Phase Space Tunneling for Operators with Symbols in a Gevrey Class*

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Abstract

Phase space tunneling and exponential decay of eigenfunctions in phase space are well known for operators with symbols which are analytic in some neighbourhood of the real axis. This can be used to prove an adiabatic theorem of exponential order if one assumes the Hamiltonian to depend analytically on time [8]. However to study compactly supported switching processes one has to weaken the analyticity assumptions. Here we examine non-analytic symbols with Gevrey class regularity and show that we get an exponential decay of the corresponding eigenfunctions with respect to $\hbar^{1/a}$ as $\hbar \to 0$, where $a > 1$. The loss of regularity causes a slower decay in $\hbar$. The analysis is done using the methods of A. Martinez and its generalization by S. Nakamura. An upper bound for the rate of decay is given.

1 Introduction

This paper adapts the techniques of exponential weighted phase space estimates as studied in [7], [8] and [10] to a case where there is no analyticity, but regularity in the sense of Gevrey classes. Such a case is of interest, if one regards for example its application to the adiabatic theorem in quantum

*This paper is a rewritten part of the authors dissertation [5] under the supervision of Prof. R. Seiler, TU-Berlin mechanics, which is understood in [8] as a phase space tunneling phenomena. From this point of view the consideration of non-analytic symbols is directly connected to the study of non-analytic switching processes in adiabatic theory, (which in particular allows to consider compactly supported switching functions).

Let $T : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^{2n})$ be the Bargmann transform, defined by

$$Tu(x,\xi) = (2\mu)^{n/4}(2\pi\hbar)^{-3n/4}e^{ix\xi/(2\hbar)}\int e^{i(x-y)\xi/\hbar-\mu(x-y)^2/(2\hbar)}u(y)\,dy\,. \quad (1.1)$$

Tu can be understood as a representation of the wave function u in phase space (see [1]). Let $P = p(x, \hbar D_x)$ be a pseudo differential operator with symbol p and $\psi \in C^\infty(\mathbb{R}^{2n}, \mathbb{R})$ be a weight function, which is bounded with all its derivatives. If the symbol p is analytic in a strip $S_\omega = \{|\operatorname{Im} x| < \omega_1, |\operatorname{Im}\xi| < \omega_2\}$ and $\sup|\partial_x\psi| < \omega_1$, $\sup|\partial_\xi\psi| < \omega_2$, the main estimate in the theory of phase space tunneling ([7], [8], [10]) is given by

$$\begin{aligned} \langle e^{\psi/\hbar}Tu, e^{\psi/\hbar}TPu\rangle &= \langle e^{\psi/\hbar}Tu, p(x-\partial_\mu\psi, \xi + i\mu\partial_\mu\psi)e^{\psi/\hbar}Tu\rangle \\ &\quad +\mathcal{O}(\hbar)\|e^{\psi/\hbar}Tu\|^2\,, \end{aligned} \quad (1.2)$$

where $\partial_\mu := \mu^{-1}\partial_x + i\partial_\xi$. Consider an eigenfunction u with $Pu = 0$. It can be shown that (1.2) implies a decay of Tu in phase space: For any compact $K \subset \{p(x,\xi) \neq 0\}$ one has

$$\|Tu\|_{L^2(K)} = \mathcal{O}(e^{-\delta/\hbar}) \quad (1.3)$$

as $\hbar \to 0$, where $\delta = \delta(K) > 0$ can be related explicitly to the geometry of K and p.

In this paper we follow closely the techniques of [10] and concentrate mainly on the difficulties which arise when the symbol p is only C^∞: What replaces the analytic continuation of p to the strip S_ω in (1.2) and what replaces the bounds of $\partial_x\psi$ and $\partial_\xi\psi$? How to deal with a non-vanishing $\overline{\partial}_z p$? What is the substitute of (1.3)?

The explicit calculus of the next sections is sometimes a little bit technical, but the main idea is as follows: Let $p \in C^\infty(\mathbb{R})$. If $\|\partial^\alpha p\| = \mathcal{O}(b^{|\alpha|}\alpha!)$, p is analytic in the strip $S_{1/b} = \{|\operatorname{Im} z| < 1/b\}$. Replacing this condition by $\|\partial^\alpha p\| = \mathcal{O}(b^{|\alpha|}\alpha!^a)$ with $a > 1$ we loose analyticity (p is called of Gevrey class a). Nevertheless b rests an important quantity for measuring how "good" one can extend p to the complex domain. In a similar way to [9] we will define such an almost analytic extension to a strip S_ω by a a, b and ω dependent cutting of a formal Taylor series of p. The cutting point is chosen to minimize $\overline{\partial}_z p$ in S_ω. This produces an additional error term (exponentially small with respect to $\omega^{1/(a-1)}$ as $\omega \to 0$) which has to be controlled. By choosing ω $\hbar$-dependent in an appropriate way, we end up with an estimate in the sense of (1.2) where we have to replace $e^{\psi/\hbar}$ by $e^{\psi/\hbar^{1/a}}$ and explicitly give a and b dependent bounds for $\partial_x\psi$ resp. $\partial_\xi\psi$.

The plan of the paper is as follows: Section 2 introduces Gevrey class functions and their extensions to the complex domain. Some properties are shown and an estimate of their $\overline{\partial}_z$-derivative is made. In section 3 the key estimate in the sense of (1.2) is formulated and proven. Section 4 contains a Gevrey class version of the estimate (1.3) on the decay of eigenfunctions in the classically forbidden region of the phase space.

2 Functions of Gevrey class type

For our purpose it is suitable to define Gevrey classes in a slightly different way than normally done in literature. We will see the reason for doing so in a moment, but let us first recall some basic facts:

The smoothness of a C^∞ function can be controlled by the growth of its derivatives. The Gevrey classes interpolate between the pure C^∞ case and the analytic case. We will define them for functions in $C^\infty(\mathbb{R}^n, \mathbb{R})$ and will use the usual multi index notation. In addition we set for $\alpha, \beta \in \mathbb{N}^n$ and $c \in \mathbb{R}$

$$\begin{aligned}
\mathbb{1} &:= (1, \dots, 1) \in \mathbb{N}^n\,, \\
\alpha^c &:= \alpha_1^c \cdot \ldots \cdot \alpha_n^c\,, \\
\alpha^\beta &:= \alpha_1^{\beta_1} \cdot \ldots \cdot \alpha_n^{\beta_n}\,, \\
0^0 &:= 1\,, \\
\alpha \le \beta &:\Leftrightarrow \alpha_i \le \beta_i \text{ for all } i = 1, \dots, n\,.
\end{aligned}$$

From Stirling's formula $k! = (k/e)^k \sqrt{2\pi k}\ (1 + \mathcal{O}(1/k))$ we deduce for $\alpha \in \mathbb{N}^n$

$$\alpha! \le \text{const}\, \frac{\alpha^\alpha}{e^{|\alpha|}} (\alpha + \mathbb{1})^{1/2}\,. \tag{2.1}$$

Definition 2.1 *Let $a \in [1, \infty)$ and $b > 0$. We denote by $\Gamma^{a,b}(\mathbb{R}^n, \mathbb{R})$ the set of functions $f \in C^\infty(\mathbb{R}^n, \mathbb{R})$ satisfying for some $c_0 > 0$*

$$\|\partial^\alpha f\|_\infty \le c_0(\alpha + \mathbb{1})^{c_0} b^{|\alpha|} \alpha!^a \tag{2.2}$$

for all $\alpha \in \mathbb{N}^n$. $\Gamma^{a,b}$ is called Gevrey class of index (a, b), $\Gamma^a = \bigcup_{b>0} \Gamma^{a,b}$ is called Gevrey class of index a.

Remarks:

- Definitions of Gevrey classes may be found in literature using $(|\alpha|!)^a$, $|\alpha|^{a|\alpha|}$ or $\Gamma(a|\alpha|+1)$ instead of $\alpha!^a$. If one only regards Gevrey classes of index a all these definitions coincide. This is a consequence of Stirling's formula and $|\alpha|! \le n^{|\alpha|}\alpha!$. See [12].

- Definition 2.1 is a refinement of the classical definition found in literature (see e.g. [2], [3], [4], [6], [11]). We are using a second index b for classification, and we are adding a term $(\alpha + \mathbb{1})^{c_0}$. This term makes $\Gamma^{a,b}$ an algebra, stable under differentiation (see Proposition 2.1 below). The classical definition gives an algebra which is only stable with respect to the parameter a but not with respect to b. Absorbing $(\alpha + \mathbb{1})^{c_0}$ by enlarging the constant b shows that our refined definition coincides with the classical one if one only considers classes of single index a. The refinement is necessary since we want to control the parameter b in our estimates.

