Twistor spinors on Lorentzian symmetric spaces

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Abstract

We solve the twistor equation on all indecomposable Lorentzian symmetric spaces explicitly.

1 Introduction

Let (M^n, g) be an oriented semi-Riemannian spin manifold with the spinor bundle S. The *twistor operator* $\mathcal{D}$ is defined as the composition of the spinor derivative ∇^S with the projection p onto the kernel of the Clifford multiplication μ

$$\mathcal{D} : \Gamma(S) \stackrel{\nabla^S}{\to} \Gamma(T^*M \otimes S) \stackrel{g}{=} \Gamma(TM \otimes S) \stackrel{p}{\to} \Gamma(\ker \mu).$$

The solutions of the conformally invariant equation $\mathcal{D}\varphi = 0$ are called *twistor spinors*. Twistor spinors were introduced by R.Penrose in General Relativity (see [Pen67], [PR86], [NW84]). They are related to Killing vector fields in semi-Riemannian supergeometry (see [ACDS97]). In the last years essential results concerning the geometry of Riemannian spin manifolds admitting twistor spinors were obtained by A.Lichnerowicz, T.Friedrich, K.Habermann, H-B.Rademacher, W.Kuehnel and other authors. For a survey on the literature cf. [Fri97]. In the Lorentzian setting there was established a relation between a special class of solutions of the twistor equation and the Fefferman spaces occuring in CR-geometry (cf. [Lew91], [Bau97]).
Let $\mathcal{T}(M^n, g)$ denote the space of all twistor spinors of (M, g). It is known that

$$\dim \mathcal{T}(M^n, g) \leq 2 \cdot 2^{\left[\frac{n}{2}\right]}$$

(see [BFGK91]). If (M^n, g) is conformally flat and simply connected, then one has $\dim \mathcal{T}(M^n, g) = 2 \cdot 2^{\left[\frac{n}{2}\right]}$. In the present paper we determine the twistor spinors on all indecomposable Lorentzian symmetric spaces expilicity. In particular, we prove:

1. If (M^n, g) is an indecomposable non-conformally flat Lorentzian symmetric spin manifold of dimension $n \geq 3$, then each twistor spinor is parallel and $\dim \mathcal{T}(M^n, g) = q \cdot 2^{\left[\frac{n}{2}\right]}$, where $q = \frac{1}{2}, \frac{1}{4}$ or 0, depending on the fundamental group $\pi_1(M)$ and on the spin structure of (M^n, g).

2. If (M^n, g) is an indecomposable conformally flat Lorentzian symmetric spin manifold of dimension $n \geq 3$ and non-constant sectional curvature, then $\dim \mathcal{T}(M^n, g) = q \cdot 2^{\left[\frac{n}{2}\right]}$, where $q = 2, \frac{3}{2}, 1, \frac{3}{4}$ or 0 , depending on $\pi_1(M)$ and on the spin structure.

3. If (M^n, g) is a Lorentzian symmetric spin manifold of dimension $n \geq 3$ and constant sectional curvature, then $\dim \mathcal{T}(M^n, g) = q \cdot 2^{[\frac{n}{2}]}$, where $q = 2, 1$, or 0, depending on $\pi_1(M)$ and on the spin structure.

The calculations are based on the fact, that the indecomposable Lorentzian symmetric spaces are completely classified (see [CW70]).

2 Lorentzian symmetric spaces

Let us first recall the description of Lorentzian symmetric spaces. A connected semi-Riemannian manifold (M, g) is called *indecomposable*, if there is no proper, nondegenerate subspace of T_xM invariant under the action of the holonomy group $\mathrm{Hol}_x(g)$. Each simply connected semi-Riemannian symmetric space is isometric to a product $M_0 \times M_1 \times \ldots \times M_r$, where M_i, $i = 1, \ldots, r$, are indecomposable simply connected semi-Riemannian symmetric spaces of dimension ≥ 2 and M_0 is semi-Euclidean.
Let (M^n, g) be a Lorentzian symmetric space. By $G(M)$ we denote the group of transvections of (M^n, g) and by $\mathfrak{g}$ its Lie algebra. One has the following structure result:

Theorem 1 ([CW70])
Let (M^n, g) be an indecomposable Lorentzian symmetric space of dimension $n \geq 2$. Then the Lie algebra $\mathfrak{g}$ of the transvection group of (M^n, g) is eigther semi-simple or solvable.

Let $\underline{\lambda} = (\lambda_1, \ldots, \lambda_{n-2})$ be an $(n-2)$-tupel of real numbers $\lambda_j \in \mathbb{R}\backslash\{0\}$ and let us denote by $M^n_{\underline{\lambda}}$ the Lorentzian space $M^n_{\underline{\lambda}} := (\mathbb{R}^n, g_{\underline{\lambda}})$, where

$$(g_{\underline{\lambda}})_{(s,t,x)} := 2ds\, dt + \sum_{j=1}^{n-2} \lambda_j x_j^2\, ds^2 + \sum_{j=1}^{n-2} dx_j^2.$$

If $\underline{\lambda}_\pi = (\lambda_{\pi(1)}, \ldots, \lambda_{\pi(n-2)})$ is a permutation of $\underline{\lambda}$ and $c > 0$, then $M^n_{\underline{\lambda}}$ is isometric to $M^n_{c\underline{\lambda}_\pi}$.

Theorem 2 ([CW70], [CP80])
Let (M^n, g) be an indecomposable solvable Lorentzian symmetric space of dimension $n \geq 3$. Then (M^n, g) is isometric to $M^n_{\underline{\lambda}}/A$, where $\underline{\lambda} \in (\mathbb{R}\backslash\{0\})^{n-2}$ and A is a discrete subgroup of the centralizer $Z_{I(M_{\underline{\lambda}})}(G(M_{\underline{\lambda}}))$ of the transvection group $G(M_{\underline{\lambda}})$ in the isometry group $I(M_{\underline{\lambda}})$ of $M^n_{\underline{\lambda}}$.

For the centralizer $Z_{\underline{\lambda}} := Z_{I(M_{\underline{\lambda}})}(G(M_{\underline{\lambda}}))$ it is known:

Theorem 3 ([CK78])
Let $\underline{\lambda} = (\lambda_1, \ldots, \lambda_{n-2})$ be a tupel of non-zero real numbers.

1. *If there is a positive λ_i or if there are two numbers λ_i, λ_j such that $\frac{\lambda_i}{\lambda_j} \notin \mathbb{Q}^2$, then $Z_{\underline{\lambda}} \simeq \mathbb{R}$ and $\varphi \in Z_{\underline{\lambda}}$ if and only if $\varphi(s, t, x) = (s, t + \alpha, x)$, $\alpha \in \mathbb{R}$.*

2. *Let* $\lambda_i = -k_i^2 < 0$ *and* $\frac{k_i}{k_j} \in \mathbb{Q}$ *for all* $i, j \in \{1, \ldots, n-2\}$. *Then* $\varphi \in Z_{\underline{\lambda}}$ *if and only if*
$$\varphi(s,t,x) = (s+\beta, t+\alpha, (-1)^{m_1} x_1, \ldots, (-1)^{m_{n-2}} x_{n-2}),$$
where $\alpha \in \mathbb{R}$, $m_1, \ldots, m_{n-2} \in \mathbb{Z}$ *and* $\beta = \frac{m_i \cdot \pi}{k_i}$ *for all* $i = 1, \ldots, n-2$.

Let us denote by $S_1^n(r)$ the pseudo-sphere

$$S_1^n(r) := \left\{ x \in \mathbb{R}^{n+1,1} \mid \langle x, x \rangle_{n+1,1} = -x_1^2 + x_2^2 + \ldots + x_{n+1}^2 = r^2 \right\} \subset \mathbb{R}^{n+1,1}$$

and by $H_1^n(r)$ the pseudo-hyperbolic space

$$H_1^n(r) := \left\{ x \in \mathbb{R}^{n+1,2} \mid \langle x, x \rangle_{n+1,2} = -x_1^2 - x_2^2 + x_3^2 + \ldots + x_{n+1}^2 = -r^2 \right\} \subset \mathbb{R}^{n+1,2}$$

with the Lorentzian metrics induced by $\langle \cdot, \cdot \rangle_{n+1,1}$ and $\langle \cdot, \cdot \rangle_{n+1,2}$, respectively.

