The Weierstrass representation of spheres in $\mathbf{R}^3$, the Willmore numbers, and soliton spheres

Taimanov I. A.*
Institute of Mathematics,
630090 Novosibirsk, Russia
e-mail: taimanov@math.nsc.ru

Abstract

The Weierstrass representation for spheres in $\mathbf{R}^3$ and, in particular, effective construction of immersions from data of spectral theory origin is discussed. These data are related to Dirac operators on a plane and on an infinite cylinder and these operators are just representations of Dirac operators acting in spinor bundles over the two-sphere which is naturally obtained as a completion of a plane or of a cylinder. Spheres described in terms of Dirac operators with one-dimensional potentials on a cylinder are completely studied and, in particular, for them a lower estimate of the Willmore functional in terms of the dimension of the kernel of the corresponding Dirac operator on a two-sphere is obtained. It is conjectured that this estimate is valid for all Dirac operators on spheres and some reasonings for this conjecture are discussed. In Appendix a criterion distinguishing Weierstrass representations, of universal coverings of compact surfaces of higher genera, converted into immersions of compact surfaces is given.

*Present address: Fachbereich Mathematik, SFB 288, Technische Universität Berlin, 10623 Berlin, Deutschland, e-mail: taimanov@sfb288.math.tu-berlin.de

§1. Introduction

In the present article we consider Weierstrass representations of spheres in $\mathbf{R}^3$. An existence of a global Weierstrass representation for any compact oriented surface of genus $g \geq 1$ has been established in [19, 20] and this proof, in fact, works for spheres also. Being mostly interested in relations of these representations to the spectral theory and in possibilities to apply the spectral theory to differential geometry ([17, 18]), we preferred to consider the case of spheres separately because in this case the spectral theory of Dirac operators

$$\mathcal{D} = \begin{pmatrix} 0 & \partial \\ -\bar{\partial} & 0 \end{pmatrix} + \begin{pmatrix} U & 0 \\ 0 & U \end{pmatrix} \tag{1}$$

is more developed.

In §2 we consider two different representations of spheres: as a plane completed by adding a point at infinity (a plane representation) and as a cylinder completed by adding a couple of "infinities" (a cylindric representation). For both of them we describe the data of Weierstrass representations which are the potential of a representation and a "wave function" ψ satisfying the equation

$$\mathcal{D}\psi = 0 \tag{2}$$

and some analytic conditions. These data of spectral theory origin are in one-to-one correspondence with immersed spheres in $\mathbf{R}^3$ (Theorems 1-4). This gives a straightforward procedure for constructing immersions in terms of zero-eigenfunctions of Dirac operators on a plane and on an infinite two-dimensional cylinder.

In §3 we consider spheres with one-dimensional potentials which means that in some cylindric representation the potential U of $\mathcal{D}$ depends on one variable. We prove that a sphere of revolution is uniquely reconstructed from the potential only (Theorem 5), describe all spheres with one-dimensional potentials in terms of the Jost functions (Theorem 6), and prove that

$$\mathcal{W}(\Sigma) \geq 4\pi \left(\dim_{\mathbf{H}} \operatorname{Ker} \mathcal{D}\right)^2 \tag{3}$$

where $\mathcal{W}$ is the Willmore functional and $\mathcal{D}$ is a Dirac operator acting on a spinor bundle over a sphere Σ (Theorem 7). We conjecture that this estimate is valid for all spheres.

In §4 we consider a special class of spheres with one-dimensional potentials – spheres with soliton (or reflectionless) potentials. This integrable case gives many interesting examples, and, in particular, for each N an equality in (3) is achieved exactly at some special soliton spheres, the Dirac spheres ([16]).

In Appendix A we give a criterion distinguishing immersions, of universal coverings of compact surfaces of higher genera, converted into immersions of compact surfaces.

Since in §3 and §4 we extensively use methods of the inverse scattering problem for one-dimensional Dirac operators on a line and this subject is not well familiar for geometers, Appendix B contains a brief exposition of this problem and especially some facts which we need.

We dedicate the present article to our teacher, S. P. Novikov, on his 60th birthday.

§2. The Weierstrass representation of spheres

2.1. The local Weierstrass representation.

First, recall the local Weierstrass representation. It is based on the following two facts:

Lemma A. (Eisenhart ([9], see also comments in [19])) *Let W be a simply connected domain in $\mathbf{C}$, $z_0 \in W$, and let a vector function $\psi = (\psi_1, \psi_2) : W \to \mathbf{C}^2$ satisfy (2) where $\mathcal{D}$ is of the form (1) and its potential $U(z, \bar{z})$ is real-valued. Then the following formulas*

$$
\begin{aligned}
X^1(z,\bar{z}) &= \frac{i}{2}\int_{z_0}^{z} \left((\bar{\psi}_2^2 + \psi_1^2)dz' - (\bar{\psi}_1^2 + \psi_2^2)d\bar{z}' \right), \\
X^2(z,\bar{z}) &= \frac{1}{2}\int_{z_0}^{z} \left((\bar{\psi}_2^2 - \psi_1^2)dz' - (\bar{\psi}_1^2 - \psi_2^2)d\bar{z}' \right), \qquad (4) \\
X^3(z,\bar{z}) &= \int_{z_0}^{z} (\psi_1\bar{\psi}_2 dz' + \bar{\psi}_1\psi_2 d\bar{z}')
\end{aligned}
$$

define an immersion of W into $\mathbf{R}^3$ with the induced metric $D(z,\bar{z})^2 dz d\bar{z}$ of the form

$$D(z,\bar{z}) = |\psi_1(z,\bar{z})|^2 + |\psi_2(z,\bar{z})|^2$$

and the Gauss curvature and the mean curvature are

$$K(z,\bar{z}) = -\frac{4}{D(z,\bar{z})^2}\partial\bar{\partial}\log D(z,\bar{z}) \quad \text{and} \quad H(z,\bar{z}) = 2\frac{U(z,\bar{z})}{D(z,\bar{z})}.$$

Lemma B. ([17]) *Let W be a domain in $\mathbf{C}$ and let $X : W \to \mathbf{R}^3$ be a conformal immersion of W into $\mathbf{R}^3$: $z \to X(z,\bar{z}) = (X^1(z,\bar{z}), X^2(z,\bar{z}), X^3(z,\bar{z}))$. Assume that $\partial X^3/\partial z \neq 0$ near $z_0 \in W$. Then near z_0 the functions*

$$\psi_1(z,\bar{z}) = \sqrt{-\partial\Phi(z,\bar{z})}, \quad \psi_2(z,\bar{z}) = \sqrt{\bar{\partial}\Phi(z,\bar{z})}, \qquad (5)$$

with

$$\Phi(z,\bar{z}) = X^2(z,\bar{z}) + iX^1(z,\bar{z}),$$

satisfy (2) with $U(z,\bar{z}) = H(z,\bar{z})D(z,\bar{z})/2$, where H is the mean curvature and $D^2 dz d\bar{z}$ is the metric of the surface $X(W) \subset \mathbf{R}^3$.

A globalization of this representation requires introducing spinor bundles, generated by ψ, over closed oriented surfaces and considering the operator (1) as acting on them. This had been shown in [17] and an existence of a global Weierstrass representation has been proved for any C^3-regular compact oriented surface of genus $g \geq 1$ (see Theorem 2 in [20]). The proof of it consists of continuing the sections (5) over the whole spinor bundle and uses the following lemma whose proof is contained in the proof of Theorem 2 from [20]:

Lemma 1 *Let $z_0 \in W$ be a nondegenerate critical point of the function X^3 defined on a domain $W \subset \mathbf{C}$ conformally immersed into $\mathbf{R}^3$. Then near z_0 the branches of (5) are correctly defined as one-valued functions and do not ramify at z_0.*

It is clear that if

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} : W \to \mathbf{C}$$

satisfies (2), then

$$\psi^* = \begin{pmatrix} \bar{\psi}_2 \\ -\bar{\psi}_1 \end{pmatrix} \tag{6}$$

also satisfies (2). Hence for any $\lambda, \mu \in \mathbf{C}$, such that $|\lambda|^2 + |\mu|^2 \neq 0$, the vector function

$$\Psi_{\lambda,\mu} = \lambda\psi + \mu\psi^* = \lambda \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} + \mu \begin{pmatrix} \bar{\psi}_2 \\ -\bar{\psi}_1 \end{pmatrix} : W \to \mathbf{C}$$

satisfies (2). Consider all immersions

$$X_{\lambda,\mu} : W \to \mathbf{R}^3$$

given via (4) by $\Psi_{\lambda,\mu}$ and normalize them by the condition

$$X_{\lambda,\mu}(z_0) = 0$$

where a point z_0 is fixed. Denote $\partial X^j_{1,0}/\partial z$ by N^j. It is shown by straightforward computations that

$$\begin{aligned}
\frac{\partial X^1_{\lambda,\mu}}{\partial z} &= \frac{(\lambda^2 + \bar{\lambda}^2 + \mu^2 + \bar{\mu}^2)}{2} \cdot N^1 + \frac{i(-\lambda^2 + \bar{\lambda}^2 + \mu^2 - \bar{\mu}^2)}{2} \cdot N^2 + i(\bar{\lambda}\bar{\mu} - \lambda\mu) \cdot N^3, \\
\frac{\partial X^2_{\lambda,\mu}}{\partial z} &= \frac{i(\lambda^2 - \bar{\lambda}^2 + \mu^2 - \bar{\mu}^2)}{2} \cdot N^1 + \frac{(\lambda^2 + \bar{\lambda}^2 - \mu^2 - \bar{\mu}^2)}{2} \cdot N^2 + (\bar{\lambda}\bar{\mu} + \lambda\mu) \cdot N^3, \\
\frac{\partial X^3_{\lambda,\mu}}{\partial z} &= i(\mu\bar{\lambda} - \lambda\bar{\mu}) \cdot N^1 + (-\lambda\bar{\mu} - \mu\bar{\lambda}) \cdot N^2 + (\lambda\bar{\lambda} - \mu\bar{\mu}) \cdot N^3.
\end{aligned} \tag{7}$$

From (7) we derive

Lemma 2 *1) The transformation $\Psi_{1,0} \to \Psi_{r,0}$, with $r \in \mathbf{R}$, generates the homothety*

$$(X^1, X^2, X^3) \to (r^2 X^1, r^2 X^2, r^2 X^3)$$

of the immersed surface ;

2) For $|\lambda|^2 + |\mu|^2 = 1$ the formulas (7) define the isomorphism

$$\rho : \{|\lambda|^2 + |\mu|^2 = 1, \lambda, \mu \in \mathbf{C}\}/\{\pm 1\} \to SO(3)$$

and the immersion $X_{\lambda,\mu}$ is a transformation of $X_{1,0}$ by the rotation $\rho(\lambda, \mu)$.

For instance,

$$\rho(\cos\varphi, \sin\varphi) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos 2\varphi & \sin 2\varphi \\ 0 & -\sin 2\varphi & \cos\varphi \end{pmatrix},$$

$$\rho(e^{i\theta}, 0) = \begin{pmatrix} \cos 2\theta & \sin 2\theta & 0 \\ -\sin 2\theta & \cos 2\theta & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$\rho(\cos\tau, i\sin\tau) = \begin{pmatrix} \cos 2\tau & 0 & \sin 2\tau \\ 0 & 1 & 0 \\ -\sin 2\tau & 0 & \cos 2\tau \end{pmatrix}.$$

2.2. A plane representation.

For constructing a global Weierstrass representation of a two-sphere immersed into $\mathbf{R}^3$ we may consider a sphere as a plane completed by a point at infinity or as an infinite cylinder completed by two infinities. We analyze both possibilities and start with a *plane representation*.

Let Σ be a 2-sphere immersed into $\mathbf{R}^3$.

