Cartan Spinor Bundles on Manifolds. *

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1 Introduction.

Spinor fields and Dirac operators on Riemannian manifolds M^n can be introduced by means of a reduction of the frame bundle from the structure group $O(n)$ to one of the groups $Spin(n)$, $Pin(n)$ or $Spin^{\mathbb{C}}(n)$. In any case the existence of a corresponding reduction imposes topological restrictions on the manifold M^n. The aim of the present paper is the construction of spinor bundles of Cartan type over certain non-orientable manifolds. The bundles under consideration do not split into subbundles invariant under the action of the Clifford algebra and therefore they are not induced from $Pin^{\mathbb{C}}$ structure of the manifold. Moreover, we study the case of the real projective space $\mathbb{RP}^n$ and its spinor bundle in more detail. These bundles over $\mathbb{RP}^n$ admit a suitable metric connection ∇^S and the corresponding Dirac operator. In particular it turns out that there are Killing spinors in the constructed twisted spinor bundle over $\mathbb{RP}^n$ for all dimensions.

Let us fix some notations. In case $n \equiv 0 \, mod \, 2$ we denote by $\gamma : \mathrm{Cliff}^{\mathbb{C}}(n) \xrightarrow{\sim} \mathrm{End}(\Delta_n)$ the isomorphism between the Clifford algebra and the algebra of endomorphisms of the space Δ_n of all Dirac spinor. In this way we obtain the so called *Dirac representation* γ of the Clifford algebra. In case $n \equiv 1 \, mod \, 2$ we have the *Pauli representation* $\gamma : \mathrm{Cliff}^{\mathbb{C}}(n) \to \mathrm{End}(\Delta_n)$. Denote by $\alpha : \mathrm{Cliff}^{\mathbb{C}}(n) \to \mathrm{Cliff}^{\mathbb{C}}(n)$ the canonical involution. Then

$$\gamma \oplus (\gamma \circ \alpha) : \mathrm{Cliff}^{\mathbb{C}}(n) \xrightarrow{\sim} \mathrm{End}(\Delta_n) \oplus \mathrm{End}(\Delta_n)$$

is an isomorphism and the corresponding representation of the Clifford algebra in $\Delta_n \oplus \Delta_n$ is the so called *Cartan representation* (see [Fr2], [Tr]).

2 The construction of spinor bundles of Cartan type.

We consider *odd-dimensional* Riemannian manifolds M^{2k+1} only. A spinor bundle of Cartan type is a 2^{k+1}-dimensional complex Clifford bundle S realizing at any point the Cartan representation. $\gamma \oplus (\gamma \circ \alpha) : \mathrm{Cliff}^{\mathbb{C}}(\mathbb{R}^{2k+1}) \xrightarrow{\sim} \mathrm{End}(\Delta_{2k+1} \oplus \Delta_{2k+1})$ of the Clifford algebra (see [Tr]). Any spinor bundle of Cartan type over an orientable manifold splits into two Clifford subbundles. Indeed, the volume form $dM^{2k+1} = e_1 \cdot \ldots \cdot e_{2k+1} \in \mathrm{Cliff}(T(M^{2k+1}))$ commutes with the Clifford multiplication. Since

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$$\alpha(e_1 \cdot \ldots \cdot e_{2k+1}) = -e_1 \cdot \ldots \cdot e_{2k+1}$$

dM^{2k+1} acts on S with two different eigenvalues. Therefore dM^{2k+1} defines a splitting of the bundle S into two Clifford subbundles.

Consider a simply-connected, odd-dimensional Riemannian spin manifold M^{2k+1} and denote by S its spinor bundle. Any isometry $\gamma : M^{2k+1} \to M^{2k+1}$ admits two lifts $\hat{\gamma}_\pm : S \to S$ into the spinor bundle S. We once again describe the construction of $\hat{\gamma}_\pm$. The differential $d\gamma$ (resp. $(-d\gamma)$ – M is odd-dimensional !) maps the frame bundle into itself if γ preserves the orientation (resp. does not preserve the orientation). Moreover, $d\gamma$ (or $(-d\gamma)$) lifts into the spin structure of M^{2k+1} and defines two lifts $\hat{\gamma}_\pm$. The lifts $\hat{\gamma}_\pm$ commute (anticommute) with the Clifford multiplication, i.e.

$$\hat{\gamma}_\pm(X \cdot \varphi) = \deg(\gamma) d\gamma(X) \cdot \hat{\gamma}_\pm(\varphi)$$

(see [Fr1]).

