# Integrable deformations of affine surfaces via Nizhnik–Veselov–Novikov equation

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**Abstract**

It is shown that deformations of affine conormal according to the Nizhnik–Veselov–Novikov equation generate, via the Lelieuvre's formula, integrable deformations of affine surfaces. The total affine mean curvature is preserved by these deformations. It is shown that a stationary point of these deformations for centro-affine surfaces is given by the proper affine sphere.

## 1 Introduction

Theory of deformations of surfaces was an important part of the classical differential geometry (see e.g. [1,2]). Different types of deformations have been studied. New results in this field have been discussed, for instance, in [3,4].

Recently a new class of integrable deformations of surfaces has been proposed in [5]. It is based on the "construction" of surfaces via the linear equations with two independent variables and then on the generation of their deformations via the corresponding 2+1-dimensional integrable (soliton) equations. Generalized Weierstrass formulae and corresponding deformations via the modified Veselov–Novikov equation was one of the particular cases of this general scheme, considered in [5]. Several subsequent interesting results have been obtained in [6–10]. It occurs that the modified Veselov–Novikov deformations represent themself the subclass of integrable deformations of generic deformations of conformal immersions [10]. Other particular example discussed in [5] has been given by the Lelieuvre's formulae and corresponding deformations via the Nizhnik–Veselov–Novikov (NVN) equation.

In the present paper we show that these Lelieuvre–NVN deformations provide us integrable deformations of affine surfaces, the NVN wave-functions being the components of the affine conormal. They are characterized by an infinite set of preserved functionals. The total affine mean curvature is the simplest of them. Formulae for deformations of coordinates of affine surfaces are derived. It is shown that normal components of deformations are proportional to cubic forms of a surface. For ruled surfaces the particular deformations have zero normal components and, hence, they are reduced to a reparametrization of a surface.

Surfaces which correspond to stationary point of NVN deformations are described. For centro-affine surfaces it is given by proper affine sphere. Deformations of a special class of affine surfaces are generated by the Korteweg-de Vries equation. Correspondingly particular class of affine spheres is provided by stationary solutions of the Korteweg-de Vries equation.

The paper is organized as follows. The Lelieuvre's formula is presented in section 2. The Nizhnik–Veselov–Novikov (NVN) equation is described in section 3. Deformations of affine surfaces via the NVN equations are formulated in section 4. Normal and tangent parts of such deformations are considered in section 5. In section 6 it is shown that stationary points of the NVN deformations are given by affine spheres. Particular class of deformations via the Korteweg-de Vries equation is discussed in section 7.

## 2 Lelieuvre's formulae

Let $\mathbf{f}$: $M \rightarrow A_3$ be an affine surface in $A_3$. Assume that the affine Blasche metric $G$ in indefinite. Then in asymptotic coordinate system $(x, y)$ one has $G = 2F dx dy$ and $\Delta \mathbf{f} = \frac{2}{F} \mathbf{f}_{xy}$ where $\Delta$ is the Laplacian and $\mathbf{f}_{xy} \equiv \frac{\partial^2 \mathbf{f}}{\partial x \partial y}$ (see e.g. [11]). For the positive-define affine metric in the complex coordinates $z = x + iy$ ($\{x, y\}$ is an isothermal coordinate system) one has $G = 2F dz d\bar{z}$, $\Delta \mathbf{f} = \frac{2}{F} \mathbf{f}_{z\bar{z}}$. So in both cases we have

$$G = 2F d\xi d\eta, , \qquad \Delta \mathbf{f} = \frac{2}{F} \mathbf{f}_{\xi\eta} \tag{1}$$

where $\xi, \eta$ are asymptotic coordinates in the indefinite case and $\xi = z$, $\eta = \bar{z}$ in the positive-define case. The corresponding cubic forms of a surface are $A d\xi^3$, $B d\eta^3$ and the Pick invariant is $J = \frac{AB}{F^3}$ where $A$ and $B$ are some functions. Let $\boldsymbol{\nu}$ be a standard affine conormal defined by the conditions

$$\langle \boldsymbol{\nu} \cdot \mathbf{N} \rangle = 1\,, \tag{2}$$

$$\langle d\mathbf{f} \cdot \boldsymbol{\nu} \rangle = 0 \tag{3}$$

where $\mathbf{N}$ is an affine normal and $\langle \cdot \rangle$ is a standard bilinear form.