Proposition 2.1 *$\Gamma^{a,b}(\mathbb{R}^n, \mathbb{R})$ is an algebra, stable under differentiation and translation.*

Proof: We only show

$$\begin{aligned} f \in \Gamma^{a,b} \quad &\Rightarrow \quad \partial^\beta f \in \Gamma^{a,b} \quad (\beta \in \mathbb{N}^n)\,, \\ f, g \in \Gamma^{a,b} \quad &\Rightarrow \quad f \cdot g \in \Gamma^{a,b}\,. \end{aligned}$$

It suffices to prove the proposition for $\beta = (1, 0, \dots, 0)$. So we can estimate

$$\begin{aligned} \|\partial^{\alpha+\beta} f\|_\infty &\le c_0(\alpha + \beta + \mathbb{1})^{c_0} b^{|\alpha|+|\beta|}(\alpha + \beta)!^a \\ &\le c_0 b 2^{c_0}(\alpha + \mathbb{1})^{c_0+a} b^{|\alpha|}\alpha!^a \end{aligned}$$

and

$$
\begin{aligned}
&\|\partial^\alpha fg\|_\infty \\
&\leq \; c_0\tilde{c}_0 \sum_{\beta\leq\alpha} \frac{\alpha!}{\beta!(\alpha-\beta)!}(\beta+\mathbb{1})^{c_0}(\alpha-\beta+\mathbb{1})^{\tilde{c}_0} b^{|\beta|} b^{|\alpha-\beta|} \beta!^a (\alpha-\beta)!^a \\
&\leq \; c_0\tilde{c}_0 (\alpha+\mathbb{1})^{c_0+\tilde{c}_0} b^{|\alpha|} \alpha!^a \sum_{\beta\leq\alpha} \left(\frac{\alpha!}{\beta!(\alpha-\beta)!}\right)^{1-a} .
\end{aligned}
$$

Since $\frac{\alpha_j!}{\beta_j!(\alpha_j-\beta_j)!} \geq 1$ and $1-a<0$ we get

$$
\|\partial^\alpha fg\|_\infty \leq c_0\tilde{c}_0(\alpha+\mathbb{1})^{c_0+\tilde{c}_0+1} b^{|\alpha|} \alpha!^a . \quad \blacksquare
$$

In the analytic case $a=1$ a function $f \in \Gamma^{1,b}(\mathbb{R}^n,\mathbb{R})$ can be extended analytically to the strip $S_\omega := \{z \in \mathbb{C}^n \mid |\operatorname{Im} z_j| < \omega \text{ for all } j\}$ with $\omega \leq 1/b$. In the same spirit as in [9] we will define an almost analytic extension for all $a \in [1,\infty)$ in such a way that $\overline{\partial}_z f$ is minimized.

Definition 2.2 *Let $a \in [1,\infty)$, $b,\omega > 0$ and $f \in \Gamma^{a,b}(\mathbb{R}^n,\mathbb{R})$. Then for $z = t+is \in S_\omega$ we define the almost analytic extension*

$$
(\mathcal{F}^{a,b,\omega} f)(t+is) := \sum_{|\alpha|\leq N_{a,b,\omega}} \frac{i^{|\alpha|} s^\alpha}{\alpha!} \partial^\alpha f(t) \tag{2.3}
$$

with

$$
N_{a,b,\omega} := \left[(b\omega)^{-1/(a-1)}\right] \tag{2.4}
$$

for $a>1$ and $N_{1,b,\omega} := \infty$, where $[\,\cdot\,]$ denotes the integer part and $s^\alpha = s_1^{\alpha_1} \cdot \ldots \cdot s_n^{\alpha_n}$.

Remark: For $a=1$ the series (2.3) converges for $z \in S_{1/b}$ and gives the usual analytic extension.

The following proposition gives an estimate for all derivatives of an antiholomorphic derivative of the almost analytic extension $\mathcal{F}^{a,b,\omega}$. The proof shows in which sense the cutoff (2.4) was chosen optimal.

Proposition 2.2 *Let $a \in (1,\infty)$, $b > 0$ and $f \in \Gamma^{a,b}(\mathbb{R}^n,\mathbb{R})$. Let furthermore $\alpha,\beta \in \mathbb{N}^n$. Then there exists constants $C_0, C_1 > 0$, such that for all $j \in \{1,\dots,n\}$ and $\omega > 0$ sufficiently small*

$$\sup_{z\in S_\omega} |\partial_t^\alpha \partial_s^\beta \overline{\partial}_{z_j} \mathcal{F}^{a,b,\omega} f(z)| \le C_0 \omega^{-C_1} \exp(-\frac{B}{\omega^{1/(a-1)}}) \tag{2.5}$$

with $z = t + is$ and $B = (a-1)b^{-1/(a-1)}$. Hence, for any $\delta > 0$

$$\sup_{z\in S_\omega} |\partial_t^\alpha \partial_s^\beta \overline{\partial}_{z_j} \mathcal{F}^{a,b,\omega} f(z)| = \mathcal{O}(\exp(-\frac{B-\delta}{\omega^{1/(a-1)}})) \tag{2.6}$$

as $\omega \to 0$.

Proof: Within this proof $N_{a,b,\omega}$ will be simply denoted by N. First we treat the case $\alpha = \beta = 0$. Inserting $2\overline{\partial}_{z_j} = \partial_{t_j} + i\partial_{s_j}$ in (2.3) we see that only terms of order N do not cancel. We get

$$2\overline{\partial}_{z_j} \mathcal{F}^{a,b,\omega} f(t+is) = \sum_{|\alpha|=N} \frac{i^{|\alpha|} s^\alpha}{\alpha!} (\partial_j \partial^\alpha f(t))\,, \tag{2.7}$$

where $\partial_j \equiv \partial_{t_j}$ denotes the j-th partial derivative with respect to t_j. Using $t + is \in S_\omega$ and the definition of $\Gamma^{a,b}$ we can estimate

$$|\frac{s^\alpha}{\alpha!}(\partial_j \partial^\alpha f(t))| \le c_0 b\, 2^{c_0} (\alpha + 1\!\!1\,)^{c_0 + a} (b\omega)^{|\alpha|} \alpha!^{a-1}\,.$$

With (2.1) and $|\alpha| = N$, $\alpha_j \le N$, $\alpha^\alpha \le (|\alpha| 1\!\!1\,)^\alpha = |\alpha|^{|\alpha|}$ this implies

$$|\frac{s^\alpha}{\alpha!}(\partial_j \partial^\alpha f(t))| \le \text{const}\,(N+1)^{n(c_0+a+1/2)} (be^{1-a}\omega)^N N^{(a-1)N}\,. \tag{2.8}$$

Let $c := be^{1-a}\omega$. We will now estimate $c^N N^{(a-1)N} =: \exp(g(N))$. The minimum of

$$g(x) = x(\ln c + (a-1)\ln x)$$

is given by

$$\begin{aligned} x_0 &= e^{-1} c^{-1/(a-1)} = (b\omega)^{-1/(a-1)}\,, \\ g(x_0) &= -\frac{a-1}{(b\omega)^{1/(a-1)}} = -\frac{B}{\omega^{1/(a-1)}}\,. \end{aligned}$$

Since we have chosen $N=[x_0]$ and because of $N\sim\omega^{-1/(a-1)}$ for $\omega\to 0$ we can find constants $\tilde C_0>0$ and $\tilde C_1>0$ such that (2.8) implies for $|\alpha|=N$ and $z=t+is\in S_\omega$

$$|\frac{s^\alpha}{\alpha!}(\partial_j\partial^\alpha f(t))|\le \tilde C_0\omega^{-\tilde C_1}\exp(-\frac{B}{\omega^{1/(a-1)}})\,.$$

Finally using (2.7) and $\#\{\alpha\in\mathbb{N}^n\mid|\alpha|=N\}\le(N+1)^n$ and again $N\sim\omega^{-1/(a-1)}$ we get the result (2.5) for $\alpha=\beta=0$. For ω sufficiently small we can absorb negative powers of ω by making the constant B smaller. This implies (2.6).

The case $\alpha,\beta\neq 0$ is proven in a similar way: Let $\mathcal{F}^{a,b,\omega}f$ be denoted by $\tilde f$. From (2.7) it follows

$$\partial_t^\alpha\overline{\partial}_{z_j}\tilde f=\overline{\partial}_{z_j}(\widetilde{\partial^\alpha f})\,.$$

Using the stability of $\Gamma^{a,b}$ under differentiation we obtain the same estimate (2.8) for $\partial_t^\alpha\overline{\partial}_{z_j}\tilde f$ instead of $\overline{\partial}_{z_j}\tilde f$. So we are left to estimate only derivatives with respect to s. For $\omega\to 0$ we get

$$\begin{aligned}\sup_{z\in S_\omega}|\partial_s^\beta\overline{\partial}_{z_j}\tilde f(z)| &= \sup_{z\in S_\omega}|\sum_{|\alpha|=N}\frac{i^{|\alpha|}(\partial_s^\beta s^\alpha)}{\alpha!}(\partial_{t_j}\partial^\alpha f(t))|\\ &= \mathcal{O}(\omega^{-|\beta|})\sup_{z\in S_\omega}|\overline{\partial}_{z_j}\tilde f(z)|\\ &= \mathcal{O}(\omega^{-|\beta|}\exp(-\frac{B-\delta}{\omega^{1/(a-1)}}))\\ &= \mathcal{O}(\exp(-\frac{B-\tilde\delta}{\omega^{1/(a-1)}}))\,.\quad\blacksquare\end{aligned}$$

Remark: We want to point out, that for $a\to 1$ Proposition 2.2 reduces to $\overline{\partial}_{z_j}\tilde f(z)=0$ for $z\in S_\omega$, $\omega<1/b$. In fact, fixing $\omega<1/b$ we get

$$\ln\tilde a-(1/\tilde a)\ln(b\omega)\to+\infty\qquad(\tilde a\to 0+)\,.$$

Using the definition of B it follows

$$\lim_{a\to 1+}\exp(-\frac{B}{\omega^{1/(a-1)}})=\lim_{\tilde a\to 0+}\exp(-\frac{\tilde a}{(b\omega)^{1/\tilde a}})=0\,.$$

In [9] almost analytic extensions are constructed by a "smooth" cut of the formal power series. This implies that the extension to a strip S_ω is smooth with respect to the width ω of the strip. So the extension of f tends to f if $\omega \to 0$.