Theorem 4 ([CLP+90], [Wol84])
Let (M^n, g) *be an indecomposable semi-simple Lorentzian symmetric space of dimension* $n \geq 3$. *Then* (M^n, g) *has constant sectional curvature* $k \neq 0$. *Therefore, it is isometric to* $S_1^n(r)/_{\{\pm I\}}$ *or* $S_1^n(r)$, $(k = \frac{1}{r^2} > 0)$, *or to a Lorentzian covering of* $H_1^n(r)/_{\{\pm I\}}$, $(k = -\frac{1}{r^2} < 0)$.

3 Spinor representation

For concrete calculations we will use the following realization of the spinor representation. Let $\mathrm{Cliff}_{n,k}$ be the Clifford algebra of $(\mathbb{R}^n, -\langle \cdot, \cdot \rangle_k)$, where $\langle \cdot, \cdot \rangle_k$ is the scalar product $\langle x, y \rangle_k := -x_1 y_1 - \ldots - x_k y_k + x_{k+1} y_{k+1} + \ldots + x_n y_n$. For the canonical basis $(e_1, \ldots, e_n)$ of $\mathbb{R}^n$ one has the following relations in $\mathrm{Cliff}_{n,k}$: $e_i \cdot e_j + e_j \cdot e_i = -2\varepsilon_j \delta_{ij}$, where $\varepsilon_j = \begin{cases} -1 & j \leq k \\ 1 & j > k \end{cases}$. Denote $\tau_j = \begin{cases} i & j \leq k \\ 1 & j > k \end{cases}$ and

$$U = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad V = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad T = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.$$

If $n = 2m$ is even, we have an isomorphism

$$\phi_{2m,k} : \mathrm{Cliff}^{\mathbb{C}}_{2m,k} \xrightarrow{\sim} M(2^m; \mathbb{C})$$

given by the Kronecker product

$$\begin{array}{rcll} \phi_{2m,k}(e_{2j-1}) & = & \tau_{2j-1} & E \otimes \ldots \otimes E \otimes U \otimes T \otimes \ldots \otimes T \\ \phi_{2m,k}(e_{2j}) & = & \tau_{2j} & E \otimes \ldots \otimes E \otimes V \otimes \underbrace{T \otimes \ldots \otimes T}_{j-1} \end{array} . \tag{1}$$

If $n = 2m+1$ is odd and $k < n$, we have the isomorphism

$$\phi_{2m+1,k} : \mathrm{Cliff}^{\mathbb{C}}_{2m+1,k} \xrightarrow{\sim} M(2^m; \mathbb{C}) \oplus M(2^m; \mathbb{C})$$

given by

$$\begin{aligned}\Phi_{2m+1,k}(e_j) &= (\Phi_{2m,k}(e_j)\,,\,\Phi_{2m,k}(e_j)), \quad j=1,\ldots,2m \\ \Phi_{2m+1,k}(e_n) &= \tau_n(i\,T\otimes\cdots\otimes T\,,\,-i\,T\otimes\cdots\otimes T).\end{aligned} \tag{2}$$

Let $\mathrm{Spin}(n,k)\subset\mathrm{Cliff}_{n,k}$ be the spin group. The spinor representation is given by

$$\kappa_{n,k}=\hat{\phi}_{n,k}|_{\mathrm{Spin}(n,k)}:\mathrm{Spin}(n,k)\longrightarrow\mathrm{GL}(\mathbb{C}^{2^{[\frac{n}{2}]}}),$$

where $\hat{\phi}_{2m,k}=\phi_{2m,k}$ and $\hat{\phi}_{2m+1,k}=pr_1\circ\phi_{2m+1,k}$. We denote this representation by $\Delta_{n,k}$. If $n=2m$, $\Delta_{2m,k}$ splits into the sum $\Delta_{2m,k}=\Delta^+_{2m,k}\oplus\Delta^-_{2m,k}$, where $\Delta^\pm_{2m,k}$ are the eigenspaces of the endomorphism $\phi_{2m,k}(e_1\cdot\ldots\cdot e_{2m})$ to the eigenvalue $\pm i^{m+k}$. Let us denote by $u(\delta)\in\mathbb{C}^2$ the vector $u(\delta)=\frac{1}{\sqrt{2}}\binom{1}{-\delta i}$, $\delta=\pm1$, and let

$$u(\delta_1,\ldots,\delta_m)=u(\delta_1)\otimes\ldots\otimes u(\delta_m)\qquad \delta_j=\pm1. \tag{3}$$

Then $(u(\delta_1,\ldots,\delta_m)\,|\,\prod_{j=1}^m\delta_j=\pm1)$ is an orthonormal basis of $\Delta^\pm_{2m,k}$ with respect to the standard scalar product of $\mathbb{C}^{2^m}$.

4 General properties of twistor spinors

In this section we recall some properties of twistor spinors which we will need in the following calculations. For proofs see [BFGK91]. Let (M^n,g) be an oriented semi-Riemannian spin manifold of dimension $n\geq3$. We denote by S the spinor bundle of (M^n,g), by $\nabla^S:\Gamma(S)\to\Gamma(T^*M\otimes S)$ the spinor derivative given by the Levi-Civita connection of (M^n,g) and by $D:\Gamma(S)\to\Gamma(S)$ the Dirac operator on S. Let $p:TM\otimes S\to TM\otimes S$ be the projection onto the kernel of the Clifford multiplication μ. The *twistor operator*

$$\mathcal{D}:\Gamma(S)\overset{\nabla^S}{\to}\Gamma(T^*M\otimes S)\overset{g}{=}\Gamma(TM\otimes S)\overset{p}{\to}\Gamma(\ker\mu)$$

is locally given by

$$\mathcal{D}\varphi=\sum_{j=1}^n\varepsilon_js_j\otimes(\nabla^S_{s_j}\varphi+\frac{1}{n}s_j\cdot D\varphi),$$

where $(s_1,\ldots,s_n)$ is a local orthonormal basis and $\varepsilon_j=g(s_j,s_j)=\pm1$. A spinor field $\varphi\in\Gamma(S)$ is called *twistor spinor* if $\mathcal{D}\varphi=0$.

Proposition 1 *For a spinor field $\varphi\in\Gamma(S)$ the following conditions are equivalent:*

1. *φ is a twistor spinor.*
2. *φ satisfies the so-called twistor equation*

$$\nabla^S_X\varphi+\frac{1}{n}X\cdot D\varphi=0 \tag{4}$$

for all vector fields X.

3. *There exists a spinor field $\psi \in \Gamma(S)$ such that*
$$\psi = g(X,X)X \cdot \nabla^S_X \varphi \tag{5}$$
for all vector fields X with $|g(X,X)| = 1$.

The dimension of the space $\mathcal{T}(M^n, g)$ of all twistor spinors is conformally invariant and bounded by
$$\dim \mathcal{T}(M^n, g) \leq 2 \cdot 2^{[\frac{n}{2}]}.$$
If (M^n, g) is simply connected and conformally flat, $\dim \mathcal{T}(M^n, g) = 2 \cdot 2^{[\frac{n}{2}]}$.
In particular, the twistor spinors on the semi-Euclidean space $\mathbb{R}^{n,k} := (\mathbb{R}^n, \langle\cdot,\cdot\rangle_{n,k})$ are the functions
$$\mathcal{T}(\mathbb{R}^{n,k}) = \left\{ \varphi \in C^\infty(\mathbb{R}^{n,k}, \Delta_{n,k}) \mid \varphi(x) = u + x \cdot v \,;\; u, v \in \Delta_{n,k} \right\}.$$

Let R be the scalar curvature and Ric the Ricci curvature of (M^n, g). $K : TM \to TM$ denotes the $(1,1)$-Schouten tensor of (M^n, g)
$$K(X) = \frac{1}{n-2} \left(\frac{R}{2(n-1)} X - \mathrm{Ric}(X) \right).$$

Furthermore, let W be the $(4,0)$-Weyl tensor of (M, g) and let us denote by the same symbol the corresponding $(2,2)$-tensor field $W : \Lambda^2 M \to \Lambda^2 M$. Then we have

Proposition 2 *Let $\varphi \in \Gamma(S)$ be a twistor spinor. Then*
$$\nabla^S_X D\varphi = \frac{n}{2} K(X) \cdot \varphi \,, \tag{6}$$
$$W(\eta) \cdot \varphi = 0 \,, \tag{7}$$
for all vector fields X and 2-forms η.

