Discrete surfaces of constant mean curvature via dressing

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1 Introduction

The last ten years have seen large progress in the investigation of surfaces of constant mean curvature. In particular, for CMC-surfaces without umbilics many important new results were obtained. Starting with the work of Wente [20], CMC-tori were first constructed and then classified in terms of algebraic geometric data [1, 18, 2, 10, 14].

In the last three years, Bobenko and Pinkall introduced the notion of discrete CMC-surfaces [3, 4]. They showed, that these surfaces, defined by elementary geometric properties, are related to the solutions of an integrable discretization of the elliptic sinh-Gordon equation in the same way, as usual CMC-surfaces without umbilics are related to solutions of the sinh-Gordon equation itself.

This connection was established using the Lax representation of the sinh-Gordon equation, which is also at the starting point of the dressing method for integrable systems. Consequently, Pedit and Wu [17] gave a method to construct discrete CMC-surfaces, i.e., solutions of the discretized sinh-Gordon equation, by dressing of a discretized vacuum solution, a discrete cylinder.

For general conformal CMC-immersions, the investigation of symmetries of the CMC-surface in $\mathbb{R}^3$ was started in [6]. There it was also shown, that, with the exception of cylinders, there are no periodic surfaces in the dressing orbit of the standard cylinder.

On the other hand, in [9] it was shown, that there is a more general definition of a dressing action, which gives all solutions of finite type from the standard cylinder. In particular, all CMC-tori, i.e., doubly periodic CMC-surfaces, are in this generalized dressing orbit. In [5], this was used to give an alternative derivation of Pinkall and Sterling's classification of CMC tori.

Since the surfaces which were constructed by Pedit and Wu are all in the dressing orbit of a discretized version of the standard cylinder, it seemed natural to adapt the discussion of [6, 5] to discrete CMC-surfaces. This is the goal of this paper.

We start in Section 2 with the definition of the discrete (standard) cylinder as an example of a discrete CMC-surface in the sense of Bobenko and Pinkall [3, 4] (Definition 2.4). This will also lead to the introduction of so called extended frames for discrete CMC-surfaces. These definitions will describe the discrete analogues of CMC-surfaces without umbilics in an isothermic parametrization. Then we introduce the dressing action on the cylinder to generate a large class of such discrete CMC-surfaces along the lines of [17].

In Section 3, we will introduce and investigate the notion of a symmetry of a discrete CMC-surface. This definition is modelled after the definition of translational symmetries for continuous CMC-surfaces as given in [5]. We will compute the transformation properties of several geometric quantities related to the discrete CMC-surface under a symmetry. The most important result here is Theorem 3.3 which gives the transformation equation of the extended frame of a discrete CMC-surface.

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In Section 4 we will investigate the periodicity conditions for discrete CMC-surfaces in more detail. We will characterize all discrete, periodic CMC-surfaces in the generalized dressing orbit of the standard cylinder in terms of rational functions, similar to [5, Theorems 3.6,3.7].

In Section 5, we will introduce a hyperelliptic Riemann surface, which allows us, as in the continuous case, to construct discrete periodic surfaces from algebro-geometric data.

2 The r-dressing orbit of the discretized cylinder

In this section we will define the discrete cylinder, the starting point of our investigations, and its dressing orbit. This section follows [17] closely.

2.1 Let us recall the formula for the standard cylinder (see [6, Section 3.7]): The extended frame of the standard cylinder is given by

$$F(z,\overline{z},\lambda) = e^{(\lambda^{-1}z-\lambda\overline{z})A} = e^{x(\lambda^{-1}-\lambda)A+iy(\lambda^{-1}+\lambda)A}, \qquad \lambda\in S^1, \tag{2.1.1}$$

where we have used complex coordinates $z = x+iy$, $\overline{z} = x-iy$. From this definition it follows, that

$$U(x,y,\lambda) = F^{-1}\partial_x F = (\lambda^{-1}-\lambda)A \tag{2.1.2}$$
$$V(x,y,\lambda) = F^{-1}\partial_y F = i(\lambda+\lambda^{-1})A. \tag{2.1.3}$$

The Euler method for the discretization of continuous dynamical systems suggests the following discretization of the cylinder: Define the matrices $U^\circ, V^\circ \in \mathbf{SU}(2)_\sigma$ by

$$U^\circ(\lambda) = \frac{1}{\Delta_+}(I + r_1(\lambda^{-1}-\lambda)A), \tag{2.1.4}$$
$$V^\circ(\lambda) = \frac{1}{\Delta_-}(I + ir_2(\lambda^{-1}+\lambda)A), \tag{2.1.5}$$

where r_1 and r_2 are real, positive constants, $r_1, r_2 \in \mathbb{R}^+$, and ($\lambda\in S^1$)

$$\Delta_+ = \sqrt{\det(I + r_1(\lambda^{-1}-\lambda)A)} = \sqrt{1 - r_1^2(\lambda^{-1}-\lambda)^2}, \tag{2.1.6}$$
$$\Delta_- = \sqrt{\det(I + ir_2(\lambda^{-1}+\lambda)A)} = \sqrt{1 + r_2^2(\lambda^{-1}+\lambda)^2}. \tag{2.1.7}$$

Here, we fix Δ_+ and Δ_- by requiring, that they take positive real values on S^1. This is possible, since for $\lambda = e^{i\theta}\in S^1$ we have

$$\Delta_+ = \sqrt{1+4r_1^2\sin^2\theta}\in\mathbb{R}, \qquad \Delta_+\geq 1, \tag{2.1.8}$$
$$\Delta_- = \sqrt{1+4r_2^2\cos^2\theta}\in\mathbb{R}, \qquad \Delta_-\geq 1. \tag{2.1.9}$$

From this we immediately get

Proposition: *For each pair $r_1, r_2 \in \mathbb{R}^+$, the functions Δ_+ and Δ_- defined above are even in λ and real and positive on S^1.*

The functions $\Delta_+^2(\lambda)$ and $\Delta_-^2(\lambda)$ are rational and can therefore be extended meromorphically to $\mathbb{C}\mathrm{P}_1$.

Lemma: *For each real constant $r_1 > 0$, Δ_+^2 has precisely four simple zeores, which are located at λ_+, $-\lambda_+$, λ_+^{-1} and $-\lambda_+^{-1}$ on the real axis, where*

$$\lambda_+ = \frac{1}{2r_1} + \sqrt{1+\frac{1}{4r_1^2}} > 1. \tag{2.1.10}$$

For each real constant $r_2 > 0$, Δ_-^2 has precisely four simple zeores, which are located at $i\lambda_-$, $-i\lambda_-$, $i\lambda_-^{-1}$ and $-i\lambda_-^{-1}$ on the imaginary axis, where

$$\lambda_- = \frac{1}{2r_2} + \sqrt{1 + \frac{1}{4r_2^2}} > 1, \tag{2.1.11}$$

In the limits we have

$$\begin{aligned} \lim_{r_1 \to 0} \lambda_+ &= \infty, \qquad \lim_{r_1 \to \infty} \lambda_+ = 1, \\ \lim_{r_2 \to 0} \lambda_- &= \infty, \qquad \lim_{r_2 \to \infty} \lambda_- = 1. \end{aligned} \tag{2.1.12}$$

Proof: Both Δ_+^2 and Δ_-^2 are rational with numerators of degree 4, i.e., they can have at most 4 zeroes with multiplicity. If $\lambda_+ \in \mathbb{C}$ is a zero of $\Delta_+^2 = 1 - r_1^2(\lambda^{-1} - \lambda)^2$, then obviously also $-\lambda_+$, λ_+^{-1} and $-\lambda_+^{-1}$ are zeroes of Δ_+^2. If λ_+ is real and different from 1, then these zeroes are real and distinct. A direct calculation using

$$\Delta_+^2 = (1 + r_1(\lambda^{-1} - \lambda))(1 - r_1(\lambda^{-1} - \lambda)) \tag{2.1.13}$$

shows, that Eq. (2.1.10) gives a zero of Δ_+^2 on the real axis. In the same way, the statement for Δ_-^2 is proved using

$$\Delta_-^2 = (1 + ir_2(\lambda^{-1} + \lambda))(1 - ir_2(\lambda^{-1} + \lambda)). \tag{2.1.14}$$

The limits (2.1.12) follow directly from Eqs. (2.1.10) and (2.1.11). □

2.2 As in the continuous case (see e.g. [5, Section 2.2]), we will interprete the λ-dependent matrices as taking values in a certain loop group.

For each real constant r, $0 < r < 1$, let $\Lambda_r \mathbf{SL}(2,\mathbb{C})_\sigma$ denote the group of smooth maps $g(\lambda)$ from C_r, the circle of radius r, to $\mathbf{SL}(2,\mathbb{C})$, which satisfy the twisting condition

$$g(-\lambda) = \sigma(g(\lambda)), \tag{2.2.1}$$

where $\sigma : \mathbf{SL}(2,\mathbb{C}) \to \mathbf{SL}(2,\mathbb{C})$ is the group automorphism of order 2, which is given by conjugation with the Pauli matrix

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.2.2}$$

The Lie algebras of these groups, which we denote by $\Lambda_r \mathbf{sl}(2,\mathbb{C})_\sigma$, consist of maps $x : C_r \to \mathbf{sl}(2,\mathbb{C})$, which satisfy a similar twisting condition as the group elements

$$x(-\lambda) = \sigma_3 x(\lambda) \sigma_3. \tag{2.2.3}$$

In order to make these loop groups complex Banach Lie groups, we equip them, as in [8], with some H^s-topology for $s > \frac{1}{2}$. Elements of these twisted loop groups are matrices with off-diagonal entries which are odd functions, and diagonal entries which are even functions in the parameter λ. All entries are in the Banach algebra $\mathcal{A}_r$ of H^s-smooth functions on C_r.

Furthermore, we will use the following subgroups of $\Lambda_r \mathbf{SL}(2,\mathbb{C})_\sigma$: Let $\mathbf{B}$ be a subgroup of $\mathbf{SL}(2,\mathbb{C})$ and $\Lambda_{r,B}^+ \mathbf{SL}(2,\mathbb{C})_\sigma$ be the group of maps in $\Lambda_r \mathbf{SL}(2,\mathbb{C})_\sigma$, which can be extended to holomorphic maps on

$$I^{(r)} = \{\lambda \in \mathbb{C}; |\lambda| < r\}, \tag{2.2.4}$$

the interior of the circle C_r, and take values in $\mathbf{B}$ at $\lambda = 0$. Analogously, let $\Lambda_{r,B}^- \mathbf{SL}(2,\mathbb{C})_\sigma$ be the group of maps in $\Lambda_r \mathbf{SL}(2,\mathbb{C})_\sigma$, which can be extended to the exterior

$$E^{(r)} = \{\lambda \in \mathbb{C}\mathrm{P}_1; |\lambda| > r\} \tag{2.2.5}$$

of C_r and take values in $\mathbf{B}$ at $\lambda = \infty$. If $\mathbf{B} = \{I\}$ (based loops) we write the subscript $*$ instead of $\mathbf{B}$, if $\mathbf{B} = \mathbf{SL}(2,\mathbb{C})$ we omit the subscript for Λ entirely.

Also, by an abuse of notation, we will denote by $\Lambda_r\mathbf{SU}(2)_\sigma$ the subgroup of maps in $\Lambda_r\mathbf{SL}(2,\mathbb{C})_\sigma$, which can be extended holomorphically to the open annulus

$$A^{(r)} = \{\lambda \in \mathbb{C}; r < |\lambda| < \frac{1}{r}\} \tag{2.2.6}$$

and take values in $\mathbf{SU}(2)$ on the unit circle. As usual, we will set

$$\Lambda\mathbf{SU}(2)_\sigma = \bigcup_{0<r<1} \Lambda_r\mathbf{SU}(2)_\sigma. \tag{2.2.7}$$

Corresponding to these subgroups, we analogously define Lie subalgebras of $\Lambda_r\mathbf{sl}(2,\mathbb{C})_\sigma$.

We quote the following results from [15] and [8]:

(i) For each solvable subgroup $\mathbf{B}$ of $\mathbf{SL}(2,\mathbb{C})$, which satisfies $\mathbf{SU}(2)\cdot\mathbf{B} = \mathbf{SL}(2,\mathbb{C})$ and $\mathbf{SU}(2)\cap\mathbf{B} = \{I\}$, multiplication

$$\Lambda_r\mathbf{SU}(2)_\sigma \times \Lambda^+_{r,B}\mathbf{SL}(2,\mathbb{C})_\sigma \longrightarrow \Lambda_r\mathbf{SL}(2,\mathbb{C})_\sigma$$

is a diffeomorphism onto. The associated splitting

$$g = Fg_+ \tag{2.2.8}$$

of an element g of $\Lambda_r\mathbf{SL}(2,\mathbb{C})_\sigma$, s.t. $F \in \Lambda_r\mathbf{SU}(2)_\sigma$ and $g_+ \in \Lambda^+_{r,B}\mathbf{SL}(2,\mathbb{C})_\sigma$ will be called Iwasawa decomposition. In the following, we will fix the group $\mathbf{B}$ as the group of upper triangular 2×2-matrices with real positive entries on the diagonal.

(ii) Multiplication

$$\Lambda^-_{r,*}\mathbf{SL}(2,\mathbb{C})_\sigma \times \Lambda^+_r\mathbf{SL}(2,\mathbb{C})_\sigma \longrightarrow \Lambda_r\mathbf{SL}(2,\mathbb{C})_\sigma$$

is a diffeomorphism onto the open and dense subset $\Lambda^-_{r,*}\mathbf{SL}(2,\mathbb{C})_\sigma \cdot \Lambda^+_r\mathbf{SL}(2,\mathbb{C})_\sigma$ of $\Lambda_r\mathbf{SL}(2,\mathbb{C})_\sigma$, called the "big cell" [19]. The associated splitting

$$g = g_-g_+ \tag{2.2.9}$$

of an element g of the big cell, where $g_- \in \Lambda^-_{r,*}\mathbf{SL}(2,\mathbb{C})_\sigma$ and $g_+ \in \Lambda^+_r\mathbf{SL}(2,\mathbb{C})_\sigma$, will be called Birkhoff factorization.

Proposition: *Let $r_1, r_2 \in \mathbb{R}^+$ and define*

$$0 < r_{\min}(r_1,r_2) = \max(\lambda_+^{-1}, \lambda_-^{-1}) < 1. \tag{2.2.10}$$

For each $r_{\min}(r_1,r_2) < r < 1$, the matrices U° and V° defined in (2.1.4) and (2.1.5) are in the twisted loop group $\Lambda_r\mathbf{SU}(2)_\sigma$. In particular, they can be extended holomorphically to $A^{(r_{\min})} = \{\lambda \in \mathbb{C}\mathrm{P}_1 |\ r_{\min} < |\lambda| < \frac{1}{r_{\min}}\}$.

