The twistor space of a Lorentzian manifold

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Abstract

We describe the twistor space of a Lorentzian 4-manifold as projective spinor bundle and study the properties of natural geometric constructions on the twistor space.

Contents

1 Introduction

Subject of this work is the twistor space of a 4-dimensional oriented Lorentzian manifold and the properties of different constructions on this twistor space.

At the end of the 70th, twistor theorie was developed by R. Penrose [Pen77] and Atiyah/Hitchin/Singer [AHS78] and was applied to problems in mathematical physics, e.g. the classification of instantons, and in differential geometry, e.g. in surface theorie.

The twistor construction assigns to a 4-dimensional Riemannian manifold (M, g) one 6-dimensional fibre bundle $\mathcal{A}(M)$ over M with a natural almost complex structure on it. This fibre bundle is the set of all orthogonal complex structures of the tangent space to M and is associated with the orthogonal frame bundle, $\mathcal{A}(M) = SO(M) \times_{\mathbf{SO}(4)} \mathbf{SO}(4)/\mathbf{U}(2)$.

In the work of [AHS78], there was shown that the twistor space is a complex analytical manifold if and only if the 4-dimensional Riemannian base manifold is self-dual. In works of Friedrich/Kurke [FK82], Friedrich/Grunewald [FG83] and Hitchin [Hit81], Riemannian metrics on the twistor space $\mathcal{A}(M)$ were analysed. In the cases $M^4 = \mathbf{S}^4,\ \mathbf{P}^2(\mathbf{C})$, exactly one natural Einstein-Kaehlerian metric exists on the twistor spaces $\mathcal{A}(M) = \mathbf{P}^3(\mathbf{C}),\ F(1,2)$.

One more application of the twistor construction is the interpretation of a spinor field ϕ of a Riemannian spin manifold M, that is solution of the equation $\mathcal{D}\phi = \mathrm{pr}_{\ker\mu} \circ \nabla^S\phi = 0$, as a holomorphic section in a 1-dimensional complex vector bundle $\mathcal{H} = [S^- \backslash 0] \times_{\mathbf{C}^*} \mathbf{C}$ over the twistor space $\mathcal{A}(M)$ (see [BFGK91]).

Now, the twistor space of a 4-dimensional oriented Lorentzian manifold $(M^{4,1}, g)$ can be defined as the 6-dimensional fibre bundle $\mathcal{Z}(M) = SO(M) \times_{\mathbf{SO}(4,1)} P$, which consists of all null directions in the tangent space to M. The twistor space $\mathcal{Z}(M)$ is a fibre bundle of 2-spheres like in the case of a Riemannian 4-manifold. This idea fits in with the orignal idea of R. Penrose.

In this paper, it will be shown that one may understand the twistor space $\mathcal{Z}(M)$ as the positive (resp. negative) projective spinor bundle $\mathbf{P}(S^+)$ $(\mathbf{P}(S^-))$, which exists over every oriented semi-Riemannian 4-manifold. This interpretation is also true for the twistor space $\mathcal{A}(M)$ in the Riemannian case, so that one can think of this construction of the twistor space in the Lorentzian case as the natural equivalent to the twistor construction in Riemannian geometry.

The twistor space $\mathcal{Z}(M)$ admits in a natural way an optical geometry. This construction is as follows. Each projective spinor $[\psi] \in \mathbf{P}(S^+)$ can be canonical identified with an orthogonal optical structure in the tangent space at a point in $M^{4,1}$. The horizontal lifts of these optical structures to the tangent space $T\mathcal{Z}(M)$ together with the natural complex structure in the fibres of the 2-sphere bundle $\mathcal{Z}(M)$ define an optical geometry on the twistor space. Comparing this construction with the construction of Hermitian geometry on the twistor space $\mathcal{A}(M)$ in Riemannian case, again one may understand these two constructions as analogous.

A comparable theorem to the well known theorem of integrability in [AHS78] states, that the optical geometry on $\mathcal{Z}(M)$ is integrable if and only if the Lorentzian manifold $M^{4,1}$ is conformally flat. Anyway, the comparison of twistor theorie and its applications between the Riemannian case and the Lorentzian case is very productive and some interesting unsolved questions arise (comp. [Nur96]).

In section 2 and section 3, it will first be given the basic concepts for the investigation of the twistor space. Section 2 investigates the Riemannian curvature tensor of Lorentzian 4-manifolds and explains the notion of Petrov typs. Section 3 is concerned with the mentioned geometric structures that naturally appear on the twistor space, the optical geometry and the CR-geometry. In section 4 the twistor space is interpreted as projective spinor bundle.

The following sections investigate different constructions on the twistor space. Section 5 is concerned with the optical geometry. The mentioned theorem of integrability will be proofed in terms of differential geometric notation. The twistor spaces with optical structure of the three space forms $\mathbf{R}_1^4$, $\mathbf{S}_1^4$ and $\mathbf{H}_1^4$ are discussed in detail.

In section 6, it will be discussed twistor spinors ϕ, $\mathcal{D}\phi = 0$, in the Lorentzian case. It will be shown that there is an interpretation of twistor spinors in terms of the optical geometry on the twistor space. The idea of this interpretation bases on the fact that one can interprets twistor spinors in Riemannian geometry as holomorphic sections in the twistor space. Moreover, in the situation that the base manifold is conformally flat, there is locally a correspondence between twistor spinors and CR-functions.

One final remark: on the twistor space $\mathcal{Z}(M)$ doesn't exist a natural construction of an almost complex structure and there isn't a natural semi-Riemannian metric. But the choice of a timelike vector field on a time-orientable Lorentzian manifold M, this means the reduction of the frame bundle $SO(M)$ to the structure group $\mathbf{SO}(3)$, gives the possibility of defining an almost complex structure and a semi-Riemannian metric of index 1. The idea of such a construction of an almost complex structure is given in [MS94].

2 Riemannian curvature tensor and Petrov types

In this section, the Riemannian curvature tensor R of a 4-dimensional Lorentzian manifold is investigated. The traceless part of the Riemannian curvature tensor, the Weyl tensor W, is classified by Petrov types and curvature properties of some kinds of Lorentzian manifolds are calculated.

2.1 Riemannian curvature tensor and Weyl tensor of a 4-dimensional Lorentzian manifold

Let $(M^{4,1}, g)$ be a 4-dimensional and oriented Lorentzian manifold (index $g = 1$). TM denotes the tangent space to M and $\Lambda^i M$ $(0 \leq i \leq 4)$ is the bundle of alternating i-forms on TM. Let $(s_1, \ldots, s_4)$ denote a local positive oriented (pseudo)-orthonormal frame in M with $g(s_1, s_1) = -1$ and $(\sigma^1, \ldots, \sigma^4)$ denotes the dual frame. $\epsilon_k := g(s_k, s_k)$. The Lorentzian metric g induces metrics $(\cdot, \cdot)$ on the bundles $\Lambda^i M$ of i-forms. The definition of these metrics is given by, $\xi, \eta \in \Lambda^i_x M,\ x \in M$,

$$(\xi, \eta)_x = \sum_{1 \leq j_1 < \ldots < j_i \leq 4} \epsilon_{j_1} \cdot \ldots \cdot \epsilon_{j_i} \xi(s_{j_1 x}, \ldots, s_{j_i x}) \cdot \eta(s_{j_1 x}, \ldots, s_{j_i x}).$$

Let dM be the volume element of M. The Hodge star operator $* : \Lambda^i M \to \Lambda^{4-i} M$ is defined by

$$\xi \wedge *\eta = -(\xi, \eta) dM, \quad \xi, \eta \in \Lambda^i M.$$

On 2-forms, the Hodge operator $* : \Lambda^2 M \to \Lambda^2 M$ has the following properties:

(H1) $*$ is a bundle morphism,

(H2)

$$* \, (\sigma^1 \wedge \sigma^2) = \sigma^3 \wedge \sigma^4, \qquad * \, (\sigma^3 \wedge \sigma^4) = -\sigma^1 \wedge \sigma^2,$$

$$* \, (\sigma^1 \wedge \sigma^3) = \sigma^4 \wedge \sigma^2, \qquad * \, (\sigma^4 \wedge \sigma^2) = -\sigma^1 \wedge \sigma^3,$$
$$* \, (\sigma^1 \wedge \sigma^4) = \sigma^2 \wedge \sigma^3, \qquad * \, (\sigma^2 \wedge \sigma^3) = -\sigma^1 \wedge \sigma^4,$$

(H3) $*^2 = -id$,

(H4) $*$ is self-adjoint with respect to $(\cdot, \cdot)$.

