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Anyonic Interpretation of Virasoro Characters and the Thermodynamic Bethe Ansatz

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Abstract

Employing factorized versions of characters as products of quantum dilogarithms corresponding to irreducible representations of the Virasoro algebra, we obtain character formulae which admit an anyonic quasi-particle interpretation in the context of minimal models in conformal field theories. We propose anyonic thermodynamic Bethe ansatz equations, together with their corresponding equation for the Virasoro central charge, on the base of an analysis of the classical limit for the characters and the requirement that the scattering matrices are asymptotically phaseless.

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1 Introduction

It is well known, that in 1+1 dimensions spin as well as statistics are entirely matters of convention. The first observation concerning the spin is due to Wigner [1], who found that one particle states always acquire a form such that they transform as scalars with respect to the action of the Lorentz group. The second observation concerning the spin is due to Swieca et al [2], who provided an explicit form for a transformation which relates particles obeying different types of statistics. In general it is possible to make these correspondences consistent for the bosonic or fermionic situation, whereas for more exotic, i.e. anyonic, statistics this task is far more complicated.

These statements hold for conformal field theories as well as for massive models. A natural question which arises, is how these features reflect itself in the analysis, which relates conformal and massive models, i.e. in the thermodynamic Bethe ansatz (TBA) [3, 4, 5]. Unexpectedly, it will turn out that, contrary to the fermionic and bosonic situations, in the anyonic context matters simplify drastically. Clearly, in order to derive the TBA-equations one should carry out an analysis analogously to the seminal work by Yang and Yang [3]. However, to obtain a first insight it is also possible to take a shortcut and follow a different route which reaches the aim quicker.

Recently the observation [6, 7] was made, that the conventional character formulae for irreducible representations of the Virasoro algebra of Feign-Fuchs and Rocha-Caridi [8, 9] can be expressed also in an alternative form. From a mathematical point of view, these equivalent formulations correspond to generalizations of the famous identities of Rogers-Ramanujan and Schur [10]. From a physical point of view they are remarkable, since they lead to an interpretation in terms of fermionic rather than bosonic quasi-particles. Furthermore, they may be exploited in order to obtain the TBA-equations from their classical limit [11, 6, 12, 7]. It is this connection on which we will base our investigation. More precisely, for the proposed anyonic versions of the characters, which are equivalent to the fermionic as well as the bosonic formulae, we assume this connection also to be valid. This assumption is sustained by our analysis.

Our approach exploits the fact, that, as first observed by Rocha-Caridi [9], and further developed by Christe [13], certain characters factorize in terms of infinite products (see also [14, 15, 16]). Technically, it turns out to be extremely useful to exploit the properties of the so-called quantum dilogarithm [17, 18] in order to deal with such expressions. In particular the computation of the classical limit becomes extremely simple in this language.

Our manuscript is organized as follows: In section 2 we make some general remarks concerning the scattering matrices and the TBA-equations in the context of generalized, i.e. anyonic, statistics. In section 3 we prove that certain characters factorize as products of quantum dilogarithms and investigate their classical limit. In section 4 we propose the generalized thermodynamic Bethe ansatz equation on the base of a saddle point analysis and compare with the predictions of section 3. In section 5 we propose some anyonic type character formulae and discuss concrete examples of them together with the explicit construction of some anyonic quasi-particle states. We state our conclusions and further outlook in section 6.

2 On generalized statistics

In order to motivate our considerations we recall the transformation proposed in [2]. Assuming, for instance, that $b^\dagger(\theta)$ * is a creation operator obeying the equal-time commutation relations for a free boson one may easily show that

$$a^\dagger(\theta) = b^\dagger(\theta) \exp\left(2\pi i s \int_\theta^\infty d\theta' b^\dagger(\theta') b(\theta') \right) \tag{1}$$

obeys exchange relations related to more exotic (i.e. anyonic) statistics whenever s is taken not to be an integer or half integer

$$a^\dagger(\theta) a^\dagger(\theta') = e^{2\pi i s \epsilon(\theta-\theta')} a^\dagger(\theta') a^\dagger(\theta) \quad . \tag{2}$$

Here we have $\epsilon(\theta) = \Theta(\theta) - \Theta(-\theta)$, where $\Theta(\theta)$ is the usual step-function. Similar relations also hold for annihilation operators upon replacing $b^\dagger(\theta)$ with $b(\theta)$ and $a^\dagger(\theta)$ with $a(\theta)$. For anyonic statistics these transformations have to be taken with caution, since as it was already pointed out in [2], they will lead immediately to several problems concerning locality, probability measurements etc. However, these issues are not of our concern here, since for our purposes we do not need an explicit representation for these operators. For an introductory account we refer the reader for instance to some lectures by Montonen [19].

It is interesting to note, that transformations of the type (1) generalize further and serve as an easy tool to construct explicit realizations for the Zamolodchikov algebra [20], which is generated by some asymptotic creation operators $Z^\dagger(\theta)$.

What the scattering matrices concerns, these transformations may be incorporated as follows. The factorized n-particle scattering matrix is defined via the relation

$$Z_n^\dagger(\theta_n) \ldots Z_1^\dagger(\theta_1) \left|0\right\rangle_{\text{out}} = \prod_{1 \leq i < j \leq n} S_{ij}(\theta_i - \theta_j) Z_1^\dagger(\theta_1) \ldots Z_n^\dagger(\theta_n) \left|0\right\rangle_{\text{in}} \,, \tag{3}$$

with $\mathrm{Re}(\theta_1) > \ldots > \mathrm{Re}(\theta_n)$. $\left|0\right\rangle_{\text{in/out}}$ is the vacuum. It follows immediately from this, that if a state involves two particles which possess the same rapidity (which is strictly to be understood as a limit $\theta_i \to \theta_j$), $S_{ij}(0)$ will account only

*As usual in this context, the two-momentum of a particle is parameterized by its rapidity $p = m(\cosh\theta, \sinh\theta)$.

for the statistics of the particles. Sometimes a statistics factor, say σ, is explicitly extracted from the S-matrix such that $S_{ij}(0) = 1$ always holds. We shall adopt the convention that this factor is incorporated into $S_{ij}(\theta)$. Regarding the scattering matrix in (3) as the one which results from the bootstrap analysis, it was suggested in [21, 22] to introduce instead a matrix

$$\hat{S}_{ij}(\theta) = S_{ij}(\theta)\left(\exp(-2\pi i\Delta_{ij}^{+})\Theta(\theta) + \exp(-2\pi i\Delta_{ij}^{-})\Theta(-\theta)\right), \tag{4}$$

where $\Delta_{ij}^{\pm}$ is the asymptotic phase of $S_{ij}(\theta)$

$$\lim_{\theta\to\pm\infty} S_{ij}(\theta) = \exp\left(2\pi i\Delta_{ij}^{\pm}\right). \tag{5}$$

