On continuous Hochschild homology and cohomology groups

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Abstract

Using the method of continuous projective resolutions established by Alain Connes we calculate the continuous Hochschild homology and cohomology groups of the Fréchet algebra of smooth functions on a manifold M. The (co)homology groups are allowed to take values in the Fréchet space $\mathcal{E}$ of smooth sections of a vector bundle E over M and in the strong dual $\mathcal{E}^*$. Moreover we show that in the cohomological case the differential and continuous Hochschild cohomologies are naturally isomorphic.

1 Introduction

Let M be a manifold and $E \to M$ a smooth vector bundle. Denote by $\mathcal{A}$ the algebra of smooth functions on M, by $\mathcal{E}$ the Fréchet space of smooth sections of E, and by $\mathcal{E}^*$ its strong dual. Since $\mathcal{A}$ with its natural Fréchet topology and the spaces $\mathcal{E}$ and $\mathcal{E}^*$ are all (nuclear) locally convex complete vector spaces we can write down the (co)chain complexes

$$0 \longleftarrow \mathcal{E}^{(*)} \longleftarrow \mathcal{A}\hat{\otimes}\mathcal{E}^{(*)} \longleftarrow \mathcal{A}\hat{\otimes}\mathcal{A}\hat{\otimes}\mathcal{E}^{(*)} \longleftarrow \ \dots\,, \tag{1}$$

and

$$0 \longrightarrow \operatorname{Hom}_{\mathcal{A}}\left(\mathcal{A}, \mathcal{E}^{(*)}\right) \longrightarrow \operatorname{Hom}_{\mathcal{A}}\left(\mathcal{A}\hat{\otimes}\mathcal{A}, \mathcal{E}^{(*)}\right) \longrightarrow \operatorname{Hom}_{\mathcal{A}}\left(\mathcal{A}\hat{\otimes}\mathcal{A}\hat{\otimes}\mathcal{A}, \mathcal{E}^{(*)}\right) \longrightarrow \ \dots\,, \tag{2}$$

where $\hat{\otimes}$ means the completed projective tensor product (cf. [7]), and $\operatorname{Hom}_{\mathcal{A}}$ the continuous $\mathcal{A}$-linear homomorphisms. The complexes (1) and (2) give rise to the continuous Hochschild (co)homology groups ${}_{\mathrm{c}}\mathrm{H}_*(\mathcal{A}, \mathcal{E})$, ${}_{\mathrm{c}}\mathrm{H}_*(\mathcal{A}, \mathcal{E}^*)$, ${}_{\mathrm{c}}\mathrm{H}^*(\mathcal{A}, \mathcal{E})$ and ${}_{\mathrm{c}}\mathrm{H}^*(\mathcal{A}, \mathcal{E})$ all of which will be calculated in this letter. Their relevance lies among other in deformation theory, where in particular the second Hochschild cohomology group ${}_{\mathrm{c}}\mathrm{H}^2(\mathcal{A}, \mathcal{A})$ provides important information about the topological deformation theory of $\mathcal{A}$ (see for example [1, 6, 3]).

Besides the continuous Hochschild cohomology one can consider the cohomology coming from differential cochains, i.e. cochains $C \in \operatorname{Hom}\left(\mathcal{A}^{\hat{\otimes}}, \mathcal{E}^{(*)}\right)$ which are given by differential operators in each argument. We denote these differential Hochschild cohomology groups by ${}_{\mathrm{d}}\mathrm{H}\left(\mathcal{A}, \mathcal{E}^{(*)}\right)$. Recall that there is a canonical morphism ${}_{\mathrm{d}}\mathrm{H}\left(\mathcal{A}, \mathcal{E}^{(*)}\right) \to$

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${}_{\mathrm{c}}\mathrm{H}\left(\mathcal{A},\mathcal{E}^{(*)}\right)$, as every differential operator is continuous with respect to the Fréchet topology of $\mathcal{A}$. We will show that the canonical map ${}_{\mathrm{d}}\mathrm{H}\left(\mathcal{A},\mathcal{E}^{(*)}\right) \to {}_{\mathrm{c}}\mathrm{H}\left(\mathcal{A},\mathcal{E}^{(*)}\right)$ is an isomorphism indeed.