These properties are proofed by using definition and $dM = \sigma^1 \wedge \sigma^2 \wedge \sigma^3 \wedge \sigma^4$. The isometry group $\mathbf{SO}(T_xM)$ of (T_xM, g_x), $x \in M$, is isomorphic to the isometry group $\mathbf{SO}(4,1)$ of the Minkowski space $\mathbf{R}^4_1 := (\mathbf{R}^4, <,>_1)$. The spaces $\Lambda^i_x M$ are in natural $\mathbf{SO}(T_xM)$-moduls. It holds: $*A = A* \ \forall A \in \mathbf{SO}(T_xM)$. If g and $\lambda^2 g$ ($\lambda : M \to \mathbf{R}$, $\lambda > 0$) are conformally equivalent metrics on $M^{4,1}$ then $*_g = *_{\lambda^2 g}$. This means that the Hodge operator $*$ is invariant to conformal change of the metric (see [Fri81]).
$\tilde{\sigma} = (\sigma^1 \wedge \sigma^2, \sigma^1 \wedge \sigma^3, \sigma^1 \wedge \sigma^4, \sigma^3 \wedge \sigma^4, \sigma^4 \wedge \sigma^2, \sigma^2 \wedge \sigma^3)$ is an orthonormal basis in $\Lambda^2 M$. Let

$$\Lambda^2_t M := Span\{\sigma^1 \wedge \sigma^2, \sigma^1 \wedge \sigma^3, \sigma^1 \wedge \sigma^4\}$$

be the timelike subspace of $\Lambda^2 M$, that means $(\cdot, \cdot)$ is negative definite on $\Lambda^2_t M$ and

$$\Lambda^2_s M := Span\{\sigma^3 \wedge \sigma^4, \sigma^4 \wedge \sigma^2, \sigma^2 \wedge \sigma^3\}$$

is the spacelike subspace of $\Lambda^2 M$, $(\cdot, \cdot)$ is positive definite on $\Lambda^2_s M$. In the basis $\tilde{\sigma}$, the Hodge operator $*$ has the form $[*] = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$. Such a basis is named a Hodge basis. Let ∇ be the Levi-Civita connection of $(M^{4,1}, g)$ and

$$R(U,V)W = \nabla_U \nabla_V W - \nabla_V \nabla_U W - \nabla_{[U,V]} W$$

denotes the Riemannian curvature tensor. The curvature tensor R has the properties

(R1) $R(U,V)W = -R(V,U)W$,

(R2) $R(U,V)W + R(V,W)U + R(W,U)V = 0$,

(R3) $g(R(U,V)W, Z) = -g(R(U,V)Z, W)$,

(R4) $g(R(U,V)W, Z) = g(R(W,Z)U, V), \quad U, V, W, Z \in TM$.

As a $(2,2)$-tensor $R : \Lambda^2 M \to \Lambda^2 M$, the curvature tensor is given as

$$R(\sigma^i \wedge \sigma^j) = \sum_{k<l} \epsilon_k \cdot \epsilon_l R_{ijkl} \sigma^k \wedge \sigma^l,$$

where $R_{ijkl} := g(R(s_i, s_j)s_k, s_l)$. Because of the property (R4), it follows that $R : \Lambda^2 M \to \Lambda^2 M$ is self-adjoint with respect to $(\cdot, \cdot)$. In the Hodge basis $\tilde{\sigma}$, the curvature tensor $R : \Lambda^2 M \to \Lambda^2 M$ is of the form

$$[R] = \begin{pmatrix} -A & -B \\ B^t & C \end{pmatrix} = \begin{pmatrix} -R_{1212} & -R_{1312} & -R_{1412} & -R_{3412} & -R_{4212} & -R_{2312} \\ -R_{1213} & -R_{1313} & -R_{1413} & -R_{3413} & -R_{4213} & -R_{2313} \\ -R_{1214} & -R_{1314} & -R_{1414} & -R_{3414} & -R_{4214} & -R_{2314} \\ R_{1234} & R_{1334} & R_{1434} & R_{3434} & R_{4234} & R_{2334} \\ R_{1242} & R_{1342} & R_{1442} & R_{3442} & R_{4242} & R_{2342} \\ R_{1223} & R_{1323} & R_{1423} & R_{3423} & R_{4223} & R_{2323} \end{pmatrix}.$$

It follows from the properties (R1)-(R4) that $A, C \in M(3,\mathbf{R})$ are symmetric matrices and the trace of $B \in M(3,\mathbf{R})$ vanishes, $trB = 0$. The matrices $-A : \Lambda_t^2 M \to \Lambda_t^2 M$ and $C : \Lambda_s^2 M \to \Lambda_s^2 M$ are self-adjoint.
The Ricci tensor is defined by

$$Ric(U,X) = tr(V \mapsto -R(U,V)X) = \sum_{k=1}^{4} \epsilon_k \cdot R(U, s_k, s_k, X).$$

The scalar curvature is

$$S = trRic = \sum_{k=1}^{4} \epsilon_k R_{kk} = \sum_{i,j} \epsilon_i \epsilon_j R_{ijji},$$

where $R_{kl} := Ric(s_k, s_l)$.
Let $(V^{4,1}, <,>_1)$ be a 4-dimensional and oriented Lorentzian vector space and let $*_V$ denote the Hodge operator. The vector space of algebraic curvature tensors of V, these are the tensors with the properties (R1)-(R4), is denoted by $\mathcal{R}V$. $\mathcal{R}V$ is a $\mathbf{SO}(V^{4,1})$-modul and splits in the $\mathbf{SO}(V^{4,1})$-irreducible subspaces $\mathcal{W}V$, $\mathcal{L}V$ and $\mathcal{U}V$ (see [Bes87, p.98-p.101]):

$$\begin{aligned} \mathcal{W}V &= \{K \in \mathcal{R}V \mid *_V K = K *_V, trK = 0\}, \\ \mathcal{L}V &= \{K \in \mathcal{R}V \mid *_V K = -K *_V\}, \\ \mathcal{U}V &= \{K \in \mathcal{R}V \mid K = -kId, k \in \mathbf{R}\}. \end{aligned}$$

On an oriented Lorentzian manifold M with Riemannian curvature tensor R, we have the following relations (see [Bes87]):

$$\begin{array}{rcl} R_x \in \mathcal{W}T_xM \quad \forall x \in M & \Leftrightarrow & M \text{ is Ricci flat } (trR = 0), \\ R_x \in \mathcal{U}T_xM \quad \forall x \in M & \Leftrightarrow & M \text{ has constant sectional curvature } k, \\ R_x \in \mathcal{U}T_xM + \mathcal{W}T_xM \ \forall x \in M & \Leftrightarrow & *R = R* \Leftrightarrow M \text{ is Einstein space } (Ric = \frac{S}{4}g), \\ R_x \in \mathcal{U}T_xM + \mathcal{L}T_xM \ \forall x \in M & \Leftrightarrow & M \text{ is conformally flat.} \end{array}$$

The curvature tensor R of $(M^{4,1}, g)$ splits as follows in its irreducible parts of the $\mathbf{SO}(T_xM)$-representations $\mathcal{R}T_xM$, $x \in M$:

$$R = W + L + U = (R - g \star l) + \frac{1}{2} g \star (\frac{S}{4} g - Ric) - \frac{S}{24} g \star g,$$

where $l = \frac{1}{2}(\frac{S}{6}g - Ric)$ is the Schouten tensor. The Kulkarni-Nomizu product $\star$ of two symmetric tensors h, b is given by

$$\begin{aligned} (h \star b)(U,V,W,Z) := \quad & h(U,W)b(V,Z) + h(V,Z)b(U,W) \\ & - h(U,Z)b(V,W) - h(V,W)b(U,Z) \end{aligned}$$

and satisfies the properties (R1)-(R4). In the basis $\tilde{\sigma}$, $[R] = \begin{pmatrix} -A & -B \\ B^t & C \end{pmatrix}$, the relations (see [Bes87])

$$\begin{array}{rl} R_x \in \mathcal{W}T_xM \ \forall x \in M & \Leftrightarrow -A = C, \ B = B^t, \ trA = 0, \\ R_x \in \mathcal{L}T_xM \ \forall x \in M & \Leftrightarrow A = C, \ B = -B^t, \\ R_x \in \mathcal{U}T_xM \ \forall x \in M & \Leftrightarrow -A = C = -kI, \ B = 0, \end{array}$$

