Discrete Integrable Systems and Geometry

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1 Introduction

Long before the theory of solitons, geometers used integrable equations to describe various special curves, surfaces etc. Nowadays this field of research takes advantage of using both geometrical intuition and methods of soliton theory in order to study *integrable geometries*, i.e. geometries described by integrable systems. *Analytic* methods developed in the soliton theory (finite-gap integration, Riemann-Hilbert problem etc.) were successfully applied to obtain new results in differential geometry *in the large*. More simple *algebraic* methods (Lax representation, Bäcklund transformation etc.) are used to study *local* properties of integrable geometries.

Recently it was found that integrable discretizations (i.e. those described by discrete integrable systems) of integrable geometries have natural properties. Looking for proper definitions of integrable nets and investigation of their geometrical and algebraical properties has become a popular field of research in the 1990's. Numerous examples in Euclidean, conformal, affine and projective geometries were found and investigated. A collection of achievements in this field can be found in the book [5].

In the first part of the present talk[1] it is shown how the loop group interpretation of the Bäcklund transformations implies the existence of the corresponding discrete integrable geometries. In the second part (sections 3,4) discrete integrable analogues of elastic curves and of the Lagrange top are defined.

2 Discrete surfaces via Bäcklund-Darboux (BD) transformations

One of the most fundamental properties of surfaces described by integrable equations is the existence of the Bäcklund transformation. It is well known [13] that these transformations coincide with the Darboux or dressing transformations, which were analytically [14, 9] and algebraically [11] studied in the theory of solitons. We show how the proper interpretation of the Bäcklund transformations yields definitions of discrete integrable surfaces.

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Analytic description. Dressing transformation.

Let $G[\lambda]$ be a loop group and $(x,y)\mapsto \Psi(\lambda,x,y)\in G[\lambda]$ be a smooth map. Its logarithmic derivatives

$$U=\Psi_x\Psi^{-1},\ V=\Psi_y\Psi^{-1}\in g[\lambda] \tag{1}$$

lie in the corresponding loop algebra $g[\lambda]$ and satisfy the compatibility condition

$$U_y-V_x+[U,V]=0. \tag{2}$$

For applications in differential geometry Ψ is a moving frame and λ describes the *associated family* - a one parameter family of deformations preserving geometrical properties.

Interpreted as the dressing transformation the BD-transformation is described as follows [9]. For any solution $\Psi\in G[\lambda]$ of (1) one can construct another solution of (1) with $\tilde{U},\tilde{V}$ by an algebraic transformation

$$\Psi\to\tilde{\Psi}=D\Psi, \tag{3}$$

with an appropriate mapping $(x,y)\mapsto D(\lambda,x,y)\in G[\lambda]$. We call D the Darboux matrix. It is determined by the condition that the points

$$\Lambda=\{\lambda_1,\dots,\lambda_N\},$$

where the matrix $D(\lambda_i)$ degenerates, and the kernels

$$k_i=\ker\tilde{\Psi}(\lambda_i)$$

of $\tilde{\Psi}(\lambda_i)$ are independent of (x,y). Provided these conditions are satisfied, $\tilde{U},\tilde{V}$ have the same dependence on λ as U,V and $\tilde{\Psi}$ describes the same geometry.

The dressing interpretation of the BD-transformation naturally yields the permutability theorem. Indeed, let us consider the transformation with the following dressing data

$$\Lambda=\{\lambda_1,\lambda_2\},\qquad K=\{k_1,k_2\}.$$

Let $D_{\lambda_1,k_1'}(\lambda)$ be the Darboux matrix with the only degeneration point λ_1:

$$D_{\lambda_1,k_1'}(\lambda_1)=0,\qquad k_1'=\ker D_{\lambda_1,k_1'}(\lambda_1).$$

The function $\tilde{\Psi}$ with the data Λ,K can be constructed in two different ways

$$\tilde{\Psi}(\lambda)=D_{\lambda_2,k_2''}(\lambda)D_{\lambda_1,k_1'}(\lambda)\Psi(\lambda)=D_{\lambda_1,k_1''}(\lambda)D_{\lambda_2,k_2'}(\lambda)\Psi(\lambda),$$

where

$$k_1'=\Psi(\lambda)k_1,\ k_2'=\Psi(\lambda)k_2,\ k_1''=D_{\lambda_1,k_2'}\Psi(\lambda_1)k_1,\ k_2''=D_{\lambda_2,k_1'}\Psi(\lambda_2)k_2.$$