The motivation for this note comes from the work of GEORGES PINCZON [6] and SIMONE GUTT [3], who have initiated the comparison of differential and continuous Hochschild cohomologies and the study of its implications for the deformation theory of algebras of smooth functions on a manifold. In [6] G. PINCZON showed that the differential and continuous Hochschild cohomologies of the algebra of smooth functions on $\mathbb{R}^n$ are naturally equivalent and isomorphic to the space of smooth totally antisymmetric multivector fields. In the paper [3] S. GUTT proved by using an argument of OMORI, MAEDA and YOSHIOKA [5] that for any smooth manifold M and $\mathcal{A}=\mathcal{C}^\infty(M)$ the second cohomology groups ${}_{\mathrm{d}}\mathrm{H}^2(\mathcal{A},\mathcal{A})$ and ${}_{\mathrm{c}}\mathrm{H}^2(\mathcal{A},\mathcal{A})$ are naturally isomorphic. In this work we generalize the results of G. PINCZON and S. GUTT to all Hochschild cohomology groups of arbitrary manifolds.

Another motivation for our paper comes from the well-known and by now classical Hochschild-Kostant-Sternberg Theorem [4]. Though it was expected to be true also in a topological context for algebras of smooth functions on a manifold, to our knowledge a full proof and exact statement of this fact seems not to have been given in the literature. We fill this gap in our letter as well.

2 Continuous projective resolutions

In his celebrated article *Noncommutative Differential Geometry* [2] ALAIN CONNES discussed the continuous Hochschild cohomology of topological algebras and showed that it can be calculated by a continuous projective resolution of the topological algebra under consideration. Using the methods established in [2] we will succeed in comparing the continuous and differential Hochschild cohomologies of algebras of smooth functions. But before dulging into this let us recall some definitions and results from [2] which are relevant for the sequel.

Let $\mathcal{B}$ be a complete locally convex topological algebra. A **topological module** over $\mathcal{B}$ then is a locally convex vector space $\mathcal{M}$ which has the structure of a $\mathcal{B}$-module such that $\mathcal{B}\times\mathcal{M}\ni(b,\xi)\mapsto b\xi\in\mathcal{M}$ is continuous. The module $\mathcal{M}$ is called **topologically projective** if it is a direct summand of a topological module of the form $\mathcal{M}'=\mathcal{B}\hat{\otimes}W$, where W is a complete locally convex vector space.

By a **continuous projective resolution** we mean an exact sequence of projective $\mathcal{B}$-modules and $\mathcal{B}$-linear continuous maps

$$0 \longleftarrow \mathcal{M} \xleftarrow{\varepsilon} \mathcal{R}_0 \xleftarrow{\delta_1} \mathcal{R}_1 \xleftarrow{\delta_2} \mathcal{R}_2 \longleftarrow \ \dots \tag{3}$$

admitting a $\mathbb{C}$-linear continuous homotopy

$$s_i : \mathcal{R}_i \to \mathcal{R}_{i+1},\ \delta_{i+1}s_i + s_{i-1}\delta_i = \mathrm{id},\ \forall i\in\mathbb{N}^*. \tag{4}$$

Let $\mathcal{A}$ be a complete locally convex topological algebra and $\mathcal{B}=\mathcal{A}\hat{\otimes}_\pi\mathcal{A}^\circ$, where $\mathcal{A}^\circ$ is the topological algebra opposite to $\mathcal{A}$. Then the sequence

$$\mathcal{A} \xleftarrow{\Delta^*} \mathcal{B} \xleftarrow{\delta_1} \mathcal{B}\hat{\otimes}\mathcal{A} \xleftarrow{\delta_2} \dots \xleftarrow{\delta_n} \mathcal{B}\hat{\otimes}\underbrace{\mathcal{A}\hat{\otimes}\dots\hat{\otimes}\mathcal{A}}_{k-\text{times}} \xleftarrow{\delta_{n+1}} \dots \tag{5}$$

with $\delta_k((1\otimes 1)\otimes a_1\otimes ...\otimes a_k) = (a_1\otimes 1)\otimes(a_2\otimes ...\otimes a_k)+$

$$+\sum_{j=1}^{k-1}(-1)^j(1\otimes 1)\otimes(a_1\otimes ...\otimes a_j\, a_{j+1}\otimes ...\otimes a_k)+(-1)^k(1\otimes a_k)\otimes(a_1\otimes ...\otimes a_{k-1}) \tag{6}$$

comprises a continuous projective resolution of the topological $\mathcal{B}$-module $\mathcal{A}$. We will call it the standard resolution of $\mathcal{A}$ with components $\mathcal{R}_{\mathrm{st},k} = \mathcal{B}\hat{\otimes}\mathcal{A}\hat{\otimes}...\hat{\otimes}\mathcal{A}$.