For the Blasche metric of the type (1) $\boldsymbol{\nu}$ obeys the equations [11]

$$\langle \mathbf{f}_{\xi\xi} \cdot \boldsymbol{\nu} \rangle = \langle \mathbf{f}_{\eta\eta} \cdot \boldsymbol{\nu} \rangle = 0 \tag{4}$$

and

$$\boldsymbol{\nu}_{\xi\eta} + FH\boldsymbol{\nu} = 0 \tag{5}$$

where $H$ is an affine mean curvature. The corresponding Gauss–Codazzi equations are of the form

$$\begin{aligned} \mathbf{f}_{\xi\xi} &= \frac{F_\xi}{F}\,\mathbf{f}_\xi + \frac{A}{F}\,\mathbf{f}_\eta\,, \\ \mathbf{f}_{\xi\eta} &= F\mathbf{N}\,, \\ \mathbf{f}_{\eta\eta} &= \frac{B}{F}\,\mathbf{f}_\xi + \frac{F_\eta}{F}\,\mathbf{f}_\eta \end{aligned} \tag{6}$$

plus two equations for $\mathbf{N}$.

Lelieuvre's formula for indefinite metric in asymptotic coordinates is (see e.g. [11])

$$\mathbf{f}(x) = \int_\Gamma^x (\boldsymbol{\nu} \wedge \boldsymbol{\nu}_{x'}\, dx' - \boldsymbol{\nu} \wedge \boldsymbol{\nu}_{y'}\, dy') \tag{7}$$

while in the positive-define metric case it is

$$\mathbf{f}(x) = \int_\Gamma^x (\boldsymbol{\nu} \wedge \boldsymbol{\nu}_{y'}\, dx' - \boldsymbol{\nu} \wedge \boldsymbol{\nu}_{x'}\, dy') \tag{8}$$

where $\{x, y\}$ is the isothermal coordinate system.

Using the variables $\xi, \eta$ introduced above, we can rewrite the formulae (7) and (8) as a single formula:

$$\mathbf{f}(\xi,\eta) = \sigma \int_\Gamma^{(\xi,\eta)} (\boldsymbol{\nu} \wedge \boldsymbol{\nu}_{\xi'}\, d\xi' - \boldsymbol{\nu} \wedge \boldsymbol{\nu}_{\eta'}\, d\eta') \tag{9}$$

where $\sigma = 1$ for indefinite metric and $\sigma = \sqrt{-1}$ for positive-define metric.

Note also that (see [11])

$$F = \sigma \langle \boldsymbol{\nu} \cdot (\boldsymbol{\nu}_\xi \wedge \boldsymbol{\nu}_\eta) \rangle = \sigma \det|\boldsymbol{\nu}, \boldsymbol{\nu}_\xi, \boldsymbol{\nu}_\eta|\,.$$

# 3 Nizhnik–Veselov–Novikov equation

The general Nizhnik–Veselov–Novikov (NVN) equation has the form

$$p_t = \alpha p_{\xi\xi\xi} + \beta p_{\eta\eta\eta} + 3\alpha (w_{\xi\xi} p)_\xi + 3\beta (w_{\eta\eta} p)_\eta, \tag{10}$$

$$w_{\xi\eta} = p\,.$$

This equation has been derived in [12] at the case of real-valued $\xi$, $\eta$ and arbitrary constants $\alpha$, $\beta$ and in [13] at the case $\xi = z$, $\eta = \bar{z}$, $\alpha = \beta = 1$. The NVN equation (10) is equivalent to the compatibility condition for the linear system [12, 13]

$$\psi_{\xi\eta} + p\psi = 0\,, \tag{11}$$

$$\psi_t = \alpha \psi_{\xi\xi\xi} + \beta \psi_{\eta\eta\eta} + 3\alpha w_{\xi\xi} \psi_\xi + 3\beta w_{\eta\eta} \psi_\eta\,.$$

The NVN equation (10) is integrable by the inverse spectral transform (IST) method, and has all properties typical for soliton equations (see e.g. [14,15]). Wide classes of explicit solutions of the NVN equation have been constructed (see e.g. [16,17,18]). Equation (10) has an infinite set of integrals of motion $C_n$. The simplest of them is

$$C_1 = \iint p\,(\xi, \eta)\, d\xi d\eta\,. \tag{12}$$

The NVN equation (10) is, obviously, the superposition of the simplier equations

$$\begin{gathered} p_{t_1} = p_{\xi\xi\xi} + 3(w_{\xi\xi} p)_\xi\,, \\ w_{\xi\eta} = p \end{gathered} \tag{13}$$

and

$$\begin{gathered} p_{t_2} = p_{\eta\eta\eta} + 3(w_{\eta\eta} p)_\eta\,, \\ w_{\xi\eta} = p\,. \end{gathered} \tag{14}$$

