Orthogonal nets and Clifford algebras

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Summary. A Clifford algebra model for Möbius geometry is presented. The notion of Ribaucour pairs of orthogonal systems in arbitrary dimensions is introduced, and the structure equations for adapted frames are derived. These equations are discretized and the geometry of the occuring discrete nets and sphere congruences is discussed in a conformal setting. This way, the notions of "discrete Ribaucour congruences" and "discrete Ribaucour pairs of orthogonal systems" are obtained — the latter as a generalization of discrete orthogonal systems in Euclidean space. The relation of a Cauchy problem for discrete orthogonal nets and a permutability theorem for the Ribaucour transformation of smooth orthogonal systems is discussed.

1. Introduction

Triply orthogonal systems in Euclidean 3-space and Ribaucour sphere congruences in the conformal 3-sphere were intensively studied around the turn of the century — among others by great geometers like Darboux [8], Bianchi, Guichard [14] (cf.[22]), and Blaschke [2]. After this first wave of work, a calm period followed — until recently, when relations between (n-dimensional) orthogonal systems and integrable system methods stimulated new interest. The corresponding equations have been integrated by the $\bar\partial$-method [23] and by the finite-gap integration theory [21], and the Ribaucour transformation for triply orthogonal systems have been studied in [13]. In differential geometry, there are relations with the theory of conformally flat hypersurfaces (cf.[15]) and (the Darboux transformation) of isothermic surfaces (cf.[16],[19],[3],[17]). Circular nets as a natural geometrical discretization of triply orthogonal coordinate systems have been introduced in [4] and generalized for higher dimensions in [7]. Geometric

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and analytic integrability of these nets were investigated in [7], [10] and [20]. Also, discrete Ribaucour sphere congruences have been defined and studied in [4], [18] and [6].

Both notions, the one of "orthogonal systems" as well as that of "Ribaucour congruences", are conformally invariant, i.e. invariant under Möbius transformations of the ambient space. Nevertheless, in all the papers mentioned above (besides [15]), orthogonal systems were treated in a Euclidean setting — even though certain aspects of the theory, e.g. the Ribaucour transformation for orthogonal systems, seem to arise more naturally in a conformal setting. Thus, we hope to make a small contribution in the growing flood of publications in this field: the purpose of this paper is to introduce a method that allows the treatment of smooth and discrete "Ribaucour pairs of orthogonal nets" in Möbius geometry[1)]. Moreover, we expect this method not only to be advantageous in the theory of discrete nets but also for the treatment of global questions in higher dimensional Möbius geometry (cf.[16]).

First, we give a short introduction to n-dimensional Möbius geometry — in particular, we introduce a model using the Clifford algebra $\mathcal{C}$ of $(n+2)$-dimensional Minkowski space $\mathbb{R}^{n+2}_1$. To obtain geometric interpretations for certain Clifford algebra elements, we use the fact that the Clifford algebra $\mathcal{C} \cong \Lambda$ is isomorphic to the Grassmann algebra as a vector space. On these Möbius geometric objects, the spin group $\mathrm{Spin}(\mathbb{R}^{n+2}_1)$ acts by Möbius transformations. Then, we discuss (smooth) orthogonal systems and Ribaucour congruences in this setting. As a generalization, we introduce the notion of "Ribaucour pairs of orthogonal systems" and derive the structure equations for "adapted frames".

These frame equations can easily be discretized: analyzing the occuring discrete nets and "discrete sphere congruences", we obtain definitions and characterizations for "discrete Ribaucour congruences" and "discrete Ribaucour pairs of orthogonal nets". The latter generalizes the notion of discrete orthogonal nets in Euclidean space (cf.[4],[7],[20]). The occurence of certain (Bäcklund, Darboux, Ribaucour) transformations is crucial in the relation of differential geometry and integrable system theory (cf.[5]). In the last part, we discuss the relation of a Cauchy problem for (Ribaucour pairs of) discrete orthogonal systems (cf.[7],[9]) and a general version of the permutability theorem for the Ribaucour transformation of (smooth) orthogonal systems (cf.[8],[13]). In fact, this permutability theorem shows how discrete orthogonal nets can be obtained by repeatedly applying Ribaucour transformations to a given orthogonal system.

2. Möbius geometry and Clifford algebra

In this section we will give a sketchy introduction to Möbius geometry — especially, we intend to elaborate the Clifford algebra model for Möbius geometry and to relate it to the classical models. Classically, the conformally compactified Euclidean space $\mathbb{R}^n \cup \{p_\infty\}$ is considered the underlying space for the group

1) In particular, we also obtain a geometrical interpretation of the spinor Lax-representation [3] for isothermic surfaces.

of Möbius transformations [2] — those transformations that map spheres (and planes — that are considered as spheres containing the point p_∞ at infinity) to spheres. It can be shown that the Möbius group is generated by inversions

$$p \mapsto m + \tfrac{r^2}{|p-m|^2}(p-m)$$

at spheres (here, m is the center and r the radius of a sphere) and reflections at planes. Via stereographic projection, $\mathbb{R}^n \cup \{p_\infty\}$ can be identified with the n-sphere S^n which is then embedded as an absolute quadric in $\mathbb{R}P^{n+1}$. This allows to consider Möbius geometry as a subgeometry of projective geometry — thus, to linearize the Möbius group: any Möbius transformation of S^n extends to a projective transformation of $\mathbb{R}P^{n+1}$ that fixes the absolute quadric S^n. For example, an inversion extends to a polar reflection of $\mathbb{R}P^{n+1}$. In this model, hyperspheres $s \subset S^n$ are given as intersections of projective hyperplanes with $S^n \subset \mathbb{R}P^{n+1}$. This way, they can be identified with points in the "outer space[2)]" $(\mathbb{R}P^{n+1})_o$ of S^n via polarity — a sphere $s \subset S^n$ is identified with the center of the cone that touches S^n in s.

On the space of homogeneous coordinates of $\mathbb{R}P^{n+1}$ there is a Lorentz scalar product — unique up to scaling — such that its isotropic lines are exactly the points of S^n: light cone vectors and unit vectors,

$$\begin{array}{rcl} p \in L^{n+1} & := & \{v \in \mathbb{R}^{n+2}_1 \,|\, \langle v, v\rangle = 0\} \\ s \in S^{n+1}_1 & := & \{v \in \mathbb{R}^{n+2}_1 \,|\, \langle v, v\rangle = 1\}, \end{array}$$

represent points and hyperspheres in S^n, respectively (cf.[2]). Incidence of a point and a hypersphere — polarity in $\mathbb{R}P^{n+1}$ — is encoded as orthogonality: $p \in s$ if and only if $\langle p, s\rangle = 0$. And, a polar reflection in $\mathbb{R}P^{n+1}$, an inversion in S^n, is represented by an ordinary orthogonal reflection $p \mapsto \pm(p - 2\langle p, s\rangle s)$. In this model, the metric subgeometries — in particular Euclidean geometry — of Möbius geometry are easily described: if $n_k \in \mathbb{R}^{n+2}_1$ is a vector with $k = -\langle n_k, n_k\rangle$, then the quadric $Q_k := \{p \in L^{n+1} \,|\, \langle p, n_k\rangle = 1\}$ has constant sectional curvature k; the stereographic projection $S^n \to \mathbb{R}^n$ becomes the central projection $Q_1 \to Q_0$ along the light cone generators.