The following proposition is a slight generalization of a result of CONNES ([2], Lemma 43.).

Proposition 2.1 *Let* $(\mathcal{R}_k, \delta_k)$ *be a continuous projective resolution of the* $\mathcal{B}$*-module* $\mathcal{A}$*, and* $\mathcal{M}$ *a topological* $\mathcal{A}$*-module. Then the continuous Hochschild* (*co*)*homology* ${}_{\mathrm{c}}\mathrm{H}_k(\mathcal{A},\mathcal{M})$ (*resp.* ${}_{\mathrm{c}}\mathrm{H}^k(\mathcal{A},\mathcal{M})$) *of* $\mathcal{A}$ *with values in the topological module* $\mathcal{M}$ *coincides with the homology of the chain complex*

$$0 \longleftarrow \mathcal{R}_0\hat{\otimes}\mathcal{M} \xleftarrow{\delta_{1*}} \mathcal{R}_1\hat{\otimes}\mathcal{M} \xleftarrow{\delta_{2*}} \mathcal{R}_2\hat{\otimes}\mathcal{M} \longleftarrow \ ... \, , \tag{7}$$

respectively the cohomology of the cochain complex

$$\mathrm{Hom}_{\mathcal{B}}(\mathcal{R}_0,\mathcal{M}) \xrightarrow{\delta_1^*} \mathrm{Hom}_{\mathcal{B}}(\mathcal{R}_1,\mathcal{M}) \xrightarrow{\delta_2^*} \mathrm{Hom}_{\mathcal{B}}(\mathcal{R}_2,\mathcal{M}) \longrightarrow \ ... \, . \tag{8}$$

Let us assume that the manifold M has a nowhere vanishing vector field $Y : M \to TM$, and that one has chosen an affine connection for M with exponential function exp. Then one can find a smooth function $\chi : M\times M \to [0,1]$ such that χ is identical to 1 on a neighborhood of the diagonal of $M\times M$ and $\exp_b^{-1}(a)$ is well-defined for all $(a,b)\in \operatorname{supp}\chi$. Now consider the pull-back bundles $E_k = \mathrm{pr}_2(\Lambda^k T^*_{\mathbb{C}}M)$, where $\Lambda^k T^*_{\mathbb{C}}M$ is the exterior product of the complexified cotangent bundle of M and $\mathrm{pr}_2 : M\times M\to M$ the projection on the second coordinate. As E_1^* is the pull-back of the complexified tangent bundle $T_{\mathbb{C}}M$ via pr_2 the vector field

$$X(a,b) = \chi(a,b)\ \exp_b^{-1}(a) + i\,(1-\chi(a,b))\,Y(b) \tag{9}$$

comprises a section of E_1^* which does not vanish outside the diagonal of $M\times M$.

Proposition 2.2 (CONNES [2], p. 127ff) *Let* M *be a smooth manifold with a nowhere vanishing vector field* $Y : M\to TM$*, and define the bundles* E_k *and the section* X *like above. Then the sequence*

$$\mathcal{C}^\infty(M) \xleftarrow{\Delta^*} \mathcal{C}^\infty(M\times M) \xleftarrow{i_X} \mathcal{C}^\infty(M\times M,E_1) \xleftarrow{i_X} ... \xleftarrow{i_X} \mathcal{C}^\infty(M\times M,E_n) \longleftarrow 0 \tag{10}$$

comprises a continuous projective resolution for the module $\mathcal{C}^\infty(M)$ *over* $\mathcal{C}^\infty(M\times M)$*. We call it the Connes resolution and denote its components by* $\mathcal{R}_{\mathrm{Co},k} = \mathcal{C}^\infty(M\times M,E_k)$*. A chain map from* $\mathcal{R}_{\mathrm{Co}}$ *to the standard resolution* $\mathcal{R}_{\mathrm{st}}$ *of* $\mathbb{C}^\infty(M)$ *over the identity map* $\mathrm{id}_{\mathbb{C}^\infty(M)}$ *is given by*

$$F\omega(a,b,x_1,...,x_k) =< X(x_1,b)\wedge ...\wedge X(x_k,b),\omega(a,b) > \quad \forall a,b,x_i\in M \text{ and } \omega\in\mathcal{R}_{\mathrm{Co},k}. \tag{11}$$