The expression (4) has the virtue, that it compensates the asymptotic phases and creates a non-trivial phase at $\theta \to 0$, i.e. the anyonic situation. Thus, one observes, that for the scattering matrix the different statistics may be implemented more easily than for the asymptotic creation and annihilation operators. It was argued [21, 22], that from a physical point of view S-matrices which possess a non-trivial asymptotic phase should be regarded rather as auxiliary objects. However, these type of S matrices usually emerge from a bootstrap analysis (see below). On the other hand, scattering matrices of the type (4) should be considered as the genuine physical quantities, since they lead to the correct physical properties, in particular the statistics. From a purely technical point of view they are however less suitable because of their non-trivial analytic properties.

Most of the known scattering matrices related to integrable models in 1+1 dimensions exhibit the feature that these asymptotic phases are non-trivial. Typical examples for diagonal S-matrices, to which we shall be referring more below, are for instance the ones of affine Toda field theories with real coupling constants related to simply laced Lie algebras [23]

$$S_{ij}(\theta) \; = \; \prod_{p=1}^{h}\left\{2p - \frac{c(i)+c(j)}{2}\right\}_{\theta}^{-\frac{1}{2}\lambda_i\cdot\sigma^p\gamma_j}. \tag{6}$$

The $\{\}_\theta$ are building blocks consisting out of sinh-functions, i.e. $\{x\}_\theta = [x]_\theta/[x]_{-\theta}$, $[x]_\theta = (\langle x+1\rangle_\theta \langle x-1\rangle_\theta)/(\langle x+1-B\rangle_\theta \langle x-1+B\rangle_\theta)$ and $\langle x\rangle_\theta = \sinh\frac{1}{2}\left(\theta + \frac{i\pi x}{h}\right)$. $B(\beta) = \frac{2\beta^2}{4\pi+\beta^2}$ is the effective coupling constant which takes values between 0 and 2 when β is taken to be purely real. h denotes the Coxeter number and σ a particular Coxeter element, $c(i) = \pm 1$ the colour values related to the bicolouration of the Dynkin diagram, λ_i a fundamental weight and γ_i is $c(i)$ times a simple root. Whenever the coupling constant dependent part is omitted the S-matrix is referred to as minimal.

The scattering matrix (6) has the property $S_{ij}(0) = (-1)^{\delta_{ij}}$. For the phases one obtains in these cases

$$\Delta_{ij}^{\pm} = \pm\left(\frac{\delta_{ij}}{2} - \left(C^{-1}\right)_{ij}\right) \tag{7}$$

for the minimal models (C denoting here the Cartan matrix of the corresponding Lie algebra) and

$$\Delta^{\pm}_{ij} = \mp \frac{\delta_{ij}}{2} \tag{8}$$

for the full theory [5, 24]. The Sine-Gordon model, as the first non-trivial example for a non-diagonal S-matrix also exhibits these features. The S-matrix reads [25]

$$S_{SG}(\theta) = \frac{S_0(\theta)}{xp - p^{-1}x^{-1}} \begin{pmatrix} xp - p^{-1}x^{-1} & 0 & 0 & 0 \\ 0 & p - p^{-1} & x - x^{-1} & 0 \\ 0 & x - x^{-1} & p - p^{-1} & 0 \\ 0 & 0 & 0 & xp - p^{-1}x^{-1} \end{pmatrix} \tag{9}$$

where $x = \exp(8\pi\theta/\gamma)$, $p = \exp(-i8\pi^2/\gamma)$ and

$$S_0(\theta) = \exp\left(i \int_0^\infty \frac{dt}{t} \frac{\sin\frac{t\theta}{\pi} \sinh t(\frac{\gamma}{16\pi} - \frac{1}{2})}{\sinh\frac{t\gamma}{16\pi} \cosh\frac{t}{2}} \right) . \tag{10}$$

Thus we obtain from this (see e.g. [22])

$$\lim_{\theta\to\pm\infty} S_{SG}(\theta) = \pm i p^{\pm\frac{1}{2}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & p^{\mp 1} & 0 \\ 0 & p^{\mp 1} & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} . \tag{11}$$

A testing ground for S-matrices in general is the thermodynamic Bethe ansatz analysis. This analysis provides information about the consistency of an S-matrix and leads ultimately to the information to which model a massive theory flows in the conformal limit. Or vice versa, regarding the massive model as a perturbation of some conformal field theory in the spirit of [26], it provides the information about the conformal ancestor of a given massive theory. The key equations of the thermodynamic Bethe ansatz [3, 4, 5] related to diagonal S-matrices, which determine the pseudo-energies ε_i are

$$\varepsilon_i = \pm \sum_{j=1}^{r} N_{ij} \ln(1 \pm e^{-\varepsilon_j}) \quad , \tag{12}$$

where the upper and lower signs correspond to the fermionic and bosonic versions, respectively. r is the number of particles and the matrix N_{ij} is related to the phases introduced above

$$N_{ij} = \Delta^{-}_{ij} - \Delta^{+}_{ij} \quad . \tag{13}$$

The central charge of the Virasoro algebra of the underlying conformal field theory is given in the following form

$$c = \frac{6}{\pi^2} \sum_{i=1}^{r} \left\{ \begin{array}{ll} L\left(\frac{1}{1+e^{\varepsilon_i}}\right) & \text{fermionic} \\ L\left(e^{-\varepsilon_i}\right) & \text{bosonic} \end{array} \right. . \tag{14}$$

Here $L(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^2} + \frac{1}{2}\ln x \ln(1-x)$ denotes the Rogers dilogarithm [27].

Recently the remarkable observation was made, that for particular models the fermionic TBA-equations also result from a limiting procedure when considering characters of the Virasoro algebra related to certain conformal field theories. It will be our goal to extent this analysis to generalized, i.e. anyonic, statistics.