Proof: It is clear, that

$$\det U^\circ(\lambda) = \det V^\circ(\lambda) = 1. \tag{2.2.11}$$

By Proposition 2.1, Δ_+ and Δ_- are real and do not vanish for $\lambda \in S^1$. Thus, U° and V° take values in $\mathbf{SU}(2)$ for $\lambda \in S^1$. Since Δ_+U° and Δ_-V° can be extended holomorphically to $\mathbb{C}^* = \mathbb{C}\setminus\{0\}$ and since, by Lemma 2.1, Δ_+^2 and Δ_-^2 have no zeroes in $A^{(r_{\min})}$, where $r_{\min}(r_1,r_2)$ is given by Eq. (2.2.10), we get, that U° and V° can be extended holomorphically to $A^{(r_{\min})}$. In addition, since

Δ_+ and Δ_- are even in λ, we have, that U° and V° satisfy the twisting condition (2.2.1). Thus, for each $r_{\min}(r_1, r_2) < r < 1$, U° and V° are in $\Lambda_r \mathbf{SU}(2)_\sigma$. □

2.3 We will define a map $F^\circ : \mathbb{Z}^2 \to \Lambda\mathbf{SU}(2)_\sigma$ by

$$F^\circ_{mn}(\lambda) = U^\circ(\lambda)^m V^\circ(\lambda)^n. \tag{2.3.1}$$

By Proposition 2.2 and since U° and V° commute, for all $r_{\min}(r_1, r_2) < r < 1$, F° is a well defined map from $\mathbb{Z}^2$ to $\Lambda_r \mathbf{SU}(2)_\sigma$. Using Sym's formula, as in the continuous case, we get a "discrete surface" $\Psi^\circ_{mn} : \mathbb{Z}^2 \to \mathbb{R}^3$ in the spinor representation $J : \mathbb{R}^3 \to \mathbf{su}(2)$, $\mathbf{r} \mapsto -\frac{i}{2}\boldsymbol{\sigma}\mathbf{r}$:

$$J(\Psi^\circ_{mn}) = \frac{\partial}{\partial\theta}|_{\theta=0} F^\circ_{mn}(\lambda = e^{i\theta}) \cdot F^\circ_{mn}(1)^{-1} + \frac{i}{2} F^\circ_{mn}(1)\sigma_3 F^\circ_{mn}(1)^{-1}. \tag{2.3.2}$$

2.4 For each $r_1, r_2 \in \mathbb{R}^+$, the discretized cylinder provides an example of a discrete CMC-surface, as was shown in [17]. Let us define, what we understand by a discrete CMC-surface:

Definition: Let Ψ_{mn} be a map from $\mathbb{Z}^2$ into $\mathbb{R}^3$. The map Ψ_{mn} is called a *discrete CMC-surface* iff the following holds: There exists a map F_{mn} from $\mathbb{Z}^2$ into $\Lambda\mathbf{SU}(2)_\sigma$, s.t. for two real positive constants r_1, r_2:

$$U_{mn}(\lambda) = F^{-1}_{mn}F_{m+1,n} = \frac{1}{\Delta_+}\begin{pmatrix} \alpha_{mn} & \lambda^{-1}r_1 p_{mn} - \lambda r_1 p^{-1}_{mn} \\ \lambda^{-1}r_1 p^{-1}_{mn} - \lambda r_1 p_{mn} & \overline{\alpha}_{mn} \end{pmatrix}, \tag{2.4.1}$$

$$V_{mn}(\lambda) = F^{-1}_{mn}F_{m,n+1} = \frac{1}{\Delta_-}\begin{pmatrix} \beta_{mn} & i\lambda^{-1}r_2 q_{mn} + i\lambda r_2 q^{-1}_{mn} \\ i\lambda^{-1}r_2 q^{-1}_{mn} + i\lambda r_2 q_{mn} & \overline{\beta}_{mn} \end{pmatrix}, \tag{2.4.2}$$

where p_{mn}, q_{mn} are positive real constants, α_{mn} and β_{mn} are complex constants, and Ψ_{mn} is given in terms of F_{mn} by Sym's formula (2.3.2). The constants r_1, r_2 will be called *lattice constants*. The set of maps $F_{mn} : \mathbb{Z}^2 \to \Lambda\mathbf{SU}(2)_\sigma$, for which U_{mn} and V_{mn} are of the form (2.4.1) and (2.4.2), respectively, will be denoted by $\mathcal{F}(r_1, r_2)$. Its elements will be called *extended frames*.

Clearly, for each $r_1, r_2 \in \mathbb{R}^+$, the map F° is in $\mathcal{F}(r_1, r_2)$ with $p_{mn} = q_{mn} = 1$. We call the associated CMC-immersion Ψ° the standard cylinder in $\mathcal{F}(r_1, r_2)$.

Remark: It can easily be calculated (see (3.1.3)), that the edge vectors $L^\circ_{mn} = J(\Psi^\circ_{m+1,n} - \Psi^\circ_{mn})$ are independent of m, n. Thus, $\Psi^\circ : \mathbb{Z}^2 \to \mathbb{R}^3$ is a discrete analogue of a ruled surface which is generated by parallel lines, i.e., a discrete analogue of a cylinder over a plane curve. However, the reader should be aware that, unlike the continuous case, for general lattice constants r_1, r_2, the standard cylinder will not close up in one direction in the parameter space.

The motivation for Definition 2.4 comes from the following

Theorem: *Let $\Psi_{mn} : \mathbb{Z}^2 \to \mathbb{R}^3$ be a discrete CMC-surface with extended frame F_{mn}. Let us define U_{mn} and V_{mn} as in (2.4.1) and (2.4.2). Then for α_{mn} and β_{mn} we get*

$$|\alpha_{mn}|^2 = 1 - r_1^2(p_{mn} - p^{-1}_{mn})^2, \tag{2.4.3}$$

$$|\beta_{mn}|^2 = 1 - r_2^2(q_{mn} - q^{-1}_{mn})^2. \tag{2.4.4}$$

Furthermore, $p = p_{mn}$, $q = q_{mn}$, $\alpha = \alpha_{mn}$, $\beta = \beta_{mn}$, $p' = p_{m,n+1}$, $\alpha' = \alpha_{m,n+1}$, $q' = q_{m+1,n}$ and $\beta' = \beta_{m+1,n}$ satisfy

$$pp' = qq' \tag{2.4.5}$$

$$\left(\frac{q}{p} + \frac{p}{q}\right)\alpha' = i\frac{r_1}{r_2}\left(\frac{p}{p'} - \frac{p'}{p}\right)\overline{\beta} + \left(\frac{q'}{p} + \frac{p}{q'}\right)\overline{\alpha}, \tag{2.4.6}$$

$$\left(\frac{q}{p} + \frac{p}{q}\right)\beta' = i\frac{r_2}{r_1}\left(\frac{q'}{q} - \frac{q}{q'}\right)\overline{\alpha} + \left(\frac{q'}{p} + \frac{p}{q'}\right)\overline{\beta}, \tag{2.4.7}$$

and the discrete sinh-*Gordon equation*

$$\alpha'\beta - \beta'\alpha = ir_1r_2(p'q + q'p - (p'q)^{-1} - (q'p)^{-1}). \tag{2.4.8}$$

Proof: Eqs. (2.4.3) and (2.4.4) follow from $\det U_{mn} = \det V_{mn} = 1$. Eqs. (2.4.5)–(2.4.8) are obtained by a direct calculation from the integrability condition

$$U_{mn}V_{m+1,n} = V_{mn}U_{m,n+1}. \tag{2.4.9}$$

□

Remark: 1. By (2.4.5) we can define

$$p_{mn} = e^{-\frac{1}{2}(\omega_{mn}+\omega_{m+1,n})}, \tag{2.4.10}$$

$$q_{mn} = e^{-\frac{1}{2}(\omega_{mn}+\omega_{m,n+1})}. \tag{2.4.11}$$

The equations (2.4.6), (2.4.7) and (2.4.8) then transform into those of Pedit and Wu [17, Theorem 4.1] for the real variables ω_{mn}. Bobenko and Pinkall first used the form of the Lax operators (2.4.1),(2.4.2) to arrive at the integrable discretization (2.4.8) of the sinh-Gordon equation and to derive the elementary geometric properties of the discretized CMC-surfaces.

2. It should be noted that in the discrete case there is no natural definition of an associated family of discrete CMC-surfaces. The reason for this loss of structure, compared to the continuous case, is the fact that discrete CMC-surfaces are actually discrete isothermic CMC-surfaces [4]. In the continuous case the parametrization is isothermic only for real Hopf differential (we only consider constant Hopf differential, i.e., no umbilics). If we choose $E \equiv 1$, as it was done in the definition of the standard cylinder above, then these surfaces are the ones corresponding to $\lambda = \pm 1$ (see the appendix of [7]). Consequently, in the discrete case as we present it here, the Hopf differential is always normalized to $E \equiv 1$, and the associated family is reduced to the two surfaces for $\lambda = 1$ and $\lambda = -1$, which are congruent.

Lemma: *For $r_1, r_2 \in \mathbb{R}^+$, let $F_{mn} \in \mathcal{F}(r_1, r_2)$ be an extended frame. Define $U_{mn}(\lambda)$ and $V_{mn}(\lambda)$ by (2.4.1),(2.4.2) and $r_{\min}(r_1, r_2)$ by (2.2.10). If*

$$F_{00}(\lambda) = I \text{ for all } \lambda \in S^1, \tag{2.4.12}$$

then for all $(m, n) \in \mathbb{Z}^2$,

1. *F_{mn}, U_{mn} and V_{mn} can be extended holomorphically to $A^{(r_{\min})}$,*
2. *$\Delta_+ U_{mn}$, $\Delta_- V_{mn}$ and $\Delta_+^{|m|}\Delta_-^{|n|}F_{mn}$ can be extended holomorphically to $\mathbb{C}^*$.*

Proof: We prove first the second statement: By (2.4.1) and (2.4.2), $\Delta_+ U_{mn}$ and $\Delta_- V_{mn}$ can be continued holomorphically to $\mathbb{C}^*$ and the same holds for $\Delta_+ U_{mn}^{-1}$ and $\Delta_- V_{mn}^{-1}$. From this and the initial condition (2.4.12) it follows by induction in m and n, that also $\Delta_+^{|m|}\Delta_-^{|n|}F_{mn}$ can be extended holomorphically to $\mathbb{C}^*$, which proves the second statement.

The first statement ist now a simple consequence of the fact, that Δ_+ and Δ_- have no zeroes on $A^{(r_{\min})}$, i.e., Δ_+^{-1} and Δ_-^{-1} can be extended holomorphically to $A^{(r_{\min})}$. □

Definition: An extended frame $F_{mn} \in \mathcal{F}(r_1, r_2)$ will be called *normalized*, if it satisfies (2.4.12). The subset of normalized extended frames in $\mathcal{F}(r_1, r_2)$ will be denoted by $\mathcal{F}_0(r_1, r_2)$.

We will show in Section 3.3, that w.l.o.g. we can restrict our attention to normalized extended frames.

Lemma 2.4 shows, that for arbitrary $0 < r_{\min}(r_1, r_2) < r < 1$ and $(m, n) \in \mathbb{Z}^2$, U_{mn} and V_{mn} as well as the normalized extended frame $F_{mn}(\lambda)$ are in $\Lambda_r \mathbf{SU}(2)_\sigma$.

2.5 For later use we introduce the following antiholomorphic involution

$$\tau : \lambda \mapsto \overline{\lambda}^{-1}. \tag{2.5.1}$$

Geometrically speaking, τ is the reflection at the unit circle in $\mathbb{C}\mathrm{P}_1$. For a map $g(\lambda)$ from a subset of $\mathbb{C}\mathrm{P}_1$ to $\mathbf{SL}(2, \mathbb{C})$ we define

$$g^*(\lambda) = \overline{g(\tau(\lambda))}^{\top}. \tag{2.5.2}$$

Thus, if $F \in \Lambda_r \mathbf{SL}(2, \mathbb{C})_\sigma$ is defined and holomorphic on $A^{(r)}$, then $F \in \Lambda_r \mathbf{SU}(2)_\sigma$ is equivalent to

$$F^* = F^{-1}. \tag{2.5.3}$$

For a scalar function $f(\lambda)$ we set

$$f^*(\lambda) = \overline{f(\tau(\lambda))}. \tag{2.5.4}$$

If f is defined and holomorphic on a τ-invariant neighbourhood of S^1, then f is real on S^1 iff $f^* = f$.

2.6 As in the continuous case, we can use the dressing action of the group $\Lambda_r^+ \mathbf{SL}(2, \mathbb{C})_\sigma$ on $\Lambda_r \mathbf{SU}(2)_\sigma$ to generate new surfaces from old ones.

For $r_1, r_2 \in \mathbb{R}^+$, let $F^\circ : \mathbb{Z}^2 \to \Lambda_r \mathbf{SU}(2)_\sigma$ be an arbitrary normalized extended frame in $\mathcal{F}(r_1, r_2)$, not necessarily the discrete cylinder. For arbitrary $r_{\min}(r_1, r_2) < r < 1$ choose $h_+ \in \Lambda_r^+ \mathbf{SL}(2, \mathbb{C})_\sigma$ and define a map $F : \mathbb{Z}^2 \to \Lambda_r \mathbf{SU}(2)_\sigma$ by the Iwasawa splitting

$$h_+(\lambda) F^\circ_{mn}(\lambda) = F_{mn}(\lambda) p_+(m, n, \lambda), \tag{2.6.1}$$

where p_+ taking values in $\Lambda_r^+ \mathbf{SL}(2, \mathbb{C})_\sigma$ is chosen such that $F_{mn}(\lambda) = I$ for all $\lambda \in S^1$. We have the following

Theorem: *Let $r_1, r_2 \in \mathbb{R}^+$ and $F^\circ \in \mathcal{F}_0(r_1, r_2)$ be a normalized extended frame. Then for arbitrary $0 < r_{\min}(r_1, r_2) < r < 1$ and $h_+ \in \Lambda_r^+ \mathbf{SL}(2, \mathbb{C})_\sigma$, $F : \mathbb{Z}^2 \to \Lambda_r \mathbf{SU}(2)_\sigma$ defined by (2.6.1) is in $\mathcal{F}_0(r_1, r_2)$.*

Proof: We define the following matrices

$$U_{mn} = F_{mn}^{-1} F_{m+1,n} = p_+(m, n, \cdot) U^\circ_{mn} p_+(m+1, n, \cdot)^{-1}, \tag{2.6.2}$$

$$V_{mn} = F_{mn}^{-1} F_{m,n+1} = p_+(m, n, \cdot) V^\circ_{mn} p_+(m, n+1, \cdot)^{-1}, \tag{2.6.3}$$

where

$$U^\circ_{mn} = (F^\circ_{mn})^{-1} F^\circ_{m+1,n} = \frac{1}{\Delta_+} \begin{pmatrix} \alpha^\circ_{mn} & \lambda^{-1} r_1 p^\circ_{mn} - \lambda r_1 (p^\circ_{mn})^{-1} \\ \lambda^{-1} r_1 (p^\circ_{mn})^{-1} - \lambda r_1 p^\circ_{mn} & \overline{\alpha^\circ_{mn}} \end{pmatrix} \tag{2.6.4}$$

and

$$V^\circ_{mn} = (F^\circ_{mn})^{-1} F^\circ_{m,n+1} = \frac{1}{\Delta_-} \begin{pmatrix} \beta^\circ_{mn} & i\lambda^{-1} r_2 q^\circ_{mn} + i\lambda r_2 (q^\circ_{mn})^{-1} \\ i\lambda^{-1} r_2 (q^\circ_{mn})^{-1} + i\lambda r_2 q^\circ_{mn} & \overline{\beta^\circ_{mn}} \end{pmatrix}. \tag{2.6.5}$$

Then, U_{mn} and V_{mn} are in $\Lambda_r \mathbf{SU}(2)_\sigma$.