Fix a pair of points $\infty_\pm$ on Σ and define a pair of charts with conformal parameters z, on $\mathbf{C} \approx \Sigma \setminus \infty_+$, and

$$u = -\frac{1}{z},$$

on $\mathbf{C} \approx \Sigma \setminus \infty_-$, such that $z(\infty_-) = 0$ and $u(\infty_+) = 0$. We have

$$du = \frac{dz}{z^2}, \quad \frac{\partial}{\partial u} = z^2 \frac{\partial}{\partial z}.$$

Introduce also the following functions

$$\tilde{\psi}_1(u,\bar{u}) = \sqrt{-\partial_u (X^2 + iX^1)}, \quad \tilde{\psi}_2(u,\bar{u}) = \sqrt{\partial_{\bar{u}}(X^2 + iX^1)}.$$

Now we arrive at the definition

Definition 1 *A sphere Σ, immersed into $\mathbf{R}^3$, possesses a global (plane) Weierstrass representation if there exist real potentials $U(z,\bar{z})$ and $\tilde{U}(u,\bar{u})$ and the vector functions $\psi(z,\bar{z})$ and $\tilde{\psi}(u,\bar{u})$ defined on a covering of Σ by a pair of charts with parameters z and $u = z^{-1}$ such that*

1)

$$\left\{ \begin{array}{l} \tilde{U}(u,\bar{u}) = |z|^2 \cdot U(z,\bar{z}), \\ \tilde{\psi}_1(u,\bar{u}) = z \cdot \psi_1(z,\bar{z}), \\ \tilde{\psi}_2(u,\bar{u}) = \bar{z} \cdot \psi_2(z,\bar{z}) \end{array} \right. \tag{8}$$

for $z = -1/u \in \mathbf{C}$;

2) the vector functions ψ and $\tilde{\psi}$ satisfy (2) for the Dirac operators with corresponding potentials U and $\tilde{U}$ and for a suitable choice of coordinates in $\mathbf{R}^3$ define by (4) an immersion of Σ.

Consider the analytic conditions met by the Dirac operator and the spinor sections ψ (8) corresponding to an immersion of a two-sphere into $\mathbf{R}^3$.

First, notice that $\tilde{D}(u,\bar{u})^2 du\, d\bar{u} = D(z,\bar{z})^2 |z|^4 du\, d\bar{u}$ near ∞_+ which implies that

$$D(z,\bar{z}) = \frac{C}{|z|^2} + O\left(\frac{1}{|z|^3}\right), \text{ with } C = \text{const} \neq 0.$$

and

$$U(z,\bar{z}) = \frac{U_+}{|z|^2} + O\left(\frac{1}{|z|^3}\right) \text{ with } U_+ = \text{const} \tag{9}$$

as $z \to \infty$. Therefore we conclude

$$|\psi(z,\bar{z})|^2 = |\psi_1(z,\bar{z})|^2 + |\psi_2(z,\bar{z})|^2 = O\left(\frac{1}{|z|^2}\right). \tag{10}$$

In fact, taking into account that the point $\infty_+ \in \Sigma$ is regular the last equality is refined as follows

$$|\psi(z,\bar{z}|^2 = \frac{C_+}{|z|^2} + O\left(\frac{1}{|z|^3}\right), \text{ with } C_+ \neq 0, \text{ as } z \to \infty. \tag{11}$$

Assume that Σ is C^3-regularly immersed into $\mathbf{R}^3$. By using the general position argument, we choose coordinates in $\mathbf{R}^3$ such that all critical points of the function X^3 defined on Σ are nondegenerate. Take a point $\infty_+ \in \Sigma$ and introduce a conformal parameter z on $\mathbf{C} \approx \Sigma \setminus \infty_+$. By Lemma 1, the branches of (5) do not ramify anywhere and are correctly defined on $\mathbf{C}$. Now from Lemma B and preceding conversations imply

Theorem 1 *Every C^3-regular two-sphere Σ immersed into $\mathbf{R}^3$ possesses a global ("plane") Weierstrass representation and the functions $U(z,\bar{z})$ and $\psi(z,\bar{z})$ defined on $\Sigma \setminus \infty_+$ satisfy (9) and (11).*

In fact, the conditions (2), (9), and (11) distinguish the data of Weierstrass representations of spheres.

Theorem 2 *Let $U(z,\bar{z})$ be a continuous function and satisfy (9) and let ψ be a solution to (2) such that $(|\psi_1|^2 + |\psi_2|^2)$ vanishes nowhere on $\mathbf{C}$ and (11) holds. Then ψ defines via (4) an immersion of $\mathbf{C}$ completed to a C^2-regular immersion of S^2 into $\mathbf{R}^3$. Moreover U and ψ and the functions $\tilde{U}$ and $\tilde{\psi}$ constructed from them by (8) form the data of a plane Weierstrass representation of the immersed sphere.*

Proof of Theorem 2.

By the definition of the local Weierstrass representation, ψ defines up to translations an immersion of $X : \mathbf{C} \to \mathbf{R}^3$. Normalize this immersion by $X(i) = 0 \in \mathbf{R}^3$.

Construct the functions $\tilde{U}$ and $\tilde{\psi}$ from U and ψ by (8) and notice that they satisfy the following equation

$$\left[\begin{pmatrix} 0 & \partial_u \\ -\partial_{\bar{u}} & 0 \end{pmatrix} + \begin{pmatrix} \tilde{U} & 0 \\ 0 & \tilde{U} \end{pmatrix}\right] \begin{pmatrix} \tilde{\psi}_1 \\ \tilde{\psi}_2 \end{pmatrix} = 0.$$

It follows from (8) and (9) that $\tilde{U}$ is a real-valued continuous function on the whole complex plane parameterized by $u \in \mathbf{C}$. Hence $\tilde{\psi}$ also defines up to translations an immersion of $\tilde{X} : \mathbf{C} \to \mathbf{R}^3$. Normalize this immersion by $\tilde{X}(i) = 0 \in \mathbf{R}^3$.

Since $u = z^{-1}$, we have

$$(\bar{\psi}_2^2 + \psi_1^2)dz - (\bar{\psi}_1^2 + \psi_2^2)d\bar{z} = (\bar{\tilde{\psi}}_2^2 + \tilde{\psi}_1^2)du - (\bar{\tilde{\psi}}_1^2 + \tilde{\psi}_2^2)d\bar{u},$$

$$(\bar{\psi}_2^2 - \psi_1^2)dz - (\bar{\psi}_1^2 - \psi_2^2)d\bar{z} = (\bar{\tilde{\psi}}_2^2 - \tilde{\psi}_1^2)du - (\bar{\tilde{\psi}}_1^2 - \tilde{\psi}_2^2)d\bar{u},$$

$$\psi_1\bar{\psi}_2 dz + \bar{\psi}_1\psi_2 d\bar{z} = \tilde{\psi}_1\bar{\tilde{\psi}}_2 du + \bar{\tilde{\psi}}_1\tilde{\psi}_2 d\bar{u}$$

on $\mathbf{C}^* = \mathbf{C} \setminus \{0\}$. These formulas imply that

$$X(z) = \tilde{X}\left(-\frac{1}{z}\right) \quad \text{for } z \in \mathbf{C}^*.$$

Hence, X and $\tilde{X}$ coincide on $\mathbf{C}^*$ and each of them is regularly continued onto the corresponding "infinity point", on $u = 0$ and $z = 0$. By (11), $\tilde{X}$ is regular at $u = 0$.

This proves the theorem.

A nice feature of this theorem is that the closedness problem consisting in distinguishing immersions of planes which are converted into immersions of compact surfaces reduces for spheres to the conditions (9) and (11) which can be easily checked. For surfaces of higher genera this problem is more complicated (see Appendix A).

If we have a solution ψ to (2) such that $|\psi|$ decays slower than $|z|^{-1}$ as $z \to \infty$ then nevertheless we may construct by using (4) an immersion of a sphere into $\mathbf{R}^3$ but it would not be regular and would have a peak singularity at the "infinity". If $|\psi|$ decays faster than $|z|^{-1}$ than we would have a branch point at "infinity". This also occurs when $|\psi(z,\bar{z})| = 0$ at $z \in \mathbf{C}$.

Let now admit branch points and consider more general situation.

Denote by $\mathcal{E}$ a $\mathbf{C}^2$-bundle

$$\mathbf{C}^2 \to \mathcal{E} \to S^2$$

whose sections ψ satisfy (8). This is a spinor bundle obtained as a square root of the complexified tangent bundle.

Theorem 3 *Let U satisfy (8) and (9). Then*

1) $\mathcal{D}$ acts on sections of $\mathcal{E}$;

2) solutions to (2) satisfying (10) are in one-to-one correspondence with zero-eigenfunctions of $\mathcal{D}$;

2) the kernel of $\mathcal{D}$ is finite-dimensional and moreover it is even-dimensional.

The first and second statements are evident. Since $\mathcal{D}$ is elliptic its kernel is finite-dimensional. We know that there exists an automorphism $*$, of the kernel, given by (6). Since $(\psi^*)^* = -\psi$, the kernel splits into two-dimensional subspaces invariant under $*$ and therefore $\dim_{\mathbf{C}} \operatorname{Ker} \mathcal{D} = 2n$ with n integer [1]. Each section $\psi \in \operatorname{Ker} \mathcal{D}$ generates via (4) an immersed sphere which may have branch points.

2.3. A cylindric representation.

Starting with a plane representation we also introduce a *cylindric representation*, i.e., a representation of an immersed sphere as an immersed cylinder completed by two points.

Put $z' = \log z = x' + iy'$ and take a cylinder $\mathcal{Z} = \mathbf{C}/i\mathbf{Z}$ with a conformal parameter z' given modulo $2\pi i$.

Let (U, ψ) be the data of a plane representation. We have

$$dz' = \frac{1}{z} dz, \quad \frac{\partial}{\partial z'} = z \frac{\partial}{\partial z}.$$

Consider the following functions

$$\hat{D}(z, \bar{z}') = |z| D(z, \bar{z}), \quad \hat{U}(z, \bar{z}') = |z| U(z, \bar{z}) \tag{12}$$

and

$$\hat{\psi}_1(z', \bar{z}') = \sqrt{z}\psi_1(z, \bar{z}), \quad \hat{\psi}_2(z', \bar{z}') = \sqrt{\bar{z}}\psi_2(z, \bar{z}). \tag{13}$$

By straightforward computations it is obtained that

Lemma 3 *1) The functions $\hat{\psi}$ satisfy the equation*

$$\left[\begin{pmatrix} 0 & \partial_{z'} \\ -\partial_{\bar{z}'} & 0 \end{pmatrix} + \begin{pmatrix} \hat{U} & 0 \\ 0 & \hat{U} \end{pmatrix} \right] \begin{pmatrix} \hat{\psi}_1 \\ \hat{\psi}_2 \end{pmatrix} = 0. \tag{14}$$

[1] F. Pedit and U. Pinkall proposed to treat $\mathcal{E}$ as a quaternion vector bundle and to treat $\operatorname{Ker} \mathcal{D}$ as a vector space over quaternions identifying $*$ with a multiplication by $\mathbf{j} \in \mathbf{H}$ ([10]). In this case $\dim_{\mathbf{H}} \operatorname{Ker} \mathcal{D} = n$.

2) There are the asymptotics

$$\hat{U} = \frac{U_\pm}{e^{|x'|}} + O\left(\frac{1}{e^{2|x'|}}\right), \quad |\hat{\psi}_1|^2 + |\hat{\psi}_2|^2 = \frac{C_\pm}{e^{|x'|}} + O\left(\frac{1}{e^{2|x'|}}\right), \tag{15}$$

with $U_\pm$ and $C_\pm$ constants such that $C_\pm \neq 0$ as $x' \to \pm\infty$.

Now it is clear how to derive from Theorems 1 and 2 the following result

Theorem 4 *1) Any vector function $\hat{\psi}$ satisfying (14) and (15) defines via (4) an immersion of $\mathcal{Z}$ into $\mathbf{R}^3$, which is completed to a regular immersion of a two-sphere.*

2) For any two-sphere Σ which is C^3-immersed into $\mathbf{R}^3$ and any pair of distinct points $\infty_\pm \in \Sigma$ there exists an immersion of a cylinder $\mathcal{Z} = \Sigma \setminus \{\infty_\pm\}$ such that

a) this immersion is defined, for a suitable choice of coordinates in $\mathbf{R}^3$, via (4) by functions $\hat{\psi}$ and $\hat{U}$ satisfying (14) and (15);

b) by adding a point to each end of the cylinder this immersion is completed to a regular immersion of Σ.