In our case (see Definition 2.2) we are cutting the series at a point $N_{a,b,\omega}$ which is the integer part of $(b\omega)^{-1/(a-1)}$. So it does not depend smoothly on ω. The fixed cutting point has the advantage to simplify many calculations, but we have to pay the price that for fixed $\omega > 0$

$$\sup_{t\in\mathbb{R}} |\mathcal{F}^{a,b,\omega} f(t+is) - f(t)| \to 0 \qquad (s \to 0) \tag{2.9}$$

is not true. Nevertheless the extension $\mathcal{F}^{a,b,\omega}$ is well behaved in the sense Proposition 2.3 below. To obtain a result of the form of (2.9) a more refined version of an almost analytic extension is needed. Details can be found in [5]. It uses a smooth cutoff of the power series where the point of cutoff depends on s.

Proposition 2.3 *Let $a \in (1,\infty)$, $b > 0$ and $f \in \Gamma^{a,b}(\mathbb{R}^n, \mathbb{R})$. Then for all $\delta > 0$ there exists an $\omega_0 > 0$ such that for all $\omega < \omega_0$*

$$|\mathcal{F}^{a,b,\omega} f(t+is) - f(t)| < \delta \qquad (t+is \in S_\omega)\,.$$

Proof: Within this proof $N_{a,b,\omega}$ will be simply denoted by N. Equation (2.4) implies $N \to \infty$ if $\omega \to 0$, so we have only to show that

$$d := \mathcal{F}^{a,b,\omega} f(t+is) - f(t) = \sum_{0<|\alpha|\le N} \frac{i^{|\alpha|} s^\alpha}{\alpha!} \partial^\alpha f(t)$$

becomes small for large N. Using Definition 2.2, $|s_j| \le \omega$, $b\omega \le N^{-a+1}$ and Stirlings formula we get

$$\begin{aligned}
|d| &\le \sum_{0<|\alpha|\le N} c_0 (\alpha + \mathbb{1})^{c_0} N^{-|\alpha|(a-1)} \alpha!^{a-1} \\
&\le \text{const} \sum_{0<|\alpha|\le N} (\alpha+\mathbb{1})^{c_0+1/2} (Ne)^{-|\alpha|(a-1)} \alpha^{(a-1)\alpha} \\
&\le \text{const} \sum_{0<|\alpha|\le N} (|\alpha|+n)^{n(c_0+1/2)} \left(\frac{|\alpha|}{Ne}\right)^{(a-1)|\alpha|} \\
&\le \text{const} \sum_{k=1}^{N} (k+n)^{n(c_0+3/2)} \left(\frac{k}{Ne}\right)^{(a-1)k} .
\end{aligned} \tag{2.10}$$

For the last step we used $\#\{\alpha \in \mathbb{N}^n \mid |\alpha| = k\} \leq (k+1)^n$. Let us choose $c > 1$ and $k_0 \in \mathbb{N}$ (depending on a and n) such that

$$ce^{-a+1} < 1$$

and

$$(k+n)^{n(c_0+3/2)} < c^k \qquad (k \geq k_0)\,.$$

Splitting the sum of (2.10) into parts $1 \leq k < k_0$ and $k_0 \leq k \leq N$ we see that the first part tends to zero if $N \to \infty$ and the second part can be estimated by

$$\sum_{k=k_0}^{N} c^k \left(\frac{k}{Ne}\right)^{(a-1)k} \leq \sum_{k=0}^{\infty} (ce^{-a+1})^k < \infty$$

On the other hand each term of the second sum tends to zero for $N \to \infty$, so the sum itself tends to zero. ■

3 Exponential weighted microlocal estimates

3.1 Symbol classes

The following definition will fix symbol classes, quantization and classes of pseudo differential operators which we use in this paper.

Definition 3.1 *For $m \in \mathbb{N}$ a function $p \in C^\infty(\mathbb{R}^n_x \times \mathbb{R}^n_\xi, \mathbb{R})$ is called a symbol $p \in \mathcal{S}^m$ if for all $\alpha \in \mathbb{N}^{2n}$ there exists constants $c_\alpha > 0$ such that for all $x, \xi \in \mathbb{R}^{2n}$ and $\hbar > 0$*

$$|\partial^\alpha_{x,\xi} p(\hbar; x, \xi)| \leq c_\alpha \hbar^m\,.$$

The Weyl quantization of $p \in \mathcal{S}^m$ is given by

$$Op^W(p)u(x) := \frac{1}{(2\pi\hbar)^n} \int\limits_{\mathbb{R}^{2n}} e^{i(x-y)/\hbar} p(\hbar; \tfrac{x+y}{2}, \xi) u(y)\, dy\, d\xi \qquad (3.1)$$

for all $u \in L^2(\mathbb{R}^n)$.

Remark: A priori the integral in (3.1) does not make sense for $u \in L^2(\mathbb{R}^n)$. It is defined in the sense of an oscillating integral for $u \in C_0^\infty(\mathbb{R}^n)$, and $\mathrm{Op}^W(p)$ can afterwards be extended to $L^2(\mathbb{R}^n)$.

Sometimes we need generalized symbols $p(\hbar; x, y, \xi)$ leading to pseudo differential operators

$$\operatorname{Op}(p)u(x) := \frac{1}{(2\pi\hbar)^n} \int\limits_{\mathbb{R}^{2n}} e^{i(x-y)/\hbar} p(\hbar; x, y, \xi) u(y)\, dy\, d\xi\,. \tag{3.2}$$

We keep the notation $\mathcal{S}^m$ for this kind of symbols which means that in Definition 3.1 the space variable $x \in \mathbb{R}^n$ has to be replaced by $(x, y) \in \mathbb{R}^{2n}$. The following (well known) proposition shows how to calculate a Weyl symbol associated to a generalized symbol.

Proposition 3.1 *Let $p \in \mathcal{S}^m$. Then there exists a symbol $q \in \mathcal{S}^m$ such that*

$$Op(p)u = Op^W(q)u \qquad (u \in L^2(\mathbb{R}^n))$$

and for all $N \in \mathbb{N}$

$$q(\hbar; x, \xi) = \sum_{|\alpha| \le N-1} \frac{\hbar^{|\alpha|}}{i^{|\alpha|}\alpha!} \partial_y^\alpha \partial_\eta^\alpha p(\hbar; x + \tfrac{y}{2}, x - \tfrac{y}{2}, \xi - \eta)\big|_{y=\eta=0} + r_N(\hbar; x, \xi)$$

with $r_N \in \mathcal{S}^{m+N}$.

3.2 Estimates for pseudo differential operators with symbol in a Gevrey class

The following proposition is the main technical tool to prove a non-analytic version of the key estimate (1.2). It contains all the parts which are related to our Gevrey class assumptions. The statement can be summarized as follows:

A pseudo differential operator with exponential weighted symbol $e^{g(x,y)/\hbar^{1/a}} p$, can be rewritten (modulo $\mathcal{O}(\hbar^\infty)$) as an operator with bounded symbol q. The essential assumption is the boundedness of the first derivatives of g, where the bounds depend on the chosen Gevrey class of p. The leading order of q (w.r.t. $\hbar$) is explicitly given in terms of p and g, all other terms can be calculated with help of Proposition 3.1.

