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COVERING GROUPS OF THE GAUGE GROUP FOR THE STANDARD ELEMENTARY PARTICLE MODEL

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Abstract. We determine all Lie groups compatible with the gauge structure of the Standard Elementary Particle Model (SM) and their representations. The groups are specified by congruence equations of quantum numbers. By comparison with the experimental results, we single out one Lie group G_{SM} and show that this choice implies certain old and new correlations between the quantum numbers of the SM quantum fields as well as some hitherto unknown group theoretical properties of the Higgs mechanism. *PACS.* 11.30, 02.20, 12.10.

1. Introduction

1.1. **The Lie algebra of the Standard Model.** The usual and well known choice for the Lie algebra of internal symmetries within the Standard Elementary Particle Model (SM) is

$$\mathfrak{g}_{SM} \;=\; \mathfrak{u}(1)_Y \;\oplus\; \mathfrak{su}(2)_T \;\oplus\; \mathfrak{su}(3)_C\,, \tag{1}$$

where we denoted by Y, T and C the internal symmetries hypercharge, isospin and colour respectively. For many purposes, the knowledge of the *local* structure of the theory, that is the Lie algebra $\mathfrak{g}_{SM}$ is sufficient. For example, the dynamics of any quantum field theory (Feynman rules) depend only on them. However, some investigations of the gauge structure of a QFT require knowledge of the *global* structure, a fact we are well accustomed with from the manifold side. Since a standard result of Lie theory states that a Lie group is *not* globally determined by its Lie algebra, the main idea of this paper is to find where in the gauge structure of the SM the Lie *group* of internal symmetries appears, which of the possible choices for it is the "true" Lie group G_{SM} of the standard model, and what further conclusions this choice implies. Since we always want particle multiplets to be finite dimensional, we restrict our attention to *compact* Lie groups.

1.2. **Results.** After compiling the necessary mathematical definitions and prerequisites in Section 2, we give a complete classification of all compact Lie groups compatible with the internal gauge structure of the SM (Thm. 1) and a practical description of their representations via congruence equations between quantum numbers ("integrability conditions", Lemma 1), thus generalizing and correcting some former results by L. O'Raifeartaigh [1],[2] and J. Hucks [3].

By comparison with the well known particle content of the SM (Table 1), exactly one of these Lie groups (called G_{SM}) is singled out for a "minimal" description in Section 4.1; we furthermore show that its representation ring is generated by these particles together with their antiparticles. Assuming from there on that G_{SM} is indeed the Lie group of the Standard Model, we can then show that confined quark states have necessarily integer electric charge (Lemma 2). Futhermore, this choice implies that although the hypercharge y of a particle may be fractional, its product with the dimensions of the isospin and colour representations is always an integer. Equivalently, the sum over all electric charges inside any particle multiplet is necessarily integral (Thm. 2, Cor. 1).

We then show that G_{SM} is actually isomorphic to $\mathrm{S(U(2)\times U(3))}$ (Section 5.1) as well as to the Kronecker product of SU(2) and SU(3) with canonical scalar U(1) action (Section 5.2),

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thus allowing a second proof of Theorem 2 in Remark 10. We close with the surprising fact that for electroweak U(2), the usual definition for electric charge yields a complement of $\mathrm{SU}(2)_T$, whereas hypercharge does not. This decomposition is equivalent to the Gell-Mann-Nishijima Formula (Section 5.3).

2. Notations and mathematical prerequisites

2.1. **Cyclic groups, tori and hypercharge.** For finite cyclic groups, we will always use the notation

$$\mathcal{C}_n \;=\; \left\{\zeta_n^m \;\middle|\; m=1,\dots,n\right\}, \quad \zeta_n := \exp\frac{2\pi i}{n}\,. \tag{2}$$

The irreducible unitary characters of U(1) are given by a discrete quantity λ (sometimes called *winding number*)

$$\chi_\lambda(e^{i\theta}) \;=\; e^{i\lambda\theta},\ \lambda\in\mathbf{Z} \tag{3}$$

where we will succinctly write the corresponding U(1)-module as $\mathbf{C}_\lambda$. For historical reasons, physicists label $\mathrm{U}(1)_Y$-representations not by the integer λ, but rather by "hypercharge" $y=\lambda/6$. The universal covering group $\mathbf{R}$ of U(1) has the smooth unitary irreducible representations $x\mapsto e^{ix\Lambda}$, $\Lambda\in\mathbf{R}$, which can only give characters of the factor group $\mathbf{R}/\mu\mathbf{Z}$, $\mu\in\mathbf{R}^\times$, when the condition $\mu\Lambda\in 2\pi\mathbf{Z}$ holds. Thus the mere transition from the noncompact real line $\mathbf{R}$ to the compact group U(1) accounts already for the discreteness of the corresponding labeling parameter for its representations (compare this with [3]).

2.2. **Irreducible representations of $\mathfrak{su}(n)$.** We shall describe any irreducible unitary (necessarily finite dimensional) representation ϱ of $\mathfrak{su}(n)$ by its highest weight, written as a linear combination with non-negative integral coefficients of some choice of fundamental weights, and refer to these coefficients as the *Dynkin indices* of the representation ϱ. If we denote by V the n-dimensional standard representation of $\mathfrak{su}(n)$, we know that the representation ϱ with Dynkin indices $(a_1,\dots,a_{n-1})$ will appear as a subrepresentation of the tensor product

$$T_\varrho \;=\; \mathsf{S}^{a_1}V\otimes\mathsf{S}^{a_2}(\textstyle\bigwedge^2 V)\otimes\cdots\otimes\mathsf{S}^{a_{n-1}}(\textstyle\bigwedge^{n-1}V)\subset V^{\otimes r},\ r=\sum_{j=1}^{n-1} ja_j\,. \tag{4}$$

Remark 1. *Size of a representation.* We will refer to the number r as the *size* of the representation and denote its congruence class modulo dim V by $\bar r$. In terms of Young diagrams, r corresponds to the number of squares in a diagram. Physically speaking, we will prove that the representations of the center of G_{SM} will be completely determined by this number. Notice that the element $\zeta_n\cdot 1_n$ of $\mathrm{SU}(n)$ will act on the tensor representation T_ϱ or any other subrepresentation of $V^{\otimes r}$ as multiplication by $\zeta_n^{\bar r}$; in particular, this implies that the trivial representation can only appear in T_ϱ ofequation (4) if $\bar r=0$ (the converse, of course, being false).

