# Modular Wedge Localization and the d=1+1 Formfactor Program

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**Abstract**

In this paper I continue the study of the new framework of modular localization and its constructive use in the nonperturbative d=1+1 Karowski-Weisz-Smirnov formfactor program. Particular attention is focussed on the existence of semilocal generators of the wedge-localized algebra without vauum polarization (FWG-operators) which are closely related to objects fulfilling the Zamolodchikov-Faddeev algebraic structure. They generate a "thermal Hilbert space" and allow a profound understanding of the equivalence of the KMS conditions with the so-called cyclicity equation for formfactors which was known to be closely related to crossing symmetry properties. The modular setting also gives rise to interesting new constructive ideas on "free" d=2+1 anyons and plektons.

## 1 Introduction

The historical roots of "Modular Localization" as a kind of inversion of the famous Bisognano-Wichmann observation [1] and its relation with the mathematical Tomita-Takesaki modular theory [2] of von Neumann algebras in "general position", their thermal KMS structure [3], as well as their constructive power for low-dimensional QFT in general and the formfactor problem in particular, has already been highlighted in previous work of the author [4]. In addition there is a recent paper [5] in which, similar to the present work but different in one important aspect (see section 4) it was attempted to relate the thermal KMS property to the crossing symmetry and in particular to its specific form of the cyclicity equation arising in the formfactor problem.

My original interest in this subject arose from the attempt to understand the loss of unicity when one passes from the unique $(m,s)$-Wigner representation to free fields: there are as many free fields in the $(m,s)$-Fock space as there are intertwiners between the $D^{(s)}(R(\Lambda,\Lambda^{-1}p))$ Wigner rotation matrices and the $D^{[n_+,n_-]}(\Lambda)$ finite dimensional representation matrices of the Lorentz group

[10]. All of these infinitely many fields *describe the same spacetime physics,* even though most of them are not "Eulerian"[1] or Lagrangian. The net of local algebras associated with the $(m, s)$-Wigner representation is again unique, and the method of its direct construction (bypassing the free fields) is based on modular localization. Although this has been described in the cited literature, we repeat the main arguments in the next section in order to keep this paper reasonably self-contained. The new localization modular applied to wave functions is more suitable for QFT than the old one of Newton and Wigner [3], who adapted the Born (probability) localization of Schrödinger theory to the Wigner representation spaces for elementary relativistic systems which by passing certain localization tests are elevated to the notion of elementary particles. In a similar vein as the Klein paradox, the difference in localization (lack of relativistic covariance and violation of Einstein causality of the N-W localization) shows the limitations of analogies between wave functions of relativistic particles and Schrödinger wave functions.

Modular localization in the context of the Wigner theory requires us to interpret the complex representation space as a "twice as large" real Hilbert space with an appropriately defined real orthogonal (or equivalently symplectic) inner product. The real orthogonality physically means causally "opposite" in the sense of quantum theoretical localization and this concept in Fock space passes directly to the von Neumann notion of compatibility of measurements in the sense of commutants of von Neumann algebras. The modular localization of subspaces of the Wigner representation space preempts Einstein causality as well as the thermal (and probably also the entropical) quantum aspects of localization and the issue of crossing symmetry.

The third section is a continuation of the second section on modular aspects of "Wignerism" [4]; but this time with a stronger emphasis on thermal KMS (Hawking, Unruh) properties of wedge localization and on crossing symmetry.

In the context of interacting theories one obtains a framework which is totally intrinsic and characteristic of local quantum physics (LQP). Concepts which are shared with nonrelativistic quantum physics and played a crucial role in perturbation theory, as the interaction picture, time-ordering, euclidean path integrals etc., *are not required.* In fact they woud be unnatural in this approach. This conspicuous absence of what is usually considered to be *the* typical aspects of QFT and their replacement by conceptually profound modular concepts (without change of any physical principle) is the most startling change of paradigm for somebody who has grown up with textbook QFT ( including myself). No-Go theorems, as the Haag theorem (an obstruction against the validity of the field theoretic interaction picture), are not valid for analogous modular concepts, since one has unitary equivalence between the wedge-restricted incoming (free) and interacting algebras. In the terminology explained in section 4 there

[1] "Eulerian" means that, similar as for the Dirac equation, the tranformation law of the field follows from the structure of an (differential) equation of motion. Eulerian fields are usually also Lagrangian. The Wigner representation theory was, among other things, introduced just to avoid the problem of ambiguities in equations of motions and their proliferation in the aftermath of the discovery of the Dirac equation.

is no Haag-theorem obstruction against the existence of the "modular Møller operator".

There is of course one construction in the literature which commits a similar "heresy" and *also does not use* the above *standard* text book *formalism: the bootstrap formfactor approach* to factorizable d=1+1 models of Karowski and Weisz [6] as axiomatized and extended by Smirnov [7]. This has generated the erroneous impression with some physicist that there are two different subjects: standard textbook kind of QFT, and 2-dimensional factorizable- (as well as chiral conformal-) QFT which contributed to loss of unicity in Local Quantum Physics (LQP). This paper is also intended to counteract such impressions.

Despite claims to the contrary, no relativistic QFT has ever allowed a physically consistent modifications in the direction of momentum space cutoffs or regularizations. This becomes particularily evident in cases where one has obtained a good conceptual and mathematical control as for factorizable models. It is simply impossible to manufacture a factorizing model with such a modification without wrecking its physical interpretation altogether; construction method and principles of LQP are inexorably related. If one wants to have models outside of relativistic quantum physics, but yet with a similar particle interpretation and scattering theory (but without Einstein causality and relativistic covariance), one has to find a lattice version. This is a highly nontrivial task in its own right; the derivation of cluster properties (and hence of Haag-Ruelle) scattering theory without being able to use Einstein causality is more difficult [8]. In particular our modular localization concepts would not apply to lattice theories. Hence the methods used in this article are totally characteristic of local quantum physics; they are wrecked by cutoffs, regulators or elementary lengths. The main advantage of looking at lattice models is that, unlike in the present work, one does not have to make a deep and unusual conceptual investment at the start and one is therefore able to use computers without knowing much about the conceptual structure of LQP. It is only by the time one reaches the problem of scaling limits and their analytic control that one realizes that the dispense from conceptual thinking was at best a temporary and tentative one.

The main part of this paper is the fourth section where we relate modular localization with factorizing models. A useful illustration, despite its physical austerity, is supplied by the Federbush model [23][24]. Its explicitly known solution is similar to the Ising model QFT [26], but its rapidity independent S-matrix depends continuously on a coupling strength. The reason why it did not hitherto appear in the factorizing list is that the S-matrix bootstrap was restricted to parity conserving theories. The main message from this model and the principles of algebraic QFT is that the simplicity of factorizable models does not so much show up in the generalized formfactors of "field coordinates" and correlation functions (which, apart from the Ising theory and the Federbush model, remained too complicated for explicit analytic calculations), but rather in the computation of the (net of) wedge algebras and the ensuing double cone algebras. The net of wedge algebras uniquely fixes the total net, and since the wedge generators turn out to be determined uniquely by the factorizing S-matrix, the so-called inverse problem of QFT has a unique solution in the context of factorizing models. These algebras and their mutual relations are sufficient for exploring and characterizing the intrinsic physical content of a real time QFT[2], since the S-matrix and perhaps some distinguished Noether current fields (energy momentum tensor, charge currents) are not only sufficient mathematical characteristics, but are also the only experimentally accessible observables in high energy physics. On the other hand in statistical mechanics, also the individual correlation functions of certain order/disorder fields are measurable and for those one has to work harder than for the structure of the algebras of real time QFT. Fortunately the computation of e.g. the spectrum of critical indices does not require such difficult calculations, since by passing to the structurally richer noncommutative real time side it becomes part of the superselection/spin-statistics discussion of algebraic QFT which relates the critical indices to the statistical phases [9]).

The concluding remarks offer a scenario for higher dimensional modular constructions in the presence of interactions. In that case one does not have an S-matrix bootstrap program which could be *separated from the rest of QFT*. Although algebraic QFT offers some hints, one its still very far from covering these ideas with an analytically accessible formalism.

Mathematically the problems at hand are a special case of the so called *inverse problem of modular theory* i.e. the reconstruction of a von Neumann algebra from its modular data [16]. Only in its more physical setting where it passes to the *inverse problem of QFT*, namely the question whether *physically admissable scattering matrices possess a unique local QFT*, the uniqueness can be established under reasonable physical assumptions. This will be shown in a subsequent paper [17]. For a mathematical and conceptual background I refer to [33] and to some notes [42] on "Nonperturbative Approach to Local Quantum Physics". There the reader also finds (in chapter 2) an account of the unpublished work [13] adapted to the present setting.

# 2 Liberation from Free Field Coordinates

As explained elsewhere [4], one may use the Wigner representation theory for positive energy representations in order to construct fields from particle states. For $d = 3 + 1$ space-time dimensions there are two families of representation: $(m, s)$ and $(0, h)$. Here $m$ is the mass and designates massive representation and $s$ and $h$ are the spins resp. the helicities $h$. These are invariants of the representations ("Casimirs") which refer to the Wigner "little" group; in the first case to SU(2) in which case $s =$ (half) integer, and for $m = 0$ to the little group (fixed point group of a momentum $\neq 0$ on the light cone) $\tilde{E}(2)$ which

[2]The constructive aspect of the formfacor approach to renormalizable factorizing models and the present new modular method reveals more structure than the old method of constructive QFT with respect to e.g. $\phi_2^4$ which was modelled on Lagrangian perturbation theory. The new construction is not impeded by worse than canonical short distance behaviour. In fact the issue of short distance behavior beyond the limits set by causality does not enter the nonperturbative modular construction.

is the two-fold covering of the euclidean group in the plane. The zero mass representations in turn split into two families. For the "neutrino-photon family" $(0,h)$ the little group has a non-faithful representation (the "translative" part is trivially represented) whereas for Wigner's "continuous spin representation" $(0,H)$ the representation is faithful and infinite component (i.e. the helicity components range over a Hilbert space $H$). It describes an elementary relativistic system whose localization properties turn out to be worse than those required of standard zero mass particles.

In the massive case, the transition to covariant fields is most conveniently done with the help of intertwiners between the Wigner spin $s$ representations $D^{(s)}(R(\Lambda,p))$ which involve the $\Lambda,p$ dependent Wigner rotation $R$ and the finite dimensional covariant representation of the Lorentz-group $D^{[A,B]}$

$$u(p)D^{(s)}(R(\Lambda,p)) = D^{[A,B]}(\Lambda)u(\Lambda^{-1}p) \tag{1}$$

The only restriction is:

$$\mid A-B \mid \leq s \leq A+B \tag{2}$$

which leaves infinitely many $A,B$ (half integer) choices for a given $s$. Here the $u(p)$ intertwiner is a rectangular matrix consisting of $2s+1$ column vectors $u(p,s_3), s_3=-s,...,+s$ of length $(2A+1)(2B+1)$. Its explicit construction using Clebsch-Gordan methods can be found in Weinberg's book [10]. Analogously there exist antiparticle (opposite charge) $v(p)$ intertwiners: $D^{(s)*}(R(\Lambda,p) \longrightarrow D^{[A,B]}(\Lambda)$. The covariant field is then of the form:

$$\begin{aligned}\psi^{[A,B]}(x) \quad &= \quad \frac{1}{(2\pi)^{3/2}}\int(e^{-ipx}\sum_{s_3}u(p_1,s_3)a(p_1,s_3)+ \\ &\quad +e^{ipx}\sum_{s_s}v(p_1,s_3)b^*(p_1,s_3))\frac{d^3p}{2\omega}\end{aligned} \tag{3}$$

where $a(p)$ and $b^*(p)$ are annihilation (creation) operators in a Fock space for particles (antiparticles). The bad news (only at first sight, as it fortunately turns out) is that we lost the Wigner unicity: there are now infinitely many $\psi^{[A,B]}$ fields with varying $A,B$ but all belonging to the same $(m,s)$-Wigner representation and living in the same Fock space. Only one of these fields is "Eulerian" (examples: for $s=\frac{1}{2}$ Dirac, for $s=\frac{3}{2}$ Rarita-Schwinger) i.e. the transformation property of $\psi$ is a consequence of the nature of a linear field equation which is derivable by an action principle from a Lagrangian. Non-Eulerian fields as e.g. Weinberg's $D^{[j,0]}+D^{[0,j]}$ fields for $j\geq\frac{3}{2}$, cannot be used in a canonical quantization scheme or in a formalism of functional integration because the corresponding field equations have more solutions than allowed by the physical degrees of freedom ( in fact some of them have tachyonic solutions). On the other hand the use of formula (3) with the correct $u,v$ intertwiners in a (on-shell) Stueckelberg-Bogoliubov-Shirkov-Epstein-Glaser approach (which is different from off-shell approaches as euclidean functional integrals) does not require the existence of free bilinear Lagrangians based on causality is however legitimate. Naturally from the point of view of the Wigner theory, which is totally intrinsic and does not use quantization ideas, there is no preference of Eulerian over non Eulerian fields in real time QFT.