hold. The decomposition

$$
\begin{aligned}
[R] &= \begin{pmatrix} -A & -B \\ B^t & C \end{pmatrix} = [W] + [L] + [U] \\
&= \begin{pmatrix} \frac{-A+C}{2} - tr\frac{-A+C}{6} I_3 & \frac{-B-B^t}{2} \\ \frac{B+B^t}{2} & \frac{-A+C}{2} - tr\frac{-A+C}{6} I_3 \end{pmatrix} + \begin{pmatrix} \frac{-A-C}{2} & \frac{-B+B^t}{2} \\ \frac{-B+B^t}{2} & \frac{A+C}{2} \end{pmatrix} \\
&\quad + tr\frac{-A+C}{6} I_6.
\end{aligned}
$$

is obtained. W is called the Weyl tensor of the Lorentzian manifold $(M^{4,1}, g)$. As a $(2,2)$-tensor, the Weyl tensor $W : \Lambda^2 M \to \Lambda^2 M$ is given by

$$W(\sigma^i \wedge \sigma^j) = \sum_{k,l} \epsilon_k \epsilon_l W_{ijkl} \sigma^k \wedge \sigma^l,$$

where $W_{ijkl} := W(s_i, s_j, s_k, s_l)$. As a $(3,1)$- tensor, the Weyl tensor W is invariant to conformal change of the metric g.

2.2 Curvature properties of some kinds of Lorentzian manifolds

In (2.1), the following statements were already fixed:
In the case that $(M^{4,1}, g)$ has constant sectional curvature k, it holds:

$$R = -k \cdot id, \; Ric = 3k \cdot g \text{ and } S = 12k.$$

(M, g) is conformally flat if and only if in any Hodge basis,

$$[R] = \begin{pmatrix} -A & -B \\ -B & A \end{pmatrix} + cI_6, \; c \in \mathbf{R},$$

where $A \in M(3, \mathbf{R})$ is symmetric and $B \in M(3, \mathbf{R})$ is anti-symmetric. For a suitable Hodge basis, the matrix A has diagonal form.
(M, g) is Einstein space if and only if the curvature tensor takes in any Hodge basis the form

$$[R] = \begin{pmatrix} -A & -B \\ B & -A \end{pmatrix}$$

and $A, B \in M(3, \mathbf{R})$ symmetric, $trB = 0$.
If, in addition, (M, g) is Ricci flat then $trA = 0$. Again the matrix A takes diagonal form in a suitable Hodge basis.

Now to the curvature properties of Lorentz warped products:
Let (N, h) be a 3-dimensional oriented Riemannian manifold and $\mathbf{R}^1_1 = (\mathbf{R}, - <,>)$ be the real numbers with negative standard metric. Let $f > 0$ be a smooth function on $\mathbf{R}^1_1$. The Lorentz warped product $M := \mathbf{R}^1_1 \times_f N$ is an oriented and time-oriented 4-dimensional Lorentzian manifold, the metric is given by

$$g = -\rho^*(<,>) + (f \circ \rho)^2 \cdot \sigma^* h,$$

where $\rho : M \to \mathbf{R}^1_1$ and $\sigma : M \to N$ are the natural projections (see [O'N83, chap. 7]). For $(\hat{s}_2, \hat{s}_3, \hat{s}_4)$ a local positive oriented orthonormal frame in $U \subset N$, $T := \frac{\partial}{\partial t} \in \Gamma(T\mathbf{R}^1_1)$ and $s_i := \frac{1}{f}\hat{s}_i \in \Gamma(TM|_{\mathbf{R}\times U})$, the local frame (T, s_2, s_3, s_4) on M is positive oriented and orthonormal. Let $S(t) := \rho^{-1}(t)$, $t \in \mathbf{R}$, be the spacelike slices in M. The field T is positive in the time-orientation and orthogonal to $TS(t) \subset TM$.
Let $X, Y, Z \in \Gamma(TM)$ be tangent to $S(t)$ $\forall t$. R^N, Ric^N and S^N denote the lifts of the Riemannian curvature tensor, the Ricci tensors and the scalar curvature on (N, h) to M.
For the Riemannian curvature tensor R of a Lorentz warped product, we have the relations (see [O'N83, p. 210]):

$$R(X,Y)Z = R^N(X,Y)Z - \left(\frac{f'}{f}\right)^2 (g(X,Z)Y - g(Y,Z)X),$$

$$R(X,T)T = -\frac{f''}{f}X,\ R(X,Y)T = 0,\ R(X,T)Y = -\frac{f''}{f}g(X,Y)T.$$

In the Hodge basis of $\Lambda^2 M$, belonging to a local orthonormal frame $(s_1 = T, s_2, s_3, s_4)$ on M, the curvature tensor R has the form

$$[R] = \begin{pmatrix} -\frac{f''}{f}I_3 & 0 \\ 0 & [R^N] - (\frac{f'}{f})^2 I_3 \end{pmatrix}.$$

For the Ricci tensor,

$$Ric(T,T) = -3\frac{f''}{f},\ Ric(T,X) = 0 \text{ and}$$

$$Ric(X,Y) = Ric^N(X,Y) + (2\left(\frac{f'}{f}\right)^2 + \frac{f''}{f})g(X,Y)$$

holds. For the scalar curvature, it holds

$$S = 6(\frac{f''}{f} + \left(\frac{f'}{f}\right)^2) + \frac{S^N}{f^2}.$$

In the situation that N has constant sectional curvature k, this is equivalent to N is Einstein space, since $\dim N = 3$, the warped product $M = \mathbf{R}^1_1 \times_f N$ is called a Robertson-Walker spacetime. A Lorentz warped product M is conformally flat if and only if (N, h) has constant sectional curvature k. In this case, the easier relations

$$R(X,Y)Z = -(\left(\frac{f'}{f}\right)^2 + \frac{k}{f^2})(g(X,Z)Y - g(Y,Z)X),$$

$$R(X,T)T = -\frac{f''}{f}X,\ R(X,Y)T = 0,\ R(X,T)Y = -\frac{f''}{f}g(X,Y)T$$

hold. With respect to a Hodge basis ($s_1 = T$), we have

$$[R] = \begin{pmatrix} -\frac{f''}{f} I_3 & 0 \\ 0 & -\frac{f'^2+k}{f^2} I_3 \end{pmatrix},$$

$$Ric(T,T) = -3\frac{f''}{f}, \ Ric(T,X) = 0,$$

$$Ric(X,Y) = (2\left(\frac{f'}{f}\right)^2 + \frac{2k}{f^2} + \frac{f''}{f})g(X,Y)$$

and

$$S = 6(\left(\frac{f'}{f}\right)^2 + \frac{k}{f^2} + \frac{f''}{f}).$$

For N with constant sectional curvature k and warping function $f = 1$, the manifold $M = \mathbf{R}^1_1 \times_f N$ is called Einstein statistic universe. Then

$$[R] = \begin{pmatrix} 0 & 0 \\ 0 & -kI_3 \end{pmatrix}, \ <Ric>= 2 \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & k & 0 & 0 \\ 0 & 0 & k & 0 \\ 0 & 0 & 0 & k \end{pmatrix} \text{ and } S = 6k.$$

Let $\mathbf{S}^4_1$ denote the 4-dimensional pseudosphere (see [O'N83, chap. 4] and section (5.2)). For $N = \mathbf{S}^3$ the sphere and $f = \cosh t$, the Lorentz warped product $M = \mathbf{R}^1_1 \times_f N = \mathbf{R}^1_1 \times_f \mathbf{S}^3$ is isometric to the pseudosphere $\mathbf{S}^4_1$. It holds:

$$[R] = \begin{pmatrix} -\frac{\cosh t}{\cosh t} I_3 & 0 \\ 0 & -\frac{\sinh^2 t+1}{\cosh^2 t} I_3 \end{pmatrix} = \begin{pmatrix} -I_3 & 0 \\ 0 & -I_3 \end{pmatrix},$$

$$Ric = 3g \text{ and } S = 12.$$

The flat spacetime is $\mathbf{R}^4_1 = \mathbf{R}^1_1 \times_1 \mathbf{R}^3$.
Let $\mathbf{H}^4_1$ denote the pseudohyperbolic space (see [O'N83, chap. 4] and section (5.2)) and let $\mathbf{H}^3$ be the 3-dimensional hyberbolic space. The space $\mathbf{H}^4_1$ is isometric to the warped product $\mathbf{H}^3 \times_f \mathbf{S}^1_1$ with $f = \cosh r$, where r is the radius in polar coordinates of $\mathbf{H}^3$ and $\mathbf{S}^1_1$ is the 1-sphere with negative standard metric. $\mathbf{H}^4_1$ has constant sectional curvature $k = -1$. The Lorentz warped product $]0, \pi[\times_{\sin t} \mathbf{H}^3$ has also constant sectional curvature $k = -1$. This means that $\mathbf{H}^4_1$ is locally isometric to a Lorentz warped product.