Proposition 3.2 *Let $a \in (1, \infty)$, $b > 0$ and $P = Op(p)$ be a pseudo differential operator with symbol $p \in \mathcal{S}^m$ satisfying*

$$\partial_x^\alpha \partial_y^\beta p(x, y, \cdot) \in \Gamma^{a,b}(\mathbb{R}^n_\xi, \mathbb{R})$$

for all $\alpha, \beta \in \mathbb{N}^n$ and $x, y \in \mathbb{R}^n$. Let furthermore $\Gamma_0 > 0$ and $g \in C^\infty(\mathbb{R}^{2n}_{x,y})$ be a weight function satisfying

1. *all derivatives of* g *are bounded,*

2. $g(x,x) = 0 \quad (x \in \mathbb{R}^n)$,

3. $|\partial_x g(x,y)| < \kappa$, $|\partial_y g(x,y)| < \kappa \quad (x,y \in \mathbb{R}^n)$ *with* $\kappa := b^{-1/a}\left(\frac{a-1}{\Gamma_0}\right)^{1-1/a}$,

4. $|g(x,y)| \leq (\sqrt{n})^{-1/a}\kappa\,\Gamma_0\,(1 + \frac{1}{a}\ln\frac{|x-y|}{\Gamma_0}) \quad (|x-y| \geq \Gamma_0)$,

then there exists a symbol $q \in \mathcal{S}^m$ *(the Weyl-symbol related to* b *of equation (3.12)) such that*

$$\|Op(e^{g(x,y)/\hbar^{1/a}}p) - Op^W(q)\|_{\mathcal{B}(L^2(\mathbb{R}^n))} = \mathcal{O}(\hbar^\infty)\,. \tag{3.3}$$

Furthermore

$$q(\hbar; x, \xi) - \mathcal{F}^{a,b,\omega}p(\hbar, x, x, \xi + i\hbar^{1-1/a}\tilde{g}(x)) \in \mathcal{S}^{m+1}\,, \tag{3.4}$$

where $\mathcal{F}^{a,b,\omega}p$ *is the almost analytic extension of* p *(see Definition 2.2) with respect to the variable* ξ*, defined in a strip* S_ω *with* $\omega = \hbar^{1-1/a}\kappa$ *and*

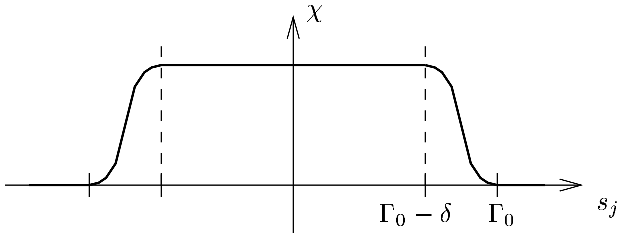
$$\tilde{g}(x) = \tfrac{1}{2}(\partial_1 g(x,x) - \partial_2 g(x,x))\,.$$

Proof: Let $u \in C_0^\infty(\mathbb{R}^n)$ and choose $\delta > 0$ in such a way that

$$(a-1)^{1-1/a}\Gamma_0^{1/a} < a(\Gamma_0 - \delta)^{1/a}\,. \tag{3.5}$$

This is always possible since for all $a > 1$ we can show by an elementary calculation that $(a-1)^{1-1/a} < a$.

Let $\chi \in C_0^\infty(\mathbb{R}^n)$ be a cutoff function with $\operatorname{supp}\chi \subset \{|s| < \Gamma_0\}$, $0 \leq \chi \leq 1$ and $\chi(s) = 1$ for $|s| < \Gamma_0 - \delta$.



We split the oscillating integral

$$\mathrm{Op}(e^{g(x,y)/\hbar^{1/a}}p)u = (2\pi\hbar)^{-n} \int\limits_{\mathbb{R}^{2n}} e^{i(x-y)\xi/\hbar + g(x,y)/\hbar^{1/a}} p(\hbar; x, y, \xi) u(y)\, dy\, d\xi$$

into two parts by inserting $(1-\chi)(x-y)$ and $\chi(x-y)$ respectively. These parts will be denoted by $A_1 u$ and $A_2 u$.

Examination of A_1:
The critical points of the oscillating phase are given by $x = y = 0$. Using $(1-\chi)(x-y)$ we integrate only over non-critical points. So we can use the operator

$$L := -i\hbar \frac{x-y}{|x-y|^2} \partial_\xi$$

for partial integration. For all $N \in \mathbb{N}$ we have

$$(1 - L^N) e^{i(x-y)\xi/\hbar} = 0\,.$$

Partial integration leads to

$$\begin{aligned} A_1 u &= \\ &= (2\pi\hbar)^{-n} \int\limits_{\mathbb{R}^{2n}} e^{i(x-y)\xi/\hbar + g(x,y)/\hbar^{1/a}} (1-\chi)(x-y) (L^t)^N p(\hbar; x, y, \xi) u(y)\, dy\, d\xi \\ &=: (2\pi\hbar)^{-n} \int\limits_{\mathbb{R}^{2n}} e^{i(x-y)\xi/\hbar} q_N(\hbar; x, y, \xi) u(y)\, dy\, d\xi \end{aligned}$$

with $L^t = -L$. In the next step we will make a suitable choice of $N = N(\hbar)$ such that the symbol q_N satisfies

$$|\partial^\alpha_{x,y,\xi} q_N(\hbar; x, y, \xi)| \le c_\alpha \hbar^l \tag{3.6}$$

for all $l \in \mathbb{N}$ and a finite number of derivatives $\alpha \in \mathbb{N}^{3n}$, $|\alpha| < M$. Using the theorem of Calderon-Vaillancourt this implies

$$\|A_1\|_{\mathcal{B}(L^2(\mathbb{R}^n))} = \mathcal{O}(\hbar^\infty)\,. \tag{3.7}$$

For the proof of (3.6) we use the fact that the derivatives of q_N with respect to x, y and ξ act on

1. $(1-\chi)(x-y)$,
2. $e^{g(x,y)/\hbar^{1/a}}$ and
3. $(L^t)^N p(\hbar; x, y, \xi)$.

By assumption, all the derivatives of χ and g are bounded so that the first term can be estimated by a constant and the second term by

$$\mathrm{const}\, \hbar^{-M/a} e^{g(x,y)/\hbar^{1/a}} .$$

To estimate the third term we first commute the x, y and ξ derivatives with $(L^t)^N$. By the assumptions on p and the stability of the Gevrey classes, we get

$$f(x,y,\cdot) := \partial^\alpha_{x,y,\xi} p(\hbar; x, y, \cdot) \in \Gamma^{a,b}(\mathbb{R}^n_\xi) .$$

Therefore we can estimate the derivatives of f (all of Nth order) in

$$|(L^t)^N f| \le \frac{\hbar^N}{|x-y|^{2N}} \sum_{j_1,\dots,j_N=1}^{n} |x_{j_1} - y_{j_1}| \cdot \ldots \cdot |x_{j_N} - y_{j_N}|\, |\partial_{\xi_{j_1}} \cdots \partial_{\xi_{j_N}} f|$$

by

$$c_0 (\beta + \mathbb{1})^{c_0} b^{|\beta|} \beta!^a \le c_0 (N+1)^{nc_0} b^N N!^a . \tag{3.8}$$

Writing $d_j := x_j - y_j$ we get

$$\begin{aligned} |d|^{-N} & \sum_{j_1,\dots,j_N=1}^{n} |d_{j_1}| \cdots |d_{j_N}| = \\ &= \left(\sum_{j_1,\dots,j_N=1}^{n} d^2_{j_1} \cdots d^2_{j_N} \right)^{-1/2} \sum_{j_1,\dots,j_N=1}^{n} |d_{j_1}| \cdots |d_{j_N}| \\ &\le \Big(\#\{ j \in \mathbb{N}\{0\}^N \mid 1 \le j_k \le n \} \Big)^{1/2} \;=\; n^{N/2} . \end{aligned} \tag{3.9}$$

(3.8), (3.9) and Stirlings formula imply

$$\begin{aligned} |(L^t)^N f| &\le \frac{\hbar^N}{|x-y|^N} c_0 (N+1)^{nc_0} (\sqrt{n} b)^N N!^a \\ &\le c_0 (N+1)^{nc_0 + 1/2} \frac{\hbar^N}{|x-y|^N} (\sqrt{n} b e^{-a})^N N^{aN} \\ &=: c_0 (N+1)^{nc_0 + 1/2} \Big(\frac{\Gamma_0 - \delta}{|x-y|} \Big)^N \exp(h(N)) \end{aligned}$$

with

$$h(N) = N \ln\Big(\frac{\sqrt{n} b \hbar N^a}{(\Gamma_0 - \delta) e^a}\Big)\,.$$

The minimum of h is given by

$$\begin{aligned} N_0 &= \Big(\frac{\Gamma_0 - \delta}{\sqrt{n} b \hbar}\Big)^{1/a}\,, \\ h(N_0) &= -a\Big(\frac{\Gamma_0 - \delta}{\sqrt{n} b \hbar}\Big)^{1/a}\,. \end{aligned}$$

Choosing $N = [N_0]$ and using $N \sim \hbar^{-1/a}$ as $\hbar \to 0$ we can estimate for small $\hbar$:

$$|(L^t)^N f| \leq \mathrm{const}\, \hbar^{-\mathrm{const}} \exp\Bigg(- \Big(a + \ln\frac{|x-y|}{\Gamma_0 - \delta}\Big)\Big(\frac{\Gamma_0 - \delta}{\sqrt{n} b \hbar}\Big)^{1/a}\Bigg)\,.$$

Summarizing the estimates for the three terms in consideration we get

$$\begin{aligned} |\partial^\alpha q_N| &\leq \mathrm{const}\, \hbar^{-\mathrm{const}} (1-\chi)(x-y) \;\times \\ &\times\; \exp\Bigg(\Big(g(x,y) - \Big(a + \ln\frac{|x-y|}{\Gamma_0 - \delta}\Big)\Big(\frac{\Gamma_0 - \delta}{\sqrt{n} b}\Big)^{1/a}\Big)\hbar^{-1/a}\Bigg) \end{aligned} \tag{3.10}$$