It turns out that the above family of fields corresponding to $(m,s)$ constitute the linear part of the associated "Borchers class" [3]. For bosonic fields the latter is defined as:

$$B(\psi) = \left\{ \chi(s) \mid [\chi(x), \psi(y)] = 0, \ (x-y)^2 < 0 \right\} \tag{4}$$

If we only consider cyclic (with respect to the vacuum) relatively local fields, then we obtain transitivity of the causality for the resulting fields [18]. This class depends only on $(m,s)$ and is generated by the Wick-monomials of $\psi$. A mathematically and conceptually more manageable object which is manifestly independent of the chosen $(m,s)$ Fock-space field, is the local von Neumann algebra $\mathcal{A}(\mathcal{O})$ generated by $\psi$:

$$\mathcal{O} \rightarrow \mathcal{A}(\mathcal{O}, \psi) = \mathcal{A}(\mathcal{O}, \chi) \tag{5}$$

Here $\chi \sim \psi$ is any cyclic (locally equivalent) field in the same Borchers class of $\psi$.

Now we have reached our first goal: the lack of uniqueness of local $(m,s)$ fields is explained in terms of the arbitrariness in the choice of "field coordinates" which generate the same net of von Neumann algebras. According to the physical interpretation in algebraic QFT this means that the physics does not depend on the concretely chosen (cyclic action on the vacuum assumed) field. It is difficult to understand why the majority of physicists who familiarized themselves with the tools of modern differential geometry never felt the need to do the analogous step of liberating oneself from "field coordinates" in favor of more intrinsic concepts. The gains in understanding are considerable: from the insensitivity of the scattering matrix against changes of the interpolating fields to the understanding that a genuinly nonperturbative approach as the present one does not distinguish individual fields[3] to name just a few.

Since algebraic QFT shuns inventions and favors discoveries, it is deeply satisfying that there are arguments to the extend that every causal net fulfilling certain spectral properties is automatically "coordinatizable". For chiral conformal theories there exists even a rigorous proof [11]. So one can be confident that the physical content has not been changed as compared to the standard Wightman approach which in turn was abstracted from a synthesis of the Heisenberg-Pauli canonical quantization of classical Lagrangian field theory with the Wigner representation theory of the Poincaré group. The use of local field coordinates tends to make geometric localization properties of the algebras manifest. But only if there exist *pointlike covariant generators* which create charged states (counter example: for Maxwellian charges they do not exist; the physical electron field is noncompactly localized) the localization can be encoded into classical smearing function. The localization concept is "maximally

[3]The introduction of a basis in field space causes technical problems whose solution are not necessary for establishing the existence of a model.

classical" for the free Weyl and CAR algebras which in fact are just *function algebras with a noncommutative product structure.* For these special cases the differential geometric concepts as fibre bundles may be directly used in local quantum physics. Outside of these special context where quantum localization can be described in terms of support properties of classical functions, the only sufficiently general reliable concepts are those based on von Neumann algebra methods of algebraic QFT. In that case the quantum localization may deviate from the classical (differential-) geometric concepts [33] and the use of "field coordinates" is less useful than that of local nets. This however does not mean that geometrical concepts are useless, but rather that they cannot be imposed but must be derived as a consequence of local quantum physics.

In the following we describe a way to construct the interaction-free nets directly [4], thus bypassing the use of field coordinates altogether. We use the d=3+1 Wigner $(m, s)$-representations as an illustrative example. In case of charged particles (particles$\neq$antiparticles) we double the Wigner representation space:

$$H = H_{Wig}^{p} \oplus H_{Wig}^{\bar{p}} \tag{6}$$

in order to incorporate the charge conjugation operation as an operation involving the particle-antiparticle-flip. On this extended Wigner space one can act with the full Poincaré group (where those reflections which change the direction of time are antiunitarily represented). For the modular localization in a wedge we only need the standard L-boost $\Lambda(\chi)$ and the standard reflection $r$ which (by definition) are associated with the $t-x$ wedge:

$$\delta^{i\tau} \equiv \pi_{Wig}(\Lambda(\chi = 2\pi\tau)) \tag{7}$$

$$j \equiv \pi_{Wig}(r) \tag{8}$$

These operators have a simple action on the p-space (possibly) doubled Wigner wave functions, in particular:

$$(j\psi)(p) \simeq \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \bar{\psi}(p_0, p_1, -p_2, -p_3) \tag{9}$$

By functional calculus we form $\delta^{\frac{1}{2}}$ and define:

$$s \equiv j\delta^{\frac{1}{2}} \tag{10}$$

This unbounded antilinear densely defined operator $s$ is involutive on its domain: $s^2 = 1$. Its -1 eigenspace[4] is a real closed subspace $H_R$ of $H$ which allows the following characterization of the domain of $s$ :

$$\begin{aligned} dom(s) &= H_R + iH_R \\ s(h_1 + ih_2) &= -h_1 + ih_2 \end{aligned} \tag{11}$$

[4]It does not matter whether we take the + or - sign for the characterization of modular localization since we can convert one into the other via multiplication with $i$ and the dense domains $H_R(W) + iH_R(W)$ are the same. This is one of the rare occasions where a sign error remains without serious consequences.

Defining:

$$H_R(W) \equiv U(g)H_R, \;\; W = gW_{stand} \tag{12}$$

where g is an appropriate Poincaré transformation, one finds the following theorem (D. Guido[5], private communication):

**Theorem 1** *$H_R(W)$ is an isotonous net of real Hilbert spaces i.e. $H_R(W_1) \subsetneq H_R(W_2)$ if $W_1 \subsetneq W_2$.*

Its proof, which follows from a theorem by Borchers [12] will nor be given here

If we now define:

$$H_R(\mathcal{O}) \equiv \bigcap_{W\supset\mathcal{O}} H_R(W) \tag{13}$$

then it is easily seen (even without the use of the u,v-intertwiners) that the spaces $H_R(\mathcal{O}) + iH_R(\mathcal{O})$ are still dense in $H_{Wig}$ and that the formula:

$$s(\mathcal{O})(h_1 + ih_2) \equiv -h_1 + ih_2 \tag{14}$$

defines a closed involutive operator with a polar decomposition:

$$s(\mathcal{O}) = j(\mathcal{O})\delta(\mathcal{O})^{\frac{1}{2}} \tag{15}$$

Although now $j(O)$ and $\delta(\mathcal{O})^{i\tau}$ have no obvious geometric interpretation, there is still a bit of geometry left, as the following theorem shows:

**Theorem 2** *The $H_R(\mathcal{O})$ form an orthocomplemented net of closed real Hilbert spaces, i.e. the following "duality" holds:*

$$H_R(\mathcal{O}') = H_R(\mathcal{O})' = iH_R^{\perp}(\mathcal{O}). \tag{16}$$

Here $\mathcal{O}'$ denotes the causal complement, $H_R^{\perp}$ the real orthogonal complement in the sense of the inner product $Re\,(\psi,\varphi)$ and $H_R'$ is the symplectic complement in the sense of $\mathrm{Im}\,(\psi,\varphi)$ . Representing $\mathcal{O}'$ as a union of double cones, one defines $H_R(\mathcal{O}')$ by additivity.

The direct construction of the interaction-free algebraic bosonic net for $(m, s = integer)$ is now achieved by converting the "premodular" theory of real subspaces of the Wigner space into the Tomita-Takesaki modular theory for nets of von Neumann algebras using the Weyl functor [13]:

The application of the Weyl functor $-$ to the net of real spaces:

$$H_R(\mathcal{O}) \xrightarrow{-} \mathcal{A}(\mathcal{O}) \equiv alg\,\{W(f)\,|\,f \in H_R(\mathcal{O})\,\} \tag{17}$$

[5] In the unpublished Brunetti-Guido-Longo approach this theorem was established for general positive energy representations of the Poincaré group. In fact these authors established the equivalence of the positivity with the spatial isotony in their more general setting.

leads to a net of von Neumann algebras in $\mathcal{H}_{Fock}$ which are in "standard position" with respect to the vacuum state. The ensuing a modular theory restricted to the Fock vacuum $\Omega$ is geometric:

$$\begin{aligned} \Gamma(s) &= S, \;\; SA\Omega = A^*\Omega, \;\; A \in \mathcal{A}(W) \\ S &= J\Delta^{\frac{1}{2}}, \;\; J = \Gamma(j), \;\; \Delta^{i\tau} = \Gamma(\delta^{i\tau}) \end{aligned} \tag{18}$$

The proof of this theorem uses the functorial formalism of [13]. It should be evident from the derivation that the wedge localization concept in Fock obtained in this functorial way from the Wigner theory only holds for interaction free situations. The Fock space is also important for interacting QFT, but in that case the wedge localization enters via scattering theory as in section 4, and not just through Wigner's representation theory.

Clearly the $W$- or $\mathcal{O}$- indexing of the Hilbert spaces corresponds to a localization concept via modular theory. Specifically $H_R(\mathcal{O}) + iH_R(\mathcal{O})$ is a certain closure (in the graph topology of the operator $\Delta^{\frac{1}{2}}$) of the one particle component of the Reeh-Schlieder domain belonging to the localization region $\mathcal{O}$. Although for general localization region the modular operators are not geometric, there is one remaining geometric statement which presents itself in the form of an algebraic duality property [14]:

$$\mathcal{A}(\mathcal{O}') = \mathcal{A}(\mathcal{O})', \quad \textit{Haag Duality} \tag{19}$$

Here the prime on the von Neumann algebra has the standard meaning of commutant. In the following we make some schematic additions and completions which highlight the modular localization concept for more general interaction-free theories different from the massive bosonic case.

- In the case of $m \neq 0, s =$ halfinteger, the Wigner theory produces a mismatch between the "quantum" ( in the sense of the commutant) and the "geometric" opposite of $H_R(W)$, which however is easily taken care of by an additional factor $i$ (interchange of symplectic complement with real orthogonal complement). This (via the physical localization property) requires the application of the CAR-functor instead of the CCR-functor, as well as the introduction of the well-known Klein transformation $K$ which corrects the above mismatch in Fock space:

  $$\begin{aligned} J = K\mathcal{F}_{CAR}(ij)K^{-1} \\ \mathcal{A}(\mathcal{O}') = K\mathcal{A}(\mathcal{O})'K^{-1} \end{aligned} \tag{20}$$

  where the $K$ is the twist operator of the "twisted" Haag Duality [3] and $j$ is related to the TCP-operator as before.