It follows from Theorem 1 and Theorem 4 that

Corollary 1 *Every C^3-regular two-sphere Σ immersed into $\mathbf{R}^3$ possesses a cylindric Weierstrass representation.*

We mention above that $\psi(z, \bar{z})$ are sections of $\mathcal{E}$. The formulas (12) and (13) show that vector functions $\hat{\psi}$ on $\mathcal{Z}$ meeting the conditions (15) and

$$\hat{\psi}(x', y' + 2\pi) = -\hat{\psi}(x', y') \tag{16}$$

are sections of $\mathcal{E}$ and these formulas just establish an equivalence between two different representations of $\mathcal{E}$. Moreover, these formulas also establish the equivalence between Dirac operators and, therefore, we have

Corollary 2 *Solutions $\hat{\psi}$ to (14) satisfying (16) and*

$$|\hat{\psi}|^2 = O\left(\frac{1}{e^{|x'|}}\right) \qquad \text{as } x' \to \pm\infty$$

form the kernel of $\mathcal{D}$. The dimension of $Ker\mathcal{D}$ is finite and even.

We also mention that each section from this kernel generates via (4) an immersed sphere which may have branch points.

§3. Spheres with one-dimensional potentials

3.1. The spectral data for one-dimensional potentials.

In this chapter we consider spheres admitting cylinder representations with one-dimensional real-valued potentials $U(x)$. This means that being defined on an infinite cylinder $\mathcal{Z} = \{(x, y) : -\infty < x < \infty, 0 \leq y \leq 2\pi\}$ the potential $U(z)$ depends on x only. We assume that $U(x)$ decays exponentially:

$$U(x) = O\left(\frac{1}{e^{|x|}}\right) \qquad \text{as } x \to \pm\infty.$$

For $U = 0$ solutions to (2) are linear combinations of

$$\begin{pmatrix} 0 \\ \exp(ik\bar{z}) \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \exp(ilz) \\ 0 \end{pmatrix}$$

with $z = x + iy$ and $k, l \in \mathbf{C}$. Of course, such functions are defined on $\mathcal{Z}$ if and only if $k, l \in 2\pi\mathbf{Z}$. Hence we look for solutions to (2) of the form

$$\psi(x,y) = \exp(ky)\varphi(x,y), \quad \varphi(x, y+2\pi) = \varphi(x,y), \tag{17}$$

i.e., we consider solutions which are defined on the universal covering $\mathbf{R}^2$ of $\mathcal{Z}$ and satisfy the periodicity condition

$$\psi(x, y+2\pi) = \mu\psi(x,y)$$

with μ a constant. For this ansatz the equation (2) reduces to

$$\left[\begin{pmatrix} 0 & \partial \\ -\bar{\partial} & 0 \end{pmatrix} + \begin{pmatrix} U & 0 \\ 0 & U \end{pmatrix} - \frac{1}{2}\begin{pmatrix} 0 & ik \\ ik & 0 \end{pmatrix}\right]\varphi = 0. \tag{18}$$

and, decomposing its solutions into Fourier series in y

$$\varphi(x,y) = \sum_{m \in \mathbf{Z}} \varphi_m(x) e^{imy},$$

we conclude that each $\varphi_m(x)$ satisfies (18) with $k + im$ substituted for k. Hence for studying all solutions to (2) of the form (17) it is enough to study solutions to

$$\left[\begin{pmatrix} 0 & \partial_x \\ -\partial_x & 0 \end{pmatrix} + \begin{pmatrix} 2U & 0 \\ 0 & 2U \end{pmatrix} - \begin{pmatrix} 0 & ik \\ ik & 0 \end{pmatrix}\right]\varphi = 0 \tag{19}$$

depending on x only.

This problem has been studied in its relation to soliton equations (see, for instance, [1, 6, 12, 21]) and we give the brief summary of results which we need in the sequel [2].

Since $U(x)$ decays exponentially as $|x| \to \infty$, we have

Summary.

The potential $U(x)$ of an operator

$$L = \begin{pmatrix} 0 & \partial_x \\ -\partial_x & 0 \end{pmatrix} + \begin{pmatrix} 2U & 0 \\ 0 & 2U \end{pmatrix}$$

is uniquely reconstructed from the spectral data which are

i) the reflection coefficient $R(k) = b(k)/a(k)$, with $k \in \mathbf{R} \setminus \{0\}$,

ii) the poles $\kappa_1, \dots, \kappa_N$ of the transmission coefficient $T(k)$ with $\mathrm{Im}\, k > 0$,

iii) some additional quantities $\lambda_1, \dots, \lambda_N \in \mathbf{C}$, attached to κ_j.

A reconstruction procedure is given by the Marchenko equations (52) and (56).

The poles of $T(k)$ are simple and correspond to exponentially decaying solutions to (19). For each pole κ_j every such solution is a multiple of $\varphi_1^+(x, \kappa_j)$ which is a unique solution to the equation

$$\varphi(x) = \begin{pmatrix} 0 \\ e^{i\kappa_j x} \end{pmatrix} + \int_x^{+\infty} \begin{pmatrix} 0 & -e^{-i\kappa_j(x-x')} \\ e^{i\kappa_j(x-x')} & 0 \end{pmatrix} \cdot 2U(x') \cdot \varphi(x')\, dx'.$$

[2] A detailed exposition is given in Appendix B.

Since $U(x)$ is real-valued,

a) κ_j are symmetric with respect to the imaginagy axis, if κ_j and $\kappa_l = -\overline{\kappa_j}$ are different poles of $T(k)$ then $\lambda_j = \bar{\lambda}_l$, and if $\mathrm{Re}\,\kappa_m = 0$ then $\lambda_m \in \mathbf{R}$;

b) $R(k) = \overline{R(-k)}$.

We recall the definition of the Kruskal integrals:

$$I_n(U) = \int_{-\infty}^{+\infty} U(x) q_n(x)\,dx,$$

where

$$q_1(x) = U(x)$$

and the other quantities $q_j(x)$ are defined by the recursion relation

$$q_{j+1}(x) = -i\frac{dq_j(x)}{dx} - 4U(x)\sum_{m=1}^{j-1} q_m(x) q_{j-m}(x).$$

The first of them is obviously the squared L_2-norm of $U(x)$:

$$I_1(U) = \int_{-\infty}^{+\infty} U^2(x)\,dx.$$

These quantities are related to the spectral data via the trace formulas (see [6], the formulas (7.20) and (7.21) in chapter 1 of Part I [3]):

$$I_n(U) = -\frac{1}{4\pi}\int_{-\infty}^{+\infty} \log(1-|b(k)|^2)(-2k)^{n-1}\,dk + \frac{i2^{n-2}}{n}\sum_{j=1}^{N}(\bar{\kappa}_j^n - \kappa_j^n) \tag{20}$$

where $b(k)$ is the ratio of $R(k)$ and $T(k)$ and satisfies the inequality

$$0 \leq |b(k)| < 1.$$

For $n = 1$ we have

$$\int_{-\infty}^{+\infty} U^2(x)\,dx = -\frac{1}{4\pi}\int_{-\infty}^{+\infty}\log(1-|b(k)|^2)\,dk + \sum_{j=1}^{N}\mathrm{Im}\,\kappa_j \tag{21}$$

and we conclude that

$$\int_{-\infty}^{+\infty} U^2(x)\,dx \geq \sum_{j=1}^{N}\mathrm{Im}\,\kappa_j \tag{22}$$

and an equality in (22) is achieved exactly at *reflectionless* potentials, i.e., $b(k) \equiv 0$ which is equivalent to $R(k) \equiv 0$.

3.2. Construction of spheres with one-dimensional potentials. A reconstruction of a sphere of revolution from its potential.

By Corollary 1, every sphere Σ regularly immersed into $\mathbf{R}^3$ possesses a cylindric Weierstrass representation. In this subchapter we describe spheres which admits cylindric representations with potentials depending on x only. The simplest and most important examples are spheres of revolution ([18]).

[3] In [6] these formulas are written in terms of κ, $\psi(x)$, and λ which, in our notation, are (-4), $-iU(x)$, and $-2k$, respectively.

Let $U(x)$ be a potential of an immersed sphere Σ. By Lemma 4, it decays exponentially and we may apply the spectral theory of L exposed in 3.1 and Appendix B. In particular, all exponentially decaying solutions to (18) are linear combinations of $\varphi_1^+(x,\kappa_j)$ and their $*$-transforms (6).

It is known that for any sphere of revolution Σ there exists a cylindric representation such that its potential is one-dimensional and Σ is immersed into $\mathbf{R}^3$ via (4) where ψ has the form

$$\psi(x,y) = \varphi(x)e^{iy/2}$$

(see [18]). Let $U(x)$ be the potential of this representation of Σ. Then we have

$$\psi_{\lambda,\mu}(x,y) = \lambda\left(\varphi_1^+(x,i/2)e^{iy/2}\right) + \mu\left(\varphi_1^+(x,i/2)e^{iy/2}\right)^*$$

where $\lambda,\mu \in \mathbf{C}$. By Lemma 2, for different λ and μ such immersions are transformed one into another by homotheties of spheres and their rigid motions in $\mathbf{R}^3$. Since the potentials of different Weierstrass representations are reconstructed one from another by the formulas (8) and (12), we conclude

Theorem 5 *Any sphere of revolution without branch points is uniquely defined (up to homotheties of the sphere and rigid motions in $\mathbf{R}^3$) by the potential of any of its Weierstrass representations.*

The condition on absence of branch points is added just for the following reason. Notice that we may consider the linear combination

$$\psi_{\lambda,\mu,\kappa}(x,y) = \lambda\left(\varphi_1^+(x,\kappa)e^{\kappa y}\right) + \mu\left(\varphi_1^+(x,\kappa)e^{\kappa y}\right)^*$$

with $\kappa = in/2$ and $n > 1$. Then the sphere constructed from $\psi_{\lambda,\mu,\kappa}$ via (4) would be an n-sheeted covering of a sphere of revolution with branch points at the infinities.

For general spheres with one-dimensional potentials the statement of Theorem 5 does not valid.

Let $\kappa_1,\dots,\kappa_N$ be the poles, of the transmission coefficient $T(k)$, coming into the spectral data of $U(x)$ and divide them into three groups

$$\kappa_j = \frac{in_j}{2} \text{ with } n_j \text{ an odd positive integer for } 1 \leq j \leq L,$$

$$\kappa_j = \frac{in_j}{2} \text{ with } n_j \text{ an even positive integer for } L+1 \leq j \leq M,$$

and κ_j is not of the form $in/2$ with n integer for $j \geq M+1$. Put

$$\psi_j(x,y) = \varphi_1^+(x,\kappa_j)e^{\kappa_j y}.$$

It is clear that for $j \geq M+1$ the functions ψ_j and ψ_j^* are no periodic and no antiperiodic in y. Therefore squares of linear combinations of such functions are not defined on $\mathcal{Z}$ and do not generate via (4) immersions of cylinders.

By Corollary 2, since ψ_j and ψ_j^* satisfy (16) for $j \leq L$, they are sections of $\mathcal{E}$ and we conclude

Lemma 4 *$Ker\mathcal{D}$ is spanned by $\psi_j(x,y)$ and $\psi_j^*(x,y)$ where $j \leq L$.*

Any linear combination $\psi(x,y)$ of ψ_j and ψ_j^* for $L+1 \le j \le M$ also generate via (4) an immersion of a sphere. It is easy to see that if for all ψ_j and ψ_k^* coming into this combination the frequencies n_j are represented in the form

$$n_j = 2^k l_j$$

with l_j odd integers then the immersion would be a 2^k-sheeted covering over its image with branch points of order 2^k at infinities. The potential of the representation of a covered sphere given by the function $\psi'(x,y) = \psi(x/2^k, y/2^k)$ would be $U'(x) = U(x/2^k)$. Otherwise the immersion would have branch points of odd order at the infinities.