2.3 Petrov types and principal null directions

We are now going to classify the Weyl tensor W by Petrov types. The Petrov type is an algebraic property of the Weyl tensor. Later on, we will obtain a more geometric interpretation of Petrov types by principal null directions. This section is mainly following the description of Petrov types in [O'N95, chap. 5] (see also [Bes87]).

The Hodge operator $*$ has the property $*^2 = -id$. It follows that $*$ is a complex structure on the bundle $\Lambda^2 M$ and $(\Lambda^2 M, *)$ is a complex vector bundle. The multiplication with i is given by $i\xi := *\xi,\ \xi \in \Lambda^2 M$. Then $\tilde{\sigma}^{\mathbf{C}} = (\sigma^1 \wedge \sigma^2, \sigma^1 \wedge \sigma^3, \sigma^1 \wedge \sigma^4)$ is a complex basis in $\Lambda^2 M$. The Weyl tensor $W : \Lambda^2 M \to \Lambda^2 M$ has the property $*W = W*$ (see (2.1)). Therefore, $W : (\Lambda^2 M, *) \to (\Lambda^2 M, *)$ is a complex linear endomorphism. With respect to the Hodge basis $\tilde{\sigma}^{\mathbf{C}}$, we have

$$[W]^{\mathbf{C}} = A + iB,\ A, B \in M(3, \mathbf{R}),\ tr(A + iB) = 0 \text{ and } A + iB \text{ symmetric.}$$

Each complex linear mapping is classified by Jordan canonical forms and this is the way how the Petrov type of the Weyl tensor $W_x,\ x \in M$, is defined. (Remember that the geometric multiplicity of an eigenvalue λ is the dimension of its eigenspace E_λ.)

Definition 2.1 *The Weyl tensor W has in $x \in M^{4,1}$ the Petrov type:*

Type 0	$:\Leftrightarrow$	$W_x = 0$
Type I	$:\Leftrightarrow$	*W_x has three distinct eigenvalues*
Type D	$:\Leftrightarrow$	*W_x is diagonalizable with exactly two distinct eigenvalues*
Type II	$:\Leftrightarrow$	*W_x has exactly two distinct eigenvalues, each of geometric multiplicity 1*
Type N	$:\Leftrightarrow$	*W_x has the unique eigenvalue 0 and its geometric multiplicity is 2*
Type III	$:\Leftrightarrow$	*W_x has the unique eigenvalue 0 and its geometric multiplicity is 1*

If the Weyl tensor W has in every point $x \in M$ the same Petrov type, we say that the Lorentzian manifold M is of that Petrov type.

Obviously, the Weyl tensor W $(trW = 0)$ has in any point $x \in M$ exactly one of the stated Petrov types. Because of the conformal invarianz of the Hodge operator and the Weyl tensor, the Petrov type of W is conformally invariant.
Let

$$b : \Lambda^2 M \times \Lambda^2 M \to \mathbf{C}, \quad \xi, \eta \mapsto (\xi, \eta) - i(\xi, *\eta).$$

It follows from the self-adjointniss of $*$ that b is a complex symmetric bilinear form. The canonical forms for $<b>^{\mathbf{C}}$ and $[W]^{\mathbf{C}}$ in various Petrov types look as follows (see [O'N95, chap.5.4] and [Bes87, p.100]):

$$W \text{ has type I} \ :\ <b>^{\mathbf{C}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},\ [W]^{\mathbf{C}} = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix},\ \begin{matrix} \lambda_i \neq \lambda_j \\ \sum \lambda_i = 0 \end{matrix}$$

$$W \text{ has type D} \ :\ <b>^{\mathbf{C}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},\ [W]^{\mathbf{C}} = \begin{pmatrix} -2\lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix}, \lambda \neq 0$$

$$W \text{ has type II} \quad : \quad <b>^{\mathbf{C}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \ [W]^{\mathbf{C}} = \begin{pmatrix} -2\lambda & 0 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix}, \lambda \neq 0$$

$$W \text{ has type N} \quad : \quad <b>^{\mathbf{C}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \ [W]^{\mathbf{C}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$W \text{ has type III} \quad : \quad <b>^{\mathbf{C}} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \ [W]^{\mathbf{C}} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

(For $[W]^{\mathbf{C}}$ in Jordan canonical form the bilinear form b isn't necessarily diagonalizable.)
Now, we will explain what are principal null directions of W:
We say that a 2-form $\beta \in \Lambda^2 M$ is decomposable if it is the wedge product of two 1-forms $\alpha_1, \alpha_2 \in \Lambda^1 M$, $\beta = \alpha_1 \wedge \alpha_2$.

Lemma 2.1 ([O'N95, chap. 5.2]) *Let $0 \neq \beta \in \Lambda^2_x M$, $x \in M$, be a 2-form.*

1. *β is null ($(\beta, \beta) = 0$) and decomposable $\Leftrightarrow$*
   $\exists\, n, \alpha \in \Lambda^1_x M$, n null, α spacelike and $0 \neq \alpha \perp n$ so that
   $\beta = n \wedge \alpha$ holds. n is up to multiples uniquely determined.

2. *Let $\beta = n \wedge \alpha_1$ be a null and decomposable 2-form.*
   Then $*\beta = n \wedge \alpha_2$, where $n, \alpha_1, \alpha_2 \in \Lambda^1_x M$
   are mutually orthogonal.

3. *β is decomposable and null $\Leftrightarrow$ $b(\beta, \beta) = 0$.*

Each decomposable and null 2-form $\beta = n \wedge \alpha \in \Lambda^2_x M$ corresponds in a natural way to one null direction $N(\beta) = \mathbf{R} \cdot n \subset T_x M \overset{g}{\cong} T^*_x M$. Let $0 \neq z \in \mathbf{C}$. From Lemma 2.1, it follows $N(z\beta) = N(\beta) = \mathbf{R} \cdot n$. We can picture a decomposable 2-form as a plain in $T_x M$, $x \in M$. A null and decomposable 2-form can be interpreted as a plain tangent to the nullcone $\Lambda^{T_x M}$. The intersection of this plain and the nullcone $\Lambda^{T_x M}$ is just the null direction that corresponds to the 2-form by the mapping N.

Definition 2.2 *A b-null 2-form $\beta \in \Lambda^2_x M$, $x \in M$, $(b(\beta, \beta) = 0)$ such that $b(W\beta, \beta) = 0$, is called a principal null 2-form of W. The null direction $N(\beta) \subset T_x M$ is a principal null direction and $0 \neq n \in N(\beta)$ is a principal null vector of W.*

Let β be a b-null eigenvector of W to the eigenvalue λ. Because of $b(W\beta, \beta) = \lambda b(\beta, \beta) = 0$, the 2-form β is principal null.
Now, we define the multiplicity of a principal null direction.

Definition 2.3 *The principal null 2-form $\beta \in \Lambda^2_x M,\ x \in M$, has multiplicity*

$m = 1$ *if β isn't an eigenvalue of W,*
$m = 2$ *if $W\beta = \lambda\beta$ with $\lambda \neq 0$,*
$m = 3$ *if $W\beta = 0$ and $\dim_{\mathbf{C}} \ker W_x = 1$,*
$m = 4$ *if $W\beta = 0$ and $\dim_{\mathbf{C}} \ker W_x = 2$.*

If $\beta \in \Lambda^2 M$ has multiplicity m, we say that β is m-principal null, $N(\beta)$ is a m-principal null direction and $n \in N(\beta)$ is a m-principal null vector.

In the case of $W \neq 0$, the multiplicity of each principal null direction is uniquely determined.