- For $m = 0, h =$(half)integer, as a consequence of the nonfaithful representation of the zero mass little group $E(2)$ (the two-dimensional euclidean group or rather its two-fold covering), the set of possible $u - v$ intertwiners is limited by the selection rule: $|A - B| = \pm h$. This means on the one hand that there are e.g. no covariant intertwiners which lead to $D^{\left[\frac{1}{2},\frac{1}{2}\right]}$ (vector-potential of classical Maxwell theory), $D^{\left[\frac{1}{2},\frac{1}{2}\right]}\otimes(D^{\left[\frac{1}{2},0\right]}+D^{\left[0,\frac{1}{2}\right]})$ (Rarita-Schwinger potential for massless particles), gravitational potentials etc. On the other hand, all local bilinear expressions in the allowed covariant intertwiners which could serve as an inner product vanish and hence cannot be used in order to rewrite the Wigner inner product (for e.g. $h=1$ any local inner product in terms of field strength intertwiners $F_{\mu\nu}(p)$ vanishes as a consequence of the mass shell condition $p^2=0$). A reasonable compromise consists in relaxing on strict L-covariance and compact (double cone) modular localization, but retaining the relation with the Wigner inner product. One then may describe the Wigner space in terms of a vectorpotential which depends in addition on a spacelike direction $e$ and which has the following "affine" Lorentz transformation:

$$\begin{aligned}\left(U(\Lambda)A\right)_\mu(p,e) &= \Lambda_\mu^\nu A_\nu(\Lambda^{-1}p,\Lambda^{-1}e) \\ &= \Lambda_\mu^\nu A_\nu(\Lambda^{-1}p,e)+p_\mu G(p,\Lambda,e)\end{aligned} \tag{21}$$

where the "gauge" contribution $G$ by which one has to re-gauge (in order to refer to the original spacelike polarization vector $e$) is a nonlocal term which follows from the above definitions. Using the x-t-boost for the definition of $\delta^{it}$ and defining $j$ with the help of the TCP operation as before, one again obtains a wedge localized real subspace $H_R(W_{st})$ which contains vectorpotentials with $e$ pointing into the wedge. This space is the same as if we would have constructed the real modular subspace of the Wigner wave function space of right and left hand polarized photons without vectorpotentials. After applying the Weyl functor, we again obtain a covariant net of wedge algebras which is described in terms of slightly nonlocal semiinfinite stringlike vectorpotential field coordinates whose relation to the local $F_{\mu\nu}(x)$ field strength can be shown to be given by:

$$A_\mu(x,e)=\int_0^\infty e^\nu F_{\mu\nu}(x-es)ds \tag{22}$$

If we now define the modular localization subspaces as before by starting from the wedge region, we find that the (smoothened versions of) the vectorpotentials are members of these subspaces (or their translates) as long as the spacelike directions $e$ point inside the wedges. They are lost if we form the localization spaces belonging to e.g. double cone regions. Hence these stringlike localized vector potentials appear in a natural way in our modular localization approach for the wedge regions. Whereas the natural use of such nonpointlike objects in a future interacting theory based on modular localization may be possible, the present formulation of gauge theories within any known perturbative scheme is based on pointlike fields in an Hilbert space involving ghosts and nonunitary (pseudounitary) Lorentz-transformations instead of the previous affine action in Wigner space. In this way one formally keeps the classical point localization through the process of deformation by interaction and the ghost removal is only done at the end of the calculation, i.e. the ghosts act like a catalyser for forcing theories involving spin$\geq 1$ vector mesons into the standard renormalizable framework. Such catalyzers which leave no intrinsic observable mark in the physical results may be OK. in chemistry, but they go a bit against the Bohr-Heisenberg spirit of removing all nonobservable aspects from the formalism. algebraic QFT therefore considers these BRS cohomological techniques as successful but preliminary.

# 3 Thermal Aspects of Modular Localization

In modular theory the dense set of vectors which are obtained by applying (local) von Neumann algebras in standard position to the standard (vacuum) vector forms a core for the Tomita operator $S$. The domain of $S$ can then be described in terms of the +1 (or -1) closed real subspace of $S$. In terms of the "premodular" objects $s$ in Wigner space and the modular Tomita operators $S$ in Fock space we introduce the following nets of wedge-localized dense subspaces:

$$H_R(W) + iH_R(W) = dom(s) \subset H_{Wigner} \tag{23}$$

$$\mathcal{H}_R(W) + i\mathcal{H}_R(W) = dom(S) \subset \mathcal{H}_{Fock} \tag{24}$$

These dense subspaces become Hilbert spaces in their own right if we use the graph norm of the Tomita operators. For the $s$-operators in Wigner space we have:

$$\begin{aligned} (f,g)_{Wigner} &\rightarrow (f,g)_G \equiv (f,g)_{Wig} + \overline{(sf,sg)}_{Wig} \\ &= (f,g)_{Wig} + (f,\delta g)_{Wig} \end{aligned} \tag{25}$$

The graph topology insures that the wave functions are strip-analytic in the wedge rapidity $\theta$:

$$\begin{aligned} p_0 &= m(p_\perp)\cosh\theta, \;\; p_1 = m(p_\perp)\sinh\theta, \;\; m(p_\perp) = \sqrt{m^2 + p_\perp^2} \\ & \quad strip: 0 < Imz < \pi, \quad z = \theta_1 + i\theta_2 \end{aligned} \tag{26}$$

where this "G-finiteness" (25)is precisely the analyticity prerequisite for the validity of the KMS property for the two-point function. For scalar Bosons we have for the Wigner inner product restricted to the wedge :

$$(f,g)^W_{Wig} = \left\langle A(\overline{\hat{f}})A^*(\hat{g})\right\rangle_0 \overset{KMS}{=} \left\langle A^*(\hat{g})\Delta A(\overline{\hat{f}})\right\rangle_0 \tag{27}$$

$$\overset{CCR}{=} \left[A^*(\hat{g}), A(\delta\overline{\hat{f}})\right] + (f,\delta g)^W_{Wig}, \;\; \delta = e^{2\pi K} \tag{28}$$

$$\curvearrowright (f,g)^W_{Wig} \equiv (f,g)_{K,T=2\pi} = \left[A^*(\hat{g}), A(\frac{\delta}{1-\delta}\overline{\hat{f}})\right] \tag{29}$$

Here $K$ denotes the infinitesimal generator of the L-boost and we used a field theoretic notation $A^*(\hat{g})$ for a wedge localized smeared scalar complex free field of the type (3) (linear in $\hat{g}$ with $supp.\hat{g} \in W$ ) in order to emphasize the typical thermal denominator in the c-number commutator on the right hand side and not only implicit as the required restriction of the wave functions to the wedge region on the left hand side. Of course the c-number commutator may be rewritten in terms of p-space Wigner wave functions for particles and ($\delta^{\frac{1}{2}}$-transformed) antiparticles in such a way that the localization restriction is guarantied by the property that the resulting expression is finite if the wave functions are finite in the sense of the graph norm. With the localization temperature in this way having been made manifest, the only difference between localization temperatures and heat bath temperatures (for a system enclosed in a box) on the level of field algebras in Fock space corresponds to the difference between hyperfinite type $III_1$ and type $I$ von Neumann algebras. On the level of the generators of the modular group this should correspond to a difference in their spectra. The fact that the boost $K$ with a two-sided spectrum appears instead of the one-sided bounded Hamiltonian $H$ reveals one difference between the two situations. More explicitly for the heat bath temperature of a Hamiltonian dynamics the modular operator $\delta = e^{-2\beta\mathbf{H}}$ is bounded on one particle wave functions, whereas the unboundedness of $\delta = e^{2\pi K}$ in (28) enforces the localization (strip analyticity) of the Wigner wave functions i.e. the boost does not permit a KMS state on the full algebra.

In fact localization temperatures are inexorably linked with unbounded modular symmetry operators and they are always equal (in our natural units) to $2\pi$ and this is independent of whether they have a classical (Killing vectors in the Hawking-Unruh situation) or quantum (localization as in the present discussion) origin. Any attempt to deform these temperatures will necessarily destroy the relation with local quantum physics.

The generalization to fermions as well as to particles of arbitrary spin is easily carried out. The differences between $K$ and $H$ also leads to somewhat different energy distribution functions for small energies so that Boson $K$-energy distributions may appear as those of $H$ heat bath Fermions . In this context one is advised to discuss matters of statistics not in Fourier space, but rather in spacetime, where they have their unequivocal physical interpretation.

One may of course consider KMS state on the same $C^*$-algebra with a different Hawking $K$-temperature than $2\pi$; however, such a situation cannot be obtained by a localizing restriction. Mathematically $C^*$-algebras to different $K$-temperatures are known to belong to different folia (in this case after von Neumann closure to unitarily inequivalent $III_1$-algebras) of the same $C^*$-algebra. Or equivalently, a scaled modular operator $\Delta^{\alpha i\tau}$ cannot be the modular operator of the same theory at a different temperature as it would be the case for type I algebras. Localization temperatures are not freely variable.

For those readers who are familiar with Unruh's work we mention that the Unruh Hamiltonian is different from $K$ by a factor $\frac{1}{a}$ where $a$ is the acceleration.

More generally we may now consider matrix elements of wedge-localized operators between wedge localized multiparticle states. Then the KMS property allows to move the wedge localized particle state as an antiparticle with the analytically continued rapidity $\theta + i\pi$ from the ket to the bra. The simplest illustration is the two-particle matrix element of a free current of a charged scalar field $j_\mu(x) =: \phi^* \overset{\leftrightarrow}{\partial}_\mu \phi :$ smeared with the wedge supported function $\hat{h}$ (but any other free field composite would also serve):

$$\begin{aligned} &\left\langle 0 \left| \int j_\mu(x)\hat{h}(x)d^4x \right| f, \delta^{\frac{1}{2}} g^c \right\rangle \overset{KMS}{=} \left\langle 0 \left| \left(\Delta\phi^*(\delta^{\frac{1}{2}}\overline{g})\Delta^{-1}\right)^* \int j_\mu(x)\hat{h}(x)d^4x \right| f \right\rangle \\ = &\left\langle \bar{g} \left| \int j_\mu(x)\hat{h}(x)d^4x \right| f \right\rangle \end{aligned} \tag{30}$$

Here $\delta^{\frac{1}{2}} g^c$ is the charged transformed antiparticle in the Wigner wave function $g$ at the analytically continued rapidity $\theta + i\pi$, whereas $\hat{g}$ denotes as before the wedge-localized spacetime smearing function whose mass shell restricted Fourier transform corresponds to the boundary value of the analytically continuable Wigner wave function $g$. Moving the left hand operator to the left vacuum changes the antiparticle charge to the particle charge. Since the $H_R(W) + iH_R(W)$ complex localization spaces are dense in the Wigner space, the momentum space kernel for both sides of (30) takes the familiar form

$$\langle p' \left| j_\mu(0) \right| p\rangle = \underset{z\to\theta+i\pi}{anal.cont.} \left\langle 0 \left| j_\mu(0) \right| p, p'(z) \right\rangle \tag{31}$$

where $p'(z)$ is the rapidity parametrization of above (26). This famous crossing symmetry, which in the free theory is an obvious identity, is known to hold also in each perturbative order of renormalizable interacting theories. But it has never been derived in sufficient generality within a nonperturbative framework of QFT. It is to be thought of as a kind of on shell momentum space substitute for Einstein causality and locality (and its strengthened form, called Haag duality). As such it played an important role in finding a candidate for a nonperturbative S-matrix of the famous Veneziano dual model. Although it stood in this indirect way on the cradle of string theory, the recent string theoretic inventions seem to pay little attention to these physical origins. It is my conviction that if the schism of string theory as compared to QFT is ever going to be undone, a deeper understanding of crossing symmetry will play a key role.