Equation (13) is equivalent to the compatibility condition of the system

$$\begin{gathered} \psi_{\xi\eta} + p\psi = 0\,, \\ \psi_{t_1} = \psi_{\xi\xi\xi} + 3w_{\xi\xi}\psi_\xi \end{gathered} \tag{15}$$

while equation (14) is associated with the system

$$\begin{gathered} \psi_{\xi\eta} + p\psi = 0\,, \\ \psi_{t_2} = \psi_{\eta\eta\eta} + 3w_{\eta\eta}\psi_\eta\,. \end{gathered} \tag{16}$$

Equation (13) and (14) are compatible. Equation (10) is, obviously, the superposition $p_t = \alpha p_t + \beta p_t$. In the case $\eta = \bar{\xi}$ and real-valued $p$ one has only equation (10) with $\alpha = \beta$.

There is an infinite hierarchy of integrable equations associated with the NVN equation. They are equivalent to the compatibility conditions for the systems

$$\begin{gathered} \psi_{\xi\eta} + p\psi = 0\,, \\ \psi_{t_{1n}} = \prod_{k=1}^{2\eta} (\partial_\xi + u_k)\, \psi_\xi \end{gathered} \tag{17}$$

or

$$\begin{gathered} \psi_{\xi\eta} + p\psi = 0\,, \\ \psi_{t_{2n}} = \prod_{k=1}^{2\eta} (\partial_\eta + v_k)\, \psi_\eta \end{gathered} \tag{18}$$

where $u_k$, $v_k$ are functions and $n$ is an arbitrary integer.

The whole NVN hierarchy of equations has the same properties as the NVN equation. For instance, integrals of motion $C_n$ (in particular, $C_1$ (12)) are integrals of motion for the whole hierarchy.

Note also that the NVN equation (10) can be represented in the form [18] ($\alpha = \beta$)

$$(e^{\varphi})_t = (\partial_\xi e^{\varphi} \partial_\eta^{-1} e^{-2\varphi} \partial_\xi + \partial_\eta e^{\varphi} \partial_\xi^{-1} e^{-2\varphi} \partial_\eta)(e^{2\varphi}\varphi_{\xi\eta} + e^{3\varphi}) \tag{19}$$

where $p = e^{\varphi}$. Thus, solutions of the Tzitzeica equation $e^{2\varphi}\varphi_{\xi\eta} + e^{3\varphi} = \text{const}$ provide solutions of the stationary NVN equation.

# 4 NVN deformations of affine surfaces

In the paper [5] one of the authors has suggested to use the NVN equation (10) to build a class of integrable deformations of surfaces in $R^3$ constructed by the Lelieuvre's formula.

We will show here that the NVN equation is more relevant for the theory of deformations in affine geometry.

Thus, we start with the Lelieuvre's formula (9). The conormal $\boldsymbol{\nu}$ obeys the equation

$$\boldsymbol{\nu}_{\xi\eta} + p\,\boldsymbol{\nu} = 0 \tag{20}$$

where $p = FH$.

We assume that all quantities in (20) and, consequently, the coordinates $\mathbf{f}$ depend on the deformation parameter $t$. We consider deformations for which there is a function $w$ such that

$$\boldsymbol{\nu}_{t_1} = \boldsymbol{\nu}_{\xi\xi\xi} + 3w_{\xi\xi}\boldsymbol{\nu}_\xi \tag{21}$$

or, alternatively,

$$\boldsymbol{\nu}_{t_2} = \boldsymbol{\nu}_{\eta\eta\eta} + 3w_{\eta\eta}\boldsymbol{\nu}_\eta\,. \tag{22}$$