We have the following relations between the Petrov type of the Weyl tensor and its principal null directions:

Proposition 2.2 ([O'N95, chap. 5.5]) *Let $0 \neq W_x$ be the Weyl tensor in $x \in M$.*

W_x has type I	$\Leftrightarrow$	*W_x has four simple principal null directions*
W_x has type II	$\Leftrightarrow$	*W_x has one 2-principal null direction and two simple principal null directions*
W_x has type D	$\Leftrightarrow$	*W_x has two 2-principal null directions*
W_x has type III	$\Leftrightarrow$	*W_x has one 3-principal null direction and one simple principal null direction*
W_x has type N	$\Leftrightarrow$	*W_x has one 4-principal null direction*

Proposition 2.3 ([O'N95, chap. 5.5]) *Let $W_x \neq 0$ and let $k \in T_x M$ be a null vector.*

1. *k is principal null to W $\Leftrightarrow$ $W(k,x,k,y) = 0 \ \ \forall x, y \perp k$*
2. *k is m-principal null, $m \geq 2$ $\Leftrightarrow$ $W(k,x,k,\cdot) = 0 \ \ \forall x \perp k$*
3. *k is m-principal null, $m = 3,4$ $\Leftrightarrow$ $W(k,x,\cdot,\cdot) = 0 \ \ \forall x \perp k$*
4. *k is 4-principal null $\Leftrightarrow$ $\iota_k(W) = 0$*

Lemma 2.4 *Let $k \in T_x M,\ x \in M$, be a null vector and let (k,l,x,y) be an arbitrary real null tetrad to k, that means a basis of $T_x M$ so that*

$$< g_x >= \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Then, the vector k is principal null if and only if $R_{kxky} = 0$ and $R_{kxkx} = R_{kyky}$.

Proof: We have

$$R = W + g * l = W + \frac{S}{12} g * g - \frac{1}{2} g * Ric.$$

Since $k, x, y \in TM$ are mutually orthogonal and $g(k,k) = 0$, we have:
$R_{kxky} = W_{kxky},\ R_{kxkx} = W_{kxkx} - \frac{1}{2} Ric(k,k),\ R_{kyky} = W_{kyky} - \frac{1}{2} Ric(k,k)$. Then

$$\begin{aligned} Ric(k,k) &= g(k,k) \cdot R_{kkkk} + g(l,l) \cdot R_{kllk} + g(x,x) \cdot R_{kxxk} + g(y,y) \cdot R_{kyyk} \\ &= R_{kxxk} + R_{kyyk}. \end{aligned}$$

And we have $\quad R_{kxky} = 0, R_{kxkx} = R_{kyky} \Leftrightarrow\ W_{kxky} = 0,\ W_{kxkx} = W_{kyky} = 0.$

The assertion follows from proposition 2.3 1. $\square$

Conclusion: Let $(s_1, \ldots, s_4)$ be a local orthonormal frame on M. $s_1 + s_2$ is principal null if and only if

$$R_{1413} + R_{2413} + R_{1423} + R_{2423} = 0,$$

$$R_{1414} + R_{2414} + R_{1424} + R_{2424} - R_{1313} - R_{2313} - R_{1323} - R_{2323} = 0.$$

3 CR-geometry and optical geometry

In this section, we give in short the essential facts of CR-geometry and optical geometry. The concept of CR-geometry is certainly more common then the concept of optical geometry. In a later section optical structures appear in context with the twistor space of a Lorentzian manifold. In spezial conditions optical structures are closely related to CR-structures. In mathematical physics optical structures were studied in context with the Maxwell equations.
A detailed introduction to CR-geometry is given in [Jac90]. The optical geometry is treated in [Tra85], [RT87], [Nur96] or [Nur97].

3.1 CR-structures

First recall the definition of an almost complex structure.

Definition 3.1 *An almost complex structure on a real C^∞-manifold M^{2n} is a bundle morphism $J : TM \to TM$ such that $J^2 = -id$. The almost complex structure J is integrable if*

$$N(X,Y) := J([X, JY] + [JX, Y]) - [JX, JY] + [X,Y] = 0 \quad \forall\ X, Y \in \Gamma(TM).$$

N is called the Nijenhuis tensor. Then we say that J is a complex structure on M.

The choice of an almost complex structure J on M is equivalent to a reduction of the $\mathbf{GL}(2n, \mathbf{R})$-frame bundle $L(M)$ over M to the group $\mathbf{GL}(n, \mathbf{C})$. The theorem of Newlander-Nierenberg (see e.g. [KN69]) states that J is a complex structure on M if and only if M admits complex coordinates. That means, there is a system $\{(U_i, \tilde{s}_i) : i \in I\}$ of local coordinates on M so that the $\tilde{s}_i,\ i \in I$, are local sections in the reduced $\mathbf{GL}(n, \mathbf{C})$-principal fibre bundle over M.

Definition 3.2 *A complex almost CR-structure on a real C^∞-manifold M^{2n+1} is a complex subbundle*

$$T_{1,0} \subset \mathbf{C} \otimes TM \text{ with } \dim_{\mathbf{C}} T_{1,0} = n,\ T_{1,0} \cap \overline{T}_{1,0} = \{0\}.$$

The complex almost CR-structure $T_{1,0}$ is integrable if

$$[\Gamma(T_{1,0}), \Gamma(T_{1,0})] \subset \Gamma(T_{1,0}) \quad (\Leftrightarrow [X,Y] \in \Gamma(T_{1,0}) \ \forall X, Y \in \Gamma(T_{1,0})).$$

Then $T_{1,0}$ is a complex CR-structure on M.
Notation: $T_{0,1} := \overline{T}_{1,0}$.

Definition 3.3 *A real almost CR-structure on a C^∞-manifold M^{2n+1} is a real subbundle $H \subset TM$ and a linear mapping $J : H \to H$ such that $\dim_{\mathbf{R}} H = 2n$ and $J^2 = -id|_H$. (H, J) is integrable if*

$$[JX, Y] + [X, JY] \in \Gamma(H), \ J([JX, Y] + [X, JY]) = [JX, JY] - [X, Y] \ \forall X, Y \in \Gamma(H).$$

Then (H, J) is a real CR-structure.

These two definitions of CR-structures are equivalent in the sense that there is a unique correspondence between real and complex almost CR-structures:
Let $T_{1,0}$ be a complex almost CR-structure. Then

$$H := Re(T_{1,0} \oplus T_{0,1}), \ J : H \to H, \ J(V + \overline{V}) = i(V - \overline{V}), \ V \in T_{1,0},$$

is a real almost CR-structure.
For (H, J) a real almost CR-structure,

$$T_{1,0} := \{X - iJX | \ X \in H\} = \{V \in \mathbf{C} \otimes H | \ JV = iV\}$$

is a complex almost CR-structure.
The integrability conditions are equivalent:

$$\begin{gathered}
[\Gamma(T_{1,0}), \Gamma(T_{1,0})] \subset \Gamma(T_{1,0}) \\
\Leftrightarrow \\
J([X - iJX, Y - iJY]) = i[X - iJX, Y - iJY], \\
[X - iJX, Y - iJY] \in \Gamma(T_{1,0}) \quad \forall X, Y \in \Gamma(H) \\
\Leftrightarrow \\
J([X,Y] - [JX, JY] - i[X, JY] - i[JX, Y]) = \\
i([X,Y] - [JX, JY] - i[X, JY] - i[JX, Y]), \\
[X,Y] - [JX, JY] \in \Gamma(H), \ [X, JY] + [JX, Y] \in \Gamma(H) \quad \forall X, Y \in \Gamma(H) \\
\Leftrightarrow \\
[JX, Y] + [X, JY] \in \Gamma(H), \\
J([JX, Y] + [X, JY]) = [JX, JY] - [X, Y] \quad \forall X, Y \in \Gamma(H).
\end{gathered}$$

In the following, we will use both definitions of almost CR-structures. Also, we call a manifold M^{2n+1} with an almost CR-structure an almost CR-manifold.

Remarks:

1. In the case of $\dim M = 3$, the integrability conditions of an almost CR-structure (H, J) on M trivially hold.

2. If H is an involutive distribution, the integral manifolds of H are almost complex manifolds.

3. Let $(M, T_{1,0})$ and $(\hat{M}, \hat{T}_{1,0})$ be two (almost) CR-manifolds.
An diffeomorphism $f : M \to \hat{M}$ such that $f_*(T_{1,0}) = \hat{T}_{1,0}$ is called a CR-diffeomorphism and we say, the (almost) CR-structures on M and $\hat{M}$ are equivalent.
Let $m_1 \in M$ and $m_2 \in \hat{M}$ be points of the almost CR-manifolds. If there exist neighborhoods $U_1 \ni m_1$ and $U_2 \ni m_2$ of these points such that there is a CR-diffeomorphism between $U_1 \subset (M, T_{1,0})$ and $U_2 \subset (\hat{M}, \hat{T}_{1,0})$ than we say that the (almost) CR-structures in the points $m_1 \in M$ and $m_2 \in \hat{M}$ are locally equivalent.
On a manifold M^{2n+1} CR-structures exist that aren't equivalent even not locally in a point.