2.3. **Irreducible isospin and colour representations.** The representations of the rank-1 Lie algebra $\mathfrak{su}(2)_T$ can be labeled by one Dynkin index; again for historical reasons, physicists use instead half its Dynkin index $t\in\frac12\mathbf{N}_0$. For $\mathfrak{su}(3)_C$, the Cartan subalgebra is two dimensional and one may choose the standard gluon fields g and $\bar b$ as fundamental weights with corresponding Dynkin indices i and j, which are exactly the colour charges listed in Table 1. The sizes of any $\mathfrak{su}(2)_T$- or $\mathfrak{su}(3)_C$-representation will be denoted by r_T or r_C, respectively. We denote the electric charge in multiples of e be q and will will assume the Gell-Mann-Nishijima formula $q=y+t_3$ to be valid throughout this article.

Remark 2. *Quark confinement.* The postulate of quark confinement that *any physical quark state must be a color singlet* means that both colour charges i, j have to vanish. Since physical states appear in tensor representations like T_ϱ or any other subrepresentation of $V^{\otimes r}$, Remark 1 implies that such an $\mathfrak{su}(3)$ tensor representation can contain a colour singlet only if $r_C\equiv 0 \bmod 3$.

3. Determining all Lie groups with Lie algebra $\mathfrak{g}_{SM}$

3.1. **Determining all (compact) Lie groups of a reductive Lie algebra.** The universal (connected) covering group $\tilde{G}$ of any reductive $\mathbf{R}$-Lie algebra $\mathfrak{g}$ is the direct product of its commutator subgroup and the connected component of the identity of its center:

$$\tilde{G} \;\cong\; \tilde{G}_c \times \mathcal{Z}(\tilde{G})_0,\;\; \mathcal{Z}(\tilde{G})_0 \cong\; \mathbf{R}^n.$$

For the full center, this means

$$\mathcal{Z}(\tilde{G}) \;\cong\; \mathcal{Z}(\tilde{G}_c) \times \mathcal{Z}(\tilde{G})_0,$$

where $\mathcal{Z}(\tilde{G}_c)$ is a finite group which can be determined from the root data of $\mathfrak{g}_c$. The general connected Lie group with Lie algebra $\mathfrak{g}$ is then of the form $H = \tilde{G}/D$, where D is a discrete subgroup of $\mathcal{Z}(\tilde{G})$, and compact if and only if D contains a subgroup isomorphic to $\mathbf{Z}^n$. A representation of $\tilde{G}$ gives a representation of the factor group H exactly in those cases where D operates trivially. Since the elements of D are central, they act as multiplication with some scalar on the $\tilde{G}$-module under consideration; therefore, D acts trivially if and only if this scalar is equal to 1 for all elements of D. Because of Schur's Lemma, any representation π of $\tilde{G}$ can be described in terms of representations of its center and of its commutator subgroup. The condition that D acts trivially then leads to congruence equations between the character of the center and the size of the representation of the semisimple part.

3.2. **General description of all compact Lie groups of $\mathfrak{g}_{SM}$.** Instead of $\mathfrak{g}_{SM}$, we may consider the slight generalization

$$\mathfrak{g}(p,q) \;=\; \mathfrak{u}(1) \oplus \mathfrak{su}(p) \oplus \mathfrak{su}(q),$$

where p and q are two different prime numbers. Its universal covering group is

$$\tilde{G}(p,q) \;\cong\; \mathbf{R} \times \mathrm{SU}(p) \times \mathrm{SU}(q) \text{ with center } \mathcal{Z}(\tilde{G}(p,q)) \;\cong\; \mathbf{R} \times \mathcal{C}_p \times \mathcal{C}_q\,.$$

Then any infinite discrete central subgroup $D \leq \mathcal{Z}(\tilde{G}(p,q))$ leads to a compact factor group

$$\tilde{G}(p,q)/D =: E(p,q) \tag{5}$$

with the same Lie algebra as $\mathfrak{g}(p,q)$. We define D_1 to be the intersection of D with the $\mathbf{R}$-factor of D, i. e. $D \cap (\mathbf{R} \times \{e\} \times \{e\}) = D_1 \times \{e\} \times \{e\}$. It is no loss of generality to assume $D_1 = \mathbf{Z}$; this means that the homomorphism $\tilde{G}(p,q) \to E(p,q)$ factors through the map

$$\varphi :\; \tilde{G}(p,q) \longrightarrow G(p,q) := \mathrm{U}(1) \times \mathrm{SU}(p) \times \mathrm{SU}(q)$$

which sends (μ, a, b) to $(e^{2\pi i\mu}, a, b)$. Thus the image $\varphi(D)$ of D in $G(p,q)$ satisfies

$$G(p,q)/\varphi(D) \cong E(p,q).$$

If $z = (e^{2\pi i\mu}, \zeta_p^m, \zeta_q^n) \in \varphi(D)$, then $z^{pq} = (e^{2\pi i\mu pq}, 1, 1) = (1,1,1)$ because of our choice of D_1. Thus μ must be an integer multiple of $1/pq$, which amounts to saying that the group order of $\varphi(D)$ must be a divisor of pq. This gives us