Since $p_0(m,n) = p_+(m,n,\lambda=0)$ takes values in the solvable Lie group $\mathbf{B}$, we can write

$$p_0(m,n) = \begin{pmatrix} e^{\omega_{mn}} & 0 \\ 0 & e^{-\omega_{mn}} \end{pmatrix}, \tag{2.6.6}$$

where $\omega_{mn} \in \mathbb{R}$. The matrix $\lambda\Delta_+ U_{mn}$ is holomorphic in the vicinity of $\lambda = 0$ and takes the value

$$p_0(m,n) \begin{pmatrix} 0 & r_1 p^\circ_{mn} \\ r_1 (p^\circ_{mn})^{-1} & 0 \end{pmatrix} p_0(m+1,n)^{-1} \tag{2.6.7}$$

at $\lambda = 0$. Thus, U_{mn} is of the form

$$U_{mn}(\lambda) = \lambda^{-1} \frac{1}{\Delta_+} \begin{pmatrix} 0 & r_1 p_{mn} \\ r_1 p_{mn}^{-1} & 0 \end{pmatrix} + \tilde{U}_{mn}(\lambda), \tag{2.6.8}$$

with

$$p_{mn} = p^\circ_{mn} e^{\omega_{mn} + \omega_{m+1,n}} \in \mathbb{R}^+. \tag{2.6.9}$$

and $\tilde{U}_{mn}$ holomorphic in λ. Since $U_{mn} \in \Lambda_r \mathbf{SU}(2)_\sigma$, we have $\overline{U_{mn}(\lambda)}^\top = U_{mn}(\lambda)^{-1}$ for $\lambda \in S^1$. Substituting this into (2.6.8) and expanding $\tilde{U}_{mn}$ as a power series in λ, we get $\tilde{U}_{mn}(\lambda) = R_{mn} + S_{mn}\lambda$, where, by the twisting condition,

$$R_{mn} = \frac{1}{\Delta_+} \begin{pmatrix} \alpha_{mn} & 0 \\ 0 & \overline{\alpha}_{mn} \end{pmatrix} \tag{2.6.10}$$

for some complex constant α_{mn}, and

$$S_{mn} = \frac{1}{\Delta_+} \begin{pmatrix} 0 & -r_1 p_{mn}^{-1} \\ -r_1 p_{mn} & 0 \end{pmatrix}. \tag{2.6.11}$$

This shows, that U_{mn} is of the form (2.4.1). By a similar argument, Eq. (2.4.2) follows with $\beta_{mn} \in \mathbb{C}$ and

$$q_{mn} = q^\circ_{mn} e^{\omega_{mn} + \omega_{m,n+1}} \in \mathbb{R}^+. \tag{2.6.12}$$

Therefore, and since F_{mn} satisfies (2.4.12) by construction, F_{mn} is an extended frame. □

Theorem 2.6 shows, that for arbitrary $r_1, r_2 \in \mathbb{R}^+$, the groups $\Lambda_r^+ \mathbf{SL}(2,\mathbb{C})_\sigma$, $r_{\min}(r_1,r_2) < r < 1$ act on the set $\mathcal{F}_0(r_1,r_2)$ of normalized extended frames with lattice constants r_1 and r_2. We call this action the r-dressing action on $\mathcal{F}_0(r_1,r_2)$.

3 Symmetries of discrete CMC-surfaces

3.1 We will now define the symmetry group of a discrete CMC-surface:

Definition: Let $\Psi_{mn} : \mathbb{Z}^2 \to \mathbb{R}^3$ be a discrete CMC-surface, then we denote by $\mathrm{Sym}(\Psi_{mn})$ the additive group of all pairs $(k,l) \in \mathbb{Z}^2$, s.t.

$$\Psi_{m+k,n+l} = \tilde{T}\Psi_{m,n} \tag{3.1.1}$$

for all $(m,n) \in \mathbb{Z}^2$, where $\tilde{T} \in \mathrm{OAff}(\mathbb{R}^3)$, the set of proper Euclidean motions of $\mathbb{R}^3$. By $\mathrm{Per}(\Psi_{mn})$ we denote the subgroup of $\mathrm{Sym}(\Psi_{mn})$ of all pairs $(k,l) \in \mathbb{Z}^2$, for which Eq. (3.1.1) is satisfied with $\tilde{T} = \mathrm{id}$. We will usually identify the elements of $\mathrm{Sym}(\Psi_{mn})$ and $\mathrm{Per}(\Psi_{mn})$ with the corresponding translations in $\mathbb{Z}^2$.

We will be mainly concerned with the transformation properties of the extended frame F_{mn} under translations in $\mathrm{Sym}(\Psi_{mn})$. A first step in this direction is the following

Lemma: *Let Ψ_{mn} and $\hat{\Psi}_{mn}$ be two discrete CMC-surfaces with the same lattice constants (r_1, r_2). Let p_{mn}, q_{mn}, $\hat{p}_{mn}$ and $\hat{q}_{mn}$ be the corresponding maps from $\mathbb{Z}^2$ to $\mathbb{R}^+$, defined by (2.4.1),(2.4.2). If Ψ_{mn} and $\hat{\Psi}_{mn}$ differ only by a proper Euclidean motion, i.e., if $\hat{\Psi}_{mn} = \tilde{T}\Psi_{mn}$ for all $(m,n) \in \mathbb{Z}^2$, then*

$$\hat{p}_{mn} = p_{mn}, \qquad \hat{q}_{mn} = q_{mn}, \qquad \text{for all } (m,n) \in \mathbb{Z}^2. \tag{3.1.2}$$

Proof: Using Sym's formula and the Eqs. (2.4.1),(2.4.2) we get for the edge vectors $L_{mn} = J(\Psi_{m+1,n} - \Psi_{mn})$ and $R_{mn} = J(\Psi_{m,n+1} - \Psi_{mn})$ of Ψ the following expressions:

$$L_{mn} = -2ip_{mn}\mathrm{Ad}(F_{mn}(1)) \begin{pmatrix} r_1^2(p_{mn} - p_{mn}^{-1}) & r_1\alpha_{mn} \\ r_1\overline{\alpha}_{mn} & -r_1^2(p_{mn} - p_{mn}^{-1}) \end{pmatrix}, \tag{3.1.3}$$

$$R_{mn} = -\frac{2i}{1+4r_2^2}q_{mn}\mathrm{Ad}(F_{mn}(1)) \begin{pmatrix} r_2^2(q_{mn} + q_{mn}^{-1}) & ir_2\beta_{mn} \\ -ir_2\overline{\beta}_{mn} & -r_2^2(q_{mn} + q_{mn}^{-1}) \end{pmatrix}. \tag{3.1.4}$$

From these equations we get with (2.4.3), (2.4.4),

$$\det(L_{mn}) = 4r_1^2p_{mn}^2, \tag{3.1.5}$$

$$\det(R_{mn}) = \frac{4r_2^2}{1+4r_2^2}q_{mn}^2. \tag{3.1.6}$$

If $\hat{\Psi}_{mn}$ and Ψ_{mn} differ only by a proper Euclidean motion, then in the spinor representation we have

$$\hat{L}_{mn} = UL_{mn}U^{-1}, \qquad \hat{R}_{mn} = UR_{mn}U^{-1}, \tag{3.1.7}$$

for some unitary matrix U, where $\hat{L}_{mn}$ and $\hat{R}_{mn}$ are the edge vectors of $\hat{\Psi}_{mn}$. Therefore, $\det(\hat{L}_{mn}) = \det(L_{mn})$, $\det(\hat{R}_{mn}) = \det(R_{mn})$, whence $\hat{p}_{mn}^2 = p_{mn}^2$, $\hat{q}_{mn}^2 = q_{mn}^2$. Since p_{mn}, q_{mn}, $\hat{p}_{mn}$ and $\hat{q}_{mn}$ take values in $\mathbb{R}^+$, the claim follows. □

Corollary: *Let Ψ_{mn} be a discrete CMC-surface and let $(k,l) \in \mathrm{Sym}(\Psi_{mn})$. If we define p_{mn} and q_{mn} by Eqs. (2.4.1),(2.4.2), then $p_{m+k,n+l} = p_{mn}$ and $q_{m+k,n+l} = q_{mn}$.*

Proof: If we set $\hat{\Psi}_{mn} = \Psi_{m+k,n+l}$, $\hat{p}_{mn} = p_{m+k,n+l}$, $\hat{q}_{mn} = q_{m+k,n+l}$, then the proof follows immediately from Lemma 3.1. □

3.2 From the proof of Lemma 3.1 and the spinor representation it is clear that

$$\mathrm{tr}(L_{mn}^2) = -2\det(L_{mn}) = (4r_1p_{mn})^2 \tag{3.2.1}$$

and

$$\mathrm{tr}(R_{mn}^2) = -2\det(R_{mn}) = \left(\frac{4r_2}{\sqrt{1+4r_2^2}}q_{mn}\right)^2 \tag{3.2.2}$$

are the squared lengths of the edge vectors of the discrete CMC-surface. Therefore, for fixed r_1 and r_2, p_{mn} and q_{mn} play the role of a discrete metric for the surface. We make the following natural

Definition: Let $\mathbb{R}_0^+$ be the set of non-negative real numbers. The *metric of a discrete surface* $\Psi_{mn} : \mathbb{Z}^2 \to \mathbb{R}^3$ is the map $g : \mathbb{Z}^2 \to \mathbb{R}_0^+ \times \mathbb{R}_0^+$, defined by

$$g(m,n) = (|\hat{\Psi}_{m+1,n} - \hat{\Psi}_{mn}|, |\hat{\Psi}_{m,n+1} - \hat{\Psi}_{mn}|) \tag{3.2.3}$$

Two discrete surfaces $\Psi_{mn}, \hat{\Psi}_{mn} : \mathbb{Z}^2 \to \mathbb{R}^3$ will be called *isometric* iff their metrics are the same, i.e., iff for all $(m,n) \in \mathbb{Z}^2$,

$$|\hat{\Psi}_{m+1,n} - \hat{\Psi}_{mn}| = |\Psi_{m+1,n} - \Psi_{mn}|, \qquad |\hat{\Psi}_{m,n+1} - \hat{\Psi}_{mn}| = |\Psi_{m,n+1} - \Psi_{mn}|. \tag{3.2.4}$$

Using this definition, Corollary 3.1 above just states (see [6, Corollary 2.6]) that elements of $\mathrm{Sym}(\Psi_{mn})$ act as self-isometries of the surface. We will therefore say, that the discrete surface Ψ_{mn} has periodic metric, if $\mathrm{Sym}(\Psi_{mn})$ contains a nontrivial element $(k,l) \neq (0,0)$.

3.3 Now we derive the transformation formula for an extended frame and the Lax operators U_{mn} and V_{mn} under a symmetry transformation of Ψ_{mn}. This will also show the ambiguity in the definition of an extended frame for a given discrete CMC-surface.

Lemma: *Let $r_1, r_2 \in \mathbb{R}^+$ and let Ψ_{mn} and $\hat{\Psi}_{mn}$ be two discrete CMC-surfaces with extended frames F_{mn} and $\hat{F}_{mn}$ in $\mathcal{F}(r_1, r_2)$, respectively. Let U_{mn}, V_{mn} be defined by Eqs. (2.4.1),(2.4.2) and let $\hat{U}_{mn}$ and $\hat{V}_{mn}$ be the analogous matrices for $\hat{F}_{mn}$. If Ψ_{mn} and $\hat{\Psi}_{mn}$ are related by a proper Euclidean motion $\tilde{T}$ on $\mathbb{R}^3$, i.e., if $\hat{\Psi}_{mn} = \tilde{T}\Psi_{mn}$ for all $(m,n) \in \mathbb{Z}^2$, then*

$$\hat{F}_{mn}(\lambda) = \chi(\lambda) F_{mn}(\lambda) k^{(-1)^{m+n+1}}, \tag{3.3.1}$$

where $\chi \in \Lambda\mathbf{SU}(2)_\sigma = \hat{F}_{00} k F_{00}^{-1}$ and $k \in \mathbf{U}(1) \subset \mathbf{SU}(2)$ is diagonal and λ-independent.

In addition, if $\tilde{T}(x) = R(x) + t$, where $R : \mathbb{R}^3 \to \mathbb{R}^3$ is a rotation and $t \in \mathbb{R}^3$ a translation, then

$$J(R(x)) = \chi(k,l,1) J(x) \chi(k,l,1)^{-1}, \qquad J(t) = \frac{i}{2}\frac{\partial}{\partial\theta}|_{\theta=0} \chi(k,l,\lambda = e^{i\theta}) \cdot \chi(k,l,1)^{-1}. \tag{3.3.2}$$

Proof: If Ψ_{mn} and $\hat{\Psi}_{mn}$ are related by a proper Euclidean motion, then we already know by Lemma 3.1, that $\hat{p}_{mn} = p_{mn}$ and $\hat{q}_{mn} = q_{mn}$ for all $(m,n) \in \mathbb{Z}^2$. This shows, with Eqs. (2.4.3) and (2.4.4), that

$$|\hat{\alpha}_{mn}| = |\alpha_{mn}|, \qquad |\hat{\beta}_{mn}| = |\beta_{mn}|. \tag{3.3.3}$$

If we define the edge vectors L_{mn}, R_{mn}, $\hat{L}_{mn}$ and $\hat{R}_{mn}$ as in the proof of Lemma 3.1, then we know, that the scalar products $\langle L_{mn}, R_{mn}\rangle$ and $\langle \hat{L}_{mn}, \hat{R}_{mn}\rangle$ are equal for both surfaces. In the spinor representation, this gives with Eqs. (3.1.3) and (3.1.4)

$$\overline{\hat{\alpha}}_{mn}\hat{\beta}_{mn} - \hat{\alpha}_{mn}\overline{\hat{\beta}}_{mn} = \overline{\alpha}_{mn}\beta_{mn} - \alpha_{mn}\overline{\beta}_{mn}. \tag{3.3.4}$$

Together with (3.3.3) this yields

$$\hat{\alpha}_{mn} = e^{i\phi_{mn}}\alpha_{mn}, \qquad \hat{\beta}_{mn} = e^{i\phi_{mn}}\beta_{mn}, \tag{3.3.5}$$

where $\phi_{mn} \in [0, 2\pi)$. Now we invoke (2.4.6) and (2.4.7). This gives

$$\phi_{m+1,n} = \phi_{m,n+1} = -\phi_{mn}, \tag{3.3.6}$$

which shows that

$$\hat{\alpha}_{mn} = e^{i(-1)^{m+n}\phi}\alpha_{mn}, \qquad \hat{\beta}_{mn} = e^{i(-1)^{m+n}\phi}\beta_{mn}, \tag{3.3.7}$$

where we have set $\phi = \phi_{00}$. By looking at (2.4.1),(2.4.2), we get

$$\hat{U}_{mn}(\lambda) = k^{(-1)^{m+n}} U_{mn} k^{(-1)^{m+n}}, \tag{3.3.8}$$

where

$$k = \begin{pmatrix} e^{i\frac{\phi}{2}} & 0 \\ 0 & e^{-i\frac{\phi}{2}} \end{pmatrix}. \tag{3.3.9}$$

Now, U_{mn} and V_{mn} ($\hat{U}_{mn}$ and $\hat{V}_{mn}$) determine F_{mn} ($\hat{F}_{mn}$) up to left multiplication with an element of $\Lambda\mathbf{SU}(2)_\sigma$.

Since $\tilde{F}_{mn} = F_{mn}k^{(-1)^{m+n+1}}$ and $\hat{F}_{mn}$ both satisfy

$$\begin{aligned} \tilde{F}_{mn}^{-1}\tilde{F}_{m+1,n} &= \hat{F}_{mn}^{-1}\hat{F}_{m+1,n} = \hat{U}_{mn}, \\ \tilde{F}_{mn}^{-1}\tilde{F}_{m,n+1} &= \hat{F}_{mn}^{-1}\hat{F}_{m,n+1} = \hat{V}_{mn}, \end{aligned} \tag{3.3.10}$$

we get Eq. (3.3.1). Finally, Eq. (3.3.2) can be easily derived from Sym's formula (2.3.2). □

Corollary: *Let $\Psi_{mn} : \mathbb{Z}^2 \to \mathbb{R}^3$ be a discrete CMC-surface with lattice constants r_1, r_2. Let $\hat{F}_{mn}(\lambda), F_{mn}(\lambda) \in \mathcal{F}(r_1, r_2)$ be two extended frames for Ψ_{mn}. Then there exists $k \in \mathbf{U}(1)$ and $\chi(\lambda) \in \Lambda\mathbf{SU}(2)_\sigma$, s.t.*

$$\hat{F}_{mn}(\lambda) = \chi(\lambda)F_{mn}(\lambda)k^{(-1)^{m+n+1}}, \tag{3.3.11}$$

where

$$\chi(1) = \pm I, \qquad \frac{\partial}{\partial\theta}|_{\theta=0}\chi(\lambda = e^{i\theta}) = 0. \tag{3.3.12}$$

Proof: If we set $\hat{\Psi}_{mn} = \Psi_{mn}$, then the proof follows immediately from Lemma 3.3. □

Theorem: *Let $\Psi_{mn} : \mathbb{Z}^2 \to \mathbb{R}^3$ be a discrete CMC-surface with extended frame $F_{mn} \in \mathcal{F}(r_1, r_2)$, $r_1, r_2 \in \mathbb{R}^+$. Then the following are equivalent:*

1. $(k, l) \in \mathrm{Sym}(\Psi_{mn})$,

2. *the equation*

$$F_{m+k,n+l}(\lambda) = \chi(k, l, \lambda)F_{mn}(\lambda)k^{(-1)^{m+n+1}}, \tag{3.3.13}$$

holds, where $\chi(k, l, \lambda) \in \Lambda\mathbf{SU}(2)_\sigma$ can be extended holomorphically to $A^{(r_{\min})}$, and $k \in \mathbf{U}(1)$ is a diagonal λ-independent matrix.