Embedding the Minkowski space $\mathbb{R}^{n+2}_1$ into its Clifford algebra $\mathcal{C}$ additionally provides a useful algebraic structure. As generators of $\mathcal{C}$, we consider a pseudo orthogonal basis $(e_0, e_1, \dots, e_n, e_\infty)$ of $\mathbb{R}^{n+2}_1$:

$$\begin{array}{rcccl} -e_i^2 & = & e_0 e_\infty + e_\infty e_0 & = & 1 \\ e_0^2 = e_\infty^2 & = & e_i e_j + e_j e_i & = & 0 \end{array}$$

for $1 \leq i \neq j \leq n$. Thus, the group $\mathrm{Spin}(\mathbb{R}^{n+2}_1) = \{\Pi_{j=1}^{2m} s_j \,|\, s_j \in S^{n+1}_1\}$ acts as a double cover of the isometry group $SO_1(n+2)$ on $\mathbb{R}^{n+2}_1$ via $p \mapsto \Phi^{-1} p \Phi$: any orientation preserving Möbius transformation is represented as an (even) composition of inversions at spheres,

$$p \mapsto \pm sps, \quad s \in S^{n+1}_1.$$

[2)] The "inner space" of $S^n \subset \mathbb{R}P^{n+1}$ can be defined as the set of those points *not* lying on any (real) tangent line of S^3, the outer space is its complement in $\mathbb{R}P^{n+1} \setminus S^n$.

For our considerations, it will turn out crucial that the spin group and its Lie algebra $\mathfrak{spin}(\mathbb{R}_1^{n+2}) = \Lambda^2$ are both described as subspaces of the same space $\mathcal{C} \cong \Lambda = \oplus_{i=0}^{n+2}\Lambda^i$ — here, we identify the Clifford algebra $\mathcal{C}$ with the Grassmann algebra over $\mathbb{R}_1^{n+2}$ as vector spaces. As the spin group acts on Λ^1 via adjoint action, the Lie algebra acts on Λ^1 via the adjoint representation, $[v,\phi] = v\phi - \phi v$ for $v \in \Lambda^1$ and $\phi \in \Lambda^2$. In particular, for $0 \leq i,j,k \leq \infty$, $i \neq j$,

$$[e_k, e_ie_j] = 2(\langle e_j, e_k\rangle e_i - \langle e_i, e_k\rangle e_j). \tag{1}$$

The grading $\mathcal{C} \cong \Lambda = \oplus_{i=0}^{n+2}\Lambda^i$ of the Grassmann algebra provides geometric interpretations for elements of the Clifford algebra: clearly, points and hyperspheres in the conformal n-sphere S^n are represented by vectors $p, s \in \Lambda^1$ with $p^2 = 0$ and $s^2 = -1$, respectively. And, as an m-dimensional sphere s in S^n can be described as the intersection of $n-m$ (orthogonal) hyperspheres $s_1, \dots, s_{n-m} \in S_1^{n+1} \subset \Lambda^1$, we can identify it with the spacelike plane $\mathrm{span}\{s_1, \dots, s_{n-m}\} \subset \mathbb{R}_1^{n+2}$, i.e.

$$s = s_1 \wedge \ldots \wedge s_{n-m} \in \Lambda^{n-m} \quad \text{with} \quad |s|^2 := \det(\langle s_i, s_j\rangle) > 0.$$

Thus, *pure*[3] spacelike $(n-m)$-vectors $s \in \Lambda^{n-m}$ represent m-spheres. Choosing $\varepsilon := (e_0 - e_\infty) \wedge e_1 \wedge \ldots \wedge e_n \wedge (e_0 + e_\infty) \in \Lambda^{n+2}$, the linear isomorphism[4]

$$\Lambda^{n-m} \ni v \mapsto \varepsilon v \in \Lambda^{m+2}$$

maps pure $(n-m)$-vectors to pure $(m+2)$-vectors — reversing the signature since $|\varepsilon v|^2 = |\varepsilon|^2|v|^2 = -|v|^2$. Consequently, pure timelike $(m+2)$-vectors can be interpreted as m-spheres, too. In particular,

Lemma. *$m+2$ points $p_i \in L^{n+1}$, $1 \leq i \leq m+2$, in general position determine a unique m-sphere s containing all points, $p_i \in s$,*

$$s = p_1 \wedge \ldots \wedge p_{m+2} \in \Lambda^{m+2} \cong \Lambda^{n-m},$$

and the $m+2$ points lie on a $(m-1)$-sphere if and only if $p_1 \wedge \ldots \wedge p_{m+2} = 0$.

In case four points $p_1, \dots, p_4 \in \Lambda^1$ are not concircular, the Λ^4-part of the quantity $p_1p_2p_3p_4 + p_4p_3p_2p_1$ determines a unique 2-sphere that contains the four points. Considering this 2-sphere as a Riemann sphere, we can determine the (complex) cross ratio of the four points:

Lemma. *Given four points $p_1, \dots, p_4 \in \Lambda^1$, $p_i^2 = 0$, in the conformal n-sphere their cross ratio is given by the quantity*[5]

$$r = r_0 + r_4 = \tfrac{p_1p_2p_3p_4 + p_4p_3p_2p_1}{(p_1p_4+p_4p_1)(p_2p_3+p_3p_2)} \in \Lambda^0 \oplus \Lambda^4 \ : \tag{2}$$

[3] Pure 2-vectors can be detected by the fact that $s = s_1 \wedge s_2$ for $s \in \Lambda^2$ if and only if $s \wedge s = 0$, or $s^2 \in \mathbb{R}$. We use this fact in the table below. Note that for pairwise orthogonal $s_1, \dots, s_k \in \Lambda^1$, the exterior product coincides with the Clifford product, and $(\Pi_{i=1}^k s_i)^2 = (-1)^{\frac{k(k-1)}{2}}\Pi_{i=1}^k s_i^2$.