Theorem 1. *There are exactly nine families of possibilities for $\varphi(D)$, namely:*

$$\begin{array}{llllll}
(\mathrm{I}): & \langle(1,1,1)\rangle =: \mathcal{I} & & & & \\
(\mathrm{P}_1): & \langle(1,\zeta_p,1)\rangle =: \mathcal{P}_1, & (\mathrm{P}_2^{(m)}): & \big\langle(\zeta_p^{-1},\zeta_p^m,1)\big\rangle & =: & \mathcal{P}_2^{(m)} \\
(\mathrm{Q}_1): & \langle(1,1,\zeta_q)\rangle =: \mathcal{Q}_1, & (\mathrm{Q}_2^{(n)}): & \big\langle(\zeta_q^{-1},1,\zeta_q^n)\big\rangle & =: & \mathcal{Q}_2^{(n)} \\
(\mathrm{PQ}_1): & \langle(1,\zeta_p,\zeta_q)\rangle =: \mathcal{PQ}_1, & (\mathrm{PQ}_2^{(m)}): & \big\langle(\zeta_p^{-1},\zeta_p^m,\zeta_q)\big\rangle & =: & \mathcal{PQ}_2^{(m)} \\
(\mathrm{PQ}_3^{(n)}): & \big\langle(\zeta_q^{-1},\zeta_p,\zeta_q^n)\big\rangle =: \mathcal{PQ}_3^{(n)}, & (\mathrm{PQ}_4^{(m,n)}): & \big\langle(\zeta_p^{-1}\zeta_q^{-1},\zeta_p^m,\zeta_q^n)\big\rangle & =: & \mathcal{PQ}_4^{(m,n)}
\end{array}$$

where the indices m and n can take any of the values

$$m \in \mathcal{M} := \{1,\ldots,[(p-1)/2]\},\quad n \in \mathcal{N} := \{1,\ldots,[(q-1)/2]\}\,.$$

The respective group orders are

$$|\mathcal{I}| = 1,\ |\mathcal{P}_1| = |\mathcal{P}_2^{(m)}| = p,\ |\mathcal{Q}_1| = |\mathcal{Q}_2^{(n)}| = q,\ |\mathcal{PQ}_1| = |\mathcal{PQ}_2^{(m)}| = |\mathcal{PQ}_3^{(n)}| = |\mathcal{PQ}_4^{(m,n)}| = pq;$$

as a group, any nontrivial $\varphi(D)$ is thus isomorphic to either $\mathcal{C}_p$, $\mathcal{C}_q$ or $\mathcal{C}_{pq} \cong \mathcal{C}_p \times \mathcal{C}_q$.

Proof. The justification that these are exactly all occuring possibilities for $\varphi(D)$ is elementary for every given group order. We shall therefore not treat them all in detail but just mention that one has to (repeatedly) use the following facts:

1. our choice of D_1 rules out any elements of the form $(\zeta_{pq}^l, 1, 1) \in \varphi(D)$ other than $(1, 1, 1)$;
2. complex conjugation is an outer automorphism of $\mathrm{U}(1)$ and thus does not modify the factor group; we may therefore replace any $\zeta \in \mathrm{U}(1)$ by its inverse ζ^{-1} without changing the factor group;
3. any power of ζ_p or ζ_q different from 1 is a primitive pth respective qth root of unity;
4. the Chinese Remainder Theorem implies that for any $n \in \mathcal{N}$, we can find a coset $a + pq\mathbf{Z}$ such that $a \equiv 1 \bmod p$ and $a \equiv n \bmod q$; thus, $\zeta_p^a = \zeta_p$, whereas $\zeta_q^a = \zeta_q^n$. Of course, this may be equally applied to a situation with p and q exchanged.

□

We thus have the surprising result that all groups $\varphi(D)$ are cyclic, which was in general not true for the initial group D.

Now consider any representation of $\mathfrak{g}(p,q)$, that is, a triple consisting of a $\mathfrak{u}(1)$-representation of parameter $\lambda \in \mathbf{Z}$, a $\mathfrak{su}(p)$-representation of size r_p and a $\mathfrak{su}(q)$-representation of size r_q. Recall that these are in one-to-one correspondence with representations of $G(p,q)$. Any element $z = (a, b, c) \in \varphi(D)$ will then map under this representation to $(a^\lambda, b^{r_p}, c^{r_q})$, and because of the isomorphism (5) and the general remarks in Section 3.1, this image will operate trivially if and only if the product $s = a^\lambda b^{r_p} c^{r_q}$ of its factors is equal to 1. Since all groups $\varphi(D)$ turned out to be cyclic, it is enough to test this condition on the generating elements listed in Theorem 1. We will then get for every possible $\varphi(D)$ a necessary and sufficient congruence equation relating the parameters λ, r_p, r_q, m and n. One easily verifies that these are:

Lemma 1. *A representation of $\mathfrak{g}(p,q)$ can be lifted to a representation of $G(p,q)/\varphi(D)$ with $\varphi(D)$ of one of the types listed in Theorem 1 if and only if the corresponding congruence equation of the same type as given below holds:*

$$\begin{array}{llll}
(\mathrm{I}): & - & & \\
(\mathrm{P}_1): & r_p \equiv 0 \bmod p, & (\mathrm{P}_2^{(m)}): & mr_p \equiv \lambda \bmod p, \\
(\mathrm{Q}_1): & r_q \equiv 0 \bmod q, & (\mathrm{Q}_2^{(n)}): & nr_q \equiv \lambda \bmod q, \\
(\mathrm{PQ}_1): & qr_p + pr_q \equiv 0 \bmod pq, & (\mathrm{PQ}_2^{(m)}): & qmr_p + pr_q \equiv q\lambda \bmod pq, \\
(\mathrm{PQ}_3^{(n)}): & qr_p + pnr_q \equiv p\lambda \bmod pq, & (\mathrm{PQ}_4^{(m,n)}): & qmr_p + pnr_q \equiv (p+q)\lambda \bmod pq.
\end{array}$$