Deformations (21), (22) are compatible with equation (20) if

$$p_{t_1} = p_{\xi\xi\xi} + 3(w_{\xi\xi}\,p)_\xi\,, \tag{22}$$

$$w_{\xi\eta} = p$$

and

$$p_{t_2} = p_{\eta\eta\eta} + 3(w_{\eta\eta}\,p)_\eta\,, \tag{24}$$

$$w_{\xi\eta} = p\,.$$

Equations (23) and (24) are representable in the local form

$$w_{t_1\eta} = w_{\eta\xi\xi\xi} + 3w_{\xi\xi}\,w_{\xi\eta}\,, \tag{25}$$

$$w_{t_2\xi} = w_{\xi\eta\eta\eta} + 3w_{\eta\eta}\,w_{\xi\eta}\,. \tag{26}$$

Using the relations

$$\mathbf{f}_\xi = \sigma\, \boldsymbol{\nu} \wedge \boldsymbol{\nu}_\xi\,, \quad \mathbf{f}_\eta = -\sigma\, \boldsymbol{\nu} \wedge \boldsymbol{\nu}_\eta \tag{27}$$

and equations (21) and (22), one can show that

$$\sigma\,(\boldsymbol{\nu} \wedge \boldsymbol{\nu}_\xi)_{t_1} = \mathbf{f}_{\xi\xi\xi\xi} + 3(w_{\xi\xi}\,\mathbf{f}_\xi)_\xi - 3\sigma\,(\boldsymbol{\nu}_\xi \wedge \boldsymbol{\nu}_{\xi\xi})_\xi\,, \tag{28}$$

$$\sigma\,(\boldsymbol{\nu} \wedge \boldsymbol{\nu}_\eta)_{t_1} = -\mathbf{f}_{\eta\xi\xi\xi} - 3(w_{\xi\xi}\,\mathbf{f}_\xi)_\eta - 3(p\,\mathbf{f}_\xi)_\xi + 6p_\xi\,\mathbf{f}_\xi\,, \tag{29}$$

$$\sigma\,(\boldsymbol{\nu} \wedge \boldsymbol{\nu}_\xi)_{t_2} = \mathbf{f}_{\xi\eta\eta\eta} + 3(w_{\eta\eta}\,\mathbf{f}_\eta)_\xi + 3(p\,\mathbf{f}_\eta)_\eta - 6p_\eta\,\mathbf{f}_\eta\,, \tag{30}$$

$$\sigma\,(\boldsymbol{\nu} \wedge \boldsymbol{\nu}_\eta)_{t_2} = -\mathbf{f}_{\eta\eta\eta\eta} - 3(w_{\eta\eta}\,\mathbf{f}_\eta)_\eta - 3\sigma\,(\boldsymbol{\nu}_\eta \wedge \boldsymbol{\nu}_{\eta\eta})_\eta\,. \tag{31}$$

Then with the use of the relations (28)–(31) one gets

$$\frac{\partial\,\mathbf{f}}{\partial t_1} = \mathbf{f}_{\xi\xi\xi} + 3w_{\xi\xi}\,\mathbf{f}_\xi + \int\limits_{\Gamma}^{(\xi\eta)} \{-3\sigma\,(\boldsymbol{\nu}_\xi \wedge \boldsymbol{\nu}_{\xi\xi})_\xi\, d\xi + [3(p\,\mathbf{f}_\xi)_\xi - 6p_\xi\mathbf{f}_\xi]\, d\eta\}\ , \tag{32}$$

$$\frac{\partial\,\mathbf{f}}{\partial t_2} = \mathbf{f}_{\eta\eta\eta} + 3w_{\eta\eta}\,\mathbf{f}_\eta + \int\limits_{\Gamma}^{(\xi\eta)} \{[(3p\,\mathbf{f}_\eta)_\eta - 6p_\eta\,\mathbf{f}_\eta]d\xi + 3\sigma(\boldsymbol{\nu}_\eta \wedge \boldsymbol{\nu}_{\eta\eta})_\eta\, d\eta\}\ . \tag{33}$$

Further, since

$$-3\sigma(\boldsymbol{\nu}_\xi \wedge \boldsymbol{\nu}_{\xi\xi})_\eta = 3\sigma p\,\boldsymbol{\nu} \wedge \boldsymbol{\nu}_{\xi\xi} - 3\sigma p_\xi\,\boldsymbol{\nu} \wedge \boldsymbol{\nu}_\xi = 3p\,\mathbf{f}_{\xi\xi} - 3p_\xi\,\mathbf{f}_\xi \tag{34}$$

and

$$3\sigma(\boldsymbol{\nu}_\eta \wedge \boldsymbol{\nu}_{\eta\eta})_\xi = -3\sigma p\,\boldsymbol{\nu} \wedge \boldsymbol{\nu}_{\eta\eta} + 3\sigma p_\eta\,\boldsymbol{\nu} \wedge \boldsymbol{\nu}_\eta = 3p\,\mathbf{f}_{\eta\eta} - 3p\,\mathbf{f}_\eta$$