For $\Gamma_0 - \delta \leq |x-y| \leq \Gamma_0$ the assumptions 2 and 3 imply

$$\begin{aligned} |g(x,y)| &= |g(x,y) - g(x,x)| \;\leq\; \sup_t |\partial_t g(x,t)|\, |x-y| \\ &< \kappa \Gamma_0 \;=\; (a-1)^{1-1/a} (\Gamma_0/b)^{1/a}\,. \end{aligned}$$

Using (3.5) we get

$$|g(x,y)| < a\Big(\frac{\Gamma_0 - \delta}{b}\Big)^{1/a} \leq \Big(a + \ln\frac{|x-y|}{\Gamma_0 - \delta}\Big)\Big(\frac{\Gamma_0 - \delta}{\sqrt{n} b}\Big)^{1/a}\,.$$

Considering points $|x-y| \geq \Gamma_0$ we can use assumption 4 and again estimate (3.5). This leads to

$$\begin{aligned} |g(x,y)| &\leq (\sqrt{n})^{-1/a} \kappa\, \Gamma_0 \Big(1 + \frac{1}{a} \ln\frac{|x-y|}{\Gamma_0}\Big) \\ &< \Big(\frac{\Gamma_0 - \delta}{\sqrt{n} b}\Big)^{1/a}\Big(a + \ln\frac{|x-y|}{\Gamma_0 - \delta}\Big)\,. \end{aligned}$$

So there exists an $\varepsilon > 0$ such that the exponential term in (3.10) can be replaced by $e^{-\varepsilon/\hbar^{1/a}}$. This term decays faster in $\hbar$ as any power of $\hbar$ does, so we get estimate (3.6) which implies (3.7).

Examination of A_2:
We write

$$A_2 u = (2\pi\hbar)^{-n} \int\limits_{\mathbb{R}^{2n}} \chi(x-y) e^{i(x-y)(\xi - i\hbar^{1-1/a} d(x,y))/\hbar} p(\hbar; x, y, \xi) u(y) \, dy \, d\xi$$

with

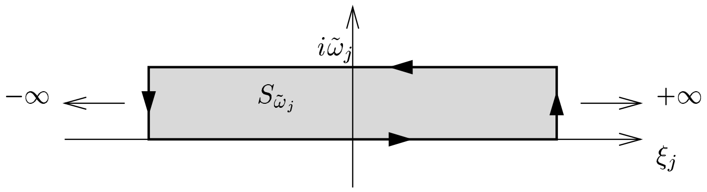
$$d(x,y) := \frac{1}{2} \int\limits_0^1 \big(\partial_1 g(y + t(x-y), y) - \partial_2 g(x, x + t(y-x)) \big) \, dt \, .$$

Using $g(x,x) = 0$ this implies

$$(x-y) \cdot d(x,y) = g(x,y) \, .$$

For fixed x and y we make the following change of path of integration:

$$\xi \mapsto \xi + i\hbar^{1-1/a} d(x,y) =: \xi + i\tilde{\omega}(x,y) \, .$$



Let $I_{\tilde{\omega}} := [0, \tilde{\omega}_1] \times \ldots \times [0, \tilde{\omega}_n]$ and $S_{\tilde{\omega}} := \mathbb{R}^n_t \times I_{\tilde{\omega}} \subset \mathbb{C}^n$. By assumption 3 we have

$$|d(x,y)| \leq \tfrac{1}{2} (\sup |\partial_x g| + \sup |\partial_y g|) < \kappa$$

or

$$|\tilde{\omega}| < \hbar^{1-1/a} \kappa = \omega \, .$$

Let $\tilde{p} := \mathcal{F}^{a,b,\omega} p$ be the extension of p to the strip $\xi \in S_\omega \supset S_{|\tilde{\omega}|}$. A_2 has to be understood as an oscillating integral. This implies that the parts $\operatorname{Re} \xi \to \pm\infty$, $\operatorname{Im} \xi \in I_{\tilde{\omega}}$ of a contour integral over $\partial S_{\tilde{\omega}_j}$ vanish. On the other hand such a contour integral can be replaced by an integral over $S_{\tilde{\omega}_j}$ where

$\tilde{p}$ has to be replaced by its anti-holomorphic derivative $\overline{\partial}_{\xi_j}\tilde{p}$. Successively shifting $\xi_1 \mapsto \xi_1 + i\tilde{\omega}_1$ up to $\xi_n \mapsto \xi_n + i\tilde{\omega}_n$ we obtain:

$$
\begin{aligned}
A_2 u &= (2\pi\hbar)^{-n} \int\limits_{\mathbb{R}^{2n}} e^{i(x-y)\xi/\hbar} \chi(x-y)\tilde{p}(\hbar; x, y, \xi + i\tilde{\omega}) u(y)\, dy\, d\xi \\
&+ (2\pi\hbar)^{-n} \int\limits_{\mathbb{R}^n_y} \int\limits_{\mathbb{R}^n_\xi} \int\limits_0^{\tilde{\omega}_1} e^{i(x-y)(\xi + i(s_1, 0, \dots, 0) - i\tilde{\omega})/\hbar} \chi(x-y) \times \\
&\qquad \times \overline{\partial}_{\xi_1}\tilde{p}(\hbar; x, y, \xi + i(s_1, 0, \dots, 0)) u(y)\, ds_1\, dy\, d\xi \\
&+ (2\pi\hbar)^{-n} \int\limits_{\mathbb{R}^n_y} \int\limits_{\mathbb{R}^n_\xi} \int\limits_0^{\tilde{\omega}_2} e^{i(x-y)(\xi + i(\tilde{\omega}_1, s_2, 0, \dots, 0) - i\tilde{\omega})/\hbar} \chi(x-y) \times \\
&\qquad \times \overline{\partial}_{\xi_2}\tilde{p}(\hbar; x, y, \xi + i(\tilde{\omega}_1, s_2, 0, \dots, 0)) u(y)\, ds_2\, dy\, d\xi \\
&+ \dots + \\
&+ (2\pi\hbar)^{-n} \int\limits_{\mathbb{R}^n_y} \int\limits_{\mathbb{R}^n_\xi} \int\limits_0^{\tilde{\omega}_n} e^{i(x-y)(\xi + i(\tilde{\omega}_1, \dots, \tilde{\omega}_{n-1}, s_n) - i\tilde{\omega})/\hbar} \chi(x-y) \times \\
&\qquad \times \overline{\partial}_{\xi_n}\tilde{p}(\hbar; x, y, \xi + i(\tilde{\omega}_1, \dots, \tilde{\omega}_{n-1}, s_n)) u(y)\, ds_n\, dy\, d\xi \\
&=: A_{20}u + A_{21}u + \dots + A_{2n}u\,.
\end{aligned}
$$

With $s^{(j)} := (\tilde{\omega}_1, \dots .\tilde{\omega}_{j-1}, s_j, 0, \dots, 0)$ we can interpret A_{2j} as a pseudo differential operator with symbol

$$
a_j(\hbar; x, y, \xi) = \int\limits_0^{\tilde{\omega}_j} e^{(x-y)(\tilde{\omega} - s^{(j)})/\hbar} \chi(x-y) \overline{\partial}_{\xi_j}\tilde{p}(\hbar; x, y, \xi + is^{(j)})\, ds_j
$$

for $j = 1, \dots, n$.

In the same manner as already done for A_1 we will now estimate the derivatives of these symbols up to a finite order M. Application of the theorem of Calderon-Vaillancourt afterwards gives

$$
\sum_{j=1}^{n} \|A_{2j}\|_{\mathcal{B}(L^2(\mathbb{R}^n))} = \mathcal{O}(\hbar^\infty)\,.
$$

First we notice that $\tilde{\omega}_j$ depends on x and y. So the boundaries of the integrals defining our symbols are functions of x and y. In view of

$$F(x) = \int_0^{g(x)} f(x,s)\, ds \;\Rightarrow\; F'(x) = \int_0^{g(x)} \partial_x f(x,s)\, ds + f(x, g(x))g'(x)\,,$$

we notice that derivatives of a_j with respect to x, y and ξ can be written as derivatives of their integrands, some of them evaluated at $\tilde{\omega}_j(x,y)$, sometimes multiplied by derivatives of $\tilde{\omega}_j$ and integrated over some s_j.