□

Because these congruence equations yield criteria when a given Lie algebra representation can be lifted to a representation of some Lie group, we will call them *integrability conditions* (IC). Some of the factor groups we can immediately identify with well known Lie groups, namely

$$\begin{array}{llcl}
(\mathrm{I}): & G(p,q) & \cong & \mathrm{U}(1) \times \mathrm{SU}(p) \times \mathrm{SU}(q) \\
(\mathrm{P}_1): & G(p,q)/\mathcal{P}_1 & \cong & \mathrm{U}(1) \times \mathrm{PSU}(p) \times \mathrm{SU}(q) \\
(\mathrm{P}_2^{(1)}): & G(p,q)/\mathcal{P}_2^{(1)} & \cong & \mathrm{U}(p) \times \mathrm{SU}(q) \\
(\mathrm{Q}_1): & G(p,q)/\mathcal{Q}_1 & \cong & \mathrm{U}(1) \times \mathrm{SU}(p) \times \mathrm{PSU}(q) \\
(\mathrm{Q}_2^{(1)}): & G(p,q)/\mathcal{Q}_2^{(1)} & \cong & \mathrm{SU}(p) \times \mathrm{U}(q) \\
(\mathrm{PQ}_1): & G(p,q)/\mathcal{PQ}_1 & \cong & \mathrm{U}(1) \times \mathrm{PSU}(p) \times \mathrm{PSU}(q) \\
(\mathrm{PQ}_2^{(1)}): & G(p,q)/\mathcal{PQ}_2^{(1)} & \cong & \mathrm{U}(p) \times \mathrm{PSU}(q) \\
(\mathrm{PQ}_3^{(1)}): & G(p,q)/\mathcal{PQ}_3^{(1)} & \cong & \mathrm{PSU}(p) \times \mathrm{U}(q)
\end{array}$$

where PSU(n) denotes as usual SU(n) modulo its center $\mathcal{Z} \cong \mathcal{C}_n$. We will prove later that

$$(\mathrm{PQ}_4^{(1,1)}): \quad G(p,q)/\mathcal{PQ}_4^{(1,1)} \quad \cong \quad \mathrm{S}(\mathrm{U}(p) \times \mathrm{U}(q)),$$

which can be realized as the elements $(a, b) \in \mathrm{U}(p) \times \mathrm{U}(q)$ satisfying the additional condition $\det a \det b = 1$, or via a suitably chosen Kronecker product.

Remark 3. *Interpretation of the parameters m and n.* Every value of m yields a possible identification of $\mathcal{C}_p$ in U(1) with $\mathcal{C}_p$ in SU(p); $m = 1$ corresponds to the identification we are familiar with by its realization in U(p) (and correspondingly for n and U(q)). The "twisted" versions of U(p) we get for $m \neq 1$ are not isomorphic; they do not appear for the SM, for then we have $\mathcal{M} = \mathcal{N} = \{1\}$.

Remark 4. *Simplification of integrability conditions.* Since p and q are coprime, each of the four integrability conditions which are congruence equations $\bmod\, pq$ holds exactly if the same relation is true $\bmod\, p$ and $\bmod\, q$ simultaneously; these relations in turn may be further simplified by eliminating the multiples of $p \bmod p$ or of $q \bmod q$ and dividing by any remaining factors coprime to p or q, respectively. We get:

$$(\mathrm{PQ}_1): \quad \left\{ \begin{array}{rcl} r_p & \equiv & 0 \bmod p \\ r_q & \equiv & 0 \bmod q \end{array} \right\} \qquad (\mathrm{PQ}_2^{(m)}): \quad \left\{ \begin{array}{rcl} mr_p & \equiv & \lambda \bmod p \\ r_q & \equiv & 0 \bmod q \end{array} \right\}$$

$$(\mathrm{PQ}_3^{(n)}): \quad \left\{ \begin{array}{rcl} r_p & \equiv & 0 \bmod p \\ nr_q & \equiv & \lambda \bmod q \end{array} \right\} \qquad (\mathrm{PQ}_4^{(m,n)}): \quad \left\{ \begin{array}{rcl} mr_p & \equiv & \lambda \bmod p \\ nr_q & \equiv & \lambda \bmod q \end{array} \right\}$$

Remark 5. *Interpretation of integrability conditions.* For an $\mathfrak{su}(n)$-representation of size r_n, the set of points of the weight lattice which satisfy the congruence relation $r_n \equiv \lambda \bmod n$ for some integer λ is the root lattice in case $\lambda \equiv 0 \bmod n$ and a translate of it otherwise.

Remark 6. *Comparison with results by other authors.* Comparing these results in the case $p = 2$ and $q = 3$ with the classification (without proof) in O'Raifeartaigh's book [2, p. 55 ff.], we see that his $G(p,q)/\mathcal{C}_{p+q}$ does not appear in the classification above; his further study of the subject as well as a footnote in [3] show that he must have meant $G(p,q)/\left\langle(\zeta_p^{-1}\zeta_q^{-1}, \zeta_p, \zeta_q)\right\rangle$ instead.

4. Integrability conditions and the Standard Model

4.1. **The "minimal" Lie group of the Standard Model.** The results above can be directly applied to the Standard Model (i.e., $p = 2$, $q = 3$). The size of an $\mathrm{SU}(2)_T$-representation is exactly $r_T = 2t$, that of an $\mathrm{SU}(3)_C$-representation $r_C = i + 2j$ (cf. Section 2.2). Every of the nine families of compact Lie groups with Lie algebra $\mathfrak{g}_{SM}$ has only one member, which is why we will drop the superscripts m and n from now on. A glimpse at Table 1 shows that experimentally, conditions (P_1) and (Q_1) are not satisfied, and thus so are all conditons implying them, that is, (PQ_1), (PQ_2) and (PQ_3). Ignoring the empty condition (I), this leaves us with the three possibilities (P_2), (Q_2) and (PQ_4), the latter being exactly the union of the former two. These conditions do actually hold,

$$(\mathrm{PQ}_4 \text{ a, b}) \qquad \lambda \equiv r_T \bmod 2 \quad \text{and} \quad \lambda \equiv r_C \bmod 3,$$

making it possible to define $G_{SM}(p,q) := G(p,q)/\left\langle(\zeta_p^{-1}\zeta_q^{-1}, \zeta_p, \zeta_q)\right\rangle$ and see that

$$G_{SM} \;:=\; G_{SM}(2,3) \;=\; G(2,3)/\left\langle(-\zeta_3^{-1}, -1, \zeta_3)\right\rangle$$

is in this sense the "minimal" Lie group of the Standard Model, that is, the smallest Lie group able to explain the experimental evidence. A very attractive property of this group is that it establishes hypercharge as the link between isospin and colour, thus relating quantities which are independent in the traditional choice $G(2,3)$.