Let Γ be a discrete subgroup of isometries and suppose that Γ acts freely on M^{2k+1}. We consider the spinor bundle of Cartan type

$$S^* = S \oplus S$$

over M^{2k+1}. The Clifford multiplication in S^* is given by the formula

$$X \cdot \begin{pmatrix} \varphi \\ \psi \end{pmatrix} = \begin{pmatrix} X \cdot \varphi \\ -X \cdot \psi \end{pmatrix} \qquad X \in T(M^{2k+1})$$

where $X \cdot \varphi$ denotes the Clifford multiplication in the spinor bundle S. Any isometry $\gamma \in \Gamma$ admits the following 4 lifts into the bundle S^*:

$$\begin{pmatrix} \hat{\gamma}_\pm & 0 \\ 0 & \hat{\gamma}_\pm \end{pmatrix} \qquad \text{in case } \gamma \text{ preserves the orientation.}$$

$$\begin{pmatrix} 0 & \hat{\gamma}_\pm \\ \hat{\gamma}_\pm & 0 \end{pmatrix} \qquad \text{in case } \gamma \text{ reverses the orientation.}$$

Any of these lifts commutes with the Clifford multiplication in the bundle S^*.

We introduce now a family of automorphisms $\Pi(\Theta_1, \Theta_2, \varepsilon_1, \varepsilon_2, \gamma)$ of the bundle S^* depending on four parameters $0 \le \Theta_1 \le 2\pi$, $0 \le \Theta_2 \le 2\pi$ and $\varepsilon_1, \varepsilon_2 \in \{\pm 1\}$. If γ preserves the orientation of the manifold we define

$$\Pi(\Theta_1, \Theta_2, \varepsilon_1, \varepsilon_2, \gamma) = \begin{pmatrix} e^{i\Theta_1}\hat{\gamma}_{\varepsilon_1} & 0 \\ 0 & e^{i\Theta_2}\hat{\gamma}_{\varepsilon_2} \end{pmatrix},$$

otherwise let us introduce the automorphism

$$\Pi(\Theta_1, \Theta_2, \varepsilon_1, \varepsilon_2, \gamma) = \begin{pmatrix} 0 & e^{i\Theta_2}\hat{\gamma}_{\varepsilon_2} \\ & \\ e^{i\Theta_1}\hat{\gamma}_{\varepsilon_1} & 0 \end{pmatrix}.$$

Suppose that $\varepsilon : \Gamma \to \mathrm{Aut}(S^*)$ is a homomorphism such that for any $\gamma \in \Gamma$ the automorphism $\varepsilon(\gamma)$ coincides with one of the lifts $\Pi(\Theta_1, \Theta_2, \varepsilon_1, \varepsilon_2, \gamma)$ of γ into S^*. Then

$$\bar{S} = S^*/\varepsilon(\Gamma)$$

is a Clifford bundle of Cartan type over the Riemannian manifold $\bar{M}^{2k+1} = M^{2k+1}/\Gamma$. The bundle $\bar{S}$ splits into two parts invariant under the action of the Clifford bundle $\mathrm{Cliff}(T(M^{2k+1}))$ if and only if $\bar{M}^{2k+1}$ is orientable. For a non-orientable manifold $\bar{M}^{2k+1}$ the non-splitting bundle $\bar{S}$ is not induced from a $Pin^{\mathbb{C}}(2k+1)$ structure of $\bar{M}^{2k+1}$.

Example: $\Gamma = \mathbb{Z}_p$.