[4] On non isotropic pure $(n-m)$-vectors, this is Hodge duality, up to sign.

[5] Note, that r does not depend on the choice of representative for the points p_i.

up to complex conjugation, $r_0 + i|r_4|$ is the complex cross ratio of the four points on that sphere as a Riemann sphere — since $r_0 + i|r_4|$ is clearly invariant under Möbius transformations $p_i \mapsto \Phi^{-1} p_i \Phi$ it suffices to check the formula using the identification $\mathbb{C} \ni z = x + iy \cong |z|^2 e_0 + x e_1 + y e_2 + e_\infty \in L^{n+1}$, which is a straightforward calculation. Moreover,

Corollary. *Four points $p_1, \dots, p_4 \in L^{n+1}$ in the conformal n-sphere are concircular if and only if their cross ratio (2) is real i.e. $r_4 = 0$.*

For reference, we have summarized the essentials of the previous discussions in Table 1 where, for simplicity of notation, we restrict to the case $n = 3$.

3. Ribaucour pairs of orthogonal systems

In this section, we are going to discuss briefly the geometry of smooth orthogonal systems and Ribaucour congruences (cf.[22],[15]) — in particular, we will derive suitable frame equations. Later, we will use discretizations of these frame equations to obtain definitions for discrete analogs of smooth orthogonal systems and Ribaucour congruences — and, more general, of Ribaucour pairs of orthogonal systems.

Definition (orthogonal system). *A system of n 1-parameter families of hypersurfaces in $\mathbb{R}^n$ is called an orthogonal system if any two hypersurfaces from different families intersect orthogonally.*

Obviously, the notion of an orthogonal system is invariant under Möbius transformations of the ambient space $\mathbb{R}^n$; or, more general, it is invariant under conformal changes of the ambient space's metric. Consequently, we will focus on conformally invariant properties of orthogonal systems — even though, we will do some calculations in a Euclidean setting where it seems more convenient.

Parametrizing the n families of orthogonal hypersurfaces one obtains an orthogonal coordinate system $(t_1, \dots, t_n) : V \subset \mathbb{R}^n \to \mathbb{R}^n$ resp. a parametrization $f : U \subset \mathbb{R}^n \to \mathbb{R}^n \subset L^{n+1}$ with $\frac{\partial}{\partial t_i} f \perp \frac{\partial}{\partial t_j} f$ for any pair $1 \leq i \neq j \leq n$. Thus, given an orthogonal system f, we can introduce an *adapted framing* $\Phi : U \to \mathrm{Spin}(\mathbb{R}^{n+2}_1)$ with

$$\begin{array}{rcl} f & = & \Phi^{-1} e_0 \Phi \\ \frac{\partial}{\partial t_i} f =: f_i & = & l_i \cdot \Phi^{-1} e_i \Phi, \quad 1 \leq i \leq n \end{array} \tag{3}$$

where $(e_0, e_1, \dots, e_n, e_\infty)$ denotes any pseudo orthonormal basis of $\mathbb{R}^{n+2}_1$ with $\mathbb{R}^n \cong \{y \in L^{n+1} \,|\, y e_\infty + e_\infty y = 1\}$ and $l_i : U \to \mathbb{R}$ are *Lamé's functions* [22]. In terms of the adapted frame Φ, the fact that $f : U \to \mathbb{R}^n$ takes values in Euclidean space is encoded by $e_\infty = \Phi^{-1} e_\infty \Phi$.

In our investigations, a key role will be played by the following

Theorem (Dupin). *In an orthogonal system in $\mathbb{R}^n$, the intersection of $n-1$ hypersurfaces from different families is a curvature line for any of the intersecting hypersurfaces.*

Table. 1. Möbius geometry in different models

$\mathbb{R}^3 \cong \mathrm{Im}\mathbb{H}$	$S^3 \subset \mathbb{R}P^4$	$L^4 \subset \mathbb{R}^5_1$	$\mathcal{C} \cong \Lambda = \oplus_{i=0}^5 \Lambda^i$
point: $p \in \mathbb{R}^3$	point on S^3: $p \in S^3$	lightlike vector: $p \in L^4$	nilpotent vector: $p \in \Lambda^1, p^2 = 0 : L^4$
sphere: $s \subset \mathbb{R}^3$	point outside S^3: $s \in (\mathbb{R}P^4)_o$	unit vector: $s \in S^4_1$	anti-involutive vector: $s \in \Lambda^1, s^2 = -1 : S^4_1$
plane: $t \subset \mathbb{R}^3$	... containing p_∞: $t \in T_{p_\infty} S^3$	... perpendicular to p_∞: $t \in S^4_1, \langle t, p_\infty \rangle = 0$	... anti-symmetric to p_∞: $t \in S^4_1, \frac{tp_\infty + p_\infty t}{2} = 0$
circle: $c \subset \mathbb{R}^3$	line outside S^3: $c \subset (\mathbb{R}P^4)_o$	spacelike plane: $c \in G_+(2,3) =: C^6_2$	anti-involutive bivector: $c \in \Lambda^2, c^2 = -1 : C^6_2$
line: $l \subset \mathbb{R}^3$	... containing p_∞: $l \subset T_{p_\infty} S^3$	... perpendicular to p_∞: $l \in C^6_2, \langle l, p_\infty \rangle = 0$	... symmetric to p_∞: $l \in C^6_2, \frac{lp_\infty - p_\infty l}{2} = 0$
incidence: $p \in s$	polarity: $s \in T_p S^3$	orthogonality: $\langle p, s \rangle = 0$	anti-symmetry: $ps + sp = 0$
incidence: $p \in c$	polarity: $c \subset T_p S^3$	orthogonality: $\langle p, s \rangle = 0$	symmetry: $pc - cp = 0$
intersection: $s_1 \cap_\perp s_2$	polarity: $s_1 \in \mathrm{Pol}[s_2]$	orthogonality: $\langle s_1, s_2 \rangle = 0$	anti-symmetry: $s_1 s_2 + s_2 s_1 = 0$
... angle: $s_1 \cap_\varphi s_2$		scalar product: $\langle s_1, s_2 \rangle = \cos\varphi$	real part: $\frac{s_1 s_2 + s_2 s_1}{2} = \cos\varphi$
intersection: $c \cap_\perp s$	polarity: $c \subset \mathrm{Pol}[s]$	orthogonality: $\langle s, c \rangle = 0$	symmetry: $cs - sc = 0$
cross ratio: $\frac{(p_1 - p_2)}{(p_2 - p_3)} \frac{(p_3 - p_4)}{(p_4 - p_1)}$		cross ratio ($l_{ij} = \langle p_i, p_j \rangle$): $\frac{l_{12} l_{34} + l_{14} l_{23} - l_{13} l_{24} + \sqrt{\lvert (l_{ij}) \rvert}}{2 l_{14} l_{23}}$	cross ratio: $\frac{p_1 p_2 p_3 p_4 + p_4 p_3 p_2 p_1}{(p_1 p_4 + p_4 p_1)(p_2 p_3 + p_3 p_2)}$
inversion: $p \mapsto m - \frac{r^2}{p - m}$	polar reflection ...	reflection: $p \mapsto \pm(p - 2\langle p, s \rangle s)$	reflection: $p \mapsto \pm sps$