We would like to illustrate the logic of our argument by a more familiar example. To the spin Lie algebra $\mathfrak{su}(2)$, there correspond exactly two compact Lie groups, namely SU(2) and PSU(2) $\cong$ SO(3); the size of a $\mathfrak{su}(2)$-representation being twice its spin $j \in \frac{1}{2}\mathbf{N}$, the former has empty integrability condition and the latter has $2j \equiv 0 \bmod 2$, which is equivalent to the condition that j be an integer. Historically, the experimental evidence of half integer spins led to the conclusion that the "right" Lie group has to be SU(2) and not SO(3). For the Standard Model, we experience the reverse situation: assume one would had thought that SU(2) is the spin Lie group and that experiment had only yielded integer spin values. Then the logical conclusion would have been that SO(3) is a better choice for the acting Lie group than SU(2) is. In the same vein, we suggest that G_{SM} is a better choice for the inner gauge group of the Standard Model than the usual choice $G(2,3)$ is.

From now on, we will *postulate* that G_{SM} is the gauge group of the Standard Model and investigate the conclusions implied by this choice.

The representation ring of G_{SM} has five generators, one possible choice for them being

$$R(G_{SM}) \;\cong\; \mathbf{Z}[V\otimes\mathbf{C}_3,\, W\otimes\mathbf{C}_{-2},\, \textstyle\bigwedge^2 W\otimes\mathbf{C}_{-4},\, \mathbf{C}_6,\, \mathbf{C}_{-6}]$$

where V (W) is the 2- (3)-dimensional defining representation of SU(2) (SU(3)). They correspond to the winding numbers and representation sizes

$$(\lambda, r_T, r_C) \;=\; (3,1,0),\ (-2,0,1),\ (-4,0,2),\ (6,0,0),\ (-6,0,0)$$

and may be identified with the following lepton and quark fields as introduced in Table 1:

$$R(G_{SM}) \;\cong\; \mathbf{Z}[l(x),\, d(x),\, u^*(x),\, e^*(x),\, e(x)].$$

Remark 7. *Antiparticles.* The transition from any particle multiplet to its antiparticle multiplet is made by taking the dual representation. In our case, this amounts to replacing the hypercharge by its negative and reversing the order of the Dynkin indices. From a group theoretical point of view, it is clear that the dual representation can always be formed. But we may also deduce the integrability condition for the dual representation by the following short argument, thus proving that conditions (PQ_4) also hold for the antiparticles which were not listed in Table 1: assume $\lambda \equiv r_n \bmod n$ for a SU(n)-representation of size r_n. By taking its negative, we get $-\lambda \equiv -r_n \bmod n$. Now remember that r_n was defined as $\sum j a_j$; since of course $-j$ is congruent to $n-j \bmod n$, we may rewrite this as $-\lambda \equiv \sum (n-j)a_j \bmod n$. But then the righthand side is exactly the size of the representation with reversed order of Dynkin indices.

4.2. **Consequences for the electric charge.**

As an example of the severe restrictions the relations (PQ_4) impose on admissible G_{SM}-representations, consider a bound quark state. As explained in Remark 2, these can only appear in representations with $r_C \equiv 0 \bmod 3$.

Lemma 2. *For any (usually non irreducible) G_{SM}-representation, the condition $r_C \equiv 0 \bmod 3$ is equivalent to integer electric charge for all particles contained in its irreducible subrepresentations.*

Proof. The statement is an easy consequence of the relations (PQ_4). We will only show one direction; the converse may be proved in a similar way. Remember that y was defined as $\lambda/6$. Then $r_C \equiv 0 \bmod 3$ together with $r_C \equiv \lambda \bmod 3$ implies $2y \in \mathbf{Z}$. Using the Clebsch-Gordan formula, we see that any irreducible multiplet of an $\mathfrak{su}(2)$-representation of size r_T will have a highest weight $2t$ of same parity. Thus the other congruence relation $r_T \equiv \lambda \bmod 2$ implies $t + 3y \in \mathbf{Z}$ for all such t, which together with $2y \in \mathbf{Z}$ gives $t + y \in \mathbf{Z}$. Since $t_3 = t, t-1, \ldots, -t$ is an integer or a half-integer precisely when t is, we thus have $y + t_3 \in \mathbf{Z}$ for all values of t_3. □

Remark 8. *Electric charges.* For studying anomalies, an important quantity is the sum of the electric charges inside a particle multiplet of given chirality. Unfortunately, this number is ill suited for representation theoretical studies, since it does not correspond to any characteristic quantity. However, given any $\mathfrak{su}(2)$-representation with highest weight $2t$, the Gell-Mann-Nishijima formula $q = y + t_3$ yields upon summation over $t_3 = t, t-1, \ldots, -t+1, -t$ the

Quantum field	y	$\lambda = 6y$	$r_T = 2t$	$r_C = i + 2j$	d_T	d_C	$\lambda d_T d_C$
l. h. quarks $q(x)$	$1/6$	1	1	1	2	3	6
r. h. u-quarks $u(x)$	$2/3$	4	0	1	1	3	12
r. h. d-quarks $d(x)$	$-1/3$	-2	0	1	1	3	-6
l. h. leptons $l(x)$	$-1/2$	-3	1	0	2	1	-6
r. h. leptons $e(x)$	-1	-6	0	0	1	1	-6
Hypercharge $B_\mu(x)$	0	0	0	0	1	1	0
Isospin $W_\mu(x)$	0	0	2	0	3	1	0
Colour $G_\mu(x)$	0	0	0	3	1	8	0
Higgs $h(x)$	$-1/2$	-3	1	0	2	1	-6