Suppose Γ is generated by one involution $\gamma : M^{2k+1} \to M^{2k+1}$ with the lifts $\hat{\gamma}_{\pm}$. If γ preserves the orientation and $\hat{\gamma}_{\pm}^2 = \mathrm{Id}$ the manifold $\bar{M}^{2k+1} = M^{2k+1}/\mathbb{Z}_2$ admits a spin structure. In case $\hat{\gamma}_{\pm}^2 = \text{-}\mathrm{Id}$ $\bar{M}^{2k+1}$ is orientable, but does not admit a spin structure. Nevertheless we can construct a spinor bundle using the homomorphism

$$\varepsilon(\gamma) = \begin{pmatrix} i\hat{\gamma}_+ & 0 \\ 0 & i\hat{\gamma}_+ \end{pmatrix}.$$

If γ does not preserve the orientation we define the homomorphism $\varepsilon : \mathbb{Z}_2 \to \mathrm{Aut}(S^*)$ by the formula

$$\varepsilon(\gamma) = \begin{pmatrix} 0 & \hat{\gamma}_+ \\ \hat{\gamma}_+ & 0 \end{pmatrix} \qquad \text{if } \hat{\gamma}_+^2 = \mathrm{Id}$$

$$\varepsilon(\gamma) = \begin{pmatrix} 0 & i\hat{\gamma}_+ \\ i\hat{\gamma}_+ & 0 \end{pmatrix} \qquad \text{if } \hat{\gamma}_+^2 = \text{-Id}.$$

A similar construction for an arbitrary cyclic group $\mathbb{Z}_p$ yields the result:

Let $\bar{M}^{2k+1}$ be a Riemannian manifold with fundamental group $\pi_1(\bar{M}^{2k+1}) = \mathbb{Z}_p$ and suppose that the universal covering M^{2k+1} admits a spin structure. Then over $\bar{M}^{2k+1}$ there exists a spinor bundle $\bar{S}$ of Cartan type. The bundle $\bar{S}$ splits into two subbundles invariant under the action of the Clifford bundle $\mathrm{Cliff}(T(\bar{M}^{2k+1}))$ *if and only if $\bar{M}^{2k+1}$ is orientable.*

Let $D : \Gamma(S) \to \Gamma(S)$ be the Dirac operator acting on sections of the spinor bundle S over M^{2k+1}. The Dirac operator $\mathfrak{D}$ of the spinor bundle S^* of Cartan type is given by

$$\mathfrak{D} = \begin{pmatrix} D & 0 \\ 0 & -D \end{pmatrix}.$$

Therefore the eigenspaces $E_\lambda(\mathfrak{D})$ consists of pairs $\begin{pmatrix} \varphi \\ \psi \end{pmatrix}$ of eigenspinors of D, i.e.

$$E_\lambda(\mathfrak{D}) = E_\lambda(D) \oplus E_{-\lambda}(D).$$

The group Γ acts on $E_\lambda(\mathfrak{D})$ by the matrices $\varepsilon(\gamma)$ and the eigenspace of the Dirac operator $\bar{\mathfrak{D}}$ on $\bar{M}^{2k+1}$ coincides with the subspace of $\varepsilon(\Gamma)$-invariant pairs in $E_\lambda(\mathfrak{D})$. In particular, in case $\Gamma = \mathbb{Z}_2$ and $\bar{M}^{2k+1}$ is non-orientable we have only one transformation

$$\varepsilon(\gamma) = \begin{pmatrix} 0 & \hat{\gamma}_+ \\ \hat{\gamma}_+ & 0 \end{pmatrix} \quad \text{or} \quad \varepsilon(\gamma) = \begin{pmatrix} 0 & i\hat{\gamma}_+ \\ i\hat{\gamma}_+ & 0 \end{pmatrix}.$$

The map $E_\lambda(D) \ni \varphi \longmapsto \begin{pmatrix} \varphi \\ \hat{\gamma}_+(\varphi) \end{pmatrix} \in E_\lambda(\bar{\mathfrak{D}})$ (resp. $\varphi \longmapsto \begin{pmatrix} \varphi \\ i\hat{\gamma}_+(\varphi) \end{pmatrix}$) defines an isomorphism between the corresponding eigenspaces. Therefore the spectrum of the operator $(\bar{\mathfrak{D}}, \bar{S}, \bar{M}^{2k+1})$ coincides with the spectrum of the operator $(D, S; M^{2k+1})$.