Examples:

A. Canonical CR-structure on a hypersurface $M^{2n+1} \subset \mathbf{C}^{n+1}$

Let $M^{2n+1} \subset \mathbf{C}^{n+1}$ be a $(2n+1)$-dimensional real C^∞-submanifold. The complex structure of $\mathbf{C}^{n+1}$ induces on M^{2n+1} a canonical CR-structure that is given by

$$T_{1,0} := Span_{\mathbf{C}}\{\frac{\partial}{\partial z_1}, \ldots, \frac{\partial}{\partial z_{n+1}}\} \cap \mathbf{C} \otimes TM,$$

where $(z_1, \ldots, z_{n+1})$ are the complex coordinates in $\mathbf{C}^{n+1}$. Then

$$T_{0,1} = Span_{\mathbf{C}}\{\frac{\partial}{\partial \overline{z}_1}, \ldots, \frac{\partial}{\partial \overline{z}_{n+1}}\} \cap \mathbf{C} \otimes TM,$$

$$T_{1,0} \cap T_{0,1} = \{0\} \text{ and } \dim_{\mathbf{C}} T_{1,0} = n.$$

From

$$[\Gamma(T_{1,0}), \Gamma(T_{1,0})] \subset \Gamma(\mathbf{C} \otimes TM)$$

and

$$[\Gamma(T_{1,0}), \Gamma(T_{1,0})] \subset Span_{\mathbf{C}}\{\frac{\partial}{\partial z_i} : i = 1, \ldots, n+1\}$$

follows the integrability $[\Gamma(T_{1,0}), \Gamma(T_{1,0})] \subset \Gamma(T_{1,0})$. The corresponding real CR-structure is given by

$$(H = TM \cap J(TM),\ J = J^{\mathbf{C}^{n+1}}|_H).$$

1. $\mathbf{R}^{2n+1} \subset \mathbf{C}^{n+1}$:
Let

$$(x_1, y_1, \ldots, x_n, y_n, x_{n+1}) \in \mathbf{R}^{2n+1} \mapsto (x_1 + iy_1, \ldots, x_n + iy_n, x_{n+1}) \in \mathbf{C}^{n+1}$$

be the imbedding. The induced CR-structure is given by

$$H = Span\{\frac{\partial}{\partial x_1}, \frac{\partial}{\partial y_1}, \ldots, \frac{\partial}{\partial x_n}, \frac{\partial}{\partial y_n}\} \text{ and}$$

$$J : H \to H,\ J(\frac{\partial}{\partial x_i}) = \frac{\partial}{\partial y_i},\ J(\frac{\partial}{\partial y_i}) = -\frac{\partial}{\partial x_i},\ i = 1, \ldots, n.$$

2. $\mathbf{S}^3 \subset \mathbf{C}^2$ (see [Jac90, p. 14]):
   In $p = (x_1, y_1, x_2, y_2) \in \mathbf{S}^3$ the induced CR-structure is given by

$$H_p = Span\{(x_2, -y_2, -x_1, y_1), (y_2, x_2, -y_1, -x_1)\} \text{ and}$$

$$J_p((x_2, -y_2, -x_1, y_1)) = (y_2, x_2, -y_1, -x_1),$$
$$J_p((y_2, x_2, -y_1, -x_1)) = -(x_2, -y_2, -x_1, y_1).$$

B. CR-structure on the unit sphere bundle of a 3-dimensional oriented Riemannian manifold

First, we consider a 3-dimensional oriented Euclidean vector space V^3. Let $l \in V^3$ be a unit vector and $H_l = (\mathbf{R}l)^\perp$ the orthogonal complement in V. On the subspace H_l is given a complex structure J_l by scalar product and orientation: let (l, h_1, h_2) be an oriented orthonormal basis in V then $J_l : H_l \to H_l$ is given by

$$J_l(h_1) = -h_2, \quad J_l(h_2) = h_1.$$

The complex structure J_l is defined independently of the choice of the oriented orthonormal basis.
In this way unit vectors $l \in V^3$ and 2-dimensional subspaces $H_l \subset V^3$ with complex structure J_l correspond. For simplicity, we say that (H_l, J_l) is a CR-structure on the vector space V^3.
Now, let (N, h) be a 3-dimensional oriented Riemannian manifold. $SO(N)$ denotes the bundle of positive oriented orthonormal frames. The tangent space of N is

$$TN = SO(N) \times_{\mathbf{SO}(3)} \mathbf{R}^3.$$

The set of unit vectors in Euclidean space is $\mathbf{S}^2 \subset \mathbf{R}^3$,

$$\mathbf{S}^2 \cong \mathbf{SO}(3)/\mathbf{SO}(2),\ A \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \mapsto A\mathbf{SO}(2).$$

The unit sphere bundle $Z(N)$ is given by

$$Z(N) := SO(N) \times_{\mathbf{SO}(3)} \mathbf{SO}(3)/\mathbf{SO}(2) = SO(N)/\mathbf{SO}(2) \subset TM.$$

Further, let $\pi : Z(N) \to N$ be the natural projection and

$$TZ(N) = T_V Z(N) \oplus T_H Z(N)$$

the decomposition of the tangent bundle into the horizontal and the vertical bundles with respect to the Levi-Civita connection on the frame bundle $SO(N)$. Now, on the bundle $Z(N)$, we can construct in a natural way an almost CR-structure:

The type of the fibres $\mathbf{SO}(3)/\mathbf{SO}(2) \cong \mathbf{S}^2$ admits a natural complex structure J, given in the point $A\mathbf{SO}(2)$ by

$$J(\frac{d}{dt}\Big|_{t=0} A \exp tE_{12}\mathbf{SO}(2)) = -\frac{d}{dt}\Big|_{t=0} A \exp tE_{13}\mathbf{SO}(2),$$

$$J(\frac{d}{dt}\Big|_{t=0} A \exp tE_{13}\mathbf{SO}(2)) = \frac{d}{dt}\Big|_{t=0} A \exp tE_{12}\mathbf{SO}(2),$$

where $\{E_{ij} : i < j\}$ denotes the standard basis of $\mathbf{so}(3)$. The complex structure J is invariant to the $\mathbf{SO}(3)$-action on $\mathbf{S}^2$, hence J induces complex structures J^V on the fibres of the bundle $Z(N)$.
Let $l \in Z(N)$ be an unit vector in $T_{\pi(l)}N$. The vector l gives the CR-structure (H_l, J_l) in $T_{\pi(l)}N$. The horizontal lift (H_l^*, J_l^*) of (H_l, J_l) is a CR-structure in the vector space $T_{Hl}Z(N) \cong T_{\pi(l)}N$. Thus for every element $l \in Z(N)$, the CR-structure

$$(H_l^* \oplus T_{Vl}Z(N), J_l^* \oplus J^V)$$

is given in the vector space $T_lZ(N)$. Altogether, we have defined an almost CR-structure on the bundle $Z(N)$ that we denote by $(H^{Z(N)}, J^{Z(N)})$. This almost CR-structure on $Z(N)$ is always integrable. This will become clear in a later section when we give a similar proof for the integrability theorem of the optical structure on the twistor space.

Now, we describe explicitly the canonical CR-structure on the bundle of unit spheres $Z(N)$ in the case of the sphere $N = \mathbf{S}^3$ and the hyperbolic space $N = \mathbf{H}^3$.

