Integrable discretizations of the Euler top

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Abstract

Discretizations of the Euler top sharing the integrals of motion with the continuous time system are studied. Those of them which are also Poisson with respect to the invariant Poisson bracket of the Euler top are characterized. For all these Poisson discretizations a solution in terms of elliptic functions is found, allowing a direct comparison with the continuous time case. We demonstrate that the Veselov–Moser discretization also belongs to our family, and apply our methods to this particular example.

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1 Introduction

The subject of integrable discretizations of integrable dynamical systems is slightly more than twenty years old. During the first decade of its existence several different approaches were proposed for discretizing soliton equations [AL], [H], [DJM], [QNCV], [NCW], but neither of them dealt with integrable systems of the classical mechanics. The first examples in this important subarea seem to appear about ten years ago [V], [S]. The Veselov's paper contained discrete time versions of such classical integrable systems as the Euler case of the rigid body motion and the Neumann system. The algebraic construction behind these examples was later elaborated in more detail in [MV]. However, the nature of these exceptionally beautiful examples remains somewhat mysterious. They still resist to be included in any of the existing general frameworks for integrable discretizations (cf. [DLT]). The origin of the Veselov–Moser's discretizations, their place in the corresponding continuous time hierarchies, their relations to other existing examples, and several further points remain to be clarified. The problem of comparing the explicit solutions found in [V], [MV] with the classical continuous time counterparts was also left open (despite the fact that on the level of equations of motion the continuous limit is easy to perform).

The present work may be considered as a sort of an extensive comment on [V], [MV] in the part concerned with the Euler top. We do not close all the open problems mentioned above, but we do hope to bring some light into some of them. In particular, we find formulas for the solution of the discrete time Euler top in the form which makes the comparison with the continuous time case immediate. Moreover, the Veselov–Moser system turns out to be by no means the only reasonable discretization of the Euler top. We introduce a whole family of discretizations sharing integrals of motion with the continuous time system, and characterize those of them which share also the underlying invariant Poisson structure. In this context the Veselov–Moser system becomes just one particular case, and not the most simple one.

It has to be mentioned that, unlike [V], [MV], our procedure for obtaining explicit solutions is rather pedestrian; it does not use such advanced tools as spectral theory of difference operators [V] or Baker–Akhiezer functions [MV]. Actually, our construction is based on the addition formulas for elliptic functions, and could be invented already by Jacobi. However, as a heuristic tool for finding it we used the Lax representation of the Euler top in su(2) with a spectral parameter on an elliptic curve, which was certainly unknown in the times of Euler and Jacobi.

2 The Euler top

The famous Euler's equations describing the motion of a rigid body with the fixed center of mass (Euler top) read [G]:

$$\begin{aligned} \dot{M}_1 &= \left(\frac{1}{B}-\frac{1}{C}\right) M_2 M_3 \,, \\ \dot{M}_2 &= \left(\frac{1}{C}-\frac{1}{A}\right) M_3 M_1 \,, \\ \dot{M}_3 &= \left(\frac{1}{A}-\frac{1}{B}\right) M_1 M_2 \,. \end{aligned} \tag{1}$$

Here $\mathbf{M}=(M_1,M_2,M_3)^{\mathrm{T}}$ is the kinetic momentum vector in the coordinate system attached firmly to the body; the axes of this system coincide with the principal axes of inertia, and the numbers $A,B,C>0$ are the corresponding moments of inertia.

The vector form of the equations (1) is:

$$\dot{\mathbf{M}} = \mathbf{M}\times\mathbf{\Omega}(\mathbf{M}) \,, \tag{2}$$

where the vector of the angular velocity is introduced:

$$\mathbf{\Omega}(\mathbf{M}) = \left(\frac{M_1}{A},\frac{M_2}{B},\frac{M_3}{C}\right)^{\mathrm{T}} \,. \tag{3}$$

The equations (1) are Hamiltonian with respect to the following Poisson bracket on $\mathbb{R}^3(\mathbf{M})$:

$$\{M_1,M_2\}=M_3\,, \qquad \{M_2,M_3\}=M_1\,, \qquad \{M_3,M_1\}=M_2\,. \tag{4}$$

with the Hamilton function $H(\mathbf{M})=E(\mathbf{M})/2$, where

$$E(\mathbf{M}) = \frac{M_1^2}{A}+\frac{M_2^2}{B}+\frac{M_3^2}{C} \,. \tag{5}$$

The bracket (4) is degenerate and has one Casimir function

$$M^2(\mathbf{M}) = M_1^2+M_2^2+M_3^2 \,. \tag{6}$$

Generic symplectic leaves of the bracket (4) are two–dimensional spheres $M^2(\mathbf{M})=\mathrm{const}$, hence each system Hamiltonian with respect to this bracket is integrable in the Liouville–Arnold sense; in particular, the Euler top is integrable.

An explicit solution to the equations (1) can be given in terms of the Jacobi elliptic functions [G]. Suppose for the sake of definiteness that

$$A>B>C>0\,, \tag{7}$$

then

$$\frac{1}{A} \le \frac{E}{M^2} \le \frac{1}{C} \,.$$

The formulas for the solution look different depending on whether this quantity is greater or smaller than $1/B$:

$$M_1(t) = a\left\{ \begin{array}{c} \mathrm{cn}\,\nu(t-t_0) \\ \mathrm{dn}\,\nu(t-t_0) \end{array} \right\}, \quad M_2(t) = b\,\mathrm{sn}\,\nu(t-t_0), \quad M_3(t) = c\left\{ \begin{array}{c} \mathrm{dn}\,\nu(t-t_0) \\ \mathrm{cn}\,\nu(t-t_0) \end{array} \right\}. \tag{8}$$

Here and below in similar situations the upper expressions in curly brackets refer to the case $\frac{1}{B} < \frac{E}{M^2} \le \frac{1}{C}$, while the lower ones refer to the case $\frac{1}{A} \le \frac{E}{M^2} < \frac{1}{B}$. The module k of the elliptic functions in (8) is given by:

$$k^2 = \left\{ \begin{array}{c} \dfrac{(A-B)(M^2-CE)}{(B-C)(AE-M^2)} \\ \\ \dfrac{(B-C)(AE-M^2)}{(A-B)(M^2-CE)} \end{array} \right\}, \tag{9}$$

so that we have always $0 < k^2 < 1$. The coefficients a, b, c are defined by:

$$a^2 = A\,\frac{M^2-CE}{A-C}\,, \quad b^2 = \left\{ \begin{array}{c} B\,\dfrac{M^2-CE}{B-C} \\ \\ B\,\dfrac{AE-M^2}{A-B} \end{array} \right\}, \quad c^2 = C\,\frac{AE-M^2}{A-C}\,. \tag{10}$$

Finally, the frequency ν is given by:

$$\nu^2 = \left\{ \begin{array}{c} \dfrac{(B-C)(AE-M^2)}{ABC} \\ \\ \dfrac{(A-B)(M^2-CE)}{ABC} \end{array} \right\}. \tag{11}$$

In the case $\frac{E}{M^2} = \frac{1}{B}$ the elliptic functions degenerate to the hyperbolic ones, and we have:

$$M_1(t) = \frac{a}{\cosh\,\nu(t-t_0)}\,, \quad M_2(t) = b\,\tanh\,\nu(t-t_0)\,, \quad M_3(t) = \frac{c}{\cosh\,\nu(t-t_0)}\,,$$

where

$$a^2 = \frac{A(B-C)}{B(A-C)}\,M^2\,, \quad b^2 = M^2\,, \quad c^2 = \frac{C(A-B)}{B(A-C)}\,M^2\,,$$

and

$$\nu^2 = \frac{(A-B)(B-C)}{AB^2C} M^2 \ .$$

In all three cases the numbers a, b, c, ν satisfy the condition $abc\nu > 0$. In what follows, we shall denote by $\nu(M^2, E)$ the positive square root of the function given in (11), and hence we must have $abc > 0$.