We thus constructed non-splitting spinor bundles over non-orientable manifolds $\bar{M}^{2k+1}$ via coverings. We mention that other situations give similarly bundles with the described properties. For example, any immersion of $\bar{M}^{2k+1}$ into an oriented spin manifold N^{2k+2} also defines a spinor bundle of Cartan type over $\bar{M}^{2k+1}$. Indeed, let $f : \bar{M}^{2k+1} \to N^{2k+2}$ be the immersion and consider the Dirac spinor bundle S of N^{2k+2}. Then the induced bundle $\bar{S} = f^*(S)$ inherits the structure of a spinor bundle of Cartan type over $\bar{M}^{2k+1}$.

3 The Spinor Bundle over $\mathbb{RP}^n$.

We start with the n-dimensional sphere

$$S^n = \{x \in \mathbb{R}^{n+1} : |x| = 1\}$$

and we denote by S^* the $2^{\left[\frac{n+1}{2}\right]}$-dimensional bundle

$$S^* = S^n \times \Delta_{n+1}.$$

The tangent bundle of the sphere

$$T(S^n) = \{(x,t) \in S^n \times \mathbb{R}^{n+1} : \langle x, t \rangle = 0\}$$

acts by Clifford multiplication

$$\mu((x,t) \otimes (x,\varphi)) := (x, t \cdot \varphi)$$

on S^*. Therefore S^* is a Clifford bundle over S^n. Let $g : S^n \to S^n, g(x) = -x$ be the antipodal map. Its differential $dg : T(S^n) \to T(S^n)$ acts on a tangent vector $(x, t) \in T(S^n)$ by

$$dg(x, t) = (-x, -t).$$

Let us define a lift $\hat{g}$ of the antipodal map into the bundle S^* by the formula

$$\hat{g}(x, \varphi) = (-x, x \cdot \varphi).$$

Then $\hat{g}$ is again an involution, $(\hat{g})^2 = \mathrm{Id}$. Moreover, $\hat{g}$ respects the Clifford multiplication, i.e. the diagram

$$\begin{tikzcd} T(S^n) \otimes S^* \arrow[r, "\mu"] \arrow[d, "(dg)\otimes\hat{g}"] & S^* \arrow[d, "\hat{g}"] \\ T(S^n) \otimes S^* \arrow[r, "\mu"] & S^* \end{tikzcd}$$

commutes. Indeed, given $(x, t) \otimes (x, \varphi) \in T(S^n) \otimes S^*$ we have

$$\mu \circ (dg \otimes \hat{g})((x, t) \otimes (x, \varphi)) = \mu((-x, -t) \otimes (-x, x \cdot \varphi)) = (-x, -t \cdot x \cdot \varphi).$$

An the other hand,

$$\hat{g}\mu((x, t) \otimes (x, \varphi)) = \hat{g}((x, t \cdot \varphi)) = (-x, x \cdot t \cdot \varphi).$$

Since x and t are orthogonal vectors in $\mathbb{R}^{n+1}$ we have in the Clifford algebra the relation

$$x \cdot t + t \cdot x = 0,$$

i.e. the mentioned diagram commutes.

Over the real projective space $\mathbb{RP}^n = S^n/g$ we define the bundle

$$S = S^*/\hat{g}.$$

The tangent bundle $T(\mathbb{RP}^n)$ can be identified with $T(\mathbb{RP}^n) = T(S^n)/dg$ and the above discussed property of the lift $\hat{g} : S^* \to S^*$ implies that we obtain a well-defined Clifford multiplication

$$\mu : T(\mathbb{RP}^n) \otimes S \to S.$$

At any point of $\mathbb{RP}^n$ the bundle S realizes the Dirac (in case $n \equiv 0 \, mod \, 2$) or the Cartan (in case $n \equiv 1 \, mod \, 2$) representation of the n-dimensional Clifford algebra.