Since by the definition, a vector function ψ coming into a cylindric representation belongs to $\mathrm{Ker}\,\mathcal{D}$, we summarize these conversations as follows

Theorem 6 *Let* $\mathbf{a} = (a_1, \dots, a_{2L}) \in \mathbf{C}^{2L} \setminus \{0\}$. *Then the function*

$$\psi_{\mathbf{a}}(x,y) = a_1\psi_1(x,y) + \ldots + a_L\psi_L(x,y) + a_{L+1}\psi_1^*(x,y) + \ldots + a_{2M}\psi_L^*(x,y) \quad (23)$$

defines via (4) an immersed sphere $\Sigma_{\mathbf{a}}$ *in* $\mathbf{R}^3$.

If there exist non-zero coefficients a_j *and* a_k *such that* $(j-k) \neq \pm L$ *then* $\Sigma_{\mathbf{a}}$ *is not a sphere of revolution.*

These spheres are exactly the spheres which have $U(x)$ *as a potential of some of their cylindric Weierstrass representations.*

3.3. The Willmore functional via the trace formula and the Willmore numbers.

Consider the following problem

Problem 1 *How to estimate the dimension of* $Ker\mathcal{D}$ *?*

For a Dirac operator with a one-dimensional potentials (in some cylindric representation of a sphere) the trace formula (21) enables us to give a precise estimate.

Indeed, by Lemma 4, $\mathrm{Ker}\,\mathcal{D}$ is spanned by $\varphi_1^+(x)e^{\kappa_j y}$ and $(\varphi_1^+(x)e^{\kappa_j y})^*$ with κ_j of the form

$$\kappa_j = \frac{i(2n_j+1)}{2} \quad (24)$$

with n_j nonnegative integers. By (21) and (22), we have

$$\int_{\infty}^{+\infty} U^2(x)\,dx \ge \frac{1}{2}\sum_{j=1}^{L}(2n_j+1) \quad (25)$$

and an equality in (25) is achieved exactly in the case then $U(x)$ is a reflectionless potential and the whole discrete spectrum with $\mathrm{Im}\,\kappa > 0$ consists of eigenvalues of the form (24). We conclude

Theorem 7 *Let* $\mathcal{D}$ *be a Dirac operator (1) on* $\mathcal{E}$ *with a one-dimensional potential* $U(x)$ *in some cylindric representation (for instance, a Dirac operator generating an immersion of a sphere of revolution). If*

$$\frac{\dim_{\mathbf{C}} Ker\mathcal{D}}{2} = \dim_{\mathbf{H}} Ker\mathcal{D} \ge N,$$

then

$$\int_{-\infty}^{+\infty} U^2(x)\,dx \ge \frac{N^2}{2}. \quad (26)$$

Indeed, any level $\kappa = i(2n+1)/2$ may be filled just by one eigenfunction of the form $\varphi_1^+(x,\kappa_j)e^{\kappa_j y}$ (see Appendix B) and, given $N = \dim_{\mathbf{H}} \mathrm{Ker}\,\mathcal{D}$, the left-hand side in (26) achieves its minimal possible value if just first N levels are filled. This means that

$$\frac{1}{2}\sum_{j=1}^{N}(2n_j+1) = \frac{1}{2}(1+3+\ldots+(2N-1)) = \frac{N^2}{2}.$$

This proves the theorem.

An example of the Dirac spheres, constructed by U. Pinkall and J. Richter ([16]), shows that for each N an equality in (26) is achieved at the potential of such sphere and therefore

Corollary 3 *For any N the estimate (26) is precise and an equality is achieved at*

$$U_N(x) = \frac{N}{2\cosh x}.$$

We discuss the spectral data of such potentials in §4.

To rewrite (26) for general spheres it needs to integrate the left-hand side over y also and obtain an integral over S^2 (by (8) and (12) this integral is correctly defined for any representation):

$$\int_\Sigma U^2(z,\bar{z})\,dx\wedge dy \geq \pi N^2. \tag{27}$$

We would like to conjecture that

Conjecture 1 *The estimate (27) holds for any Dirac operator on $\mathcal{E}$.*

In fact, by Theorem 4, the dimension of $\mathrm{Ker}\,\mathcal{D}$ measures the dimension of a family of isopotential spheres in $\mathbf{R}^3$. If there exists $\psi \in \mathrm{Ker}\,\mathcal{D}$ such that ψ vanishes nowhere then it generates an immersion of a sphere without branch points. It is also known that for any compact surface Σ immersed via (4) the value of the Willmore functional $\mathcal{W}$, an integral of a squared mean curvature, is given by

$$\mathcal{W}(\Sigma) = \int H^2\,d\mu = 4\int_\Sigma U^2(z,\bar{z})\,dx\wedge dy \tag{28}$$

(see [17]), i.e., a multiple of the squared L_2-norm of the potential U. We have

$$\int_\Sigma (H^2-K)\,d\mu = \int_\Sigma \left(\frac{k_1-k_2}{2}\right)^2 d\mu \geq 0$$

where k_j are the principal curvatures and K is the Gauss curvature of Σ. By the Gauss–Bonnet theorem, for spheres

$$\int_\Sigma K\,d\mu = 4\pi$$

and this implies

$$\int_\Sigma H^2\,d\mu \geq 4\pi.$$

Therefore we conclude that

$$\int_\Sigma U^2(z,\bar{z})\,dx\wedge dy \geq \pi.$$

But this is just the inequality (27) for $N = 1$ and an equality is achieved at $U_1(x)$ which is the potential of the unit sphere ([18]). Recalling the recent result of F. Pedit and U. Pinkall obtained by methods of the so-called quaternionic algebraic geometry ([15]), we conclude

Proposition 1 *Assume that there exists a zero-eigenfunction ψ of $\mathcal{D}$ such that ψ defines via (4) a regular immersion of S^2 into $\mathbf{R}^3$. Then Conjecture 1 is valid*
1) for $N = 1$ (Gauss–Bonnet);
2) for $N = 2$ (Pedit–Pinkall).

For such operators the estimate (27) in terms of $\mathcal{W}$ takes the form

$$\mathcal{W}(\Sigma) \geq 4\pi N^2. \tag{29}$$

We show in §4 the dimension of $\mathrm{Ker}\,\mathcal{D}$ can not be estimated from below in terms of the Willmore functional.

This treatment of (27) fits into a general approach to estimates for the Willmore functional based on the Weierstrass representation (see [20] where the spectral approach for the Willmore conjecture for tori is introduced). The right-hand sides of (29), *the Willmore numbers*, measure not only the existence of a sphere with given value of the Willmore functional but the dimension of a family of isopotential spheres. It looks natural that for given dimension of $\mathrm{Ker}\,\mathcal{D}$ the Willmore functional has to attain its minimal possible value on a very symmetric operator which has to have a one-dimensional potential, i.e., to be of the form covered by Theorem 7. The analogous idea is discussed in [20] for the Willmore conjecture for tori.

§4. Soliton spheres

4.1. Solving the Marchenko equations for reflectionless potentials.

We call a sphere reflectionless if it admits a Weierstrass representation with a one-dimensional reflectionless potential $U(x)$. This means that the reflection coefficient $R(k)$ of $U(x)$ vanishes identically, i.e., $R(k) \equiv 0$, and the spectral data of $U(x)$ are just

1) a half, of a discrete spectrum of L with $U(x)$ its potential, lying in the upper-half plane: $\kappa_1, \dots, \kappa_N$; this spectrum is symmetric with respect to the imaginary axis;

2) some quantities λ_j corresponding to κ_j such that if $\mathrm{Re}\,\kappa_j = 0$ then $\lambda_j \in \mathbf{R}$ and if κ_k and $\kappa_l = -\bar{\kappa}_k$ do not coincide then $\lambda_k = \bar{\lambda}_l$.

The potential $U(x)$ is reconstructed from the spectral data via the Marchenko equations (see Appendix B). For reflectionless potentials solutions to these equations can be found explicitly. We explain this procedure following [12].

Given the spectral data for a reflectionless potential, consider the following ansatz:

$$B_j(x, y) = \langle B_j(x) | T(y) \rangle$$

where

$$T(z) = (e^{i\kappa_1 z}, \dots, e^{i\kappa_N z})$$

and

$$\langle u|v \rangle = u_1 v_1 + \dots + u_N v_N$$

is a standard inner product. Then Ω takes the form

$$\Omega(x + y) = \langle \Psi(x) | T(y) \rangle$$

where
$$\Psi(z) = (-\lambda_1 e^{i\kappa_1 z}, \ldots, -\lambda_N e^{i\kappa_N z}).$$
In terms of these functions the Marchenko equations (52) are written as
$$\langle B_2(x)|T(y)\rangle + \int_x^{+\infty} \langle B_1(x)|T(x')\rangle\langle\Psi(x')|T(y)\rangle\, dx' = 0$$
and
$$\langle\Psi(x)|T(y)\rangle - \langle B_1(x)|T(y)\rangle + \int_x^{+\infty} \langle B_2(x)|T(x')\rangle\langle\Psi(x')|T(y)\rangle\, dx' = 0.$$
Introduce the matrix
$$M(x) = \int_x^{+\infty} |T(x')\rangle\langle\Psi(x')|\, dx'$$
(here we use Dirac's notation treating the inner product as a product of a bra vector $\langle u|$ and a ket vector $|v\rangle$, [4]), rewrite the Marchenko equations as
$$\langle B_2(x) + B_1(x)M(x)|T(y)\rangle = \langle\Psi(x) - B_1(x) + B_2(x)M(x)|T(y)\rangle = 0$$
and finally arrive at the following form of them
$$B_2(x) + B_1(x)M(x) = \Psi(x) - B_1(x) + B_2(x)M(x) = 0. \tag{30}$$
The entries of $M(x)$ are simply computed
$$M_{jk}(x) = \int_x^{+\infty} T_j(x')\Psi_k(x')\, dx' = \frac{\lambda_k}{i(\kappa_j+\kappa_k)} e^{i(\kappa_j+\kappa_k)x}$$
and (30) implies that
$$B_1(x) = \Psi(x)\cdot(1+M^2(x))^{-1}$$
and
$$B_2(x) = -B_1(x)\cdot M(x).$$
By (55), we derive
$$U(x) = -\langle\Psi(x)\cdot(1+M^2(x))^{-1}|T(x)\rangle.$$
Now represent $B_1(x,x)$ and $B_2(x,x)$ as follows
$$B_1(x,x) = -\mathrm{Tr}\left[\frac{dM(x)}{dx}\cdot(1+M^2(x))^{-1}\right],$$
$$B_2(x,x) = \mathrm{Tr}\left[\frac{dM(x)}{dx}\cdot M(x)\cdot(1+M^2(x))^{-1}\right].$$
It follows from (55) and (57) that
$$2U^2(x) + i\frac{dU(x)}{dx} = \frac{d}{dx}\mathrm{Tr}\left[\frac{dM(x)}{dx}(M(x)+i)(1+M^2(x))^{-1}\right],$$

and, since $1 + M^2(x) = (1 + iM(x))(1 - iM(x))$, we have

$$2U^2(x) + i\frac{dU(x)}{dx} = \frac{d}{dx}\mathrm{Tr}\left[\frac{d(1+iM(x))}{dx}(1+iM(x))^{-1}\right].$$

Using the well-known identity

$$\frac{d}{dx}\log\det A(x) = \mathrm{Tr}\left(\frac{dA(x)}{dx}\cdot A^{-1}(x)\right),$$

we obtain

$$2U^2(x) + i\frac{dU(x)}{dx} = \frac{d^2}{dx^2}\log\det(1+iM(x)).$$

Since $U(x)$ is real-valued and fast decaying, we have

$$U(x) = \frac{d}{dx}\mathrm{Im}\,\log\det(1+iM(x)) = \frac{d}{dx}\arctan\frac{\mathrm{Im}\,\det(1+iM(x))}{\mathrm{Re}\,\det(1+iM(x))}.$$

For reflectionless potentials $\varphi_1^+(x,k)$ is simply written. Put

$$W(x,k) = \int_x^{+\infty} T(x')e^{ikx'}\,dx' = \left(\frac{i}{\kappa_1+k}e^{i(\kappa_1+k)x}, \dots, \frac{i}{\kappa_N+k}e^{i(\kappa_N+k)x}\right).$$

and, by (47), obtain

$$\varphi_1^+(x,k) = \begin{pmatrix} \langle\Psi(x)\cdot(1+M^2(x))^{-1}|W(x,k)\rangle \\ e^{ikx} - \langle\Psi(x)\cdot(1+M^2(x))^{-1}M(x)|W(x,k)\rangle \end{pmatrix}. \tag{31}$$

4.2. Construction and properties of reflectionless spheres.

We consider some explicit examples of reflectionless spheres which are spheres with N-soliton potentials.