Finally, we recall two possibilities to obtain new manifolds with twistor spinors from a given one.
Let $(\tilde{M}^n, \tilde{g})$ be a simply connected parallelizable semi-Riemannian manifold and let $A \subset I(\tilde{M}, \tilde{g})$ denote a discrete subgroup of orientation preserving isometries of $(\tilde{M}, \tilde{g})$. We trivialize the spin structure of $(\tilde{M}, \tilde{g})$ with respect to a fixed global orthonormal basis field $\mathfrak{a} = (\mathfrak{a}_1, \ldots, \mathfrak{a}_n)$. For $\gamma \in A$ we denote by $\Gamma(x) \in SO(n,k)$ the matrix of $d\gamma_x$ with respect to $\mathfrak{a}(x)$ and $\mathfrak{a}(\gamma(x))$. Then there are two lifts $\tilde{\Gamma}^\pm$ of Γ into $Spin(n,k)$

$$\begin{tikzcd} & Spin(n,k) \arrow[d, "\lambda"] \\ \tilde{M} \arrow[ur, "\tilde{\Gamma}^\pm"] \arrow[r, "\Gamma"'] & SO(n,k) \end{tikzcd}$$

Let $\mathcal{E}(A)$ be the set of all left actions of A on $\tilde{M} \times Spin(n,k)$ such that

$$\varepsilon(\gamma)(x,a) = (\gamma(x), \varepsilon(\gamma,x)\cdot a) \qquad \text{and} \qquad \varepsilon(\gamma,x) = \tilde{\Gamma}(x)^{\pm}.$$

This set of left actions $\mathcal{E}(A)$ corresponds to the set of spin structures of the oriented semi-Riemannian manifold $M = \tilde{M}/A$. The spinor bundle on M corresponding to the spin structure $\varepsilon \in \mathcal{E}(A)$ is given by

$$S_\varepsilon = \left(\tilde{M} \times \Delta_{n,k}\right) \Big/_{\varepsilon}$$

where $\varepsilon(\gamma)(x,v) = (\gamma(x), \varepsilon(\gamma,x)\cdot v)$ for all $\gamma \in A$, $(x,v) \in \tilde{M} \times \Delta_{n,k}$. Hence, the spinor fields on M corresponding to $\varepsilon \in \mathcal{E}(A)$ are given by the ε-invariant functions

$$\Gamma(S_\varepsilon) \;=\; C^\infty(\tilde{M}, \Delta_{n,k})^{\varepsilon} := \left\{\varphi \in C^\infty(\tilde{M}, \Delta_{n,k}) \;\mid\; \varphi(\gamma(x)) = \varepsilon(\gamma,x)\cdot\varphi(x)\right\},$$

and for the twistor spinors on M with the spin structure ε we have

Proposition 3 *The twistor spinors on $M = \tilde{M}/A$ with respect to the spin structure $\varepsilon \in \mathcal{E}(A)$ are given by*

$$\mathcal{T}((M,g),\varepsilon) = \{\varphi \in \mathcal{T}(\tilde{M},\tilde{g}) \;\mid\; \varphi \textit{ ist } \varepsilon\textit{-invariant }\}.$$

Let (M^{n+1}, g) be a semi-Riemannian spin manifold with spinor bundle S_M and let $F^n \subset M^{n+1}$ be a non-degenerate oriented hypersurface in M^{n+1}. We denote by $\eta : F \to TM$ the Gauss map of F, $\kappa(\eta) := g(\eta,\eta) = \pm 1$. It is well known, that the spinor bundle S_F of $(F, g|_F)$ with respect to the spin structure on F induced by the embedding, is isomorphic to $S_M|_F$ in case of even dimension n and to $S_M^{\pm}|_F$ in case of odd dimension n. Using this identification the Clifford multiplication and the spinor derivative are expressed by

$$\begin{aligned} X\cdot(\varphi|_F) &= (X\cdot\varphi)|_F \\ \nabla_X^{S_F}(\varphi|_F) &= \left(\nabla_X^{S_M^{(\pm)}}\varphi + \frac{1}{2}\kappa(\eta)\,\nabla_X^M \eta\cdot\eta\cdot\varphi\right)\Big|_F, \end{aligned}$$

where $\varphi \in \Gamma(S_M^{(\pm)}), X \in TF$ and $\varphi|_F$ always means the spinor field in $\Gamma(S_F)$ corresponding to φ with respect to the above mentioned isomorphism.

Proposition 4 *If $F^n \subset M^{n+1}$ is an umbilic hypersurface and $\varphi \in \Gamma(S_M^{(\pm)})$ is a twistor spinor on M, then $\varphi|_F \in \Gamma(S_F)$ is a twistor on F.*

<u>Proof:</u> Let $\lambda \in C^\infty(F)$ be the function satisfying $\nabla_X^M \eta = \lambda X$. Then

$$\nabla_X^{S_F}(\varphi|_F) = \left(\nabla_X^{S_M^{(\pm)}}\varphi + \frac{1}{2}\kappa(\eta)\lambda X\cdot\eta\cdot\varphi\right)\Big|_F.$$

If $\varphi \in \Gamma(S_M^{(\pm)})$ is a twistor spinor, from Proposition 1, (4), follows

$$\kappa(X)\,X\cdot\nabla_X^{S_M^{(\pm)}}\varphi = \frac{1}{n+1}D_M^{(\pm)}\varphi$$

for each $X \in TF$ with $\kappa(X) = g(X,X) = \pm 1$. Hence,

$$\kappa(X)\, X \cdot \nabla^{S_F}_X (\varphi|_F) = \left(\frac{1}{n+1} D_M^{(\pm)} \varphi - \frac{1}{2} \kappa(\eta) \lambda\, \eta \cdot \varphi \right) \Big|_F .$$

The right hand side is independend of $X \in TF$. Therefore, according to Proposition 1, (5), $\varphi|_F$ is a twistor spinor on F. $\square$

5 Twistor spinors on indecomposable, non-conformally flat Lorentzian symmetric spaces

Let us first consider the simply connected solvable Lorentzian symmetric space $M^n_{\underline{\lambda}} = (\mathbb{R}^n, g_{\underline{\lambda}})$, where

$$(g_{\underline{\lambda}})_{(s,t,x)} = 2\, ds\, dt + \sum_{j=1}^{n-2} \lambda_j x_j^2\, ds^2 + \sum_{j=1}^{n-2} dx_j^2 ,$$

and $\underline{\lambda} = (\lambda_1, \ldots, \lambda_{n-2})$, $\lambda_i \in \mathbb{R}\backslash\{0\}$, $n \geq 3$. Let $\Lambda_0 := - \sum\limits_{j=1}^{n-2} \lambda_j$. We fix the following global orthonormal basis on $M^n_{\underline{\lambda}}$:

$$\begin{array}{rcll}
\mathfrak{a}_{\overline{0}}(y) & := & \frac{\partial}{\partial s}(y) - \frac{1}{2} \left(\sum\limits_{j=1}^{n-2} \lambda_j x_j^2 + 1 \right) \frac{\partial}{\partial t}(y) & \\
\mathfrak{a}_0(y) & := & \frac{\partial}{\partial s}(y) - \frac{1}{2} \left(\sum\limits_{j=1}^{n-2} \lambda_j x_j^2 - 1 \right) \frac{\partial}{\partial t}(y) & \\
\mathfrak{a}_j(y) & := & \frac{\partial}{\partial x_j}(y) & j = 1, \ldots, n-2,
\end{array}$$

where $y = (s, t, x_1, \ldots, x_{n-2}) \in M^n_{\underline{\lambda}}$. The vector field $V(y) := \frac{\partial}{\partial t}(y)$ is isotropic and parallel. The Ricci tensor of $M^n_{\underline{\lambda}}$ is given by

$$\mathrm{Ric}\,(X) = \Lambda_0 \cdot g_{\underline{\lambda}}(X, V) V,$$

the scalar curvature R vanishes. Therefore, the Schouten tensor satisfies

$$K(X) = -\tfrac{1}{n-2} \Lambda_0 \cdot g_{\underline{\lambda}}(X, V)\, V. \tag{8}$$