one, finally, obtains the following evolution equations for $\mathbf{f}$ compatible with (27) and the evolution of $\boldsymbol{\nu}$:

$$\frac{\partial\mathbf{f}}{\partial t_1} = \mathbf{f}_{\xi\xi\xi} + 3w_{\xi\xi}\,\mathbf{f}_\xi - 3\sigma\,\boldsymbol{\nu}_\xi \wedge \boldsymbol{\nu}_{\xi\xi}\,, \tag{35}$$

$$\frac{\partial\mathbf{f}}{\partial t_2} = \mathbf{f}_{\eta\eta\eta} + 3w_{\eta\eta}\,\mathbf{f}_\eta + 3\sigma\boldsymbol{\nu}_\eta \wedge \boldsymbol{\nu}_{\eta\eta}\,. \tag{36}$$

Note that, although $\boldsymbol{\nu}$ defines $\mathbf{f}$ up to a translational constant the formulae (35),(36) provide canonical way to deform $\mathbf{f}$ along with an evolving $\boldsymbol{\nu}$. Using the identities (34), i.e.

$$\sigma\,(\boldsymbol{\nu}_\xi \wedge \boldsymbol{\nu}_{\xi\xi})_\eta = -w_{\xi\eta}\,\mathbf{f}_{\xi\xi} + w_{\xi\xi\eta}\,\mathbf{f}_\xi\,, \tag{37}$$

$$\sigma(\boldsymbol{\nu}_\eta \wedge \boldsymbol{\nu}_{\eta\eta})_\xi = w_{\xi\eta}\,\mathbf{f}_{\eta\eta} - w_{\xi\eta\eta}\,\mathbf{f}_\eta\,,$$

we can rewrite deformation (35)–(36) in the form which envolves only coordinates:

$$\mathbf{f}_{t_1\eta} = \mathbf{f}_{\xi\xi\xi\eta} + 3w_{\xi\xi}\,\mathbf{f}_{\xi\eta} + 3w_{\xi\eta}\,\mathbf{f}_{\xi\xi}\,, \tag{38}$$

$$\mathbf{f}_{t_2\xi} = \mathbf{f}_{\eta\eta\eta\xi} + 3w_{\eta\eta}\,\mathbf{f}_{\eta\xi} + 3w_{\xi\eta}\,\mathbf{f}_{\eta\eta}\,. \tag{39}$$

Recall that $w$ obeys equation (25) or (26).

The formulae (35), (36) or (38), (39) define deformations of surfaces. One can straightforwardly check that these deformations preserve the conditions (4) and (5). So they convert asymptotic lines into asymptotic lines.

The NVN deformations (38), (39), (25), (26) preserve an infinite set of functionals. The simplest of them is the total affine mean curvature $C = \int p d\xi d\eta = \int HF d\xi d\eta$. Note that,due to the relation $HF + JF = -(lnF)_{\eta\xi}$, this functional is equal ( up to sign and total Gaussian curvature ) to the total Pick invariant: $C = -\int JF d\xi d\eta + K_{tot}$.

Thus, we have

THEOREM. The NVN deformations (38), (37), (25), (26) provide us integrable deformations of affine surfaces which preserve infinite set of functional, the simplest of which is the total affine mean curvature.

Previously found explicit solutions of the NVN equation (see e.g [12,13,15–17]) allow us to find an infinite family of deformations given by explicit formulae.

The whole NVN hierarchy of equations gives rise to the infinite hierarchy of deformations of affine surfaces.

Note that the Nizhnik equation provides us deformations of surfaces with indefinite metric while the Veselov-Novikov equation is relevant for the case of positive -defined metric.