$\equiv \bmod 2$

$\equiv \bmod 3$

TABLE 1. Quantum numbers of the elementary fields, stated in an way appropriate for representation theory. One then checks easily that the indicated congruence relations hold.

relation $\sum q = (2t+1)y = d_T y$. In order to get the total electric charge for a $G_{SM}(p,q)$-multiplet, we have to multiply by the dimension of the colour-representation, thus obtaining

$$\sum_{G_{SM}(p,q)\ \text{rep.}} q = \ y d_T d_C \, .$$

The righthand side makes sense for any values of p and q and is accessible to group theoretical arguments as the following theorem shows.

Theorem 2. *Given a representation of $G_{SM}(p,q)$, the dimensions of the corresponding representations of* $\mathrm{SU}(p)$ *and* $\mathrm{SU}(q)$ *and the winding number satisfy the relation*

$$pq \mid d_p d_q \lambda .$$

By the preceding remark and because $y = \lambda/6 = \lambda/pq$, it is clear that Theorem 2 immediately implies for the SM:

Corollary 1. *If G_{SM} is the gauge group of the internal SM symmetries, then the sum over all electric charges inside any of its representations (particle multiplets) has to be an integer:*

$$\sum_{G_{SM}(p,q)\ \text{rep.}} q \ \in \mathbf{Z}.$$

Remark 9. *Covering groups and anomalies.* An easy consequence of the requirement of anomaly freedom in the SM is the condition that the sum of $\lambda d_p d_q$ over all fields of a given chirality has to be exactly zero. By choosing G_{SM} instead of $G(2,3)$ as the SM Lie group, Theorem 2 implies that these quantities are congruent $0 \bmod 6$; thus, this choice does not interfere with the anomaly freedom of the SM.

Proof. (of Theorem 2) The formula for the dimension of an SU(p)-representation with Dynkin indices $(a_1, \dots, a_{n-1})$ is

$$d_p = \prod_{0\le r<s<p} \left(\sum_{j=r+1}^{s} \tilde{a}_j \right) \cdot \prod_{l=1}^{p} \frac{1}{(l-1)!}$$

with $\tilde{a}_j = a_j + 1$ and its analog for SU(q). We show the stronger relations $p \,|\, \lambda d_p$ and $q \,|\, \lambda d_q$, which imply the assertion since p and q are two different prime numbers. Because of the symmetry of the problem, it is enough to show the assertion for p. Relation (PQ$_4$ a) implies that the difference of λ and r_p is a multiple of p and it is therefore enough to show $p \,|\, r_p d_p$. The factorials in the denominator of d_p contain only factors $< p$ and can thus be ignored, leaving us with the claim

$$p \text{ divides } r_p \cdot \prod_{0\le r<s\le p-1} \left(\sum_{j=r+1}^{s} \tilde{a}_j \right). \tag{6}$$

Case $p = 2$: Equation (6) is reduced to $2 \mid r(r+1)$, r the size of the representation, and this is of course always true.
Case $p \neq 2$: Now we have

$$\sum_1^{p-1} j\tilde{a}_j = \sum_1^{p-1} j(a_j+1) = \sum_1^{p-1} ja_j + \sum_1^{p-1} j = r_p + \frac{p(p-1)}{2}.$$

Since p is an odd prime number, $2 \mid p-1$. Therefore p divides the last term and it disappears mod p:

$$\sum_1^{p-1} j\tilde{a}_j \equiv r_p \bmod p.$$

Rewrite the negative of this last expression as

$$-\sum_1^{p-1} j\tilde{a}_j \equiv \sum_1^{p-1} (p-j)\tilde{a}_j \equiv \left\{ \begin{array}{l} \quad 0+ \\ +(0+\tilde{a}_1)+ \\ +(0+\tilde{a}_1+\tilde{a}_2)+ \\ +\dots+ \\ +(0+\tilde{a}_1+\dots+\tilde{a}_{p-1})\,. \end{array} \right.$$

If all p terms on the righthand side are different modulo p, their sum is congruent to $\frac{p(p-1)}{2}$ mod p and therefore equal to 0 mod p by the argument above, and we have thus $r_p \equiv 0 \bmod p$. If not, there exist indices $0 \le r < s < p$ satisfying

$$0 + \tilde{a}_1 + \dots + \tilde{a}_r = 0 + \tilde{a}_1 + \dots + \tilde{a}_s \bmod p, \text{ thus } \sum_{j=r+1}^{s} \tilde{a}_j \equiv 0 \bmod p,$$

and d_p is divisible by p. □

Although only based on simple number theoretical steps, this proof has the disadvantage of not showing any deeper structure. We will now give two explicit realizations of the group G_{SM}. The first one, very popular, is based on a direct sum construction and extensively used in connection with "Grand Unified Theories". The second one, less known, will make use of a Kronecker product and give a nice representation theoretical proof of Theorem 2.