If crossing symmetry is really a general property of local QFT, a conjecture (only proven in perturbation theory, as mentioned before) which nobody seems to doubt, then it should be the on shell manifestation of the off shell KMS property (originating from causality via the Reeh-Schlieder property) for modular wedge localization. In fact we will show in the next section that the main step towards a deeper understanding of crossing symmetry is the existence of on-shell (polarization-free) operators $\int F(x)f(x)dx$ $(suppf \in W)$ which generate the wedge algebra. In the case of d=1+1 factorizing models their mass shell Fouriertransforms satisfy the Zamolodchikov-Faddeev algebraic relations[6]

[6]As will become clear in the next section, although these operators are nonlocal, they

in the momentum space rapidity [15], and the derivation of crossing symmetry is similar (albeit more involved) to the previous free field derivation [4].

In fact the previous very special free field formalism may be generalized into two directions:

- interacting fields
- curved spacetime

As mentioned before, low-dimensional interacting theories will be discussed in the next section. For the generalization to curved space time (e.g. the Schwarzschild black hole solution) it turns out that only the existence of a bifurcated horizon together with a certain behavior near that horizon (symmetry-improving restriction with "surface gravitation") [19][20] is already sufficient in order to obtain the thermal Hawking-Unruh aspect. In the standard treatment one needs isometries in spacetime i.e. classical horizons defined in terms of Killing vectors. The idea of modular localization suggests to consider also e.g. double cones for which there is no spacetime isometry but only an isometry in $H_{Wigner}$ or $\mathcal{H}_{Fock}$. Of course such enlargements of spaces (here from the finite dimensional spacetime to the infinite dimensional test function or quantum state space) for obtaining a better formulation and a generalization of a problem are commonplace in modern mathematics, particularly in noncommutative geometry. Here the guiding idea is that one replaces the ill-defined isometries by a geometrically "fuzzy" but well-defined symmetry transformations in quantum space, which only near the horizon looses its spacetime fuzziness. The candidates for these nongeometric symmetries are the modular automorphisms of von Neumann algebras of arbitrary space time regions together with suitable faithful states from the local folium of admissable states. Although the restriction of the global vacuum state is in that folium, it is not always the appropriate state for the construction of the modular automorphism.

In this context one obtains a good illustration by the (nongeometric) modular theory of, e.g., the double cone algebra of a massive free field. From the folium of states one may want to select that vector, with respect to which the algebra has a least fuzzy (most geometric) behavior under the action of the modular group. Appealing to the net subtended by spheres $S$ at time t=0 one realizes that algebras localized in these spheres are independent of the mass. Since $m = 0$ leads to a geometric modular situation[7] for the pair $(\mathcal{A}_{m=0}(S), \Omega_{m=0})$, and since the nonlocality of the modular group of the massive theory in the "wrong" (massless) vacuum $(\mathcal{A}_{m\neq 0}(C(S)), \Omega_{m=0})$ in the subtended double cone $C(S)$ is only the result of the fuzzy propagation inside the light cone (the breakdown of Huygens principle or the "reverberation" phenomenon caused by the mass), the fuzziness of the modular group for this pair together with its asymptotically geometric action near the horizon is a pure propagation phenomenon, i.e., can

generate the wedge localized algebra, and as a consequence the modular KMS formalism is applicable to them.

[7]The modular group is a one-parametric subgroup of the conformal group.

be understood in terms of the deviation from Huygens principle. In view of the recent micro-local spectrum condition, one expects such an action also in the correct vacuum i.e. for $(A_{m\neq 0}(C(S)),\Omega_{m\neq 0})$ in fact the hope is that the generator of the modular group[8] is a pseudo-differential operators [21] whose leading symbol is identical to the modular generator in the conformal massless limit. In this case the asymptotically local action near the light-like horizon will continue to hold. In order to avoid the pathology of the d=1+1 scalar zero mass field, one should use for the above consideration a massive free spinor field whose massless limit gives a two-component field with the first component only depending on the left light cone and the second on the right hand light cone. In fact the above observation is very much related to the zero mass theory results from Sewell's symmetry-increasing restriction axiomatics [19] for the Wightman theory on the d=1+1 light ray horizon (which was adapted to the algebraic setting in [20]). It is my conviction that all the recent speculations about the quantum version of Bekenstein's classical entropy and in particular the horizon ("holographic") aspects of the associated degrees of freedom are manifestations of modular properties of generic nonperturbative QFT which however are overlooked in the Lagrangian quantization method. I hope to return to this and similar subjects in a separate paper with Wiesbrock [17].

The Hilbert space setting of modular localization offers also a deeper physical understanding of the universal domain $\mathcal{D}$ which plays a rather technical role in the Wightman framework [18] In the modular localization approach the necessity for such a domain appears if one wants to come from the (net of) localization spaces which receive their natural topology from the (graphs) net of Tomita operators $\bar{S}(\mathcal{O})$ to a net of (unbounded) polynomial algebras $\mathcal{P}(\mathcal{O})$ such that:

$$dom\ \bar{S}(\mathcal{O})\cap\mathcal{D}=\mathcal{P}(\mathcal{O})\Omega=dom\ \mathcal{P}(\mathcal{O}); \tag{32}$$

This domain is of course also expected to be equal to $\mathcal{A}(\mathcal{O})\Omega$. Here we used a more precise notation which distinguishes between the operator $S$ defined on the core $\mathcal{A}(0)\Omega$ and its closure $\bar{S}$ which is defined on $\mathcal{H}_R(\mathcal{O})+i\mathcal{H}_R(\mathcal{O})$. The apparent connection between the Wightman domain and the the representation theory of a partially geometric but mostly "hidden" ($\simeq$ not visible by quantization methods) representation theory of an infinite dimensional symmetry group in local quantum physics is a truely amazing newly emerging structure.

# 4 Modular Wedge-Localization and Factorizing Theories

In this section we will show that the modular localization can be used as a starting point for constructing for interacting theories. This constructive approach is presently most clear in the case of factorizing d=1+1 QFT's to which we will limit ourselves. We remind the reader that a "factorizable" model in the

[8] The modular group for the correct vacuum differ from the previous situation by a Connes cocycle [3].

intrinsic physical interpretation [33][22] of algebraic QFT means that the long distance limit of the S-matrix (which only consists of the two-particle elastic part since the higher elastic processes factorize in terms of two particle data and therefore automatically fulfills the Yang-Baxter relation as a physical consistency condition) defines a QFT model in its own right, which inherits the name "factorizable" from the S-matrix. It has no real (on-shell) particle creation, but it does possess a rich "virtual" (off-shell ) particle structure i.e. it is a full-fledged QFT with nontrivial vacuum polarization caused by interactions. It should be viewed as being the simplest (no real creation) representative of a vast equivalence class of complicated (with real particle creation) models which share the same particle content together with the same long distance S-Matrix $S_{lim}$. This notion of factorizable is better suited for the present use in local quantum physics than the traditional "integrability" which is defined via quantization of conserved currents. The idea is that the factorizable models are the simplest representatives in a long-distance equivalence class of massive theories which have the same superselection charge structure and the same long distance limit for their S-matrix. In other words each general massive d=1+1 field theory in addition to its well-known membership to a short distance conformal universality class has also a long distance asymptotic companion which has the same superselection sectors ($\simeq$ same particle structure or incoming Fock space), but vastly simplified dynamics associated to a factorizing S-matrix. In d=3+1 this distinguished representative reduces to a free field with the same superselection structure as the other members of the equivalence class. This intrinsic understanding without imposing conservation laws or even Yang-Baxter structures (but rather obtaining them from consistency of the long distance limits of scattering operators) takes away some of the freak appearance of factorizable models and gives them an enhanced significance as the simplest representative in a class of general models with the same charge superselection structure.

The field theoretic understanding of the bootstrap-formfactor program up to now remained largely a collection of more or less plausible (but nevertheless very successful) recipes [7] and leaves a certain amount of conceptual clarity to be desired. Our main concern in this paper is therefore the elaboration of a framework which positions the formfactor-bootstrap program within general QFT in such a way that it can be used as a theoretical laboratory for the latter.

All applications of modular localization to interacting theories are based on the observation that in asymptotically complete theories with a mass gap, the full interaction resides in the Tomita operator $J(W)$, whereas the modular group $\Delta^{i\tau}(W)$ for wedges (being equal to Lorentz boosts) as well as the continuous Poincaré group transformations is "blind" against interactions (the physical representation of the Poincaré group are already correctly defined on the free incoming states). In fact analyzing QFT with the help of scattering theory, the interaction resides in those disconnected parts of the Poincaré group which involve antiunitary time reflections. The Tomita $J$ for the standard wedge inherits this from the TCP operator. To be more explicit the Haag Ruelle scattering theory together with the asymptotic completeness easily yield (for each wedge):

$$J = S_s J_0, \quad \Delta^{i\tau} = \Delta_0^{i\tau} \tag{33}$$

where the subscript 0 refers to the free incoming situation and we have omitted the reference to the particular wedge. Although this is a rather easy consequence of the TCP invariance of the scattering operator [18] together with the recognition that the rotation by which the TCP operator differs from the Tomita involution $J$ [3] is independent of the interaction, the fact that the S-matrix is a relative modular invariant has, according to my best knowledge, hitherto not been appreciated. It is very important here to emphasize that this interpretation of modular data in terms of scattering theories exists *only for wedge regions*. It enters via the TCP invariance and is not just a consequence of Haag-Ruelle scattering theory, since the latter cannot be formulated within wedges. At this point we differ from the approach in [5] which seeks to extract the cyclic relation associated with the crossing symmetry of formfactors by applying scattering theory to the wedge algebra of the interacting local fields. Without going into a detailed critical evaluation of some conceptual errors and misunderstandings one encounters in various recent attempts to go beyond the setting of Smirnov and Karowski et al., we state that there is no *direct* link between the off-shell KMS property and the on-shell crossing symmetry of the S-matrix and formfactors (basically since time-dependent scattering theory cannot be restricted to a wedge). One rather has to construct new, hitherto unknown (wedge-localized, but nevertheless on-shell) generators, in terms of which the KMS condition for the wedge becomes equivalent to the crossing symmetry of the scattering matrix. In fact the cyclic form factor equation is then equivalent to the KMS relation for the mixed correlation functions containing in addition to the new on-shell generators one local operator which may be either a local field or an operator from a double cone subalgebra inside the wedge. These nonlocal on-shell generators[9] are free of additional vacuum polarization contributions to the one particle state vector. Their rapidity space creation- and annihilation- components are forming the Zamolodchikov-Faddeev algebra. We will call them (vacuum-polarization-) free wedge generators $F(x)$, abbreviated FWG, because they are on-shell operators. They are in some sense a nonlocal generalization of free fields and agree with the latter in case of absence of interactions, but they do generate the interacting wedge algebra (and therefore "semilocal"). These objects had been introduced already in previous publications of the author [22][33]. Here we will present them in more details and also use the opportunity to clarify some earlier loose ends and remove possible causes of misunderstandings. The reader who studies carefully the content of the present paper and its planned continuation [17] will realize that the conceptual framework of local quantum physics is extremely tight and does not allow other uses of the Z-F algebraic structure than that for modular localization. The issue is primarily that of the

[9] For this reason they cannot be used to construct better than wedge localizations in the standard way via sharpening the support properties of test functions. Rather compactly localized algebras (as double cone algebras) have to be constructed with the help of intersections and they woud have new generators which, different from the wedge algebra, would be off-shell.

derivation of the Z-F algebra from the general physical principles of LQP and not the invention and postulation of new ad hoc algebraic and representation structures around the Z-F algebra.

