{pmatrix} + \begin{pmatrix} U & 0 \\ 0 & V \end{pmatrix} \tag{25}$$

with U and V periodic with respect to a rank two lattice $\Lambda \subset \mathbf{C}$. For this operator an analog of Theorem 3 holds and L is called finite-zone (on the zero energy level) if the normalization of its zero Floquet spectrum is a compact Riemann surface with two points removed. But we will call L finite-zone if it is as follows. [2]

[2] It is clear that the equivalence of these definitions for smooth potentials can be justified by perturbation theory ([13]).

Proposition 1 *Let Γ be a compact Riemann surface of genus g, $\infty_\pm$ be a pair of distinct points on Γ, $k_\pm^{-1}$ be local parameters near these points such that $k_\pm^{-1}(\infty_\pm) = 0$, and D be a nonspecial effective divisor of degree $g+1$ on $\Gamma \setminus \{\infty_\pm\}$, i.e., $D = P_1 + \ldots + P_{g+1}$ with $P_i \in \Gamma \setminus \{\infty_\pm\}$. Then*

1. There exists a unique vector function $\psi(z, \bar{z}, P) = (\psi_1, \psi_2)$, with $z \in \mathbf{C}$, such that ψ is meromorphic in P on $\Gamma \setminus \{\infty_\pm\}$ and has poles only in D,

$$\psi = \exp(k_+ z) \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} \xi_{11}^+ / k_+ \\ \xi_{21}^+ / k_+ \end{pmatrix} + O(k_+^{-2}) \right] \quad \text{as } P \to \infty_+, \tag{26}$$

and

$$\psi = \exp(k_- \bar{z}) \left[\begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} \xi_{11}^- / k_- \\ \xi_{21}^- / k_- \end{pmatrix} + O(k_-^{-2}) \right] \quad \text{as } P \to \infty_-. \tag{27}$$

2. Moreover there exists a unique operator L of the form (25) such that $L\psi = 0$. The potentials of L are as follows

$$U = -\xi_{21}^+, \quad V = \xi_{11}^-. \tag{28}$$

This proposition is a particular case of the general theorem on uniqueness of the Baker-Akhieser function ([13]).

Fix a basis $\alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g$ for $H_1(\Gamma)$ such that the intersection form is

$$\alpha_j \circ \beta_k = \delta_{jk}, \quad \alpha_j \circ \alpha_k = \beta_j \circ \beta_k = 0.$$

To this basis corresponds a unique basis of holomorphic differentials $\omega_1, \ldots, \omega_g$ normalized by the condition

$$\int_{\alpha_k} \omega_j = \delta_{jk}.$$

Define now the period matrix Ω and the theta function of Γ as follows

$$\Omega_{jk} = \int_{\beta_k} \omega_j,$$

$$\vartheta(u) = \sum_{N \in \mathbf{Z}^g} \exp \pi i((\Omega N, N) + 2(N, u)),$$

where $u \in \mathbf{C}^g$. Fixing a point $P_0 \in \Gamma \setminus \{\infty_\pm\}$, define also the Abel map from Γ into its Jacobian variety $J(\Gamma) = \mathbf{C}^g / \{M + \Omega N : M, N \in \mathbf{Z}^g\}$:

$$A(P) = (\int_{P_0}^{P} \alpha_1, \ldots, \int_{P_0}^{P} \alpha_g).$$

Denote by $\eta_l^\pm$ a unique meromorphic differential having a single pole at $\infty_\pm$ with the Laurent part $dk_\pm^l$ and normalized by the condition

$$\int_{\alpha_j} \eta_l^\pm = 0 \quad \text{for } j = 1, \ldots, g.$$

To every $\eta_l^\pm$ corresponds the β-period vector $U_l^\pm$:

$$(U_l^\pm)^j = \frac{1}{2\pi i} \int_{\beta_j} \eta_l^\pm.$$

There exist effective divisors $Q_1+\ldots+Q_g$ and $R_1+\ldots+R_g$ such that there are linear equivalences

$$D=P_1+\ldots+P_{g+1} \quad\sim\quad Q_1+\ldots+Q_g+\infty_-,$$

$$D=P_1+\ldots+P_{g+1} \quad\sim\quad R_1+\ldots+R_g+\infty_+.$$

Put $Q_{g+1}=\infty_-$ and $R_{g+1}=\infty_+$ and denote $(A(Q_1)+\ldots+A(Q_g))$ and $(A(R_1)+\ldots A(R_g))$ by $A(Q)$ and $A(R)$, respectively.