5. Realizations of $G_{SM}(p,q)$ and Second Proof of Theorem 2

5.1. **Additive realization of $G(p,q)$ and its quotient groups.** Embed $\mathrm{SU}(p)$ as upper left, $\mathrm{SU}(q)$ as lower right block in $\mathrm{SU}(p+q)$ and write such elements succintly as pairs (a,b). For realizing G_{SM} as a subgroup of $\mathrm{U}(p+q)$, the requirement that ζ_p should be mapped to $(\zeta_p^1 \cdot 1_p, 1_q)$ and ζ_q to $(1_p, \zeta_q^1 \cdot 1_q)$ can only be achieved by a map of the form

$$e^{it} \longmapsto (e^{ikt} 1_p, e^{ilt} \cdot 1_q) \quad k,\, l \in \mathbf{Z}$$

where k and l satisfy $k \in q\mathbf{Z} \cap (1+p\mathbf{Z})$ and $l \in p\mathbf{Z} \cap (1+q\mathbf{Z})$. According to the Chinese Remainder Theorem, there always exist cosets $k + pq\mathbf{Z}$ and $l + pq\mathbf{Z}$ fulfilling these conditions. Since $p+q$ is coprime to p and q, we can further assume $pk + qk \equiv 0 \bmod (p+q)$ and still get solutions which will now be cosets mod $pq(p+q)$, thus making it possible to impose trace zero condition upon the $\mathfrak{u}(1)$-generator (for $p=2$ and $q=3$, a possible choice is $k=3$ and $l=-2$). This achieves to show the isomorphism between $G_{SM}(p,q)$ and $\mathrm{S}(\mathrm{U}(p) \times \mathrm{U}(q))$.

5.2. **Multiplicative realization of G_{SM}.** We now discuss a realization of G_{SM} using the Kronecker product of matrix groups (tensor product of the underlying vector spaces).

For $g \in \mathrm{SU}(p)$, $h \in \mathrm{SU}(q)$, let $g \otimes h$ act on the tensor product $V \otimes W$ of their standard modules. Then the image of (ζ_p, ζ_q) is

$$(\zeta_p \cdot 1_p) \otimes (\zeta_q \cdot 1_q) \;=\; \zeta_{pq}^{p+q} \cdot 1_{pq}.$$

By defining the action of $\mathrm{U}(1)$ on $V \otimes W$ as scalar multiplication, we get a natural identification of $\mathcal{C}_p \times \mathcal{C}_q \subset \mathrm{SU}(p) \times \mathrm{SU}(q)$ with $\mathcal{C}_{pq} \subset \mathrm{SU}(pq)$, because ζ_{pq}^{p+q} is always a primitive pqth root of unity. To check that it really satisfies the integrability conditions $(\mathrm{PQ}_4^{(1,1)})$, we first notice that for representations with sizes r_p, r_q and winding number λ, the following diagram has to commute:

$$\begin{array}{ccc}
(\zeta_p,\, \zeta_q) & \longrightarrow & \zeta_{pq}^{p+q} \\
{\scriptstyle r_p,\, r_q}\big\downarrow & & \big\downarrow{\scriptstyle \lambda} \\
(\zeta_p^{r_p},\, \zeta_q^{r_q}) & \xrightarrow{\;!\;} & \zeta_{pq}^{\lambda(p+q)}
\end{array}$$

which is equivalent to the requirement that the mapping relation denoted by ! holds. But

$$(\zeta_p^{r_p},\, \zeta_q^{r_q}) \;\longmapsto\; \zeta_p^{r_p} 1_p \otimes \zeta_q^{r_q} 1_q \;=\; \zeta_{pq}^{q r_p + p r_q} 1_{pq} \;\stackrel{!}{=}\; \zeta_{pq}^{\lambda(p+q)} 1_{pq},$$

means exactly

$$q r_p + p r_q \;\stackrel{!}{\equiv}\; \lambda(p+q) \bmod pq\,, \tag{7}$$

which we recognize to be the integrability condition $(\mathrm{PQ}_4^{(1,1)})$. Thus, the Kronecker product of $\mathrm{SU}(p)$ and $\mathrm{SU}(p)$ with a natural action of $\mathrm{U}(1)$ by scalar multiplication is isomorphic to $G_{SM}(p,q)$.

Remark 10. *Second proof of Theorem 2.* We may reprove Theorem 2 with a purely representation theoretical argument, without even having to know the formula for the dimensions of the representation.

Indeed, if we have $pq \mid d_p d_q$, then there is nothing to prove. If not, assume for example that p does not divide d_p. The image of ζ_p under an $\mathrm{SU}(p)$-representation is the matrix $\zeta_p^{r_p} \cdot 1_{d_p}$; its determinant has to be 1, and since p did not divide d_p, this can only be the case if $\zeta_p^{r_p} = 1$. But then the images of all pth roots of unity have to act trivially under any λ-representation, which amounts to saying that λ is a multiple of p. The same argument holds for q.

5.3. **Non-standard decompositions of compact connected groups.** We are interested in finding Cartan subgroups H of any compact connected reductive Lie group G which are a direct product of their intersections with the commutator subgroup and the center. Certainly, every complement of the commutator subgroup yields such a Cartan subgroup:

Lemma 3 (Direct decompositions of Cartan subgroups)**.** *Let G be a compact connected reductive Lie group and $\mathcal{K}$ a complement of its commutator subgroup G_c, i. e., we have a semidirect decomposition $G \cong G_c \rtimes \mathcal{K}$. Then any Cartan subgroup H of G is the direct product of its intersection H_c with G_c and $\mathcal{K}$, i. e., $H \cong H_c \times \mathcal{K}$, $H_c = G_c \cap H$.* □

5.4. **Application to $G = \mathrm{U}(n)$ and electroweak interactions.** For the Standard Model, it will turn out that these purely group theoretical considerations have the property of singling out exactly those symmetries which remain after spontaneous symmetry breakdown and Higgs mechanism. We take the liberty of assuming colour confinement, that is, G_{SM} is reduced to the subgroup $G = \mathrm{U}(2)$ of the electroweak forces. Since we thus only need to find complements of the commutator subgroup $G_c = \mathrm{SU}(n)$ of $G = \mathrm{U}(n)$, we refer the reader to the general results by K. H. Hofmann and H. Scheerer [4, Kor. 8], [5, Lemma] without stating them here for the verification that the following construction yields indeed all desired complements.