3 Discretizations

Our aim here is to find *integrable discretizations* of the Euler top.

Definition 1 *An integrable discretization of the Euler top* (2) *is a one–parameter family of diffeomorphisms* $F : \mathbb{R}^3 \times [0, \epsilon) \mapsto \mathbb{R}^3$,

$$\widehat{\mathbf{M}} = F(\mathbf{M}, h) \ , \tag{12}$$

such that each one of them $F(\cdot, h) : \mathbb{R}^3 \mapsto \mathbb{R}^3$ *is Poisson with respect to the bracket* (4)*, has two integrals of motion* M^2 *and* E*, i.e.*

$$M^2(F(\mathbf{M}, h)) = M^2(\mathbf{M}) \ , \qquad E(F(\mathbf{M}, h)) = E(\mathbf{M}) \ , \tag{13}$$

and the following asymtotics hold:

$$F(\mathbf{M}, h) = \mathbf{M} + h\mathbf{M} \times \mathbf{\Omega}(\mathbf{M}) + o(h) \ , \quad h \to 0 \ . \tag{14}$$

Actually, our maps always will be defined by *implicit* equations of motion:

$$\widehat{\mathbf{M}} - \mathbf{M} = hf(\mathbf{M}, \widehat{\mathbf{M}}, h) \ . \tag{15}$$

Proposition 1 *Let* $f : \mathbb{R}^3 \times \mathbb{R}^3 \times [0, \epsilon) \mapsto \mathbb{R}^3$ *be a* C^1*–function in each argument such that*

$$f(\mathbf{M}, \mathbf{M}, 0) = \mathbf{M} \times \mathbf{\Omega}(\mathbf{M}) \ .$$

Then for h *small enough the equation* (15) *defines a local diffeomorphism* (12) *satisfying* (14)*.*

Proof. This follows immediately from the implicit function theorem for all h satisfying $\det(I - h\partial f/\partial\widehat{\mathbf{M}}) \neq 0$, i.e. for all h small enough. ∎

When speaking about discretizations, it is natural to think about $\mathbf{M}$ in (12) as about sequences $\mathbf{M}_m : \mathbb{Z} \mapsto \mathbb{R}^3$ approximating solutions $\mathbf{M}(t) : \mathbb{R} \mapsto \mathbb{R}^3$ of the Euler top (2) in the sense that $\mathbf{M}_m \approx \mathbf{M}(mh)$. In this context the formula (12) takes the form of the difference equation

$$\mathbf{M}_{m+1} = F(\mathbf{M}_m, h) \ ,$$

and, similarly, the formula (15) has to be thought of as

$$\mathbf{M}_{m+1} - \mathbf{M}_m = hf(\mathbf{M}_m, \mathbf{M}_{m+1}, h) \ .$$

The approximation property $\mathbf{M}_m = \mathbf{M}(mh) + O(h)$ holds on finite time intervals and is assured by (14) (this is a standard fact from the numerical analysis).

4 Elliptic coordinates and Poisson discretizations

Now we would like to find a manageable criterium for a map (12) to be Poisson with respect to the bracket (4). The corresponding statement becomes rather transparent in a new coordinate system in $\mathbb{R}^3(\mathbf{M})$. The corresponding change of variables

$$\Phi : \ (M^2, E, \varphi) \mapsto (M_1, M_2, M_3)$$

is suggested by the formulas (8) for a solution of the Euler top. In the above formula $(M_1, M_2, M_3) \in \mathbb{R}^3$, and $(M^2, E, \varphi) \in \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{T}$, where $\mathbb{T} = \mathbb{R}/(4K\mathbb{Z})$, and $K = K(k^2)$ is the full elliptic integral of the first kind corresponding to the value of k^2 given in (9). The formulas for Φ are different in two regions of $\mathbb{R}^3$ separated by the two planes

$$\frac{E(\mathbf{M})}{M^2(\mathbf{M})} = \frac{1}{B} \quad \Leftrightarrow \quad \left(\frac{1}{B} - \frac{1}{A}\right) M_1^2 = \left(\frac{1}{C} - \frac{1}{B}\right) M_3^2 \ . \tag{16}$$

Each one of the sets $\left\{\frac{E(\mathbf{M})}{M^2(\mathbf{M})} > \frac{1}{B}\right\}$ and $\left\{\frac{E(\mathbf{M})}{M^2(\mathbf{M})} < \frac{1}{B}\right\}$ consists of two sectors, and each one of these sectors is bounded by two half-planes.

The *elliptic coordinates* in $\mathbb{R}^3(\mathbf{M})$ are introduced by:

$$M_1 = a \left\{ \begin{array}{c} \mathrm{cn}\,\varphi \\ \mathrm{dn}\,\varphi \end{array} \right\}, \quad M_2 = b\,\mathrm{sn}\,\varphi, \quad M_3 = c \left\{ \begin{array}{c} \mathrm{dn}\,\varphi \\ \mathrm{cn}\,\varphi \end{array} \right\}. \tag{17}$$

with a, b, c given by (10), and the modulus k^2 of the elliptic functions given by (9). On the subset $\left\{\frac{E}{M^2} > \frac{1}{B}\right\}$ the sign of c coincides with the sign of M_3 (which is constant in each one of two sectors), and the signs of a, b satisfy the condition $\mathrm{sign}(ab) = \mathrm{sign}(c)$. Similarly, on the subset $\left\{\frac{E(\mathbf{M})}{M^2(\mathbf{M})} < \frac{1}{B}\right\}$ the sign of a coincides with the sign of M_1, while the signs of b, c satisfy $\mathrm{sign}(bc) = \mathrm{sign}(a)$. Finally, on the four half-planes described by (16) the elliptic coordinates are defined by continuity according to

$$M_1 = \frac{a}{\cosh\,\varphi} \ , \quad M_2 = b\,\tanh\,\varphi \ , \quad M_3 = \frac{c}{\cosh\,\varphi} \ ,$$

the signs of a, c being the same as the signs of M_1, M_3, respectively, and $\mathrm{sign}(b) = \mathrm{sign}(ac)$.

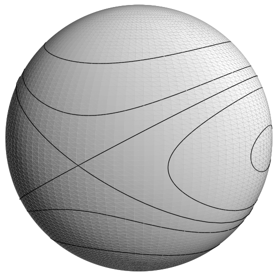

Figure 1: Different energy levels for constant M^2

Proposition 2 *The formulas* (17) *define a valid change of variables (a local diffeomorphism) near each point of* $\mathbb{R}^3$.