Denote by δ the Riemann constants vector defined as follows: for generic $u\in J(\Gamma)$ the multi-valued function $\vartheta(A(P)-u)$ vanishes exactly at g points $S_1,\ldots,S_g$ such that $u+\delta=A(S_1)+\ldots+A(S_g)$. We also take ε, an odd half-period of ϑ , i.e., $\vartheta(\varepsilon)=0$ and $2\varepsilon\equiv 0$ on $J(\Gamma)$.

Proposition 2 *The function ψ from Proposition 1 takes the form*

$$\psi_1(z,\bar z,P)=\exp\left(z(\int_{P_0}^P\eta_1^+-a_1^+)+\bar z(\int_{P_0}^P\eta_1^--b_1^-)\right)\cdot$$

$$\frac{\vartheta(A(P)+F_1(z,\bar z))}{\vartheta(A(P)+\delta-A(Q))}\cdot\frac{\vartheta(A(\infty_+)+\delta-A(Q))}{\vartheta(A(\infty_+)+F_1(z,\bar z))}\cdot \tag{29}$$

$$\cdot\frac{\prod^{g+1}\vartheta(\varepsilon+A(P)-A(Q_j))\cdot\vartheta(\varepsilon+A(\infty_+)-A(P_j))}{\prod^{g+1}\vartheta(\varepsilon+A(P)-A(P_j))\cdot\vartheta(\varepsilon+A(\infty_+)-A(Q_j))},$$

and

$$\psi_2(z,\bar z,P)=\exp\left(z\int_{P_0}^P(\eta_1^+-b_1^+)+\bar z(\int_{P_0}^P\eta_1^--a_1^-)\right)\cdot$$

$$\frac{\vartheta(A(P)+F_2(z,\bar z))}{\vartheta(A(P)+\delta-A(R))}\cdot\frac{\vartheta(A(\infty_-)+\delta-A(R))}{\vartheta(A(\infty_-)+F_2(z,\bar z))}\cdot \tag{30}$$

$$\cdot\frac{\prod^{g+1}\vartheta(\varepsilon+A(P)-A(R_j))\cdot\vartheta(\varepsilon+A(\infty_-)-A(P_j))}{\prod^{g+1}\vartheta(\varepsilon+A(P)-A(P_j))\cdot\vartheta(\varepsilon+A(\infty_-)-A(R_j))},$$

where the constants $a_1^\pm$ and $b_1^\pm$ are defined as follows

$$\int_{P_0}^P\eta_1^\pm-a_1^\pm=k_\pm+O(k_\pm^{-1})\ \textit{near}\ \infty_\pm\quad\textit{and}\quad\int_{P_0}^P\eta_1^\pm-b_1^\pm=O(k_\mp^{-1})\ \textit{near}\ \infty_\mp \tag{31}$$

and

$$F_1(z,\bar z)=U_1^+z+U_1^-\bar z+\delta-A(Q),$$

$$F_2(z,\bar z)=U_1^+z+U_1^-\bar z+\delta-A(R).$$

The potentials U and V take the form

$$U=c_1\exp\left(z(a_1^+-b_1^+)+\bar z(b_1^--a_1^-)\right)\frac{\vartheta(A(\infty_+)+F_2(z,\bar z))}{\vartheta(A(\infty_-)+F_2(z,\bar z))} \tag{32}$$

with

$$c_1=-\frac{\prod^{g+1}\vartheta(\varepsilon+A(\infty_-)-A(P_j))}{\prod^{g+1}\vartheta(\varepsilon+A(\infty_+)-A(P_j))\cdot\vartheta(\varepsilon+A(\infty_-)-A(R_j))}.$$

$$\prod_{j=1}^{g} \vartheta(\varepsilon + A(\infty_+) - A(R_j)) \cdot \sum (U_1^+)^j \frac{\partial \vartheta(\varepsilon)}{\partial u^j}$$

and

$$V = c_2 \exp\left(z(b_1^+ - a_1^+) + \bar{z}(a_1^- - b_1^-)\right) \frac{\vartheta(A(\infty_-) + F_1(z,\bar{z}))}{\vartheta(A(\infty_+) + F_1(z,\bar{z}))} \tag{33}$$

with

$$c_2 = \frac{\prod^{g+1} \vartheta(\varepsilon + A(\infty_+) - A(P_j))}{\prod^{g+1} \vartheta(\varepsilon + A(\infty_-) - A(P_j)) \cdot \vartheta(\varepsilon + A(\infty_+) - A(Q_j))} \cdot$$

$$\prod_{j=1}^{g} \vartheta(\varepsilon + A(\infty_-) - A(Q_j)) \cdot \sum (U_1^-)^j \frac{\partial \vartheta(\varepsilon)}{\partial u^j}.$$

Here we assume that the paths joining P_0 and small neighborhoods of infinities and the paths coming in definition of $A(P)$ and $A(\infty_\pm)$ are the same and for some homotopy classes of them the expansions (31) hold.