For a maximal torus H_c of $\mathrm{SU}(n)$ we make the usual choice of the diagonal matrices in $\mathrm{SU}(n)$. Furthermore, the center $\mathcal{Z}_0$ of G is $\mathrm{U}(1)$ and connected, thus giving $\mathcal{Z}_0 \cap H_c = \mathcal{Z}(\mathrm{SU}(n)) \cong \mathcal{C}_n$. The construction of all complements uses a continuous group morphism $f:\ \mathcal{Z}_0 \to H_c$

$$e^{i\omega}\cdot 1_n \overset{f}{\longmapsto} \begin{pmatrix} e^{ik_1\omega} & & 0 \\ & \ddots & \\ 0 & & e^{ik_n\omega} \end{pmatrix},\quad k_l \in \mathbf{Z}\ \forall l = 1,\dots,n$$

satisfying the conditions that the determinant be equal to 1

$$\sum_{l=1}^{n} k_l = 0 \tag{8}$$

and that f be equal to the identity map on $\mathcal{Z}(\mathrm{SU}(n))$

$$k_1,\dots,k_n \ \equiv\ 1 \bmod n. \tag{9}$$

We then have the representatives

$$\{f(z)^{-1}z : z \in \mathcal{Z}_0\} \;=\; \left\{ \begin{pmatrix} e^{i(1-k_1)\omega} & & 0 \\ & \ddots & \\ 0 & & e^{i(1-k_n)\omega} \end{pmatrix} \;:\; \omega \in [0,2\pi[\right\}$$

of a class of complements of G_c. By substituting $1 - k_i = z_i n$, $z_i \in \mathbf{Z}$, condition (9) holds automatically and since we then have $\sum k_i = n(1 - \sum z_i)$, eq. (8) implies $\sum z_i = 1$:

$$\mathcal{K}(z_1,\dots,z_n) \;:=\; \{f(z)^{-1}z : z \in \mathcal{Z}_0\} \;=\; \left\{ \begin{pmatrix} e^{iz_1 n\omega} & & 0 \\ & \ddots & \\ 0 & & e^{iz_n n\omega} \end{pmatrix} =: k(\omega) \;:\; \begin{matrix} \omega \in [0,2\pi[\\ \sum z_i = 1 \end{matrix} \right\}$$

The solutions for $(z_1,\dots,z_n)$ are, up to permutations,

$$(z_1,\dots,z_n) \;=\; (1,0,\dots,0),\ (1-z,z,0,\dots,0),\ (1-z-z',z,z',0,\dots,0)\dots$$

and each of them defines a complement of $\mathrm{SU}(n)$ isomorphic to $\mathrm{U}(1)$ with the product

$$(g,k)(g',k') \longmapsto (g\cdot kg'k^{-1}, kk') \quad \forall g,g' \in \mathrm{SU}(n),\ k,k' \in \mathcal{K}(z_1,\dots,z_n)\ . \tag{10}$$

Because of $k^{-1}(\omega) = k(-\omega)$, we have for $g' = (g'_{ij})_{ij}$, $1 \le i,j \le n$:

$$kg'k^{-1} \;=\; \left(e^{i\omega n(z_i - z_j)} g'_{ij}\right)_{ij}\ ,$$

and $kg'k^{-1}$ is certainly equal to g' if g' is diagonal. In this case the composition law (10) becomes

$$(g,k)(g',k') \longmapsto (gg',kk') \;,$$

which leads to the following direct product in $\mathrm{U}(n)$:

$$H_c \times \mathcal{K}(z_1,\dots,z_n) \;<\; \mathrm{SU}(n) \;\rtimes\; \mathcal{K}(z_1,\dots,z_n) \;=\; \mathrm{U}(n)\,.$$

Since $H_c \times \mathcal{K}$ is an n-parameter abelian subgroup of G, it is necessarily a maximal torus.

For $n = 2$, the Cartan subgroups of $\mathrm{SU}(2)$ are one-dimensional and the complements depend only on one parameter $z \in \mathbf{Z}$:

$$H_c \;=\; \left\{ \begin{pmatrix} e^{i\omega} & 0 \\ 0 & e^{-i\omega} \end{pmatrix} \right\}, \qquad \mathfrak{H} \;= \left\{ \omega h,\;\; h = i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\}$$

$$\mathcal{K}_z \;=\; \left\{ \begin{pmatrix} e^{i2z\alpha} & 0 \\ 0 & e^{i2(1-z)\alpha} \end{pmatrix} \right\}, \quad \mathfrak{K}_z \;= \left\{ \alpha k_z,\;\; k_z = i\begin{pmatrix} 2z & 0 \\ 0 & 2(1-z) \end{pmatrix} \right\}$$

Physically speaking, h is the generator of the third component of isospin and thus has eigenvalue t_3. Hypercharge is identified with the center of $\mathrm{U}(2)$. But then the relation

$$1_2 + h \;=\; k_1 \;\Leftrightarrow\; y + t_3 \;=\; \kappa_1 \;, \tag{11}$$

where κ_1 is the eigenvalue of k_1, allows us to identify κ_1 according to the Gell-Mann-Nishijima-formula with the electromagnetic charge q:

$$H_c \times \mathcal{K}_1 < \mathrm{SU}(2) \rtimes \mathcal{K}_1 \;\Leftrightarrow\; \mathrm{U}(1)_T \times \mathrm{U}(1)_Q < \mathrm{SU}(2) \rtimes \mathrm{U}(1)_Q.$$

This decomposition may be a hint *why* the transition from hypercharge to electric charge is necessary in the Standard Model. It has the property that any element of $\mathrm{U}(2)$ may be uniquely written as an element of $\mathrm{SU}(2) \rtimes \mathrm{U}(1)_Q$, whereas this was true only up to central elements for $\mathrm{SU}(2) \cdot \mathrm{U}(1)_Y$. Of course, we cannot explain the breakdown of isospin symmetry in this way, since we left the semisimple part untouched. Conversely, we can observe empirically: the Higgs mechanism singles out exactly the complement of the semidirect part in these special decompositions and forgets the semisimple part afterwards.

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