4.2.1. $N = 1$, $\kappa_1 = \frac{i}{2}$**.**

In this case

$$M(x) = -\lambda e^{-x}$$

where $\lambda = \lambda_1 \in \mathbf{R}\setminus\{0\}$ and we obtain

$$U(x,\lambda) = \frac{\lambda e^{-x}}{1+\lambda^2 e^{-2x}}.$$

Since for an immersed surface $U(x)$ is defined up to a sign, we assume that $\lambda = e^{-a} > 0$ and derive $U(x,\lambda) = U_1(x+a)$ with

$$U_1(x) = \frac{1}{2\cosh x}.$$

This is the potential of a round sphere and it is easily checked that $\varphi_1^+(x,i/2)e^{iy/2}$ defines it via (4).

4.2.2. $N = 2$, $\kappa_1 = \frac{i}{2}$, $\kappa_2 = \frac{3i}{2}$**.**

A general potential corresponding to this data is

$$U(x,\lambda_1,\lambda_2) = \frac{144\lambda_1 e^{-x} + 144\lambda_2 e^{-3x} + 36\lambda_1^2\lambda_2 e^{-5x} + 4\lambda_1\lambda_2^2 e^{-7x}}{144 + 144\lambda_1^2 e^{-2x} + 72\lambda_1\lambda_2 e^{-4x} + 16\lambda_2^2 e^{-6x} + \lambda_1^2\lambda_2^2 e^{-8x}}$$

which for $\lambda_1 = 2, \lambda_2 = 6$ takes the form

$$U_2(x) = \frac{1}{\cosh x}.$$

4.2.3. The potentials of the Dirac spheres.

U. Pinkall and J. Richter had constructed the Dirac spheres without using the the representation theory ([16]). We mentioned above that for these spheres the estimate (27) is precise. Their potentials are

$$U_N(x) = \frac{N}{2\cosh x}.$$

We show above how $U_2(x)$ is obtained via the inverse scattering method and it is clear from the trace formula (21) and Theorem 7 that

Proposition 2 *The discrete spectrum of $U_N(x)$ consists of $\pm\frac{(2j-1)i}{2}$ with $j \leq N$.*

For giving a complete their description of these potentials it needs to find the coefficients $\lambda_1, \dots, \lambda_N$. But let us recall that there exists an infinite family of soliton equations, *the modified Korteweg–de Vries hierarchy* of nonlinear equations, such that

1) the m-th mKdV equation has the form

$$\frac{\partial}{\partial t_m} U = \frac{\partial^{2m-1}}{\partial x^{2m-1}} U + \dots,$$

preserves the spectrum of L, and transforms $R(k)$ and λ_j as follows

$$\lambda_j \to \lambda_j \cdot \exp(i2^{2m-1}\kappa_j t_m), \quad R(k) \to R(k) \cdot \exp(i2^{2m-1}kt_m);$$

2) all flows generated by these equation pairwise commute.

Stationary solutions to linear combinations of these flows satisfy to *the Novikov equations* ([14]):

$$\left(a_1 \frac{\partial}{\partial t_1} + \dots + a_m \frac{\partial}{\partial t_m}\right) U(x, t_1, \dots) = 0.$$

These facts are exposed in [1, 6, 12, 21]. It follows from these formulas that reflectionless potentials are exactly fast decaying solutions to the Novikov equations (for the mKdV hierarchy) and, given $\kappa_1, \dots, \kappa_N$, the mKdV-orbit of a reflectionless potential consists of solutions to these equations with given $a_1, \dots, a_m$.

Proposition 3 *Every reflectionless potential with $\kappa_j = \frac{(2j-1)i}{2}$, where $1 \leq j \leq N$, is obtained from $U_N(x)$ by the mKdV-deformations.*

4.2.4. The Dirac spheres as rational spheres.

A nice property of the Dirac spheres is that they are described in terms of rational functions. Indeed, return back to a plane representation of spheres. It means that we represent S^2 as a complex plane $\mathbf{C}$ completed by a point at infinity. A conformal parameter Z on $\mathbf{C}$ is related with x and y, coming in (23) and (31), as $Z = e^{x+iy}$ and, by (13), a spinor field $\Psi(Z, \bar{Z})$ defining via (4) a plane representation of a sphere is

$$\Psi_1(Z, \bar{Z}) = \frac{1}{e^{(x+iy)/2}} \psi_1(x, y), \quad \Psi_2(Z, \bar{Z}) = \frac{1}{e^{(x-iy)/2}} \psi_2(x, y).$$

Now it is easy to see that Ψ is a rational function of Z and $\bar{Z}$.

4.2.5. Soliton deformations of reflectionless spheres.

Each mKdV-flow $U(x,t)$ deforms the eigenfunctions $\varphi_1^+(x,k)$ via quite simple differential equations which are linear in φ_1^+. This gives a deformation of a sphere defined by any linear combination (23). These deformations are called the mKdV-deformations of spheres and they are reductions (for one-dimensional potentials) of more general the modified Novikov–Veselov deformations introduced in [11] as deformations of surfaces locally presented via (4).

Here we only mention some interesting facts:

1) If $N \geq 2$, then generically a function of the form (23) defines not a sphere of revolution and moreover a sphere admitting no S^1-isometries. Nevertheless, soliton deformations of this sphere are described by $1+1$-dimensional soliton equations;

2) The Kruskal integrals are first integrals for all mKdV-flows and for reflectionless spheres they are presented in terms of $\varkappa_1, \dots, \varkappa_N$ by the trace formulas (20);

3) The mKdV-deformations preserve closedness of spheres (for tori this has been proved in [18]).

Generically the values of these first integrals as well as the modified Novikov–Veselov deformations depend on a choice of a conformal parameter on a sphere and on a surface of revolution there exists a distinguished parameter (see [18]).

4.2.6. Reflectionless spheres with $\dim_{\mathbf{H}} \mathrm{Ker}\,\mathcal{D} = 1$ and with large values of the Willmore functional.

A construction of such spheres is simple: take $\varkappa_1 = i/2, \varkappa_2 = -a+it$, and $\varkappa_3 = -a+it$, and fix admissible λ_j. Then for each $t > 0$ take a sphere of revolution Σ_t given by $\varphi_1^+(x, i/2)$. By (21), we have

$$\mathcal{W}(\Sigma_t) = 4\pi + 8\pi t.$$

Therefore we conclude

Proposition 4 *For any $C > 0$ there exists a regular sphere Σ in $\mathbf{R}^3$ such that* $\dim_{\mathbf{H}}$ *$Ker\mathcal{D} = 1$, Σ is defined via (4) by a zero-eigenfunction of $\mathcal{D}$, and $\mathcal{W}(\Sigma) > C$.*

We do not check the regularity condition but it is easy using (31).

4.2.7. Deformations of reflectionless spheres via deformations of the spectrum.

We consider a simple example which relates to the previous one (see 4.2.6) and do that only for demonstrating such deformations.

Take a reflectionless potential $U(x,t$ whose spectral data are $\varkappa_1, \dots, \varkappa_{N+2}$ and $\lambda_1, \dots, \lambda_{N+2}$ with $\varkappa_{N+1} = -a+it, \varkappa_{N+2} = a+it$ and $\lambda_{N+1} = \alpha t, \lambda_{N+2} = \bar{\alpha} t$ with $a, t \in \mathbf{R}$. Take a linear combination (23), of eigenfunctions of L with eigenvalues of the form $(2m+1)i/2$. Since $\varphi_1^+(x, \varkappa_{N+1})$ and $\varphi_1^+(x, \varkappa_{N+2})$ do not come into this combination, then for each t this combination defines a sphere Σ_t immersed into $\mathbf{R}^3$.

Consider the limit $t \to 0$. Then it is easy to see that

$U(x,t) \to U(x,0)$ with the spectral data $\varkappa_1, \dots, \varkappa_N$ and $\lambda_1, \dots, \lambda_N$, and Σ_t tends to a sphere Σ_0 immersed into $\mathbf{R}^3$.

It is quite sure from the construction that the spectral data of L depends on $U(x)$ continuously but we do not know are any estimates for stability of the inverse scattering problem for Dirac operators obtained or not. Hence we may propose the following treating the Dirac spheres as "constrained" Willmore only on a physical rigor level.

Let Σ be a Dirac sphere corresponding to an N-soliton potential and assume that the eigenfunctions corresponding to all levels of discrete spectrum come into a linear combination (23) defining Σ. Consider small perturbations of the sphere preserving the class of spheres with one-dimensional potentials. Such perturbations reflect in small perturbations of the potential and therewith in small perturbations of the discrete spectrum. These perturbations may result in appearance of new eigenvalues and perturbations of $R(k)$ which transform $U(x)$ into non-reflectionless potential. But the discrete spectrum $\kappa_j = (2j-1)i/2$ has to be preserved because it is all coming in the representation of the sphere. Now it follows from (21) and (28) that all such perturbations have to increase the value of the Willmore functional. By the completeness argument, we conclude that also for all Dirac spheres and moreover for all spheres such that each eigenvalue of L takes the form $(2m+1)i/2$. Therefore we have

soliton spheres, such that each eigenvalue of L is of the form $(2m+1)i/2$, are critical points of the Willmore functional restricted onto the class of spheres with one-dimensional potentials.

§5. Final remarks

1) In the present paper we mostly use a cylindric representation which enables us to apply well-developed inverse spectral theory for one-dimensional Dirac operators.

In a plane representation we may treat the two-dimensional Zakharov–Shabat problem by the $\bar\partial$ method and the nonlocal Riemann problem. In this case we consider the spectral data related to one level of energy E:

$$\mathcal{D}\psi = E\psi,$$

and in [3] the reconstruction problem has been solved for positive E and assuming that the L_2-norm of U is sufficiently small. The last condition is required for a unique solvability of the integral equations which are analogs of (38). For geometric reasons we have to consider this problem for $E = 0$ and for potentials with sufficiently large L_2-norms, $||U||^2_{L_2} \geq \pi$ (the Gauss–Bonnet theorem, (28)), and the vanishing of E itself leads to appearance of logarithmic singularities of the Green–Faddeev functions coming into the kernels of these integral equations.

In a cylindric representation we may introduce analogs of (38) as follows.