To estimate the derivatives of a_j we use the facts that

- all derivatives of χ and $\tilde{\omega} = \hbar^{1-1/a} d$ are bounded,
- by definition of the extension $\mathcal{F}^{a,b,\omega}$ we have

  $$\partial_\xi^\alpha (\mathcal{F}^{a,b,\omega} p(\hbar; x, y, \xi + is^{(j)})) = \mathcal{F}^{a,b,\omega}(\partial_\xi^\alpha p)(\hbar; x, y, \xi + is^{(j)}))\,.$$

  and the same for ∂_x^α und ∂_y^α. In view of the stability of the Gevrey classes we can use Proposition 2.2 to estimate the anti-holomorphic derivatives of $\partial^\alpha \tilde{p}$.
- $|\tilde{\omega} - s^{(j)}| \leq |\tilde{\omega}|$ for all $s_j \in [0, \tilde{\omega}_j]$. Since $\chi(x-y) = 0$ if $|x-y| \geq \Gamma_0$ this implies

  $$\chi(x-y) e^{(x-y)(\tilde{\omega} - s^{(j)})/\hbar} \leq \chi(x-y) e^{|x-y|\, |\tilde{\omega} - s^{(j)}|/\hbar} \leq e^{\Gamma_0 |\tilde{\omega}|/\hbar}\,.$$
- derivatives of the exponential terms produce at most M negative powers of $\hbar$.

So we get for all $|\alpha| \leq M$ and $\hbar$ sufficiently small:

$$\begin{aligned} |\partial_{x,y,\xi}^\alpha a_j| &\leq \text{const}\, \hbar^{-M} e^{\Gamma_0 |\tilde{\omega}|/\hbar} \omega^{-\text{const}} \exp\Big(-\frac{B}{\omega^{1/(a-1)}}\Big) \\ &\leq \text{const}\, \hbar^{-\text{const}} \exp\Big(\Gamma_0 |\tilde{\omega}|/\hbar - (a-1)^{1-1/a} \Big(\frac{\Gamma_0}{b\hbar}\Big)^{1/a}\Big). \quad (3.11) \end{aligned}$$

For (3.11) we used $\tilde{\omega}_j = \mathcal{O}(\hbar^{1-1/a})$, $\omega = \hbar^{1-1/a}\kappa$, the definition of κ and the definition of B from Proposition 2.2.

On the other hand, we have

$$\Gamma_0 |\tilde{\omega}|/\hbar < \Gamma_0 \hbar^{-1/a} \kappa = (a-1)^{1-1/a} \Big(\frac{\Gamma_0}{b\hbar}\Big)^{1/a}\,.$$

Therefore (as we did for A_1) the exponent of (3.11) can be estimated by a negative constant $-\varepsilon$, and this exponential decay also absorbs negative powers of $\hbar$. Finally we get

$$|\partial^\alpha_{x,y,\xi} a_j| \leq \text{const}\, \hbar^l$$

for all $l \in \mathbb{N}$ and thus

$$\sum_{j=1}^n \|A_{2j}\|_{\mathcal{B}(L^2(\mathbb{R}^n))} = \mathcal{O}(\hbar^\infty)\,.$$

Examination of A_{20}:
We rewrite A_{20} as a pseudo differential operator

$$A_{20}u = (2\pi\hbar)^{-n} \int e^{i(x-y)\xi/\hbar} b(\hbar; x, y, \xi) u(y)\, dy\, d\xi$$

with symbol

$$b(\hbar; x, y, \xi) = \chi(x-y)\, \mathcal{F}^{a,b,\omega} p(\hbar; x, y, \xi + i\hbar^{1-1/a} d(x,y))\,. \tag{3.12}$$

By the assumptions on p we have $b \in \mathcal{S}^m$ (the derivatives of d are bounded). Let $q \in \mathcal{S}^m$ be the associated Weyl-symbol for b, i.e. $A_{20}u = \mathrm{Op}^W(q)u$ (see Proposition 3.1). Then the leading order of q is given by

$$b(\hbar; x, x, \xi) = \mathcal{F}^{a,b,\omega} p(\hbar; x, x, \xi + i\hbar^{1-1/a} d(x,x))$$

with

$$d(x,x) = \frac{1}{2}(\partial_1 g(x,x) - \partial_2 g(x,x))\,.$$

This corresponds with (3.4).

Putting together all the terms, we finally obtain

$$\mathrm{Op}(e^{g(x,y)/\hbar^{1/a}} p)u = \mathrm{Op}^W(q)u + (A_1 + \sum_{j=1}^n A_{2j})u$$

with $\|A_1 + \sum_{j=1}^n A_{2j}\|_{\mathcal{B}(L^2(\mathbb{R}^n))} = \mathcal{O}(\hbar^\infty)$. This finishes the proof of Proposition 3.2. ■

Remark: Proposition 3.2 can be slightly generalized: If the weight function g actually only depends on $m < n$ components of the variables x and y

respectively, $\sqrt{n}$ in assumption 4 can be replaced by $\sqrt{m}$. The proof has to be modified by using a cutoff function χ which only depends on these m components. Furthermore the integration by parts has to be done using an operator L containing only derivatives w.r.t. to these m components. This is sufficient since $g(x, y)$ is constant with respect to the other $n - m$ variables such that the corresponding components of $\partial_x g$ and $\partial_y g$ do not cause an exponential increase which otherwise has to be compensated.

3.3 Microlocal Estimates

This section contains the main result of this paper. It is the Gevrey class version of the key estimate (1.2) in phase space tunneling. It can be understood as an Agmon estimate in phase space. Section 4 contains an application obtaining exponential decay (in $\hbar^{1/a}$) for eigenfunctions of pseudo differential operators inside the classically forbidden region.

Theorem 3.1 *Let $a \in (1, \infty)$, $b > 0$ and $P = Op^W(p)$ be a pseudo differential operator with symbol $p \in \mathcal{S}^m \cap \Gamma^{a,b}(\mathbb{R}^{2n})$. Let furthermore $\Gamma_0 > 0$ and $g \in C^\infty(\mathbb{R}^{2n})$ be a weight function satisfying*

- *all derivatives of g are bounded,*
- $\sup\limits_{(x,\xi)\in\mathbb{R}^{2n}} |\partial g(x, \xi)| < (\sqrt{2n})^{-1/a}\kappa$ *with* $\kappa = b^{-1/a}\left(\frac{a-1}{\Gamma_0}\right)^{1-1/a}$,
- $\operatorname{supp} \partial g \subset \{(x, \xi) \in \mathbb{R}^{2n} \mid |(x, \xi)| \le \Gamma_0/2\}$,

then there exist $q \in \mathcal{S}^m$ and $R_N \in \mathcal{B}(L^2(\mathbb{R}^n))$ (for all $N \in \mathbb{N}$) such that for all $u \in \mathcal{S}(\mathbb{R}^n)$ and $\hbar$ sufficiently small

$$\langle e^{g/\hbar^{1/a}}Tu, e^{g/\hbar^{1/a}}TPu\rangle = \langle e^{g/\hbar^{1/a}}Tu, (q(\hbar; x, \xi) + R_N)e^{g/\hbar^{1/a}}Tu\rangle \qquad (3.13)$$

with $\|R_N\| = \mathcal{O}(\hbar^N)$,

$$q(\hbar; x, \xi) - \mathcal{F}^{a,b,\omega}p(\hbar; x - \hbar^{1-1/a}\tilde{g}_\mu(x, \xi), \xi + i\hbar^{1-1/a}\mu\tilde{g}_\mu(x, \xi)) \in \mathcal{S}^{m+1} \qquad (3.14)$$

and $\omega = \hbar^{1-1/a}\kappa$. $\mathcal{F}^{a,b,\omega}p$ is the almost analytic extension to the strip S_ω given by Definition 2.2 in dimension $2n$ and $\tilde{g}_\mu$ is given by

$$\tilde{g}_\mu := \mu^{-1}\partial_x g + i\partial_\xi g\,.$$

Remarks

- (3.13) and (3.14) imply (for $m = 0$)

$$\begin{aligned}&\langle e^{g/\hbar^{1/a}}Tu, e^{g/\hbar^{1/a}}TPu\rangle = \\ &\quad \langle e^{g/\hbar^{1/a}}Tu, \mathcal{F}^{a,b,\omega}p(\hbar; x - \hbar^{1-1/a}\tilde{g}_\mu, \xi + i\hbar^{1-1/a}\mu\tilde{g}_\mu)e^{g/\hbar^{1/a}}Tu\rangle \\ &\quad +\mathcal{O}(\hbar)\|e^{g/\hbar^{1/a}}Tu\|^2\,,\end{aligned}$$

  which should be compared to formula (1.2) of the analytic case.

- Notice that by the assumptions on the bounds of ∂g we have $|\operatorname{Im}(\hbar^{1-1/a}\tilde{g}_\mu)| < \omega$ and $|\operatorname{Im}(i\hbar^{1-1/a}\tilde{g}_\mu)| < \omega$ so that the expression (3.14) is well defined.