This implies the permutability theorem

$$D_{\lambda_2,k_2''}(\lambda)D_{\lambda_1,k_1'}(\lambda) = D_{\lambda_1,k_1''}(\lambda)D_{\lambda_2,k_2'}(\lambda). \tag{4}$$

Let us fix $(x, y) = (x_0, y_0)$ and introduce a $\mathbb{Z}^2$-family of permutable BD-transformations

$$\mathcal{U}_{n,m} = D_{\mu_n,p_{n,m}}(\lambda), \qquad \mathcal{V}_{n,m} = D_{\nu_m,q_{n,m}}(\lambda), \tag{5}$$

where p, q are the corresponding kernels.

Equation (4) in the loop group $G[\lambda]$ becomes

$$\mathcal{U}_{n,m+1}\mathcal{V}_{n,m} = \mathcal{V}_{n+1,m}\mathcal{U}_{n,m} \tag{6}$$

and can be interpreted as a discrete analog of the frame equation (2). The discrete net is given by the frame $\Phi : \mathbb{Z}^2 \to G[\lambda]$, which satisfies

$$\Phi_{n+1,m} = \mathcal{U}_{n,m}\Phi_{n,m}, \qquad \Phi_{n,m+1} = \mathcal{V}_{n,m}\Phi_{n,m}$$

For the values μ_n, ν_m of the spectral parameter and the kernels $p_{n,m}, q_{n,m}$ one obtains an integrable difference equation, which should be treated[2] as a discretization of (2).

Remark. The construction above can be trivially generalized for any splitting

$$\Lambda = \bigcup_{i=1}^{N} \Lambda_i, \; K = \bigcup_{i=1}^{N} K_i, \; \Lambda_i = \{\lambda_1, \ldots, \lambda_{N_i}\}, \; K_i = \{k_1, \ldots, k_{N_i}\}$$

of the dressing data. The cases $N = 1$ and $N = 2$ correspond to discrete curves and surfaces respectively. For $N > 2$ we obtain an N-dimensional integrable net. This net can be also interpreted as a sequence of the BD-transformations of an integrable net of lower dimension.

The suggested method applied to the classical case of surfaces with constant negative Gaussian curvature yields a definition of the discrete K-surfaces. To show this we recall some well-known properties of the Bäcklund transformation [2]. Let

$$(x, y) \mapsto r(x, y) \in \mathbb{R}^3$$

be an asymptotic line parametrization of a K-surface and $\tilde{r}(x, y)$ its Bäcklund transformed which is also asymptotic line parametrized.

[2] Specifying parameters one can obtain the differential equation (2) from (5) as a limit

$$\epsilon \to 0, \; \mathcal{U} \to I + \epsilon U, \; \mathcal{V} \to I + \epsilon V.$$

The net Φ approximates its smooth limit. Note that the net Φ *does not* approximate the original surface $\Psi(\lambda, x, y)$. We have discretize not a particular surface but a particular class of surfaces.

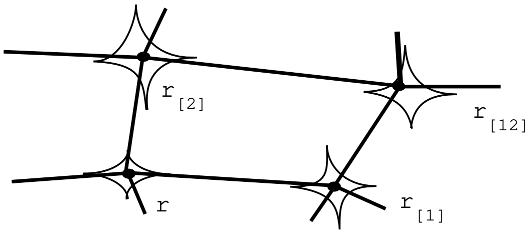

Figure 1: Discrete K-surfaces via permutability of two Bäcklund transformations

Geometric description. Bäcklund transformation of K-surfaces.

For a given surface r there exists a two-parametric $\{\mu, \phi\}$ family of BD-transformations $r \to \tilde{r}$, parametrized by the length $\|\tilde{r} - r\| = \mu$, which is independent of (x, y) and the angle ϕ between the tangent vector $\tilde{r} - r$ and the asymptotic x-line at some point (x_0, y_0).