No physical interpretation is yet known for the modular objects of compactly localized algebras of e.g. double cones. One expects in that case, that different from the wedge case, not only the reflection $J$, but also the modular group $\Delta^{i\tau}$ will depend on the interaction. As will be seen later, the double cone algebras can be constructed from the wedge algebras.

In order not to change the traditional notation $S$ for the Tomita involutions, we use the subscript $s$ whenever we mean the S-matrix of scattering theory. The most convenient form for the previous equation which puts the modular aspect of scattering in evidence is:

$$S = S_s S_0 \tag{34}$$

where $S$ and $S_0$ are the antiunitary Tomita operators and $S_s$ is the unitary scattering operator. Therefore the scattering operator in relativistic QFT has two rather independent aspects: it is a global operator in the sense of large time limits of scattering theory, and it has a modular localization interpretation in measuring the deviation of $J$ or $S$ from their free field values $J_0, S_0$ i.e. *it is a relative modular invariant.* This modular aspect is characteristic of local quantum physics and has no counterpart in nonrelativistic theory or quantum mechanics.

The modular subspace of $\mathcal{H}_{Fock} \equiv \mathcal{H}_{in}$ for the standard wedge is characterized in terms of the following equations[10]:

$$\begin{aligned} S_s S_0 \mathcal{H}_R &= \mathcal{H}_R \\ S_s S_0 \psi &= \psi, \;\; \psi \in \mathcal{H}_R \end{aligned} \tag{35}$$

There is one more important idea which is borrowed from scattering theory namely the existence of a "modular Møller operator" [22] $U$ which is related to the S-matrix as:

$$S_s = U J_0 U^* J_0 \tag{36}$$

This correspond to the well-known standard formula $S_s = (\Omega^{out})^* \Omega^{in}$. Writing the S-matrix in terms of an Hermitian phase matrix $\eta$ as $S = e^{i\eta}$ it is not difficult to find an $U$ in terms of $\eta$. Note however that the Haag-Ruelle scattering theory (as well as its more formal but better known LSZ predecessor) in local quantum physics does not provide a Møller isometry between Heisenberg states and incoming states because the scattering state space and the space for the interacting fields are identical since in local quantum physics, different from the nonrelativistic rearrangement scattering theory, one does not introduce a separate space of asymptotic fragments.

The idea of introducing such an object into our modular approach comes from the unitary equivalence of the interacting and the free hyperfinite type

[10] The distiction between the $\pm$sign is not very important since the multiplication with $i$ converts one real subspace into the other.

$III_1$ wedge algebras:

$$\mathcal{A}(W) = U\mathcal{A}^{in}(W)U^* \tag{37}$$

We demand the $U$-invariance of the vacuum $U\Omega = \Omega$ and (if this should not turn out to be already a consequence) of the one-particles space $U\mathcal{H}^{(1)} = \mathcal{H}^{(1)}$ . The physical idea behind this is that whereas the vacuum and the one-particle states cannot be resolved from the rest of the energy-momentum spectrum in compact regions as e. g. double cones, the semiinfinite wedge region, which is left invariant by the associated Lorentz-boosts, does allow such a resolution. Note that this situation is different from the problem of unitary equivalence of the canonical equal time commutation relations in the free versus the interacting case. A unitary equivalence in this case (of an algebra belonging to a region with trivial spacelike complement) would be forbidden by Haag's theorem [3] on the nonexistence of the interaction picture in QFT.

The above characterization of $U$ may be replaced by a slightly more convenient one[11] in terms of an intertwining property between modular operators:

$$US_0 = SU \tag{38}$$

In fact the property (37) is equivalent to (38) if the vacuum requirement is added in both cases. It is also equivalent to relation (36) if one uses the fact that a $U$ which leaves the vacuum invariant, has to commute with the modular group [27].

In terms of localized spaces, the $U$ has the property:

$$U\mathcal{H}_R^{in}(W) = \mathcal{H}_R(W) \tag{39}$$

Unfortunately one cannot conclude from this transformation of spaces (39) (which is the only property derivable from (38)) in favor of the validity of (37) and therefore the spatial modular property is too weak for the construction of the wedge algebra. In order to find another method for constructing $\mathcal{A}(W)$, we first study a simple model.

Between the two simplest possibilities, the Ising field theory with $S_s^{(2)} = -1$ and the (non-parity invariant) Federbush model with $S_{I,II}^{(2)} = e^{i\pi g}$ we chose the latter because it allows also a Lagrangian interpretation and hence is simpler to describe to readers familiar with the Lagrangian quantization approach to QFT. The model consists in coupling two species of Dirac fermions via a (parity violating) current-pseudocurrent coupling [23][24]:

$$\mathcal{L}_{int} = g : j_\mu^I j_\nu^{II} : \varepsilon^{\mu\nu}, \quad j_\mu =: \bar{\psi}\gamma_\mu\psi : \tag{40}$$

One easily verifies that:

$$\begin{aligned}
\psi_I(x) &= \psi_I^{(0)}(x) \vdots e^{ig\Phi_{II}^{(l)}(x)} \vdots & (41)\\
\psi_{II}(x) &= \psi_{II}^{(0)}(x) \vdots e^{ig\Phi_I^{(r)}(x)} \vdots \\
\psi_{I,II}^{(0)}(x) &= \frac{1}{\sqrt{2\pi}} \int \left( e^{-ipx} a_{I,II}(\theta) + e^{ipx} b_{I,II}^*(\theta) \right) d\theta & (42)
\end{aligned}$$

[11] I am indebted to H.-W. Wiesbrock for emphasizing this intertwining property as the most convenient definition of $U$.

where $\Phi^{(l,r)} = \int_{x' \lessgtr x} j_0 dx'$ is a potential of $j_{\mu 5}$ i.e. $\partial_\mu \Phi \sim \varepsilon_{\mu\nu} j^\nu = j_{\mu 5}$ and the superscript $l, r$ refers to whether we choose the integration region for the line integral on the spacelike left or right of $x$. The triple ordering is needed in order to keep the closest possible connection with classical geometry and localization and in particular to maintain the validity of the classical field equation in the quantum theory; for its meaning and its conversion into the standard Fermion Wick-ordering we refer to the above papers. This conceptually simpler triple ordering can be recast into the form of the analytically (computational) simpler standard Fermion Wick-ordering in terms of the (anti)particle creation/annihilation operators $a^{\#}_{I,II}(\theta), b^{\#}_{I,II}(\theta)$. Although in this latter description the classical appearance of locality is lost, the quantum exponential do still define local Fermi-fields[24]; in the case of relative commutation of $\psi_I$ with $\psi_{II}$ the contributions from the exponential (disorder fields) compensate. This model belongs to the simplest class of factorizing models (those with rapidity independent S-matrix) and its explicit construction via the formfactor program is almost identical to that of the massive Ising field theory [26]. The reason why it does not appear under this approach in the literature is that the bootstrap classification was limited to strictly parity conserving theories. For our present purposes it serves as the simplest nontrivial illustration of new concepts arising from modular localization.

Despite the involved looking local fields (41), the wedge algebras are easily shown to be of utmost simplicity:

$$\begin{aligned} \mathcal{A}(W) &= alg\left\{\psi_I^{(0)}(f)U_{II}(g), \psi_{II}^{(0)}(h); suppf, h \in W\right\} \\ \mathcal{A}(W') &= \mathcal{A}(W)'_{Klein} = alg\left\{\psi_I^{(0)}(f), \psi_{II}^{(0)}(h)U_I(g); suppf, h \in W'\right\} \end{aligned} \tag{43}$$

i.e. the two wedge-localized algebras ($W$ denotes the right standard wedge) are generated by free fields "twisted" by global $U(1)$-symmetry transformation of angle $g$ (coupling constant) and the subscript "Klein" denotes the well-known Klein transformation associated with the $2\pi$ Fermion rotation. The right hand side follows from the observation that with $x$ restricted to $W$, one may replace the exponential in $\psi_I$ in (41) (which represents a left half space rotation) by the full rotation since the exponential of the right half-space charge is already contained in the right free fermion algebra etc. The following unitarily equivalent description of the pair $A(W), A(W')$ has a more symmetric appearance under the extended parity symmetry $\psi_I(t,x) \leftrightarrow \psi_{II}(t,-x)$:

$$\begin{aligned} \mathcal{A}(W) &= alg\left\{\psi_I^{(0)}(f)U_{II}(\frac{g}{2}), \psi_{II}^{(0)}(h)U_I(-\frac{g}{2}); suppf, h \in W\right\} \\ \mathcal{A}(W') &= alg\left\{\psi_I^{(0)}(h)U_{II}(\frac{g}{2}), \psi_{II}^{(0)}(f)U_I(-\frac{g}{2}); suppf, h \in W'\right\} \end{aligned} \tag{44}$$

The verification of all these properties is elementary, and no modular theory is needed. The computation [24] of the scattering matrix $S_s$ from (41) is most conveniently done by Haag-Ruelle scattering theory [3]:

$$S_s \left|\theta_1^I, \theta_2^{II}\right\rangle = S_s^{(2)} \left|\theta_1^I, \theta_2^{II}\right\rangle = e^{i\pi g} \left|\theta_1^I, \theta_2^{II}\right\rangle \tag{45}$$

$$S_s^{(n)} = \prod_{pairings} S_s^{(2)}$$

These formulae (including antiparticles) can be collected into an operator expression [24] :

$$S_s = \exp i\pi g \int \rho_I(\theta_1)\rho_{II}(\theta_2)\varepsilon(\theta_1 - \theta_2)d\theta_1 d\theta_2 \tag{46}$$

Where $\rho_{I,II}$ are the momentum space charge densities in the rapidity parametrization.

The surprising simplicity of the wedge algebra of this model as compared to, say, its double cone algebras consists in the fact that one can choose on-shell generators. We will show that modular wedge localization for factorizing models always leads to on shell generators for the localized spaces, though for rapidity dependent S-matrices they do not generate the wedge algebras.

It would now be easy to solve the n-particle modular localization equation[12]:

$$\begin{aligned} S\mathcal{H}_R^{(n)}(W) &= \mathcal{H}_R^{(n)}(W);\ \mathcal{H}_R(W) = \oplus_n \mathcal{H}_R^{(n)}(W) \qquad (47) \\ \mathcal{H}_R^{(n)}(W) &= \left\{ \int F(\theta_1, \theta_2, ..., \theta_n) \left|\theta_1, \theta_2, ..., \theta_n\right\rangle d\theta_1 d\theta_2 ... d\theta_n \mid F \in H_{strip}^{(n)} \right\} \end{aligned}$$

Here $H_{strip}^{(n)}$ denotes the closure of the space of square integrable function which allow an analytic continuation into the strip $0 < Imz_i < \pi,\ i = 1...n$ for ordered $\mathrm{Im}\, z's$. This issue of analytic properties and boundary conditions becomes much clearer on the level of the later introduced operators which generate the wedge algebra (rather then on the level of spatial modular theory (47)). Then the analytic multivariable strip properties and the appropriate boundary conditions reflect the KMS and algebraic commutation properties of the wedge generators. This follows from the abstract modular setting and one finds a detailed presentation in [28]. In analogy to Wightman functions in x-space, the $n!$ different boundary prescriptions $Imz_{i_1} > Imz_{i_2} > .... > Imz_{i_n} \longrightarrow 0$ yield the generally $n!$ different boundary values $F(\theta_{i_1}, \theta_{i_2}, ...\theta_{i_n})$ which are described by one masterfunction $F$. Similar statements hold for the boundary values on the upper rim. In the free case i.e. for $S_s = 1$, one just recovers the Bose/Fermi statistics but with the Federbush S-matrix the space consists of strip-analytic functions which are a solution of a Riemann-Hilbert boundary problem. The general solution of this problem (i.e. the characterization of the subspace $\mathcal{H}_R(W)$ within the full multiparticle wave function space) may be presented as a product of special solution of the Riemann-Hilbert problem with the general solution of the interaction free problem in $\mathcal{H}_R(W)^{in}$. A convenient way to write the solution consists in using auxiliary operators $Z$ as generators for a basis of state