Proof. It can be verified by a direct calculation that in the region $\left\{\frac{E(\mathbf{M})}{M^2(\mathbf{M})} \neq \frac{1}{B}\right\}$ we have:

$$\frac{\partial(M_1, M_2, M_3)}{\partial(M^2, E, \varphi)} = -\frac{1}{4\nu(M^2, E)}\,\mathrm{dn}^2\,\varphi \;\neq\; 0\;,$$

and that the partial derivatives of which this Jacobian is composed allow a continuation to the boundary $\left\{\frac{E(\mathbf{M})}{M^2(\mathbf{M})} = \frac{1}{B}\right\}$. ■

Now it is obvious that an arbitrary map (12) having two integrals of motion $M^2(\mathbf{M})$ and $E(\mathbf{M})$, can be cast, in the elliptic coordinates (M^2, E, φ), in the form

$$\widehat{M}^2 = M^2\;, \quad \widehat{E} = E\;, \quad \widehat{\varphi} = \widehat{\varphi}(M^2, E, \varphi)\;. \tag{18}$$

Proposition 3 *The map* (12) *having two integrals of motion* $M^2(\mathbf{M})$ *and* $E(\mathbf{M})$ *is Poisson with respect to the bracket* (4) *iff in the elliptic coordinates* (M^2, E, φ) *it takes the form*

$$\widehat{M}^2 = M^2\;, \quad \widehat{E} = E\;, \quad \widehat{\varphi} = \varphi + g(M^2, E) \tag{19}$$

with the function g *not depending on* φ.

Proof. To prove this statement, we have to calculate the Poisson bracket (4) in the coordinates (M^2, E, φ). The corresponding formulas read:

$$\{M^2, E\} = \{M^2, \varphi\} = 0\;, \qquad \{E, \varphi\} = -2\nu(M^2, E)\;. \tag{20}$$

Indeed, the function M^2 is a Casimir function, hence it Poisson commutes with both E and φ. To calculate the bracket $\{E, \varphi\}$, we substitute (away from the boundary $\frac{E}{M^2} = \frac{1}{B}$) the expressions (17) into an arbitrary one of the formulas (4), and after straightforward calculations arrive at the expression given above.

Actually, the concrete expression for $\{E, \varphi\}$ is not essential for the proof of our proposition. The only important thing is that this Poisson bracket does not depend on φ. Indeed, we have:

$$\{\widehat{E}, \widehat{\varphi}\} = \frac{\partial \widehat{\varphi}}{\partial \varphi} \{E, \varphi\} \; .$$

Since the Poisson bracket $\{E, \varphi\}$ depends only on M^2, E which are integrals of motion, we see that the necessary and sufficient condition for our map to be Poisson reads $\partial\widehat{\varphi}/\partial\varphi = 1$, which is equivalent to the last equation in (19). ■

5 Special discretizations

We derive now a family of discretizations of the Euler top. Our derivation is based on a Lax representation with a spectral parameter for the Euler top. We prefere to work with a su(2) Lax representation, since our experience in discretizing various geometric structures (see, for example, [BP]) conviced us that this procedure may be performed most straightforwardly when applied to su(2) Lax formulations.

To find a suitable Lax representation for the Euler top, we use a stationary version of the Lax representation of the chiral field model due to Cherednik [Ch]. Set

$$M(u) = \frac{1}{2i} \sum_{k=1}^{3} M_k w_k(u) \sigma_k \; , \tag{21}$$

where σ_k $(k = 1, 2, 3)$ are the Pauli matrices, and $w_k(u)$ are the following elliptic functions:

$$w_1(u) = \rho \, \frac{1}{\mathrm{sn}(u, \kappa)} \; , \quad w_2(u) = \rho \, \frac{\mathrm{dn}(u, \kappa)}{\mathrm{sn}(u, \kappa)} \; , \quad w_3(u) = \rho \, \frac{\mathrm{cn}(u, \kappa)}{\mathrm{sn}(u, \kappa)} \; . \tag{22}$$

Here the parameter ρ and the module κ of the elliptic functions are defined by

$$\rho^2 = \frac{A - C}{AC} \; , \qquad \kappa^2 = \frac{C(A - B)}{B(A - C)} \; . \tag{23}$$

Further, for a vector $\mathbf{V} = (V_1, V_2, V_3)^{\mathrm{T}} \in \mathbb{R}^3$ set:

$$V(u) = \frac{1}{2i} \sum_{k=1}^{3} V_k w_k(u - u_0) \sigma_k \; , \tag{24}$$

where the point u_0 is chosen so that

$$w_1(u_0) = A^{-1/2}, \qquad w_2(u_0) = B^{-1/2}, \qquad w_3(u_0) = C^{-1/2}. \tag{25}$$

Consider now the Lax equation

$$\dot{M}(u) = [M(u), V(u)] \; . \tag{26}$$

With the help of identities

$$w_j(u)w_k(u-u_0) = w_j(u_0)w_l(u-u_0) - w_k(u_0)w_l(u) \; , \tag{27}$$

where (j,k,l) is a permutation of $(1,2,3)$, one sees that the matrix equation (26) is equivalent to the set of the following two equations:

$$\dot{\mathbf{M}} = \mathbf{M} \times \mathbf{\Omega}^{1/2}(\mathbf{V}) \; , \tag{28}$$
$$0 = \mathbf{V} \times \mathbf{\Omega}^{1/2}(\mathbf{M}) \; . \tag{29}$$

Here the second equation is equivalent to $\mathbf{V} = \gamma\, \mathbf{\Omega}^{1/2}(\mathbf{M})$ with some $\gamma \in \mathbb{R}$, which, being substituted in the first equation, results in

$$\dot{\mathbf{M}} = \gamma\, \mathbf{M} \times \mathbf{\Omega}(\mathbf{M}) \; . \tag{30}$$

Obviously, for a nonvanishing function $\gamma = \gamma(t)$ the latter equation is nothing but a time reparametrization of the Euler top (2). In this sense (26) is a Lax representation of the Euler top.

Discretizing the above construction in time, we introduce the matrix

$$\mathcal{V}(u) = I + \frac{h}{2i} \sum_{k=1}^{3} V_k w_k(u-u_0)\sigma_k \; , \tag{31}$$

and consider instead of (26) the discrete time Lax equation

$$\widehat{M}(u) = \mathcal{V}^{-1}(u) M(u) \mathcal{V}(u) \; . \tag{32}$$

(this equation may be interpreted as a stationary version of the lattice chiral field model by [NP]). Representing the latter equation as $\mathcal{V}(u)\widehat{M}(u) = M(u)\mathcal{V}(u)$ and using the identities (27), we find that our matrix equation is equivalent to the set of the following three equations:

$$\widehat{\mathbf{M}} - \mathbf{M} = \frac{h}{2} (\mathbf{M} + \widehat{\mathbf{M}}) \times \mathbf{\Omega}^{1/2}(\mathbf{V}) \; , \tag{33}$$
$$0 = \mathbf{V} \times \mathbf{\Omega}^{1/2}(\mathbf{M} + \widehat{\mathbf{M}}) \; , \tag{34}$$

and

$$\sum_{k=1}^{3} (\widehat{M}_k - M_k) V_k w_k(u) w_k(u-u_0) = 0 \; . \tag{35}$$

Now (34) is equivalent to

$$\mathbf{V}=\frac{1}{2}\gamma\,\mathbf{\Omega}^{1/2}(\mathbf{M}+\widehat{\mathbf{M}}) \tag{36}$$

with some $\gamma\in\mathbb{R}$, which, being substituted in (33), results in

$$\widehat{\mathbf{M}}-\mathbf{M}=\frac{1}{4}h\gamma\,(\mathbf{M}+\widehat{\mathbf{M}})\times\mathbf{\Omega}(\mathbf{M}+\widehat{\mathbf{M}})\;. \tag{37}$$

It remains to notice that, plugging (36), (37) into (35), we bring the latter equation to the form

$$\sum_{k=1}^{3}w_k(u)w_k(u-u_0)w_k(u_0)\Big(w_l^2(u_0)-w_j^2(u_0)\Big)=0\;,$$

which is automatically satisfied due to the identity

$$w_l(u)w_l(u-u_0)w_j(u_0)-w_j(u)w_j(u-u_0)w_l(u_0)=w_k(u_0)\Big(w_l(u_0)-w_j(u_0)\Big)\;. \tag{38}$$

So, the matrix equation (32) is equivalent to (37) with some $\gamma\in\mathbb{R}$, the vector $\mathbf{V}$ being given by (36).