The proof of Theorem 3.1 follows closely the outline of the corresponding proof found in [10] for the analytic case. It can be divided into three steps which will be formulated in the next three lemmata. Only the second step uses regularity conditions on the symbol p and has to be redone for the case of Gevrey classes. For this step we use the results of section 3.2. For the proof of the other two steps we refer to [10]. Using the notation $D_x := -i\partial_x$ we can formulate:

Lemma 3.3 *Let $p \in \mathcal{S}^m$ and $P = Op^W(p)$, then*

$$TP = \tilde{P}T \tag{3.15}$$

with $\tilde{P} = Op^W(\tilde{p}) = \tilde{p}(\hbar; x, \xi, \hbar D_x, \hbar D_\xi)$ and

$$\tilde{p}(\hbar; x, \xi, x^*, \xi^*) = p(\hbar; x/2 - \xi^*, \xi/2 + x^*) \in \mathcal{S}^m\,. \tag{3.16}$$

Remark: Note that the operator $\tilde{P}$ located to the left of T, acts on $L^2(\mathbb{R}^{2n}_{x,\xi})$. So the symbol $\tilde{p}$ is a function of phase space variables $(x, \xi, x^*, \xi^*) \in \mathbb{R}^{2n} \times \mathbb{R}^{2n}$ where the star denotes the corresponding dual variables.
Proof: Equation (3.15) follows from an explicit calculation as done in [10]. ■

Lemma 3.4 *Let $a \in (1, \infty)$, $b > 0$, $\tilde{p} \in \mathcal{S}^m$ and $\tilde{P} = Op^W(\tilde{p})$ with*

$$\partial_x^\alpha \tilde{p}(x, \cdot) \in \Gamma^{a,b}(\mathbb{R}^n_\xi)$$

for all $\alpha \in \mathbb{N}^n$ und $x \in \mathbb{R}^n$. Let furthermore $\Gamma_0 > 0$ and $g \in C^\infty(\mathbb{R}^n_x)$ be a weight function satisfying

- *all derivatives of g are bounded,*
- $\sup_{x\in\mathbb{R}^n} |\partial g(x)| < (\sqrt{n})^{-1/a}\kappa$ *with κ as defined in Theorem 3.1,*
- $\operatorname{supp}\partial g \subset \{x \in \mathbb{R}^n \mid |x| \leq \Gamma_0/2\}$,

then there exist $\tilde{q} \in \mathcal{S}^m$, $\tilde{Q} := Op^W(\tilde{q})$ and $R_N \in \mathcal{B}(L^2(\mathbb{R}^n))$ (for all $N \in \mathbb{N}$) such that for all $\hbar$ sufficiently small

$$e^{g/\hbar^{1/a}}\tilde{P}e^{-g/\hbar^{1/a}} = \tilde{Q} + R_N$$

where $\|R_N\| = \mathcal{O}(\hbar^N)$ and

$$\tilde{q}(\hbar;x,\xi) - \mathcal{F}^{a,b,\omega}\tilde{p}(\hbar;x,\xi + i\hbar^{1-1/a}\partial g(x)) \in \mathcal{S}^{m+2} \tag{3.17}$$

with $\omega = \hbar^{1-1/a}\kappa$.

Proof: We use

$$e^{g/\hbar^{1/a}}\tilde{P}e^{-g/\hbar^{1/a}} = \mathrm{Op}(e^{(g(x)-g(y))/\hbar^{1/a}}\tilde{p})$$

and apply Proposition 3.2 with weight function $\tilde{g}(x,y) := g(x) - g(y)$ and symbol $p(x,y,\xi) := \tilde{p}(\frac{x+y}{2},\xi)$. The assumptions on g assure that $\tilde{g}$ fulfills the assumptions of Proposition 3.2. To verify assumption 4 we can estimate

$$|\tilde{g}(x,y)| \leq \max_{\substack{|x|\leq\Gamma_0/2\\|y|\leq\Gamma_0/2}} |g(x)-g(y)| \leq \max_{|t|\leq\Gamma_0/2} |\partial g(t)|\Gamma_0 < (\sqrt{n})^{-1/a}\kappa\Gamma_0\,.$$

Proposition 3.2 then implies

$$\|e^{g/\hbar^{1/a}}\tilde{P}e^{-g/\hbar^{1/a}} - \mathrm{Op}^W(\tilde{q})\|_{\mathcal{B}(L^2(\mathbb{R}^n))} = \mathcal{O}(\hbar^\infty)\,.$$

with

$$\tilde{q}(\hbar;x,\xi) - \mathcal{F}^{a,b,\omega}\tilde{p}(\hbar,x,\xi + i\hbar^{1-1/a}\partial\tilde{g}(x))) \in \mathcal{S}^{m+1}\,. \tag{3.18}$$

To show that the difference (3.18) is in $\mathcal{S}^{m+2} \subset \mathcal{S}^{m+1}$ first notice that $\tilde{q}$ is given by the Weyl symbol that corresponds to $b(\hbar,x,y,\xi)$ of equation (3.12). Since here $p(x,y,\xi) = \tilde{p}(\frac{x+y}{2},\xi)$ it follows that $b(x,y,\xi)$ is symmetrically w.r.t. to an exchange of x and y. Thus the term of order $\hbar^{m+1}$ vanishes according to Proposition 3.1. ■

Lemma 3.5 *Let* $\tilde{q} \in \mathcal{S}^m$ *and* $\tilde{Q} = Op^W(\tilde{q}) = \tilde{q}(\hbar; x, \xi, \hbar D_x, \hbar D_\xi)$. *Let furthermore* $g \in C^\infty(\mathbb{R}^{2n}_{x,\xi})$ *be a weight function with bounded derivatives, then there exist* $q = q(\hbar; x, \xi) \in \mathcal{S}^m$ *and* $R_N \in \mathcal{B}(L^2(\mathbb{R}^n))$ *(for all* $N \in \mathbb{N}$*) such that*

$$\langle e^{g/\hbar^{1/a}} Tu, \tilde{Q} e^{g/\hbar^{1/a}} Tu \rangle = \langle e^{g/\hbar^{1/a}} Tu, (q(\hbar; x, \xi) + R_N) e^{g/\hbar^{1/a}} Tu \rangle \qquad (3.19)$$

with

$$\begin{aligned} &q(\hbar; x, \xi) - && (3.20) \\ &- \tilde{q}(\hbar; x, \xi, \xi/2 - \mu\hbar^{1-1/a}\partial_\xi g(x, \xi), -x/2 + \mu^{-1}\hbar^{1-1/a}\partial_x g(x, \xi)) \in \mathcal{S}^{m+1} \end{aligned}$$

and $\|R_N\| = \mathcal{O}(\hbar^N)$.

Remark: In (3.19) q acts as multiplication operator: Located to the left of T it acts on phase space $\mathbb{R}^{2n} \times \mathbb{R}^{2n}$ with (x, ξ) being position variables.
Proof: The proof can be adopted from [10]. There the lemma is formulated using a weight $e^{\psi/\hbar}$. Analyticity of q is not used. Just setting

$$\psi = \hbar^{1-1/a} g$$

we can follow step by step the arguments in [10]. Since $a > 1$ all derivatives of ψ are uniformly bounded for $\hbar < 1$. ■

Proof of Theorem 3.1:
To estimate $\langle e^{g/\hbar^{1/a}} Tu, e^{g/\hbar^{1/a}} TPu \rangle$ we first use Lemma 3.3 to commute P with T:

$$TP = \tilde{P} T \, .$$

Then we apply Lemma 3.4 in dimension $2n$ with variables $\tilde{x} = (x, \xi)$ and $\tilde{\xi} = (x^*, \xi^*)$. The assumptions on p and $g(x, \xi)$ of Theorem 3.1 assure that $\tilde{p}$ and $g(\tilde{x})$ fulfill the assumptions of Lemma 3.4:

- For g this is trivially seen.
- Since $\tilde{p}(\hbar; \tilde{x}, \cdot)$ is obtained by shifting the arguments of $p(\hbar; \cdot, \cdot)$ by $\tilde{x}_1/2$ resp. $\tilde{x}_2/2$ and $p \in \Gamma^{a,b}(\mathbb{R}^{2n})$ by assumption of Theorem 3.1, we get, using the stability of $\Gamma^{a,b}$ under differentiation and translation,

$$\partial^\alpha_{\tilde{x}} \tilde{p}(\hbar; \tilde{x}, \cdot) \in \Gamma^{a,b}(\mathbb{R}^{2n}_{\tilde{\xi}}) \, .$$

Finally we apply Lemma 3.5: We rewrite $\tilde{Q}$ obtained from Lemma 3.4 acting on $e^{g/\hbar^{1/a}}T$ as a multiplication operator q plus an error term R_N. This leads to equation (3.13).

The error terms from Lemma 3.4 and 3.5 sum up to R_N from Theorem 3.1. Combining equations (3.16), (3.17) and (3.20) we end up with equation (3.14). ■

A statement similar to (3.13) of Theorem 3.1 can be formulated for the norm $\|e^{g/\hbar^{1/a}}TPu\|$. We shall use this result in the next section.