For the Weyl tensor $W : \Lambda^2 M_{\underline{\lambda}} \to \Lambda^2 M_{\underline{\lambda}}$ one has

$$\begin{array}{rcll}
W(\mathfrak{a}_{\overline{0}} \wedge \mathfrak{a}_j) & = & W(\mathfrak{a}_0 \wedge \mathfrak{a}_j) \ = \ (\lambda_j + \frac{1}{n-2} \Lambda_0)\, V \wedge \mathfrak{a}_j, & j = 1, \ldots, n-2 \\
W(\mathfrak{a}_\alpha \wedge \mathfrak{a}_\beta) & = & 0 & \text{for all other indices } \alpha, \beta,
\end{array} \tag{9}$$

where $TM_{\underline{\lambda}}$ is identified with $T^*M_{\underline{\lambda}}$ using the metric $g_{\underline{\lambda}}$. In particular, $M_{\underline{\lambda}}$ is conformally flat if and only if $\underline{\lambda} = (\lambda, \ldots, \lambda)$, $\lambda \in \mathbb{R}\backslash\{0\}$.
Since $M_{\underline{\lambda}}$ is simply connected, it has an uniquely determined spin structure. We trivialize this spin structure using the global orthonormal basis $(\mathfrak{a}_{\overline{0}}, \mathfrak{a}_0, \mathfrak{a}_1, \ldots, \mathfrak{a}_{n-2})$ and identify the spinor fields with the smooth functions $C^\infty(M_{\underline{\lambda}}, \Delta_{n,1})$. The spinor derivative is defined by

$$\nabla^S_X\varphi = X(\varphi) + \frac{1}{2}\sum_{1\leq k<l\leq n} \varepsilon_k\varepsilon_l\, g(\nabla^{LC}_X s_k, s_l)\, s_k\cdot s_l\cdot\varphi,$$

where $(s_1,\ldots,s_n)$ is a local orthonormal basis and $\varepsilon_j = g(s_j,s_j) = \pm 1$. This gives for the spinor derivative on $M_{\underline{\lambda}}$

$$\nabla^S_{\frac{\partial}{\partial t}}\varphi = \tfrac{\partial}{\partial t}\varphi \tag{10}$$

$$\nabla^S_{\frac{\partial}{\partial s}}\varphi = \tfrac{\partial}{\partial s}\varphi + \tfrac{1}{2}\sum_{j=1}^{n-2}\lambda_j x_j \mathfrak{a}_j\cdot V\cdot\varphi \tag{11}$$

$$\nabla^S_{\frac{\partial}{\partial x_j}}\varphi = \tfrac{\partial}{\partial x_j}\varphi \tag{12}$$

$$\nabla^S_{\mathfrak{a}_{\overline{0}}}\varphi = \mathfrak{a}_{\overline{0}}(\varphi) + \tfrac{1}{2}\sum_{j=1}^{n-2}\lambda_j x_j \mathfrak{a}_j\cdot V\cdot\varphi \tag{13}$$

$$\nabla^S_{\mathfrak{a}_0}\varphi = \mathfrak{a}_0(\varphi) + \tfrac{1}{2}\sum_{j=1}^{n-2}\lambda_j x_j \mathfrak{a}_j\cdot V\cdot\varphi. \tag{14}$$

The vector space $\Delta_{n,1}$ is isomorphic to $\Delta_{n-2,0}\otimes\mathbb{C}^2$. Let us denote by Δ_V the subspace

$$\Delta_V := \Delta_{n-2,0}\otimes\mathbb{C}u(-1) \quad \subset \Delta_{n,1}.$$

Using the formulas (1), (2) and (3) one obtains that a spinor field $\varphi\in C^\infty(M_{\underline{\lambda}},\Delta_{n,1})$ satisfies $V\cdot\varphi = 0$ if and only if the image of φ lies in Δ_V.

Proposition 5 *The space $\mathcal{P}(M_{\underline{\lambda}})$ of parallel spinors of $M_{\underline{\lambda}}$ is*

$$\mathcal{P}(M_{\underline{\lambda}}) = \{\varphi\in C^\infty(M_{\underline{\lambda}},\Delta_{n,1}) \mid \varphi = \mathit{constant}\in\Delta_V\}.$$

In particular, $\dim\mathcal{P}(M_{\underline{\lambda}}) = \frac{1}{2}\cdot 2^{[\frac{n}{2}]}$.

Proof: From (10) - (12) it follows that $\varphi\in C^\infty(M_{\underline{\lambda}},\Delta_{n,1})$ is parallel if and only if φ depends only on s and

$$\tfrac{\partial\varphi}{\partial s} = -\tfrac{1}{2}\sum_{j=1}^{n-2}\lambda_j x_j\, \mathfrak{a}_j\cdot V\cdot\varphi. \tag{15}$$

Therefore,

$$0 = \tfrac{\partial^2\varphi}{\partial x_k\partial s} = -\tfrac{1}{2}\lambda_k\mathfrak{a}_k\cdot V\cdot\varphi.$$

Since $\lambda_k\neq 0$ and $\mathfrak{a}_k$ is space-like, this yields $V\cdot\varphi = 0$. Hence, because of (15), φ has to be constant. □

Proposition 6 *Let $M_{\underline{\lambda}}$ be non-conformally flat. Then each twistor spinor on $M_{\underline{\lambda}}$ is parallel. In particular,*

$$\dim\mathcal{T}(M_{\underline{\lambda}}) = \tfrac{1}{2}\cdot 2^{[\frac{n}{2}]}.$$

Proof: Let $\varphi \in C^\infty(M_{\underline{\lambda}}, \Delta_{n,1})$ be a twistor spinor. Then according to (7) of Proposition 2 $W(\eta)\cdot\varphi = 0$ for each 2-form η. Using (9) we obtain

$$\left(\lambda_j + \tfrac{1}{n-2}\Lambda_0\right)\mathfrak{a}_j \cdot V \cdot \varphi = 0 \qquad j = 1, \ldots, n-2.$$

Since $M_{\underline{\lambda}}$ is not conformally flat and $\mathfrak{a}_j$ is space-like, it follows

$$0 = V \cdot \varphi \tag{16}$$

Furthermore, we have

$$\nabla^S_X D\varphi \overset{(6)}{=} \tfrac{n}{2} K(X)\cdot\varphi.$$

Using (8) and (16) we obtain

$$\nabla^S_X D\varphi = -\tfrac{n}{2(n-2)}\Lambda_0\, g_{\underline{\lambda}}(X,V)V\cdot\varphi = 0.$$

Hence, $D\varphi$ is parallel. From Proposition 5 follows, that $D\varphi$ is constant and $V\cdot D\varphi = 0$. Then the twistor equation and (10) yield

$$0 = \nabla^S_V\varphi + \tfrac{1}{n}V\cdot D\varphi = \tfrac{\partial}{\partial t}(\varphi).$$

Therefore, φ does not depend on t. Moreover, the twistor equation implies that

$$-\mathfrak{a}_{\overline{0}}\cdot\nabla^S_{\mathfrak{a}_{\overline{0}}}\varphi = \mathfrak{a}_0\cdot\nabla^S_{\mathfrak{a}_0}\varphi = \frac{1}{n}D\varphi.$$

Then the formulas (13), (14) and (16) give

$$D\varphi = -n\mathfrak{a}_{\overline{0}}\cdot\left(\tfrac{\partial}{\partial s} - \tfrac{1}{2}\Big(\sum_{j=1}^{n-2}\lambda_j x_j^2 + 1\Big)\tfrac{\partial}{\partial t}\right)(\varphi) = n\mathfrak{a}_0\cdot\left(\tfrac{\partial}{\partial s} - \tfrac{1}{2}\Big(\sum_{j=1}^{n-2}\lambda_j x_j^2 - 1\Big)\tfrac{\partial}{\partial t}\right)(\varphi).$$

Since φ does not depend on t we obtain

$$2D\varphi = n(\mathfrak{a}_0 - \mathfrak{a}_{\overline{0}})\cdot\tfrac{\partial}{\partial s}(\varphi) = nV\cdot\tfrac{\partial}{\partial s}(\varphi) = n\tfrac{\partial}{\partial s}(V\cdot\varphi) \overset{(16)}{=} 0.$$

Therefore, φ is harmonic and the twistor equation implies that φ is parallel. $\square$

Now, let (M^n, g) be a non-conformally flat, non-simply connected, indecomposable Lorentzian symmetric space of dimension $n \geq 3$. Then, according to the Theorems 1,2 and 4, (M^n, g) is isometric to $M^n_{\underline{\lambda}}/A$, where A is a discrete subgroup of the centralizer $Z_{\underline{\lambda}} := Z_{I(M_{\underline{\lambda}})}(G(M_{\underline{\lambda}}))$.