C. CR-structure on the homogeneous spaces $\mathbf{SO}(4)/\mathbf{SO}(2)$ and $\mathbf{SO}^+(4,1)/\mathbf{SO}(2)$

1. $\mathbf{SO}(4)$-invariant CR-structure on $\mathbf{SO}(4)/\mathbf{SO}(2)$:
   Let be given the sphere $\mathbf{S}^3$ as homogeneous space:

$$\mathbf{S}^3 \cong \mathbf{SO}(4)/\mathbf{SO}(3),\ A\begin{pmatrix}1\\0\\0\\0\end{pmatrix} \leftrightarrow A\mathbf{SO}(3), \quad A \in \mathbf{SO}(4).$$

   Let denote τ the $\mathbf{SO}(4)$-action on $\mathbf{S}^3$. It is

$$\tilde{e}_0 = \{(0,1,0,0), (0,0,1,0), (0,0,0,1)\}$$

   an oriented orthonormal basis in $T_{(1,0,0,0)}\mathbf{S}^3$. The frame bundle $SO(\mathbf{S}^3)$ can be identified with $\mathbf{SO}(4)$ by

$$SO(\mathbf{S}^3) \cong \mathbf{SO}(4), \quad d\tau_A(\tilde{e}_0) \leftrightarrow A$$

   and the bundle of unit spheres over $\mathbf{S}^3$ is naturally diffeomorphic to the homogeneous symmetric space

$$Z(\mathbf{S}^3) \cong \mathbf{SO}(4)/\mathbf{SO}(2).$$

The frame bundle $SO(\mathbf{S}^3)$ over $\mathbf{S}^3$ is trivial and therefore, $Z(\mathbf{S}^3)$ is diffeomorphic to $\mathbf{S}^3 \times \mathbf{S}^2$.
The canonical CR-structure on $Z(\mathbf{S}^3) \cong \mathbf{SO}(4)/\mathbf{SO}(2)$ is by construction invariant to the $\mathbf{SO}(4)$-action on $Z(\mathbf{S}^3)$. On $\mathbf{SO}(4)/\mathbf{SO}(2)$, we can describe this $\mathbf{SO}(4)$-invariant CR-structure in the following way:
Let be
$\{E_{ij} :\ 1 \leq i < j \leq 4\}$ the standard basis in $\mathbf{so}(4)$,
$i : \mathbf{SO}(2) \hookrightarrow \mathbf{SO}(4)$, $\begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \mapsto \exp tE_{34}$ the imbedding,
$\underline{K} := Span\{E_{12}, E_{13}, E_{14}\} \subset \mathbf{so}(4)$ an $Ad\mathbf{SO}(3)$-invariant subspace,

$$\underline{H} := Span\{E_{13}, E_{14}, E_{23}, E_{24}\} \subset \mathbf{so}(4)$$

an $Ad\mathbf{SO}(2)$-invariant subspace and
$J : \underline{H} \to \underline{H}$ an $Ad\mathbf{SO}(2)$-equivariant complex structure on the subspace $\underline{H} \subset \mathbf{so}(4)$ given by

$$E_{13} \mapsto -E_{14},\ E_{14} \mapsto E_{13},\ E_{23} \mapsto -E_{24},\ E_{24} \mapsto E_{23}.$$

The horizontal subbundle in $T\mathbf{SO}(4)$ with respect to the Levi-Civita connection of the standard sphere $\mathbf{S}^3 \cong \mathbf{SO}(4)/\mathbf{SO}(3)$ is given by

$$T_H\mathbf{SO}(4) = \mathbf{SO}(4) \times \underline{K} \subset T\mathbf{SO}(4),\ \left.\frac{d}{dt}\right|_{t=0} A \exp tk = (A, k).$$

The canonical CR-structure on $\mathbf{SO}(4)/\mathbf{SO}(2) \cong Z(\mathbf{S}^3)$ is given by

$$H^{Z(\mathbf{S}^3)} = \mathbf{SO}(4) \times_{Ad\mathbf{SO}(2)} \underline{H} \subset T(\mathbf{SO}(4)/\mathbf{SO}(2)),\ [A, h] \leftrightarrow \left.\frac{d}{dt}\right|_{t=0} A \exp th\mathbf{SO}(2),$$

$$J^{Z(\mathbf{S}^3)} : H^{Z(\mathbf{S}^3)} \to H^{Z(\mathbf{S}^3)},\ J^{Z(\mathbf{S}^3)}([A, h]) = [A, Jh],$$

$[A, h] = \hat{h} \in H^{Z(\mathbf{S}^3)}$, $(A \in \mathbf{SO}(4), h \in H)$. It holds: $[E_{13}, E_{23}] = E_{12}$. Therefore, the distribution $H^{Z(\mathbf{S}^3)}$ isn't involutive.

2. $\mathbf{SO}^+(4,1)$-invariant CR-structure on $\mathbf{SO}^+(4,1)/\mathbf{SO}(2)$:
   Let denote $\mathbf{SO}^+(4,1)$ the connected component of the unit in the group $\mathbf{SO}(4,1)$. The hyperbolic space $\mathbf{H}^3$ is described as homogeneous space by $\mathbf{SO}^+(4,1)/\mathbf{SO}(3)$. The frame bundle $SO(\mathbf{H}^3)$ is identified with the group $\mathbf{SO}^+(4,1)$. The bundle of unit spheres is diffeomorphic to

$$Z(\mathbf{H}^3) \cong \mathbf{SO}^+(4,1)/\mathbf{SO}(2).$$

   The hyperbolic space $\mathbf{H}^3$ is diffeomorphic to $\mathbf{R}^3$ and the bundle $Z(\mathbf{H}^3) \cong \mathbf{R}^3 \times \mathbf{S}^2$ is trivial.
   The canonical CR-structure on $Z(\mathbf{H}^3) \cong \mathbf{SO}^+(4,1)/\mathbf{SO}(2)$ is invariantly defined

to the $\mathbf{SO}^+(4,1)$-action on $Z(\mathbf{H}^3)$. On $\mathbf{SO}^+(4,1)/\mathbf{SO}(2)$, the $\mathbf{SO}^+(4,1)$-invariant CR-structure $(H^{Z(\mathbf{H}^3)}, J^{Z(\mathbf{H}^3)})$ is described in the following way:
Let be

$\{E_{ij} : 1 \leq i < j \leq 4\}$ the standard basis in $\mathbf{so}(4,1)$, $(E_{ij} = \begin{pmatrix} & \cdot & & \cdot & \\ \cdot & 0 & \cdot & -\epsilon_j & \cdot \\ & \cdot & & \cdot & \\ \cdot & \epsilon_i & \cdot & 0 & \cdot \\ & \cdot & & \cdot & \end{pmatrix})$,

$i : \mathbf{SO}(2) \hookrightarrow \mathbf{SO}^+(4,1)$, $\begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \mapsto \exp tE_{34}$ the imbedding,

$\underline{K} := Span\{E_{12}, E_{13}, E_{14}\} \subset \mathbf{so}(4,1)$ an $Ad\mathbf{SO}(3)$-invariant subspace,

$$\underline{H} := Span\{E_{13}, E_{14}, E_{23}, E_{24}\} \subset \mathbf{so}(4,1)$$

an $Ad\mathbf{SO}(2)$-invariant subspace and
$J : \underline{H} \to \underline{H}$ an $Ad\mathbf{SO}(2)$-equivariant complex structure on the subspace $\underline{H} \subset \mathbf{so}(4,1)$ given by

$$E_{13} \mapsto -E_{14},\ E_{14} \mapsto E_{13},\ E_{23} \mapsto -E_{24},\ E_{24} \mapsto E_{23}.$$

The horizontal bundle in $T\mathbf{SO}^+(4,1)$ with respect to the Levi-Civita connection of the hyperbolic space $\mathbf{H}^3 \cong \mathbf{SO}^+(4,1)/\mathbf{SO}(3)$ is given by

$$T_H\mathbf{SO}^+(4,1) = \mathbf{SO}^+(4,1) \times \underline{K} \subset T\mathbf{SO}^+(4,1).$$

The canonical CR-structure on $\mathbf{SO}^+(4,1)/\mathbf{SO}(2) \cong Z(\mathbf{H}^3)$ is given by

$$H^{Z(\mathbf{H}^3)} = \mathbf{SO}^+(4,1) \times_{Ad\mathbf{SO}(2)} \underline{H} \subset T(\mathbf{SO}^+(4,1)/\mathbf{SO}(2)),$$

$$J^{Z(\mathbf{H}^3)} : H^{Z(\mathbf{H}^3)} \to H^{Z(\mathbf{H}^3)},\ J^{Z(\mathbf{H}^3)}([A,h]) = [A, Jh],$$

$[A,h] = \hat{h} \in H^{Z(\mathbf{H}^3)}$, $(A \in \mathbf{SO}^+(4,1), h \in H)$.

Remark: Let $T_{1,0}$ be a CR-structure on M^{2n+1}. If (locally) exists an imbedding of $(M^{2n+1}, T_{1,0})$ into $\mathbf{C}^{n+1}$, so that $T_{1,0}$ is induced in the canonical way then we say the CR-structure $T_{1,0}$ is (locally) realized on a hypersurface in $\mathbf{C}^{n+1}$. There are CR-structures, which aren't even locally realizable (see [Jac90]).