However, the equation (37) does not completely define the discretization due to arbitrariness of γ. In what follows we shall consider this equation with γ being a certain fixed function on $\mathbf{M}$, $\widehat{\mathbf{M}}$. In other words, the subject of our further investigations will consist of discretizations governed by implicit equations of motion of the following special form:

$$\widehat{\mathbf{M}}-\mathbf{M}=\frac{1}{4}h\gamma(\mathbf{M},\widehat{\mathbf{M}},h)\,(\mathbf{M}+\widehat{\mathbf{M}})\times\mathbf{\Omega}(\mathbf{M}+\widehat{\mathbf{M}})\;. \tag{39}$$

We call them *special discretizations*. It is obvious that if

$$\gamma:\,\mathbb{R}^3\times\mathbb{R}^3\times[0,\epsilon)\mapsto\mathbb{R}_+$$

is a C^1–function of each argument satisfying

$$\gamma(\mathbf{M},\mathbf{M},0)=1\;, \tag{40}$$

then the equation (39) fulfills the conditions of Proposition 1, and therefore defines a map (12).

In components, special discretizations may be presented as:

$$\widehat{M}_1-M_1 \;=\; \frac{1}{4}h\gamma(\mathbf{M},\widehat{\mathbf{M}},h)\,\left(\frac{1}{B}-\frac{1}{C}\right)(M_2+\widehat{M}_2)(M_3+\widehat{M}_3)\;, \tag{41}$$

$$\widehat{M}_2-M_2 \;=\; \frac{1}{4}h\gamma(\mathbf{M},\widehat{\mathbf{M}},h)\,\left(\frac{1}{C}-\frac{1}{A}\right)(M_3+\widehat{M}_3)(M_1+\widehat{M}_1)\;, \tag{42}$$

$$\widehat{M}_3-M_3 \;=\; \frac{1}{4}h\gamma(\mathbf{M},\widehat{\mathbf{M}},h)\,\left(\frac{1}{A}-\frac{1}{B}\right)(M_1+\widehat{M}_1)(M_2+\widehat{M}_2)\;. \tag{43}$$

6 Poisson property of special discretizations

We investigate now the integrability properties of special discretizations (which are naturally expected due to the existence of a Lax representation with a spectral parameter).

Proposition 4 *Maps defined by equations of motion* (39) *possess* $M^2(\mathbf{M})$ *and* $E(\mathbf{M})$ *as integrals of motion.*

Proof. It follows from (39) that

$$\langle \widehat{\mathbf{M}} - \mathbf{M}, \mathbf{M} + \widehat{\mathbf{M}} \rangle = 0 \quad \text{and} \quad \langle \widehat{\mathbf{M}} - \mathbf{M}, \mathbf{\Omega}(\mathbf{M} + \widehat{\mathbf{M}}) \rangle = 0 \;, \tag{44}$$

or, equivalently,

$$\langle \widehat{\mathbf{M}}, \widehat{\mathbf{M}} \rangle = \langle \mathbf{M}, \mathbf{M} \rangle \quad \text{and} \quad \langle \widehat{\mathbf{M}}, \mathbf{\Omega}(\widehat{\mathbf{M}}) \rangle = \langle \mathbf{M}, \mathbf{\Omega}(\mathbf{M}) \rangle \;. \tag{45}$$

This proves our statement. ■

Actually, this statement can be almost inverted. Namely, consider an arbitrary discretization having $M^2(\mathbf{M})$ and $E(\mathbf{M})$ as integrals of motion. Then the pairs $(\mathbf{M}, \widehat{\mathbf{M}})$ satisfy (45), which is equivalent to (44). If, in addition, $(\mathbf{M} + \widehat{\mathbf{M}}) \times \mathbf{\Omega}(\mathbf{M} + \widehat{\mathbf{M}}) \neq 0$, then there exists a real number $\gamma = \gamma(\mathbf{M}, \widehat{\mathbf{M}})$ such that (39) holds.

In what follows we shall need also the folowing technical and obvious statement.

Lemma 1 *The set of fixed points of the maps* (12) *defined by* (39) *coincides with the set of the points* $\mathbf{M}$ *for which* $(\mathbf{M}+\widehat{\mathbf{M}}) \times \mathbf{\Omega}(\mathbf{M}+\widehat{\mathbf{M}}) = 0$*, and this coincides with the union of the coordinate axes*

$$\{M_1 = M_2 = 0\} \cup \{M_2 = M_3 = 0\} \cup \{M_3 = M_1 = 0\} \;.$$

For all other points $\mathbf{M}$*, at least two of the three expressions* $M_1 + \widehat{M}_1$*,* $M_2 + \widehat{M}_2$*,* $M_3 + \widehat{M}_3$ *do not vanish.*

Now we find conditions for a map defined by (39) to be Poisson. To this end notice that the pairs $(\mathbf{M}, \widehat{\mathbf{M}}) = \big(\mathbf{M}, F(\mathbf{M}, h)\big)$ belong to the subset of $\mathbb{R}^3(\mathbf{M}) \times \mathbb{R}^3(\widehat{\mathbf{M}})$ singled out by the conditions $M^2(\mathbf{M}) = M^2(\widehat{\mathbf{M}})$ and $E(\mathbf{M}) = E(\widehat{\mathbf{M}})$. The elements of this subset may be parametrized by the quadruples $(M^2, E, \varphi, \widehat{\varphi})$ according to

$$\mathbf{M} = \Big(a \left\{ \begin{array}{c} \mathrm{cn}\,\varphi \\ \mathrm{dn}\,\varphi \end{array} \right\},\; b\,\mathrm{sn}\,\varphi,\; c \left\{ \begin{array}{c} \mathrm{dn}\,\varphi \\ \mathrm{cn}\,\varphi \end{array} \right\} \Big) \;, \tag{46}$$

$$\widehat{\mathbf{M}} = \Big(a \left\{ \begin{array}{c} \mathrm{cn}\,\widehat{\varphi} \\ \mathrm{dn}\,\widehat{\varphi} \end{array} \right\},\; b\,\mathrm{sn}\,\widehat{\varphi},\; c \left\{ \begin{array}{c} \mathrm{dn}\,\widehat{\varphi} \\ \mathrm{cn}\,\widehat{\varphi} \end{array} \right\} \Big) \;. \tag{47}$$

Equivalently, we can use the following coordinates: $(M^2, E, \bar{\varphi}, \Delta\varphi)$, where

$$\bar{\varphi} = \frac{\varphi + \widehat{\varphi}}{2}\,, \qquad \Delta\varphi = \frac{\widehat{\varphi} - \varphi}{2}\,. \tag{48}$$

Let us denote on the above mentioned subset:

$$\gamma(\mathbf{M}, \widehat{\mathbf{M}}, h) = \Gamma(M^2, E, \bar{\varphi}, \Delta\varphi, h)\,. \tag{49}$$

