Class Operators as Intertwining Maps into the Group Algebra

Aleksander Strasburger

Department of Mathematical Methods of Physics,
University of Warsaw
Hoża 74,
00-682 Warszawa, Poland
e-mail: alekstra@fuw.edu.pl

Abstract

With the aim of completing the previous study by A. Orłowski and the author concerning intertwining maps between induced representations and conjugation representation, termed here weighted class operators, we compute the latter explicitely for the conjugation representation arising from the regular representation in the group algebra of a compact group. To that efect a theorem of Wigner–Eckart type for weighted class operators obtained from matrix coefficients of irreducible representations of a compact group is proved. Also the previous construction of weighted class operators is reviewed and extended to the case of locally compact groups rather then just compact ones.

1 Introduction

In recent papers [1, 2] A. Orłowski and the author have investigated, in the context of the theory of representations of compact groups, a certain construction extending naturally that of the class operator. Let us remind that the class operators, as they are defined in the context of the theory of finite groups, are elements of the group algebra which are sums of group elements belonging to a particular conjugacy class. They span an abelian subalgebra of the group algebra and the knowledge of this

[1]1991 *Mathematics Subject Classification.* Primary 22A25, 22D15, 22D30, Secondary 20C35

subalgebra (including its multiplicative structure) is tantamount to the knowledge of the representation theory of the group (see [3] and also the begining of the Section 3 of the present paper, where we elaborate slightly on this point). The extention alluded to above consists in replacing class sums by operators obtained by integrating the class functions against conjugation operators of the representation — the class operator corresponds to the class function being identically 1. Here the term class function describes a function whose domain is a fixed conjugacy class. In view of an interpretation of such functions as (non-necessarily positive) weights assigned to points of the conjugacy class, the operators obtained this way were called in [2] weighted class operators.

In the mentioned papers some properties of this construction, focusing on properties of the map assigning weighted class operators to class functions, were investigated for compact groups. However since the concept is expected to play a significant role for the non-compact case as well, we describe here the relevant construction for locally compact rather then just for compact groups.

2 The construction of weighted class operators

Given any locally compact topological space X we shall denote by $\mathcal{K}(X)$ the space of continuous compactly supported functions on X, endowed with the topology of uniform convergence on compact subsets.

If G is a unimodular locally compact group and V a topological vector space then by a representation of G on V we shall mean a homomorphism $T\colon G \to GL(V)$ into the group $GL(V)$ of continuous invertible linear maps on V, which is continuous with respect to the strong operator topology. In such a case we shall simply say (T, V) is a representation of G. In the most cases we shall deal with unitary representations (not necessarily irreducible) of G on a Hilbert space V by what we shall mean a homomorphism $T\colon G \to U(V)$, where $U(V) \subset GL(V)$ is the group of unitary automorphisms of V.

If $L(V)$ denotes the space of continuous linear operators on V with the strong topology, then the conjugation $L(V) \ni A \mapsto T(g)AT(g^{-1}) \in L(V)$ defines a continuous representation of G on $L(V)$ which will be called the conjugation representation defined by (T, V).

If $f \in \mathcal{K}(X)$ and $g_0 \in G$ is an arbitrary, but fixed, element of the group, then we define

$$T(f;\, g_0) = \int_G f(x)T(x)T(g_0)T(x^{-1})\, dx \tag{1}$$

where the integration is performed with respect to the Haar measure dx on G. It can easily be shown by the standard methods (cf. e.g. [4, 5]) that the integral (1) converges in the sense of the strong topology in $L(V)$. Thus for each fixed $g_0 \in G$ we have a linear mapping

$$\mathcal{K}(G) \ni f \mapsto T(f;\, g_0) \in L(V). \tag{2}$$

By λ, resp. ρ, we shall denote the left, resp. right, regular representation of G in $\mathcal{K}(G)$, where the action of an element $g \in G$ is defined as the mapping $f \mapsto \lambda(g)f$, resp. $f \mapsto \rho(g)f$, where

$$\lambda(g)f(x) = f(g^{-1}x), \quad \text{resp.} \quad \rho(g)f(x) = f(xg), \quad x \in G. \tag{3}$$

A straightforward computation gives the following relations

$$\begin{aligned} T(g)T(f;g_0)T(g)^{-1} &= T(\lambda(g)f;g_0), && (4)\\ T(\rho(h)f;g_0) &= T(f;g_0), \qquad \text{for each } h \in Z_0, && (5) \end{aligned}$$

where $Z_0 = Z(g_0) \subset G$ denotes the centralizer of g_0 in G defined by $h \in Z_0 \iff hg_0 = g_0h$. Note Z_0 is a closed subgroup of G.

Recall now that the conjugacy classes in G can be regarded as G-homogeneous spaces in the following way. For any element $g_0 \in G$ let $C_0 = C(g_0) = \{xg_0x^{-1} \mid x \in G\}$ be its conjugacy class in G. The map $G \ni x \mapsto xg_0x^{-1} \in C_0$ is surjective and constant on the left Z_0 cosets in G, hence it induces a bijection $G/Z_0 \ni xZ_0 \mapsto xg_0x^{-1} \in C_0$ of G/Z_0 with the conjugacy class C_0, such that the left action of G on G/Z_0 corresponds to the action by conjugation on C_0. We shall use this bijection to transfer topology and in the Lie case also the manifold structure from G/Z_0 to the conjugacy class C_0. The reason for doing so is that the subspace topology inherited by C_0 from G in general is rather complicate d, in particular C_0 need not be closed in G, while the topology, and if aplicable also the manifold structure of the homogeneous space G/Z_0 are much simpler. However, if G is compact, then all its conjugacy classes are compact subsets of G and the topology transferred from G/Z_0 by means of the above map is the same as their subspace topology.