Let us consider the even-dimensional case, $n = 2k$. The Dirac representation of the Clifford algebra decomposes into two irreducible representations with respect to the action of the even part $\mathrm{Cliff}_o^{\mathbb{C}}(n)$ of the Clifford algebra. Therefore the bundle S decomposes at any fixed point as a Clifford module too. However, globally the bundle S over $\mathbb{RP}^n$ does not split as a $\mathrm{Cliff}_o(T(\mathbb{RP}^n))$-bundle. Indeed suppose that there exists a decomposition $S = S_1 \oplus S_2$ invariant under the action of the Clifford bundle. Then we obtain a corresponding decomposition $S^* = S_1^* \oplus S_2^*$ of the bundle S^* over S^n. The spin module Δ_{n+1} decomposes in a unique way as a $\mathrm{Cliff}_o(T_x S^n)$-module

$$\Delta_{n+1} = \Delta^+_{n+1}(x) \oplus \Delta^-_{n+1}(x).$$

We describe this decomposition in an explicit way. Let $e_1, \ldots, e_n$ an orthonormal basis of the tangent space $T_x(S^n)$ and consider the element

$$f = i^{n/2}\, e_1 \cdot \ldots \cdot e_n : \Delta_{n+1} \to \Delta_{n+1}.$$

Then $f^2 = \mathrm{Id}$ and Δ_{n+1} decomposes into the eigenspaces of $f, \Delta_{n+1} = \Delta^+_{n+1}(x) \oplus \Delta^-_{n+1}$ with

$$\Delta^{\pm}_{n+1}(x) = \{\varphi \in \Delta_{n+1} : \quad f(\varphi) = \pm\varphi\}.$$

Now the volume form $e_1 \cdot \ldots \cdot e_n \cdot x$ of $\mathbb{R}^{n+1}$ acts on Δ_{n+1} by multiplication

$$e_1 \cdot \ldots \cdot e_n \cdot x_{|\Delta_{n+1}} = \alpha_{n+1}.$$

Then $f = i^{n/2}\, e_1 \cdot \ldots \cdot e_n = -i^{n/2}\, \alpha_{n+1} x$ and the subspaces $\Delta^{\pm}_{n+1}(x)$ are given by

$$\Delta^{\pm}_{n+1}(x) = \{\varphi \in \Delta_{n+1} : \ \pm x\varphi = i^{n/2}\, \alpha_{n+1}\varphi\}.$$

The bundles S_1^* and S_2^* coincide therefore with the subbundles $\Delta^{\pm}_{n+1}(x)$. Since $S_i^*(i = 1, 2)$ are induced bundles by the covering $S^n \to \mathbb{RP}^n = S^n/g$ they are invariant under the lift $\hat{g} : S^* \to S^*$. However, the lift $\hat{g}$ defined by the Clifford multiplication does not preserve the decomposition $\Delta_{n+1} = \Delta^+_{n+1}(x) \oplus \Delta^-_{n+1}(x)$, a contradiction.

We discuss now the odd-dimensional case, $n = 2k + 1$. Again we start with an orthonormal basis $e_1, \ldots, e_n \in T_x(S^n)$ and we introduce the automorphism

$$f = e_1 \cdot \ldots \cdot e_n : \Delta_{n+1} \to \Delta_{n+1}.$$

Then we have

$$f^2 = (-1)^{k+1} \quad , \quad f \cdot x = -x \cdot f$$

and

$$f \cdot t = t \cdot f \quad \text{for all tangent vectors } t \in T_x(S^n).$$

We decompose Δ_{n+1} into the eigenspaces of f

$$\Delta_{n+1} = \Delta^+_{n+1}(x) \oplus \Delta^-_{n+1}(x) \quad , \quad \Delta^{\pm}_{n+1}(x) = \{\varphi \in \Delta_{n+1} : e_1 \cdot \ldots \cdot e_n\varphi = \pm i^{k+1}\varphi\}.$$

Since g commutes with all tangent vectors $t \in T_x(S^n)$ the subspaces $\Delta^{\pm}_{n+1}(x)$ are invariant under the action of the Clifford algebra $\mathrm{Cliff}(T_x(S^n))$. The lift $\hat{g}$ preserves now $(n = 2k+1)$ the decomposition $\Delta_{n+1} = \Delta^+_{n+1}(x) \oplus \Delta^-_{n+1}(x)$, i.e. the Clifford bundle S over $\mathbb{RP}^n$ decomposes into two Clifford subbundles.