3 Properties of continuous Hochschild (co)homology

Theorem 3.1 *Let M be a smooth manifold and $\mathcal{M}$ a topological module over $\mathcal{C}^\infty(M)$. Then the antisymmetrization*

$$\alpha : \mathcal{R}_{\mathrm{st}} \to \mathcal{R}_{\mathrm{st}}, \quad (a \otimes a') \otimes a_1 \otimes \ldots \otimes a_k \mapsto (a \otimes a') \otimes \frac{1}{k!} \sum_{\sigma \in \mathrm{S}_k} a_{\sigma(1)} \otimes \ldots \otimes a_{\sigma(k)} \tag{12}$$

induces identity morphisms

$$\alpha_* = \alpha_{M*} : {}_{\mathrm{c}}\mathrm{H}_*(\mathcal{C}^\infty(M), \mathcal{M}) \to {}_{\mathrm{c}}\mathrm{H}_*(\mathcal{C}^\infty(M), \mathcal{M}) \tag{13}$$

$$\alpha^* = {\alpha_M}^* : {}_{\mathrm{c}}\mathrm{H}^*(\mathcal{C}^\infty(M), \mathcal{M}) \to {}_{\mathrm{c}}\mathrm{H}^*(\mathcal{C}^\infty(M)\mathcal{M}) \tag{14}$$

in Hochschild (co)homology, or in other words any Hochschild (co)cycle is (co)homologous to its antisymmetrization.

PROOF: First let us assume that M is an open subset in some $\mathbb{R}^n$. Then in particular M possesses a nowhere vanishing smooth vector field $Y : M \to TM$. Thus, by Proposition 2.2 the chain map $F : \mathcal{R}_{\mathrm{Co}} \to \mathcal{R}_{\mathrm{st}}$ defined by Eq. (11) induces isomorphisms $F_* : \mathrm{H}_*(\mathcal{R}_{\mathrm{Co}} \hat{\otimes}_{\mathcal{C}^\infty(M^2)} \mathcal{M}) \to \mathrm{H}_*(\mathcal{R}_{\mathrm{st}} \hat{\otimes}_{\mathcal{C}^\infty(M^2)} \mathcal{M})$ and $F^* : \mathrm{H}^*\left(\mathrm{Hom}_{\mathcal{C}^\infty(M^2)}(\mathcal{R}_{\mathrm{st}}, \mathcal{M})\right) \to \mathrm{H}^*\left(\mathrm{Hom}_{\mathcal{C}^\infty(M^2)}(\mathcal{R}_{\mathrm{Co}}, \mathcal{M})\right)$. But by definition of F and α the relation $\alpha \circ F = F$ is true, so $\alpha_* F_* = F_*$ and $F^* \alpha^* = F^*$ holds. Hence the maps α_* and α^* have to be identity morphisms.

Now, if M is an arbitrary manifold, there exists an embedding $i : M \to \mathbb{R}^N$ of M in some Euclidean space $\mathbb{R}^N$. Choose a tubular neighborhood V of M in $\mathbb{R}^N$, and let $p : V \to M$ be the canonical projection. Denote the standard resolutions of $\mathcal{C}^\infty(M)$ and $\mathcal{C}^\infty(V)$ by $\mathcal{R}^M_{\mathrm{st}}$ resp. $\mathcal{R}^V_{\mathrm{st}}$. Finally let $\mathcal{M}^V$ be the topological $\mathcal{C}^\infty(V^2)$-module $\mathcal{C}^\infty(V^2) \hat{\otimes}_{\mathcal{C}^\infty(M^2)} \mathcal{M}$. Then we have natural embeddings