Consequently, restricting a parametrization $f : U \to I\!R^n$ of an orthogonal system to any of the (coordinate) hypersurfaces $t_i = const$ provides a curvature line parametrization of the hypersurface[6]. Moreover, an adapted framing (3) restricts to a principal framing of any of the hypersurfaces of an orthogonal system: $n_i := \Phi^{-1}e_i\Phi$ yields a unit normal field for the hypersurface $t_i = const$ and the remaining directions $\Phi^{-1}e_j\Phi$ are its principal directions with curvatures $\kappa_{ij} = -\frac{1}{l_il_j}\frac{\partial}{\partial t_i}l_j$, i.e $[e_i, \Phi_j\Phi^{-1}] = -\kappa_{ij}l_je_j$ where $\Phi_j := \frac{\partial}{\partial t_j}\Phi$. Thus (cf.(1)),

$$\Phi_j \quad = \quad l_j(e_\infty + \tfrac{1}{2}\textstyle\sum_{i=1}^n \kappa_{ij}e_i)e_j \cdot \Phi\,. \tag{4}$$

In terms of a parametrization $f : U \to L^{n+1}$ of an orthogonal system, a conformal change of the ambient space's metric can be modeled by a "conformal deformation" $f \mapsto e^uf$. For an adapted frame (3), this has the effect that the second point of intersection of the spheres $\Phi^{-1}e_j\Phi$ is generally not constant any more, i.e. that these spheres cannot be interpreted as planes in a Euclidean space any longer: $[e_\infty, \Phi_j\Phi^{-1}] = \frac{1}{2}l_j\sum_{i=1}^n \sigma_{ij}e_i$ with the (symmetric) Schouten tensor (σ_{ij}) of the induced metric $\sum_{i=1}^n l_i^2dt_i$. Now,

$$\Phi_j \quad = \quad l_j[(e_\infty + \tfrac{1}{2}\textstyle\sum_{i=1}^n \gamma_{ij}e_i)e_j + e_0(\tfrac{1}{2}\textstyle\sum_{i=1}^n \sigma_{ij}e_i)]\cdot\Phi\,. \tag{5}$$

The other key concept we are going to examine is the one of a

Definition (Ribaucour congruence). *A hypersurface is said to envelope a sphere congruence (an $(n-1)$-parameter family of spheres) if, at every point, the hypersurface has first order contact with a sphere of the congruence, and a sphere congruence is said to be Ribaucour if the curvature lines on its two envelopes do correspond.*

A more technical description for the envelopes of a sphere congruence and for Ribaucour congruences will prove useful (cf.[2],[15]):

Lemma. *$f : M^{n-1} \to L^{n+1}$ envelopes a sphere congruence $s : M^{n-1} \to S_1^{n+1}$ if and only if $\langle s, f\rangle = 0$ and $\langle ds, f\rangle = 0$*

Thus, the two envelopes $f, \hat{f} : M^{n-1} \to L^{n+1}$ of a (regular) sphere congruence $s : M^{n-1} \to S_1^{n+1}$ can be interpreted as its two isotropic normal fields — that are uniquely determined up to rescalings. And, since the principal curvature directions of the two envelopes coincide with the principal directions of s with respect to f and $\hat{f}$, respectively, as normal fields,

Lemma. *$s : M^{n-1} \to S_1^{n+1}$ is a Ribaucour congruence if and only if its normal bundle is flat, i.e. if $d\langle df, \hat{f}\rangle = 0$ for its two envelopes $f, \hat{f} : M^{n-1} \to L^{n+1}$.*

Consequently, the two isotropic normal fields of a Ribaucour congruence — its envelopes — can be normalized to be *parallel* sections of the normal bundle. If we additionally *assume* the existence of principal curvature line coordinates we

[6] Note, that higher dimensional hypersurfaces usually do not carry curvature line coordinates.

obtain two orthogonal systems of codimension 1 in the conformal n-sphere, with $\frac{\partial}{\partial t_i}f // \frac{\partial}{\partial t_i}\hat{f} // \frac{\partial}{\partial t_i}s$. Thus, an *adapted frame* $\Phi : U \subset I\!R^{n-1} \to \mathrm{Spin}(I\!R_1^{n+2})$,

$$\begin{array}{c} \Phi^{-1}e_0\Phi = f, \quad \Phi^{-1}e_n\Phi = s, \quad \Phi^{-1}e_\infty\Phi = \hat{f}, \\ \text{and} \qquad \Phi^{-1}e_i\Phi // \frac{\partial}{\partial t_i}f, \quad 1 \leq i \leq n-1, \end{array} \tag{6}$$

can be introduced. Again, Dupin's theorem holds, i.e. $[e_i, \Phi_j\Phi^{-1}] \wedge e_j = 0$ for $0 \leq i \leq \infty$ and $1 \leq j \leq n-1$, $j \neq i$: and consequently, we find

$$\Phi_j \;\; = \;\; (l_j e_\infty + \tfrac{1}{2}\textstyle\sum_{i=1}^{n-1}\gamma_{ij}e_i + \tfrac{1}{2}a_j e_n + \hat{l}_j e_0)e_j \cdot \Phi \tag{7}$$

with suitable functions $l_j, \hat{l}_j, a_j, \gamma_{ij} : U \to I\!R$, $\gamma_{jj} = 0$. A rather remarkable property of such a "Ribaucour pair of orthogonal systems" motivates a definition in case of arbitrary codimension: any two corresponding subnets $t_i = const$ of f and $\hat{f}$ do not only envelope the sphere congruence s but also $s_i := \Phi^{-1}e_i\Phi$ since $s_i \perp \frac{\partial}{\partial t_j}f, \frac{\partial}{\partial t_j}\hat{f}$ for $j \neq i$ — and hence, both subnets envelope the congruence $s \cdot s_i : U \to \Lambda^2$ of $(n-2)$-spheres. Moreover, $\frac{\partial}{\partial t_j}s \perp s_i$ for $j \neq i$ which shows that both subnets carry well defined curvature lines (independent of the normal direction) — that, as before, do correspond on both envelopes and are given by the coordinate directions t_j. Clearly, these facts hold true for any pair of corresponding subnets of any dimension. Thus,

Definition. *A map $f : M^m \to S^n$ into the conformal n-sphere is said to envelope a congruence of m-spheres $s : M^m \to \Lambda^{n-m}$ if, at every point $p \in M^m$, f has first order contact with the corresponding sphere $s(p)$;*
a congruence of m-spheres $s : M^m \to \Lambda^{n-m}$ is called Ribaucour if it has two envelopes with well defined curvature lines that do correspond; and
two nets $f, \hat{f} : U \subset I\!R^m \to S^n$ in the conformal n-sphere are said to form a Ribaucour pair of orthogonal systems if any corresponding k-dimensional subnets, $1 \leq k \leq m$, envelope a Ribaucour congruence of k-spheres.