1. case: There exist $i \in \{1, \ldots, n-2\}$ such that $\lambda_i > 0$ or (i,j) such that $\frac{\lambda_i}{\lambda_j} \notin \mathbb{Q}^2$. Then $Z_{\underline{\lambda}} \simeq \mathbb{R} = \{\gamma_\alpha \mid \gamma_\alpha(s,t,x) = (s, t+\alpha, x),\ \alpha \in \mathbb{R}\}$.
Let $\gamma \in A$. With respect to the global basic $(\mathfrak{a}_{\overline{0}}, \mathfrak{a}_0, \mathfrak{a}_1, \ldots, \mathfrak{a}_{n-2})$ the differential $d\gamma_y$ corresponds to the matrix $\Gamma(y) \equiv E \in SO(n,1)$. Hence, $\tilde{\Gamma}^{\pm}(y) = \pm 1 \in Spin(n,1)$. Therefore, we have 2 spin structures on $M = M_{\underline{\lambda}}/A$ corresponding to the homomorphisms $\mathrm{Hom}(A;\mathbb{Z}_2)$. If $\varepsilon \in \mathrm{Hom}(A,\mathbb{Z}_2)$ is not trivial, there are no non-trivial

ε-invariant constant spinor fields on $M_{\underline{\lambda}}$. From the Propositions 3,5 and 6 it follows that the twistor spinors on $M = M_{\underline{\lambda}}/A$ are given by

$$\mathcal{T}(M_{\underline{\lambda}}/A, \varepsilon) = \left\{ \begin{array}{ll} \{\varphi \in C^\infty(M, \Delta_V) \mid \varphi \text{ const}\} & \varepsilon \text{ trivial} \\ \{0\} & \varepsilon \text{ non-trivial.} \end{array} \right.$$

2. case: Let $\lambda_j = -k_i^2 < 0$ and $\frac{k_i}{k_j} \in \mathbb{Q}$ for all $i, j = 1, \dots, n-2$. Then

$$\begin{aligned} Z_{\underline{\lambda}} &\simeq \left\{ \gamma_{\underline{m},\alpha} \;\middle|\; \begin{array}{l} \gamma_{\underline{m},\alpha}(s,t,x) := (s+\beta, t+\alpha, (-1)^{m_1}x_1, \dots, (-1)^{m_{n-2}}x_{n-2}); \\ \text{where } \alpha \in \mathbb{R},\ \underline{m} = (m_1, \dots, m_{n-2}) \in \mathbb{Z}^{n-2}, \\ \beta = \pi \cdot \frac{m_i}{k_i} \ \ i = 1, \dots, n-2 \end{array} \right\} \\ &\simeq \mathbb{Z} \oplus \mathbb{R}. \end{aligned}$$

A discrete subgroup $A_{\underline{m},\alpha} \subset Z_{\underline{\lambda}}$ is generated by $\gamma_{\underline{m},0}$ and $\gamma_{0,\alpha}$. Let us suppose that $\sum_{i=1}^{n-2} m_i$ is even since otherwise $M_{\underline{\lambda}}/A_{\underline{m},\alpha}$ is not orientable. $(d\gamma_{\underline{m},\alpha})_y$ corresponds to the matrix

$$\Gamma(y) = \begin{pmatrix} 1 & & & & 0 \\ & 1 & & & \\ & & (-1)^{m_1} & & \\ & & & \ddots & \\ 0 & & & & (-1)^{m_{n-2}} \end{pmatrix}.$$

Hence $\tilde{\Gamma}^\pm(y) = \pm e_1^{m_1} \cdot \ldots \cdot e_{n-2}^{m_{n-2}}$. Let $m_{i_1}, \dots, m_{i_s}$ be the odd elements in the tupel $\underline{m}$ $(s \in 2\mathbb{Z})$, and let us denote by $\omega_{\underline{m}} \in Spin(n-2,0) \subset Spin(n,1)$ the element

$$\omega_{\underline{m}} = e_{i_1} \cdot \ldots \cdot e_{i_s}.$$

Then because of $\omega_{\underline{m}}^2 = (-1)^{\frac{s}{2}}$, $\omega_{\underline{m}}$ is an involution on $\Delta_{n-2,0}$ if $s \equiv 0(4)$ and an almost complex structure if $s \equiv 2(4)$. The eigenspaces of $\omega_{\underline{m}}$ to the eigenvalues ± 1 and $\pm i$, respectively, have the same dimension (see formulas (1) and (2)).
The manifold $M_{\underline{\lambda}}/A_{\underline{0},\alpha}$, $\alpha \neq 0$, has 2 spin structures and the twistor spinors are given as in case 1. $M_{\underline{\lambda}}/A_{\underline{m},0}$, $\underline{m} \neq \underline{0}$, has 2 spin structures, described by the homomorphisms $\varepsilon_\pm \in \text{Hom}\,(A_{\underline{m},0}, Spin(n-2,0))$ given by $\varepsilon_\pm(\gamma_{\underline{m},0}) = \pm\omega_{\underline{m}}$. Then, according to the Propositions 3,5 and 6 the twistor spinors on $M = M_{\underline{\lambda}}/A_{\underline{m},0}$ are

$$\mathcal{T}(M_{\underline{\lambda}}/A_{\underline{m},0}, \varepsilon_\pm) = \left\{ \varphi \in C^\infty(M_{\underline{\lambda}}, \Delta_{n,1}) \;\middle|\; \begin{array}{l} \varphi(y) \equiv v \otimes u(-1); \\ \text{where } v \in \Delta_{n-2,0} \text{ and } \omega_{\underline{m}} \cdot v = \pm v \end{array} \right\}.$$

Hence,

$$\dim \mathcal{T}(M_{\underline{\lambda}}/A_{\underline{m},0}\,, \varepsilon_\pm) = \left\{ \begin{array}{ll} 0 & \text{if } s \equiv 2(4) \\ \frac{1}{4} \cdot 2^{[\frac{n}{2}]} & \text{if } s \equiv 0(4). \end{array} \right.$$

The manifold $M_{\underline{\lambda}}/A_{\underline{m},\alpha}$, $\underline{m} \neq 0, \alpha \neq 0$, has 4 spin structures corresponding to the homomorphisms $\varepsilon \in \text{Hom}\,(A_{\underline{m},\alpha}; Spin(n,1))$ given by $\varepsilon(\gamma_{\underline{m},0}) = \pm\omega_{\underline{m}}$, $\varepsilon(\gamma_{0,\alpha}) = \pm 1$.

Hence,

$$\mathcal{T}(M_{\underline{\lambda}}/A_{\underline{m},\alpha}\,,\varepsilon) \;=\; \begin{cases} \{0\} & \varepsilon(\gamma_{0,\alpha}) = -1 \\ & \\ \mathcal{T}(M_{\underline{\lambda}}/A_{\underline{m},0}\,,\varepsilon_{\pm}) & \begin{array}{l}\varepsilon(\gamma_{0,\alpha}) = 1 \\ \varepsilon(\gamma_{\underline{m},0}) = \pm\omega_{\underline{m}}.\end{array} \end{cases}$$

Summing up, we have

Proposition 7 *Let (M^n, g) be an indecomposable, non-conformally flat Lorentzian symmetric spin manifold of dimension $n \geq 3$. Then each twistor spinor is parallel and the dimension of the space of twistor spinors is*

$$\dim \mathcal{T}(M^n, g) = q \cdot 2^{[\frac{n}{2}]},$$

where $q = 0, \frac{1}{4}$ or $\frac{1}{2}$, depending on the fundamental group $\pi_1(M)$ and on the spin structure.