A proof of Proposition 2.

The formulas for ψ are verified by using the periodicity properties of theta functions and the Riemann theorem on zeroes of theta functions ([6]).

Derive (32) and (33). For instance, near ∞_- the function ψ_1 decomposes into the product $\vartheta(\varepsilon + A(P) - A(\infty_-)) \cdot H(z,\bar{z},P) \cdot \exp(k_-\bar{z})$ and it is known that $\partial A(P)/\partial k_-^{-1} = U_1^-$ at ∞_-. Hence, we have

$$\xi_{11}^- = H(z,\bar{z},\infty_-) \cdot \sum (U_1^-)^j \frac{\partial \vartheta(\varepsilon)}{\partial u^j}.$$

This proves Proposition 2.

Proposition 3 *Let the spectral data $(\Gamma, \infty_\pm, k_\pm, D)$ of a finite-zone operator L (25) (see Proposition 1) satisfy the following conditions*

i) there exists a holomorphic involution $\sigma : \Gamma \to \Gamma$ such that

i1) $\sigma(\infty_\pm) = \infty_\pm$ and $\sigma(k_\pm) = -k_\pm$;

i2) there exists a meromorphic differential ω on Γ with zeroes at $D + \sigma(D)$ and with two poles at $\infty_\pm$ with the principal parts $(\pm k_\pm^2 + O(k_\pm^{-1}))dk_\pm^{-1}$;

ii) there exists an antiholomorphic involution $\tau : \Gamma \to \Gamma$ such that

ii1) $\tau(\infty_\pm) = \infty_\mp, \tau(k_\pm) = -\bar{k}_\mp$;

ii2) there exists a meromorphic differential $\tilde{\omega}$ on Γ with zeroes at $D + \tau(D)$ and with two poles at $\infty_\pm$ with the principal parts $(k_\pm^2 + O(k_\pm^{-1}))dk_\pm^{-1}$.

Then L takes the form (2) with a real potential U, i.e., $U = V = \bar{U}$.

A proof of Proposition 3.

Consider the meromorphic differential $\psi_1(P)\psi_2(\sigma(P))\omega$ with poles only at $\infty_\pm$. The sum of its residue equals $\xi_{21}^+ + \xi_{11}^-$. Since it vanishes, by (28), this implies $U = V$.

Now consider the differentials $\psi_1(P)\overline{\psi_1(\tau(P))}\tilde{\omega}$ and $\psi_2(P)\overline{\psi_2(\tau(P))}\tilde{\omega}$ and, computing the sums of their residues as above, we conclude $U = \bar{U}$ and $V = \bar{V}$.

This proves the proposition.

Proposition 4 *1) Given a spectral data $(\Gamma, \infty_\pm, k_\pm, D)$ meeting the conditions of Proposition 3, for any n-tuple of points $Q_1, \dots, Q_n \in \Gamma \setminus \{\infty_\pm, P_1, \dots, P_g\}$ and any*

n-tuple of constants $a_1, \dots, a_n \in \mathbf{C}$ the vector function $\psi(z,\bar z) = a_1\psi(z,\bar z,Q_1) + \dots + a_n\psi(z,\bar z,Q_n)$ defines via (3) a surface immersed into $\mathbf{R}^3$.

2) If $\psi(z+\gamma) = \pm\psi(z)$ for $z \in \mathbf{C}$ and $\gamma \in \Lambda$, with $\Lambda \subset \mathbf{C}$ a rank two lattice, then the immersion converts into an immersion of the torus $\mathbf{C}/\Lambda$ if and only if

$$\int_{\mathbf{C}/\Lambda} \psi_1^2 dz \wedge d\bar z = \int_{\mathbf{C}/\Lambda} \psi_2^2 dz \wedge d\bar z = \int_{\mathbf{C}/\Lambda} \psi_1 \bar\psi_2 dz \wedge d\bar z = 0. \tag{34}$$

A proof of Proposition 4.