In particular, the coordinate lines in the nets of a Ribaucour pair of orthogonal systems intersect pairwise orthogonal since they arise as curvature lines — in case $m = n$ as curvature lines of lower dimensional subnets. We also explicitely allow one of the nets, f or $\hat{f}$, to degenerate: this way, the classical orthogonal systems in Euclidean n-space (coupled with the point at infinity) appear as special cases of Ribaucour pairs of orthogonal systems.

Finally, we want to characterize adapted frames for Ribaucour pairs of orthogonal systems: we call $\Phi : U \subset I\!R^m \to \mathrm{Spin}(I\!R_1^{n+2})$ an *adapted frame* if

$$f = \Phi^{-1}e_0\Phi, \quad \hat{f} = \Phi^{-1}e_\infty\Phi,$$

and if $s_i := \Phi^{-1}e_i\Phi$ are the principal directions of f and $\hat{f}$ for $1 \leq i \leq m$ and form a parallel orthonormal basis of $\{f, \hat{f}, df(TU)\}^\perp$ for $m+1 \leq i \leq n$. Then,

Proposition. *$\Phi : U \subset I\!R^m \to \mathrm{Spin}(I\!R_1^{n+2})$ is an adapted frame for a Ribaucour pair $f, \hat{f} : U \to L^{n+1}$ of orthogonal systems in the conformal n-sphere iff*

$$\Phi_j \;\; = \;\; (l_j e_\infty + \tfrac{1}{2}\textstyle\sum_{i=1}^{m}\gamma_{ij}e_i + \tfrac{1}{2}\sum_{i=1}^{n-m}a_{ij}e_i + \hat{l}_j e_0)e_j \cdot \Phi \tag{8}$$

for $1 \leq j \leq m$, and suitable functions $l_i, \hat{l}_i, \gamma_{ij}, a_{ij} : U \to I\!R$, $\gamma_{ii} = 0$.

At this point, we are prepared to formulate the discrete frame equations that we will use to define (algebraically) the discrete analogs for Ribaucour pairs of orthogonal systems:

4. Discrete orthogonal systems and Ribaucour congruences

The basic idea is to discretize the frame equations we derived in the previous section, i.e. to discretize the first order Taylor expansion of the adapted frames $\Phi : U \to \mathrm{Spin}(\mathbb{R}^{n+2}_1)$: for infinitesimal $0 \simeq \varepsilon \in \mathbb{R}$, $t \in U$ and the standard direction vectors $t_j = (\delta_{1j}, \dots, \delta_{nj})$, we have

$$\Phi(t + \varepsilon t_j) \simeq \Phi(t) + \Phi_j(t) \cdot \varepsilon = [1 + \varepsilon \Phi_j \Phi^{-1}(t)] \cdot \Phi(t).$$

A discrete analog of this equation can easily be formulated if 1 and $\Phi_j\Phi^{-1}$ can be combined to take values in $\mathrm{Spin}(\mathbb{R}^{n+2}_1)$ — since $\Phi(t + \varepsilon t_j)$ and $\Phi(t)$ lie in $\mathrm{Spin}(\mathbb{R}^{n+2}_1)$ the product $\Phi(t+\varepsilon t_j)\Phi^{-1}(t) \simeq [1+\varepsilon\Phi_j\Phi^{-1}(t)]$ should, too. Thus, if $[\varepsilon_{j1} + \varepsilon_{j2}\Phi_j\Phi^{-1}(t)] \in \mathrm{Spin}(\mathbb{R}^{n+2}_1)$ for some *finite* numbers $\varepsilon_{j1}, \varepsilon_{j2} \in \mathbb{R}$ we can just use the *sructure* of $[\varepsilon_{j1} + \varepsilon_{j2}\Phi_j\Phi^{-1}(t)]$ to define the discrete analogs of the frame equations.

Having this in mind, an examination of the frame equations (4) for an adapted frame of an orthogonal system in *Euclidean ambient space* $\mathbb{R}^n$ as well as (7) for an adapted frame of a Ribaucour congruence in the conformal n-sphere — or, more general, the frame equations (8) for a Ribaucour pair of orthogonal systems in the conformal n-sphere — can be discretized following that approach: here, *any* combination $[1+\varepsilon\Phi_j\Phi^{-1}]$ with $\varepsilon \in \mathbb{R}$ is a *pure* bivector — and, consequently, a suitable normalization takes values in $\mathrm{Spin}(\mathbb{R}^{n+2}_1)$. The situation is different in the case of the adapted frame equations (5) for a single orthogonal system in the conformal n-sphere: in that case, it will be hard to control whether a combination $1 + \varepsilon\Phi_j\Phi^{-1}$ is a pure bivector, and therefore, whether a suitable normalization can ever take values in $\mathrm{Spin}(\mathbb{R}^{n+2}_1)$.

Thus, we will stick to the case of Ribaucour pairs of orthogonal systems: here, the structure equations for a discrete frame $\Phi : \Gamma \subset \mathbb{Z}^m \to \mathrm{Spin}(\mathbb{R}^{n+2}_1)$ will be of the form

$$\Phi(t + t_j) = e_j s_j(t + \tfrac{1}{2}t_j) \cdot \Phi(t) \tag{9}$$

with suitable vector functions $s_j : \Gamma_j \to S^{n+1}_1$ — where we use the notation $t + \frac{1}{2}t_j$ for the edges in j-direction of Γ, and Γ_j for the lattice formed by these edges. With this ansatz for Φ, we examine the geometry of the two maps

$$\begin{array}{l} F := \Phi^{-1}e_0\Phi : \ \Gamma \to L^{n+1} \\ \hat{F} := \Phi^{-1}e_\infty\Phi : \Gamma \to L^{n+1} \end{array} \tag{10}$$