6 Twistor spinors on indecomposable conformally flat Lorentzian symmetric spaces of non-constant sectional curvature

According to the Theorems 1, 2 and 4 there are two isometry classes of indecomposable, conformally flat, simply connected Lorentzian symmetric spaces of dimension $n \geq 3$ and non-constant sectional curvature, namly

$$M^n_{\pm} := (\mathbb{R}^n, g_{\pm}),$$

where
$$(g_{\pm})_{(s,t,x)} = 2ds\, dt \pm ||x||^2 ds^2 + \sum_{j=1}^{n-2} dx_j^2.$$

Knowing that $\dim \mathcal{T}(M^n_{\pm}) = 2 \cdot 2^{[\frac{n}{2}]}$ we want to describe the twistor spinors explicitly. Let $w_1, w_2, w_3, w_4 \in \Delta_{n-2,0}$ and let us denote by $\varphi_{w_1,w_2,w_3,w_4} \in C^\infty(M^n_{\pm}, \Delta_{n,1})$ the following smooth functions

$$\begin{aligned} \varphi_{w_1,w_2,w_3,w_4}(s,t,x) \;:=\; & (\mp f'(s) w_3 - f(s) w_4 + x \cdot w_1) \otimes u(1) \\ & + [-2w_1 t + w_2 + x \cdot (f(s) w_3 + f'(s) w_4)] \otimes u(-1), \end{aligned}$$

where
$$f(s) = \begin{cases} \sinh(s) & \text{for } M^n_+ \\ \sin(s) & \text{for } M^n_-. \end{cases}$$

Proposition 8 *The twistor spinors on $M^n_{\pm}$ are*

$$\mathcal{T}(M^n_{\pm}) = \{\varphi_{w_1,w_2,w_3,w_4} \mid w_1, w_2, w_3, w_4 \in \Delta_{n-2,0}\}.$$

Proof: We use the identification

$$\begin{array}{rcc} \Delta_{n,1} \simeq \Delta_{n-2,0} \otimes \Delta_{2,1} & \xrightarrow{\sim} & \Delta_{n-2,0} \oplus \Delta_{n-2,0} \\ \varphi = \varphi_1 \otimes u(1) + \varphi_2 \otimes u(-1) & \longmapsto & (\varphi_1, \varphi_2). \end{array}$$

Then according to (1) and (2) the Clifford multiplication corresponds to

$$\begin{array}{l} X \cdot \varphi = (-X \cdot \varphi_1, X \cdot \varphi_2) \qquad \text{if } X \in \text{ span}\,(\mathfrak{a}_1, \ldots, \mathfrak{a}_{n-2}), \\ \mathfrak{a}_{\overline{0}} \cdot \varphi = (-\varphi_2, -\varphi_1) \quad , \quad \mathfrak{a}_0 \cdot \varphi = (-\varphi_2, \varphi_1) \quad , \quad V \cdot \varphi = (0, 2\varphi_1). \end{array}$$

For the spinor derivative we obtain

$$\begin{array}{l} \nabla^S_{\mathfrak{a}_{\overline{0}}} \varphi = (\mathfrak{a}_{\overline{0}}(\varphi_1), \mathfrak{a}_{\overline{0}}(\varphi_2) \pm x \cdot \varphi_1) \quad , \quad \nabla^S_{\mathfrak{a}_0} \varphi = (\mathfrak{a}_0(\varphi_1), \mathfrak{a}_0(\varphi_2) \pm x \cdot \varphi_1), \\ \nabla^S_{\mathfrak{a}_k} \varphi = (\mathfrak{a}_k(\varphi_1), \mathfrak{a}_k(\varphi_2)) \quad , \quad k = 1, \ldots, n-2. \end{array}$$

Let $\varphi = (\varphi_1, \varphi_2)$ be a twistor spinor on $M^n_\pm$. Then, according to Proposition 1, there exists a spinor field $\psi = (\psi_1, \psi_2)$ on $M^n_\pm$ such that

$$(\psi_1, \psi_2) = \mathfrak{a}_k \cdot \nabla^S_{\mathfrak{a}_k} \varphi = (-\mathfrak{a}_k \cdot \mathfrak{a}_k(\varphi_1), \mathfrak{a}_k \cdot \mathfrak{a}_k(\varphi_2)) \tag{17}$$

for each $k = 1, 2, \ldots, n-2$. Therefore, $\varphi_1(s,t,\cdot)$ and $\varphi_2(s,t,\cdot)$ are twistor spinors on the Euclidean space $\mathbb{R}^{n-2}$. Hence,

$$\varphi_i(s,t,x) = u_i(s,t) + x \cdot v_i(s,t) \quad , \quad i = 1, 2,$$

where $u_i, v_i : \mathbb{R}^2 \to \Delta_{n-2,0}$. From (17) follows

$$\psi_1(s,t,x) = v_1(s,t) \quad \text{and} \quad \psi_2(s,t,x) = -v_2(s,t).$$

Furthermore, $\psi = (\psi_1, \psi_2)$ satisfies

$$(\psi_1, \psi_2) = -\mathfrak{a}_{\overline{0}} \cdot \nabla^S_{\mathfrak{a}_{\overline{0}}} \varphi = \mathfrak{a}_0 \cdot \nabla^S_{\mathfrak{a}_0} \varphi.$$

Therefore,

$$\begin{aligned} v_1 &= \left(\tfrac{\partial}{\partial s} - \tfrac{1}{2}\,(\pm||x||^2 + 1)\,\tfrac{\partial}{\partial t}\right)(u_2 + x \cdot v_2) \,\pm\, x \cdot (u_1 + x \cdot v_1) && (18) \\ v_1 &= -\left(\tfrac{\partial}{\partial s} - \tfrac{1}{2}\,(\pm||x||^2 - 1)\,\tfrac{\partial}{\partial t}\right)(u_2 + x \cdot v_2) \,\mp\, x \cdot (u_1 + x \cdot v_1) && (19) \\ -v_2 &= \left(\tfrac{\partial}{\partial s} - \tfrac{1}{2}\,(\pm||x||^2 + 1)\,\tfrac{\partial}{\partial t}\right)(u_1 + x \cdot v_1) && (20) \\ -v_2 &= \left(\tfrac{\partial}{\partial s} - \tfrac{1}{2}\,(\pm||x||^2 - 1)\,\tfrac{\partial}{\partial t}\right)(u_1 + x \cdot v_1). && (21) \end{aligned}$$

(18) + (19) gives $2v_1 = -\frac{\partial}{\partial t}u_2 - x \cdot \frac{\partial}{\partial t}v_2$ and after differentation with respect to x_k $0 = -\mathfrak{a}_k \cdot \frac{\partial}{\partial t}v_2$. Hence,

$$\frac{\partial}{\partial t}v_2 = 0 \quad \text{and} \quad v_1 = -\frac{1}{2}\frac{\partial}{\partial t}u_2. \tag{22}$$

Using this, we obtain from (19) - (18)

$$0 = \mp x \cdot u_1 - \frac{\partial}{\partial s}u_2 - x \cdot \frac{\partial}{\partial s}v_2. \tag{23}$$

Differentiation and (22) show that

$$u_1 = \mp \tfrac{\partial}{\partial s} v_2 \quad \text{and} \quad \tfrac{\partial}{\partial t} u_1 = 0. \tag{24}$$

Inserting this in (23) and using (22) we obtain

$$\tfrac{\partial}{\partial s} u_2 = 0 \quad \text{and} \quad \tfrac{\partial}{\partial s} v_1 = 0. \tag{25}$$