3 Factorized Characters and Quantum Dilogarithms

The bosonic realization [8, 9] of the character of an irreducible representation of the Virasoro algebra related to the Hilbert space of the minimal models $\mathcal{M}(s,t)$ [28] with central charge $c = 1 - \frac{6(s-t)^2}{s\,t}$ and highest weight $h_{n,m} = \frac{(nt-ms)^2-(s-t)^2}{4\,s\,t}$ reads

$$\chi^{s,t}_{n,m}(q) = \frac{q^{h_{n,m}-\frac{c}{24}}}{(q)_\infty} \sum_{k=-\infty}^{\infty} q^{stk^2}\left(q^{k(nt-ms)} - q^{k(nt+ms)+nm}\right) \tag{15}$$

$$= \frac{q^{h_{n,m}-\frac{c}{24}}}{(q)_\infty}\hat{\chi}^{s,t}_{n,m}(q) = q^{h_{n,m}-\frac{c}{24}} \sum_{k=0}^{\infty} c_k q^k\,. \tag{16}$$

Irreducibility also demands that s and t are co-prime and $1 \leq n \leq s-1$ and $1 \leq m \leq t-1$. We have used the standard abbreviation for Euler's function $(q)_m = \prod_{k=1}^{m}(1-q^k)$ and $(q)_0 = 1$. In the following analysis we shall be exploiting the relation of the expressions (15) to the quantum dilogarithm whose defining relations are

$$\ln_q(\theta) := \prod_{k=0}^{\infty}(1-e^{2\pi i\theta}q^k) \;= \exp\sum_{k=1}^{\infty}\frac{1}{k}\frac{e^{2\pi i\theta k}}{q^k-1}\,. \tag{17}$$

Taking $q = e^{2\pi i\tau}$, we assume that $\mathrm{Im}(\tau) > 0$ in order to guarantee the convergence of (17). We see from (17) that $\ln_q(\theta)$ is a pseudo-double-periodic function

$$\ln_q(\theta+1) = \ln_q(\theta) \qquad \text{and} \qquad \ln_q(\theta+\tau) = \frac{1}{1-e^{2\pi i\theta}}\ln_q(\theta) \quad . \tag{18}$$

It follows easily from this that

$$\ln_q(\theta) = \sum_{k=0}^{\infty}\frac{(-1)^k q^{\frac{k(k-1)}{2}} e^{2\pi i\theta k}}{(q)_k} \tag{19}$$

and

$$\frac{1}{\ln_q(\theta)} = \sum_{k=0}^{\infty}\frac{e^{2\pi i\theta k}}{(q)_k} \quad . \tag{20}$$

The function $\ln_q(\theta)$ was coined as quantum dilogarithm because in its classical limit $\tau \to 0$ (τ may be thought of as $\hbar$ or $1/L$, with L being the size of the system (see section 5)) the singular term involves Euler's dilogarithm $Li_2(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^2}$ [27]

$$\lim_{\tau\to 0} \ln_q(\theta) \;=\; \exp\left\{ \frac{1}{2\pi i \tau}\; Li_2\left(e^{2\pi i\theta}\right) \;+\; \mathcal{O}\left(1\right) \right\} \quad . \tag{21}$$

This follows directly from (17). A relation between characters and the quantum dilogarithm in the form of a sum as (15) is most easily established by re-expressing them in terms of theta functions, which in turn are composed of three quantum dilogarithms. However, this type of relation is of no concern to us here, instead we will seek to factorize the quantity $\hat{\chi}^{s,t}_{n,m}(q)$ in the following form

$$\hat{\chi}^{s,t}_{n,m}(q) = \prod_{k=1}^{N} \ln_{q^b}(a_k) \qquad . \tag{22}$$

The advantage of this realization is twofold. On one hand it will serve us to obtain anyonic versions of the characters and on the other it may be used to carry out the classical limit effortless. Introducing the S-modular transformed version of q, i.e. $\hat{q} = \exp(-2\pi i/\tau)$, we carry out the classical limit $q \to 1^-$ (and hence $\hat{q} \to 0^+$) by means of (21)

$$\lim_{\tau\to 0} \chi^{s,t}_{n,m}(q) = \hat{q}^{\frac{Li_2(1)}{4\pi^2}\left(\frac{N}{b}-1\right)} = \hat{q}^{\frac{1}{24}\left(\frac{N}{b}-1\right)} \quad . \tag{23}$$

The -1 in (23) occurs due to the fact that $(q)_\infty$, which appears in (15), coincides with $\ln_q(\tau)$ and therefore its limit is also ruled by (21).

It is clear, that in general the classical limit for expressions like (15) is quite non-trivial to carry out directly. However, one may exploit the behaviour of these formulae under the modular transformation. It is well known, that the characters transform under the S-modular transformation in the following general form [14]

$$\chi^{s,t}_{n,m}(q) = \sum_{n',m'} C^{n'm'}_{nm}\, \chi^{s,t}_{n',m'}(\hat{q}) \quad . \tag{24}$$

The $C^{n'm'}_{nm}$ are explicitly known constants. In the classical limit we obtain

$$\lim_{\tau\to 0} \chi^{s,t}_{n,m}(q) \;=\; \sum_{n',m'} \hat{q}^{h_{n',m'}-\frac{c}{24}}\, C^{n'm'}_{nm} \tag{25}$$

$$\;=\; C^{\bar{n}\bar{m}}_{nm} \hat{q}^{-\frac{c_{eff}}{24}} \left(1 + \sum_{n',m'\neq \bar{n},\bar{m}} \hat{q}^{h_{n',m'}-h_{\bar{n},\bar{m}}}\, C^{n'm'}_{nm} / C^{\bar{n}\bar{m}}_{nm} \right) . \tag{26}$$

Here we have introduced the so-called effective central charge

$$c_{eff} = c - 24 h_{\bar{n},\bar{m}} = 1 - \frac{6(\bar{n}t - \bar{m}s)^2}{s\; t} \quad , \tag{27}$$

where $h_{\bar{n},\bar{m}}$ denotes the lowest of all conformal weights in the model. Comparison with (23) yields the following constraint on the number of factors N and the constant b

$$c_{eff} = 1 - \frac{N}{b} \ , \tag{28}$$

for every character which factorizes in the form of (22). These formulae provide a criterion which serves to decide whether any given character of the general Feigin-Fuchs/Rocha-Caridi form (15) may be factorized. From a mathematical point of view it provides an existence criterion for certain Rogers-Ramanujan-Schur type identities [10, 29, 30].