Vector fields V on the projective space $\mathbb{RP}^n$ can be identified with maps $V : S \to \mathbb{R}^{n+1}$ such that

$$V(-x) = -V(x) \quad , \quad \langle V(x), x \rangle = 0 \quad x \in S^n .$$

The bundle S over $\mathbb{RP}^n$ arises from $S^* = S^n \times \Delta_{n+1}$ and the identification $\hat{g}(x, \varphi) = = (-x, x \cdot \varphi)$. Therefore a section $\Phi \in \Gamma(S; \mathbb{RP}^n)$ is a map $\Phi : S^n \to \Delta_{n+1}$ with the property

$$\Phi(-x) = x \cdot \Phi(x) \ . \tag{$*$}$$

The formula

$$(\nabla^S_V \Phi)(x) = d\Phi(V)(x) + \frac{1}{2} V(x) \cdot x \cdot \Phi(x)$$

defines a covariant derivative

$$\nabla^S : \Gamma(T(\mathbb{RP}^n)) \times \Gamma(S) \to \Gamma(S).$$

Indeed, a simple calculation shows that $\nabla^S_V \Phi$ has the invariance property $(*)$ in case Φ and V satisfy the corresponding transformation rules. Moreover, the covariant derivative ∇^S is compatible with the Clifford multiplication, i.e.

$$\nabla^S_V (W \cdot \Phi) = (\nabla_V W) \cdot \Phi + W \cdot (\nabla^S_V \Phi),$$

and ∇^S preserves the hermitian metric of the bundle S. To sum up, we see that the triple $(S, \nabla^S, \langle , \rangle)$ is a Dirac bundle over $\mathbb{RP}^n$ (see [LM] for the general definition of a Dirac bundle). A direct calculation yields the formula

$$R^S(V, W)\Phi = \frac{1}{2} \left(W \cdot V + \langle V, W \rangle \right) \cdot \Phi$$

for the curvature tensor R^S of the connection ∇^S.

Remark: Let us compare the above defined covariant derivative ∇^S with the usual covariant derivative ∇ in the usual spin bundle of a hypersurface in $\mathbb{R}^{n+1}$. The formula for ∇ is

$$\nabla_V \Phi = d\Phi(V) + \frac{1}{2} \mathrm{II}\,(V) \cdot \vec{N} \cdot \Phi$$

i.e. over the sphere S^n the introduced Dirac bundle coincides with the usual spinor bundle of the sphere.

The Dirac operator $D = \sum\limits_{i=1}^{n} e_i \nabla^S_{e_i}$ acting on sections $\Gamma(S; \mathbb{RP}^n)$ is related to the Laplace operator $\Delta = - \sum\limits_{i=1}^{n} \nabla^S_{e_i} \nabla^S_{e_i} - \sum\limits_{i=1}^{n} \mathrm{div}\,(e_i) \nabla^S_{e_i}$ by the well-known formula

$$D^2 = \Delta + R$$

where the endomorphism R (in case of a general Dirac bundle) is given by

$$R = \frac{1}{2}\sum_{j,k} e_j e_k R^S(e_j, e_k).$$

In our situation we simply obtain

$$D^2 = \Delta + \frac{1}{4}n(n-1) = \Delta + \frac{\tau}{4}$$

where τ is the scalar curvature of $\mathbb{RP}^n$. The bundle S over $\mathbb{RP}^n$ admits Killing spinors, i.e. spinor fields Φ satisfying the differential equation