Find $G(z,k)$ satisfying

$$\left[\begin{pmatrix} 0 & \partial \\ -\bar\partial & 0 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} 0 & ik \\ ik & 0 \end{pmatrix}\right] G(z,k) = \delta(z).$$

Look for them in the form

$$G(z,k) = \begin{pmatrix} 0 & -e^{-ikx}h_1(z) \\ e^{ikx}h_2(z) & 0 \end{pmatrix}$$

where $\bar\partial h_1(z) = \partial h_2(z) = \delta(z)$. For that consider the Fourier decompositions of h_1 and h_2:

$$h_j(z) = \int_{-\infty}^{+\infty} d\lambda \sum_{n\in\mathbf{Z}} h^j_{\lambda,n} e^{i\lambda x + iny},$$

and keeping in mind that

$$
\frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{e^{i\lambda x}}{i\lambda+n}d\lambda=\begin{cases} e^{-nx} & \text{for } n>0, x>0 \\ -e^{-nx} & \text{for } n<0, x<0 \\ 0 & \text{for } n<0, x>0 \text{ or } n>0, x<0 \end{cases},
$$

obtain

$$
h_1(z)=\frac{1}{\pi}\left(\sigma_1(x)+\theta(x)\frac{e^{-z}}{1-e^{-z}}+(\theta(x)-1)\frac{e^{z}}{1-e^{z}}\right),
$$

$$
h_2(z)=\frac{1}{\pi}\left(\sigma_2(x)+\theta(x)\frac{e^{-\bar z}}{1-e^{-\bar z}}+(\theta(x)-1)\frac{e^{\bar z}}{1-e^{\bar z}}\right).
$$

$$
\partial\sigma_j(x)=\delta(x).
$$

Now choosing σ_j as in Appendix B we construct analogs of (39) and (40):

$$
\Phi^-(z,k)=\begin{pmatrix} 0 & e^{-ikx} \\ e^{ikx} & 0 \end{pmatrix}-
$$

$$
\frac{1}{\pi}\int_{-\infty}^{x}dx'\int_0^{2\pi}dy'\begin{pmatrix} 0 & \frac{e^{-ik(x-x')}}{e^{z'-z}-1} \\ \frac{e^{ik(x-x')}}{1-e^{\bar z'-\bar z}} & 0 \end{pmatrix}\cdot U(z')\cdot\Phi^-(z',k)+
$$

$$
\frac{1}{\pi}\int_{x}^{+\infty}dx'\int_0^{2\pi}dy'\begin{pmatrix} 0 & \frac{e^{-ik(x-x')+(z-z')}}{e^{z-z'}-1} \\ \frac{e^{ik(x-x')+(\bar z-\bar z')}}{1-e^{\bar z-\bar z'}} & 0 \end{pmatrix}\cdot U(z')\cdot\Phi^-(z',k)
$$

and

$$
\Phi^+(z,k)=\begin{pmatrix} 0 & e^{-ikx} \\ e^{ikx} & 0 \end{pmatrix}+
$$

$$
\frac{1}{\pi}\int_{x}^{+\infty}dx'\int_0^{2\pi}dy'\begin{pmatrix} 0 & \frac{e^{-ik(x-x')}}{e^{z-z'}-1} \\ \frac{e^{ik(x-x')}}{1-e^{\bar z-\bar z'}} & 0 \end{pmatrix}\cdot U(z')\cdot\Phi^+(z',k)-
$$

$$
\frac{1}{\pi}\int_{-\infty}^{x}dx'\int_0^{2\pi}dy'\begin{pmatrix} 0 & \frac{e^{-ik(x-x')+(z'-z)}}{e^{z'-z}-1} \\ \frac{e^{ik(x-x')+(\bar z'-\bar z)}}{1-e^{\bar z'-\bar z}} & 0 \end{pmatrix}\cdot U(z')\cdot\Phi^+(z',k).
$$

The kernels of these integral equations have removable singularities at $z=0$. Indeed, for h_1 the singularity is of the form

$$
\frac{\theta(x)}{z}-\frac{\theta(x)-1}{z}=\frac{1}{z},
$$

i.e., a multiple of a fundamental solution to the Cauchy–Riemann equations. For U and Φ depending on x only, these equations after integrating by y reduce to (39) and (40).

Analogs of (41) for these equations are not quite good because for sufficiently large $|\mathrm{Im}\,k|$ their kernels are exponentially growing. Nevertheless $\mathrm{Ker}\,\mathcal{D}$ has to admit a description in terms of solutions to these equations because it is just another representation of the spectral problem (2) for an operator acting on a spinor bundle over a two-dimensional sphere.

2) The spinor field ψ on an immersed surface may be obtained also as the restriction of a parallel spinor field on $\mathbf{R}^3$ onto the surface. This recently has been

exposed in [8] and also had been pointed out to the author by Pinkall just after author's talk in Amherst (November, 1995) about the paper [17] [4].

Being geometrically invariant this treating of Lemma B does not enable us to extract spectral-theoretical properties of the Weierstrass representation which are of a global origin and to develop construction of surfaces from spectral-theoretical data as we do that in §3 and §4.

But this fits into the more general consideration of Dirac operators on hypersurfaces in $\mathbf{R}^{n+1}$ with induced spin structures. Recently it has been shown by C. Bär ([2]) that for such hypersurface M there exists at least $2^{[n/2]}$ distinct eigenvalues λ of the induced operator $\mathcal{D}$ such that

$$\lambda^2 \leq \frac{n^2}{4\mathrm{Vol}\,(M)}\mathcal{W}(M).$$

3) This work has been done when the author was a guest of SFB 288 in Technische-Universität in Berlin and was also supported by INTAS-RFBR (the Russian Foundation for Basic Researches) (grant 95-0418).

The author thanks U. Pinkall for helpful discussions.

Appendix A. Period problem for immersions of surfaces of genus ≥ 2

Recall the definition of a global Weierstrass representation of a surface Σ of genus $q \geq 2$ which is conformally equivalent to $\Sigma_0 = \mathcal{H}/\Lambda$ with $\mathcal{H}$ the Lobachevskii upper half-plane and Λ a lattice in $PSL(2,\mathbf{R})$ ([17, 20]):

Definition 2 *A sphere Σ with $g\,(> 1)$ handles, immersed into $\mathbf{R}^3$, possesses a global Weierstrass representation if there exist a real potential U and functions ψ_1 and ψ_2, defined on the universal covering of Σ, i.e., on $\mathcal{H}$, such that*

1)

$$\left\{\begin{array}{l} U(\gamma(z)) = |cz+d|^2 U(z), \\ \psi_1(\gamma(z)) = (cz+d)\psi_1(z), \\ \psi_2(\gamma(z)) = (c\bar{z}+d)\psi_2(z) \end{array}\right. \tag{32}$$

for $z \in \mathcal{H}$ and $\gamma \in \Lambda$, represented by the matrix

$$\left(\begin{array}{cc} a & b \\ c & d \end{array}\right), \quad a,b,c,d \in \mathbf{R}, \quad ad-bc=1;$$

2) the vector function ψ satisfies (2) and for a suitable choice of coordinates in $\mathbf{R}^3$ defines by (4) an immersion of Σ.

We have

Theorem 8 *Let Σ_0 be a compact oriented surfaces of genus $g \geq 2$ and let U and ψ satisfy (32) and define via (4) an immersion of the universal covering of Σ_0 into $\mathbf{R}^3$. Then this immersion converts into an immersion of Σ_0 if and only if*

$$\int_{\Sigma_0} \bar{\psi}_1^2\, d\bar{z} \wedge \omega = \int_{\Sigma_0} \psi_2^2\, d\bar{z} \wedge \omega = \int_{\Sigma_0} \bar{\psi}_1 \psi_2\, d\bar{z} \wedge \omega = 0 \tag{33}$$

for any holomorphic differential ω on Σ_0.

[4] Notice that in [17] spinors appear as a result of globalizing Weierstrass representations.

A proof of the theorem.

Take a canonical basis $\alpha_1, \dots, \alpha_g, \beta_1, \dots, \beta_g$ for $H_1(\Sigma_0)$. This means that its intersection form is

$$\alpha_j \circ \beta_k = \delta_{jk}, \quad \alpha_j \circ \alpha_k = \beta_j \circ \beta_k = 0$$

and there exists loops representing these cycles such that after cutting along them we obtain a domain M bounded by a polygon

$$\partial M = \alpha_1 \beta_1 \alpha_1^{-1} \beta_1^{-1} \dots \alpha_g \beta_g \alpha_g^{-1} \beta_g^{-1}.$$

To this basis corresponds a unique basis of holomorphic differentials $\omega_1, \dots, \omega_g$ normalized by the condition

$$\int_{\alpha_k} \omega_j = \delta_{jk}.$$

Define the period matrix Ω by

$$\Omega_{jk} = \int_{\beta_k} \omega_j.$$

This matrix is symmetric and its imaginary part is positive definite ([7]).

It follows from (4) and (32) that $\bar{\psi}_1^2 d\bar{z}, \psi_2^2 d\bar{z}$, and $\bar{\psi}_1 \psi_2 d\bar{z}$ are correctly defined 1-forms on Σ_0 and

$$\bar{\psi}_1^2 = \bar{\partial}(iX^1 - X^2), \quad \psi_2^2 = \bar{\partial}(iX^1 + X^2), \quad \bar{\psi}_1 \psi_2 = \bar{\partial} X^3. \tag{34}$$

Introduce the following vectors of translation periods

$$V_j = \int_{\alpha_j} (\bar{\psi}_1^2 \, d\bar{z} - \bar{\psi}_2^2 dz), \quad V_{g+j} = \int_{\beta_j} (\bar{\psi}_1^2 \, d\bar{z} - \bar{\psi}_2^2 dz)$$

and

$$W_j = \int_{\alpha_j} (\psi_1 \bar{\psi}_2 \, dz + \bar{\psi}_1 \psi_2 \, d\bar{z}), \quad W_{g+j} = \int_{\beta_j} (\psi_1 \bar{\psi}_2 \, dz + \bar{\psi}_1 \psi_2 \, d\bar{z}).$$

Denote by Y and Z the vectors

$$Y_j = \int_{\Sigma_0} \bar{\psi}_1^2 \, d\bar{z} \wedge \omega_j, \quad Y_{j+g} = -\overline{\int_{\Sigma_0} \psi_2^2 \, d\bar{z} \wedge \omega_j}$$

and

$$Z_j = \int_{\Sigma_0} \bar{\psi}_1 \psi_2 \, d\bar{z} \wedge \omega_j, \quad Z_{j+g} = \overline{\int_{\Sigma_0} \bar{\psi}_1 \psi_2 \, d\bar{z} \wedge \omega_j}.$$

It is evident that an immersion of $\mathcal{H}$ is converted into an immersion of Σ_0 if and only if $V = W = 0$.

Now, by the Stokes theorem and (34), we have

$$\int_{\Sigma_0} \bar{\psi}_1^2 \, d\bar{z} \wedge \omega_j = \int_{\partial M} (iX^1 - X^2) \omega_j = \sum_{k=1}^{g} \left(V_k \int_{\beta_k} \omega_j - V_{k+g} \int_{\alpha_k} \omega_j \right),$$

$$\int_{\Sigma_0} \psi_2^2 \, d\bar{z} \wedge \omega_j = \int_{\partial M} (iX^1 + X^2) \omega_j = \sum_{k=1}^{g} \left(-\bar{V}_k \int_{\beta_k} \omega_j + \bar{V}_{k+g} \int_{\alpha_k} \omega_j \right) \tag{35}$$

$$\int_{\Sigma_0} \bar{\psi}_1 \psi_2 \, d\bar{z} \wedge \omega_j = \int_{\partial M} X^3 \omega_j = \sum_{k=1}^{g} \left(W_k \int_{\beta_k} \omega_j - W_{k+g} \int_{\alpha_k} \omega_j \right).$$

Consider the $2g \times 2g$-matrix

$$\tilde{\Omega} = \begin{pmatrix} \Omega & -1 \\ \bar{\Omega} & -1 \end{pmatrix}.$$

Since $\mathrm{Im}\,\Omega$ is positive definite, $\tilde{\Omega}$ is nondegenerate. Rewrite (35) as follows

$$\tilde{\Omega} V = Y, \quad \tilde{\Omega} W = Z,$$

and conclude that $V = W = 0$ if and only if

$$Y = Z = 0. \tag{36}$$

Since ω_j form a basis for holomorphic differentials, (36) is equivalent to vanishing integrals (33) for any holomorphic differential on Σ_0.

This proves the theorem.

Proposition 4 from [20] which settles the period problem for tori may be reformulated as (33).

Appendix B. The inverse scattering problem for the one-dimensional Dirac operator

B1. The forward scattering problem.

Consider the linear problem

$$L\varphi = \begin{pmatrix} 0 & ik \\ ik & 0 \end{pmatrix} \varphi \tag{37}$$

with

$$L = \begin{pmatrix} 0 & \partial_x \\ -\partial_x & 0 \end{pmatrix} + 2 \begin{pmatrix} U & 0 \\ 0 & U \end{pmatrix}$$

and k a spectral parameter. This is the simplest reduction of the Zakharov–Shabat linear problem corresponding to the case when both potentials are equal to $U(x)$ and are real-valued.