Theorem 1 *The equations of motion of the special discretization* (39) *in the elliptic coordinates* (M^2, E, φ) *have the form* (18), *where the function* $\widehat{\varphi}(M^2, E, \varphi, h)$ *is implicitly defined by the equation*

$$\frac{\Gamma(M^2, E, \bar{\varphi}, \Delta\varphi, h)}{1 - k^2\mathrm{sn}^2(\bar{\varphi})\mathrm{sn}^2(\Delta\varphi)} = \frac{2}{h\nu(M^2, E)}\,\frac{\mathrm{sn}(\Delta\varphi)}{\mathrm{cn}(\Delta\varphi)\mathrm{dn}(\Delta\varphi)} \tag{50}$$

The map defined by (39) *is Poisson, iff the equation* (50) *may be solved for* $\Delta\varphi$ *as*

$$\Delta\varphi = \delta(M^2, E, h) = \frac{h\nu(M^2, E)}{2}\big(1 + O(h)\big) \tag{51}$$

with the function δ *not depending on* $\bar{\varphi}$.

Proof. Denoting

$$D = D(M^2, E, \bar{\varphi}, \Delta\varphi) = 1 - k^2\mathrm{sn}^2(\bar{\varphi})\mathrm{sn}^2(\Delta\varphi)\,, \tag{52}$$

we find from (46), (47) with the help of addition formulae for elliptic functions:

$$\widehat{M}_1 - M_1 = -\frac{2a}{D}\left\{\begin{array}{c} \mathrm{sn}(\bar{\varphi})\mathrm{dn}(\bar{\varphi})\mathrm{sn}(\Delta\varphi)\mathrm{dn}(\Delta\varphi) \\ k^2\mathrm{sn}(\bar{\varphi})\mathrm{cn}(\bar{\varphi})\mathrm{sn}(\Delta\varphi)\mathrm{cn}(\Delta\varphi) \end{array}\right\}, \quad \widehat{M}_1 + M_1 = \frac{2a}{D}\left\{\begin{array}{c} \mathrm{cn}(\bar{\varphi})\mathrm{cn}(\Delta\varphi) \\ \mathrm{dn}(\bar{\varphi})\mathrm{dn}(\Delta\varphi) \end{array}\right\}, \tag{53}$$

$$\widehat{M}_2 - M_2 = \frac{2b}{D}\,\mathrm{cn}(\bar{\varphi})\mathrm{dn}(\bar{\varphi})\mathrm{sn}(\Delta\varphi)\,, \quad \widehat{M}_2 + M_2 = \frac{2b}{D}\,\mathrm{sn}(\bar{\varphi})\mathrm{cn}(\Delta\varphi)\mathrm{dn}(\Delta\varphi)\,, \tag{54}$$

$$\widehat{M}_3 - M_3 = -\frac{2c}{D}\left\{\begin{array}{c} k^2\mathrm{sn}(\bar{\varphi})\mathrm{cn}(\bar{\varphi})\mathrm{sn}(\Delta\varphi)\mathrm{cn}(\Delta\varphi) \\ \mathrm{sn}(\bar{\varphi})\mathrm{dn}(\bar{\varphi})\mathrm{sn}(\Delta\varphi)\mathrm{dn}(\Delta\varphi) \end{array}\right\}, \quad \widehat{M}_3 + M_3 = \frac{2c}{D}\left\{\begin{array}{c} \mathrm{dn}(\bar{\varphi})\mathrm{dn}(\Delta\varphi) \\ \mathrm{cn}(\bar{\varphi})\mathrm{cn}(\Delta\varphi) \end{array}\right\}. \tag{55}$$

Plugging this into an arbitrary one of the equations of motion (41)– (43), we arrive after some cancellations at the equation (50). Notice that Lemma 1 assures that, away from the fixed points, these cancellations are legitime in at least one of the equations of motion.

So, we have arrived at the equation (50) of the form

$$\Psi(M^2, E, \bar{\varphi}, \Delta\varphi, h) = 0\,, \tag{56}$$

which serves for determining the function $\widehat{\varphi}(M^2,E,\varphi,h)$ for our map. The equation (56) may be rewritten as

$$\Psi\Big(M^2,E,\frac{\varphi+\widehat{\varphi}}{2},\frac{\widehat{\varphi}-\varphi}{2},h\Big)\stackrel{\rm def}{=}\widetilde{\Psi}(M^2,E,\varphi,\widehat{\varphi},h)=0\ . \tag{57}$$

Proposition 3 gives a necessary and sufficient condition $\partial\widehat{\varphi}/\partial\varphi=1$ for our map to be Poisson, and this is equivalent to the following condition:

$$\frac{\partial\widetilde{\Psi}}{\partial\widehat{\varphi}}+\frac{\partial\widetilde{\Psi}}{\partial\varphi}=0\quad{\rm on\ \ the\ \ solutions\ \ of\ \ (57)}\ . \tag{58}$$

Since, obviously,

$$\frac{\partial\widetilde{\Psi}}{\partial\widehat{\varphi}}=\frac{1}{2}\left(\frac{\partial\Psi}{\partial\bar{\varphi}}+\frac{\partial\Psi}{\partial(\Delta\varphi)}\right)\ ,\qquad \frac{\partial\widetilde{\Psi}}{\partial\varphi}=\frac{1}{2}\left(\frac{\partial\Psi}{\partial\bar{\varphi}}-\frac{\partial\Psi}{\partial(\Delta\varphi)}\right)\ ,$$

we find that the Poisson property is equivalent to the following condition:

$$\frac{\partial\Psi}{\partial\bar{\varphi}}=0\quad{\rm on\ \ the\ \ solutions\ \ of\ \ (56)}\ . \tag{59}$$

In turn, (59) assures that the solutions of (56) for $\Delta\varphi$ do not depend on $\bar{\varphi}$. ■

Corollary. In the conditions of the above theorem, solutions $\mathbf{M}_m=(M_{1,m},M_{2,m},M_{3,m})^{\rm T}$ to the difference equation

$$\mathbf{M}_{m+1}-\mathbf{M}_m=\frac{1}{4}h\gamma(\mathbf{M}_m,\mathbf{M}_{m+1},h)\,(\mathbf{M}_m+\mathbf{M}_{m+1})\times\mathbf{\Omega}(\mathbf{M}_m+\mathbf{M}_{m+1}) \tag{60}$$

in the integrable ($\equiv$ Poisson) case are given by

$$M_{1,m}=a\left\{\begin{array}{c}{\rm cn}(2m\delta+\varphi_0)\\ {\rm dn}(2m\delta+\varphi_0)\end{array}\right\},\quad M_{2,m}=b\,{\rm sn}(2m\delta+\varphi_0),\quad M_{3,m}=c\left\{\begin{array}{c}{\rm dn}(2m\delta+\varphi_0)\\ {\rm cn}(2m\delta+\varphi_0)\end{array}\right\}. \tag{61}$$

7 First examples of integrable discretizations

We use Theorem 1 to investigate the Poisson property of several discretizations. We start with the following negative result.

Proposition 5 *The function $\gamma_0(\mathbf{M},\widehat{\mathbf{M}},h)\equiv 1$ defines a non–Poisson map.*

The simplest way to find Poisson (and therefore integrable) discretizations is to assure that the left–hand side of the equation (50) does not depend on $\bar{\varphi}$.