— which will lead us to the definition of *discrete Ribaucour pairs of orthogonal systems*: first, we notice that two points $F(t)$ and $F(t + t_j)$ lie symmetric with respect to the sphere[7)]

$$S_j(t + \tfrac{1}{2}t_j) := \Phi^{-1}(t)e_j\Phi(t + t_j) = \Phi^{-1}(t + t_j)e_j\Phi(t) \ : \tag{11}$$

[7)] Note, that the second equality shows that $S_j(t + \frac{1}{2}t_j)$ depends symmetrically on the endpoints t and $t + t_j$ of the edge $t + \frac{1}{2}t_j$.

$$F(t+t_j) = S_j(t+\tfrac{1}{2}t_j)F(t)S_j(t+\tfrac{1}{2}t_j)\,,$$

i.e. $F(t+t_j)$ is obtained from $F(t)$ by inversion at the sphere $S_j(t+\frac{1}{2}t_j)$. Since the same holds true for the net $\hat{F}$ we deduce that any four points $F(t)$, $\hat{F}(t)$, $F(t+t_j)$ and $\hat{F}(t+t_j)$ lie on a circle — as a calculation of their cross ratio (2)

$$\mathrm{DV}[\hat{F}(t), F(t), F(t+t_j), \hat{F}(t+t_j)] = -\frac{1}{\langle e_0, s_j(t+\frac{1}{2}t_j)\rangle\langle e_\infty, s_j(t+\frac{1}{2}t_j)\rangle}$$

confirms — and the point pairs $(F(t), F(t+t_j))$ and $(\hat{F}(t), \hat{F}(t+t_j))$ do not seperate each other on the circle. Conversely, if two corresponding edges of F and $\hat{F}$ do not both degenerate at the same time, $F(t) \neq F(t+t_j)$ or $\hat{F}(t) \neq \hat{F}(t+t_j)$, then the corresponding sphere $S_j(t+\frac{1}{2}t_j)$ can be geometrically (i.e. up to sign) reconstructed from the four points — since they are concircular and do not seperate. As another simple consequence we find that if one of the maps F or $\hat{F}$ is constant, say $\hat{F} \equiv e_\infty$, then all the spheres S_j, $1 \leq j \leq m$, are planes, i.e. they contain e_∞, the "point at infinity": $S_j e_\infty + e_\infty S_j = 0$. In that case, $F : \Gamma \to \mathbb{R}^n = \{y \in L^{n+1} \,|\, e_\infty y + y e_\infty = 1\}$ takes values in Euclidean space.

As in smooth differential geometry, a key tool in understanding the geometry of the nets $F(\Gamma)$ and $\hat{F}(\Gamma)$ are the integrability conditions

$$e_j s_j(t+t_i+\tfrac{1}{2}t_j)\cdot e_i s_i(t+\tfrac{1}{2}t_i) = e_i s_i(t+t_j+\tfrac{1}{2}t_i)\cdot e_j s_j(t+\tfrac{1}{2}t_j) \qquad (12)$$

of the structure equations (9). Using these "Maurer-Cartan equations", it is a straightforward calculation to obtain the cross ratio (2) of an elementary quadrilateral with vertices $F(t)$, $F(t+t_i)$, $F(t+t_i+t_j)$ and $F(t+t_j)$

$$\mathrm{DV}[F(t), F(t+t_i), F(t+t_i+t_j), F(t+t_j)] = -\frac{\langle e_0, s_i(t+\frac{1}{2}t_i)\rangle\langle e_0, s_i(t+t_j+\frac{1}{2}t_i)\rangle}{\langle e_0, s_j(t+\frac{1}{2}t_j)\rangle\langle e_0, s_j(t+t_i+\frac{1}{2}t_j)\rangle}\,.$$

Since the cross ratio of four points is real if and only if the four points lie on a circle we find that not only corresponding point pairs of F and $\hat{F}$ are concircular:

Lemma. *The vertices of any elementary quadrilateral of the net F, or $\hat{F}$, as well as the endpoints of corresponding edges of F and $\hat{F}$ are concircular.*

Together with the investigations in [6] or [19] and Dupin's theorem, this might motivate the following

Corollary and Definition. *The two nets $F, \hat{F} : \Gamma^m \to S^n$ in the conformal n-sphere are discrete curvature line nets, i.e. the vertices of any elementary quadrilateral (2-cell) have real cross ratio.*

The above lemma directly generalizes to higher dimension: for example, we may consider corresponding elementary quadrilaterals (2-cells) of the nets F and $\hat{F}$. Then, by the lemma, the vertices of the elementary quadrilateral of F lie on a circle, as well as the two endpoints of an edge and their corresponding points of the net $\hat{F}$ do. These two circles are contained in a 2-sphere. Now, taking another pair of edges that shares one pair of endpoints with the former pair of edges, another circle is obtained. Since three of the points on this circle already lie on the 2-sphere, the whole circle does. Consequently, by symmetry, all the vertices of the elementary 2-cell of $\hat{F}$ also lie on that 2-sphere. A similar argument shows that the vertices of an elementary 3-cell of one of the nets, F or $\hat{F}$, lie on a 2-sphere — and, finally, the same arguments also prove similar statements for higher dimensional elementary cells. Thus, as a generalization of the previous lemma, we obtain the following

Lemma. *The vertices of any elementary k-cell of the net F, or $\hat{F}$, lie on a $(k-1)$-sphere, and the vertices of any corresponding k-cells of F and $\hat{F}$ lie on a k-sphere, for $1 \leq k \leq m$.*

This lemma motivates the definition of a discrete Ribaucour congruence and its envelopes (cf. [15], [18]), and of a discrete Ribaucour pair of orthogonal systems:

Corollary and Definition. *The two nets $F, \hat{F} : \Gamma^m \to S^n$ in the conformal n-sphere envelope a discrete Ribaucour sphere congruence*[8)]

$$S = \Phi^{-1} e_0 \wedge s_1 \wedge \ldots \wedge s_m \wedge e_\infty \Phi : \Gamma^* \to \Lambda^{m+2} \cong \Lambda^{n-m}, \tag{13}$$

i.e. all corresponding elementary m-cells of the two nets F and $\hat{F}$ lie on m-spheres $S(t + \frac{1}{2}(1, \ldots, 1))$.
Moreover, the nets form a Ribaucour pair of orthogonal nets, i.e. all corresponding elementary k-cells, $1 \leq k \leq m$, lie on k-spheres and the points of corresponding 1-cells (edges) do not seperate on the circles they lie on.