For each $k \in \mathbf{R} \setminus \{0\}$ the system (37) has a two-dimensional space of solutions. Take a matrix $\Phi(x, k)$ whose columns form a basis for this space. For $U = 0$ we take

$$\Phi_0(x, k) = \begin{pmatrix} 0 & e^{-ikx} \\ e^{ikx} & 0 \end{pmatrix}$$

For fast decaying U the matrix Φ may be defined by the integral equation

$$\Phi(x, k) = \Phi_0(x, k) - \int_{-\infty}^{+\infty} G(x - x', k) \cdot 2U(x') \cdot \Phi(x', k) \, dx' \tag{38}$$

where

$$\left[\begin{pmatrix} 0 & \partial_x \\ -\partial_x & 0 \end{pmatrix} - \begin{pmatrix} 0 & ik \\ ik & 0 \end{pmatrix} \right] G(x, k) = \delta(x),$$

i.e., $G(x,k)$ is a fundamental solution to (37) with $U=0$ and it may be taken in the form

$$G(x,k) = \begin{pmatrix} 0 & -e^{-ikx}g_1(x) \\ e^{ikx}g_2(x) & 0 \end{pmatrix}$$

with

$$\partial_x g_j(x) = \delta(x).$$

The last equality is satisfied exactly by $\theta(x)+\text{const}$ with $\theta(x)$ the Heaviside function

$$\theta(x) = \begin{cases} 1 & \text{for } x \geq 0 \\ 0 & \text{for } x < 0 \end{cases}.$$

This freedom to choose g_j enables us to define solutions to (37) converging to the free waves, given by the columns of $\Phi_0(x,k)$, as $x \to -\infty$ or $x \to +\infty$. Indeed, for $g_1(x) = g_2(x) = \theta(x)$ the system (38) takes the form

$$\Phi^-(x,k) = \begin{pmatrix} 0 & e^{-ikx} \\ e^{ikx} & 0 \end{pmatrix} - \tag{39}$$

$$\int_{-\infty}^{x} \begin{pmatrix} 0 & -e^{-ik(x-x')} \\ e^{ik(x-x')} & 0 \end{pmatrix} \cdot 2U(x') \cdot \Phi^-(x',k)\,dx'$$

and for $g_1(x) = g_2(x) = \theta(x) - 1$ it is

$$\Phi^+(x,k) = \begin{pmatrix} 0 & e^{-ikx} \\ e^{ikx} & 0 \end{pmatrix} + \tag{40}$$

$$\int_{x}^{+\infty} \begin{pmatrix} 0 & -e^{-ik(x-x')} \\ e^{ik(x-x')} & 0 \end{pmatrix} \cdot 2U(x') \cdot \Phi^+(x',k)\,dx'.$$

The solutions to (39) and (40) are called the Jost functions.

1) *For each $k \in \mathbf{R} \setminus \{0\}$ the equations (39) and (40) have unique solutions.*

These equations have the form

$$\Phi^{\pm}(x,k) = \Phi_0(x,k) + (A^{\pm} \circ \Phi^{\pm})(x,k)$$

where operators $A^{\pm}$ are of the Volterra type. Solutions to them are given by the Neumann series

$$\Phi^{\pm}(x,k) = \sum_{j=0}^{\infty} \Phi_j^{\pm}(x,k),$$

with $\Phi_j^{\pm}(x,k) = (A^{\pm} \circ \Phi_{j-1}^{\pm})(x,k)$, which uniformly converge in x on each compact interval.

Denote the j-th column of $\Phi^{\pm}(x,k)$ by $\varphi_j^{\pm}$. Each of the pairs $(\varphi_1^-, \varphi_2^-)$ and $(\varphi_1^+, \varphi_2^+)$ forms a basis for solutions to (37).

2) *The functions $\varphi_1^-(x,k)e^{-ikx}$ and $\varphi_2^+(x,k)e^{ikx}$ are analytically continued onto the lower-half plane $\operatorname{Im} k < 0$ and the functions $\varphi_2^-(x,k)e^{ikx}$ and $\varphi_1^+(x,k)e^{-ikx}$ are analytically continued onto the upper-half plane $\operatorname{Im} k > 0$.*

Without loss of generality we explain this fact for $f(x,k) = \varphi_1^-(x,k)e^{-ikx}$. This function satisfies the integral equation

$$f(x,k) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} - \int_{-\infty}^{x} \begin{pmatrix} 0 & -e^{-2ik(x-x')} \\ 1 & 0 \end{pmatrix} 2U(x') f(x',k)\,dx' \tag{41}$$

For $\operatorname{Im} k < 0$ its kernel decays exponentially as $x \to \infty$ and the Neumann series for this equation converge.

3) *Given vector functions* $\theta(x) = (\theta_1(x), \theta_2(x))$ *and* $\tau(x) = (\tau_1(x), \tau_2(x))$ *satisfying (37), the Wronskian* $W(\theta, \tau)(x) = \theta_1(x)\tau_2(x) - \theta_2(x)\tau_1(x)$ *is constant, i.e., independent of* x.

This Wronskian identity is obtained by straightforward computations and implies that

$$\det \Phi^{\pm}(x,k) = -1.$$

4) *For* $k \in \mathbf{R} \setminus \{0\}$ *the matrices* $\Phi^+(x,k)$ *and* $\Phi^-(x,k)$ *are related as*

$$\Phi^-(x,k) = S(k)\Phi^+(x,k)$$

with the scattering matrix $S(k)$ *independent of* x *with*

$$\det S(k) = 1.$$

Indeed, the columns of $\Phi^+(x,k)$ and $\Phi^-(x,k)$ form different bases for solutions to (37) and, therefore, are linearly dependent:

$$\varphi_1^-(x,k) = s_{11}(k)\varphi_1^+(x,k) + s_{12}(k)\varphi_2^+(x,k),$$

$$\varphi_2^-(x,k) = s_{21}(k)\varphi_1^+(x,k) + s_{22}(k)\varphi_2^+(x,k).$$

Since $\det \Phi^{\pm}(x,k) = -1$, we have $\det S(k) = 1$.

Denote $s_{22}(k)$ by $a(k)$ and $s_{21}(k)$ by $b(k)$.

Notice that, if $\theta(x) = (\theta_1(x), \theta_2(x))$ satisfies (37), then the function $\tilde{\theta}(x) = (\theta_2(x), -\theta_1(x))$ satisfies (37) with $-k$ substituted for k. This implies that

5) *For* $k \in \mathbf{R} \setminus \{0\}$,

$$\Phi^{\pm}(x,k) = J \cdot \Phi^{\pm}(x,-k) \cdot J \quad \text{with} \quad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{42}$$

Since $U(x)$ is real-valued, it is also clear that

$$\Phi^{\pm}(x,k) = \overline{\Phi^{\pm}(x,-k)} \quad \text{for } k \in \mathbf{R} \setminus \{0\}. \tag{43}$$

6) *The scattering matrix takes the form*

$$S(k) = \begin{pmatrix} \overline{a(k)} & -\overline{b(k)} \\ b(k) & a(k) \end{pmatrix} \qquad \text{with } |a(k)|^2 + |b(k)|^2 = 1.$$

Notice that (42) implies $\Phi^-(x,-k) = (-JS(k)J)\,\Phi^+(x,-k)$ and (43) implies $\Phi^-(x,-k) = \overline{S(k)}\Phi^+(x,-k)$ It follows from these equalities that $\overline{S(k)} = -JS(k)J$ which proves 6). The following quantities

$$T(k) = \frac{1}{a(k)}, \quad R(k) = \frac{b(k)}{a(k)}$$

are called *the transmission coefficient* and *the reflection coefficient* respectively. It is shown that $a(k)$ vanishes nowhere on $\mathbf{R} \setminus \{0\}$.

7) $T(k)$ *is analytically continued onto the upper-half plane* $\operatorname{Im} k \geq 0$.

Indeed, this follows from 2) and

$$a(k) = W(\varphi_1^+(x,k), \varphi_2^-(x,k)) = W(e^{-ikx}\varphi_1^+(x,k), e^{ikx}\varphi_2^-(x,k)). \tag{44}$$

The poles of $T(k)$ correspond to *bounded states*, i.e., to solutions to (37) which decay exponentially as $x \to \pm\infty$. These solutions are $\varphi_1^+(x,\kappa)$ or $\varphi_2^-(x,\kappa)$ where κ is a pole of $T(k)$ and, since $a(\kappa) = 0$, these functions are linearly dependent

$$\varphi_2^-(x,\kappa) = \mu(\kappa)\varphi_1^+(x,\kappa), \quad \mu(\kappa) \in \mathbf{C}. \tag{45}$$

Some computations lead to the conclusion which we only recall:

8) *$T(k)$ has only simple poles in $\mathrm{Im}\, k > 0$ and for fast decaying, for instance, for exponentially decaying, potentials there are finitely many poles of $T(k)$.*

The transform $* : (\xi_1, \xi_2) \to (\bar{\xi}_2, -\bar{\xi}_1)$ maps $\varphi_1^\pm(x,k)$ into a multiple of $\varphi_2^\pm(x,\bar{k})$. Therefore either $\varphi_1^+(x,\kappa)$ and $\varphi_2^+(x,\bar{\kappa})$ both decay exponentially as $x \to \pm\infty$ or neither of them do. This implies that

9) *The discrete spectrum of L is preserved by the complex conjugation $\kappa \to \bar{\kappa}$.*

The following quantities form *the spectral data* of L.

Spectral data:

1) the reflection coefficient $R(k)$;

2) the poles of $T(k)$: $\kappa_1, \dots, \kappa_N$;

3) the products $\lambda_j = i\gamma_j\mu_j$, where $\gamma_j = \gamma(\kappa_j)$ are the residues of $T(k)$ at κ_j and $\mu_j = \mu(\kappa_j)$ relate φ_1^+ and φ_2^- at $k = \kappa_j$ (45).

Notice, that L has a continuous spectrum $k \in \mathbf{R} \setminus \{0\}$ of multiplicity two and a discrete spectrum $\kappa_1, \dots, \kappa_N, \bar{\kappa}_1, \dots, \bar{\kappa}_N$ of multiplicity one.

Since $U(x)$ is real-valued, this reflects in "reality conditions" met by the spectral data.

First, notice that

$$\varphi_j^\pm(x,-k) = \overline{\varphi_j^\pm(x,k)} \quad \text{for } k \in \mathbf{R} \setminus \{0\}. \tag{46}$$

It follows from (44) and (46) that $a(k) = \overline{a(-k)}$ for $k \in \mathbf{R} \setminus \{0\}$. Consider now the meromorphic function $F(k) = T(k) - \overline{T(-\bar{k})}$ defined on $\{\mathrm{Im}\, k \geq 0\}$. We see that it vanishes everywhere on the boundary of $\{\mathrm{Im}\, k > 0\}$ and, therefore, $F(k) \equiv 0$. This implies that

10) The poles of $T(k)$ are symmetric with respect to the imaginary axis , i.e., if $(k_R + ik_I)$ is a pole of $T(k)$, where $k_R, k_I \in \mathbf{R}$, then $(-k_R + ik_I)$ is also the pole of $T(k)$. Moreover the residues of $T(k)$ are related as $\gamma(k_R + ik_I) = -\overline{\gamma(-k_R + ik_I)}$ and if $\Re\kappa_j = 0$ then $\mathrm{Re}\,\gamma(\kappa_j) = 0$.

Consider the analytic continuation of the functions $G_1(x,k) = \varphi_1^+(x,k) - \overline{\varphi_1^+(x,-\bar{k})}$ and $G_2(x,k) = \varphi_2^-(x,k) - \overline{\varphi_2^+(x,-\bar{k})}$ onto the same upper-half plane. Since, by (46), they identically vanish on the boundary, we conclude that $G_1(x,k) \equiv G_2(x,k) \equiv 0$. This implies that

11) The coefficients $\mu_(\kappa)$ relating, via (45), the functions $\varphi_1^+(x,k)$ and $\varphi_2^-(x,k)$ at the poles of $T(k)$ are complex conjugate, i.e. $\mu(k_R + ik_I) = \overline{\mu(-k_R + ik_I)}$, and if $\mathrm{Re}\,\kappa_j = 0$ then $\mu(\kappa_j) \in \mathbf{R}$.