Proposition 6 *The functions*

$$\gamma_1(\mathbf{M},\widehat{\mathbf{M}},h)=\frac{4M^2}{\langle \mathbf{M}+\widehat{\mathbf{M}},\mathbf{M}+\widehat{\mathbf{M}}\rangle} \tag{62}$$

and

$$\gamma_2(\mathbf{M},\widehat{\mathbf{M}},h)=\frac{4E}{\langle \mathbf{M}+\widehat{\mathbf{M}},\mathbf{\Omega}(\mathbf{M}+\widehat{\mathbf{M}})\rangle} \tag{63}$$

define Poisson maps.

Proof. Using the second expressions in (53)–(55), it is not difficult to derive the following formulas:

$$\frac{1}{4}\langle \mathbf{M}+\widehat{\mathbf{M}},\mathbf{M}+\widehat{\mathbf{M}}\rangle=\frac{a^2+c^2-b^2\mathrm{sn}^2(\Delta\varphi)}{1-k^2\mathrm{sn}^2(\bar{\varphi})\mathrm{sn}^2(\Delta\varphi)}\ ,$$

$$\frac{1}{4}\langle \mathbf{M}+\widehat{\mathbf{M}},\mathbf{\Omega}(\mathbf{M}+\widehat{\mathbf{M}})\rangle=\frac{\dfrac{a^2}{A}+\dfrac{c^2}{C}-\dfrac{b^2}{B}\mathrm{sn}^2(\Delta\varphi)}{1-k^2\mathrm{sn}^2(\bar{\varphi})\mathrm{sn}^2(\Delta\varphi)}\ .$$

Hence, the left–hand side of the equation (50) for $\gamma=\gamma_{1,2}$ does not depend on $\bar{\varphi}$. Taking into account that, according to (10),

$$a^2+c^2=M^2\ ,\qquad \frac{a^2}{A}+\frac{c^2}{C}=E\ ,$$

we find the following expressions for this left–hand side:

$$1-\alpha_{1,2}\mathrm{sn}^2(\Delta\varphi)\ ,$$

where

$$\alpha_1(M^2,E)=\frac{b^2}{M^2}\ ,\qquad \alpha_2(M^2,E)=\frac{b^2}{BE}\ . \tag{64}$$

(see (10) for the expression of b^2 through M^2 and E). Therefore, the equation (50) in these two cases is equivalent to

$$\frac{\mathrm{sn}(\Delta\varphi)}{\mathrm{cn}(\Delta\varphi)\mathrm{dn}(\Delta\varphi)}=\frac{h\nu}{2}\,\frac{1}{1-\alpha_{1,2}\mathrm{sn}^2(\Delta\varphi)}\ . \tag{65}$$

Obviously, its solutions satisfy

$$\Delta\varphi=\delta_{1,2}(M^2,E,h)=\frac{h\nu(M^2,E)}{2}\Big(1+O(h)\Big)\ .$$

8 Veselov–Moser discretization

In order to introduce the Veselov–Moser discretization, we need, first of all, the matrix notation for the Euler top equation (2). Using the well–known isomorphism between the Lie algebra $\left(\mathbb{R}^3, \times\right)$ with the Lie algebra $\left(\mathrm{so}(3), [\cdot,\cdot]\right)$ of 3×3 skew–symmetric matrices with the usual commutator, we can rewrite the equations of motion (2) also in the matrix form

$$\dot{\mathrm{M}} = [\mathrm{M}, \Omega(\mathrm{M})]\ , \tag{66}$$

where

$$\mathrm{M} = \begin{pmatrix} 0 & M_3 & -M_2 \\ -M_3 & 0 & M_1 \\ M_2 & -M_1 & 0 \end{pmatrix} \tag{67}$$

and

$$\Omega(\mathrm{M}) = \begin{pmatrix} 0 & M_3/C & -M_2/B \\ -M_3/C & 0 & M_1/A \\ M_2/B & -M_1/A & 0 \end{pmatrix}\ . \tag{68}$$

The relation between the matrices M and $\Omega = \Omega(\mathrm{M})$ may be expressed as follows:

$$\mathrm{M} = \mathrm{J}\Omega + \Omega\mathrm{J}\ , \tag{69}$$

where the entries of the diagonal matrix

$$\mathrm{J} = \mathrm{diag}(J_1, J_2, J_3) \tag{70}$$

are defined by the relations

$$A = J_2 + J_3\ , \qquad B = J_3 + J_1\ , \qquad C = J_1 + J_2\ . \tag{71}$$

A spectral parameter dependent Lax representation for (66) is:

$$(\mathrm{M} + \lambda\mathrm{J}^2)^{\bullet} = [\mathrm{M} + \lambda\mathrm{J}^2, \Omega(\mathrm{M}) + \lambda\mathrm{J}]\ . \tag{72}$$

Now we can describe the Veselov–Moser construction. The differential equation (66) is replaced by the difference one,

$$\widehat{\mathrm{M}} = \omega^{\mathrm{T}}\mathrm{M}\omega\ , \tag{73}$$

where $\omega \in \mathrm{SO}(3)$ is an orthogonal matrix related to M by means of the following relation, coming to replace, or to approximate, (69):

$$h\mathrm{M} = \omega\mathrm{J} - \mathrm{J}\omega^{\mathrm{T}}\ . \tag{74}$$

It is easy to see that the previous two relations imply also

$$h\widehat{\mathrm{M}} = \mathrm{J}\omega - \omega^{\mathrm{T}}\mathrm{J}\ . \tag{75}$$

Moser and Veselov demonstrated that, in general, the equation (74) does not determine the matrix $\omega \in \mathrm{SO}(3)$ uniquely. This non–uniqueness may be described as follows. Notice that (74), (75) are equivalent to the following factorizations of matrix polynomials:

$$\mathrm{I} - h\lambda\mathrm{M} - \lambda^2\mathrm{J}^2 = (\omega + \lambda\mathrm{J})(\omega^{\mathrm{T}} - \lambda\mathrm{J})$$

and

$$\mathrm{I} - h\lambda\widehat{\mathrm{M}} - \lambda^2\mathrm{J}^2 = (\omega^{\mathrm{T}} - \lambda\mathrm{J})(\omega + \lambda\mathrm{J})\ ,$$

respectively. Denote by S the set of 6 roots of the equation $\det(\mathrm{I} - h\lambda\mathrm{M} - \lambda^2\mathrm{J}^2) = 0$. Then to each decomposition $S = S_+ \cap S_-$ into two disjoint sets of three roots satisfying $S_+ = \bar{S}_+ = -S_-$ there corresponds a unique solution $\omega \in \mathrm{SO}(3)$ of (74) such that S_+ is the set of the roots of $\det(\omega + \lambda\mathrm{J}) = 0$, while S_- is the set of the roots of $\det(\omega^{\mathrm{T}} - \lambda\mathrm{J}) = 0$.

However, in the continuous limit, when h is supposed to be small, there exists a *unique* decomposition, for which the roots from S_- are positive real numbers $O(h)$–close to J_1^{-1}, J_2^{-1}, J_3^{-1}. This is the only decomposition for which the corresponding ω has the following asymptotics:

$$\omega = \mathrm{I} + h\Omega(\mathrm{M}) + O(h^2)\ . \tag{76}$$

Supposing h small enough, we consider from now on only this choice of ω, and demonstrate that the discretization of Veselov and Moser also belongs to the class of special discretizations described by the equation (39).