The invariance condition (5) allows us now to use the following well known technique (cf. e.g. [5]) of transferring integrals from the group G to the homogeneous space G/Z_0, thus passing from the map (2) to the map of the space of functions on the coset space G/Z_0. The transfer can be described as follows. Let dh denote the (left) Haar measure on Z_0 and assume for simplicity that there exists an invariant measure $d\mu$ on G/Z_0 such that

$$\int_G f(x)\,dx = \int_{G/Z_0} d\mu(\dot{x}) \int_{Z_0} f(xh)\,dh, \tag{6}$$

for each $f \in \mathcal{K}(G)$, where the symbol $\dot{x}$ is used to denote the coset $xZ_0 \in G/Z_0$ corresponding to $x \in G$. It is known (cf. e.g. [5]) that the mapping obtained by averaging functions on G over Z_0-cosets, to wit $f \mapsto \widetilde{f}$, where

$$\widetilde{f}(\dot{x}) = \int_{Z_0} f(xh)\,dh, \tag{7}$$

is a linear surjection of $\mathcal{K}(G)$ onto $\mathcal{K}(G/Z_0)$, carrying the left regular representation of G in $\mathcal{K}(G)$ onto the natural representation of G by left translations in $\mathcal{K}(G/Z_0)$.

We shall therefore denote the latter also by λ and note for the future use that by the natural extention by continuity (cf. below) it gives rise to the representation of G induced by the trivial (one dimensional identity) representation of Z_0. Now observing that $T(xg_0x^{-1})$ depends only on the coset $\dot{x} = xZ_0$ of x, so we can write $T(xg_0x^{-1}) = T(\dot{x})$, we use (6) to rewrite the equation (1) in the form

$$\begin{aligned} T(f;\, g_0) &= \int_G f(x)T(x)T(g_0)T(x^{-1})\, dx = \int_{G/Z_0} d\mu(\dot{x})T(\dot{x}) \int_{Z_0} f(xh)\, dh \\ &= \int_{G/Z_0} \tilde{f}(\dot{x})T(\dot{x})\, d\mu(\dot{x}). \end{aligned}$$

This shows that the mapping (2) factorises through the surjection $\mathcal{K}(G) \ni f \mapsto \tilde{f} \in \mathcal{K}(G/Z_0)$, giving rise to a map $\mathcal{K}(G/Z_0) \to L(V)$. We shall collect properties of this construction in the following form.

Proposition 1 (cf. [1, 2]) *Let for $f \in \mathcal{K}(G)$ the operator $T(f;\, g_0)$ be defined by* (1) *and for $\phi \in \mathcal{K}(G/Z_0)$ define the operator $\tilde{T}(\phi;\, g_0) \in L(V)$ by setting*

$$\tilde{T}(\phi;\, g_0) = \int_{G/Z_0} \phi(\dot{x})T(\dot{x})\, d\mu(\dot{x}). \tag{8}$$

Then the mapping

$$\tilde{T}\colon \mathcal{K}(G/Z_0) \ni \phi \mapsto \tilde{T}(\phi;\, g_0) \in L(V) \tag{9}$$

satisfies the condition of covariance with respect to the action of G,

$$T(g)\tilde{T}(\phi;\, g_0)T(g)^{-1} = \tilde{T}(\lambda(g)\phi;\, g_0), \quad \phi \in \mathcal{K}(G/Z_0), \tag{10}$$

and for every $f \in \mathcal{K}(G)$ such that $\tilde{f} = \phi$ we have

$$T(f;\, g_0) = \tilde{T}(\phi;\, g_0). \tag{11}$$

In addition, if the representation (T, V) is unitary, then the map (9) *extends by continuity to the mapping of the space $L^2(G/Z_0)$ of all (equivalence classes of) square integrable functions on G/Z_0 to the space of Hilbert–Schmidt operators on V.*

The maps (2), (9) are the generalizations of the class operator we have put forward (in the case of the compact group) in [1, 2]. We shall refer to them indifferently as a weighted class operator maps for the representation (T, V) based on the conjugacy class $C_0 = G/Z_0$. Note, however that the class operator, which should correspond to the integral $\int_{G/Z_0} T(\dot{x})\, d\mu(\dot{x})$ might be not defined itself, unless the group is compact or the representation is rather special, since in general this integral does not converge.

It is clear that the covariance condition (10) is implied by the equation (4). One also says the map (9) intertwines the action of G by λ on $\mathcal{K}(G/Z_0)$ with the conjugation representation on $L(V)$. The significance of the last part of the statement of Proposition 1 can be grasped better, if one notes the representation $(\lambda,\, L^2(G/Z_0))$ of

G is nothing else but the representation of G induced by the trivial representation of Z_0. Thus we see that the construction gives an intertwining operator between the induced representation and the conjugation representation. We shall say more to this point in the last section.

We point out the double sided character of the Proposition 1. On one side the equation (8) is closer to the original meaning of the class operator, while on the other hand the actual computations are often easier to handle on the level of the group, by the use of the formula (1), as we shall see below.