The first statement is evident. The second statement is quite clear for tori of revolution (see [20]) and has been extended onto the general case by M. Schmidt.

Let $\hat\gamma$ and $\tilde\gamma$ be generators of Λ with the basis $(\hat\gamma,\tilde\gamma)$ is positively oriented. By (3), $X^1 + iX^2 = X^1 + i\bar X^2 = i\int(\bar\psi_2^2 dz - \bar\psi_1^2 d\bar z)$. We have $d(x(\bar\psi_2^2 dz - \bar\psi_1^2 d\bar z)) = i(\bar\psi_1^2 + \bar\psi_2^2) dx \wedge dy$ and $d(y(\bar\psi_2^2 dz - \bar\psi_1^2 d\bar z)) = (\bar\psi_1^2 - \bar\psi_2^2) dx \wedge dy$. The Stokes theorem implies that

$$\int_{\mathbf{C}/\Lambda} (\bar\psi_1^2 + \bar\psi_2^2) dz' \wedge d\bar z' = -2(\mathrm{Re}\,\hat\gamma \int_z^{z+\tilde\gamma} (\bar\psi_2^2 dz' - \bar\psi_1^2 d\bar z') - \mathrm{Re}\,\tilde\gamma \int_z^{z+\hat\gamma} (\bar\psi_2^2 dz' - \bar\psi_1^2 d\bar z'))$$

and

$$\int_{\mathbf{C}/\Lambda} (\bar\psi_1^2 - \bar\psi_2^2) dz' \wedge d\bar z' = 2i(\mathrm{Im}\,\hat\gamma \int_z^{z+\tilde\gamma} (\bar\psi_2^2 dz' - \bar\psi_1^2 d\bar z') - \mathrm{Im}\,\tilde\gamma \int_z^{z+\hat\gamma} (\bar\psi_2^2 dz' - \bar\psi_1^2 d\bar z')).$$

Therefore $X^1 + iX^2$ is Λ-periodic if and only if the first two equalities from (34) hold. The equivalence of the last one to Λ-periodicity of X^3 is proven in the same manner. This proves the proposition [3].

§5. Finite-zone solutions to the modified Novikov–Veselov equations

The modified Novikov–Veselov (mNV) equations are related to the operator $\mathcal{D}$ (3) and take the form

$$\mathcal{D}_t = \mathcal{D}A + B\mathcal{D} \tag{35}$$

(the Manakov triple). The deformations of U generate the deformations of ψ, the zero-eigenfunction of $\mathcal{D}$, of the form

$$\psi_t = A\psi. \tag{36}$$

These equations had been introduced by Bogdanov in [2] and it is an observation of Konopelchenko that if a surface is defined by (3) then (35) generates via (36) a local deformation of the surface ([12]).

The first equation of this hierarchy is

$$U_t = (U_{zzz} + 3U_z V + \frac{3}{2}UV_z) + (U_{\bar z\bar z\bar z} + 3U_{\bar z}\bar V + \frac{3}{2}U\bar V_{\bar z})$$

with $V_{\bar z} = (U^2)_z$. If U depends only on one spatial variable then these equations reduce to the modified Korteweg–de Vries equations.

[3] The constructed surfaces may have singularities which are exactly at points where $|\psi_1|^2 + |\psi_2|^2 = 0$. For C^2-regular surfaces the periodicity conditions (34) are quite perspequitive because, by (7) and (8), $\psi_1^2 = -\partial(X^2 + iX^1)$, $\psi_2^2 = \bar\partial(X^2 + iX^1)$, and $\psi_1\bar\psi_2 = \partial X^3$ globally.

Proposition 5 *Let the spectral data* $(\Gamma, \infty_\pm, k_\pm, D)$ *are as in Proposition 3. Define the constants* $a_l^\pm$ *and* $b_l^\pm$ *by the following analogs of (31)*

$$\int_{P_0}^P \eta_l^\pm - a_l^\pm = k_\pm^l + O(k_\pm^{-1}) \text{ near } \infty_\pm \text{ and } \int_{P_0}^P \eta_l^\pm - b_l^\pm = O(k_\mp^{-1}) \text{ near } \infty_\mp$$

and define $F_{1l}(z,\bar z,t_l)$ *and* $F_{2l}(z,\bar z,t_l)$ *by*

$$F_{1l}(z,\bar z,t_l) = U_1^+ z + U_1^- \bar z + (U_{2l+1}^+ + U_{2l+1}^-)t_l + \delta - A(Q),$$

$$F_{2l}(z,\bar z,t_l) = U_1^+ z + U_1^- \bar z + (U_{2l+1}^+ + U_{2l+1}^-)t_l + \delta - A(R).$$