Hence, $u_2 = u_2(t)$, $v_1 = v_1(t)$, $v_2 = v_2(s)$ and $u_1 = u_1(s)$. (21) - (20) shows

$$0 = x \cdot \tfrac{\partial}{\partial t} v_1 + \tfrac{\partial}{\partial t} u_1 = x \cdot v_1'(t).$$

Therefore, we have $v_1(t) \equiv w_1 \in \Delta_{n-2}$ and, because of (22), $u_2(t) = -2tw_1 + w_2$. (21) + (20) yields

$$-2v_2 = 2u_1'(s) \mp ||x||^2\, x \cdot v_1'(t) = 2u_1'(s),$$

so that, regarding (24), $v_2(s) = \pm v_2''(s)$. Therefore,

$$v_2(s) = f(s)w_3 + f'(s)w_4 \qquad \text{and} \qquad u_1(s) = \mp f'(s)w_3 - f(s)w_4,$$

where

$$f(s) = \begin{cases} \sinh(s) & \text{for } M^n_+ \\ \sin(s) & \text{for } M^n_-. \end{cases}$$

Consequently, the twistor spinor φ is of the form $\varphi = \varphi_{w_1,w_2,w_3,w_4}$. □

Now, let (M^n, g) be an indecomposable, conformally flat non-simply connected Lorentzian symmetric space of dimension $n \geq 3$ and non-constant sectional curvature. Then (M^n, g) is isometric to M^n_+/A or to M^n_-/A, where A is a discrete subgroup of

$$Z_+ := Z_{I(M_+)}(G(M_+)) = \{\varphi_\alpha \mid \varphi_\alpha(s,t,x) = (s, t+\alpha, x)\,; \quad \alpha \in \mathbb{R}\}$$

in the first and of

$$Z_- := Z_{I(M_-)}(G(M_-)) = \left\{ \begin{array}{ll} \varphi_{m,\alpha} \mid & \varphi_{m,\alpha}(s,t,x) = (s+m\pi, t+\alpha, (-1)^m x); \\ & m \in \mathbb{Z}, \alpha \in \mathbb{R} \end{array} \right\}$$

in the second case.

1. case: $M = M^n_+/A_\alpha$, $A_\alpha = \mathbb{Z}\varphi_\alpha$. Then there are 2 spin structures corresponding to $\varepsilon \in \mathrm{Hom}(A_\alpha, \mathbb{Z}_2)$. The Propositions 3 and 8 show

$$\mathcal{T}(M,\varepsilon) = \begin{cases} \{\varphi_{0,w_2,w_3,w_4} \mid w_2, w_3, w_4 \in \Delta_{n-2,0}\} & \varepsilon = 1 \\ \{0\} & \varepsilon \neq 1. \end{cases}$$

2. case: $M = M^n_-/A_{m,\alpha}$, $A_{m,\alpha} = \langle \varphi_{m,0}, \varphi_{0,\alpha} \rangle$. If m is even and $\alpha \neq 0$, we have the same result as in case 1, since $f(s) = \sin(s)$ is $2\pi\mathbb{Z}$-invariant. If m is odd, M

is orientable only if n is even. Then M^{2k} has 2 spin structures if $\alpha = 0$ and 4 spin structures if $\alpha \neq 0$. The Propositions 3 and 8 show

$$\mathcal{T}(M_-^{2k}/A_{m,0},\varepsilon) = \begin{cases} \{0\} & n = 2k \equiv 0(4) \\ \left\{ \begin{array}{l} \varphi_{w_1,w_2,w_3,w_4} \mid \\ \qquad w_1, w_2 \in \Delta^{\pm}_{n-2,0} \\ \qquad w_3, w_4 \in \Delta^{\mp}_{n-2,0} \end{array} \right\} & \begin{array}{l} n = 2k \equiv 2(4) \\ \varepsilon(\varphi_{m,0}) = \pm e_1 \cdot \ldots \cdot e_{n-2} \end{array} \end{cases}$$

$$\mathcal{T}(M_-^{2k}/A_{m,\alpha},\varepsilon) = \begin{cases} \{0\} & \begin{array}{l} \varepsilon(\varphi_{0,\alpha}) = -1 \\ \text{or } n = 2k \equiv 0(4) \end{array} \\ \left\{ \begin{array}{l} \varphi_{0,w_2,w_3,w_4} \mid \\ \qquad w_2 \in \Delta^{\pm}_{n-2,0} \\ \qquad w_3, w_4 \in \Delta^{\mp}_{n-2,0} \end{array} \right\} & \begin{array}{l} n = 2k \equiv 2(4) \\ \varepsilon(\varphi_{0,\alpha}) = 1 \text{ and} \\ \varepsilon(\varphi_{m,0}) = \pm e_1 \cdot \ldots \cdot e_{n-2}. \end{array} \end{cases}$$

Summing up, we have in particular:

Proposition 9 *Let (M^n, g) be an indecomposable, conformally flat Lorentzian spin manifold (M^n, g) of non-constant sectional curvature and dimension $n \geq 3$. Then the dimension of the space of twistor spinors is*

$$\dim \mathcal{T}(M^n, g) = q \cdot 2^{\left[\frac{n}{2}\right]},$$

where $q = 0, \frac{3}{4}, 1, \frac{3}{2}$ or 2, depending on the fundamental group $\pi_1(M)$ and on the spin structure.

7 Twistor spinors on Lorentzian symmetric spaces of constant sectional curvature

Let $\psi_{u,v} \in C^\infty(\mathbb{R}^{n+1,k}, \Delta_{n+1,k})$ denote the twistor spinors on the pseudo-Euclidean space $\mathbb{R}^{n+1,k}$

$$\psi_{u,v}(x) := u + x \cdot v \quad , \quad u, v \in \Delta_{n+1,k}.$$

The pseudo-sphere $S_1^n(r) \subset \mathbb{R}^{n+1,1}$ and the pseudo-hyperbolic space $H_1^n(r)$ are umbilic hypersurfaces. Using the identification of the spinor bundle of the hypersurface with that of the external space (see section 4) we obtain from Proposition 4:

Proposition 10 *The twistor spinors on $S_1^n(r)$ and $H_1^n(r)$ with the induced spin structure are*

$$\mathcal{T}(S_1^n(r)) = \left\{ \psi_{u,v}|_{S_1^n(r)} \;\middle|\; \begin{array}{ll} u, v \in \Delta_{n+1,1} & \textit{if } n \equiv 0(2) \\ u \in \Delta^+_{n+1,1},\, v \in \Delta^-_{n+1,1} & \textit{if } n \equiv 1(2) \end{array} \right\}$$

$$\mathcal{T}(H_1^n(r)) = \left\{ \psi_{u,v}|_{H_1^n(r)} \;\middle|\; \begin{array}{ll} u, v \in \Delta_{n+1,2} & \textit{if } n \equiv 0(2) \\ u \in \Delta^+_{n+1,2},\, v \in \Delta^-_{n+1,2} & \textit{if } n \equiv 1(2) \end{array} \right\}.$$

The Lorentzian manifold $S^n_1(r)/_{\{\pm I\}}$ is orientable if and only if n is odd, hence let n be odd. The volume form $\omega_{n+1,1} = e_1 \cdot \ldots \cdot e_{n+1} \in Spin(n+1,1)$ satisfies $\omega^2_{n+1,1} = (-1)^{\frac{n+1}{2}+1}$. Therefore, $S^n_1(r)/_{\{\pm I\}}$ has no spin structure, if $n \equiv 3(4)$ and 2 spin structures, if $n \equiv 1(4)$. The spinor fields to these different spin structures can be identified with the invariant functions $C^\infty(S^n_1(r), \Delta^+_{n+1,1})^{\varepsilon_\pm}$, where $\varepsilon_\pm$ is the $\mathbb{Z}_2$-action given by

$$(\varepsilon_\pm(-1)\varphi)(x) = \pm\omega_{n+1,1} \cdot \varphi(-x) = \pm\varphi(-x).$$

From the Propositions 3 and 10 follows

$$\mathcal{T}(S^{4k+1}_1(r)/_{\{\pm I\}}, \varepsilon) = \begin{cases} \{\psi_{u^+,0}|_{S^{4k+1}_1} \mid u^+ \in \Delta^+_{4k+2,1}\} & \text{if } \varepsilon = \varepsilon_+ \\ \{\psi_{0,v^-}|_{S^{4k+1}_1} \mid v^- \in \Delta^-_{4k+2,1}\} & \text{if } \varepsilon = \varepsilon_-. \end{cases}$$