# 5 Normal and tangent deformations

Let us represent the deformations considered above as the superposition of normal and tangential parts:

$$\frac{\partial \mathbf{f}}{\partial t_i} = \mu_i \mathbf{N} + a_i\, \mathbf{f}_\xi + b_i\, \mathbf{f}_\eta \qquad (i = 1, 2)\,. \tag{40}$$

Since $\langle \boldsymbol{\nu} \cdot \mathbf{N} \rangle = 1$, $\langle \boldsymbol{\nu} \cdot \mathbf{f}_\xi \rangle = \langle \boldsymbol{\nu} \cdot \mathbf{f}_\eta \rangle = 0$, one gets from (40) that

$$\mu_1 = \langle \mathbf{f}_{\xi\xi\xi} \cdot \boldsymbol{\nu} \rangle - 3\sigma \langle \boldsymbol{\nu} \cdot (\boldsymbol{\nu}_\xi \wedge \boldsymbol{\nu}_{\xi\xi}) \rangle = \tag{41}$$

$$= -2\sigma \langle \boldsymbol{\nu} \cdot (\boldsymbol{\nu}_\xi \wedge \boldsymbol{\nu}_{\xi\xi}) \rangle = -2\sigma \det |\boldsymbol{\nu}, \boldsymbol{\nu}_\xi, \boldsymbol{\nu}_{\xi\xi}|$$

and

$$\mu_2 = \langle \mathbf{f}_{\eta\eta\eta} \cdot \boldsymbol{\nu} \rangle + 3\sigma \langle \boldsymbol{\nu} \cdot (\boldsymbol{\nu}_\eta \wedge \boldsymbol{\nu}_{\eta\eta}) \rangle = \tag{42}$$

$$= 2\sigma \langle \boldsymbol{\nu} \cdot (\boldsymbol{\nu}_\eta \wedge \boldsymbol{\nu}_{\eta\eta}) \rangle = 2\sigma \det |\boldsymbol{\nu}, \boldsymbol{\nu}_\eta, \boldsymbol{\nu}_{\eta\eta}|\,.$$

To calculate the functions $a_i$, $b_i$ we, first, differentiate equations (40) with respect to $\xi$ and $\eta$, then take the scalar product of equations obtained with $\boldsymbol{\nu}$ and use equations (2)-(6) and expresions (41),(42). As a result we get

$$(a_1 + 3w_{\xi\xi})\det|\boldsymbol{\nu}, \boldsymbol{\nu}_\xi, \boldsymbol{\nu}_\eta| = \det|\boldsymbol{\nu}, \boldsymbol{\nu}_\eta, \boldsymbol{\nu}_{\xi\xi\xi}| - 2\det|\boldsymbol{\nu}_\xi, \boldsymbol{\nu}_\eta, \boldsymbol{\nu}_{\xi\xi}|,, \tag{43}$$

$$b_1\det|\boldsymbol{\nu}, \boldsymbol{\nu}_\xi, \boldsymbol{\nu}_\eta| = \det|\boldsymbol{\nu}, \boldsymbol{\nu}_\xi, \boldsymbol{\nu}_{\xi\xi\xi}|,$$

$$a_2\det|\boldsymbol{\nu}, \boldsymbol{\nu}_\xi, \boldsymbol{\nu}_\eta| = -\det|\boldsymbol{\nu}, \boldsymbol{\nu}_\eta, \boldsymbol{\nu}_{\eta\eta\eta}|,$$

$$(b_2 + 3w_{\eta\eta})\det|\boldsymbol{\nu}, \boldsymbol{\nu}_\xi, \boldsymbol{\nu}_\eta| = -\det|\boldsymbol{\nu}, \boldsymbol{\nu}_\xi, \boldsymbol{\nu}_{\eta\eta\eta}| - 2\det|\boldsymbol{\nu}_\xi, \boldsymbol{\nu}_\eta, \boldsymbol{\nu}_{\eta\eta}|.$$

Now we present the NVN deformations in one more form using the Gauss–Codazzi equations. Taking into account (6), one gets

$$\frac{\partial \mathbf{f}}{\partial t_1} = A\mathbf{N} + \left(\frac{F_{\xi\xi}}{F} + 3w_{\xi\xi}\right)\mathbf{f}_\xi + \frac{A_\xi}{F}\mathbf{f}_\eta - 3\sigma\boldsymbol{\nu}_\xi \wedge \boldsymbol{\nu}_{\xi\xi}, \tag{44}$$

$$\frac{\partial \mathbf{f}}{\partial t_2} = B\mathbf{N} + \frac{B_\eta}{F}\mathbf{f}_\xi + \left(\frac{F_{\eta\eta}}{F} + 3w_{\eta\eta}\right)\mathbf{f}_\eta + 3\sigma\,\boldsymbol{\nu}_\eta \wedge \boldsymbol{\nu}_{\eta\eta}. \tag{45}$$