$$\nabla^S_V \Phi = \lambda V \cdot \Phi$$

for some $\lambda \in \mathbb{R}^1$. Indeed, suppose that Φ is a Killing spinor in $\Gamma(S; \mathbb{RP}^n)$. Then we obtain

$$R^S(V, W)\Phi = 2\lambda^2 (WV + \langle V, W\rangle) \cdot \Phi$$

using the differential equation. On the other hand, the formula for the curvature tensor implies

$$R^S(V, W)\Phi = \frac{1}{2}(WV + \langle V, W\rangle) \cdot \Phi,$$

i.e. $\lambda = \pm\frac{1}{2}$. Let us consider the spinor field

$$\Phi(x) = (1 - x)\Phi_0$$

where $\Phi_0 \in \Delta_{n+1}$ is constant. Then

$$\Phi(-x) = (1 + x)\Phi_0 = x(1 - x)\Phi_0 = x\Phi(x)$$

and therefore $\Phi \in \Gamma(S; \mathbb{RP}^n)$ is a section in the spinor bundle over $\mathbb{RP}^n$. We calculate the covaraint derivative $\nabla^S_V \Phi$:

$$\begin{aligned}
(\nabla^S_V \Phi)(x) &= -V(x) \cdot \Phi_0 + \frac{1}{2}V(x) \cdot x \cdot (1 - x)\Phi_0 = \\
&= -V(x) \cdot \Phi_0 + \frac{1}{2}V(x) \cdot (1 + x)\Phi_0 = \\
&= \frac{1}{2}V(x) \cdot (-1 + x) \cdot \Phi_0 = -\frac{1}{2}V(x) \cdot \Phi(x),
\end{aligned}$$

i.e. $\Phi(x)$ is a Killing spinor with Killing number $\lambda = -\frac{1}{2}$.

Remark: Killing spinors with Killing number $\lambda = +\frac{1}{2}$ on $\mathbb{RP}^n$ appear if we use the identification $\hat{g}_- : S^* \to S^*$ defined by the formula

$$\hat{g}(x, \varphi) = (-x, -x \cdot \varphi).$$

$\hat{g}_-$ defines in a similar way a Clifford bundle $S_- = S^*/\hat{g}_-$ whose sections are maps $\Phi : S^n \to \Delta_{n+1}$ with the property

$$\Phi(-x) = -x\Phi(x).$$

The spinor fields

$$\Phi(x) = (1+x)\Phi_0$$

are Killing spinors in the bundle S_- over $\mathbb{RP}^n$ with Killing number $\lambda = +\frac{1}{2}$.

Killing spinors on compact Riemannian spin manifolds with positive scalar curvature correspond to eigenspinors of the Dirac operator related to the smallest eigenvalue. In case $\mathbb{RP}^{4k+3}$ the first eigenvalue of the Dirac operator on usual spinors has been calculated $\left(\lambda_1 = \pm\frac{1}{2}\sqrt{\frac{n}{n-1}\tau}\right)$ and there are Killing spinors (see [F1]). More general, we can calculate the spectrum of the Dirac operator acting on sections $\Phi \in \Gamma(S;\mathbb{RP}^n)$ using the computation of the eigenvalues of D on the sphere. Let $\Phi : S^n \to \Delta_{n+1}$ be a section in the bundle S^*. We decompose Φ into $\Phi = \Phi_+ + \Phi_-$ with

$$\Phi_+(x) = \frac{\Phi(x) - x\cdot\Phi(-x)}{2} \quad , \quad \Phi_-(x) = \frac{\Phi(x) + x\cdot\Phi(-x)}{2}.$$

The sections $\Gamma(S;\mathbb{RP}^n)$ are described by the condition $\Phi_- \equiv 0$. Now we apply the calculation of the spectrum of the Dirac operator on spheres as well as the realization of eigenspinors by polynomials (see [Su], [Tr]). Imposing the additional condition $\Phi_- \equiv 0$ we obtain the spectrum of the Dirac operator acting on the sections $\Gamma(S;\mathbb{RP}^n)$ for arbitrary n.

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