Example 1 We illustrate the use of the above technique of integration on a classical example of the group $\mathbf{SU}(2)$, in order to show that the equation (8) for the function ϕ being identically 1, leads in that case to the integral

$$\frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi}\exp[i\psi(J_x\sin\theta\cos\varphi+J_y\sin\theta\sin\varphi+J_z\cos\theta)]\sin\theta\,d\theta\,d\varphi, \tag{12}$$

whose evaluation was the main issue in the papers [6, 7, 8] (also cf. [1]). The J_x, J_y, J_z are of course the infinitesimal generators of a representation of the group $\mathbf{SU}(2)$ (or the rotation group $\mathbf{SO}(3)$). In the present formalism it is clearly enough to consider the case of the identity (spin 1/2) matrix representation of $\mathbf{SU}(2)$,

$$\mathbf{SU}(2)=\left\{\begin{pmatrix} a & b\\ -\bar{b} & \bar{a}\end{pmatrix}\,\middle|\, a,\, b\in\mathbf{C},\ |a|^2+|b|^2=1\right\},$$

in which case the J's will be replaced by the Pauli matrices σ_α. The Lie algebra $su\,(2)$ of $\mathbf{SU}(2)$ is therefore the space of antihermitean traceless 2×2 matrices, which can be identified with $\mathbf{R}^3$ by means of the basis $\{i\sigma_\alpha\}_{\alpha=1}^3$.

Without loss of generality we may assume the chosen representative of a conjugacy class in $\mathbf{SU}(2)$ is of the form $g(\psi)=\exp(i\frac{\psi}{2}\sigma_3)$ with $0<\psi<2\pi$ (excluding trivial cases), so that its centralizer Z_ψ is the circle group $\mathbf{U}(1)=\{\exp(it\sigma_3)\mid t\in\mathbf{R}\}\subset\mathbf{SU}(2)$. Recall the surjective map

$$\mathbf{R}^3\ni\mathbf{x}\mapsto\exp(i\mathbf{x}\cdot\sigma)\in\mathbf{SU}(2) \tag{13}$$

where $\mathbf{x}\cdot\sigma=(x_1\sigma_1+x_2\sigma_2+x_3\sigma_3)\in su\,(2)$ and the adjoint representation $g\mapsto\mathrm{Ad}(g)$ defined by means of the relation

$$g\exp(i\mathbf{x}\cdot\sigma)g^{-1}=\exp(i\mathrm{Ad}(g)\mathbf{x}\cdot\sigma),$$

which, when the matrices of $\mathrm{Ad}(g)$ are taken with respect to the basis $\{i\sigma_\alpha\}_{\alpha=1}^3$, gives the standard covering $\mathbf{SU}(2)\to\mathbf{SO}(3)$. One knows the map (13) is injective for $|\mathbf{x}|<\pi$.

Now expressing the map $\mathbf{SU}(2)\,/Z_\psi\ni gZ_\psi\mapsto gg(\psi)g^{-1}\in\mathbf{SU}(2)$ in terms of the coordinatization given by (13) one arrives at the identification of the conjugacy class C_ψ of $g(\psi)$ with the sphere $S_\psi=\{i\frac{\psi}{2}\mathbf{n}\cdot\sigma\mid\mathbf{n}\in\mathbf{R}^3,\ |\mathbf{n}|=1\}$ of radius $|\frac{\psi}{2}|$ in

$\mathbf{R}^3 \simeq su(2)$. To see this consider $G \ni g \mapsto i\mathrm{Ad}(g)\sigma_3 \in su(2)$ — since the map is constant on Z_ψ and has the unit sphere $S \subset \mathbf{R}^3 \simeq su(2)$ as its image, it gives rise to a bijection of G/Z_ψ with S. In fact, if we parametrize $\mathbf{SU}(2)$ by the Euler angles,

$$g = g(\varphi, \theta, \psi) = g(\varphi)h(\theta)g(\psi),$$

with $g(\varphi)$, $g(\psi)$ as above and $h(\theta) = \exp(i\frac{\theta}{2}\sigma_1)$, then we have

$$\mathrm{Ad}(g(\varphi, \theta, \psi))\sigma_3 = \mathbf{n}(\varphi, \theta) \cdot \sigma,$$

where

$$\mathbf{n}(\varphi, \theta) = (\sin\theta \sin\varphi,\ \sin\theta\cos\varphi,\ \cos\theta),$$

so that $[0, 2\pi[\times[0, \pi] \ni (\varphi, \theta) \mapsto \exp(i\frac{\psi}{2}\mathbf{n}(\varphi, \theta) \cdot \sigma)$ is a parametrization of C_ψ. Since the normalized Haar measure on $\mathbf{SU}(2)$ is expressed in terms of the Euler angles by

$$\int_G f(g)\, dg = \frac{1}{16\pi^2} \int_{-2\pi}^{2\pi} \int_0^{\pi} \int_0^{2\pi} f(g(\varphi, \theta, \psi))\, \sin\theta\, d\varphi\, d\theta\, d\psi$$

it is clear that the invariant integral on C_ψ defined by (6) is given as

$$\frac{1}{4\pi} \int_0^{\pi} \int_0^{2\pi} f\Big(\exp(i\frac{\psi}{2}\mathbf{n}(\varphi, \theta) \cdot \sigma)\Big) \sin\theta\, d\varphi\, d\theta.$$

The class operator for a representation (T, V) will therefore be given by the integral

$$T(1; g(\psi)) = \frac{1}{4\pi} \int_0^{\pi} \int_0^{2\pi} T\Big(\exp(i\frac{\psi}{2}\mathbf{n}(\varphi, \theta) \cdot \sigma)\Big) \sin\theta\, d\varphi\, d\theta \tag{14}$$

where

$$T\Big(\exp(i\frac{\psi}{2}\mathbf{n}(\varphi, \theta) \cdot \sigma)\Big) = T\Big(g(\varphi)h(\theta)g(\psi)h(\theta)^{-1}g(\varphi)^{-1}\Big).$$

3 The class operator for the group algebra of a compact group

In this section we present a rigorous construction of the class operator for the (left) regular representation (the group algebra) of compact groups. To motivate the subsequent considerations we start with a brief overview of the construction of the class operator in the case of the finite group, but before doing so, let us first introduce some general notations and recall few known facts.