We shall now provide explicit examples for factorized characters of the form (22). We commence by considering the well-known Gauß-Jacobi identity (see for instance [18])

$$\sum_{k=-\infty}^{\infty} (-1)^k x^{\frac{k(k+1)}{2}} y^{\frac{k(k-1)}{2}} = \prod_{k=1}^{\infty} (1 - x^k y^k)(1 - x^{k-1} y^k)(1 - x^k y^{k-1}). \tag{29}$$

We attempt to match the left hand side of this formula with $\hat{\chi}^{s,t}_{n,m}(q)$ in the form of (15). Making the ansatz $x = q^a$ and $y = q^b$ one easily derives that the only solution† is $a = nm$ and $b = \frac{st}{2} - nm$ together with the condition $s = 2n$. We therefore obtain

$$\hat{\chi}^{2n,t}_{n,m}(q) = \prod_{l \in \mathbf{e}^{2n}} \ln_{q^{nt}}(\tau l) \quad , \tag{30}$$

where we introduced the set $\mathbf{e}^{2n} = \{nm, nt - nm, tn\}$. This is a well-known formula and may be found for instance in [14, 13]. We may also carry out a consistency check by taking the limit $\lim_{\tau \to 0} \hat{\chi}^{2n,t}_{n,m}(q)$ and compare (27) with (28). We obtain for the effective central charge

$$c_{eff} = 1 - \frac{3}{nt} = 1 - \frac{3(\bar{n}t - 2\bar{m}n)^2}{nt} \quad . \tag{31}$$

This is only possible if the quantity $\lambda = (\bar{n}t - 2\bar{m}n)^2 = 1$, which is indeed the case as a simple argument shows. The lowest value λ can take is zero, which is obviously impossible, since by construction s and t are to be co-prime. A theorem concerning the Diophantine equations in number theory (e.g. Theorem 1.3 in [30]) states, that the equation $t\bar{n} - s\bar{m} = \pm 1$ possess always an integer solution when t and s are co-prime. One may also show that (e.g. [30] p.96) $0 \leq \bar{n} < s$ and $0 \leq \bar{m} < t$. Therefore we have $\lambda = 1$, together with the restrictions on $\bar{n}$ and $\bar{m}$. Hence, equation (31) holds.‡

†More precisely, one also obtains the alternative solution with the condition $t = 2m$. However, from a physical point of view these solutions are equivalent because of the symmetry $\chi^{s,t}_{n,m}(q) = \chi^{t,s}_{m,n}(q)$.

‡Reversing here the argumentation, we have also proven a theorem from number theory by a physical reasoning as a by-product of our analysis of characters.

According to the arguments of Christe [13] there is only one other class of characters which also factorizes in the form of (22), that is $\hat{\chi}^{3n,t}_{n,m}(q)$. To our knowledge this was only established via computer [§] on a case-by-case level [13]. We prove this assertion by considering Watson's identity [33]

$$\sum_{k=-\infty}^{\infty} x^{\frac{3k^2+k}{2}} y^{3k^2}(y^{-2k}-y^{4k+1}) = \prod_{k=1}^{\infty}(1-x^k y^{2k})(1-x^k y^{2k-1})(1-x^{k-1}y^{2k-1}) \times(1-x^{2k-1}y^{4k-4})(1-x^{2k-1}y^{4k}) \quad . \tag{32}$$

Making once more an ansatz of the type $x=q^a$ and $y=q^b$ and trying to match with $\hat{\chi}^{s,t}_{n,m}(q)$, we obtain now that the only solution[¶] is $a=\frac{2}{3}st-2nm$ and $b=nm$ together with the condition $s=3n$, such that

$$\hat{\chi}^{3n,t}_{n,m}(q) = \prod_{l\in \mathbf{e}^{3n}} \ln_{q^{4nt}}(\tau l) \quad . \tag{33}$$

Here we employ the set $\mathbf{e}^{3n} = \{nm, 2nt, 2n(t-m), n(2t-m), n(2t+m), 2n(t+m), n(4t-m), 4nt)\}$.

In this case we obtain from the limit $\lim_{\tau\to 0}\hat{\chi}^{3n,t}_{n,m}(q)$ for the effective central charge

$$c_{eff} = 1-\frac{8}{4nt} = 1-\frac{2(\bar{n}t-3\bar{m}n)^2}{nt} \quad , \tag{34}$$

which is only satisfied if we have $\lambda' = (\bar{n}t-3\bar{m}n)^2 = 1$. By the same reasoning as before we deduce also in this case that indeed $\lambda'=1$.

We would like to stress, that despite the fact that characters of the type (30) and (33) may appear somewhat exotic, they are ubiquitous. All unitary minimal models, i.e. $\mathcal{M}(n,n+1)$, possess factorizable sectors. Moreover, for minimal models of the type $\mathcal{M}(2,t)$ and $\mathcal{M}(3,t)$ <u>all</u> sectors are of this form.

We conclude this section with the remark, that the argument of Christe concerning the non-existence of other factorizable characters is based on the assumption that the characters factorize precisely in the form (22). However, with some minor modifications our analysis works equally well for characters of the form $\hat{\chi}^{s,t}_{n,m}(q) = \prod_{k=1}^{N} \ln_{q^b}(a_k) / \prod_{l=1}^{N'} \ln_{q^{b'}}(a'_l)$. At first sight, such expressions occur in the work of [9, 14, 15, 16], but it turns out that they are always "reducible" to (22) (see below). It is left for future investigations to extend our considerations to characters which factorize genuinely different from (22).

[§] After submission of this manuscript, it was brought to our attention by W. Eholzer that there exist analytical arguments [31, 32]. Our analysis is close to [32].

[¶] Once more, one also obtains alternative solutions, which are equivalent because of the symmetries $\chi^{s,t}_{n,m}(q) = \chi^{t,s}_{m,n}(q) = \chi^{s,t}_{s-n,t-m}(q)$.

4 From characters to TBA

In the preceding section we have demonstrated that the classical limit $q \to 1^-$ may be carried out effortless once a character factorizes. On the other hand, there exists a method [11, 6, 12, 7] which allows to perform the limiting procedure for every type of character.