Let $\hat\psi$ *be a vector function obtained from (29) and (30) by replacing* F_1 *and* F_2 *by* F_{1l} *and* F_{2l} *and by adding* $t_l(\int_{P_)}^P(\eta_{2l+1}^+ + \eta_{2l+1}^-) - (a_{2l+1}^+ + b_{2l+1}^-))$ *and* $t_l(\int_{P_)}^P(\eta_{2l+1}^+ + \eta_{2l+1}^-) - (a_{2l+1}^- + b_{2l+1}^+))$ *to the arguments of exponents in (29) and (30). Then* $\mathcal{D}\hat\psi = 0$ *where* $U(z,\bar z,t_l)$ *is constructed by* $\hat\psi$ *via (28) and satisfies the* l*-th equation of the mNV hierarchy.*

The formula for $U(z,\bar z,t_l)$ is derived by the same substitutions.

The proof of this proposition is analogous to one for the Novikov–Veselov equation ([23]) as well as Proposition 3 is an analog of the Novikov–Veselov theorem distinguishing potential finite-zone Schrödinger operators ([22]). But there is one important difference. The form ω from Proposition 3 is not invariant under σ but antiinvariant, i.e., $\sigma^*(\omega) = -\omega$, and this implies the following. Finite-zone potentials can be written in terms of Prym theta functions of the covering $\Gamma \to \Gamma/\sigma$ but we cannot control the topological type of involution σ and cannot conclude that for nonsingular Γ it has only two fixed points. We can only say that

$$\text{genus}(\Gamma) - \text{genus}(\Gamma/\sigma) = \dim \text{Prym}(\Gamma,\sigma) \geq [\frac{\text{genus}(\Gamma)}{2}]. \qquad (37)$$

§6. The Willmore functional

The following conjecture of Willmore is well-known

For tori immersed into $\mathbf{R}^3$ *the minimum of the Willmore functional*

$$\mathcal{W}(\Sigma) = \int_\Sigma H^2 d\mu,$$

with $d\mu$ *an induced Liouville measure, equals* $2\pi^2$ *and is attained on the Clifford torus and its images under conformal transformations of* $\mathbf{R}^3$.

The Clifford torus is obtained by revolving of a circle of radius 1 around the axis lying in the same plane as the circle at distance $\sqrt{2}$ from the circle center.

Until now this conjecture is proved only in some particular cases (see the survey of them in [19]).

By (4), for a torus $\Sigma \subset \mathbf{R}^3$ the simplest Kruskal integral

$$4\int_{\mathbf{C}/\Lambda} U^2(z,\bar z)dxdy,$$

where $U : \mathbf{C}/\Lambda \to \mathbf{C}$ is the potential of its Weierstrass representation, coincides with $\mathcal{W}(\Sigma)$ ([19]). This leads us to the conjecture that *for fixed conformal classes of tori the minima of $\mathcal{W}$ are attained on tori with minimal spectral genus.*

This fits into the Willmore conjecture because the spectral genus of the Clifford torus equals zero (this is derived from its Weierstrass representation found in [20]). Moreover, it is difficult to imagine that the minimum of such variational problem is degenerated. In [19] it was conjectured that the minima of this functional are stationary with respect to the deformations generated by the first mNV equation for which it is shown in ([19]) that this deformation preserves tori. We may extend that as follows

for all equations of the mNV hierarchy the minima of $\mathcal{W}$ for fixed conformal classes are stationary with respect to induced deformations.

This implies that the Prym variety of $\Gamma \to \Gamma/\sigma$ would be one-dimensional and the mNV deformations reduce to translations of tori along themselves. By (37), this implies genus$(\Gamma) \leq 3$. For surfaces of genus 3 the dimension of the Prym variety equals 3 if σ is a hyperelliptic involution or 2 if σ has 4 fixed points (the case when this dimension equals 1 correspond to an involution without fixed points). Hence, the last conjecture implies that for minima genus$(\Gamma) \leq 2$.

The Weierstrass representation gives a physical explanation for lower bounds for $\mathcal{W}$: it is clear that for small perturbations of the zero-potential $U = 0$ the surfaces constructed in terms of zero-eigenfunctions of (2) do not convert into tori and, since for U the Willmore functional is its L_2-norm, the lower bound reflects how big a perturbation has to be to get planes converted into tori.

Final remark.

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