$$\begin{aligned} \iota_k : \;& \mathcal{R}^M_{\mathrm{st},k} \hat{\otimes}_{\mathcal{C}^\infty(M^2)} \mathcal{M} \to \mathcal{R}^V_{\mathrm{st},k} \hat{\otimes}_{\mathcal{C}^\infty(V^2)} \mathcal{M}^V, \\ & (1 \otimes 1) \otimes a_1 \otimes \ldots \otimes a_k \otimes m \mapsto (1 \otimes 1) \otimes a_1 p \otimes \ldots \otimes a_k p \otimes (1 \otimes m), \end{aligned} \tag{15}$$

$$\begin{aligned} \iota^k : \;& \mathrm{Hom}_{\mathcal{C}^\infty(M^2)}\left(\mathcal{R}^M_{\mathrm{st},k}, \mathcal{M}\right) \to \mathrm{Hom}_{\mathcal{C}^\infty(V^2)}\left(\mathcal{R}^V_{\mathrm{st},k}, \mathcal{M}^V\right), \\ & \delta \mapsto 1 \otimes (\delta \circ i^*), \end{aligned} \tag{16}$$

together with their respective left inverses or in other words splitting maps

$$\begin{aligned} \kappa_k : \;& \mathcal{R}^V_{\mathrm{st},k} \hat{\otimes}_{\mathcal{C}^\infty(V^2)} \mathcal{M}^V \to \mathcal{R}^M_{\mathrm{st},k} \hat{\otimes}_{\mathcal{C}^\infty(M^2)} \mathcal{M}, \\ & (1 \otimes 1) \otimes a_1 \otimes \ldots \otimes a_k \otimes (1 \otimes m) \mapsto (1 \otimes 1) \otimes a_1 i \otimes \ldots \otimes a_k i \otimes m, \end{aligned} \tag{17}$$

$$\begin{aligned} \kappa^k : \;& \mathrm{Hom}_{\mathcal{C}^\infty(V^2)}\left(\mathcal{R}^V_{\mathrm{st},k}, \mathcal{M}^V\right) \to \mathrm{Hom}_{\mathcal{C}^\infty(M^2)}\left(\mathcal{R}^M_{\mathrm{st},k}, \mathcal{M}\right), \\ & \delta \mapsto (i^* \otimes \mathrm{id}_{\mathcal{M}}) \circ \delta \circ p^*. \end{aligned} \tag{18}$$

By naturality of these functions we receive commutative diagrams

$$\begin{array}{ccccc} \mathcal{R}^M_{\mathrm{st},k} \hat{\otimes}_{\mathcal{C}^\infty(M^2)} \mathcal{M} & \xrightarrow{\iota_k} & \mathcal{R}^V_{\mathrm{st},k} \hat{\otimes}_{\mathcal{C}^\infty(V^2)} \mathcal{M}^V & \xrightarrow{\kappa_k} & \mathcal{R}^M_{\mathrm{st},k} \hat{\otimes}_{\mathcal{C}^\infty(M^2)} \mathcal{M} \\ {\scriptstyle \alpha_{M*}} \downarrow & & {\scriptstyle \alpha_{V*}} \downarrow & & {\scriptstyle \alpha_{M*}} \downarrow \\ \mathcal{R}^M_{\mathrm{st},k} \hat{\otimes}_{\mathcal{C}^\infty(M^2)} \mathcal{M} & \xrightarrow{\iota_k} & \mathcal{R}^V_{\mathrm{st},k} \hat{\otimes}_{\mathcal{C}^\infty(V^2)} \mathcal{M}^V & \xrightarrow{\kappa_k} & \mathcal{R}^M_{\mathrm{st},k} \hat{\otimes}_{\mathcal{C}^\infty(M^2)} \mathcal{M} \end{array} \tag{19}$$

and

$$\begin{array}{ccccc}
\mathrm{Hom}_{\mathcal{C}^\infty(M^2)}\left(\mathcal{R}^M_{\mathrm{st},k},\mathcal{M}\right) & \xrightarrow{\iota^k} & \mathrm{Hom}_{\mathcal{C}^\infty(V^2)}\left(\mathcal{R}^V_{\mathrm{st},k},\mathcal{M}^V\right) & \xrightarrow{\kappa^k} & \mathrm{Hom}_{\mathcal{C}^\infty(M^2)}\left(\mathcal{R}^M_{\mathrm{st},k},\mathcal{M}\right) \\
{\scriptstyle \alpha_M{}^*}\downarrow & & {\scriptstyle \alpha_V{}^*}\downarrow & & {\scriptstyle \alpha_M{}^*}\downarrow \\
\mathrm{Hom}_{\mathcal{C}^\infty(M^2)}\left(\mathcal{R}^M_{\mathrm{st},k},\mathcal{M}\right) & \xrightarrow{\iota^k} & \mathrm{Hom}_{\mathcal{C}^\infty(V^2)}\left(\mathcal{R}^V_{\mathrm{st},k},\mathcal{M}^V\right) & \xrightarrow{\kappa^k} & \mathrm{Hom}_{\mathcal{C}^\infty(M^2)}\left(\mathcal{R}^M_{\mathrm{st},k},\mathcal{M}\right).
\end{array} \tag{20}$$