Before investigating characters in particular, let us consider the function

$$\chi(q) = \sum_{\vec{m}=0}^{\infty} \frac{q^{\vec{m}A\vec{m}^t + \vec{m}\cdot\vec{B}}}{(q^b)_{m_1} \dots (q^b)_{m_r}} \quad . \tag{35}$$

A is assumed to be an $(r \times r)$-matrix and $\vec{B}$ is an r-component vector. Despite the suggestive notation, $\chi(q)$ does not need to be a character for the purpose of deriving an equations of TBA type. However, for particular A and $\vec{B}$ equations (35) coincide (up to the factor $q^{h_{n,m}-\frac{c}{24}}$) with certain Virasoro characters, which give rise to a remarkable fermionic quasi-particle interpretation [6, 7, 34]. Below we will demonstrate, that whenever b is taken to be an integer greater than one and $\chi(q)$ in (35) coincides with a generating function for characters, it admits an anyonic interpretation. We will now apply the method of [11, 6, 12, 7] to an expression of the type (35). Hitherto, this has been carried out for the particular situation when $b = 1, A \neq 0$ and $\vec{B} = 0$. When $\vec{B}$ only characterizes a different superselection sector it is justified to set it to zero. We will include it into our considerations. Furthermore, since we are interested in anyonic representations, we only demand b to be a positive integer.

4.1 Saddle Point Analysis

Viewing the series in (16) as a Laurent expansion and assuming that A is a positive definite matrix, a comparison between (16) and (35) together with the application of Cauchy's theorem yields

$$c_{k-1} = \sum_{\vec{m}} \oint \frac{dz}{2\pi i} \frac{z^{\vec{m}A\vec{m}^t + \vec{B}\cdot\vec{m} - k}}{(z^b)_{m_1} \dots (z^b)_{m_r}} \simeq \int d\vec{m} \oint \frac{dz}{2\pi i} \frac{z^{\vec{m}A\vec{m}^t + \vec{B}\cdot\vec{m} - k}}{(z^b)_{m_1} \dots (z^b)_{m_r}}. \tag{36}$$

As a first approximation we have changed the r sums over m_i into integrals. This step also makes possible restrictions, which typically occur for fermionic type representations (see section 5), insignificant. A natural setting which gives information about the asymptotic behaviour is the saddle point approximation or method of the steepest descent, which is well known from the theory of complex functions. In order to be able to apply the saddle point method to (36) we need the integrand to acquire the form of an exponential

$$c_{k-1} \simeq \int d\vec{m} \oint \frac{dz}{2\pi i} \exp\left[f(\vec{m}, z)\right], \tag{37}$$

with

$$f(\vec{m},z) = \left(\vec{m}A\vec{m}^t + \vec{B}\cdot\vec{m} - k\right)\ln z - \sum_{l=1}^{r}\int_0^{m_l} dt\,\ln\left(1-z^{bt}\right). \tag{38}$$

Here we have approximated once more a sum by an integral, i.e. $\ln(z)_m = \sum_{k=1}^{m}\ln\left(1-z^k\right) \simeq \int_0^m dt\,\ln\left(1-z^t\right)$. The saddle point conditions with respect to the integration in m_i and z are

$$\partial_{m_i} f(\vec{m},z)|_{m_i=n_i} = 0 \qquad \text{and} \qquad \partial_z f(\vec{m},z)|_{z_0} = 0\,. \tag{39}$$

So, $f(\vec{n},z_0)$ is the value of the function at the saddle point $(\vec{n},z_0)$. The relations resulting from the first constraining equations read

$$z^{B_i+\sum_j(A_{ij}+A_{ji})n_j} + z^{bn_i} = 1 \qquad \text{for} \quad i=1,\ldots,r\,. \tag{40}$$

These equations serve to fix $\vec{n}$. We obtain from (40)

$$\left(2\vec{n}A\vec{n}^t + \vec{B}\cdot\vec{n}\right)(\ln z)^2 = \sum_{l=1}^{r}(\ln z^{n_l})\ln(1-z^{bn_l})\,. \tag{41}$$

Substituting this relation into (38) and exploiting the following identity [27] for Roger's dilogarithm

$$(\ln z)\int_0^m dt\,\ln\left(1-z^t\right) = L(1-z^m) + \frac{1}{2}(\ln z^m)\ln(1-z^m)\,, \tag{42}$$

we derive

$$f(\vec{n},z) = -\left(k - \frac{1}{2}\vec{B}\cdot\vec{n}\right)\ln z - \frac{1}{b\ln z}\sum_{i=1}^{r}L(1-z^{bn_i})\,. \tag{43}$$

Using the formula $\frac{d}{dz}L(z) = -\frac{1}{2}\left(\frac{\ln(1-z)}{z} + \frac{\ln z}{1-z}\right)$, we obtain the remaining saddle point condition in (39)

$$k\,(\ln z_0)^2 = \frac{1}{b}\sum_{i=1}^{r}L(1-z_0^{bn_i}) - (\ln z_0)^2\left(\vec{n}A\vec{n}^t + \frac{b}{2}\sum_{i=1}^{r}n_i^2 z_0^{bn_i-B_i-\sum_j(A_{ij}+A_{ji})n_j}\right).$$

Analyzing the limit $k\to\infty$ of this relation with the help of (40)–(41), we infer that

$$\ln z_0 = -\sqrt{\mathcal{K}/k}\left(1+\mathcal{O}(k^{-1})\right) \qquad \text{with} \quad \mathcal{K} = \frac{1}{b}\sum_{i=1}^{r}L(1-z_0^{bn_i}) \quad . \tag{44}$$

That is, z_0 tends to one for large k. Thus, the asymptotics of c_k is given by

$$c_k \simeq \exp\left[f(\vec{n},z_0)\right] \simeq \exp\left(2\sqrt{k\mathcal{K}}\right)\ . \tag{45}$$

Now we are in a position to perform the classical limit. In [11, 6, 12, 7] it was observed, that once the coefficients in the Laurent expansion are of the form (45) one may approximate

$$\sum_{k=0}^{\infty} c_k q^k \simeq \int_0^{\infty} dk\, c_k q^k \simeq \exp\left(\frac{i\mathcal{K}}{2\pi\tau} + \mathcal{O}(\ln\tau)\right) . \tag{46}$$

A comparison with (26) leads to the following formula for the effective central charge

$$c_{eff} = \frac{6}{\pi^2}\mathcal{K} \quad . \tag{47}$$

4.2 The TBA equation

Introducing now the quantity $\xi_i = 1 - z_0^{bn_i}$ we obtain from (44) and (47) for the central effective charge

$$c_{eff} = \frac{6}{b\pi^2}\sum_{i=1}^{r} L\left(\xi_i\right) \quad , \tag{48}$$

where the quantities ξ_i are determined by the equations resulting from (40)

$$\xi_i^b = z_0^{bB_i}\prod_{j=1}^{r}(1-\xi_j)^{(A_{ij}+A_{ji})} \quad . \tag{49}$$

As argued in the preceding subsection z_0 tends to one and different choices for the vector $\vec{B}$, i.e. different superselection sectors, have no effect on the value of the effective central charge. Taking A to be of the form

$$A_{ij} = \frac{N_{ij}}{2} + \begin{cases} b\frac{\delta_{ij}}{2} & \text{fermionic} \\ 0 & \text{bosonic} \end{cases} \tag{50}$$

and relating ξ_i to the pseudo-energies as

$$\xi_i = \begin{cases} \frac{1}{1+e^{\varepsilon_i}} & \text{fermionic} \\ e^{-\varepsilon_i} & \text{bosonic} \end{cases} \quad , \tag{51}$$

one recovers the known fermionic and bosonic TBA-equations (12) and (14), upon choosing $b = 1$.