If $(k, l) \in \mathrm{Sym}(\Psi_{mn})$, then $(k, l) \in \mathrm{Per}(\Psi_{mn})$ iff

$$\chi(k, l, 1) = \pm I \qquad \text{and} \qquad \frac{\partial}{\partial\theta}|_{\theta=0}\chi(k, l, \lambda = e^{i\theta}) = 0. \tag{3.3.14}$$

Proof: We set $\hat{\Psi}_{mn} = \Psi_{m+k,n+l}$ and $\hat{F}_{mn}(\lambda) = F_{m+k,n+l}(\lambda)$. Then 1.⇒2. follows from Lemma 3.3. Conversely, if Eq. (3.3.13) is satisfied with χ and k_{mn} as above, then by Sym's formula (2.3.2) we have

$$\Psi_{m+k,n+l} = \chi(k, l, 1)\Psi_{mn}\chi(k, l, 1)^{-1} + \frac{i}{2}\frac{\partial}{\partial\theta}|_{\theta=0}\chi(k, l, \lambda = e^{i\theta}) \cdot \chi(k, l, 1)^{-1}. \tag{3.3.15}$$

Thus, $\Psi_{m+k,n+l}$ and Ψ_{mn} differ by a proper Euclidean motion $\tilde{T}$, which can be written as

$$\tilde{T}(x) = R(x) + t, \qquad \text{for all } x \in \mathbb{R}^3, \tag{3.3.16}$$

where $R : \mathbb{R}^3 \to \mathbb{R}^3$ is a rotation and $t \in \mathbb{R}^3$ describes a translation. In the spinor representation, R and t are given by

$$J(R(x)) = \chi(k,l,1)J(x)\chi(k,l,1)^{-1}, \qquad J(t) = \frac{i}{2}\frac{\partial}{\partial\theta}|_{\theta=0}\chi(k,l,\lambda = e^{i\theta}) \cdot \chi(k,l,1)^{-1}. \quad (3.3.17)$$

From this, 1. and the rest of the statement follow. □

From now on we normalize, analogous to the continuous case, the extended frames for the discrete CMC-surfaces by (2.4.12). Up to proper Euclidean motions we still get, by Lemma 3.3, all discrete CMC-surfaces in this way.

4 Symmetric surfaces in the r-dressing orbit of the cylinder

In this section for fixed lattice constants $r_1, r_2 \in \mathbb{R}^+$ and fixed $(k,l) \in \mathbb{Z}^2 \setminus \{(0,0)\}$, we want to classify those discrete CMC-surfaces $\Psi_{mn} : \mathbb{Z}^2 \to \mathbb{R}^3$ in the dressing orbit of the cylinder, for which $(k,l) \in \mathrm{Sym}(\Psi_{mn})$.

Here, if we talk about the dressing orbit without specifying a radius r, we always mean the orbit under the action of the union

$$\bigcup_{r_{\min}(r_1,r_2)<r<1} \Lambda_r^+ \mathbf{SL}(2,\mathbb{C})_\sigma,$$

where $r_{\min}(r_1,r_2)$ was defined in (2.2.10).

4.1 Let $F_{mn}(\lambda) \in \mathcal{F}_0(r_1,r_2)$ be defined by the dressing action of $h_+ \in \Lambda_r^+\mathbf{SL}(2,\mathbb{C})_\sigma$, $r_{\min}(r_1,r_2) < r < 1$, on the extended frame of the cylinder with lattice constants $r_1, r_2 \in \mathbb{R}^+$. I.e.,

$$h_+(\lambda)U^\circ(\lambda)^m V^\circ(\lambda)^n = F_{mn}(\lambda)p_+(n,m,\lambda), \qquad p_+(z,\lambda) \in \Lambda_{r,B}^+\mathbf{SL}(2,\mathbb{C})_\sigma. \quad (4.1.1)$$

Under the translation $(m,n) \mapsto (m+k, n+l)$, $(k,l) \in \mathbb{Z}^2$, the extended frame $F_{mn}(\lambda)$ transforms like

$$F_{m+k,n+l}(\lambda) = Q(\lambda)F_{mn}(\lambda)r_+(m,n,\lambda), \quad (4.1.2)$$

where

$$Q(\lambda) = h_+(U^\circ)^k(V^\circ)^l h_+^{-1} \quad (4.1.3)$$

and

$$r_+(m,n,\lambda) = p_+(m,n,\lambda)p_+(m+k,n+l,\lambda)^{-1}. \quad (4.1.4)$$

4.2 By Theorem 3.3 we have $(k,l) \in \mathrm{Sym}(\Psi_{mn})$ iff

$$Q(\lambda)F_{mn}(\lambda)r_+(m,n,\lambda) = \chi(\lambda)F_{mn}(\lambda)k_{mn}, \quad (4.2.1)$$

where $k_{mn} \in \mathbf{SU}(2)$ is a diagonal matrix, $\chi(\lambda) \in \Lambda\mathbf{SU}(2,\mathbb{C})_\sigma$, and by Lemma 2.4,

$$\Delta_+^{|k|}\Delta_-^{|l|}\chi(\lambda) = \Delta_+^{|k|}\Delta_-^{|l|}F_{kl}(\lambda)k_{00}^{-1} \quad (4.2.2)$$

can be extended holomorphically to $\mathbb{C}^*$. In the following we will derive further conditions on the matrix $\chi(\lambda)$.

The initial condition (2.4.12) together with Eq. (4.2.1) implies

$$\chi = Q \cdot R_+, \quad (4.2.3)$$

with

$$R_+(\lambda) = r_+(0,0,\lambda)k_{00}^{-1} \in \Lambda_r^+ \mathbf{SL}(2,\mathbb{C})_\sigma. \tag{4.2.4}$$

Thus, $F_{mn}(\lambda)$ is invariant under the r-dressing transformation with R_+,

$$R_+(\lambda)F_{mn}(\lambda) = F_{mn}(\lambda)r_+(m,n,\lambda)k_{mn}^{-1}. \tag{4.2.5}$$

Lemma: *The matrix $\chi(\lambda)$ is of the form*

$$\chi = h_+ H h_+^{-1}, \tag{4.2.6}$$

where

$$H = (U^\circ)^k (V^\circ)^l w_+ \tag{4.2.7}$$

and $w_+ = h_+^{-1} R_+ h_+$ commute with $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

Proof: Substituting (4.1.1) into (4.2.5) and rearranging terms gives

$$V^{\circ -n} U^{\circ -m} h_+^{-1} R_+ h_+ U^{\circ m} V^{\circ n} = p_+(m,n,\lambda)^{-1} r_+(m,n,\lambda) k_{mn}^{-1} p_+(m,n,\lambda). \tag{4.2.8}$$

Abbreviating

$$V_+(m,n,\lambda) = p_+(m,n,\lambda)^{-1} r_+(m,n,\lambda) k_{mn}^{-1} p_+(m,n,\lambda), \tag{4.2.9}$$

and using the definition of w_+, this is

$$V^{\circ -n} U^{\circ -m} w_+ U^{\circ m} V^{\circ n} = V_+(m,n,\lambda). \tag{4.2.10}$$

This yields that

$$Z(m,n,\lambda) = V^{\circ -n} U^{\circ -m} w_+ U^{\circ m} V^{\circ n} \in \Lambda_r^+ \mathbf{SL}(2,\mathbb{C})_\sigma, \tag{4.2.11}$$

in particular, that $Z(m,n,\lambda)$ is holomorphic on $I^{(r)}$, $r_{\min} < r < 1$, for all $m,n \in \mathbb{Z}$. Let $D = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$, then

$$DAD^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{4.2.12}$$

If we set

$$\tilde{w}_+ = D w_+ D^{-1} = \begin{pmatrix} \tilde{w}_a & \tilde{w}_b \\ \tilde{w}_c & \tilde{w}_d \end{pmatrix} \tag{4.2.13}$$

then

$$\begin{aligned} &DZ(m,n,\lambda)D^{-1} = \\ &= \begin{pmatrix} \tilde{w}_a & \tilde{w}_b \left(\frac{1+r_1(\lambda^{-1}-\lambda)}{1-r_1(\lambda^{-1}-\lambda)}\right)^m \left(\frac{1+ir_2(\lambda^{-1}+\lambda)}{1-ir_2(\lambda^{-1}+\lambda)}\right)^n \\ \tilde{w}_c \left(\frac{1-r_1(\lambda^{-1}-\lambda)}{1+r_1(\lambda^{-1}-\lambda)}\right)^m \left(\frac{1-ir_2(\lambda^{-1}+\lambda)}{1+ir_2(\lambda^{-1}+\lambda)}\right)^n & \tilde{w}_d \end{pmatrix}. \end{aligned} \tag{4.2.14}$$

If we define

$$S_{r_1}(\lambda) = \frac{1 + r_1(\lambda^{-1} - \lambda)}{1 - r_1(\lambda^{-1} - \lambda)} \tag{4.2.15}$$

and

$$T_{r_2}(\lambda) = \frac{1 + ir_2(\lambda^{-1} + \lambda)}{1 - ir_2(\lambda^{-1} + \lambda)}, \tag{4.2.16}$$

then the off-diagonal entries of $Z(m,n,\lambda)$ are of the form

$$\tilde{w}_b S_{r_1}(\lambda)^m T_{r_2}(\lambda)^n \tag{4.2.17}$$

and

$$\tilde{w}_c S_{r_1}(\lambda)^{-m} T_{r_2}(\lambda)^{-n}. \tag{4.2.18}$$

Using (2.1.13) and the notation of Lemma 2.1, we see that S_{r_1} has two simple zeroes at λ_+ and $-\lambda_+^{-1}$, and two simple poles at $-\lambda_+$ and λ_+^{-1}. Analogously, using (2.1.14), we get that T_{r_2} has two simple zeroes at $i\lambda_-$ and $-i\lambda_-^{-1}$ and two simple poles at $-i\lambda_-$ and $i\lambda_-^{-1}$.

Assume now, that $\tilde{w}_b$ and $\tilde{w}_c$ do not vanish identically. Then, for $m \neq 0$ and $n \neq 0$, the r.h.s. of (4.2.14) has a simple pole on the circle $C_{r_{\min}} \subset I^{(r)}$. This contradicts $DZ(m,n,\lambda)D^{-1} \in \Lambda_r^+ \mathbf{SL}(2,\mathbb{C})_\sigma$. Therefore, $\tilde{w}_b = \tilde{w}_c \equiv 0$ and $\tilde{w}_+$ is diagonal and commutes with $\sigma_3 = DAD^{-1}$. Thus, w_+ commutes with A.

This implies

$$V_+(m,n,\lambda) = V_+(0,0,\lambda) = w_+(\lambda). \tag{4.2.19}$$

The definition of w_+ gives

$$R_+ = h_+ w_+ h_+^{-1}. \tag{4.2.20}$$

Substituting (4.2.20) into (4.2.3) finally gives (4.2.6), with H defined by (4.2.7). Since U°, V° and w_+ commute with A, also H does. □

4.3 In the proof of Lemma 4.2 we have diagonalized A. We will use this method to derive a more convenient expression for the matrix H introduced in the lemma.

Recall, that $\mathcal{A}_r$ was the Banach algebra defining the topology of the loop groups introduced in Section 2.2. Let $\mathcal{A}_r^+$ be the subalgebra of $\mathcal{A}_r$ which consists of those functions which can be continued holomorphically to $I^{(r)}$. Lemma 4.2 together with $\det w_+ = 1$ implies (see [5, Section 3.3]), that the matrix w_+ is of the form

$$w_+ = e^{f_+ A} = \begin{pmatrix} \cosh(f_+) & \sinh(f_+) \\ \sinh(f_+) & \cosh(f_+) \end{pmatrix}, \tag{4.3.1}$$

where $f_+ \in \mathcal{A}_r^+$ is odd in λ. For $\tilde{w}_+ = Dw_+D^{-1}$ this amounts to $\tilde{w}_+ = e^{f_+\sigma_3}$. The matrix $\tilde{H} = DHD^{-1} = D(U^\circ)^k(V^\circ)^l w_+ D^{-1}$ is of the form

$$\tilde{H} = \frac{1}{\Delta_+^k \Delta_-^l} \begin{pmatrix} r_+ & 0 \\ 0 & r_- \end{pmatrix}, \tag{4.3.2}$$

where

$$r_+(\lambda) = (1 + r_1(\lambda^{-1} - \lambda))^k (1 + ir_2(\lambda^{-1} + \lambda))^l e^{f_+}, \tag{4.3.3}$$

$$r_-(\lambda) = (1 - r_1(\lambda^{-1} - \lambda))^k (1 - ir_2(\lambda^{-1} + \lambda))^l e^{-f_+}. \tag{4.3.4}$$

A short computation using $\det \tilde{H} = 1$ and the parity of f_+ shows that we can rewrite this as

$$\tilde{H} = \frac{1}{\Delta_+^{|k|} \Delta_-^{|l|}} \begin{pmatrix} \hat{p}(\lambda) & 0 \\ 0 & \hat{p}(-\lambda) \end{pmatrix}, \tag{4.3.5}$$

where

$$\hat{p}(\lambda) = (1 + \epsilon_k r_1(\lambda^{-1} - \lambda))^{|k|} (1 + i\epsilon_l r_2(\lambda^{-1} + \lambda))^{|l|} e^{f_+}, \tag{4.3.6}$$

with $k = \epsilon_k |k|$, $l = \epsilon_l |l|$. Obviously,

$$\hat{p}(\lambda) \text{ is defined and holomorphic in } I^{(r)} \setminus \{0\}. \tag{4.3.7}$$

For H this gives

Lemma: *The matrix H introduced in Lemma 4.2.1 is of the form*

$$H = \alpha I + \beta A, \qquad \alpha, \beta \in \mathcal{A}_r, \tag{4.3.8}$$

where

$$\alpha^2 - \beta^2 = 1, \tag{4.3.9}$$

$$\alpha \text{ is even in } \lambda \text{ and } \beta \text{ is odd in } \lambda. \tag{4.3.10}$$

and

$$\alpha + \beta = \frac{1}{\Delta_+^{|k|}\Delta_-^{|l|}}\hat{p}, \tag{4.3.11}$$

with $\hat{p}$ given by (4.3.6).