Hence, as α_{V*} and $\alpha_V{}^*$ induce identity morphisms on the (co)homological level, the same holds true for α_{M*} and $\alpha_M{}^*$ as well. This proves the claim. □

An essential ingrediant for calculating the Hochschild (co)homology groups of $\mathcal{C}^\infty(M)$ in $\mathcal{E}$ and $\mathcal{E}^*$ will be the natural morphisms

$$\pi: \quad \mathcal{R}_{\mathrm{st},k} \to \Omega^k_{\mathbb{C}}M, \quad (1\otimes 1)\otimes f_1\otimes ...\otimes f_k \mapsto df_1\wedge ...\wedge df_k \tag{21}$$

$$\begin{aligned}\rho: \quad & \Lambda^k T_{\mathbb{C}}M \to \mathrm{Hom}_{\mathcal{C}^\infty(M^2)}\left(\mathcal{R}_{\mathrm{st},k},\mathcal{C}^\infty(M)\right), \\ & V_1\wedge ..\wedge V_k \mapsto \left((1\otimes 1)\otimes f_1\otimes ...\otimes f_k \mapsto \tfrac{1}{k!}\textstyle\sum_{\sigma\in S_k}(-1)^k\, V_{\sigma(1)}f_1\otimes ...V_{\sigma(k)}f_k\right),\end{aligned} \tag{22}$$

as they will provide the natural representation of the Hochschild (co)homology classes we are looking for. The other essential ingrediant is given by the following lemma, which is easily proved by using the fact that $\mathcal{C}^\infty(M)\hat{\otimes}_{\mathcal{C}^\infty(M^2)}\mathcal{R}_{\mathrm{Co},k} \cong \Omega^k_{\mathbb{C}}M$.

Lemma 3.2 *In case the manifold M has a nowhere vanishing vector field $Y: M \to TM$ the following maps are all well-defined and isomorphisms:*

$$\mathcal{R}_{\mathrm{Co},k}\hat{\otimes}_{\mathcal{C}^\infty(M^2)}\mathcal{E} \ni \omega\otimes e \overset{\tau_{\mathcal{E}}}{\mapsto} \omega|_\Delta\otimes e \in \Gamma^\infty(\Lambda^k T^*_{\mathbb{C}}M\otimes E), \tag{23}$$

$$\mathcal{R}_{\mathrm{Co},k}\hat{\otimes}_{\mathcal{C}^\infty(M^2)}\mathcal{E}^* \ni \omega\otimes\lambda \overset{\tau_{\mathcal{E}^*}}{\mapsto} \omega|_\Delta\otimes\lambda \in \Omega^k_{\mathbb{C}}M\hat{\otimes}_{\mathcal{C}^\infty(M)}\mathcal{E}^*, \tag{24}$$

$$\begin{aligned}&\mathrm{Hom}_{\mathcal{C}^\infty(M^2)}(\mathcal{R}_{\mathrm{Co},k},\mathcal{E}) \ni t \overset{\eta_{\mathcal{E}}}{\mapsto} T \in \Gamma^\infty(\Lambda^k T_{\mathbb{C}}M\otimes E), \\ &\textit{with } <\omega, T>= t(\omega') \quad \textit{for } \omega\in\Omega^k_{\mathbb{C}}M \textit{ and } \Delta^*\omega' = \omega,\end{aligned} \tag{25}$$

$$\begin{aligned}&\mathrm{Hom}_{\mathcal{C}^\infty(M^2)}(\mathcal{R}_{\mathrm{Co},k},\mathcal{E}^*) \ni t \overset{\eta_{\mathcal{E}^*}}{\mapsto} T \in \left(\Gamma^\infty(\Lambda^k T^*_{\mathbb{C}}(M)\otimes E)\right)^*, \\ &\textit{with } <T, \omega\otimes e>= t(\omega')(e) \quad \textit{for } \omega\in\Omega^k_{\mathbb{C}}M, e\in\mathcal{E} \textit{ and } \Delta^*\omega' = \omega.\end{aligned} \tag{26}$$