The analysis for particular fermionic cases was carried out in [11, 6, 12, 7]. One might be surprised, that formally a fermionic type expression gives rise to a fermionic as well as a bosonic TBA-equation. Such features are however common in the TBA analysis itself, in which also bosonic statistics may correspond to fermionic boundary conditions and vice versa.

As explained in section 2, anyonic statistics may be implemented into the scattering theory by removing the asymptotic phases in the S-matrix and introducing them at $\theta = 0$. As a consequence the quantity N_{ij} (13) becomes zero. In this case a substitution of (50) into (49) leads to

$$\xi_i = \frac{1}{2} \qquad \text{and} \qquad \xi_i = 1 \tag{52}$$

as two possible versions of anyonic TBA-equations. Notice, that the anyonic TBA-equations (52) correspond precisely to the situation in which the pseudo-energies are taken to be zero. The contribution to the effective central charge from each particle is in these cases either $\frac{1}{b}$ or $\frac{1}{2b}$, according to (48) (recall that $L(1) = 2L(1/2) = \pi^2/6$). So, in some sense the anyons carry a remembrance on a fermionic or bosonic nature. Below we will see that this picture may also be related to Pauli's exclusion principle.

In both cases the complicated coupling between equations involving different types of particles has vanished. This is a virtue of the anyonic TBA-equation, in which all particles of the same kind contribute equally to the central charge.

To summarize: we have proposed a transformation either from a bosonic or fermionic type of TBA-equation to an anyonic one. Of course this transformation has to preserve the value of the central charge. Therefore, we expect the existence of character formulae of the type (35), admitting two forms, i.e. involving a matrix of the type (50) either with $N_{ij} \neq 0$ and $b = 1$ or $N_{ij} = 0$ and $b \neq 1$.

5 Anyonic Characters and States

As already mentioned, when interpreted as partition functions, the character formulae of the preceding sections possess some fermionic realizations. These representations emerge in two different types. One of them is intimately related to simply laced Lie algebras in the following sense. When A is chosen to be the inverse of the Cartan matrix and $b = 1$, (35) coincides with some characters corresponding to minimal models of conformal field theory. The number of fermions contained in the model equals the rank r of the Lie algebra in this case. The summation over $m_1, m_2, \ldots, m_r$ may be restricted in some way, indicating that certain particles may only appear in conjunction with others. From a Lie algebraic point of view this expresses usually some symmetry in the Dynkin diagram. The vector $\vec{B}$ characterizes different superselection sectors for a particular theory.

Notice that such formulae, i.e. A being the inverse of the Cartan matrix, correspond precisely to the fermionic case in (50), if the matrix N_{ij} (13) is defined via the asymptotic phases of the minimal affine Toda field theory (7). It is very interesting to note that some models possess different fermionic realizations, for example, the Ising model with $c = 1/2$ can be related either to E_8 or to A_1, which indicates the possible relevant perturbations already at the conformal level.

5.1 Anyonic Characters

According to the ideology laid down above, one might expect the existence of character formulae corresponding to the same models, which admit an anyonic quasi-particle interpretation. In this section we will demonstrate how to obtain them and show that these formulae as well relate to simply laced Lie algebras.

First of all we assume the main part of the characters, $\hat{\chi}$, to factorize in the form of (22). With $a_l = \tau l$ and l being element in the N-dimensional set $\mathbf{e}$, we obtain

$$\chi^{s,t}_{n,m}(q) = q^{h_{n,m}-\frac{c}{24}} \prod_{\substack{l=1 \\ l \notin \mathbf{e}}}^{b} \left(\ln_{q^b}(\tau l) \right)^{-1} . \tag{53}$$

Employing (20) in (53), we obtain the desired anyonic realizations

$$\chi^{s,t}_{n,m}(q) = q^{h_{n,m}-\frac{c}{24}} \sum_{\vec{k}=0}^{\infty} \frac{q^{\vec{k}\cdot\vec{B}^{s,t}_{n,m}}}{(q^b)_{k_1} \dots (q^b)_{k_{(b-N)}}} . \tag{54}$$

We have introduced here the $(b-N)$-dimensional vector $\vec{B}^{s,t}_{n,m} = \{1,2,3,\dots,b\}/\mathbf{e}$. Our particular examples of section 3 for factorizing characters (30) and (33) acquire the following form

$$\chi^{2n,t}_{n,m}(q) = q^{h_{n,m}-\frac{c}{24}} \sum_{\vec{k}} \frac{q^{\vec{k}\cdot\vec{B}^{2n,t}_{n,m}}}{(q^{nt})_{k_1} \dots (q^{nt})_{k_{nt-3}}} , \tag{55}$$

$$\chi^{3n,t}_{n,m}(q) = q^{h_{n,m}-\frac{c}{24}} \sum_{\vec{k}} \frac{q^{\vec{k}\cdot\vec{B}^{3n,t}_{n,m}}}{(q^{4nt})_{k_1} \dots (q^{4nt})_{k_{4nt-8}}} , \tag{56}$$

with $\vec{B}^{2n,t}_{n,m} = \{1,2,3,\dots,nt\}/\mathbf{e}^{2n}$ and $\vec{B}^{3n,t}_{n,m} = \{1,2,3,\dots,4nt\}/\mathbf{e}^{3n}$.