Proof: Eq. (4.3.8) follows from $[H, A] = 0$, since A is regular semisimple. Then $\det H = \alpha^2 - \beta^2 = 1$ gives (4.3.9), and (4.3.10) follows from the twisting condition (2.2.1). Finally, (4.3.11) follows from (4.3.5) by conjugation with D^{-1}. □

4.4 Let us derive further properties of the functions α and β introduced in Lemma 4.3. Eq. (4.3.8) together with (4.3.9) yields

$$H^{-1} = \alpha I - \beta A. \tag{4.4.1}$$

Since $(k, l) \neq (0, 0)$, we get from (4.2.7) and Lemma 2.1, that H has a pole in $I^{(r)}$, $r > r_{\min}(r_1, r_2)$. Therefore, by (4.3.9),

$$\beta \not\equiv 0. \tag{4.4.2}$$

Let us also define the matrix

$$\hat{H} = \Delta_+^{|k|}\Delta_-^{|l|}H = \Delta_+^{|k|}\Delta_-^{|l|}(U^\circ)^k(V^\circ)^l w_+ = \hat{\alpha}I + \hat{\beta}A, \tag{4.4.3}$$

with

$$\hat{\alpha} = \Delta_+^{|k|}\Delta_-^{|l|}\alpha, \qquad \hat{\beta} = \Delta_+^{|k|}\Delta_-^{|l|}\beta. \tag{4.4.4}$$

and

$$\det \hat{H} = \hat{\alpha}^2 - \hat{\beta}^2 = \Delta_+^{2|k|}\Delta_-^{2|l|} = (1 - r_1^2(\lambda^{-1} - \lambda)^2)^{|k|}(1 + r_2^2(\lambda^{-1} + \lambda)^2)^{|l|}. \tag{4.4.5}$$

Since Δ_+ and Δ_- are even in λ we have

$$\hat{\alpha} \text{ is even in } \lambda \text{ and } \hat{\beta} \text{ is odd in } \lambda. \tag{4.4.6}$$

Since by Theorem 3.3,

$$\hat{\chi}(\lambda) = \Delta_+^{|k|}\Delta_-^{|l|}\chi(\lambda) = h_+\hat{H}h_+^{-1} \tag{4.4.7}$$

is holomorphic on $\mathbb{C}^*$, taking the trace of $\hat{\chi}$ and $\hat{\chi}^2$ shows that $\hat{\alpha}$ and $\hat{\beta}^2$ have holomorphic extensions to $\mathbb{C}^*$. Now, since $w_+ \in \Lambda_r^+\mathbf{SL}(2, \mathbb{C})_\sigma$ and since Δ_+U° and Δ_-V° as well as $\Delta_+(U^\circ)^{-1}$ and $\Delta_-(V^\circ)^{-1}$ are holomorphic on $\mathbb{C}^*$, with meromorphic extension to $\lambda = 0$ and $\lambda = \infty$, also $\hat{H}(\lambda)$ is meromorphic on $I^{(r)}$. Thus $\hat{\chi}$ has a meromorphic extension to $\lambda = 0$, and by unitarity also to $\lambda = \infty$. This yields

$$\hat{\alpha} \text{ and } \hat{\beta}^2 \text{ are rational functions which are holomorphic on } \mathbb{C}^*. \tag{4.4.8}$$

From (4.3.11), we get

$$\hat{\alpha} + \hat{\beta} = \hat{p}, \tag{4.4.9}$$

where $\hat{p}(\lambda)$ is defined by (4.3.6). From this and (4.4.8) it follows, that

$$\hat{\beta} \text{ has a meromorphic extension to } I^{(r)}. \tag{4.4.10}$$

4.5 We now define the matrices

$$S = h_+Ah_+^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \tag{4.5.1}$$

and

$$\hat{S} = \hat{\beta} S = \begin{pmatrix} \hat{a} & \hat{b} \\ \hat{c} & \hat{d} \end{pmatrix}. \tag{4.5.2}$$

With (4.3.8) and (4.4.7), we have

$$\chi = \alpha I + \beta S \tag{4.5.3}$$

and

$$\hat{\chi} = \hat{\alpha} I + \hat{\beta} S = \hat{\alpha} I + \hat{S}. \tag{4.5.4}$$

Using (4.4.1) we get

$$\chi^{-1} = \alpha I - \beta S. \tag{4.5.5}$$

Since $\hat{\chi}$ and $\hat{\alpha}$ are meromorphic on $\mathbb{C}\mathrm{P}_1$ and holomorphic on $\mathbb{C}^*$, we get from (4.5.4)

$$\hat{a}, \hat{b}, \hat{c}, \hat{d} \text{ are rational and holomorphic on } \mathbb{C}^*. \tag{4.5.6}$$

Clearly, we have $\mathrm{tr} S = \mathrm{tr}\hat{S} = 0$, whence

$$d = -a, \qquad \hat{d} = -\hat{a}. \tag{4.5.7}$$

Also, in view of (4.4.6), the twisting condition for $\Lambda_r \mathbf{SL}(2, \mathbb{C})_\sigma$ implies

$$\hat{a}, b, c \text{ are even in } \lambda,\ a, \hat{b}, \hat{c} \text{ are odd in } \lambda. \tag{4.5.8}$$

Since $S^2 = I$, we get with (4.5.7)

$$a^2 + bc = 1 \tag{4.5.9}$$

Since $\hat{S} = \hat{\beta} S$, we also have

$$\hat{a}^2 + \hat{b}\hat{c} = \hat{\beta}^2. \tag{4.5.10}$$

The unitarity of $\chi(\lambda)$ on S^1 is in view of Section 2.5 and (4.5.5) equivalent with

$$\alpha^* = \alpha, \tag{4.5.11}$$

$$\hat{S}^* = -\hat{S}. \tag{4.5.12}$$

In particular, (4.5.12) is equivalent with

$$\hat{a}^* = -\hat{a}, \qquad \hat{b}^* = -\hat{c}. \tag{4.5.13}$$

By Proposition 2.1, we see that (4.5.11) is equivalent to

$$\hat{\alpha}^* = \hat{\alpha}, \tag{4.5.14}$$

Now we consider the squares of $\hat{\alpha}, \hat{\beta}, a, b, c$ and $\hat{a}, \hat{b}, \hat{c}$. First we note

$$\hat{\alpha}^2 \text{ and } \hat{\beta}^2 \text{ are real on } S^1. \tag{4.5.15}$$

This follows for $\hat{\alpha}^2$ from (4.5.14) and for $\hat{\beta}^2$ from (4.4.5) and Proposition 2.1. Next, (4.5.13) implies

$$\hat{a}^2 \text{ is non-positive on } S^1. \tag{4.5.16}$$

Substituting this and (4.5.13) into (4.5.10) gives

$$\hat{\beta}^2 \text{ is non-positive on } S^1. \tag{4.5.17}$$

Since $\hat{a} = \hat{\beta} a$, we know $\hat{a}^2 = \hat{\beta}^2 a^2$. In particular, by (4.5.6), (4.4.8), (4.5.16) and (4.5.17),

$$a^2 \text{ is a rational function, real and non-negative on } S^1. \tag{4.5.18}$$

Since a^2 is by definition also holomorphic at $\lambda = 0$, it follows that

$$a^2 \text{ is locally holomorphic around 0 and } \infty. \tag{4.5.19}$$

For b^2 and c^2 one argues similarly. E.g. $b^2 = \frac{\hat{b}^2}{\hat{\beta}^2}$ is clearly meromorphic on $\mathbb{C}^*$ and is also, by the definition of b, holomorphic at $\lambda = 0$. From (4.5.13) we obtain that $(b^2)^* = \frac{(\hat{b}^2)^*}{\hat{\beta}^2} = \frac{\hat{c}^2}{\hat{\beta}^2} = c^2$ is also holomorphic at $\lambda = 0$. This shows, that b^2 is meromorphic on $\mathbb{C}\mathrm{P}_1$ and thus rational. Altogether we have shown

$$b^2 \text{ and } c^2 \text{ are rational and finite at 0 and } \infty, \tag{4.5.20}$$

and

$$(b^2)^* = c^2. \tag{4.5.21}$$

Next, from (4.5.1) we see that $a(\lambda = 0) = 0$ and $b(\lambda = 0) = c(\lambda = 0)^{-1}$. Since a is odd in λ, we obtain

$$a^2 \text{ has a zero of order } 2(2n-1) \text{ for some } n > 0 \text{ at } \lambda = 0, \tag{4.5.22}$$

$$b(\lambda = 0)c(\lambda = 0) = 1. \tag{4.5.23}$$

We also note that the relations

$$\hat{a}^2 = \hat{\beta}^2 a^2, \qquad \hat{b}^2 = \hat{\beta}^2 b^2, \qquad \hat{c}^2 = \hat{\beta}^2 c^2 \tag{4.5.24}$$

show that a^2, b^2, and c^2 can have poles only where $\hat{\beta}^2$ has a zero.

Finally, from (4.5.12) we obtain $(\hat{\beta}b)^* = -(\hat{\beta}c)$. Hence (4.5.9) implies

$$\hat{\beta}^2 = \hat{\beta}^2 a^2 + \hat{\beta}b \cdot \hat{\beta}c = \hat{\beta}^2 a^2 - (\hat{\beta}b)(\hat{\beta}b)^*. \tag{4.5.25}$$

Therefore, on S^1 we obtain $\hat{\beta}^2(a^2 - 1) = |\hat{\beta}b|^2$. Since $\hat{\beta}^2$ is non-positive on S^1 by (4.5.17), and $\hat{\beta}^2 \not\equiv 0$ by (4.4.2), we have $a^2 - 1 \leq 0$. Thus,

$$0 \leq a^2(\lambda) \leq 1 \qquad \text{for } \lambda \in S^1. \tag{4.5.26}$$

4.6 In the last section we considered the matrix $S = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and we listed properties of a, b, c, d. In the rest of this paper we will characterize $\mathrm{Sym}(\Psi_{mn})$ in terms of a, b, c, d. The entries of elements of $\Lambda_r^+ \mathbf{SL}(2, \mathbb{C})_\sigma$ are elements of $\mathcal{A}_r^+$. To be sure, that to such a, b, c, d there exists an h_+ satisfying (4.5.1), producing F —and thus Ψ— for which $(k, l) \in \mathrm{Sym}(\Psi_{mn})$, we slightly extend Theorem 3.6 from [5]:

Theorem: *Let $a, b, c, d \in \mathcal{A}_R^+$, $0 < R < 1$, where a, d are odd and b, c are even. Then $S = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is of the form $S = h_+ A h_+^{-1}$ for some $0 < r \leq R$ and $h_+ \in \Lambda_r^+ \mathbf{SL}(2, \mathbb{C})_\sigma$ iff*

$$d = -a, \tag{4.6.1}$$

$$a^2 + bc = 1, \tag{4.6.2}$$

$$b(\lambda = 0) \neq 0. \tag{4.6.3}$$

Furthermore, if for some $0 < r' < R < 1$, b is the square of a holomorphic function on the closure $I^{(r')}$, then we can find such an h_+ for some $r \geq r'$.

Proof: All but the last statement are taken from Theorem 3.6 in [5]. The last statement follows from the special choice

$$h_+ = \frac{1}{\sqrt{c}} \begin{pmatrix} 1 & a \\ 0 & c \end{pmatrix} \tag{4.6.4}$$

given in the proof of this Theorem. □

4.7 Let us collect the necessary conditions we have derived in Sections 4.2–4.5:

Theorem: *Let $\Psi_{mn} : \mathbb{Z}^2 \to \mathbb{R}^3$ be a discrete CMC-surface with extended frame $F_{mn}(\lambda) \in \mathcal{F}_0(r_1, r_2)$, s.t. $F_{mn}(\lambda)$ is given by dressing the cylinder under the r-dressing (4.1.1) with some $h_+ \in \Lambda_r^+ \mathbf{SL}(2,\mathbb{C})_\sigma$, $r_{\min}(r_1, r_2) < r < 1$. Assume also, that for $(k,l) \in \mathbb{Z}^2$, $(k,l) \neq (0,0)$, $F_{m+k,n+l}(\lambda) = \chi(\lambda) F_{mn}(\lambda)$, i.e., $(k,l) \in \mathrm{Sym}(\Psi_{mn})$. Define $h_+ A h_+^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then $d = -a$ and the functions $a(\lambda)$, $b(\lambda)$, and $c(\lambda)$ are in $\mathcal{A}_r^+$ and satisfy the following conditions:*

a) *a^2, b^2, c^2 are rational,*

b) *a is odd in λ, b and c are even in λ,*

c) *$a^2 + bc = 1$.*

d) *a^2 is real on S^1 and $0 \leq a^2 \leq 1$ on S^1,*

e) *$c^2 = (b^2)^*$.*

Furthermore, there exists an odd function f_+ in $\mathcal{A}_r^+$, s.t. with $2\hat{\alpha}(\lambda) = \hat{p}(\lambda) + \hat{p}(-\lambda)$, $2\hat{\beta}(\lambda) = \hat{p}(\lambda) - \hat{p}(-\lambda)$,

$$\hat{p} = (1 + \epsilon_k r_1(\lambda^{-1} - \lambda))^{|k|} (1 + \epsilon_l r_2(\lambda^{-1} + \lambda))^{|l|} e^{f_+}, \tag{4.7.1}$$

where $k = \epsilon_k |k|, l = \epsilon_l |l|$, we have

a') *$\hat{\alpha}$ and $\hat{\beta}^2$ are rational and holomorphic on $\mathbb{C}^*$,*

b') *$\hat{\alpha}$ and $\hat{\beta}^2$ are real on S^1,*

c') *$\hat{\beta}^2$ is non-positive on S^1,*

d') *the functions $\hat{\beta}a$, $\hat{\beta}b$, and $\hat{\beta}c$ are rational and holomorphic on $\mathbb{C}^*$.*

The matrix function $\chi(\lambda)$ is given by $\Delta_+^{|k|} \Delta_-^{|l|} \chi = \hat{\alpha} I + \hat{\beta} h_+ A h_+^{-1}$.

Proof: By the results of the last sections we know for the functions a, b, c, d, defined by $S = h_+ A h_+^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$:

- $d = -a$: (4.5.7),
- a^2, b^2, c^2 are rational functions: (4.5.18), (4.5.20),
- a is odd, b and c are even in λ: (4.5.8),
- $a^2 + bc = 1$: (4.5.9),
- a^2 is real on S^1 and $0 \leq a^2 \leq 1$ on S^1: (4.5.18), (4.5.26),
- $c^2 = (b^2)^*$: (4.5.21).

Since $(k,l) \in \mathrm{Sym}(\Psi_{mn})$ we have $F_{m+k,n+l}(\lambda) = \chi(\lambda) F_{mn}(\lambda)$ by Theorem 3.3. Moreover,

$$\Delta_+^{|k|} \Delta_-^{|l|} \chi(\lambda) = \hat{\alpha} I + \hat{\beta} h_+ A h_+^{-1} \tag{4.7.2}$$

by (4.5.4). By anti-/symmetrization of (4.4.9) we get $\hat{\alpha}(\lambda) = \hat{p}(\lambda) + \hat{p}(-\lambda)$, and $\hat{\beta}(\lambda) = \hat{p}(\lambda) - \hat{p}(-\lambda)$, where $\hat{p}$ is given by (4.7.1). Furthermore,

- $\hat\alpha$ and $\hat\beta^2$ are rational and holomorphic on $\mathbb{C}^*$: (4.4.8),
- $\hat\alpha$ and $\hat\beta^2$ are real on S^1: (4.5.14), (4.5.15),
- $\hat\beta^2$ is non-positive on S^1: (4.5.17).

Finally, $\hat a = \hat\beta a$, $\hat b = \hat\beta b$, and $\hat c = \hat\beta c$ are rational and holomorphic on $\mathbb{C}^*$ by (4.5.6). □

4.8 We have seen in Section 4.6 under what conditions on a, b, c, and d we can find an $h_+ \in \Lambda_r^+\mathbf{SL}(2,\mathbb{C})_\sigma$, s.t. $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = h_+ A h_+^{-1}$. This then defines a CMC-immersion Ψ via dressing of the trivial solution with h_+. In this section we characterize those $a, b, c, d, \hat\alpha, \hat\beta$ such that a given $(k,l) \in \mathbb{Z}^2 \setminus \{(0,0)\}$ is in $\mathrm{Sym}(\Psi_{mn})$.