Theorem 2 *The Veselov–Moser discretization for h small enough may be represented in the form* (39) *with*

$$\gamma_{\mathrm{VM}}(\mathbf{M}, \widehat{\mathbf{M}}, h) = \frac{2}{1 + \sqrt{1 - \frac{1}{4}h^2||\mathbf{\Omega}(\mathbf{M} + \widehat{\mathbf{M}})||^2}}\ . \tag{77}$$

Proof. We shall need the following lemma, proof of which is put in the Appendix.

Lemma 2 *For a matrix $\omega \in \mathrm{SO}(3)$, $O(h)$–close to I, there exists a unique matrix $\mathrm{W} \in \mathrm{so}(3)$ such that $\mathrm{W} = O(h)$, and*

$$\omega = \mathrm{I} + \mathrm{W} + \frac{\gamma}{2}\mathrm{W}^2\ ,$$

where

$$\gamma = \frac{2}{1 + \sqrt{1 - |\mathrm{W}|^2}}\ .$$

Here
$$|\mathrm{W}|^2 = W_1^2 + W_2^2 + W_3^2 \quad \text{for} \quad \mathrm{W} = \begin{pmatrix} 0 & W_3 & -W_2 \\ -W_3 & 0 & W_1 \\ W_2 & -W_1 & 0 \end{pmatrix} \in \mathrm{so}(3)\ .$$

With the help of this lemma we derive the following two equations:
$$\omega - \omega^{\mathrm{T}} = 2\mathrm{W}\ , \qquad \omega + \omega^{\mathrm{T}} = 2\mathrm{I} + \gamma \mathrm{W}^2\ . \tag{78}$$

Now from (74), (75) and the first equation in (78) we find:
$$h(\mathrm{M} + \widehat{\mathrm{M}}) = (\omega - \omega^{\mathrm{T}})\mathrm{J} + \mathrm{J}(\omega - \omega^{\mathrm{T}}) = 2(\mathrm{WJ} + \mathrm{JW})\ , \tag{79}$$

hence
$$\mathrm{W} = \frac{1}{2}\, h\Omega(\mathrm{M} + \widehat{\mathrm{M}})\ . \tag{80}$$

Further, from (74), (75) and the second equation in (78) we find:
$$\begin{aligned} h(\widehat{\mathrm{M}} - \mathrm{M}) &= \mathrm{J}(\omega + \omega^{\mathrm{T}}) - (\omega + \omega^{\mathrm{T}})\mathrm{J} = \gamma[\mathrm{J}, \mathrm{W}^2] = \gamma[\mathrm{WJ} + \mathrm{JW}, \mathrm{W}] \\ &= \frac{1}{4}\gamma h^2[\mathrm{M} + \widehat{\mathrm{M}}, \Omega(\mathrm{M} + \widehat{\mathrm{M}})]\ , \end{aligned} \tag{81}$$

where
$$\gamma = \frac{2}{1 + \sqrt{1 - \frac{1}{4}h^2|\Omega(\mathrm{M} + \widehat{\mathrm{M}})|^2}}\ .$$

Using the above–mentioned isomorphism between $\left(\mathbb{R}^3, \times\right)$ and $\left(\mathrm{so}(3), [\cdot,\cdot]\right)$, we see that the theorem is proved. ■

It remains to reproduce from our point of view the result of Moser and Veselov concerning the Poisson property of their discretization. To this end, we demonstrate that the equation (50) with Γ_{VM} corresponding to γ_{VM} allows a $\bar{\varphi}$–independent solution for $\Delta\varphi$.

Theorem 3 *For the Veselov–Moser discretization the equation* (50) *is equivalent to*
$$\mathrm{sn}(\Delta\varphi)\mathrm{cn}(\Delta\varphi)\mathrm{dn}(\Delta\varphi) = \frac{h\nu}{2}\left(1 + \alpha\,\mathrm{sn}^2(\Delta\varphi) - \beta\,\mathrm{sn}^4(\Delta\varphi)\right)\ , \tag{82}$$

where
$$\alpha = \left\{ \begin{array}{c} \dfrac{2ACE - (A+C-B)M^2}{(B-C)(AE-M^2)} \\ \\ \dfrac{2ACE - (A+C-B)M^2}{(A-B)(M^2-CE)} \end{array} \right\}\ , \qquad \beta = \left\{ \begin{array}{c} \dfrac{AC}{(B-C)^2}\,\dfrac{M^2-CE}{AE-M^2} \\ \\ \dfrac{AC}{(A-B)^2}\,\dfrac{AE-M^2}{M^2-CE} \end{array} \right\}\ . \tag{83}$$

The solution of (82) *satisfies*

$$\Delta\varphi = \delta_{\rm VM}(M^2,E,h) = \frac{h\nu(M^2,E)}{2}\Big(1+O(h)\Big)\,. \tag{84}$$

Proof. For the Veselov–Moser discretization the left–hand side of the equation (50) can be calculated with the help of (77) and the second expressions in (53)–(55):

$$\frac{\Gamma(M^2,E,\bar\varphi,\Delta\varphi,h)}{1-k^2{\rm sn}^2(\bar\varphi){\rm sn}^2(\Delta\varphi)} = \frac{2}{1-k^2{\rm sn}^2(\bar\varphi){\rm sn}^2(\Delta\varphi)+\sqrt{\Big(1-k^2{\rm sn}^2(\bar\varphi){\rm sn}^2(\Delta\varphi)\Big)^2-h^2G}}\,, \tag{85}$$

where

$$G = \frac{a^2}{A^2}\left\{\begin{array}{c}{\rm cn}^2(\bar\varphi){\rm cn}^2(\Delta\varphi)\\ {\rm dn}^2(\bar\varphi){\rm dn}^2(\Delta\varphi)\end{array}\right\}+\frac{b^2}{B^2}{\rm sn}^2(\bar\varphi){\rm cn}^2(\Delta\varphi){\rm dn}^2(\Delta\varphi)+\frac{c^2}{C^2}\left\{\begin{array}{c}{\rm dn}^2(\bar\varphi){\rm dn}^2(\Delta\varphi)\\ {\rm cn}^2(\bar\varphi){\rm cn}^2(\Delta\varphi)\end{array}\right\}\,. \tag{86}$$

Hence the equation (50) in the present case is equivalent to:

$$1-k^2{\rm sn}^2(\bar\varphi){\rm sn}^2(\Delta\varphi)+\sqrt{\Big(1-k^2{\rm sn}^2(\bar\varphi){\rm sn}^2(\Delta\varphi)\Big)^2-h^2G} = h\nu\,\frac{{\rm cn}(\Delta\varphi){\rm dn}(\Delta\varphi)}{{\rm sn}(\Delta\varphi)}$$

We leave the radical alone on the left–hand side and square the resulting equation to derive:

$$-h^2G = h^2\nu^2\frac{{\rm cn}^2(\Delta\varphi){\rm dn}^2(\Delta\varphi)}{{\rm sn}^2(\Delta\varphi)} - 2h\nu\,\frac{{\rm cn}(\Delta\varphi){\rm dn}(\Delta\varphi)}{{\rm sn}(\Delta\varphi)}\Big(1-k^2{\rm sn}^2(\bar\varphi){\rm sn}^2(\Delta\varphi)\Big)\,. \tag{87}$$