If G is a compact group we shall always assume the Haar measure has been normalized so that $\int_G dg = 1$. $\widehat{G}$ will denote the set of equivalence classes of irreducible representations of G and for any $\alpha \in \widehat{G}$ n^α will stand for its dimension. Given $\alpha \in \widehat{G}$ we denote (T^α, V^α) any of its representatives and let $t^\alpha_{ij}(g)$ be the matrix elements of $T^\alpha(g)$, which we shall assume to satisfy $t^\alpha_{ij}(g^{-1}) = \overline{t^\alpha_{ji}(g)}$, the bar denoting the complex conjugation. As usual χ^α will denote the character corresponding to the class α.

Recall also that for any unitary representation (U, W) of G and any $\alpha \in \widehat{G}$ there is a uniquely determined invariant subspace $W^\alpha \subset W$, with the property that the restriction of U to W^α is a multiple of T^α — such a subspace or better the representation associated to it is called the isotypic component of type α of the representation (U, W). The decomposition of (U, W) into the direct sum of isotypic components, usually called the canonical decomposition, is unique up to an order of summands and is writen in the form

$$W = \bigoplus_{\alpha \in \widehat{G}} W^\alpha,$$

with the corresponding orthogonal projections on W^α given by

$$P^\alpha = n^\alpha \int_G \overline{\chi}^\alpha(g) U(g)\, dg.$$

3.1 A special case — finite groups

Assume now G to be finite; for any subset $X \subset G$ the number of elements of X will be denoted by $|X|$. Recall that in the finite case the group algebra $\mathcal{A}(G)$ is spanned by group elements $g \in G$, assumed to be linearly independent, and the multiplication in $\mathcal{A}(G)$ is obtained by extending the group multiplication from basis elements to the whole algebra by bilinearity. The inner product in $\mathcal{A}(G)$ is obtained by declaring the group elements to be mutually orthogonal and their norms set to be equal $|G|^{-1/2}$. Hence if $\phi = \sum_{g\in G} \phi(g) g$, $\psi = \sum_{g \in G} \psi(g) g$ are arbitrary elements of $\mathcal{A}(G)$ then we have

$$(\phi \mid \psi) = \frac{1}{|G|} \sum_{g \in G} \phi(g) \overline{\psi}(g). \tag{15}$$

It is sometimes convenient not to distinguish between the element $\phi = \sum_{g \in G} \phi(g) g$ of the group algebra $\mathcal{A}(G)$ and the coefficient function $g \mapsto \phi(g)$ describing its coefficients with respect to the basis consisting of the group elements. Note however that this is in fact an identification of $\mathcal{A}(G)$ with $\mathcal{K}(G)$ of the previous section, or in yet another (and more common in this case, cf. [9]) notation with $\mathcal{F}(G)$, the latter denoting the space of all functions on G.

Let now $C_0 \subset G$ be a conjugacy class, $g_0 \in C_0$ an arbitrary element and Z_0 its centralizer — recall $|G| = |Z_0||C_0|$. Now let

$$L_0 = \frac{1}{|C_0|} \sum_{g \in C_0} g = \frac{1}{|G|} \sum_{g \in G} g g_0 g^{-1} \tag{16}$$

be the class sum corresponding to C_0. If χ is a function on G constant on conjugacy classes, in particular the character of a representation of G, then

$$(L_0 \mid \chi) = \frac{1}{|G||C_0|} \sum_{g \in C_0} \overline{\chi}(g) = \frac{1}{|G|} \overline{\chi}(C_0), \tag{17}$$

where by a slight abuse of notation we have used $\overline{\chi}(C_0)$ to denote the common value of χ on the elements of the class C_0. As L_0 clearly belongs to the center of the group algebra $\mathcal{A}(G)$ and recalling that irreducible characters form an orthonormal (w. r. to the inner product (15)) basis of the centre of $\mathcal{A}(G)$ we see that

$$L_0 = \frac{1}{|G|} \sum_{\alpha \in \widehat{G}} \overline{\chi}^\alpha(C_0) \chi^\alpha. \tag{18}$$

Since the left multiplication in $\mathcal{A}(G)$ is just the linear extention of the left regular representation λ, the operator of the left multiplication with $\phi \in \mathcal{A}(G)$ will be denoted by $\lambda(\phi)$ and since this action of $\mathcal{A}(G)$ on itself is faithful, we can identify ϕ with $\lambda(\phi) \in L(\mathcal{A}(G))$. In terms of the coefficient function the left multiplication by $\phi \in \mathcal{A}(G)$ corresponds to the convolution with the function $\widetilde{\phi} = |G|\phi$, i.e., if $\psi = \sum_{g \in G} \psi(g) g \in \mathcal{A}(G)$, then

$$\lambda(\phi)\psi = \phi \cdot \psi = \sum_{x \in G} \frac{1}{|G|} \sum_{g \in G} \widetilde{\phi}(g) \psi(g^{-1}x) x = \sum_{x \in G} \widetilde{\phi} * \psi(x) x, \tag{19}$$

where as usual the convolution of $\kappa, \rho \in \mathcal{A}(G)$ is denoted by $\kappa * \rho$ and defined by the following equality

$$\kappa * \rho(x) = \frac{1}{|G|} \sum_{g \in G} \kappa(g) \rho(g^{-1}x).$$

In the effect the operator of multiplication with the class sum L_0, which is the class operator to be denoted $\lambda(L_0) = \lambda(L_0; g_0)$, can be written in the form

$$\lambda(L_0; g_0) = \sum_{\alpha \in \widehat{G}} \frac{1}{n^\alpha} \chi^\alpha(C_0) P^\alpha, \tag{20}$$

where P^α is given by the convolution operator $\psi \mapsto P^\alpha \psi = n^\alpha \overline{\chi}^\alpha * \psi$. We shall see presently that the equality (20) retains literally its form also for non finite compact groups, but the above proof, based on the summation over conjugacy class, cannot be adapted to that case.