A discrete map $S : \Gamma^* \to \Lambda^{n-m}$ into the space of m-spheres in the conformal n-sphere generally has no envelopes: for the notion of "envelopes" to make sense one has to require the spheres $S(t + \frac{1}{2}(\varepsilon_1, \ldots, \varepsilon_m))$, $t \in \Gamma$ and $\varepsilon_j \in \{\pm 1\}$, of each elementary m-cell in Γ^* to intersect in a point pair. In that case, it makes sense to speak of a "discrete sphere congruence". Note, that for the notion of a "discrete Ribaucour sphere congruence" we *assume* the existence of two enveloping discrete curvature line nets. Thus, the two envelopes can be reconstructed from a discrete Ribaucour congruence: since we additionally know that the endpoints of corresponding edges of the two envelopes should not seperate on the circles they lie on (cf.(11)) there is a unique ordering on the pairs of corresponding points.

From the previous lemma, it also becomes clear that — analogous to the smooth case — any corresponding subnets of a discrete Ribaucour pair of orthogonal nets form themselves a discrete Ribaucour pair.

With the equations (11), the frame Φ can as well be "integrated" from the spheres S_j, $1 \leq j \leq m$ — and, since the spheres S_j can be (up to sign) constructed from the nets F and $\hat{F}$,

Theorem. *All three, the discrete frame $\Phi : \Gamma \to \mathrm{Spin}(\mathbb{R}^{n+2}_1)$ satisfying (9), the corresponding discrete Ribaucour pair of orthogonal nets $F, \hat{F} : \Gamma \to L^{n+1}$, and the system of element spheres $S_j : \Gamma_j \to S^{n+1}_1$ $(1 \leq j \leq m)$ can be constructed from one of them. If $m < n$, they can also be reconstructed from the discrete Ribaucour congruence $S : \Gamma^* \to \Lambda^{n-m}$ of m-spheres enveloped by F and $\hat{F}$.*

8) Note, that the spheres S only depend on the elementary m-cells of Γ^m, i.e. the domain of S is the dual lattice Γ^* of Γ.

In these constructions, the frame is only determined up to a pointwise sign change. In terms of the element spheres S_j, such sign changes are reflected by a sign ambiguity, too. However, the spheres S_j have to satisfy a integrability condition that is equivalent to the Maurer-Cartan equations (12) and which restricts this sign ambiguity for the S_j:

Lemma (Maurer-Cartan equations). *Given maps $S_j : \Gamma_j \to S_1^{n+1}$ there exists a frame $\Phi : \Gamma \to \mathrm{Spin}(\mathbb{R}_1^{n+2})$ satisfying (11) for $1 \leq j \leq m$ if and only if*

$$0 = S_i(t + \tfrac{1}{2}t_i)S_j(t + t_i + \tfrac{1}{2}t_j) + S_j(t + \tfrac{1}{2}t_j)S_i(t + t_j + \tfrac{1}{2}t_i)\,. \qquad (14)$$

It follows that any four spheres $S_i(t + \frac{1}{2}t_i)$, $S_j(t + \frac{1}{2}t_j)$, $S_j(t + t_i + \frac{1}{2}t_j)$ and $S_i(t + t_j + \frac{1}{2}t_i)$ intersect in a common $(n-2)$-sphere. Thus, interpreted as points in projective $(n+1)$-space (cf. Table 1), the four spheres are collinear. Again, this picture can be directly generalized to higher dimensions:

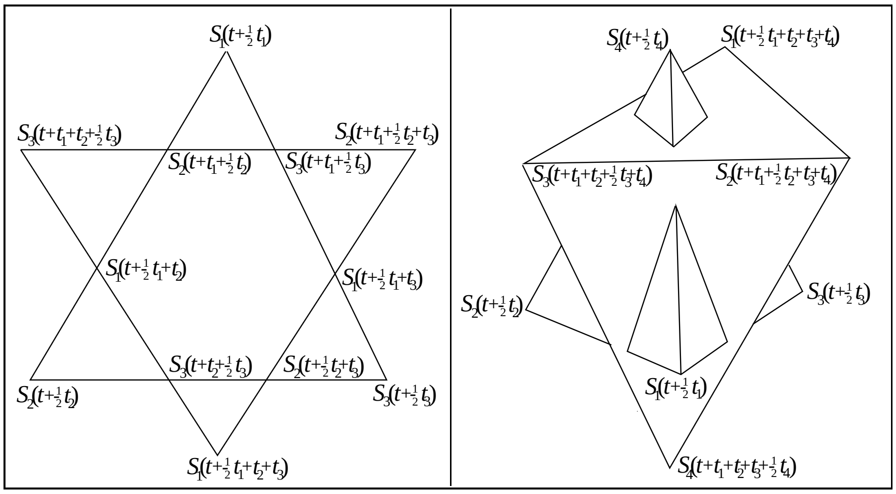

Fig. 1. Images of elementary 3- and 4-cells of F in $\mathbb{RP}^{n+1}$

for $k \geq 2$, any two elementary k-cells that belong to an elementary $(k+1)$-cell share an elementary $(k-1)$-cell. Thus, assuming that all spheres belonging to an elementary k-cell are contained in a $(k-1)$-plane of $\mathbb{RP}^{n+1}$, it follows by induction that

Lemma. *All spheres belonging to an elementary $(k+1)$-cell, $1 \leq k \leq m-1$, of the net F, or $\hat{F}$, are contained in a k-dimensional plane in $\mathbb{RP}^{n+1}$.*

This lemma provides a method to reconstruct an elementary m-cell of the net F, or $\hat{F}$, from one point and all spheres S_j belonging to those k-cells, $k \geq 2$, that contain the initial point — or, from these k-cells themselves[9] (cf. Fig. 1): since

[9] As mentioned above, a sphere S_j belonging to corresponding edges of F and $\hat{F}$ can generically be reconstructed from the four points. However, if only points of one of the nets, F or $\hat{F}$, are given the spheres cannot be uniquely reconstructed — but, for example, choosing them to be planes in a Euclidean space (i.e. $\hat{F} \equiv e_\infty$) resolves this ambiguity.

at least $(m-1)$ spheres belonging to *any* elementary $(m-1)$-cell are known, the corresponding $(m-2)$-planes in $\mathbb{R}P^{n+1}$ can be constructed. Intersecting these $(m-2)$-planes yields some of the missing spheres — which can now be used to construct all $(m-3)$-planes corresponding to the $(m-2)$-cells. This construction can be continued until all spheres are constructed as the intersection of the lines corresponding to 2-cells. The missing points of the m-cell can then be obtained by reflecting known points at the constructed spheres.