Theorem 3.1' *Under the same assumptions as in Theorem 3.1 there exists* $q \in \mathcal{S}^m$ *such that (3.13) can be replaced by*

$$\|e^{g/\hbar^{1/a}}TPu\| = \|q(\hbar; x, \xi)e^{g/\hbar^{1/a}}Tu\| + \mathcal{O}(\hbar^{m+1/2})\|e^{g/\hbar^{1/a}}Tu\| \qquad (3.21)$$

with

$$q(\hbar; x, \xi) = \mathcal{F}^{a,b,\omega}p(\hbar; x - \hbar^{1-1/a}\tilde{g}_\mu(x, \xi), \xi + i\hbar^{1-1/a}\mu\tilde{g}_\mu(x, \xi)) . \qquad (3.22)$$

Proof: Let $T_g := e^{g/\hbar^{1/a}}T$. We use Lemma 3.3 and Lemma 3.4 to obtain for arbitrary large N:

$$\begin{aligned} \|T_g Pu\| &= \|e^{g/\hbar^{1/a}}\tilde{P}Tu\| = \|(\tilde{Q} + R_N)T_g u\| \\ &= \|\tilde{Q}T_g u\| + \mathcal{O}(\hbar^N)\|T_g u\| . \end{aligned}$$

So it is sufficient to estimate

$$\|\tilde{Q}T_g u\|^2 = \langle \tilde{Q}T_g u, \tilde{Q}T_g u\rangle = \langle T_g u, \tilde{Q}^*\tilde{Q}T_g u\rangle .$$

According to the symbolic calculus we have $\tilde{Q}^*\tilde{Q} \in \mathcal{S}^{2m}$ and

$$\tilde{Q}^*\tilde{Q} = \mathrm{Op}^W(\bar{\tilde{q}})\mathrm{Op}^W(\tilde{q}) = \mathrm{Op}^W(\bar{\tilde{q}}\tilde{q}) + R_{2m+1}$$

with $R_{2m+1} \in \mathcal{S}^{2m+1}$. Now we apply Lemma 3.5 to $\mathrm{Op}^W(\bar{\tilde{q}}\tilde{q})$. Only considering the leading order terms w.r.t. $\hbar$ and using (3.16), (3.17), (3.20) we obtain:

$$\begin{aligned} \langle T_g u, \tilde{Q}^*\tilde{Q}T_g u\rangle &= \langle T_g u, ((\bar{q}q)(\hbar; x, \xi) + \tilde{R}_{2m+1})T_g u\rangle \\ &= \|q(\hbar; x, \xi)T_g u\|^2 + \mathcal{O}(\hbar^{2m+1})\|T_g u\|^2 \end{aligned}$$

with $\|\tilde{R}_{2m+1}\| = \mathcal{O}(\hbar^{2m+1})$ and q as given in (3.22). ■

4 Application

From Theorem 3.1 one gets an exponential decay of the eigenfunctions of a pseudo differential operator in the classically forbidden region. In the case of Gevrey classes this exponential decay is w.r.t. $\hbar^{1/a}$.

Definition 4.1 *Let $a \in [1, \infty)$. For $u \in L^2(\mathbb{R}^n)$ the micro support $MS_{\Gamma^a}(u)$ is the subset of phase space defined by*

$(x_0, \xi_0) \notin MS_{\Gamma^a}(u)$ iff there exist $\delta > 0$ and a neighbourhood U of (x_0, ξ_0) such that

$$Tu(x,\xi) = \mathcal{O}(e^{-\delta/\hbar^{1/a}}) \text{ uniformly in } (x,\xi) \in U\,.$$

Remark: This definition is adopted to the case of Gevrey classes by adding the index Γ^a. $a = 1$ gives the usual analytic case.

Definition 4.2 *Let $p \in \mathcal{S}^0$ be a symbol of the form*

$$p(\hbar; x, \xi) = \sum_{k=0}^{m} \hbar^k p_k(x,\xi)$$

then the characteristic of $P = Op^W(p)$ is defined by

$$Char(P) := \{(x,\xi) \in \mathbb{R}^{2n} \mid p_0(x,\xi) = 0\}\,.$$

Theorem 4.1 *Let $a \in (1, \infty)$, $b > 0$, $p \in \mathcal{S}^0 \cap \Gamma^{a,b}(\mathbb{R}^{2n})$, $P = Op^W(p)$ and $u \in L^2(\mathbb{R}^n)$ be an eigenfunction of P with zero eigenvalue:*

$$Pu = 0$$

then

$$MS_{\Gamma^a}(u) \subset Char(P)\,.$$

Remark: Theorem 4.1 is the generalization of estimate (1.3) mentioned in the introduction to the case of Gevrey classes.

Physical interpretation:

Consider for example a Schrödinger operator $P = -\hbar^2\Delta + V(x)$. Its symbol is the corresponding Hamilton function $p(x,\xi) = p_0(x,\xi) = \xi^2 + V(x)$. Theorem 4.1 tells us that the zero energy eigenfunctions are located in phase space at

$\{\xi^2 + V(x) = 0\}$. This is exactly where the classical trajectories are located. Outside this area Tu decays exponentially w.r.t. $\hbar^{1/a}$.

Proof of Theorem 4.1: Let $(x_0, \xi_0) \notin \mathrm{Char}\,(P)$, i.e. $p_0(x_0, \xi_0) \neq 0$. From continuity if follows that there exist a neighbourhood U of (x_0, ξ_0) and a constant $c > 0$ such that

$$|p_0 \restriction_U| \geq 2c\,.$$

Let $\tilde{p}_0 := \mathcal{F}^{a,b,\omega} p_0$ be the almost analytic extension of p_0 with $\omega = \hbar^{1-1/a}\kappa$ as in Theorem 3.1. We now apply Proposition 2.3. It follows that for sufficiently small $\hbar$ we can estimate

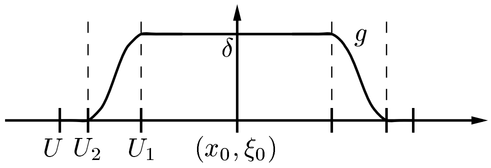
$$|\tilde{p}_0 \restriction_{\tilde{U}}| \geq c \tag{4.1}$$

with $\tilde{U} = \{(x, \xi) \in S_\omega \mid \mathrm{Re}(x, \xi) \in U\}$. We are now going to define a suitable weight function g. It will be constant in the a neighbourhood of (x_0, ξ_0) and will have sufficiently small derivatives:

Let $U_1 \subset U_2 \subset U$. For sufficiently small $\delta > 0$ we can find a function $g \in C_0^\infty(\mathbb{R}^{2n})$ with $\mathrm{supp}\, g \subset U_2$,

$$g \restriction_{U_1} = \delta \tag{4.2}$$

and $|\tilde{g}_\mu| = |\mu^{-1}\partial_x g + i\partial_\xi g| < \kappa$.



From $\omega = \hbar^{1-1/a}\kappa$ it follows that

$$\langle x - \hbar^{1-1/a}\tilde{g}_\mu(x, \xi), \xi + i\hbar^{1-1/a}\mu\tilde{g}_\mu(x, \xi)\rangle \in \tilde{U} \tag{4.3}$$

for all $(x, \xi) \in U_2$ and small $\hbar$. Now we apply Theorem 3.1' with $m = 0$. Using $Pu = 0$ we get

$$0 = \|\tilde{p}_0(x - \hbar^{1-1/a}\tilde{g}_\mu(x, \xi), \xi + i\hbar^{1-1/a}\mu\tilde{g}_\mu(x, \xi))e^{g/\hbar^{1/a}}Tu\| + \mathcal{O}(\sqrt{\hbar})\|e^{g/\hbar^{1/a}}Tu\|\,.$$

This implies

$$\|\tilde{p}_0(x - \hbar^{1-1/a}\tilde{g}_\mu(x, \xi), \xi + i\hbar^{1-1/a}\mu\tilde{g}_\mu(x, \xi))e^{g/\hbar^{1/a}}Tu\|_{L^2(U_2)} = \mathcal{O}(\sqrt{\hbar})\|e^{g/\hbar^{1/a}}Tu\|$$

and, using (4.1) and (4.3),

$$\|e^{g/\hbar^{1/a}}Tu\|_{L^2(U_2)} = \mathcal{O}(\sqrt{\hbar})\|e^{g/\hbar^{1/a}}Tu\|\,.$$

Let ${}^cU_2$ be the complement of U_2. Via bootstrap we obtain for small $\hbar$:

$$\|e^{g/\hbar^{1/a}}Tu\|_{L^2(U_2)} = \mathcal{O}(\sqrt{\hbar})\|e^{g/\hbar^{1/a}}Tu\|_{L^2({}^cU_2)}\,.$$

Finally using $g\restriction_{{}^cU_2}= 0$ and $U_1 \subset U_2$ we get

$$\|e^{g/\hbar^{1/a}}Tu\|_{L^2(U_1)} = \mathcal{O}(\sqrt{\hbar})\|Tu\|$$

and, with help of (4.2),

$$\|Tu\|_{L^2(U_1)} = \mathcal{O}(e^{-\delta/\hbar^{1/a}})\,.$$

Thus $(x_0, \xi_0) \notin \mathrm{MS}_{\Gamma^a}$ and Theorem 4.1 is proven. ■

Conclusion

A key estimate in the theory of phase space tunneling is proven for the case of a pseudo differential operator with a symbol in some Gevrey class, being hence not necessarily analytic. Explicit bounds are given for an exponential decay with respect to $\hbar^{1/a}$, where $a > 1$ depends on the smoothness of the symbol.

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