3.2 The general case

Let now G be an arbitrary compact group. In this case as the group algebra the space $L^2(G)$ of all (equivalence classes of) square integrable functions on G is taken with the multiplication given by the convolution

$$\phi * \psi(x) = \int_G \phi(g) \psi(g^{-1}x)\, dg$$

and the usual $L^2(G)$ inner product

$$(\phi \mid \psi) = \int_G \phi(g) \overline{\psi}(g)\, dg.$$

The group G acts on $L^2(G)$ unitarily by means of both left and right regular representations (cf. Eq. (3)) and again as in the finite case the left or right multiplication (i.e. convolution) can be regarded as the natural extention of the corresponding regular representation of G . The isotypic decomposition of the regular representation has the following form. The spaces $L^2(G)^\alpha = \mathrm{span}\{\overline{t^\alpha_{ij}}(g) \mid 1 \leq i,j \leq n^\alpha\} \subset L^2(G)$ are minimal subspaces of $L^2(G)$ invariant under left and right regular representation of G and the restriction of λ to $L^2(G)^\alpha$ is an n^α-fold multiple of the representation of the class α. Moreover, the normalized matrix elements $(n^\alpha)^{1/2}\overline{t^\alpha_{ij}}$ form an orthonormal basis of $L^2(G)^\alpha$.

The class operator is now, according to (8), given by the integral

$$\widetilde{\lambda}(1; g_0) = \int_{G/Z_0} \lambda(\dot{x})\, d\mu(\dot{x}) = \int_G \lambda(x g_0 x^{-1})\, dx$$

the last equality following in virtue of compactness of G. We denote it for brevity by $\lambda(C_0)$. The integral has to be evaluated pointwise for $f \in L^2(G)$, so that

$$\lambda(C_0) f(y) = \int_G \lambda(x g_0 x^{-1}) f(y)\, dx = \int_G f(x g_0^{-1} x^{-1} y)\, dx. \tag{21}$$

We compute the integral on each of the isotypic subspaces separately, so taking $f = \overline{t^\alpha_{ij}}$ we use the multiplicativity of the matrix elements to get

$$\overline{t^\alpha_{ij}}(x g_0^{-1} x^{-1} y) = \sum_{k=1}^{n^\alpha} \overline{t^\alpha_{ik}}(x g_0^{-1} x^{-1}) \overline{t^\alpha_{kj}}(y).$$

Inserting this expression into (21) we obtain

$$\lambda(C_0)\overline{t^\alpha_{ij}}(y) = \sum_{k=1}^{n^\alpha} \overline{t^\alpha_{kj}}(y) \int_G \overline{t^\alpha_{ik}}(x g_0^{-1} x^{-1})\, dx.$$

Expanding further and using the orthogonaliy of matrix elements we get after easy manipulations

$$\int_G \overline{t^\alpha_{ik}}(x g_0^{-1} x^{-1})\, dx = \frac{1}{n^\alpha} \overline{\chi^\alpha}(g_0^{-1}) \delta_{ik} = \frac{1}{n^\alpha} \chi^\alpha(g_0) \delta_{ik}.$$

Thus $\lambda(C_0)$ acts on $L^2(G)^\alpha$ as multiplication by $(n^\alpha)^{-1}\chi^\alpha(g_0)$, so we have obtained the result claimed above.

Proposition 2 *The class operator for the group algebra of a compact group G based on a conjugacy class C_0 is given by the formula*

$$\lambda(C_0) = \sum_{\alpha \in \widehat{G}} \frac{1}{n^\alpha} \chi^\alpha(C_0) P^\alpha,$$

with $\chi^\alpha(C_0)$ denoting the (common) value of χ^α on (the elements of) the class C_0.

We note in particular, that for the case of the group $\mathbf{SU}(2)$ and the class operator given by the formula (14) with T replaced by λ, this gives the following expression (cf. [6, 7, 1, 8])

$$\lambda(C_\psi) = \sum_{j=0}^{\infty} \frac{\sin(2j+1)\frac{\psi}{2}}{(2j+1)\sin\frac{\psi}{2}} P^j,$$

with P^j denoting projections on subspaces corresponding to the spin j.

4 Connection with tensor operators

In this last section we consider the connections of the above construction of weighted class operators with the concept of tensor operators which is known as being of paramount importance for physics. The general discussion, which was given in a recently published paper of A. Orłowski and the author [2], will be briefly recalled first and then suplemented by the computation of weighted class operators for the group algebra along the lines of section 3.2. As above, the discussion will be confined to the compact case (for some earlier related work, see [10, 11, 12]).

4.1 Definitions and general results on tensor operators

In a conventional formulation a tensor operator is understood to be a linearly independent set of operators $\{T_i\}$ on the vector space of a certain representation (U, H) of G, such that

$$U(g)T_iU(g)^{-1} = \sum_j D_{ji}(g)T_j, \tag{22}$$

where $D_{ij}(g)$ are matrix coefficients of a representation of the group G.