Formulae (44) and (45) first imply that

$$\mu_1 = \langle\frac{\partial \mathbf{f}}{\partial t_1}\cdot\boldsymbol{\nu}\rangle = A - 3\sigma\langle\boldsymbol{\nu}\cdot(\boldsymbol{\nu}_\xi \wedge \boldsymbol{\nu}_{\xi\xi})\rangle, \tag{46}$$

$$\mu_2 = \langle\frac{\partial \mathbf{f}}{\partial t_2}\cdot\boldsymbol{\nu}\rangle = B + 3\sigma\langle\boldsymbol{\nu}\cdot(\boldsymbol{\nu}_\eta \wedge \boldsymbol{\nu}_{\eta\eta})\rangle.$$

Comparing with (41) and (42), one gets

$$A = \sigma\langle\boldsymbol{\nu}\cdot(\boldsymbol{\nu}_\xi \wedge \boldsymbol{\nu}_{\xi\xi})\rangle = \sigma\det|\boldsymbol{\nu}, \boldsymbol{\nu}_\xi, \boldsymbol{\nu}_{\xi\xi}|, \tag{47}$$

$$B = -\sigma\langle\boldsymbol{\nu}\cdot(\boldsymbol{\nu}_\eta \wedge \boldsymbol{\nu}_{\eta\eta})\rangle = -\sigma\det|\boldsymbol{\nu}, \boldsymbol{\nu}_\eta, \boldsymbol{\nu}_{\eta\eta}|.$$

So

$$\mu_1 = -2A, \mu_2 = -2B.$$

Thus, the NVN deformations have a very particular feature, namely, their normal components are proportional to cubic forms of a surface. The deformations constructed have two simple and rather obvious properties. First, the affine minimal surfaces ($H = 0$ and ,hence, $p = 0$) remain minimal under these deformations. Second, for surfaces with zero Pick invariant J ($A$ or $B$ are equal to zero), i.e. for ruled surfaces in the case of indefinite metric one of the deformations (40) has zero normal component and , hence, it is reduced to a reparametrization of the surface. In particular, for quadrics ( $A = B = 0$) the NVN deformations of both types have only tangential components. So, quadrics are, in fact, invariant under the NVN deformations.

## 6 Affine sphere as stationary point of the NVN deformations

Here we will describe affine surfaces which are associated with stationary point of the NVN deformations. Let us consider the deformations given by (21), (23), (25), (35), (44). A stationary point is defined by the conditions

$$w_{t_1} = 0, \qquad \boldsymbol{\nu}_{t_1} = \omega \boldsymbol{\nu} \tag{48}$$

where $\omega$ is a constant. So $w$ and $\boldsymbol{\nu}$ obey the equations

$$w_{\xi\xi\xi\eta} + 3w_{\xi\xi}\, w_{\xi\eta} = 0\,, \tag{49}$$

$$\omega \cdot \boldsymbol{\nu} = \boldsymbol{\nu}_{\xi\xi\xi} + 3w_{\xi\xi}\, \boldsymbol{\nu}_{\xi}\,. \tag{50}$$

In this stationary case

$$\mathbf{f}_{t_1} = 2\omega\, \mathbf{f} \tag{51}$$

and

$$\sigma\, \boldsymbol{\nu}_{\xi} \wedge \boldsymbol{\nu}_{\xi\xi} = -\omega\, \mathbf{f} \tag{52}$$

where we put constant of integration equal to zero.

Due to (51) and (52) equation (44) takes the form

$$\omega\, \mathbf{f} + A\mathbf{N} + \left(\frac{F_{\xi\xi}}{F} + 3w_{\xi\xi}\right) \mathbf{f}_{\xi} + \frac{A_{\xi}}{F}\, \mathbf{f}_{\eta} = 0\,. \tag{53}$$

Thus, the stationary point of the NVN deformation (44) is characterized by the conditions

$$\mathbf{f} = A\mathbf{N} + \left(\frac{F_{\xi\xi}}{F} + 3w_{\xi\xi}\right) \mathbf{f}_{\xi} + \frac{A_{\xi}}{F}\, \mathbf{f}_{\eta}\,, \tag{54}$$

$$(FH)_{\xi\xi} + 3w_{\xi\xi}\, FH = 0\,, \tag{55}$$

$$w_{\xi\eta} = FH \tag{56}$$

where $A$ is given by (47) and we put $\omega = -1$. Equations (55), (56) imply

$$\left(\frac{p_{\xi\xi}}{p}\right)_{\eta} + 3p_{\xi} = 0 \tag{57}$$

for $p = FH$. In terms of $\varphi = \ln p$ equation (57) looks like as

$$\varphi_{\xi\xi\eta} + 2\varphi_{\xi}\, \varphi_{\xi\eta} + 3e^{\varphi}\varphi_{\xi} = 0$$

or, equivalently,

$$\varphi_{\xi\eta} = \alpha(\eta)\, e^{-2\varphi} - e^{\varphi} \tag{58}$$

where $\alpha(\eta)$ is an arbitrary function.