[12]The Tomita operator $S$ for Fermions is different from that of Bosons by a Klein transformation. For a special family of d=1+1 solitons the correct TCP operator has been computed by Rehren [25]. Since all the known families of factorizing models are described by Fermions and Bosons and since it is not clear whether this generalization is compatible with the factorization we will ignore this more general TCP-situation in the present context.

vectors:

$$\int d^2x_1...d^2x_n\hat{f}_n(x_1,....x_n) \quad : \quad Z_{I,II}(x_1)....Z_{I,II}(x_n):\Omega,$$
$$supp\hat{f}_n \quad \in \quad W^{\otimes n},\ \hat{f}_n\ real \tag{48}$$

where the $Z's$ are on shell operators whose frequency positive and negative momentum space components have to fulfill commutation relations which must be compatible with the boundary relations governed by products of two particle S-matrices. One immediately realizes that this leads to the Zamolodchikov-Faddeev algebra relations for the Federbush S-matrix:

$$Z_{I,II}(x) = \frac{1}{\sqrt{2\pi}}\int\left(e^{-ipx}c_{I,II}(\theta) + e^{ipx}d^*_{I,II}(\theta)\right)d\theta \tag{49}$$

where the $c$ and the corresponding anti $d$ can be formally expressed in terms of the incoming (anti)particle creation and annihilation operators:

$$\begin{aligned} c_{I,II}(\theta) &= a_{I,II}(\theta):e^{-i\pi g\int_{-\infty}^{\theta}\rho_{II,I}(\theta')d\theta'}: \\ d_{I,II}(\theta) &= b_{I,II}(\theta):e^{i\pi g\int_{-\infty}^{\theta}\rho_{II,I}(\theta')d\theta'}: \end{aligned} \tag{50}$$

with the Zamolodchikov-Faddeev relations between the type I and II particles:

$$\begin{aligned} c_I(\theta_1)c_{II}(\theta_2) &= -S^{(2)}(\theta_1-\theta_2)c_{II}(\theta_2)c_I(\theta_1) \\ c_I(\theta_1)c^*_{II}(\theta_2) &= -S^{(2)}(\theta_1-\theta_2)^{-1}c^*_{II}(\theta_2)c_I(\theta_1) \\ d_I(\theta_1)d_{II}(\theta_2) &= -S^{(2)}(\theta_1-\theta_2)d_{II}(\theta_2)d_I(\theta_1) \\ d_I(\theta_1)d^*_{II}(\theta_2) &= -S^{(2)}(\theta_1-\theta_2)^{-1}d^*_{II}(\theta_2)d_I(\theta_1) \\ d_{I,II}(\theta_1)c_{II,I}(\theta_2) &= -S^{(2)}(\theta_1-\theta_2)^{-1}c_{II,I}(\theta_2)d_{I,II}(\theta_1) \\ d_{I,II}(\theta_1)c^*_{II,I}(\theta_2) &= -S^{(2)}(\theta_1-\theta_2)c^*_{II,I}(\theta_2)d_{I,II}(\theta_1) \end{aligned} \tag{51}$$

and the free Fermion anticommutation relations between the same type. The simplicity of the model is reflected in the fact that interactions only take place between species I and II and the independence of $S^{(2)}$ on $\theta$. It is now easy to check that the solution of the wedge localization equation (47) has indeed the form (48). The use of the Z-basis takes care of the boundary conditions in an analogous way as the (anti)symmetry of the coefficient functions follows from the commutation relations of the Fermion/Boson operator basis. In particular it takes care of the inversion of the order in Z-products under Hermitian conjugation: $SA\Omega = A^*\Omega$, with $A$ an operator as in (48). as well as the coalescence of the x-space commutation relations for (49) with those of (43) or (44). The interaction does make a distinction between left and right and parity is only conserved if one also interchanges the two species. The wedge localized states fulfill the following thermal KMS condition

$$\varphi(t) = \left\langle\psi_2 \mid \Delta^{it}\psi_1\right\rangle,\ \ \varphi(t-i) = \left\langle\Delta^{it}\psi_1 \mid \psi_2\right\rangle \tag{52}$$

where $\psi_{1,2}$ are n-particle localized states of the form (48) and $\varphi$ is analytic in the open t-strip and continuous on the boundary.

A calculation of the $J$-transformed operators $Z^J = JZJ$ with

$$\begin{aligned} c^J_{I,II}(\theta) &= a_{I,II}(\theta) : e^{-i\pi g \int_\theta^\infty \rho_{II,I}(\theta')d\theta'} : \\ d^J_{I,II}(\theta) &= b_{I,II}(\theta) : e^{i\pi g \int_\theta^\infty \rho_{II,I}(\theta')d\theta'} : \end{aligned} \tag{53}$$

reveals that $Z^J$ commutes with $Z$. This is a speciality of models with rapidity independent S-matrices. In such models the wedge locality is manifest since in a (selfexplanatory notation)

$$\left[Z_I^{J\#}(\theta), Z_{II}^{\#}(\theta')\right] = 0 \tag{54}$$

and all other commutators between operators $Z^J$ and $Z$ of the same type are like those of free fields. This means that $Z^J(x)$ commutes with $Z(y)$ for $x, y \in W$ , even though the $Z^{\#}(x)'s$ are nonlocal fields.

For more general factorizing models however the verification of wedge locality of the FWG operators is more subtle. We illustrate the procedure for a factorizing model with one interacting (selfconjugate) particle. In that case the $Z$ has the following form in terms of the scattering phase shift $\vartheta$

$$\begin{aligned} Z(x) &= \frac{1}{\sqrt{2\pi}} \int \left\{e^{-ipx} Z(\theta) + h.c.\right\} d\theta, \ p = m(ch\theta, sh\theta) \\ Z(\theta) &= a(\theta) e^{-i\int_{-\infty}^{\theta} \vartheta(\theta-\theta')a^*(\theta')a(\theta')d\theta'}, \ S_{sc}(\theta) = e^{i\vartheta(\theta)} \end{aligned} \tag{55}$$

$$\begin{aligned} Z^{\#}(\theta_1)Z^{\#}(\theta_2) &= S_{sc}(\theta_1 - \theta_2)Z^{\#}(\theta_2)Z^{\#}(\theta_1), \ Z^{\#} \equiv Z \ or \ Z^* \\ Z(\theta_1)Z^*(\theta_2) &= S_{sc}^{-1}(\theta_1 - \theta_2)Z^*(\theta_2)Z(\theta_1) + \delta(\theta_1 - \theta_2) \end{aligned} \tag{56}$$

For the following we find it convenient to introduce the path notation which allows to denote rapidity space creation and annihilation operators by one symbol. We write for free fields $A(x)$ smeared with wedge supported test functions

$$\begin{aligned} A(\hat{f}) &= \int f(\theta)a(\theta) + \int f(\theta - i\pi)a(\theta - i\pi) \equiv \int_C f(\theta)a(\theta) \\ & a(\theta - i\pi) \equiv a^*(\theta), \ f(\theta - i\pi) = \bar{f}(\theta) \end{aligned} \tag{57}$$

$C$ consists of the real $\theta-$axis and the parallel path shifted down by $-i\pi$ and it is only the function $f$ which is analytic in the strip $-\pi < Im\theta < 0$ and not the operators. The analyticity of $\hat{f}$ is equivalent to the localization property. We will use the same notation for the FWG $Z(\hat{f})$. In the application to the vacuum of course only the creation contribution from the lower rim of the strip survives.

We want to prove that the on-shell $Z$ is a FWG i.e. the smeared operator $Z(\hat{f}) = \int Z(x)\hat{f}(x)d^2x$ with supp$\hat{f} \in W$ generates the $^*$-algebra of the interacting theory localized in the wedge. In formula

$$\begin{aligned}\left[JZ(\hat{f})J, Z(\hat{g})\right] &= 0,\ \operatorname{supp} \hat{f}, \hat{g} \in W \\ Z(\hat{f})\Omega &= \int d\theta \bar{f}(\theta) a^*(\theta)\Omega\end{aligned} \tag{58}$$

To prove this one first notices that the $Z(\theta)^\#$ commutes with the $JZ(\theta)^\# J$ underneath the Wick-ordering. The reason is that the exponential involve integrals over number density which extend over complementary rapidity regions. The numerical phase factors which originate from the commutation of these exponential factors with the $a^\#{}'s$ mutually compensate. There remains the contraction between the pre-exponential $a^\#{}'s$ which leads to

$$\int \bar{f}(\theta) g(\theta - i\pi) \exp i \int \delta_{sc}(\theta - \theta')n(\theta')d\theta' \tag{59}$$

Shifting the integration by $i\pi$, and using the crossing symmetry in the form: $\delta_{sc}(\xi + i\pi) = \delta_{sc}(-\xi) + 2\pi n i$, we see that this contraction is equal to that in the opposite order (which has the negative exponential). The pointwise equation $JA(-x_0, x)J = PA(x)P$ for the local field $A(x)$, with $P$ being the generator of the parity transformation, is to be replaced by $P\left(\int Z(\theta)f(\theta)d\theta\right)P \in J\mathcal{A}(W)J$, a relation which is easily checked using the explicit form of the Z-operators.

Now we come to the crucial part of the modular localization method, the exploration of consequences of the KMS condition for the Z-correlation functions. As a typical case we consider the 4-point function.

$$\begin{aligned}(\Omega, Z(f_{1'})Z(f_{2'})Z(f_2)Z(f_1)\Omega) &\equiv \langle Z(f_{1'})Z(f_{2'})Z(f_2)Z(f_1)\rangle_{therm} \\ &= \left\langle Z(f_{2'})Z(f_2)Z(f_1)Z(f_{1'}^{2\pi})\right\rangle_{therm}\end{aligned} \tag{60}$$

Each side is the sum of two terms, the direct term associated with

$$\begin{aligned}Z(f_2)Z(f_1)\Omega &= \int f_2(\theta_2 - i\pi) f_1(\theta_1 - i\pi) Z^*(\theta_1) Z^*(\theta_2)\Omega + c\ number \cdot \Omega \\ &= \int f_2(\theta_2 - i\pi) f_1(\theta_1 - i\pi) S(\theta_2 - \theta_1) a^*(\theta_1) a^*(\theta_2)\Omega + c\Omega\end{aligned} \tag{61}$$

and the analogous formula for the bra-vector. For the inner product there are two contraction terms consisting of direct and crossed contraction (in indices 1 or 2) of the $a^\# s$. Only the second one gives an S-matrix factor in the integrand. The c-number term an the left hand side cancels the direct term on the right hand side. The equality of the crossed terms on both sides gives (using the denseness of the analytic wave functions)

$$S(\theta_2 - \theta_1) = S(\theta_1 - \theta_{1'} + i\pi)\mid_{\theta_{1'} = \theta_2} \tag{62}$$

i.e. one obtains the above crossing relation for the two particle S-matrix. Higher inner products involve products of S-matrices, and it is easy to see that the KMS condition for the FWG algebra is equivalent to the crossing property of the S-matrix. The presence of additional local operators $A$ which can be a fortiori localized in the wedge does not influence the validity of the KMS condition.

$$\begin{aligned} & \langle Z(f_{1'})Z(f_{2'})...Z(f_{m'})AZ(f_n)...Z(f_2)Z(f_1)\rangle_{therm} = \\ = \quad & \left\langle Z(f_{2'})..Z(f_m)AZ(f_n)..Z(f_2)Z(f_1)Z(f_{1'}^{2\pi})\right\rangle_{therm} \end{aligned} \tag{63}$$

The rapidity space formulation of this KMS condition is (again using denseness of wave functions, the derivation is completely analogous to the previous case) the desired cyclicity relation for the formfactor of that local operator $A$ (its coefficient functions in the sense of (65) below:

$$a_n(\theta_1, \theta_2, ..., \theta_n) = a_n(\theta_2, ..., \theta_n, \theta_1 - 2\pi i) \tag{64}$$

This equation was hitherto derived as a special consequence of the crossing symmetry for factorizing systems. The crossing symmetry itself in turn follows from the LSZ scattering theory together with certain analytic assumptions ([32]) which are presently only controllable in the factorizing setting. Without the mediation of the semilocal FWG $Z^{\#}$ it would not have been possible to link the cyclicity equation with the KMS property of $\mathcal{A}(W)$ i.e. as was already stated in different words previously, the KMS relation only takes the form of the cyclicity relation in the semilocal FWG field "coordinates". Any scheme which relates the off-shell thermal KMS property with the on-shell crossing symmetry (and its manifestation as the cyclicity equation in factorizable models) must implicitly contain the FGW operators, they are the only mediators between the off-shell KMS property and the on-shell cyclicity relation. A direct relation through some kind of scattering theory within a wedge as imagined in [5] without their use is not possible.