However, recall the following elegant (although purely algebraic) definition given by L. Michel in [13].

Definition 1 *Let (S, V) and (U, H) be two representations of a group G. Then a tensor operator of type (S, V) over the space H is a (nonzero) linear mapping $T : V \to L(H)$, intertwining S with the conjugation representation defined by U on $L(H)$.*

Since the intertwining condition means that the following equality is valid

$$T(S(g)v) = U(g)T(v)U(g)^{-1}, \qquad \forall\, g \in G,\ \forall\, v \in V, \tag{23}$$

it is clear that the results of Section 2 are in fact statements about tensor operators. To describe then properly we shall need some more terminology.

A tensor operator $T : V \to L(H)$ is called irreducible if the representation (S, V) is irreducible. If α is the class of representation (S, V), then we say that T is of the type α. Two irreducible tensor operators of the same type $T_1 : V_1 \to L(H)$ and

$T_2 : V_2 \to L(H)$ are called independent if their images are different subspaces of $L(H)$ — note that the images being irreducible they can either be identical or have only zero in common.

Before going further, let us indicate that the equivalence of these two notions of tensor operators can be obtained by choosing a basis, say $\{v_i\}$, of the space V and setting $T_i = T(v_i)$. Then (22) follows trivially from (23) by taking $D_{ij}(g)$ to be the matrix coefficients of $S(g)$ defined by the usual recipe $S(g)v_i = \sum_{j=1}^{\dim V} D_{ji}(g)v_j$.

Now we can state

Theorem 1 (cf. [2]) *The weighted class operator map* (9) *is a tensor operator of the type of the induced representation* $(\lambda,\, L^2(G/Z_0))$*. The restriction of this map to any invariant and irreducible subspace* $V^\alpha \subset L^2(G/Z_0)^\alpha$*, if nonzero, is an irreducible tensor operator of type* α*.*

In order to actually construct irreducible tensor operators as weighted class operators one needs to know explicitely the canonical decomposition of the induced representation, or better still, irreducible subspaces of $L^2(G/Z_0)$. The first part of the issue is solved by the classical Frobenius reciprocity theory, cf. e.g. Section 4.3 of [5] or Chapter 8 of [4].

Let $\widehat{G}_0 \subset \widehat{G}$ be the subset consisting of (classes of) representations admitting nonzero Z_0-fixed vectors. Given $\alpha \in \widehat{G}_0$ choose $(T^\alpha,\, V^\alpha) \in \alpha$ and let $V_0^\alpha \subset V^\alpha$ be the subspace of Z_0-fixed vectors. Set $m^\alpha = \dim V_0^\alpha$ and note this number depends on the class α, but not on the choice of $(T^\alpha,\, V^\alpha) \in \alpha$. On the other hand consider the induced representation $(\lambda,\, L^2(G/Z_0))$ and let $\mathbf{i}(T^\alpha\colon \lambda)$ be the multiplicity of the representation T^α in the induced representation. Then the Frobenius reciprocity states these two numbers are equal, $m^\alpha = \mathbf{i}(T^\alpha\colon \lambda)$.

Consequently the subspace $L^2(G/Z_0)^\alpha$ is nonzero if and only if $\alpha \in \widehat{G}_0$ and in this case it has dimension $m^\alpha n^\alpha$ and contains m^α copies of representations of the class α. Thus the only irreducible tensor operators which can be constructed as weighted class operators based on the conjugacy class $C_0 = G/Z_0$ are those whose types contain the trivial (identity) representation of Z_0.

4.2 Tensor operators over the group algebra

We shall now give a description of the construction of tensor operators over the group algebra extending the method of Section 3.2. The aim is to compute the integrals

$$\widetilde{\lambda}(\widetilde{f};\, g_0) = \int_{G/Z_0} \widetilde{f}(\dot{x})\lambda(\dot{x})\, d\mu(\dot{x}) = \int_G f(x)\lambda(xg_0x^{-1})\, dx, \tag{24}$$

where f and $\widetilde{f}$ are related by (7). Since we are interested in irreducible tensor operators we can assume that $\widetilde{f}$'s belong to an irreducible subspace of $L^2(G/Z_0)^\alpha$. Such functions can be constructed in the following way. Choose an orthonormal basis $\{e_i\}$ in V^α in such a way that its first m^α vectors form a basis in V_0^α and the remaining ones span the complementary Z_0-invariant subspace and denote by $t^\alpha_{ij}(g) = \langle T^\alpha(g)e_j \mid e_i\rangle$ the matrix elements of $(T^\alpha,\, V^\alpha)$ with respect to this chosen basis (we are using here the inner product which is linear in the first variable). Note that the matrix elements $t^\alpha_{ij}(g)$ for $1 \le j \le m^\alpha$ are right Z_0-invariant functions, hence can be regarded as functions from $L^2(G/Z_0)$ and thus t^α_{ij} can be identified with their right Z_0 averages $\widetilde{t^\alpha_{ij}}$. In virtue of the Frobenius reciprocity their complex conjugates $\overline{t^\alpha_{ij}}(g)$ span the space $L^2(G/Z_0)^\alpha$.

It follows that for a given representation $(T,\, V)$ of G by setting $T^{(\alpha,j)}_i = \widetilde{T}(\overline{t^\alpha_{ij}};\, g_0)$ we obtain in general m^α sets $\{T^{(\alpha,j)}_i \mid 1 \le i \le n^\alpha\}$ of weighted class operators transforming according to the formula $V(g)T^{(\alpha,j)}_i V(g)^{-1} = \sum_k t^\alpha_{ki}(g)T^{(\alpha,j)}_k$. However, some of these sets may degenerate (i.e. consists only of the zero operator), but the author is unaware of any general criterion for nonvanishing of an intertwining operator on a given irreducible subspace. We shall not pursue this question here.