Two important properties of this transformation are:

- The vector $\tilde{r}(x, y) - r(x, y)$ lies in the intersection of the tangent planes of the surfaces r and $\tilde{r}$ at the points $r(x, y)$ and $\tilde{r}(x, y)$ respectively.
- Bianchi permutability theorem. Let $r \to r_{[1]} \to r_{[12]}$ be a sequence of the BT. By another sequence of the BT $r \to r_{[2]} \to r_{[12]}$ this sequence can be completed to a commuting diagramm (see Fig.1). Moreover

$$\|r_{[1]} - r\| = \|r_{[12]} - r_{[2]}\|, \qquad \|r_{[2]} - r\| = \|r_{[12]} - r_{[1]}\|.$$

The loop group corresponding to the K-surfaces is

$$G[\lambda] = \{\phi : \mathbb{R}_* \to SU(2) \mid \phi(-\lambda) = \sigma_3 \phi(\lambda) \sigma_3\}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The immersion function is given by the Sym formula [13, 3] $r = \Psi^{-1}\Psi_\lambda$, where one should identify $\mathbb{R}^3 \cong su(2)$.

Let $r : \mathbb{Z}^2 \to \mathbb{R}^3$ be a discrete K-surface described by the BD-transformations as above, and therefore given by

$$r_{n,m} = \Phi^{-1}_{n,m} \frac{\partial}{\partial \lambda} \Phi_{n,m}.$$

The geometrical properties of the Bäcklund transformation obviously imply that

- r is a discrete asymptotic net, i.e. for each point $r_{n,m}$ there exists a plane $\mathcal{P}_{n,m}$ (tangent plane of the original smooth surface r) such that

$$r_{n,m}, r_{n-1,m}, r_{n+1,m}, r_{n,m-1}, r_{n,m+1} \in \mathcal{P}_{n,m}.$$

- The lengths of the opposite edges of elementary quadrilaterals are equal

$$\|r_{n+1,m}-r_{n,m}\| = \|r_{n+1,m+1}-r_{n,m+1}\|,\ \|r_{n,m+1}-r_{n,m}\| = \|r_{n+1,m+1}-r_{n+1,m}\|.$$

These properties describe a discrete analog of the Chebyshev net and can be used as the definition of the discrete K-surfaces (for details see [3]). The angles ϕ corresponding to vertices of an elementary quadrilateral (corresponding to the smooth surfaces $r, r_{[1]}, r_{[2]}, r_{[12]}$) satisfy a difference equation (6), which is an integrable discretization (Hirota equation) of the sine-Gordon equation.

In a similar way the affine spheres with the indefinite Blaschke metric have been discretized in [4]. The corresponding loop group is

$$G[\lambda] = \{\phi : \mathbb{R}_* \to SL(3,\mathbb{R}) \mid Q\phi(q\lambda)Q^{-1} = \phi(\lambda), T[\phi(-\lambda)^{-1}]^T T = \phi(\lambda)\},$$

where

$$T = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad Q = \begin{pmatrix} q & 0 & 0 \\ 0 & q^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad q = e^{2\pi i/3}.$$

The discrete affine spheres turn out to be discrete affine Lorentz-harmonic[3] asymptotic nets. Again one can use these properties to define the discrete affine spheres geometrically. The corresponding discrete integrable equation is an integrable discretization of the Gauss equation of affine spheres

$$u_{xy} = e^u - e^{-2u}.$$

3 Discrete elastic curves

Let $\gamma : [0, L] \to \mathbb{R}^3$ be a framed arclength parametrized $\|\gamma'\| = 1$ curve with the frame

$$(N, B, T) : [0, L] \to SO(3),\ T = \gamma'.$$

We prefer to identify[4] the linear spaces $\mathbb{R}^3 \cong su(2)$ and we prefer to describe the frame as

$$\Phi : [0, L] \to SU(2)$$

with the tangent vector

$$T = -i\Phi^{-1}\sigma_3\Phi.$$

A framed curve is called *elastic* if it is an extremal of the functional

$$\mathcal{L} = \int_0^L (k^2 + \alpha\tau^2)dx,$$

where $k = \|T''\|$ is the curvature and

$$\tau = -i \ \mathrm{tr}\ (\sigma_3\Phi_x\Phi^{-1})$$

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[3] i.e. the vectors $r_{n+1,m+1} + r_{n,m}$ and $r_{n+1,m} + r_{n,m+1}$ are proportional