Theorem: *Let there be given three even rational functions $a^2(\lambda)$, $b^2(\lambda)$, and $c^2(\lambda)$, which for some $r_1, r_2 \in \mathbb{R}^+$ and $r_{\min}(r_1, r_2)$ defined in Proposition 2.2 satisfy the following conditions*

a) *a^2 is real on S^1 and $0 \le a^2 \le 1$ on S^1,*

b) *$c^2 = (b^2)^*$,*

c) *There exists an $r_{\min} < r < 1$, s.t. the restrictions of a^2, b^2, and c^2 to C_r are the squares of functions a, b, c in $\mathcal{A}_r^+$,*

d) *a is odd, b and c are even in λ,*

e) *$a^2 + bc = 1$,*

f) *b is the square of a holomorphic function on the closure of $I^{(r_{\min})}$.*

In addition, with r as in c), we assume that there exists an odd function f_+ in $\mathcal{A}_{r'}^+$, $r_{\min} < r' \le r$, such that for $\hat p = (1 + \epsilon_k r_1(\lambda^{-1} - \lambda))^{|k|}(1 + \epsilon_l r_2(\lambda^{-1} + \lambda))^{|l|} e^{f_+}$, $\hat\alpha(\lambda) = \frac{1}{2}(\hat p(\lambda) + \hat p(-\lambda))$, $\hat\beta(\lambda) = \frac{1}{2}(\hat p(\lambda) - \hat p(-\lambda))$, we have

a') *$\hat\alpha$ and $\hat\beta^2$ are rational and holomorphic on $\mathbb{C}^*$,*

b') *$\hat\alpha$ and $\hat\beta^2$ are real on S^1,*

c') *$\hat\beta^2$ is non-positive on S^1.*

d') *The functions $\hat\beta a$, $\hat\beta b$, and $\hat\beta c$ are rational and holomorphic on $\mathbb{C}^*$.*

Then there exists $0 < r_{\min} < r'' \le r'$ and $h_+ \in \Lambda_{r''}^+\mathbf{SL}(2,\mathbb{C})_\sigma$, such that $h_+ A h_+^{-1} = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}$. Moreover, for the extended frame $F_{mn}(\lambda)$ defined by $h_+(U^\circ)^m(V^\circ)^n = F_{mn}(\lambda)p_+(m,n,\lambda)$, $|\lambda| = r''$, we have $F_{m+k,n+l}(\lambda) = \chi(\lambda)F_{mn}(\lambda)$, where $\chi = \alpha I + \beta h_+ A h_+^{-1}$ is holomorphic on $\mathbb{C}^*$ and takes values in $\mathbf{SU}(2)$ on S^1. In particular, $(k,l) \in \mathrm{Sym}(\Psi_{mn})$ for the CMC-immersion Ψ_{mn} associated with $F_{mn}(\lambda)$ via Sym's formula.

Proof: Assume, that we have functions a^2, b^2, c^2 and $f_+, \hat p, \hat\alpha, \hat\beta$, such that a)–f), a')–d') are satisfied. We first want to apply Theorem 4.6. We set $d = -a$ and know (4.6.2) by e). Since a, b, c are defined at $\lambda = 0$ and since a is odd we have $a(0) = 0$, whence $b(0) \neq 0$. Thus, by Theorem 4.6, there exists some $r_{\min}(r_1, r_2) < r'' \le r' < 1$ and some $h_+ \in \Lambda_{r''}^+\mathbf{SL}(2,\mathbb{C})_\sigma$, s.t. $S = h_+ A h_+^{-1} = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}$. Next we consider the extended frame defined by the r-dressing $h_+(U^\circ)^m(V^\circ)^n = F_{mn}(\lambda)p_+(m,n,\lambda)$

of the cylinder. Recall that we use in this paper the unique Iwasawa splitting discussed in Section 2.2. We also set $\hat{\chi} = \Delta_+^{|k|}\Delta_-^{|l|}\chi = \hat{\alpha}I + \hat{\beta}S$. From a') and d') it follows, that $\hat{\chi}$ is defined and holomorphic on $\mathbb{C}^*$. A simple calculation using the parity of f_+ gives

$$\hat{\alpha}(\lambda)^2 - \hat{\beta}(\lambda)^2 = \hat{p}(\lambda)\hat{p}(-\lambda) = \Delta_+^{|k|}\Delta_-^{|l|}. \tag{4.8.1}$$

Thus, $\det \chi = 1$. Since $\hat{\alpha}$ is even and $\hat{\beta}$ is odd in λ, $\hat{\chi}$ and χ both satisfy the twisting condition (2.2.1), whence $\chi \in \Lambda_{r''}\mathbf{SL}(2,\mathbb{C})_\sigma$. As outlined in Section 4.5, χ is unitary on S^1, iff (4.5.14) and (4.5.12) are satisfied. But (4.5.14) follows from a'), b'), and the first part of (4.5.12) is just d'). The second condition is $(\hat{\beta}a)^* = -(\hat{\beta}a)$ and $(\hat{\beta}c)^* = -(\hat{\beta}b)$. To verify this condition we square $\hat{\beta}a$, $\hat{\beta}b$, and $\hat{\beta}c$ and obtain $((\hat{\beta}a)^2)^* = (\hat{\beta}a)^2$ and $((\hat{\beta}c)^2)^* = (\hat{\beta}b)^2$, since $\hat{\beta}^2$ and a^2 are real by b'), and $(c^2)^* = b^2$ by b). Hence $(\hat{\beta}a)^* = \pm\hat{\beta}a$ and $(\hat{\beta}c)^* = \pm\hat{\beta}b$. If $(\hat{\beta}a)^* = \hat{\beta}a$, then $\hat{\beta}a$ is real on S^1 and $\hat{\beta}^2a^2 = (\hat{\beta}a)^2$ is non-negative on S^1. But a) and c') imply that $\hat{\beta}^2a^2$ is non-positive on S^1. The only possibility for both conditions to hold is $\hat{\beta}a = 0$ on S^1. But in this case, of course, also $(\hat{\beta}a)^* = -\hat{\beta}a$ as desired. For the remaining case we consider e) and obtain $\hat{\beta}^2 = \hat{\beta}^2a^2 + (\hat{\beta}b)(\hat{\beta}c)$. If $(\hat{\beta}c)^* = +\hat{\beta}b$, then $\hat{\beta}^2 = \hat{\beta}^2a^2 + |\hat{\beta}c|^2$ on S^1. Hence $|\hat{\beta}c|^2 = \hat{\beta}^2(1-a^2)$ implies $\hat{\beta}^2 \equiv 0$ or $1 - a^2 \leq 0$. The first case is not possible in view of the form of $\hat{p}$. The second case yields in view of a), that $a^2 \equiv 1$ on S^1. Hence $a = \pm 1$ on $\mathbb{C}$, a contradiction, since $a(0) = 0$. Thus $(\hat{\beta}c)^* = -\hat{\beta}b$ as required.

Finally, we show $(k,l) \in \mathrm{Sym}(\Psi_{mn})$. To this end we multiply $\chi = \frac{1}{\Delta_+^{|k|}\Delta_-^{|l|}}h_+(\hat{\alpha}I + \hat{\beta}A)h_+^{-1}$ from the right with $h_+(U^\circ)^m(V^\circ)^n$. A simple calculation like the one leading to Lemma 4.3 yields

$$\frac{1}{\Delta_+^{|k|}\Delta_-^{|l|}}(\hat{\alpha}I + \hat{\beta}A) = (U^\circ)^k(V^\circ)^l w_+, \tag{4.8.2}$$

where

$$w_+ = \begin{pmatrix} \cosh(f_+) & \sinh(f_+) \\ \sinh(f_+) & \cosh(f_+) \end{pmatrix} \tag{4.8.3}$$

is holomorphic on $I^{(r)}$. Thus,

$$\chi h_+(U^\circ)^m(V^\circ)^n = h_+(U^\circ)^{m+k}(V^\circ)^{n+l}w_+. \tag{4.8.4}$$

Using the definition of $F_{mn}(\lambda)$ gives

$$\chi(\lambda)F_{mn}(\lambda)p_+(m,n,\lambda) = F(m+k,n+l,\lambda)p_+(m+k,n+l,\lambda)w_+. \tag{4.8.5}$$

This shows,

$$F_{m+k,n+l}(\lambda) = \chi(\lambda)F_{mn}(z,\lambda), \tag{4.8.6}$$

since the Iwasawa splitting chosen is unique and χ is unitary. □

5 Hyperelliptic Curves

For fixed lattice constants $r_1, r_2 \in \mathbb{R}^+$, let $\Psi_{mn} : \mathbb{Z}^2 \to \mathbb{R}^3$ be a CMC-immersion in the r-dressing orbit of the cylinder. I.e., if $F_{mn}(\lambda)$ is the extended frame of Ψ_{mn}, then there exists $r_{\min}(r_1,r_2) < r < 1$ and $h_+ \in \Lambda_r^+\mathbf{SL}(2,\mathbb{C})_\sigma$, such that $F_{mn}(\lambda)$ is given by (4.1.1). In this chapter we also assume, that Ψ_{mn} has a periodic metric, i.e., that there exists $(k,l) \in \mathrm{Sym}(\Psi_{mn})$, $(k,l) \neq (0,0)$.

As we have seen in the last section, we can characterize discrete CMC-surfaces with periodic metric in terms of the functions a, b, c, d given by $h_+Ah_+^{-1} = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}$. This characterization is in far reaching analogy to the description of CMC-immersions with periodic metric in the continuous case [5].

It is only natural to try to adapt also the algebro-geometric description of the periodicity conditions to the dicrete case, thereby aiming at a classification of discrete CMC-tori similar to the one of Pinkall and Sterling [18].

First we note, that the properties of the rational functions a^2, b^2, c^2 stated in Theorem 4.7 a)–e) are the same as those stated in a)–e) of [5, Theorem 3.6]. It is therefore natural to introduce a hyperelliptic surface associated to a discrete periodic CMC-surface in the same way as in the continuous case.

5.1 We define the new variable $\nu = \lambda^2$. We will regard the even rational functions $a^2(\lambda)$, $b^2(\lambda)$, $c^2(\lambda)$ as rational functions of ν. Since by b) in Theorem 4.7, a is an odd function in λ, $a^2(\lambda)$ has a zero of order $2(2n-1)$, $n > 0$, at $\lambda = 0$. Hence, as a function in ν, a^2 has a zero of odd order $2n-1$ at $\nu = 0$. Let $\nu_1, \ldots, \nu_k$ be the points in the ν-plane where a^2 has a pole of odd order, and let $\nu_{k+1}, \ldots, \nu_{k+l}$ be the points in the ν-plane away from $\nu = 0$ where a^2 has a zero of odd order.

As in the continuous case we easily get

Lemma: *None of the points $\nu_1, \ldots, \nu_{k+l} \in \mathbb{C}^*$ defined above lies on the unit circle.*

Proof: We have to show, that $a^2(\lambda)$ has neither a pole nor a zero of odd order on S^1. By d) in Theorem 4.7, we know, that a^2 has no poles on S^1. By c) and e) in Theorem 4.7, we have $(1-a^2)^2 = |b^2|^2$ on S^1, which shows that b^2 and $c^2 = (b^2)^*$ are defined on S^1 and also, that a^2, b^2, and c^2 cannot vanish simultaneously on S^1. If a^2 has a zero of odd order at $\lambda_0 \in S^1$, then $b^2(\lambda_0) \neq 0$. By d') of Theorem 4.7, $(\hat{\beta}a)^2$ and $(\hat{\beta}b)^2$ are squares of holomorphic functions on $\mathbb{C}^*$. Thus, the function $\hat{\beta}^2 = \frac{(\hat{\beta}a)^2}{a^2} = \frac{(\hat{\beta}b)^2}{b^2}$ has both a zero of odd order and of even order at λ_0. This implies $\hat{\beta} \equiv 0$, contradicting (4.4.2), □

Proposition: *Let k, l and $\nu_1, \ldots, \nu_{k+l}$ be defined as above. Then $g = \frac{1}{2}(k+l)$ is an integer and we can order the points $\nu_1, \ldots, \nu_{2g}$, such that*

$$\nu_{2n} = \tau(\nu_{2n-1}), \qquad |\nu_{2n-1}| < 1, \qquad n = 1, \ldots, g. \tag{5.1.1}$$

Proof: By Lemma 5.1, we have $|\nu_{2n-1}| \neq 1$ for $n = 1, \ldots, k+l$. Since $a^2(\nu)$ is real on S^1, we have using Section 2.5, that the set $B = \{\nu_1, \ldots, \nu_{2g}\}$ is invariant under the antiholomorphic involution $\tau : \nu \to \overline{\nu}^{-1}$. Since τ has no fixed points off the unit circle, we get that B consists of pairs $(\nu_n, \tau(\nu_n))$. This shows, that $k+l$ is even, whence $g = \frac{1}{2}(k+l)$ is an integer, and that we can order $\{\nu_1, \ldots, \nu_{2g}\}$, such that (5.1.1) holds. □

In the following we will order the points $\nu_1, \ldots, \nu_{2g}$ always such that (5.1.1) holds.

We proceed as in the continuous case by considering the algebraic equation

$$\mu^2 = \nu \prod_{k=1}^{2g} (\nu - \nu_k). \tag{5.1.2}$$

Theorem: *The plane affine curve $\tilde{\mathcal{C}}$ defined by (5.1.2) can be uniquely extended to a compact Riemann surface $\mathcal{C}$ of genus g. The meromorphic function $\nu : \tilde{\mathcal{C}} \to \mathbb{C}$ extends to a holomorphic map $\pi : \mathcal{C} \to \mathbb{C}\mathrm{P}_1$ of degree 2. The branchpoints of π are the roots of μ^2 and the point ∞.*

Proof: The proof follows immediately from [16, Lemma III.1.7] since μ^2 has odd degree. □

In other words, (5.1.2) is a (nonsingular) hyperelliptic curve, obtained by compactifying the plane affine curve $\tilde{\mathcal{C}} = \mathcal{C} \setminus \{P_\infty\}$, where $P_\infty = \pi^{-1}(\infty) \in \mathcal{C}$ is a single point.

Remark: In Proposition 5.4 it will be shown, that $a(\nu) = \sqrt{a^2(\nu)}$ can be lifted to a nonconstant meromorphic function on $\mathcal{C}$. It determines the complex structure of $\mathcal{C}$ uniquely (see [11, I.1.6], [12, Satz 8.9]).

5.2 On $\tilde{\mathcal{C}} = \mathcal{C} \setminus \{P_\infty\}$, ν and μ are holomorphic functions. Every point on $P \in \mathcal{C} \setminus \{P_\infty\}$ is determined uniquely by the values of ν and μ at P and can thus be identified with the pair $(\nu(P), \mu(P))$. Formally, we will also write $P_\infty = (\infty, \infty)$. In this notation we get $\pi(\nu, \mu) = \nu$. Let us define the hyperelliptic involution I on $\mathcal{C}$ by

$$I(\nu, \mu) = (\nu, -\mu). \tag{5.2.1}$$

A point on $\mathcal{C}$ is a branchpoint iff it is mapped by π to a branchpoint on $\mathbb{C}\mathrm{P}_1$. Clearly, the branchpoints of $\mathcal{C}$ are precisely the fixed points of I, i.e., the points $(\nu_k, 0)$, $k = 1, \ldots, 2g+1$ and P_∞. We will also write $P_0 = (0, 0)$. Locally around P_0 (P_∞), μ ($\tilde{\mu} = \mu\nu^{-(g+1)}$) and each branch of $\lambda(\nu, \mu) = \sqrt{\nu}$ ($\lambda^{-1}(\nu, \mu) = \nu^{-\frac{1}{2}}$) are local coordinates on $\mathcal{C}$, as can be easily checked using the representation of $\mathcal{C}$ given in [16, Chapter III].