Contrary to the previous free case, the sharpening of the support of the test function does not improve the localization within the wedge. This is equivalent to the statement that the reflection with $J$ does not create an operator which is localized at the geometrically mirrored support region of the test function $\hat{f}$ in the opposite wedge $W'$. It only fulfills the commutation relation with respect to the full $W'$. In fact the breakdown of parity covariance is important for the existence of such nonlocal but wedge-localized fields, since fields which are covariant under all transformations are expected to be either point local or completely delocalized (i.e. not even in a wedge). We will later see that any sharper localization requires the operator to be an infinite power series in the $Z's$ :

$$A = \sum \frac{1}{n!} \int_C ... \int_C a_n(\theta_n, \theta_{n-1}, ....\theta_1) : Z(\theta_1)....Z(\theta_n) : \tag{65}$$

where we again used the previously explained path notation. The sharper localization leads to relations between the $a_n's$ . Note that since the commutations of $Z's$ produce S-matrix phase factors, the $a_n$ must compensate this phase factors upon commuting $\theta's$.

$$a_n(\theta_1, ...\theta_i, \theta_{i+1}, ..\theta_n) = S_{sc}(\theta_i - \theta_{i+1})a_n(\theta_1, ...\theta_{i+1}, \theta_i, ..\theta_n) \tag{66}$$

In this respect the phase factors are like statistics terms. However since the $a_n$ have meromorphic properties in the multi-$\theta$ strip (actually for compact localization the meromorphy region is much bigger), these phase factors must be consistent with the univaluedness in the analytic domain. Together with other requirements, this leads to a multi-variable Riemann-Hilbert problem for the $a_n^{\prime}s$ ([32]). Since the d=1+1 double cone algebras $\mathcal{A}(\mathcal{O})$ are obtained by translations and intersections

$$\mathcal{A}(\mathcal{O}_a) = U(-\frac{a}{2})\mathcal{A}(W)U(\frac{a}{2}) \cap U(\frac{a}{2})\mathcal{A}(W')U(-\frac{a}{2}) \tag{67}$$

this sharper localization leads to additional restrictions for the $a_n^{\prime}s$. For a detailed characterization of these algebras in terms of generators I refer to forthcoming joint work [17]. It is very important to appreciate the difference between classical and quantum localization. For the FWG fields the quantum localization is defined by these intersection of algebras and cannot be replaces by the classical localization in the sense of support properties of test functions.

It is instructive to reproduce the results of Smirnov and the reformulation of Laskevich [7][29] in the present setting.

**Theorem 3** *A sufficient condition for a power series in the $Z$-fields to describe a pointlike local field $A(x)$ is in addition the previous commutation (66)and cyclicity property (64) (in case of our illustrative model without bound states) the presence of poles in the $a_n(\theta_1, \theta_2....\theta_n)$ for $\theta$-differences lying on the boundary of the strip:*

$$a_{n+2}(\vartheta + i\pi + i\varepsilon, \vartheta, \theta_1, \theta_2....\theta_n) \simeq \frac{1}{\varepsilon}\left[1 - \prod_{i=1}^{n} S(\vartheta - \theta_i)\right] a_n(\theta_1, \theta_2....\theta_n) \tag{68}$$

Here we did not specify the Lorentz transformation property of the field

$$\begin{aligned} A(x) &= \sum_n \frac{1}{n!}\int_C d\theta_1 ... \int_C d\theta_n e^{-iP(\theta_1,...\theta_n)x} a_n(\theta_n, \theta_{n-1}....\theta_1) : Z(\theta_1)....Z(\theta_n) : \\ & P = \sum p_i(\theta) \end{aligned} \tag{69}$$

If the field is irreducible with spin s one must have

$$a_n(\theta_1 + \vartheta, \theta_2 + \vartheta....\theta_n + \vartheta) = e^{-s\vartheta} a_n(\theta_1, \theta_2....\theta_n), \tag{70}$$

but in the proof this is not needed. The proof is purely structural i.e. independent of the computation or parametrization of the solutions of all these conditions.

**Proof (**following Lashkevich [29]**)**

we show that for spacelike separation we can by contour deformation transform of

$$A(x)B(y) = \sum \frac{1}{m!n!k!}\int_{-\infty}^{\infty} d^k\xi \int_C d^m\theta \int d^n\vartheta \times \tag{71}$$

$$\times \exp -i\left[P(\xi)(x-y)+iP(\theta)x-iP(\vartheta)y\right]$$
$$\times a_{m+k}(\xi_1,...,\xi_k,\theta_1,...,\theta_m)b_{n+k}(\vartheta_n,...,\vartheta_1,\xi_k-i\pi,,...,\xi_1-i\pi)$$
$$\times :Z(\theta_m)...Z(\theta_1)Z(\vartheta_1)...Z(\vartheta_n):$$

achieve the opposite order. Since the $Z's$ underneath the Wick-product do not commute (unlike the previous case of $Z^J$ with $Z$), the compensating terms must come from the $i\pi$ shift of contour, i.e. from residua of poles encircled in the process of deformation. In order to apply (48) say to $\xi_i$ together with $\theta_j$ (i.e. the pole at $\xi_i=\theta_j+i\pi$), we have to position the relevant variables at the first two places in $a_{m+k}$ by successive transpositions. Whereas the contributions from $\xi$-permutations in $a_{m+k}$ compensate those in $a_{n+k}$, from the $\theta$ one obtains the phase factor $\Pi_{l\neq i}S(\xi_l+i\pi-\theta_j)$. It is convenient to shift simultaneously $Z(\theta_j)$ to the right and denote the total resulting phase factor:

$$s(\xi_i)\equiv\Pi_{l\neq i}S(\xi_l+i\pi-\theta_j)\Pi_{l=1}^nS(\theta_j-\vartheta_l) \tag{72}$$

Now we do the contour shifting which according to (48). We only pick up poles from the coefficient function $a_{m+k}$. As a result we get the additional factor -$[1-\Pi_{l\neq i}S(\theta_j-\xi_l)\Pi_{l\neq j}S(\theta j-\theta_l)]$. Note that for space-like distances (we can place the x,y so that they are in opposite wedges) the exponential factor is damped in $0<Im\xi<\pi$. The integrand of the term $(m,n,k)$ is, apart from the mentioned two factors and the exponential function, as follows

$$a_{m+k-2}(\xi_1,..,\hat{\xi}_i,..,\xi_k,\theta_1,..,\hat{\theta}_j,..,\theta_m)b_{n+k}(\vartheta_n,..,\vartheta_1,\xi_k-i\pi,,..,\theta_j,..,\xi_1-i\pi,\theta_j)$$
$$\times :Z(\theta_m)...\widehat{Z(\theta_j)}...Z(\theta_1)Z(\vartheta_1)...Z(\vartheta_n)Z(\theta_j): \tag{73}$$

where as usual the roof sign indicates deletion of the variable or the operator. commuting the operator $Z(\theta_j)$ to the left, we obtain a phase factor which cancels the previous factor evaluated at the pole $s(\xi_i=\theta_j+i\pi)$. The important step is now the following renaming (leading to a resummation in $\sum$): $\theta_j\rightarrow\vartheta_{n+1}$, followed by $n\rightarrow n-1, m\rightarrow m+1$ and $k\rightarrow k+1$ and re-numbering $\xi$ and $\theta$

$$-\left[1-\Pi_{l\neq i}S(\theta_j-\xi_l)\Pi_{l\neq j}S(\theta j-\theta_l)\right]a_{m+k}(\xi_1,..\xi_k,\theta_1,..,\theta_m) \tag{74}$$
$$\times b_{n+k}(\vartheta_n,..\vartheta_1,\xi_k-i\pi,..\xi_1-i\pi):Z(\theta_m)..Z(\theta_j)..Z(\theta_1)Z(\vartheta_1)..Z(\vartheta_n):$$

This shifting process has to be repeated with every $\xi$ and the boundary term has to be taken into account (i.e. the remaining integration variables are placed on the boundary $\xi\rightarrow\xi+i\pi$). Using the following identity

$$\sum_{l=0}^n(-)^l\sum_{j_1<...j_l}\prod_{r=1}^l(1-t_{j_r})=\prod_{j=1}^nt_j \tag{75}$$

one arrives at the desired result

$$A(x)B(y)=B(y)A(x),\ \ (x-y)^2<0 \tag{76}$$

As mentioned before this somewhat tricky derivation is not quite in the spirit of our modular approach. The latter requires to study the double cone algebras i.e. to characterize the quantum localization properties of intersections and to derive a *necessary and sufficient* condition on the $a_n$. It turns out that this modular method simplifies the computations [17] and solves the existence problem. In the standard bootstrap-formfactor approach the existence problem is the proof that the fields do not only exist in the sense of sesquilinear forms on a dense set of states (i.e. as formfactors), but also in the sense of Wightman i.e. in correlation functions and smeared operators. Unfortunately this existence problem in the standard approach becomes inexorably linked with the technically very demanding problem of construction of a basis in field space [45]. In a forthcoming paper we will show how the proof of nontriviality of the double cone algebra bypasses the latter problem and proves the existence.

One can show that the modular construction of "free" anyons and plektons [33] in d=1+2 leads to similar mathematical problems[13]. In this case there is no scattering, but the whole construction takes place in a multiparticle space which, in contradistinction to Fermions and Bosons, has no tensor product structure in terms of Wigner spaces. The braid group commutation relation leads to a Tomita $J$ which, similar to the Federbush model, involves a constant matrix $S_{twist}$. In this case $S_{twist}$ does not carry scattering information, but is identical to the braid group representation R-matrix which appears in the exchange algebra of conformal QFT. This statistical R-matrix $S_{twist}$ has a similar effect as a Klein transformation i.e. the opposite of $\mathcal{A}(W)$ in the quantum sense of the von Neumann commutant is now *different from the geometric opposite* $A(W)' \neq A(W')$. Again the model has a rich virtual particle structure, even though no real particle is created in a scattering process. But since this time this vacuum polarization is a result of the nontrivial statistics twist $S_{twist}$, there is no reason to expect that this goes away in the nonrelativistic limit and the model passes to Schrödinger theory. The fact that the nonrelativistic limit of Fermions and Bosons leads to the Schrödinger QM is related to the *existence of relativistic free fields in Fock space.* But since the Fock space structure in d=1+2 cannot support anyons and plektons, there is good reason to expect a kind of *nonrelativistic field theory which can incorporate the virtual particle or vacuum polarization structure* which is *necessary* to *maintain the relation between spin and (plektonic) statistics in the nonrelativistic limit.* Indeed all attempts to incorporate braid group statistics into QM, ever since the time of Leinaas and Myrheim [36] [37] have only led to a deformation of (half)integer spin [40] in quantum mechanical two-particle problems, but not to bona fide nonrelativistic operators with the correct spin-statistics commutation structure which would insure consistency with the multi-particle structure. To phrase it into a more mundane fashion: ther are good reasons why nobody has succeeded to construct a QM interaction (e.g. Aharonov-Bohm like) which leads to an anyonic spin in the two-particle S-matrix and fulfills those higher particle S-

[13]The question whether anyon or plekton correlation functions allow for a Chern-Simons euclidean functional representation is, at least within the present approach, of a more academic nature.

matrix cluster properties which are necessary in order to obtain coherence with the multiparticle statistics. The attempts based on Aharonov-Bohm potentials allow only to mimic an anyonic two-particle spin. I would not expect that the scattering boundary condition for the long range A-B interaction of n-particles can be chosen in such a way that the multiparticle S-matrix fulfills the cluster property which certainly would be a prerequisite for sustaining a nonrelativistic spin-statistics connection. Like e.g. the nonrelativistic Lee model, I rather expect anyons and plektons to be only describable by a (Galilei-invariant) *nonrelativistic field theory* which maintains the vacuum polarization and this is my explanation for why the search for a consistent theory of particles with genuine braid group statistics in quantum mechanics has been unsuccessful.. At this point it is instructive to compare with a recent No-Go theorem [39].