Now, let us substitute a matrix element $\overline{t^\alpha_{kl}}$ for f in the integral (24) and evaluate the restriction of $\widetilde{\lambda}(\overline{t^\alpha_{kl}};\, g_0)$ to the subspace $L^2(G)^\sigma$. Proceeding exactly as above in the Section 3.2 we get

$$\begin{aligned}
\widetilde{\lambda}(\overline{t^\alpha_{kl}};\, g_0)\overline{t^\sigma_{ij}}(y) &= \int_G \overline{t^\alpha_{kl}}(x)\overline{t^\sigma_{ij}}(xg_0^{-1}x^{-1}y)\,dx \\
&= \sum_s \overline{t^\sigma_{sj}}(y)\int_G \overline{t^\alpha_{kl}}(x)\overline{t^\sigma_{is}}(xg_0^{-1}x^{-1})\,dx \\
&= \sum_{spr} \overline{t^\sigma_{sj}}(y)t^\sigma_{pr}(g_0)\int_G \overline{t^\alpha_{kl}}(x)\overline{t^\sigma_{ir}}(x)t^\sigma_{sp}(x)\,dx. \qquad (25)
\end{aligned}$$

Now, by the general wisdom of the canonical decomposition, one knows that the products of matrix elements of irreducible representations can be written as linear combinations of matrix elements of irreducible representations. One way of doing so is to use the so called coupling coefficients (cf. [11, 2]). We digress briefly to introduce the needed notions, in the form familiar from the treatment of the classical Clebsch–Gordan coefficients.

For $(T^\sigma,\, V^\sigma) \in \sigma$ consider the conjugation representation in $L(V^\sigma)$ and write its canonical decomposition in the form

$$L(V^\sigma) = \bigoplus_{\gamma\in\Gamma(V^\sigma)} E^\gamma = \bigoplus_{\gamma\in\Gamma(V^\sigma)} m(\sigma;\gamma)V^\gamma, \qquad (26)$$

where $\Gamma(V^\sigma) \subset \widehat{G}$ denotes the set of classes of irreducible unitary representations of G which occur in that decomposition, E^γ is the isotypic component of the class γ, and the right hand side of the equality is obtained by further decomposing E^γ into irreducibles. The multiplicity $m(\sigma;\gamma)$ of the class γ in $L(V^\sigma)$ is sometimes called $3j$ symbol and denoted $\{\sigma\overline{\sigma}\gamma\}$, cf. e. g. [10]. Following the general usage we are using $\overline{\sigma}$ to denote the complex conjugate representation, i.e. the one with the complex conjugate matrix elements. An important and much simpler situation occurs, when the canonical decomposition is multiplicity free, i.e. when $m(\sigma;\gamma) = 1$ for each $\gamma \in \Gamma(V^\sigma)$, what is the case in particular for the so called simply reducible groups of Wigner.

The coupling coefficients are related to a choice of a basis realizing the canonical decomposition. The vectors of such (orthonormal) basis are denoted e^γ_{mn}, where γ describes the classes of irreducible representations occurring in the decomposition, $1 \leq m \leq m(\sigma;\gamma)$ distinguishes between different copies of the same representation of the class γ and $1 \leq n \leq n^\gamma$ indexes vectors of a given base within a fixed copy of the representation space. In particular for fixed γ and m the set $\{e^\gamma_{mn} \mid 1 \leq n \leq n^\gamma\}$ is an orthonormal basis for an invariant subspace, on which the conjugation representation acts by a representation belonging to the class γ. Now fix an orthonormal basis $\{v_i\}$ for V^σ and let $E_{ij} \in L(V^\sigma)$ be a "matrix unit" corresponding to that basis, i.e., the linear map given by $v \mapsto E_{ij}(v) = \langle v \mid v_j\rangle v_i$. The coupling coefficients $c(\sigma i; \overline{\sigma} j \mid \gamma mn)$ give the transition between the two bases (the summation extending over the whole range of indices involved)

$$E_{ij} = \sum_{\gamma mn} c(\sigma i; \overline{\sigma} j \mid \gamma mn) e^\gamma_{mn}, \tag{27}$$

and it is a simple exercise to show validity of the equation

$$\overline{t^\sigma_{ir}}(x) t^\sigma_{sp}(x) = \sum_{\gamma mnq} \overline{c(\sigma s; \overline{\sigma} i \mid \gamma mq)}\, t^\gamma_{qn}(x) c(\sigma p; \overline{\sigma} r \mid \gamma mn). \tag{28}$$

It now follows that

$$\begin{aligned} \int_G \overline{t^\alpha_{kl}}(x) \overline{t^\sigma_{ir}}(x) t^\sigma_{sp}(x)\, dx &= \sum_{\gamma mnq} \overline{c(\sigma s; \overline{\sigma} i \mid \gamma mq)}\, c(\sigma p; \overline{\sigma} r \mid \gamma mn) \int_G \overline{t^\alpha_{kl}}(x) t^\gamma_{qn}(x)\, dx \\ &= \frac{1}{n^\alpha} \sum_m \overline{c(\sigma s; \overline{\sigma} i \mid \alpha mk)}\, c(\sigma p; \overline{\sigma} r \mid \alpha ml). \end{aligned}$$