Apparently (53) and (54) correspond to the second case in (50), since all $A_{ij} = 0$. The first case can be recovered if one considers a character of the type

$$\chi(q) = \frac{\prod_{l}^{N} \ln_{q^b}(\tau l + \frac{1}{2})}{\prod_{l'}^{N'} \ln_{q^b}(\tau l')} . \tag{57}$$

Employing then (19) and (20), we can rewrite this factorized character in the form

$$\chi(q) = \sum_{\vec{k}=0}^{\infty} \frac{q^{\frac{b}{2}({k_1}^2+\ldots+k_N^2)+\vec{k}\cdot\vec{B}}}{(q^b)_{k_1} \dots (q^b)_{k_{N+N'}}} . \tag{58}$$

Below we will encounter an example of this type.

5.2 Anyonic Quasi-particles

Anyonic quasi-particle states may be constructed analogously to fermionic states as proposed by the Stony Brook group [6, 7]. First of all one regards the characters as partition functions

$$\chi\left(q=e^{-\frac{2\pi v}{LkT}}\right) \sim Z = \sum_{\text{states}} e^{-\frac{\mathrm{E(states)}}{kT}} = \sum_{l=0}^{\infty} P(E_l) e^{-\frac{E_l}{kT}} \quad . \tag{59}$$

k denotes here Boltzmann's constant, T the temperature, L the size of a box which serves to quantize the momenta, v the speed of sound, E_l the energy of a particular level and $P(E_l)$ its degeneracy. The aim is now to bring (54) into the form of the right hand side of (59). A direct comparison then leads to the construction of quasi-particle states

$$|p_1^1, \ldots, p_1^{k_1}, \ldots, p_{b-N}^1, \ldots, p_{b-N}^{k_n}\rangle \quad , \tag{60}$$

which are in one-to-one correspondence to the states of the Virasoro algebra. The energy of each of these states can then be decomposed further into the contributions resulting from each single quasi-particle

$$E_l \;=\; \sum_{a=1}^{b-N}\sum_{i_a=1}^{k_a} e_a(p_a^{i_a}) \qquad , \qquad p_l \;=\; \sum_{a=1}^{b-N}\sum_{i_a=1}^{k_a} p_a^{i_a} . \tag{61}$$

The dispersion relation is to be understood as $e_a(p_a^i) = v\,|p_a^i|$.

Technically, one has several options to replace the factors $\frac{1}{(q)_{k_i}}$ in order to achieve formal equality between (54) and (59). In the fermionic case one makes use of the generating function [29, 30]

$$\sum_{n=0}^{\infty} \mathcal{P}(n,m) q^n \;=\; \frac{q^{\frac{1}{2}m(m-1)}}{(q)_m}, \tag{62}$$

for the number of partitions $\mathcal{P}(n,m)$ of a positive integer n into m <u>distinct</u> non-negative integers.[‖] Performing the same manipulations on the anyonic type character (54), we obtain for the anyonic single quasi-particle momenta (in units of $\frac{2\pi}{L}$)

$$p_a^i = (B_{n,m}^{s,t})_a + \frac{b}{2} - \frac{b}{2}\sum_{l=1}^{b-N} k_l + bN_a^i . \tag{63}$$

Here the N_a^i are distinct positive integers, and therefore we still have, despite the the anyonic nature of our quasi-particles, a Pauli type exclusion principle.

[‖]There exists also the possibility to employ the generating function for the number of partitions of a positive integer n into m distinct non-negative integers smaller than M, which gives rise to fermionic quasi-particle states of a different nature [7].

Since here we are considering anyonic quasi-particle states, we can relax the requirement that the partitions have to be carried out into distinct numbers. In this case we may also use the generating function

$$\sum_{n=0}^{\infty} \mathcal{Q}(n,m) q^n = \frac{1}{(q)_m} \tag{64}$$

for the number of partitions $\mathcal{Q}(n,m)$ of a positive integer n into positive integers smaller than m. In this way we obtain an alternative formula for the single quasi-particle momenta (in units of $\frac{2\pi}{L}$)

$$p_a^i = (B_{n,m}^{s,t})_a + bN_a^i \quad , \tag{65}$$

with N_a^i being some positive integers.

From a formal mathematical point of view we do not know which choice of the partition functions (62) or (64) is adequate. However, it seems plausible, that in agreement with the two possible TBA equations of section 4.2, the two cases are to be treated with different partitions.

5.3 Examples

5.3.1 The Ising Model

The Ising model is the first example in the series of unitary minimal models, i.e. $\mathcal{M}(3,4)$ with $c = 1/2$. It has only three different sectors. The anyonic E_8 related characters read

$$\chi_{1,m}^{3,4}(q) = q^{h_{1,m} - \frac{1}{48}} \sum_{\vec{k}} \frac{q^{\vec{B}_{1,m}^{3,4} \cdot \vec{k}}}{(q^{16})_{k_1} \cdots (q^{16})_{k_8}} \tag{66}$$

with

$$\vec{B}_{1,1}^{3,4} = \{2,3,4,5,11,12,13,14\} \qquad \vec{B}_{1,2}^{3,4} = \{1,3,5,7,9,11,13,15\} \qquad \vec{B}_{1,3}^{3,4} = \{1,4,6,7,9,10,12,15\} \quad . \tag{67}$$

Notice that $2\vec{B}_{1,3}^{3,4} - \vec{1}$ is precisely the set of exponents of E_8. Moreover, one observes that all vectors $\vec{B}$ possess similar properties as the exponents, namely $B_i + B_{r+1-i} = b$. This indicates that in a similar way as the exponents are related to the eigenvalues of the Coxeter transformation, these values are also related to the eigenvalues of a real matrix with order b. One can therefore conjecture that the characters of the form (54) also possess some formulation involving the Weyl group.

Similarly as the fermionic realizations possess formulations in terms of different Lie algebras, the anyonic counterparts also enjoy this property. From (66) we obtain easily the following anyonic type characters related to A_1

$$\chi_{1,2}^{3,4}(q) = q^{\frac{1}{24}} \sum_{k} \frac{q^k}{(q^2)_k} \, . \tag{68}$$

Notice that for both cases $b = \frac{h}{2} + 1$, with h being the Coxeter number.