The method advocated here by analogy to factorizing d=1+1 models of using the Wigner one particle representations combined with the correct non-tensor-product multiparticle structure from scattering theory [38] together with the present modular localization method looks very promising (but still needs to be carried out). The results should be of great practical relevance for unraveling the structure of (nonrelativistic) d=1+2 quasi-particles in many body systems with spatial layer structures [41]. If the explanation behind phenomena as the fractional quantum Hall effect or high $T_c$ superconductivity are manifestations of truely fundamental 2-d quasi-particle properties than they must be related to string-like localized charges with plektonic fusion rules leading to braid group statistics because the latter is the only fundamental property which separates low dimensional local quantum physics from the higher dimensional case. As with Fermions and Bosons, the derivation of the spin-statistics properties and the classification of the possible statistics (+internal symmetry) is done in the tighter relativistic setting, in order then to be used e.g. for the (nonrelativistic) quasi-particles in condensed matter physics and statistical mechanics. In that case the new setting has (even nonrelativitically, as was explained before) no formulation in quantum mechanics and the Laughlin wave function is a phenomenologically successful description whose possible theoretical merits still have to be found. The problem of d=1+2 plektons carries however also a significant theoretical weight quite independent of their possible role in condensed matter physics. Whenever we find a new realization of old physical principles which we did not invent but which the present framework allows, we must investigate all its physical consequences, and the zeroth step would be to construct and physically explore their dynamically simplest realization. Even if nature does not realize this structure, it will enrich our understanding to find theoretical reasons for this situation (this is the difference to inventions).

A similar idea of using an $S_{twist}$ in a constructive modular approach should also be helpful to complete the classification and construction of chiral conformal QFT's. In this connection one should recall that presently the construction of these models is done in most cases by studying the representation theory of affine algebras, in other words by ideas which are quite different from those of standard QFT. The charge-carrying fields which fulfill braid group exchange algebras are then constructed as intertwiners between the vacuum and charged representations of these algebras. It would be more in the spirit of QFT to first classify all plektonic statistics (the structure constants of the exchange algebras) and then to construct the vacuum representation (expected to be unique) of the associated exchange algebras. There is however one caveat: the *exchange algebra* (unlike the CCR or CAR algebras) *is incomplete* since the distributional behavior at coalescent points is left undetermined. This is the reason why the attempts in e.g. [43] which were based on monodromy properties are more "artistic" (in the sense of depending on hindsight and luck) than systematic. Momentum space algebras, as the Z-F algebra on the other hand, are complete. But can one use the R-matrices as a Klein twist $S_{twist}$ in momentum space using the previous method of wedge localization? Since conformal theories have an infraparticle- but not a particle-structure, this problem is not trivial. The normalization for obtaining finite form factors is infrared divergent with respect to that of finite correlation functions. We hope to present an affirmative answer in a separate work. The importance of such a new approach to chiral conformal QFT results from the fact that it places it back into the mainstream of QFT where it belongs. The statistics input in the absence of genuine interaction (chiral conformal QFT's are expected to belong to this kind since there can be no genuine coupling parameter dependent interaction on one light cone) and for given internal symmetry groups (in order to have a distinction between charged fields for current algebras and the so-called W-fields of chiral conformal field theory) should uniquely determine the vacuum representation of the completed exchange algebra. In more technical terms and only for experts on conformal QFT: the Friedan-Qiu-Shenker-quantization of the energy-momentum tensor algebra and similar quantizations for W-algebra generalizations should follow from the more fundamental DHR-Jones-Wenzl quantization which has a direct interpretation in terms of particle statistics. In this way chiral conformal QFT would change from a mathematical playground back to a valuable theoretical laboratory of QFT since it can then be formulated as simple analytic realization of principles which have validity in higher dimension.

In the remaining part of this section I would like to make some pedagogical remarks for the non-expert reader on matters of general structural properties of nonperturbative QFT. In connection with the analytic aspects of the rapidity-dependent Zamolodchikov-Faddeev algebra operators and their analogy with the x-space chiral conformal operators of the exchange algebras one often finds the erroneous concept of "analytic field operators" and "holomorphic algebras". Since their use is so widespread (the few articles where this misleading terminology is not used are rather the exception than the rule), it is interesting to ask where such ideas are coming from. I am not an expert on string theory, therefore I have limited my search to QFT. The oldest paper which could be (and was) misinterpreted as alluding to "analytic operators" $A(z), z \in \mathbf{C}$ seems to be the famous BPZ [30] paper[14] on minimal models. Although the

[14] In an older paper on conformal blocks e.g. [31] (called nonlocal components in a conformal decomposition theory with respect to the center of the universal covering), such "holomorphic" terminology was never used.

authors do not use such terminology in print, the notation used in that paper may have caused misunderstandings (and has been misunderstood by physicist whose first experience with nonperturbative QFT came through that famous work or was influenced by string theory). The truth is that *field algebras never have holomorphic properties, not even in "holomorphic algebras" of chiral conformal QFT.* The analytic properties of correlation functions and state vectors on the other hand depend entirely on the nature of states one puts on those algebras. Whereas vacuum ground states lead to the famous BHW-domain [18] (in chiral conformal QFT equal to a uniformormization region with poles for coalescing coordinates), KMS states will only lead to strip analyticity. It is in any case the state which generates the analytic continuation holomorphy properties, and the higher its symmetry property, the bigger the holomorphic regions.

It is a fact that the associated analytic Bargman-Hall-Wightman domain for a Moebius invariant vacuum state in conformally covariant QFT is larger than that of the corresponding Poincaré covariant massive theories and as a result of braid group statistics, one looses the univaluedness of these analytic continuations (but of course not the univaluedness in the real time physical localization points!). The effect of restriction of the vacuum state of massive theories to the wedge algebra yields the momentum space analytic properties which together with the factorization property lead to the rapidity being an analytic uniformization variable. Contrary to the use of field theoretic terminology in the contemporary literature neither the old nor the new BHW domains nor the analytic continuation in rapidity have anything to do with the "living space" of fields in the sense of quantum localization of operators in this article. "Localization" is indeed a property of the algebras whereas holomorphy is not.

In an attempt to attribute despite the negative previous remarks a constructive algebraic meaning to the above unfortunate but nevertheless popular terminology, one could point to the following property which has *no analogue in higher dimension* (not even in the conformal invariant limit of higher dimensional theories). Whereas generally with vacuum expectation values one can relate at most two physical theories: a noncommutative real time QFT and a commutative euclidean field theory (a candidate for a continuous statistical mechanics), a chiral conformal theory on one light cone has *infinitely many noncommutative boundary values* (and no commutative chiral boundary value) each of which defines a set of positive definite correlation functions and hence a theory. This is to say the restriction of the analytically continued correlation function defines a positive QFT not only on the circle (the standard living space of chiral theories) but also *on each boundary encircling the origin* (with the right $i\varepsilon$ Wightman boundary prescription)[15]. However all this does not legitimize "holomorphic operators" in the literal sense but rather the existence of a operator conformal QFT for each chosen boundary circumference. The reader recognizes easily that this structure is equivalent to the existence of the infinite dimensional diffeomorphism group which is related to the Virasoro algebra structure. The application of any symmetry, which does not leave

[15]This is probably what string theorist have in mind when they draw their pictures.

the vacuum reference state invariant, defines another set of positive Wightman functions which at the case at hand belong to the deformed boundary. If one prefers a geometrical to a quantum physical terminology, one may emphasize the diffeomorphism group, whereas for a physicist who prefers the setting of local quantum physics the existence of a an interesting generalization of the higher dimensional dichotomy between real time-imaginary time theories may be the interesting aspect.

# 5 General Interactions and Outlook

The factorizing models of the bootstrap formfactor approach belong to a class of models which are "real particle" (on shell) conserving but "virtual particle" (off shell) nonconserving. In the context of our modular localization approach this means that although the wedge localization Hilbert spaces can be generated by on shell FWG operators, any sharper localization as e.g. the double cone localization and in particular the state vector obtained by applying smeared pointlike local fields to the vacuum lead to "virtual particle clouds". Although absence of real particle creation according to well-known theorems is impossible in d=1+3 interacting theories, we expect the three-dimensional theories d=1+2 to form an exception if the associated particles obey genuine braid group statistics (anyons and plektons) and hence their charge-carrying field have a noncompact semiinfinite string-like extension [33]. Namely we expect the existence of "free"[16] anyons and plektons which similar to free Fermions and Bosons have no on-shell creation but (different from free Fermions and Bosons) possess a rich virtual particle structure [39].

Let me finally address the important question of whether the concept of modular localization can be expected to lead to a nonperturbative approach for d=1+3 interacting theories of Fermions and Bosons. This depends on whether it is possible to relate an admissable S-matrix with an auxiliary semilocal (wedge localized) operator $Z$ which is free of vacuum polarization and creates the interacting localized wedge spaces $\mathcal{H}_R(W)$. The existence of such an "on-shell" operator would be the correct substitute for the FWG operators in the case of non-factorizing situations. On a formal level such operators have appeared in the light cone quantization of interacting theories [44] but in that formalism the relation to the original local variables (with vacuum polarization) gets lost and without relation to local operators one has no physical interpretation beyond global spectral properties [42]. Our modular wedge localization approach on the other hand does not depend on assumptions about canonical commutation relations and therefore could serve as a rigorous substitute for the ill-defined light cone quantization.

[16] If "free" in d=1+2 implies the existence of generating fields $A$ in Fock space without vacuum polarization clouds in $A\Omega$ then plektons are never free because the vacuum polarization is needed to sustain the braid group statistics and the associated string-like extension. We use this terminology in the more liberal sense of absense of additional interactions which cause nonvanishing scattering cross sections.

A prerequisite for all modular constructions including the modular approach to the formfactor bootstrap program is its uniqueness i.e. to one admissable S-matrix there should be only one local algebraic net. This is part of the more general question whether modular data $(J, \Delta^{it})$ determine uniquely an algebra and a state. Whereas this modular inverse problem has many solutions [16], the corresponding inverse problem of QFT

$$S_{sc}\ in\ \mathcal{H}_{Fock} \stackrel{?}{\rightarrow} \{A(W)\}_{all\ wedges} \tag{77}$$

can be shown to have a unique solution even beyond the special factorizing d=1+1 models [17] if one accepts certain plausible but presently unproven) vacuum and one-particle properties of the modular Møller operator $U$ [17].

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