Assume now for a moment that M is open in some $\mathbb{R}^N$. Then we can consider the following sequences:

$$\begin{aligned}\Gamma^\infty(\Lambda^k T^*_{\mathbb{C}}M\otimes E) \to \mathcal{R}_{\mathrm{st},k}\hat{\otimes}_{\mathcal{C}^\infty(M^2)}\mathcal{E} \overset{F_*}{\to} \mathcal{R}_{\mathrm{Co},k}\hat{\otimes}_{\mathcal{C}^\infty(M^2)}\mathcal{E} \overset{\alpha_*}{\to} \\ \overset{\alpha_*}{\to} \mathcal{R}_{\mathrm{Co},k}\hat{\otimes}_{\mathcal{C}^\infty(M^2)}\mathcal{E} \overset{\pi_*}{\to} \Gamma^\infty(\Lambda^k T^*_{\mathbb{C}}M\otimes E),\end{aligned} \tag{27}$$

$$\begin{aligned}\Gamma^\infty(\Lambda^k T_{\mathbb{C}}M\otimes E) \overset{\rho_*}{\to} \mathrm{Hom}_{\mathcal{C}^\infty(M^2)}(\mathcal{R}_{\mathrm{st},k},\mathcal{E}) \overset{\alpha^*}{\to} \mathrm{Hom}_{\mathcal{C}^\infty(M^2)}(\mathcal{R}_{\mathrm{st},k},\mathcal{E}) \overset{F^*}{\to} \\ \overset{F^*}{\to} \mathrm{Hom}_{\mathcal{C}^\infty(M^2)}(\mathcal{R}_{\mathrm{Co},k},\mathcal{E}) \to \Gamma^\infty(\Lambda^k T_{\mathbb{C}}M\otimes E)\end{aligned} \tag{28}$$

and analogous ones for $\mathcal{E}^*$. In each of these sequences the composition of all its morphisms is equal to the identity map. Let us show this for (28); the proof for the other sequences is similar. Let $V = V_1\wedge ...\wedge V_k \in \Gamma^\infty(\Lambda^k T_{\mathbb{C}}M)$ and $e\in\mathcal{E} = \Gamma^\infty(E)$. Then (9) and (11) entail for $\omega\in\Omega^k_{\mathbb{C}}M$ and $\omega'\in\mathcal{R}_{\mathrm{Co},k}$ with $\Delta^*\omega' = \omega$ the relation

$$F^*\alpha^*\rho_*(e\otimes V)\,(\omega') = e\otimes\rho(V)\,(F\omega') = e\otimes <\omega, V>, \tag{29}$$

hence by (25) in Lemma 3.2 the section $e\otimes V$ is mapped to $e\otimes V$ under the composition of all morphisms in (28). Now, as α^*, F^* and ρ induce isomorphisms in cohomology, $\rho\otimes\mathrm{id}_{\mathcal{E}}$

has to be an isomorphism in cohomology as well. Moreover, as the image of $\rho\otimes\mathrm{id}_{\mathcal{E}}$ consists only of differential cocycles, the differential and continuous Hochschild cohomologies of $\mathcal{A}$ with values in $\mathcal{E}$ have to coincide. Obviously similar properties hold for the other type of Hochschild (co)homologies as well, so we receive the following theorem in case M is an open subset of $\mathbb{R}^N$.