[4] We normalize the scalar product to be $< A, B >= -\frac{1}{2}\mathrm{tr}\ AB$.

is the torsion of the frame. Admissible variations preserve $\gamma(0), \gamma(L), \Phi(0), \Phi(L)$. The Euler-Lagrange equations of elastic curves are

$$\gamma' \times \gamma'' + c\gamma' = \gamma \times a + b \Leftrightarrow T \times T'' + cT' = T \times a, \tag{7}$$
$$\tau' = 0, \tag{8}$$

where $c = \alpha\tau$ and $a \in \mathbb{R}^3$ is a Lagrange multiplier

$$\mathcal{L}_a = \mathcal{L} + < a, \int_0^L T dx > . \tag{9}$$

Let us consider the well-known smoke-ring evolution [8] of a curve

$$\gamma_t = \gamma' \times \gamma''. \tag{10}$$

The tangent flow γ' (which is a reparametrization of the curve) is one of infinitely many commuting flows of (10). Comparing (7) and (10) one can prove the following characterization of the elastic curves.

Characteristic property. *A curve is elastic iff its smoke-ring propagation (10) is a rigid motion of the curve.*

We use this characterization to define discrete elastic curves. A discrete framed arclength parametrized curve is a map $\gamma : \mathbb{Z} \to \mathbb{R}^3$ with $\|\gamma_{n+1} - \gamma_n\| = 1$. Frames $\Phi : \mathbb{Z} \to SU(2)$ are defined on edges so that the tangent vector is equal

$$T_n = \gamma_n - \gamma_{n-1} = -i\Phi_n^{-1}\sigma_3\Phi_n. \tag{11}$$

The curvature functions $\mathcal{U}_n = \Phi_{n+1}\Phi_n^{-1}$ are defined at vertices. The smoke-ring evolution of the discrete curves was defined in [6], it is given by the Ablowitz-Ladik hierarchy [1]. The first two flows are [5]

$$\frac{\partial}{\partial x}\gamma_n = \frac{T_n + T_{n+1}}{1 + < T_n, T_n + 1 >}, \qquad \frac{\partial}{\partial t}\gamma_n = 2\frac{T_n \times T_{n+1}}{1 + < T_n, T_n + 1 >}.$$

Let us call a discrete curve elastic if there exists $c \in \mathbb{R}$ such that the flow $\gamma_{n,t} + c\gamma_{n,x}$ is a rigid motion of the curve. For the tangent vector we obtain

$$\begin{aligned} & 2T_n \times \left(\frac{T_{n+1}}{1 + < T_n, T_{n+1} >} + \frac{T_{n-1}}{1 + < T_n, T_{n-1} >} \right) \\ & + c\left(\frac{T_{n+1} + T_n}{1 + < T_n, T_{n+1} >} - \frac{T_n + T_{n-1}}{1 + < T_n, T_{n-1} >} \right) = T_n \times a, \end{aligned} \tag{12}$$

which is a special case of the spin chain dynamics [7]. The torsion of the frame is constant

$$\tau_\Delta = \frac{\mathrm{tr}\ (\mathcal{U}_n T_n)}{\mathrm{tr}\ \mathcal{U}_n} = \mathrm{const.} \tag{13}$$

[5] Geometrically these flows were first introduced by U. Pinkall in 1993.

Equations (12, 13) are discrete analogues of (7, 8) and are also Lagrangian. The corresponding Lagrangian on $SU(2) \times SU(2)$ is

$$\mathcal{L}_\Delta = \sum_{n=1}^{N-1} (-(2-\alpha)\log(1+ < T_n, T_{n+1} >) - 2\alpha \log \mathrm{tr}\,(\Phi_{n+1}\Phi_n^{-1})) + < a, \sum_{n=1}^{N} T_n >, \tag{14}$$

where $a \in \mathbb{R}^3$ is a Lagrange multiplier and $c = \alpha\tau_\Delta$. Finally we come to the following