Thus, for the stationary point of NVN deformation (44) the product $p = HF$ obeys the Tzitzeica equation (58).

For centro-affine surfaces ($\langle \mathbf{f} \cdot \mathbf{f}_\xi \rangle = \langle \mathbf{f} \cdot \mathbf{f}_\eta \rangle = 0$) condition (34) implies

$$\mathbf{f} = A\mathbf{N}\,, \quad A_\xi = 0, \quad F_{\xi\xi} + 3w_{\xi\xi}\, F = 0\,. \tag{59}$$

Without lost of generality one can put $A = 1$. So **f=N**, that is $H = 1$ (see e.g. [11]). Equation (59) is reduced to (55) and we have the Tzitzeica equation (58) for the metric coefficient $F = e^{\varphi}$. So in this case the stationary point is given by proper affine sphere.

Stationary point of the NVN deformation (22), (24), (26), (36), (45) is defined by the equations

$$w_{\xi\eta\eta\eta} + 3w_{\eta\eta}\, w_{\xi\eta} = 0\,, \tag{60}$$

$$\omega\, \boldsymbol{\nu} = \boldsymbol{\nu}_{\eta\eta\eta} + 3w_{\eta\eta}\, \boldsymbol{\nu}_\eta = 0\,. \tag{61}$$

In this case the corresponding equations are, obviously given by equations (54)–(59) with the substitution $\xi \leftrightarrow \eta$, $A \leftrightarrow B$. So for $p = FH$ one again has equation (58) and in the centro-affine case the stationary point is given by the proper affine sphere.

Similar results take place also for general deformations $\boldsymbol{\nu}_t = \alpha\, \boldsymbol{\nu}_t + \beta \boldsymbol{\nu}_t$. Due to (19) the proper affine sphere is the stationary point of general NVN deformations.

# 7 Special subclass of surfaces

Let us consider a subclass of surfaces with indefinite metric for which $FH = p\left(\frac{\xi+\eta}{2}\right) = p(x)$. In this case $\boldsymbol{\nu} = \boldsymbol{\nu}(x)\exp\left[\lambda\left(\frac{\xi-\eta}{2}\right)\right]$.

Equation (5) is reduced to

$$\boldsymbol{\nu}_{xx} + p(x)\boldsymbol{\nu} = \lambda^2 \boldsymbol{\nu} \tag{62}$$

and the Lelieuvre's formula takes the form

$$\mathbf{f}\,(x, y) = \mathbf{f}_0 + \frac{1}{\lambda}\, e^{2\lambda y}\, \boldsymbol{\nu} \wedge \boldsymbol{\nu}_x \tag{63}$$

where $y = \frac{\xi-\eta}{2}$.

For this type of surface equations (13) and (14) are reduced to the famous Korteweg-de Vries (KdV) equation

$$p_t = p_{xxx} + 6pp_x \tag{64}$$

and deformation (35) takes the form

$$\frac{\partial \mathbf{f}}{\partial t} = \mathbf{f}_{xxx} + 3p\, \mathbf{f}_x - \frac{3}{\lambda}\, e^{2\lambda y}\, \boldsymbol{\nu}_x \wedge \boldsymbol{\nu}_{xx}\,. \tag{65}$$

The KdV equation and, consequently, the KdV deformations (65) preserve an infinite set of functionals

$$C_n = \int dx\, P_{2n+1}(x) \tag{66}$$

where the densities $P_n$ are defined via the recurrent relations (see e.g. [14,15]).

$$P_{m+1} = P_{mx} + \sum_{k=1}^{m-1} P_k\, P_{m-k}\,, \quad m = 1, 2, 3 \ldots\,, \tag{67}$$

$$P_1 = -P = FH\,.$$

The whole infinite KdV hierarchy generate an infinite family of integrable deformations which preserve the infinite set of functionals (66), (67). Stationary solutions of the KdV equation provide, via Lelieuvre's formula, a class of affine spheres. This and other problems will be considered in a separate paper.

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