Theorem 3.3 *Let M be a smooth manifold and $E \to M$ a smooth vector bundle over M. Denote by $\mathcal{A}$ the Fréchet algebra of smooth functions on M, by $\mathcal{E}$ the Fréchet space of smooth sections of E, and by $\mathcal{E}^*$ its strong dual. Then the continuous Hochschild (co)homology groups of $\mathcal{A}$ in $\mathcal{E}$ and $\mathcal{E}^*$ are given by the following natural sequences of isomorphisms:*

$$\Gamma^\infty(\Lambda^k T_{\mathbb{C}} M \otimes E) \to {}_{\mathrm{d}}\mathrm{H}^k(\mathcal{A},\mathcal{E}) \to {}_{\mathrm{c}}\mathrm{H}^k(\mathcal{A},\mathcal{E}), \tag{30}$$

$$\big(\Gamma^\infty(\Lambda^k T^*_{\mathbb{C}} M \otimes E)\big)^* \to {}_{\mathrm{d}}\mathrm{H}^k(\mathcal{A},\mathcal{E}^*) \to {}_{\mathrm{c}}\mathrm{H}^k(\mathcal{A},\mathcal{E}^*), \tag{31}$$

$${}_{\mathrm{c}}\mathrm{H}_k(\mathcal{A},\mathcal{E}) \to \Gamma^\infty(\Lambda^k T^*_{\mathbb{C}} M \otimes E) \tag{32}$$

$${}_{\mathrm{c}}\mathrm{H}_k(\mathcal{A},\mathcal{E}^*) \to \Omega^k_{\mathbb{C}} M \hat{\otimes}_{\mathcal{C}^\infty(M)} \mathcal{E}^*. \tag{33}$$

In particular ${}_{\mathrm{c}}\mathrm{H}_k(\mathcal{A},\mathcal{A}) \cong \Omega^k_{\mathbb{C}} M$ and ${}_{\mathrm{c}}\mathrm{H}^k(\mathcal{A},\mathcal{A}) \cong \Gamma^\infty(\Lambda^k T^*_{\mathbb{C}} M)$ which gives a topological version of the Hochschild-Kostant-Sternberg Theorem.

Remark 3.4 Alain Connes used the method of continuous projective resolutions to prove ${}_{\mathrm{c}}\mathrm{H}^k(\mathcal{A},\mathcal{A}^*) \cong \big(\Omega^k_{\mathbb{C}} M\big)^*$ for any compact manifold M. According to the theorem this relation holds for noncompact manifolds as well.

Proof: It remains to show the claim for an arbitrary manifold M. So embed M in some Euclidean space $\mathbb{R}^N$ and choose a tubular neighborhood V of M in $\mathbb{R}^N$. The canonical injection $i : M \to V$ and projection $p : V \to M$ give rise to the pullback bundle $E^V = p^*E$ and the relation $E = i^*E^V$. Note that $\mathcal{E}^V = \mathcal{C}^\infty(V)\hat{\otimes}_{\mathcal{C}^\infty(M)}\mathcal{E} \cong \Gamma^\infty(E^V)$. Now i and p induce a commutative diagram of the form

$$\begin{array}{ccccc}
\Gamma^\infty(\Lambda^k T_{\mathbb{C}} M \otimes E) & \longrightarrow & \Gamma^\infty(\Lambda^k T_{\mathbb{C}} V \otimes E^V) & \longrightarrow & \Gamma^\infty(\Lambda^k T_{\mathbb{C}} M \otimes E) \\
\downarrow & & \downarrow & & \downarrow \\
{}_{\mathrm{d}}\mathrm{H}^k(\mathcal{C}^\infty(M),\mathcal{E}) & \longrightarrow & {}_{\mathrm{d}}\mathrm{H}^k(\mathcal{C}^\infty(V),\mathcal{E}^V) & \longrightarrow & {}_{\mathrm{d}}\mathrm{H}^k(\mathcal{C}^\infty(M),\mathcal{E}) \\
\downarrow & & \downarrow & & \downarrow \\
{}_{\mathrm{c}}\mathrm{H}_k(\mathcal{C}^\infty(M),\mathcal{E}) & \longrightarrow & {}_{\mathrm{c}}\mathrm{H}_k(\mathcal{C}^\infty(V),\mathcal{E}^V) & \longrightarrow & {}_{\mathrm{c}}\mathrm{H}_k(\mathcal{C}^\infty(M),\mathcal{E}),
\end{array} \tag{34}$$

where all the horicontal compositions comprise identity morphisms. Hence, as the middle vertical arrows in (34) are isomorphisms, the other vertical arrows have to be isomorphisms as well. This proves the claim for the sequence (30). Check that one can write down diagrams analogous to (34) for the sequences (31) to (33). Arguing similarly like for (30) the theorem finally follows. □

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