Definition of discrete elastic curves. *A discrete curve*

$$\gamma : \{0, \dots, N\} \to \mathbb{R}^3 \cong su(2)$$

with a frame $\Phi : \{1, \dots, N\} \to SU(2)$ satisfying (11) is called elastic if it is an extremal of the functional (14). Admissible variations preserve $\Phi(1), \Phi(N)$ and

$$\gamma_N - \gamma_0 = \sum_{n=1}^{N} T_n.$$

In the torsion free case $\alpha = 0$ (classical elastica) the bending energy is

$$E^\Delta_{elastica} = \sum \log(1 + \tan^2 \frac{\phi_n}{2}),$$

where ϕ_n is the angle between T_n and T_{n+1}.

4 Discrete spinning top

Let us return to the smooth elastic curves and treat the arclength parameter x of the previous section as the time variable. The Lagrangian (9) can be rewritten as

$$\mathcal{L} = \int_0^L ((\Omega_1^2 + \Omega_2^2) + \alpha\Omega_3^2 + < a, T >) dx,$$

where

$$\Omega = -i \sum_{k=1}^{3} \Omega_k \sigma_k = -2\Phi'\Phi^{-1}.$$

In this form it can be identified with the Lagrangian of the symmetric spinning top. In the formula above $\Phi(x)$ and $T(x)$ describe the evolution of the frame and of the axis of the top respectively, Ω is the angular velocity vector in the moving frame of the top. The inertia tensor is $(2, 2, 2\alpha)$ and $a \in \mathbb{R}^3$ is up to a constant the gravitational field. This result is known as

Kirchhoff's kinetic analogue [10]. *The frame of an elastic curve describes the motion of a symmetric spinning top. To the motion of such a top there corresponds an elastic curve.*

Using this observation and the discrete elastic curves defined above we naturally come to the following[6]

Definition of the discrete Lagrangian top. *The motion of the discrete Lagrangian top is a map $\Phi : \mathbb{Z} \to SU(2)$ with the Lagrangian (14) on any finite interval of $\mathbb{Z}$.*

The Euler-Lagrange equations (12,13) imply for the motion of the frame

$$\begin{aligned} \mathcal{U}_n &= \frac{\operatorname{tr} \mathcal{U}_n}{2} \left(I + \frac{T_{n+1} \times T_n - \tau_\Delta (T_n + T_{n+1})}{1+ < T_n . T_{n+1} >} \right), \\ \operatorname{tr} \mathcal{U}_n &= \sqrt{\frac{1+ < T_n . T_{n+1} >}{1 + \tau_\Delta^2}}. \end{aligned}$$

Thus, for a fixed τ_Δ equation (12) describes the motion of the frame completely.

One can find the integrals of (12)

$$H =< p_n, p_n - \frac{a}{2} >, \quad M = \frac{< a, T_n \times T_{n-1} > - \frac{c}{2}}{1+ < T_n, T_{n-1} >},$$

where

$$p_n = \frac{T_n + T_{n-1}}{1+ < T_n, T_{n-1} >}.$$

Equation (12) can be rewritten as a well-defined map

$$(T_{n-1}, T_n) \to T_{n+1}$$

for the axis of the top. A result [7] is presented in Fig.2, the corresponding smooth version can be found in textbooks. The Lax representation for the discrete Lagrangian top follows from the Ablowitz-Ladik L-A pair [1].

In the case $\alpha = 0$ we obtain a Lagrangian system on $S^2 \times S^2$ - the discrete spherical pendulum.

5 Acknowledgement

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[6] Although the rotation of a rigid body about a fixed point is a classical problem of mechanics, only an integrable discretization of the Euler case is known [12].

[7] The corresponding computer programm is written by C. Gunn

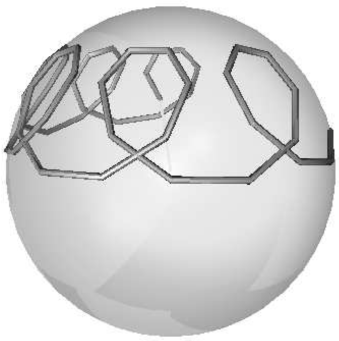

Figure 2: Evolution of the axis of the discrete spinning top

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