This construction scheme provides a method to solve the following (cf.[7])

Theorem (Cauchy problem). *Knowing the k-dimensional subnets of an m-dimensional discrete curvature line net F, $k \geq 2$, that pass through a point in all $\binom{m}{k}$ different directions, or k-dimensional subnets of discrete curvature line nets F and $\hat{F}$, that pairwise envelope discrete Ribaucour k-sphere congruences, the net F, resp. both nets F and $\hat{F}$, can be reconstructed.*

This theorem can as well be proved directly on the level of the points of an elementary m-cell — in the case $m = 3$ of lowest dimension, it can be reduced to Miguel's theorem [1] (cf.[9],[20]):

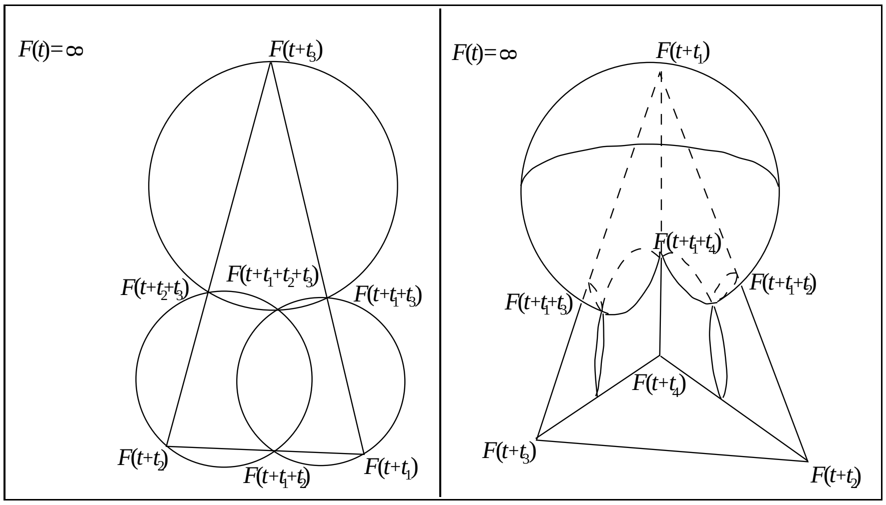

Fig. 2. Miguel's theorem

since the vertices of an elementary 3-cell lie on a 2-sphere an inversion that maps the common point $F(t)$ of all known 2-cells to infinity leaves us with a 2-dimensional picture (Fig. 2, left). By Miguel's theorem the three circles intersect in a point — the image of the unknown point of the elementary 3-cell. A similar procedure works in higher dimensions: mapping the initial point of an elementary $(k+1)$-cell to infinity, we are left with a k-simplex in Euclidean k-space. Focussing on one of the $(k-1)$-spheres (corresponding to the elementary k-cells) that contain the vertices of the simplex, the problem can be reduced by one dimension (cf. Fig. 2, right). Thus, the assumption follows by induction.

Another relation (besides relations through analogy) between the smooth and the analogous discrete theory of (Ribaucour pairs of) orthogonal nets may be established through a

5. Permutability theorem

for the Ribaucour transformation — an orthogonal system $\hat{f} : U \subset \mathbb{R}^m \to S^n$ (or, a submanifold with well defined curvature lines, i.e. with flat normal bundle) is called "Ribaucour transformation" of a given orthogonal system $f : U \to S^n$ in the conformal n-sphere if they form a Ribaucour pair. "2-dimensional versions" of this permutability theorem were known classically (cf.eg.[11]):

Theorem. *Given two Ribaucour transforms $f_1, f_2 : U \to S^3$ of a 2- or 3-dimensional orthogonal system $f : U \to S^3$ in the conformal 3-sphere, there is a 1-parameter family of orthogonal systems $f_{12} = f_{21} : U \to S^3$ that are Ribaucour transforms of both, f_1 as well as f_2. Corresponding points of the four orthogonal systems are concircular.*

Thus, a 2-dimensional discrete curvature line net $F : \mathbb{Z}^2 \to S^3$ can be obtained by repeatedly applying this permutability theorem[10] (cf.[20]): the images of one point in the original system will form a discrete orthogonal (curvature line) net in S^3. A "3-dimensional version" of this fact has recently been formulated [13]:

Theorem. *Given three Ribaucour transforms $f_1, f_2, f_3 : U \to S^3$ of a triply orthogonal system $f : U \to S^3$ as well as three systems $f_{12}, f_{23}, f_{31} : U \to S^3$ such that each f_{ij} is a Ribaucour transform of f_i and f_j at the same time, there is a unique triply orthogonal system $f_{123} : U \to S^3$ that is a Ribaucour transform of all three, f_{12}, f_{23} and f_{31}.*

Clearly, this triply orthogonal system f_{123} can be constructed as in the solution of the Cauchy problem, earlier. In fact, starting with three discrete curvature line nets, constructed according to the "2-dimensional version" of the permutability theorem as described before, the solution of the Cauchy problem provides a discrete triply orthogonal system in the conformal 3-sphere. Following this chain of thought, the Cauchy problem for discrete orthogonal nets appears as a discrete analog of a general permutability theorem for the Ribaucour transformation of smooth orthogonal systems — or, the permutability theorem appears as a smooth limit of the Cauchy problem:

Property[11]**.** *Let $1 \leq i \leq k$ and let $f_t : U \subset \mathbb{R}^m \to S^n$ be smooth orthogonal systems, corresponding to the vertices $t \in \{0,1\}^k$ of the $\binom{k}{i}$ elementary i-cells containing $0 \in \{0,1\}^k$, such that the orthogonal systems corresponding to the endpoints of an edge form a Ribaucour pair. Then, all missing smooth orthogonal systems f_t, $t \in \{0,1\}^k$ in the hypercube, can be — uniquely if $i \geq 2$ — constructed such that any edge corresponds to a Ribaucour pair. Corresponding points of four orthogonal systems of an elementary 2-cell are concircular.*

Considering a smooth orthogonal system $f : U \subset \mathbb{R}^m \to \mathbb{R}^n$ as a discrete orthogonal net with infinitesimal edge lengthes $f(t + \varepsilon t_i) - f(t) \simeq 0$ with fixed

[10] Note, that specializing this procedure to Darboux transforms of isothermic surfaces one obtains discrete isothermic nets in a similar way (cf.[6],[18]).

[11] A rigorous proof seems to be a technicality.

$\varepsilon \simeq 0$: the missing systems can be constructed by solving the Cauchy problem for the $(m+k)$-dimensional[12] net $F : \mathbb{Z}^{m+k} \to \mathbb{R}^n$ with the given smooth orthogonal systems as part of $\binom{m+k}{i}$ initial data. This might serve as a plausibility argument for the general permutability theorem for the Ribaucour transformation of orthogonal systems.

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[12] Here, we may also allow $m+k > n$ without harm.

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