Obviously, we have:

$$G = G_0(M^2,E,\Delta\varphi) - G_1(M^2,E,\Delta\varphi)\,{\rm sn}^2(\bar\varphi)\,, \tag{88}$$

where

$$G_0 = \frac{a^2}{A^2}\left\{\begin{array}{c}{\rm cn}^2(\Delta\varphi)\\ {\rm dn}^2(\Delta\varphi)\end{array}\right\}+\frac{c^2}{C^2}\left\{\begin{array}{c}{\rm dn}^2(\Delta\varphi)\\ {\rm cn}^2(\Delta\varphi)\end{array}\right\}\,, \tag{89}$$

$$G_1 = \frac{a^2}{A^2}\left\{\begin{array}{c}{\rm cn}^2(\Delta\varphi)\\ k^2{\rm dn}^2(\Delta\varphi)\end{array}\right\}+\frac{c^2}{C^2}\left\{\begin{array}{c}k^2{\rm dn}^2(\Delta\varphi)\\ {\rm cn}^2(\Delta\varphi)\end{array}\right\}-\frac{b^2}{B^2}{\rm cn}^2(\Delta\varphi){\rm dn}^2(\Delta\varphi)\,. \tag{90}$$

We shall prove that

$$G_1-k^2{\rm sn}^2(\Delta\varphi)G_0 = k^2\nu^2{\rm cn}^2(\Delta\varphi){\rm dn}^2(\Delta\varphi)\,. \tag{91}$$

This formula allows to derive from (88):

$$G = \frac{G_1}{k^2{\rm sn}^2(\Delta\varphi)}\Big(1-k^2{\rm sn}^2(\bar\varphi){\rm sn}^2(\Delta\varphi)\Big) - \nu^2\frac{{\rm cn}^2(\Delta\varphi){\rm dn}^2(\Delta\varphi)}{{\rm sn}^2(\Delta\varphi)}\,, \tag{92}$$

so that (87) is equivalent to

$$\frac{h^2 G_1}{k^2\mathrm{sn}^2(\Delta\varphi)} = 2h\nu\,\frac{\mathrm{cn}(\Delta\varphi)\mathrm{dn}(\Delta\varphi)}{\mathrm{sn}(\Delta\varphi)}\;,$$

or

$$\mathrm{sn}(\Delta\varphi)\mathrm{cn}(\Delta\varphi)\mathrm{dn}(\Delta\varphi) = \frac{h}{2\nu k^2}G_1\;.$$

This is equivalent to (82), as one easily calculates from the definition (90) that

$$G_1 = k^2\nu^2\Big(1+\alpha\,\mathrm{sn}^2(\Delta\varphi)-\beta\,\mathrm{sn}^4(\Delta\varphi)\Big)$$

with α and β as in (83). It remains to prove the relation (91). But it follows from the definitions (89), (90):

$$G_1 - k^2\mathrm{sn}^2(\Delta\varphi)G_0 = \left\{\begin{array}{c} \dfrac{a^2}{A^2}+\dfrac{k^2c^2}{C^2}-\dfrac{b^2}{B^2} \\[3ex] \dfrac{k^2a^2}{A^2}+\dfrac{c^2}{C^2}-\dfrac{b^2}{B^2}\end{array}\right\}\mathrm{cn}^2(\Delta\varphi)\mathrm{dn}^2(\Delta\varphi)\;,$$

and the expressions in the curly brackets here in both cases are equal to $k^2\nu^2$, due to (10), (9), (11). ■

9 Conclusions

In the present work we have introduced a large family of discretizations of the Euler top sharing the integrals of motion with the continuous time system. We characterized those of them which are also Poisson with respect to the invariant Poisson bracket of the Euler top. For all these Poisson discretizations we found a solution in terms of elliptic functions which allows a direct comparison with the continuous time case. We demonstrated that the Veselov–Moser discretization also belong to our family, and applied our methods to this particular example.

Let us mention some of the possible dirsections of the further progress. First of all, our construction works on the level of reduced equations of motion of the rigid body (in the Lie algebra $\mathbb{R}^3 \approx \mathrm{so}(3) \approx \mathrm{su}(2)$). It would be important to lift it to the level of the cooresponding Lie group SO(3) (or SU(2)). On a more mechanical language, we want to discretize the equations of motion in the rest frame, and not only in the frame attached firmly to the body. Notice that the Veselov–Moser construction has a variational (Lagrangian) origin, which makes it valid in the group. Another example of such a discretization admitting a variational formulation in the corresponding group is the recently found discrete time Lagrange top [B].

As a second problem, our construction has to be generalised to higher dimensions (for the general Euler–Manakov case of the multidimensional rigid body). Notice that the Veselov–Moser construction goes through in higher dimensions.

Futher, it would be important to include our construction into the general r–matrix framework, connected to factorizations in the loop groups, cf. [S2].

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A Proof of Lemma 2

Let $\omega \in \mathrm{SO}(3)$, $\omega = \mathrm{I} + O(h)$. Set $\mathrm{V} = \dfrac{\omega - \mathrm{I}}{\omega + \mathrm{I}}$, then $\mathrm{V} \in \mathrm{so}(3)$, $\mathrm{V} = O(h)$, and

$$\omega = \frac{\mathrm{I} + \mathrm{V}}{\mathrm{I} - \mathrm{V}} = \frac{2}{\mathrm{I} - \mathrm{V}} - \mathrm{I} \,.$$

But for $\mathrm{V} \in \mathrm{so}(3)$ we have: $\mathrm{V}^3 = -|\mathrm{V}|^2 \mathrm{V}$, and by induction

$$\mathrm{V}^{2k+1} = (-1)^k |\mathrm{V}|^{2k} \mathrm{V} \,, \quad \mathrm{V}^{2k+2} = (-1)^k |\mathrm{V}|^{2k} \mathrm{V}^2 \,, \quad k \geq 1 \,,$$

so that

$$\frac{\mathrm{I}}{\mathrm{I} - \mathrm{V}} = \mathrm{I} + \frac{1}{1 + |\mathrm{V}|^2}\,\mathrm{V} + \frac{1}{1 + |\mathrm{V}|^2}\,\mathrm{V}^2 \,,$$

hence

$$\omega = \mathrm{I} + \frac{2}{1 + |\mathrm{V}|^2}\,\mathrm{V} + \frac{2}{1 + |\mathrm{V}|^2}\,\mathrm{V}^2 \,.$$

Now setting

$$\mathrm{W} = \frac{2}{1 + |\mathrm{V}|^2}\,\mathrm{V} \,,$$

so that $\mathrm{W} \in \mathrm{so}(3)$, $\mathrm{W} = O(h)$, we find:

$$\omega = \mathrm{I} + \mathrm{W} + \frac{\gamma}{2}\,\mathrm{W}^2 \,, \quad \text{where} \quad \gamma = 1 + |\mathrm{V}|^2 \,.$$

We have, obviously,

$$|\mathrm{W}|^2 = \frac{4}{(1+|\mathrm{V}|^2)^2}\,|\mathrm{V}|^2 \quad \Longrightarrow \quad |\mathrm{V}|^2 = \frac{2 - |\mathrm{W}|^2 - 2\sqrt{1-|\mathrm{W}|^2}}{|\mathrm{W}|^2} \,,$$

and

$$\gamma = 1 + |\mathrm{V}|^2 = \frac{2 - 2\sqrt{1-|\mathrm{W}|^2}}{|\mathrm{W}|^2} = \frac{2}{1+\sqrt{1-|\mathrm{W}|^2}} \,.$$

This proves the lemma. ■