Inserting this into the integral (25) we get

$$\begin{aligned} \widetilde{\lambda}(\overline{t^\alpha_{kl}};\, g_0) \overline{t^\sigma_{ij}}(y) &= \frac{1}{n^\alpha} \sum_{mspr} \overline{t^\sigma_{sj}}(y) t^\sigma_{pr}(g_0) \overline{c(\sigma s; \overline{\sigma} i \mid \alpha mk)}\, c(\sigma p; \overline{\sigma} r \mid \alpha ml) \\ &= \frac{1}{n^\alpha} \sum_{sm} \overline{t^\sigma_{sj}}(y) \overline{c(\sigma s; \overline{\sigma} i \mid \alpha mk)} \Big(\sum_{pr} c(\sigma p; \overline{\sigma} r \mid \alpha ml) t^\sigma_{pr}(g_0) \Big). \end{aligned} \tag{29}$$

From that one immediately computes the matrix coefficients of the operator $\widetilde{\lambda}(\overline{t^\alpha_{kl}};\, g_0)$ obtaining this way the following result of the Wigner–Eckart type.

Theorem 2 *Let for $\sigma, \gamma \in \widehat{G}$ the coupling coefficients $c(\sigma i; \overline{\sigma} j \mid \gamma mn)$ be defined by the equation (27). For any matrix coefficient $\{t^\alpha_{kl}\}$ of the irreducible representation of G of the class α let $\widetilde{\lambda}(\overline{t^\alpha_{kl}};\, g_0)$ be the corresponding weighted class operator for the* *group algebra G defined by the equation* (24). *Then its matrix coefficients are given by the formulae of the form*

$$n^\sigma \langle \widetilde{\lambda}(\overline{t^\alpha_{kl}};\, g_0)\overline{t^\sigma_{ij}} \mid \overline{t^\gamma_{uv}}\rangle \;=\; \frac{1}{n^\alpha}\delta_{\gamma\sigma}\delta_{jv}\sum_{m=1}^{m(\sigma;\alpha)} \overline{c(\sigma u; \overline{\sigma} i \mid \alpha m k)} \times\Big(\sum_{pr} c(\sigma p; \overline{\sigma} r \mid \alpha m l)\, t^\sigma_{pr}(g_0)\Big) \tag{30}$$

Let us point out that in addition to the usual content of the Wigner–Eckart theorem we have obtained in the formula (30) an explicit and complete description of the reduced matrix elements which are given by

$$\frac{1}{n^\alpha}\sum_{pr} c(\sigma p; \overline{\sigma} r \mid \alpha m l)\, t^\sigma_{pr}(g_0)$$

depending on the chosen element g_0 determining the conjugacy class.

Acknowledgments

A preliminary version of the results contained in the paper was presented during the II International Workshop "Lie Theory and Its Applications in Physics" at the Arnold-Sommerfeld Institute of the Technical University Clausthal. Thanks are due to the organizers of the Workshop for the invitation to present these results and for creating so pleasant and inspiring atmosphere of the meeting.

The work on this paper was partially supported by State Committee for Scientific Research (SCSR) grant No 2 P03B 044 13, and also by the research project SFB 288 — *Differentialgeometrie und Quantenphysik* at the Institut für Reine Mathematik of the Humboldt University in Berlin, where the actual writing of the paper took place. The author expresses his thanks to Prof. Thomas Friedrich for the invitation and hospitality during his stay in Berlin.

References

[1] Orłowski A and Strasburger A 1994 "A Generalization of the Construction of the Class Operator", J. Phys. A **27**, 167-175; "Corrigenda," *ibid.* **27**, 4723-4724.

[2] Strasburger A and Orłowski A 1997 "Weighted class operators: a new approach to tensor operators for compact groups", *J. Math. Phys.* **38** 2720–2727.

[3] Chen J-Q 1989 *Group Representation Theory for Physicists* (Singapore: World Scientific)

[4] Robert A 1983 *Introduction to the Representation Theory of Compact and Locally Compact Groups* (Cambridge: Cambridge University Press)

[5] Wawrzyńczyk A 1985 *Group representations and special functions* (PWN and D. Reidel, Warszawa and Dordrecht)

[6] Fan H and Ren Y 1988 "New applications of the coherent state in calculating the class operator of rotation group" *J. Phys. A: Math. Gen.* **21** 1971–1976

[7] Backhouse N B 1988 "On the evaluation of the class operator for the rotation group" *J. Phys. A: Math. Gen.* **21** L1113–L1115

[8] Montaldi E and Zucchelli G 1994 "On the class operator of the SU(2) group" *Rend. Semin. Mat. Fis. Milano* **64** 217–222

[9] Miller W Jr. 1972 *Symmetry Groups and their Applications* (New York: Academic Press)

[10] van Zanten A J and de Vries E 1973 "Tensor operators of finite groups" *J. Math. Phys.* **14** 1423–1429

[11] Kasperkovitz P and Dirl R 1974 "Irreducible tensorial sets within the group algebra of a compact group" *J. Math. Phys.* **15** 1203–1210

[12] Backhouse N and Gard P 1976 "On the tensor representation for compact groups" *J. Math. Phys.* **17** 2098–2100

[13] Michel L 1970 "Application of Group Theory to Quantum Physics; Algebraic Aspects," in *Group Representations in Mathematics and Physics*, edited by V. Bargmann, Lecture Notes in Physics 6 (Springer-Verlag, Berlin), pp. 36-143.

[14] Biedenharn L C and Louck J D 1981 *Angular Momentum in Quantum Physics, The Racah–Wigner Algebra in Quantum Theory* (Reading, Mass.: Addison–Wesley)