We summarize 10) and 11) in

R1) The poles of $T(k)$ are symmetric with respect to the imaginary axis, $\lambda_j = \bar{\lambda}_l$ for a pair of symmetric poles κ_j and κ_l, and if $\mathrm{Re}\,\kappa_m = 0$ then $\lambda_m \in \mathbf{R}$.

The definition of $T(k)$ and (46) also imply that

R2) $R(k) = \overline{R(-k)}$.

The conditions R1–R2 perfectly distinguish the spectral data of real potentials.

B2. The Marchenko equations.

Introduce the following representations for $\varphi_2^-(x,k)$ and $\varphi_1^+(x,k)$ [5] :

$$\varphi_2^-(x,k) = \begin{pmatrix} e^{-ikx} \\ 0 \end{pmatrix} + \int_{-\infty}^{x} dx' \begin{pmatrix} A_1(x,x') \\ A_2(x,x') \end{pmatrix} e^{-ikx'}$$

and

$$\varphi_1^+(x,k) = \begin{pmatrix} 0 \\ e^{ikx} \end{pmatrix} + \int_{x}^{+\infty} dx' \begin{pmatrix} B_1(x,x') \\ B_2(x,x') \end{pmatrix} e^{ikx'}. \tag{47}$$

It follows from (42) that

$$\varphi_2^+(x,k) = \begin{pmatrix} e^{-ikx} \\ 0 \end{pmatrix} + \int_{x}^{+\infty} dx' \begin{pmatrix} B_2(x,x') \\ -B_1(x,x') \end{pmatrix} e^{-ikx'}. \tag{48}$$

By the definition of $T(k)$ and $R(k)$, we have

$$T(k)\varphi_2^-(x,k) = R(k)\varphi_1^+(x,k) + \varphi_2^+(x,k). \tag{49}$$

Introduce the Fourier transforms of $R(k)$ and $(T(k)-1)$:

$$\Gamma(z) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} (T(k)-1)e^{-ikz}\,dk, \qquad r(z) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} R(k)e^{-ikz}\,dk,$$

and also apply the Fourier transformation to both sides of (49), obtaining a pair of integral equations corresponding to the coefficients of φ:

$$\Gamma(x+t) + \int_{-\infty}^{x} \Gamma(x'+t)A_1(x,x')\,dx' + \theta(x+t)A_1(x,-t) = \tag{50}$$

$$\theta(-t-x)B_2(x,-t) + \int_{x}^{+\infty} r(x'-t)B_1(x,x')\,dx'.$$

and

$$\int_{-\infty}^{x} \Gamma(x'+t)A_2(x,x')\,dx' + \theta(x+t)A_2(x,-t) = \tag{51}$$

$$r(x-t) - \theta(-t-x)B_1(x,-t) + \int_{x}^{+\infty} r(x'-t)B_2(x,x')\,dx'.$$

Since $e^{-ikx}\varphi_1^+(x,k)$ and $e^{ikx}\varphi_2^-(x,k)$ converge very fast to $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, respectively, as $k \to \infty$, Im $k \geq 0$, we conclude from (44) that $(T(k)-1)$ is fast decaying and for $z < 0$ we have

$$\Gamma(z) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} (T(k)-1)e^{-ikz}\,dk = \sum_j i\gamma_j e^{-i\kappa_j z}$$

where γ_j is the residue of $T(k)$ at κ_j. If there are no poles of $T(k)$, then $\Gamma(z) = 0$ for $z < 0$ and for $x + t < 0$ the left-hand sides of (50) and (51) vanish. Otherwise for $x + t < 0$ the Fourier transform of the left-hand side of (49) is

$$\sum_{j=1}^{N} i\gamma_j e^{-i\kappa_j t}\varphi_2^-(x,\kappa_j).$$

[5]The existence of these representations is established via the Goursat equations which we discuss in B3.

Since $a(\kappa_j) = 0$, we have (see (45))

$$\varphi_2^-(x,\kappa_j) = \mu_j \varphi_1^+(x,\kappa_j), \quad \mu_j \in \mathbf{C},$$

and substituting (47) into (50) and (51) we obtain

$$\int_x^{+\infty} \sum_{j=1}^N i\gamma_j e^{-i\kappa_j t} \mu_j B_1(x,x') e^{i\kappa_j x'}\, dx' =$$

$$\theta(-t-x) B_2(x,-t) + \int_x^{+\infty} r(x'-t) B_1(x,x')\, dx'$$

and

$$\sum_{j=1}^N i\gamma_j e^{-i\kappa_j t} \mu_j e^{i\kappa_j x} + \int_x^{+\infty} \sum_{j=1}^N i\gamma_j e^{-\kappa_j t} \mu_j B_2(x,x') e^{i\kappa_j x'}\, dx' =$$

$$r(x-t) - \theta(-t-x) B_1(x,-t) + \int_x^{+\infty} r(x'-t) B_2(x,x')\, dx'.$$

where $x + t < 0$. Introducing the function

$$\Omega(z) = r(z) - \sum_{j=1}^N i\gamma_j \mu_j e^{i\kappa_j z} = r(z) - \sum_{j=1}^N \lambda_j e^{i\kappa_j z}$$

and substituting $-y$ for t, rewrite these equations as follows

$$\begin{cases} B_2(x,y) + \int_x^{+\infty} B_1(x,x')\Omega(x'+y)\, dx' = 0, \\ \Omega(x+y) - B_1(x,y) + \int_x^{+\infty} B_2(x,x')\Omega(x'+y)\, dx' = 0 \end{cases} \tag{52}$$

where $y > x$. These equations are called *the Marchenko equations* (for the Zakharov–Shabat linear problem).

Notice that $\Omega(x)$ is uniquely reconstructed from the spectral data of L and if the reality conditions R1–R2 hold then $\Omega(x)$ is real-valued for $x \in \mathbf{R}$.

B3. The inverse scattering problem.

Assume that $\Omega(z)$ is constructed from the spectral data of L and the equations (52) are solved, i.e., the functions $B_1(x,y)$ and $B_2(x,y)$ are known for $y > x$. In fact, these equations are of the Volterra type and uniquely solvable. Moreover

$$\lim_{y\to+\infty} B_1(x,y) = \lim_{y\to+\infty} B_2(x,y) = 0, \tag{53}$$

the limits of $B_j(x,y)$ as $y \to x$ are defined and we denote them by $B_j(x,x)$.

Substituting (48) into (37), we obtain a pair of equations corresponding to the rows of a matrix equation. The first of them is

$$-\int_x^{+\infty} \frac{B_1(x,y)}{\partial x} e^{-iky}\, dy + e^{-ikx} B_1(x,x) + ik \int_x^{+\infty} B_1(x,y) e^{-iky}\, dy +$$

$$2U(x) e^{-ikx} + 2U(x) \int_x^{+\infty} B_2(x,y) e^{-iky}\, dy = 0.$$

Integrating by parts

$$ik \int_x^{+\infty} B_1(x,y) e^{-iky}\, dy =$$

$$\int_x^{+\infty} \frac{\partial B_1(x,y)}{\partial y} e^{-iky}\, dy - \lim_{y\to+\infty} (B_1(x,y)e^{-iky}) + B_1(x,x)e^{-ikx},$$

we finally derive

$$\int_x^{+\infty} \left(\frac{\partial B_1(x,y)}{\partial y} - \frac{\partial B_1(x,y)}{\partial x} + 2U(x)B_2(x,y) \right) e^{-iky}\, dy +$$

$$2(U(x) + B_1(x,x))e^{-ikx} = 0$$

which implies

$$\frac{\partial B_1(x,y)}{\partial y} - \frac{\partial B_1(x,y)}{\partial x} + 2U(x)B_2(x,y) = 0 \tag{54}$$

and

$$U(x) = -B_1(x,x). \tag{55}$$

Analogously we infer that the second equation is equivalent to

$$\frac{\partial B_2(x,y)}{\partial x} + \frac{\partial B_2(x,y)}{\partial y} = -2U(x)B_1(x,y). \tag{56}$$

Substituting x for y in (56) and taking (55) in account, we derive

$$\frac{dB_2(x,x)}{dx} = 2U^2(x). \tag{57}$$

In fact, the integral representation (48) is initially derived from the Goursat equations (54) and (56) with the boundary conditions (53) and (55).

The formula (55) gives a solution to the inverse scattering problem: reconstructing the potential from the spectral data. If the conditions R1–R2 hold then $\Omega(x)$ is real-valued for $x \in R$ and the solution to (52) is also real-valued for $x, y \in \mathbf{R}$.

To complete this scheme it needs to prove that starting from the spectral data we obtain via (52) and (55) a potential $U(x)$ with the same spectral data. The main tool in proving that is the fact that (52) is just the Fourier transform of (49). A detailed analysis would enable us to distinguish the decay of a potential in terms of the data. For the one-dimensional Schrödinger operator on the line, i.e., the most similar problem to (37), this had been done in [5] (see, also [13]) where one can find a detailed study of this problem.

References

[1] Ablowitz M. J. and Segur H. Solitons and the inverse scattering method, SIAM, Philadelhia, 1981.

[2] Bär C. Extrinsic bounds for eigenvalues of the Dirac operator, Preprint, Freiburg Universität, 1997.

[3] Bogdanov L. V. On the two-dimensional Zakharov–Shabat problem, Theor. Math. Phys. **72** (1987), 790–793.

[4] Dirac P. A. M. The principles of quantum mechanics, 4th edition, Oxford University Press, 1958.

[5] Faddeev L. D. Properties of the S-matrix of the one-dimensional Schroedinger operator, Amer. Math. Soc., Transl., II Ser. **65**, 1967, pp. 139–166.

[6] Faddeev L.D. and Takhtadzhyan L. A. Hamiltonian methods in the theory of solitons, Springer, Berlin, 1987.

[7] Fay J. Theta functions on Riemann surfaces, Lecture Notes in Math., v. 352, Springer, 1973.

[8] Friedrich T. On the spinor representation of surfaces in Euclidean 3-space, Preprint no. 295, SFB 288, Technische Universität Berlin, 1997.

[9] Eisenhart L. P. A treatise on the differential geometry of curves and surfaces, Dover. Publ., New York, 1909.

[10] Kamberov G., Pedit F., and Pinkall U. Bonnet pairs and isothermic surfaces, to appear in Duke Math. J.

[11] Konopelchenko B. G. Induced surfaces and their integrable dynamics, Stud. Appl. Math., **96**, 9–52 (1996).

[12] Lamb Jr. G. L. Elements of soliton theory, John Wiley & Sons, New York, 1980.

[13] Marchenko V. A. Sturm–Liouville operators and their applications. (Russian). Naukova Dumka, Kiev, 1977.

[14] Novikov S. P. The periodic problem for the Korteweg–De Vries equation, Funct. Anal. Appl. **8**, 236–246 (1974).

[15] Pedit F. and Pinkall U., private communication.

[16] Richter J. Conformal maps of a Riemann surface into the space of quaternions, Ph. D. Thesis, Technische Universität, Berlin, 1997.

[17] Taimanov I. A. Modified Novikov–Veselov equation and differential geometry of surfaces, in: Translations of the Amer. Math. Soc., Ser. 2, V. 179, 1997, pp. 133–151.

[18] Taimanov I. A. Surfaces of revolution in terms of solitons, Annals of Global Analysis and Geometry **15**, 419–435 (1997).

[19] Taimanov I. A. Global Weierstrass representation and its spectrum, (in Russian) Uspekhi Mat. Nauk **52**:6, 187–188 (1997); translation to appear in Russian Math. Surveys, **52**:6 (1997).

[20] Taimanov I. A. The Weierstrass representation of closed surfaces in $\mathbf{R}^3$, Preprint no. 291, SFB 288, Technische Universität Berlin, 1997.

[21] Theory of solitons, Ed. S. P. Novikov. Nauka, Moscow, 1980.