Let us investigate the set of meromorphic functions on $\mathcal{C}$. By [16, Proposition 1.10], every meromorphic function on $\mathcal{C}$ can be uniquely written as

$$f(\nu, \mu) = f_1(\nu) + f_2(\nu)\mu, \tag{5.2.2}$$

with two rational functions f_1, f_2.

Remark: It is clear from the representation (5.2.2) of meromorphic functions on $\mathcal{C}$, that each rational function $f_1(\nu)$ can be lifted to a meromorphic function on $\mathcal{C}$ by setting $f(\nu, \mu) = f_1(\nu)$. Clearly, then $f \circ I = f$ for such a function.

Conversely, if $f : \mathcal{C} \to \mathbb{C}$ is meromorphic, then it can be identified with a rational function $f_1(\nu)$ iff $f_2(\nu) \equiv 0$ in (5.2.2), i.e., iff it satisfies $f \circ I = f$. We will frequently use this identification of rational functions in ν with I-invariant meromorphic functions on $\mathcal{C}$.

5.3 Let us define

$$\mathcal{S} = \pi^{-1}(S^1) = \{(\nu, \mu) \in \mathcal{C}; \nu \in S^1\}. \tag{5.3.1}$$

The set $\mathcal{S}$ is connected if g is even, and has two connected components if g is odd. Since $\mathcal{S}$ is contained in $\tilde{\mathcal{C}}$, we can identify it with a subset of $\mathbb{C}^2$. Using the antiholomorphic involution $\tau : \nu \to \overline{\nu}^{-1}$ defined in Section 2.5, we define the map $\tilde{\sigma} : \tilde{\mathcal{C}} \setminus \{P_0\} \to \tilde{\mathcal{C}} \setminus \{P_0\}$ by

$$\tilde{\sigma} : (\nu, \mu) \longmapsto (\overline{\nu}^{-1}, \overline{\nu}^{-(g+1)} \left(\prod_{i=1}^{2g} \nu_j \right)^{\frac{1}{2}} \overline{\mu}). \tag{5.3.2}$$

We will choose the sign of the square root such that the points on $\mathcal{S}$ are fixed by $\tilde{\sigma}$.

The following is well known (see e.g. [14]):

Theorem: *The map $\tilde{\sigma}$ defined by (5.3.2) can be extended to an antiholomorphic involution $\hat{\sigma}$ on $\mathcal{C}$, which preserves the points of $\mathcal{S} \subset \mathcal{C}$.*

Furthermore, $\hat{\sigma}$ commutes with the hyperelliptic involution and leaves invariant the set of branchpoints of $\mathcal{C}$.

Proof: Using Theorem 5.1 it is easily checked, that $\tilde{\sigma}$ defines an antiholomorphic involution on $\tilde{\mathcal{C}} \setminus \{P_0\}$. By using the appropriate branches of λ and λ^{-1} in the vicinity of P_0 and P_∞ on $\mathcal{C}$, we get $\tilde{\sigma}(\lambda) = \overline{\lambda}^{-1}$ in local coordinates around P_0 and P_∞, whence $\tilde{\sigma}$ extends to an antiholomorphic involution $\hat{\sigma}$ on $\mathcal{C}$, which maps P_0 to P_∞. By the choice of the square root in (5.3.2), $\hat{\sigma}$ fixes the

points on $\mathcal{S}$. $\hat{\sigma}$ clearly commutes with I. If P is a branchpoint of $\mathcal{C}$, then $I(P) = P$. Therefore, $I(\hat{\sigma}(P)) = \hat{\sigma}(I(P)) = \hat{\sigma}(P)$ and $\hat{\sigma}(P)$ is also a branchpoint. Thus, $\hat{\sigma}$ leaves invariant the set of branchpoints of $\mathcal{C}$. □

For a scalar function on $\mathcal{C}$ we also define

$$f^* = \overline{f \circ \hat{\sigma}}. \tag{5.3.3}$$

Since, by Proposition 5.3, $\hat{\sigma}$ fixes the points of $\mathcal{S}$, we get

Lemma: *Let f be a meromorphic function defined on a $\hat{\sigma}$-invariant subset of $\mathcal{C}$ which contains $\mathcal{S}$. Then f is real on $\mathcal{S}$ iff $f^* = f$ and f is purely imaginary on $\mathcal{S}$ iff $f^* = -f$.*

5.4 Let us now investigate the properties of a^2 w.r.t. $\mathcal{C}$.

Proposition: *The rational function a^2 defined in Theorem 4.7 is of the form*

$$a^2(\nu) = f(\nu)^2 \mu^2(\nu), \tag{5.4.1}$$

where f is rational and defined at $\nu = 0$. The function $a = f\mu$ is a meromorphic function on $\mathcal{C}$, which satisfies

$$a \circ I = -a \tag{5.4.2}$$

and

$$a^* = a. \tag{5.4.3}$$

Proof: By the definition of μ^2, the quotient $\frac{a^2}{\mu^2}$ is rational and has only poles and zeroes of even order. Therefore, it is the square of a rational function $f(\nu)$. Since a^2 has a zero of odd order at $\nu = 0$, $f(\nu)$ is defined at $\nu = 0$. By (5.2.2), $a = f\mu$ is a meromorphic function on $\mathcal{C}$. Since a^2 is real and non-negative on S^1, the function $a = \sqrt{a^2}$ takes real values over S^1 on $\mathcal{C}$, whence, by Lemma 5.3, (5.4.3) holds. Furthermore, $a \circ I = -f\mu = -a$ and (5.4.2) holds. □

Theorem: *Let ϕ be a meromorphic function on $\mathcal{C}$. Then ϕ is antisymmetric w.r.t. the hyperelliptic involution I, i.e. $\phi \circ I = -\phi$, iff $\phi(\nu, \mu) = f(\nu)\mu$, where f is a rational function. Furthermore, in this case the following holds:*

1. *Locally around P_0, ϕ is an odd meromorphic function of the local coordinate λ.*
2. *The product ϕa can be identified with a rational function of ν on $\mathbb{C}\mathrm{P}_1$.*

Proof: The equivalence statement follows immediately from the representation (5.2.2) of meromorphic functions on $\mathcal{C}$. Since λ is a local coordinates around P_0, and since $\lambda \circ I = -\lambda$, 1. holds. Finally, by (5.4.2), we have that ϕa is invariant under I, whence, by Remark 5.2, can be identified with a meromorphic function $(\phi a)(\nu)$ on $\mathbb{C}\mathrm{P}_1$. □

5.5 Let us also define the Riemann surface $\mathcal{C}'$ on which $\sqrt{a^2(\lambda)}$ is meromorphic. To be precise (see [16, Lemma III.1.7]), let $\mathcal{C}'$ be the hyperelliptic curve associated to the algebraic equation

$$\tilde{\mu}^2 = \prod_{i=1}^{2g} (\lambda - \sqrt{\nu_i})(\lambda + \sqrt{\nu_i}). \tag{5.5.1}$$

The holomorphic map λ extends to a holomorphic map $\pi' : \mathcal{C}' \to \mathbb{C}$ of degree 2.

As for $\mathcal{C}$ we will identify the points of $\mathcal{C}'$ with pairs $(\lambda, \tilde{\mu})$. It should be noted, that $\mathcal{C}'$ has no branchpoints over $\lambda = 0$ and $\lambda = \infty$ since a^2 is even in λ. In particular, $(\pi')^{-1}(\infty)$ consists of two different points $P_\infty^{(1)}$ and $P_\infty^{(2)}$. However, this will not cause any confusion here. By $P_0^{(1)}$ and $P_0^{(2)}$ we will denote the two covering points of $\lambda = 0$. Clearly, λ is a local coordinate around $P_0^{(1)}$ and $P_0^{(2)}$ and λ^{-1} is a local coordinate around $P_\infty^{(1)}$ and $P_\infty^{(2)}$.

Every meromorphic function $\tilde{f}$ on $\mathcal{C}'$ is of the form

$$\tilde{f}(\lambda, \tilde{\mu}) = \tilde{f}_1(\lambda) + \tilde{f}_2(\lambda)\tilde{\mu} \tag{5.5.2}$$

with two rational functions $\tilde{f}_1$ and $\tilde{f}_2$.

Remark: Let I' be the hyperelliptic involution on $\mathcal{C}'$. As in Remark 5.2, we will use the representation (5.5.2) to identify rational functions of λ with I'-invariant meromorphic functions on $\mathcal{C}'$. In addition, we have

Proposition: *There is a one-to-one correspondence between*

1. *meromorphic functions on $\mathcal{C}$ and*
2. *meromorphic functions $\tilde{f}$ on $\mathcal{C}'$ which satisfy*

$$\tilde{f}(-\lambda, \tilde{\mu}) = \tilde{f}(\lambda, -\tilde{\mu}). \tag{5.5.3}$$

Proof: In terms of λ we have $\mu^2(\lambda) = \lambda\tilde{\mu}^2(\lambda)$. We can therefore rewrite (5.2.2) as

$$f(\nu, \mu) = f_1(\nu) + f_2(\nu)\lambda\tilde{\mu} = \tilde{f}_1(\lambda) + \tilde{f}_2(\lambda)\tilde{\mu} = \tilde{f}(\lambda, \mu), \tag{5.5.4}$$

where $\tilde{f}_1(\lambda) = f_1(\nu = \lambda^2)$ and $\tilde{f}_2(\lambda) = f_2(\nu = \lambda^2)\lambda$ are rational. This shows, that f can be identified with the meromorphic function $\tilde{f}$ on $\mathcal{C}'$. Since $\tilde{f}_1$ is even and $\tilde{f}_2$ is odd in λ, also (5.5.3) is satisfied. Conversely, if $\tilde{f}$ is a meromorphic function on $\mathcal{C}'$ such that (5.5.3) is satisfied, then in (5.5.2), $\tilde{f}_1$ is even and $\tilde{f}_2$ is odd in λ. Therefore, $f_1(\nu) = \tilde{f}_1(\lambda^2)$ and $f_2(\nu) = \lambda^{-1}\tilde{f}_2(\lambda)$ are rational and define a meromorphic function f on $\mathcal{C}$ via (5.2.2). □

Corollary: *The function $a^2(\lambda)$ is the square of a meromorphic function on $\mathcal{C}'$.*

Proof: By Proposition 5.4, a^2 is the square of a meromorphic function a on $\mathcal{C}$ which is of the form $a = f\mu$, where f is rational in ν. Using the proof of Proposition 5.5 we can identify a with a meromorphic function $\tilde{a}$ on $\mathcal{C}'$ which is of the form $\tilde{a} = \tilde{f}\tilde{\mu}$, where $\tilde{f}(\lambda) = f(\nu = \lambda^2)\lambda$ is an odd rational function in λ. Then $\tilde{a}^2$ is I'-invariant and $\tilde{a}^2 = \tilde{f}^2\tilde{\mu}^2 = f^2\mu^2 = a^2$, proving the claim. □

5.6 Let us define the non-compact Riemann surface $\mathcal{C}^* = \mathcal{C} \setminus \{P_0, P_\infty\}$. We already know, that a is meromorphic on $\mathcal{C}'$ and $\mathcal{C}$, and therefore also on $\mathcal{C}^*$. Now we prove the following important result:

Theorem: *The functions $\hat{\alpha}$ and $\hat{\beta}$ are meromorphic on $\mathcal{C}$ without poles on $\mathcal{C}^*$. The functions $b = \sqrt{b^2}$ and $c = \sqrt{c^2}$ are meromorphic on $\mathcal{C}'$ without poles over 0 and ∞.*

Proof: We know by a') and d') in Theorem 4.7, that $\hat{\beta}a$, $\hat{\alpha}$, and $\hat{\beta}^2$ are even rational functions in λ, which don't have poles on $\mathbb{C}^*$. Therefore, by Remark 5.2, they can be identified with I-invariant meromorphic functions on $\mathcal{C}$ without poles on $\mathcal{C}^*$. Since also a is meromorphic on $\mathcal{C}$, we have that $\hat{\beta} = \frac{\hat{\beta}a}{a}$ is a meromorphic function on $\mathcal{C}$. Since the square $\hat{\beta}^2$ is holomorphic on $\mathbb{C}^*$, $\hat{\beta}$ has no poles on $\mathcal{C}^*$.

By Proposition 5.5, $\hat{\beta}$ can be identified with a meromorphic function on $\mathcal{C}'$. Furthermore, $\hat{\beta}b$ and $\hat{\beta}c$ are rational functions of λ. Thus, $\hat{\beta}b$, $\hat{\beta}c$, $b = \frac{\hat{\beta}b}{\hat{\beta}}$ and $c = \frac{\hat{\beta}c}{\hat{\beta}}$ are meromorphic on $\mathcal{C}'$. Since, by

Theorem 4.7, b and c are in $\mathcal{A}_r^+$, they can be continued holomorphically to $\lambda = 0$ on $\mathbb{C}\mathrm{P}_1$. By e) in Theorem 4.7, the same holds for b and c around $\lambda = \infty$. Since λ is a local coordinate around $P_0^{(1)}$ and $P_0^{(2)}$ and λ^{-1} is a local coordinate around $P_\infty^{(1)}$ and $P_\infty^{(2)}$, the functions b and c can be extended holomorphically to these points, which finishes the proof. □

The proof of the following proposition is the same as the one of [5, Prop. 4.6].

Proposition: *With $(\cdot)^*$ defined in Section 5.3, the holomorphic functions $\hat{\alpha}$ and $\hat{\beta}$ on $\mathcal{C}^*$ satisfy*

$$\hat{\alpha}^* = \hat{\alpha}, \qquad \hat{\beta}^* = -\hat{\beta}, \tag{5.6.1}$$

$$\alpha \circ I = \alpha \text{ and } \beta \circ I = -\beta. \tag{5.6.2}$$

Furthermore, the meromorphic functions α, β and b and c on $\mathcal{C}'$ satisfy

$$c = b^*, \tag{5.6.3}$$

$$\alpha \circ I' = \alpha \text{ and } \beta \circ I' = -\beta, \tag{5.6.4}$$

$$b \circ I' = -b \text{ and } c \circ I' = -c. \tag{5.6.5}$$

It is easily possible to pursue this road of deriving results analogous to the continuous case, as it was done in [5, Sections 4.8–4.9]. In particular, one can define higher commuting flows acting on discrete CMC-surfaces and finite type solutions of the discrete sinh-Gordon equation in precisely the same way as in the continuous case. The same argument as in the proof of [5, Theorem 4.9] then shows, that discrete CMC-surfaces with periodic metric belong to finite type solutions of the discrete sinh-Gordon equation. However, since there is up to now no theory of finite type solutions of the discrete sinh-Gordon equation, this does not promise any help towards our goal.

6 Algebro-geometric description of surfaces with periodic metric

For a discrete CMC-surface with periodic metric we defined in Section 5.1 a nonsingular hyperelliptic curve $\mathcal{C}$. In this section, we will show, that $\mathcal{C}$ allows us to express the periodicity conditions for discrete CMC-surfaces stated in Theorem 4.7 and Theorem 4.8 in terms of algebro-geometric data.

We will also investigate the case, that the discrete CMC-surface Ψ_{mn} under consideration does not only have a periodic or doubly periodic metric, but closes in $\mathbb{R}^3$, i.e., that the image $\{\Psi_{mn}; (m,n) \in \mathbb{Z}^2\} \subset \mathbb{R}^3$ consists of finitely many points. This case is an obvious analogue of a CMC-torus. We will call such surfaces *discrete CMC-tori*.