Now, let us consider a Lorentzian symmetric space M^n of constant negative sectional curvature. Then M^n is isometric to a Lorentzian covering of $H^n_1(r)/_{\{\pm I\}}$. Let

$$\begin{array}{rccc} \tilde{\pi}: & \widetilde{H^n_1(r)} = \mathbb{R} \times \mathbb{R}^{n-1} & \longrightarrow & H^n_1(r) \subset \mathbb{R}^{2,2} \times \mathbb{R}^{n-1} \\ & (t,x) & \longmapsto & (\sqrt{r^2 + ||x||^2}\cos t, \sqrt{r^2 + ||x||^2}\sin t, x) \end{array}$$

be the universal Lorentzian covering of $H^n_1(r)$. Let $\hat{Q}$ denote the reduction of the trivial spin structure Q of $\mathbb{R}^{n+1,2}$ to the subgroup $Spin(n,1)$ given by the Gaus map. Then $\tilde{Q} := \tilde{\pi}^*\hat{Q}$ is the uniquely determined spin structure of $\widetilde{H^n_1}(r)$. The spinor fields of $\widetilde{H^n_1}(r)$ can be identified with the smooth functions $C^\infty(\widetilde{H^n_1}(r), \Delta^{(\pm)}_{n+1,2})$, the twistor spinors are given by

$$\mathcal{T}(\widetilde{H^n_1}(r)) = \{\tilde{\psi}_{u,v} := \psi_{u,v}|_{H^n_1(r)} \circ \tilde{\pi} \;\Big|\; \psi_{u,v}|_{H^n_1(r)} \in \mathcal{T}(H^n_1(r))\}.$$

Let

$$\begin{array}{rccc} \pi_m: & N^n_m & \longrightarrow & H^n_1(r) \\ & (\sqrt{\cdot}\cos t, \sqrt{\cdot}\sin t, x) & \longmapsto & (\sqrt{\cdot}\cos(mt), \sqrt{\cdot}\sin(mt), x), \end{array}$$

$(\sqrt{\cdot} = \sqrt{r^2 + ||x||^2})$, be the Lorentzian covering of $H^n_1(r)$ with respect to $m\mathbb{Z} \subset \pi_1(H^n_1(r)) = \mathbb{Z}$, $m = 1,2,3,\ldots.$. The manifold N^n_m has 2 spin structures. The corresponding spinor fields are given by the $\varepsilon^\pm_m$-invariant functions $C^\infty(\widetilde{H^n_1(r)}, \Delta^{(\pm)}_{n+1,2})^{\varepsilon^\pm_m}$, where $\varepsilon^\pm_m$ is the $m\mathbb{Z}$-action

$$(\varepsilon^\pm_m(mz)\varphi)(t,x) = (\pm 1)^z \varphi(t + 2\pi mz, x).$$

Therefore, the twistor spinors on N^n_m are

$$\mathcal{T}(N^n_m, \varepsilon) = \begin{cases} \{\psi_{u,v}|_{H^n_1(r)} \circ \pi_m \;\Big|\; \psi_{u,v}|_{H^n_1(r)} \in \mathcal{T}(H^n_1(r))\} & \text{if } \varepsilon = \varepsilon^+_m \\ \{0\} & \text{if } \varepsilon = \varepsilon^-_m. \end{cases}$$

Finally, let us consider the manifolds $N^n_m/_{\{\pm I\}}$. Since $N^n_m/_{\{\pm I\}}$ is orientable if and only if n is odd, let n be odd. For the volume form $\omega_{n+1,2} = e_1 \cdot \ldots \cdot e_{n+1} \in Spin(n+1,2)$ we have $\omega^2_{n+1,2} = (-1)^{\frac{n+1}{2}+2}$. Therefore, there is no spin structure on $N^n_m/_{\{\pm I\}}$ if

$n \equiv 1(4)$ and there are 4 spin structures in case $n \equiv 3(4)$. The spinor fields are given by the functions $C^\infty(\widetilde{H_1^n(r)}, \Delta^+_{n+1,2})^{(\varepsilon_m^\pm, \delta^\pm)}$, invariant under the $m\mathbb{Z}$-action $\varepsilon_m^\pm$ and the $\mathbb{Z}_2$-action $\delta^\pm$ given by

$$\begin{aligned}(\delta^\pm(-1)\varphi)(t,x) &= \pm\omega_{n+1,2}\cdot\varphi(t+m\pi,-x)\\ &= \pm\varphi(t+m\pi,-x).\end{aligned}$$

Then, the twistor spinors are

$$\mathcal{T}\left(N_m^{4k+3}/_{\{\pm I\}},\varepsilon\right) = \begin{cases} \{0\} & \begin{array}{l}\varepsilon = (\varepsilon_m^-, \delta^\pm) \text{ or}\\ \varepsilon = (\varepsilon_m^+, \delta^-),\ m \equiv 0(2)\end{array}\\ \{\tilde\psi_{u^+,0} \mid u^+ \in \Delta^+_{4k+4,2}\} & \varepsilon = (\varepsilon_m^+, \delta^+)\\ \{\tilde\psi_{0,v^-} \mid v^- \in \Delta^+_{4k+4,2}\} & \varepsilon = (\varepsilon_m^+, \delta^-),\ m \equiv 1(2).\end{cases}$$

Summing up, we have in particular

Proposition 11 *Let (M^n, g) be a Lorentzian symmetric spin manifold of constant sectional curvature $k \neq 0$ and dimension $n \geq 3$, then the dimension of the space of twistor spinors is*

$$\dim\mathcal{T}(M^n,g) = q\cdot 2^{[\frac{n}{2}]},$$

where $q = 0, 1,$ or 2 depending on $\pi_1(M)$ and on the spin structure.

References

[ACDS97] D.V. Alekseevsky, V. Cortes, C. Devchand, and U. Semmelmann. Killing spinors are Killing vector fields in Riemannian supergeometry. MPI-preprint 97-29, 1997.

[Bau97] H. Baum. Strictly pseudoconvex spin manifolds, Fefferman spaces and Lorentzian twistor spinors. SFB 288-Preprint No. 250, 1997.

[BFGK91] H. Baum, T. Friedrich, R. Grunewald, and I. Kath. *Twistors and Killing Spinors on Riemannian Manifolds*, volume 124 of *Teubner-Texte zur Mathematik*. Teubner-Verlag, Stuttgart/Leipzig, 1991.

[CK78] M. Cahen and Y. Kerbrat. Champs de vecteurs conformes et transformations conformes des spaces Lorentzian symetriques. *J.Math. pures et appl.*, 57:99–132, 1978.

[CLP+90] M. Cahen, J. Leroy, M. Parker, F. Tricerri, and L. Vanhecke. Lorentzian manifolds modelled on a Lorentz symmetric space. *J. Geom. Phys.*, 7(4):571–581, 1990.

[CP80] M. Cahen and M. Parker. Pseudo-Riemannian symmetric spaces. *Mem. AMS*, 24(229):1–108, 1980.

[CW70] M. Cahen and N. Wallach. Lorentzian symmetric spaces. *Bull. AMS*, 76(3):585–591, 1970.

[Fri97] Th. Friedrich. *Dirac-Operatoren in der Riemannschen Geometrie.* Vieweg-Verlag, 1997.

[Lew91] J. Lewandowski. Twistor equation in a curved spacetime. *Class. Quant. Grav.*, 8:11–17, 1991.

[NW84] P. Nieuwenhuizen and N.P. Warner. Integrability conditions for Killing spinors. *Comm. Math. Phys.*, 93:277–284, 1984.

[Pen67] R. Penrose. Twistor algebra. *J. Math. Phys.*, 8:345–366, 1967.

[PR86] R. Penrose and W. Rindler. *Spinors and Space-time II.* Cambr. Univ. Press, 1986.

[Wol84] J.A. Wolf. *Spaces of constant curvature.* Publish or Perish, Inc., 1984.

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