5.3.2 The Tricritical Ising Model

The next model in the series of unitary minimal models is the tricritical Ising model $\mathcal{M}(4,5)$ with $c = 7/10$. Similar to the fermionic representation the anyonic version is related to E_7

$$\chi^{4,5}_{2,m}(q) = q^{h_{2,m}-\frac{7}{240}} \sum_{\vec{k}} \frac{q^{\vec{B}^{4,5}_{2,m}\cdot\vec{k}}}{(q^{10})_{k_1}\cdots(q^{10})_{k_7}} \tag{69}$$

with

$$\vec{B}^{4,5}_{2,1} = \{1,3,4,5,6,7,9\} \quad , \quad \vec{B}^{4,5}_{2,2} = \{1,2,3,5,7,8,9\} \tag{70}$$

Once more we observe the feature that $2\vec{B}^{4,5}_{2,1} - \vec{1}$ is precisely the set of exponents of E_7 and $b = \frac{h}{2}+1$.

As was already observed in [9], the characters $\chi^{4,5}_{2,m}$ admit a product representation which, in our notations, corresponds to formulae of the type (57). For instance,

$$\chi^{4,5}_{2,1}(q) = q^{\frac{49}{120}} \frac{\ln_{q^5}(\tau+\frac{1}{2})\ln_{q^5}(4\tau+\frac{1}{2})\ln_{q^5}(5\tau+\frac{1}{2})}{\ln_{q^5}(2\tau)\ln_{q^5}(3\tau)} \; . \tag{71}$$

Rewriting this expression with the help of (19) and (20), we obtain

$$\chi^{4,5}_{2,1}(q) = q^{\frac{49}{120}} \sum_{\vec{k}} \frac{q^{\frac{5}{2}(k_1^2+k_2^2+k_3^2)+\vec{k}\cdot\vec{B}^{4,5}_{2,1}}}{(q^5)_{k_1}\cdots(q^5)_{k_5}} \quad \text{with} \quad \vec{B}^{4,5}_{2,1} = \{-\frac{3}{2},\frac{3}{2},\frac{5}{2},2,3\} \; . \tag{72}$$

This provides an example of an anyonic type character involving both types of anyonic quasi-particles (see the discussion in subsection 4.2). The formula (48) now acquires the form: $c = \frac{6}{5\pi^2}(3L(1/2)+2L(1)) = 7/10$.

5.3.3 The Yang-Lee Model

The Yang-Lee model $\mathcal{M}(2,5)$ (with $c = -22/5$) serves as the simplest example of a model which is non-unitary. Its anyonic characters read

$$\chi^{2,5}_{1,m}(q) = q^{h_{1,m}+\frac{11}{60}} \sum_{\vec{k}} \frac{q^{\vec{B}^{2,5}_{1,m}\cdot\vec{k}}}{(q^5)_{k_1}(q^5)_{k_2}} \tag{73}$$

with

$$\vec{B}^{2,5}_{1,1} = \{2,3\} \quad , \quad \vec{B}^{2,5}_{1,2} = \{1,4\} \, . \tag{74}$$

Notice that precisely these characters correspond to the original identities of Rogers-Ramanujan and Schur [10]. For this case we also illustrate the working of formula (63) and present an explicit construction of the quasi-particle states of the lowest levels in table 1.

l	d	$\vec{p}(\vec{m})$			
2	1	$p_1^1(1,0)=2$			
3	1	$p_2^1(0,1)=3$			
4	1	$p_1^1(2,0)=-\frac{1}{2}$	$p_1^2(2,0)=\frac{9}{2}$		
5	1	$p_1^1(1,1)=\frac{9}{2}$	$p_1^2(1,1)=\frac{1}{2}$		
6	2	$p_2^1(0,2)=\frac{1}{2}$	$p_2^2(0,2)=\frac{11}{2}$		
		$p_1^1(3,0)=-3$	$p_1^2(3,0)=2$	$p_1^3(3,0)=7$	
7	2	$p_1^1(2,1)=7$	$p_1^2(2,1)=2$	$p_2^1(2,1)=-2$	
		$p_1^1(2,1)=-3$	$p_1^2(2,1)=7$	$p_2^1(2,1)=3$	
8	3	$p_1^1(1,2)=-3$	$p_2^1(1,2)=3$	$p_2^2(1,2)=8$	
		$p_1^1(1,2)=2$	$p_2^1(1,2)=-2$	$p_2^2(1,2)=8$	
		$p_1^1(4,0)=-\frac{11}{2}$	$p_1^2(4,0)=-\frac{1}{2}$	$p_1^3(4,0)=\frac{9}{2}$	$p_1^4(4,0)=\frac{19}{2}$
9	3	$p_2^1(0,3)=-2$	$p_2^2(0,3)=3$	$p_2^3(0,3)=8$	
		$p_1^1(3,1)=-\frac{1}{2}$	$p_1^2(3,1)=\frac{9}{2}$	$p_1^3(3,1)=\frac{19}{2}$	$p_2^1(3,1)=-\frac{9}{2}$
		$p_1^1(3,1)=-\frac{11}{2}$	$p_1^2(3,1)=\frac{9}{2}$	$p_1^3(3,1)=\frac{19}{2}$	$p_2^1(3,1)=\frac{1}{2}$

Table 1: Anyonic quasi-particle states for $\chi_{1,1}^{2,5}$

6 Conclusions

We conclude by summarizing our main results and comment on further problems. We have obtained character formulae of the Virasoro algebras which are related to simply laced Lie algebras and admit an anyonic quasi-particle interpretation. As the main technical tool we exploited the fact that some characters factorize in terms of quantum dilogarithms. Performing the classical limit we also conjectured an anyonic version of the thermodynamic Bethe ansatz.

In order to support the picture we have been presenting, it is of course highly desirable to derive [35] the TBA-equations from proper thermodynamic principles in the spirit of Yang and Yang [3]. We would like to emphasize once more that the proposed anyonic TBA-equations are by far simpler than the usual fermionic or bosonic versions. The problem of solving the set of non-linearly coupled equations to find the pseudo-energies does not occur at all in the anyonic formulation. This simplification is contrary to most situations in which exotic statistics is implemented. The explicit construction of anyonic creation and annihilation operators in momentum as well as in real space is far more complicated than the construction of their bosonic and fermionic counterparts. Also the scattering matrix, despite the fact that one may write it down easily, is an object which is difficult to handle due to its complicated analytical structure.

What the conformal side concerns, the anyonic picture of the Virasoro characters is also simpler. For instance, in comparison with the fermionic realizations, the relatively complicated restrictions on the sums have entirely vanished. It would be interesting to obtain the lattice version of the states (equivalently to [36, 37]) constructed in the previous section as well as their explicit representations in the spirit of [38, 39]. An investigation of how the integrals of motion act on these states, i.e. to find their eigenvalues like in [34], will certainly help in this direction.

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