6.1 We will first reformulate the statement of Theorem 4.7 in terms of algebro-geometric data. We start with the same assumptions as in Section 5: For fixed lattice constant $r_1, r_2 \in \mathbb{R}^+$, let $\Psi_{mn} : \mathbb{Z}^2 \to \mathbb{R}^3$ be a discrete CMC-surface with extended frame $F_{mn} \in \mathcal{F}_0(r_1, r_2)$, obtained by dressing the discrete cylinder with some $h_+ \in \Lambda_r^+ \mathbf{SL}(2,\mathbb{C})_\sigma$, $r_{\min}(r_1, r_2) < r < 1$. We also assume, that $\mathrm{Sym}(\Psi_{mn})$ contains a nontrivial element $(k,l) \neq (0,0)$. I.e., Ψ_{mn} has periodic metric and we can define the hyperelliptic curve $\mathcal{C}$ as in Section 5.1.

We introduce a standard homotopy basis for $\mathcal{C}$ which is adapted to the $\hat{\sigma}$-symmetry of $\mathcal{C}$ stated in Proposition 5.3. Let $a_1, \ldots, a_g, b_1, \ldots, b_g$, g the genus of $\mathcal{C}$, be a canonical basis of $H_1(\mathcal{C}, \mathbb{Z})$, such that the intersection numbers are given by

$$a_i a_j = 0, \qquad b_i b_j = 0, \qquad a_i b_j = \delta_{ij}, \qquad i,j = 1, \ldots, g. \tag{6.1.1}$$

For the cycles a_k we choose (see [11, VII.7.1])

$$a_k = \gamma_k - I \circ \gamma_k, \tag{6.1.2}$$

where γ_k is a curve joining the branchpoints over ν_{2k-1} and ν_{2k}, which satisfies $\hat{\sigma} \circ \gamma = -\gamma$. Then

$$I \circ a_k = -a_k \tag{6.1.3}$$

and

$$\hat{\sigma} \circ a_k = -a_k, \tag{6.1.4}$$

since $\hat{\sigma}$ and I commute. I.e., the cycles a_k are up to orientation invariant under $\hat{\sigma}$ and I. In addition, we can choose b_k such that

$$\hat{\sigma} \circ b_k = b_k - a_k + \sum_{j=1}^{g} a_j. \tag{6.1.5}$$

6.2 By Theorem 5.6, $\hat{\alpha}$ and $\hat{\beta}$ are meromorphic functions on $\mathcal{C}$ without poles on $\mathcal{C}^*$. Therefore, $\mathrm{d}\hat{\alpha}$ and $\mathrm{d}\hat{\beta}$ and $\hat{\alpha}\mathrm{d}\hat{\beta} - \hat{\beta}\mathrm{d}\hat{\alpha}$ are meromorphic one-forms on $\mathcal{C}$ without poles on $\mathcal{C}^*$. We define the meromorphic one-form ω on $\mathcal{C}$ by

$$\omega := \frac{\hat{\alpha}\mathrm{d}\hat{\beta} - \hat{\beta}\mathrm{d}\hat{\alpha}}{\Delta_+^{|k|}\Delta_-^{|l|}}. \tag{6.2.1}$$

Using (4.4.4) and (4.3.9), this can be rewritten as

$$\omega = \alpha\mathrm{d}\beta - \beta\mathrm{d}\alpha = (\alpha - \beta)\mathrm{d}(\alpha + \beta) = \frac{\mathrm{d}(\alpha + \beta)}{\alpha + \beta} \tag{6.2.2}$$

for

$$\alpha = \frac{1}{\Delta_+^{|k|}\Delta_-^{|l|}}\hat{\alpha}, \qquad \beta = \frac{1}{\Delta_+^{|k|}\Delta_-^{|l|}}\hat{\beta}. \tag{6.2.3}$$

From now on, we restrict our attention to the case that k and l are even. Since $\mathrm{Sym}(\Psi_{mn})$ is an additive group, it is clear, that for every discrete CMC-surface with periodic metric, there exist nontrivial $(k, l) \in \mathrm{Sym}(\Psi_{mn})$ with k and l even. Thus our choice of k and l even does not impose a restriction on the class of discrete CMC-surfaces under consideration. Since Δ_+^2 and Δ_-^2 are even rational functions of λ, i.e. by Remark 5.2, meromorphic functions on $\mathcal{C}$, it is clear that in this case with $\hat{\alpha}$ and $\hat{\beta}$ also α and β are meromorphic on $\mathcal{C}$.

In the following we will denote by $\mathcal{I}^{(r)}$ the set of all points on $\mathcal{C}$ which are mapped to $I^{(r)}$ by the projection π. Since the surface $\mathcal{C}$ constructed in Section 5.1 has no branchpoints in $\mathcal{I}^{(r)} \setminus \{P_0\}$, the local coordinate λ extends to all of $\mathcal{I}^{(r)}$, i.e., $\mathcal{I}^{(r)}$ is a chart domain on $\mathcal{C}$ around P_0. Similarly, we define $\mathcal{E}^{(\frac{1}{r})}$ to be the set of all points on $\mathcal{C}$ which are mapped to $\tau(I^{(r)}) = \{\nu; |\nu| > \frac{1}{r}\}$. Obviously, $\hat{\sigma}(\mathcal{I}^{(r)}) = \mathcal{E}^{(\frac{1}{r})}$. Each branch of λ^{-1} extends to a local coordinate on $\mathcal{E}^{(\frac{1}{r})}$. For a chosen branch of λ on $\mathcal{I}^{(r)}$ we fix a branch of λ^{-1} by requiring

$$\lambda^{-1}(\hat{\sigma}(P)) = \overline{\lambda(P)} \text{ for all } P \in \mathcal{I}^{(r)}. \tag{6.2.4}$$

Thus $\hat{\sigma}$ restricts to the map $\lambda \mapsto \overline{\lambda}^{-1}$ from $\mathcal{I}^{(r)}$ to $\mathcal{E}^{(\frac{1}{r})}$.

From (4.3.11) and (4.3.6), we get on $\mathcal{I}^{(r)}$:

$$\alpha + \beta = \frac{(1 + \epsilon_k r_1(\lambda^{-1} - \lambda))^{|k|}(1 + i\epsilon_l r_2(\lambda^{-1} - \lambda))^{|l|}}{(1 - r_1^2(\lambda^{-1} - \lambda)^2)^{\frac{|k|}{2}}(1 + r_2^2(\lambda^{-1} + \lambda)^2)^{\frac{|l|}{2}}} e^{f_+} = S_{r_1}(\lambda)^{\frac{k}{2}} T_{r_2}(\lambda)^{\frac{l}{2}} e^{f_+}, \tag{6.2.5}$$

where S_{r_1} and T_{r_2} were defined in Section 4.2 and f_+ is a holomorphic function on $\mathcal{I}^{(r)}$. This shows, that $\alpha + \beta$ has no pole at P_0. Therefore, by (6.2.2), ω has poles on $\mathcal{I}^{(r)}$ only where S_{r_1} and T_{r_2}

have zeroes or poles, and all of these poles are simple. Furthermore, by Proposition 2.1 and c') in Theorem 4.7,

$$(\alpha+\beta)^* = \alpha - \beta = (\alpha+\beta)^{-1}. \tag{6.2.6}$$

Therefore, $\alpha+\beta$ and ω are meromorphic on $\mathcal{E}^{(\frac{1}{r})}$.

Recall the definition of λ_+ and λ_- in Lemma 2.1. On $I^{(r)}$, S_{r_1} has precisely one zero at λ_+ and one pole at $-\lambda_+$, both are simple. Also on $I^{(r)}$, T_{r_2} has precisely one (simple) zero at $i\lambda_-$ and one (simple) pole at $-i\lambda_-$. Therefore, the representation (6.2.2) together with (6.2.5) shows, that ω has precisely 4 poles on $\mathcal{I}^{(r)}$, which are located at $\pm\lambda_+$ and $\pm i\lambda_-$ with $\lambda_\pm$ defined as in Lemma 2.1. All of these poles are simple and the residues of ω are given by

$$\begin{aligned} \mathrm{res}_{\lambda_+}\omega &= -\mathrm{res}_{-\lambda_+}\omega = \frac{k}{2}, \\ \mathrm{res}_{i\lambda_-}\omega &= -\mathrm{res}_{-i\lambda_-}\omega = \frac{l}{2}. \end{aligned} \tag{6.2.7}$$

Using the local coordinate λ^{-1} on $\mathcal{E}^{(\frac{1}{r})}$, we get another set of 4 simple poles of ω, which are located at $\pm\lambda_+^{-1}$ and $\pm i\lambda_-^{-1}$. The residues of ω at these poles are given by

$$\begin{aligned} \mathrm{res}_{-\lambda_+^{-1}}\omega &= -\mathrm{res}_{\lambda_+^{-1}}\omega = \frac{k}{2}, \\ \mathrm{res}_{-i\lambda_-^{-1}}\omega &= -\mathrm{res}_{i\lambda_-^{-1}}\omega = \frac{l}{2}. \end{aligned} \tag{6.2.8}$$

Since $\hat\alpha \mathrm{d}\hat\beta - \hat\beta \mathrm{d}\hat\alpha$ has no poles on $\mathcal{C}^*$, the form ω has by (6.2.1) no further poles on $\mathcal{C}$. We have proved the first part of the

Lemma: *The one-form ω has precisely 8 simple poles on $\mathcal{C}$ with residues given by (6.2.7) and (6.2.8). Furthermore,*

$$\hat\sigma^*\omega = -\overline{\omega} \tag{6.2.9}$$

and

$$\int_{a_k} \omega = 0, \qquad k = 1,\dots,g. \tag{6.2.10}$$

ω is an Abelian differential of the third kind, which is completely determined by (6.2.7), (6.2.8) and (6.2.10).

Proof: Note that by Propositions 2.1,5.6 and Lemma 5.3, we have

$$\alpha^* = \overline{\alpha\circ\hat\sigma} = \alpha, \qquad \beta^* = \overline{\beta\circ\hat\sigma} = -\beta. \tag{6.2.11}$$

From this it follows,

$$\hat\sigma^*\mathrm{d}\alpha = \overline{\mathrm{d}\alpha}, \qquad \hat\sigma^*\mathrm{d}\beta = -\overline{\mathrm{d}\beta}. \tag{6.2.12}$$

With this, Eq. (6.2.9) follows from (6.2.1). Since an Abelian differential of the third kind with only simple poles is completely determined by its residues and its a_k-cycles (see [11, Prop. III.3.3]), it only remains to be shown, that (6.2.10) holds.

Since ω is the logarithmic derivative of the meromorphic function $\alpha+\beta$ on $\mathcal{C}$, its integral over a closed cycle on $\mathcal{C}$ (avoiding the poles of $\alpha+\beta$) is an integer multiple of $2\pi i$. By (6.2.9) and (6.1.4), we have

$$\overline{\int_{a_k}\omega} = \int_{a_k}\overline{\omega} = -\int_{a_k}\hat\sigma^*\omega = -\int_{\hat\sigma\circ a_k}\omega = \int_{a_k}\omega. \tag{6.2.13}$$

Thus, $\int_{a_k}\omega$ is real, which implies (6.2.10). □

6.3 Let Ω_+ be an Abelian differential of the third kind on $\mathcal{C}$, which has two simple poles at λ_+ and $-\lambda_+$ in $\mathcal{I}^{(r)}$ with residue

$$\mathrm{res}_{\lambda_+}\Omega_+ = -\mathrm{res}_{-\lambda_+}\Omega_+ = 1. \tag{6.3.1}$$

Such a differential exists by [11, Theorem II.5.2]. If we require

$$\int_{a_k}\Omega_+ = 0, \qquad k = 1,\dots,g, \tag{6.3.2}$$

then Ω_+ is uniquely determined by (6.3.1) and (6.3.2) (see [11, Prop. III.3.3]). We also introduce a second Abelian differential Ω_- of the third kind on $\mathcal{C}$, which has two simple poles at $i\lambda_-$ and $-i\lambda_-$ in $\mathcal{I}^{(r)}$. It is uniquely determined by

$$\mathrm{res}_{i\lambda_-}\Omega_- = -\mathrm{res}_{-i\lambda_-}\Omega_- = 1 \tag{6.3.3}$$

and

$$\int_{a_k}\Omega_- = 0, \qquad k = 1,\dots,g. \tag{6.3.4}$$

If we define

$$\Omega_+^* = \overline{\hat\sigma^*\Omega_+}, \qquad\qquad \Omega_-^* = \overline{\hat\sigma^*\Omega_-}, \tag{6.3.5}$$

then Ω_+^* and Ω_-^* are also Abelian differentials of the third kind with simple poles at $\pm\lambda_+^{-1}$ and $\pm i\lambda_-^{-1}$ in $\mathcal{E}^{(\frac{1}{r})}$, respectively. The residues of Ω_+^* and Ω_-^* are given by

$$\mathrm{res}_{\lambda_+^{-1}}\Omega_+^* = -\mathrm{res}_{-\lambda_+^{-1}}\Omega_+^* = 1 \tag{6.3.6}$$

and

$$\mathrm{res}_{i\lambda_-^{-1}}\Omega_-^* = -\mathrm{res}_{-i\lambda_-^{-1}}\Omega_-^* = 1. \tag{6.3.7}$$

By (6.1.4) and (6.3.2) we get for $k = 1,\dots,g$:

$$\int_{a_k}\Omega_+^* = \int_{a_k}\overline{\hat\sigma^*\Omega_+} = \overline{\int_{\hat\sigma\circ a_k}\Omega_+} = -\overline{\int_{a_k}\Omega_+} = 0. \tag{6.3.8}$$

In the same way it follows from (6.3.4) that

$$\int_{a_k}\Omega_-^* = 0, \qquad k = 1,\dots,g. \tag{6.3.9}$$

Comparing (6.2.10) with (6.3.2), (6.3.4), (6.3.8), (6.3.9) and comparing (6.2.7), (6.2.8) with (6.3.1), (6.3.3), (6.3.6), (6.3.7), we see that

$$\omega = \frac{k}{2}(\Omega_+ - \Omega_+^*) + \frac{l}{2}(\Omega_- - \Omega_-^*). \tag{6.3.10}$$

Let us set

$$U_n^+ = \int_{b_n}\Omega_+, \qquad\qquad U_n^- = \int_{b_n}\Omega_-, \qquad n = 1,\dots,g. \tag{6.3.11}$$

Then with (6.1.5) and (6.3.2),(6.3.4),

$$\int_{b_n}\Omega_+^* = \overline{U_n^+}, \qquad\qquad \int_{b_n}\Omega_-^* = \overline{U_n^-}, \qquad k = 1,\dots,g. \tag{6.3.12}$$

Lemma: *The integrals U_n^+ and U_n^- of Ω_+ and Ω_- over the b_n-cycles satisfy*

$$k\mathrm{Im}(U_n^+) + l\mathrm{Im}(U_n^-) = 2\pi m_n, \qquad m_n \in \mathbb{Z}. \tag{6.3.13}$$

Proof: We can deform the cycles b_n homotopically such that none of them crosses one of the poles of ω. Then the integral of ω over b_n is an integer multiple of $2\pi i$, since it is the logarithmic derivative of the meromorphic function $\alpha + \beta$ on $\mathcal{C}$. From this and (6.3.10) the claim follows. □

The question naturally arises, if there really exist algebro-geometric data $\mathcal{C}$, s.t. the periodicity conditions are satisfied. In the continuous case this question was answered in the affirmative first by Ercolani, Knörrer, and Trubowitz for even genus $g > 0$ of the spectral curve, later by Jaggy [14] for every genus $g > 1$. It should in principle be possible to start a similar investigation for the discrete case.

There exists numerical evidence, that as in the continuous case, the conditions cannot be satisfied for genus $g = 1$ of the spectral curve. However, T. Hoffmann has constructed discrete Delaunay type surfaces using a different approach. It remains to be investigated, if these surfaces are covered by the method of this paper.

Finally, besides of existence proofs, it is an interesting problem to construct such surfaces numerically, as it was done in the continuous case by M. Heil [13]. Using theta functions it is no problem to give an explicit formula for the discrete 'immersion' Ψ. The formulae look similar to those of Bobenko, just replacing the differentials of the second kind by differentials of the third kind.

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