Definition 3.4 *A complex valued function f on M is a CR-function on $(M, T_{1,0})$ if $Lf = 0$ for all sections $L : M \to T_{0,1}$.*

The restriction of holomorphic functions provides CR-functions:

Lemma 3.1 ([Jac90, chap. 1.1]) *Let M^{2n+1} be a hypersurface in $\mathbf{C}^{n+1}$. Let F be a holomorphic function on $\mathbf{C}^{n+1}$. The restriction $f = F|_{M^{2n+1}}$ is a CR-function on M^{2n+1}.*

But not any CR-function f on $M^{2n+1} \subset \mathbf{C}^{n+1}$ is restriction of a holomorphic function: Each function $f(x_2)$ on $\mathbf{R}^3 \subset \mathbf{C}^2$ is a CR-function.
There exist holomorphic functions on $\{(z_1, z_2) : |z_1|^2 + |z_2|^2 < 1\} \subset \mathbf{C}^2$, which can be smoothly extended to the boundary $\mathbf{S}^3$, so that such a function restricted to the boundary is smooth but not real analytic. The restricted function is a CR-function on $\mathbf{S}^3$ and isn't extendible to a holomorphic function (see [Jac90, p. 17-18]).

Proposition 3.2 ([Jac90, chap. 1.1]) *Let M^{2n+1} be a real C^ω-submanifold in $\mathbf{C}^{n+1}$ and f a real analytic CR-function on M^{2n+1}. Then f is locally in every point the restriction of a holomorphic function to M^{2n+1}.*

3.2 Optical structures

Now we are going to introduce optical structures on C^∞-manifolds. First, we develop the concept on vector spaces.

Definition 3.5 *An optical structure on a vector space V^{2n} exists on*

1. *an optical flag, that means a pair (K, L) of subspaces such that* $K \subset L \subset V$ *and* $\dim K = 1,\ \dim L = 2n - 1,$

2. *a complex structure $J : L/K \to L/K$ on the $(2n-2)$-dimensional quotient space L/K.*

The optical structure is denoted by (K, L, J).

On the dual space V^*, a canonical optical structure is given. Its flag is (L^0, K^0), where the subspace $W^0 \subset V^*$ to $W \subset V$ is the set of all forms that vanish on W. Then

$$L^0 \subset K^0,\ \dim L^0 = 1 \text{ and } \dim K^0 = 2n - 1.$$

The space K^0/L^0 is canonical isomorphic to $(L/K)^*$ and the dual mapping $J^* : (L/K)^* \to (L/K)^*$ to J defines a complex structure on K^0/L^0.

Notations: *Let $0 \neq k \in K,\ 0 \neq \kappa \in L^0$.*

1. *A scalar product g on a vector space V with optical structure (K, L, J) is adapted (to the flag) if $g(k, \cdot) \wedge \kappa = 0$. That claim is equivalent to: K is null in (V, g) and L is the orthogonal complement of K.*

2. *A form $\alpha \in \Lambda V^*$ is adapted if $\iota_k \alpha = 0$ and $\kappa \wedge \alpha = 0$. For an adapted form α we have $\alpha = \kappa \wedge \beta$ with $\beta \in \Lambda(K^0)$.*
   The space of adapted i-forms ($1 \leq i \leq n$) is denoted by L^{i-1}. L^{i-1} is canonically isomorphic to $L^0 \otimes \Lambda^{i-1}(K^0/L^0)$:
   $L^{i-1} \cong L^0 \otimes \Lambda^{i-1}(K^0/L^0),\ \kappa \wedge \beta \mapsto \kappa \otimes p(\beta)$, *where $p : \Lambda K^0 \to \Lambda(K^0/L^0)$ is the canonical projection.*

3. *Let (V^{2n}, g) be a vector space with scalar product g, index $g = 1$. Let (K, L, J) be an optical structure on V^{2n} so that K is a null direction in V with respect to g and $L = K^{\perp_g}$. By $\hat{g}(l_1 + K, l_2 + K) := g(l_1, l_2)$, $l_1, l_2 \in L$, we define a positive definite scalar product on L/K. If J is orthogonal with respect to $\hat{g}$, $\hat{g}(J\cdot, J\cdot) = \hat{g}(\cdot, \cdot)$, we say that (K, L, J) is an orthogonal optical structure on (V^{2n}, g).*
   Conversely, let V^{2n} be a vector space with optical structure (K, L, J). A scalar product g with index $g = 1$ on V is optical if g is adapted to the flag (K, L) and J is orthogonal with respect to g, consequently if (K, L, J) is orthogonal with respect to g.

Now, let $(V^{4,1}, g)$ be a 4-dimensional oriented Lorentzian vector space. Let K be null in V with respect to g. A natural optical structure on V is given. Its flag is $(K, K^{\perp_g})$. The complex structure on $K^{\perp_g}/K$ is given by the the Hodge operator $*$:
Let $\alpha \in L^{i-1}$ be adapted to the flag (K, L). From $\iota_v(*\gamma) = *(\gamma \wedge g(v, \cdot))$ for all $v \in V$, $\gamma \in \Lambda V^*$, it follows that $*\alpha \in \Lambda^{4-i}V^*$ is also adapted. Therefore, the Hodge operator $*$, restricted to the adapted 2-forms L^1, is a complex structure $* : L^1 \to L^1$. Via the identification $L^1 \cong L^0 \otimes \Lambda^1(K^0/L^0)$, the dual mapping $\hat{*}$ to $*$ is a complex structure $\hat{*} : L/K \to L/K$ on L/K. $(L, K, \hat{*})$ is orthogonal.

Definition 3.6 *A real almost optical structure on a C^∞-manifold M^{2n} exists on*

1. *a flag geometry, that means a pair $(\mathcal{K}, \mathcal{L})$ of subbundles such that $\mathcal{K} \subset \mathcal{L} \subset TM$ and $\dim \mathcal{K} = 1$, $\dim \mathcal{L} = 2n - 1$,*

2. *an almost complex structure $J : \mathcal{L}/\mathcal{K} \to \mathcal{L}/\mathcal{K}$ on the $(2n-2)$-dimensional quotient bundle $\mathcal{L}/\mathcal{K}$.*

The real almost optical structure is denoted by $(\mathcal{K}, \mathcal{L}, J)$. $\mathrm{pr} : \mathcal{L} \to \mathcal{L}/\mathcal{K}$ denotes the natural projection. $\mathcal{L}/\mathcal{K}$ is also called screen space.

Definition 3.7 *A complex almost optical structure on a C^∞-manifold M^{2n} is a complex bundle $\mathcal{N}_{1,0} \subset \mathbf{C} \otimes TM$ such that*

$$\dim_{\mathbf{C}} \mathcal{N}_{1,0} = n \text{ and } \dim_{\mathbf{C}}(\mathcal{N}_{1,0} \cap \overline{\mathcal{N}}_{1,0}) = 1.$$

Notation: $\mathcal{N}_{0,1} := \overline{\mathcal{N}}_{1,0}$.

Between real and complex almost optical structures exist a one-to-one corresponce:
Let $(\mathcal{K}, \mathcal{L}, J)$ be a real optical structure. Then the bundle

$$\mathcal{N}_{1,0} := \mathrm{pr}^{-1}(\{X - iJX \in \mathbf{C} \otimes \mathcal{L}/\mathcal{K} : \ X \in \mathcal{L}/\mathcal{K}\}) \subset \mathbf{C} \otimes TM$$

is a complex almost optical structure. It holds:

$$\mathcal{N}_{0,1} = \mathrm{pr}^{-1}(\{X + iJX : \ X \in \mathcal{L}/\mathcal{K}\}) \text{ and } \mathcal{N}_{1,0} \cap \mathcal{N}_{0,1} = \mathbf{C} \otimes \mathcal{K}.$$

On the other side, let $\mathcal{N}_{1,0}$ be a complex almost optical structure so is

$$\mathcal{K} := Re(\mathcal{N}_{1,0} \cap \mathcal{N}_{0,1}), \ \mathcal{L} := Re(\mathcal{N}_{1,0} + \mathcal{N}_{0,1}),$$

$$J : \mathcal{L}/\mathcal{K} \to \mathcal{L}/\mathcal{K},\ l + \mathcal{K}_m = n + \overline{n} + \mathcal{K}_m \mapsto i(n - \overline{n}) + \mathcal{K}_m,$$

$m \in M,\ n \in \mathcal{N}_{1,0m},\ l \in \mathcal{L}_m$, a real optical structure.

Notations:

1. *Let $(\mathcal{L}^0, \mathcal{K}^0, J^*)$ be the dual almost optical structure on T^*M.*
2. *A metric tensor g on M is called adapted if $g(k, \cdot) \wedge \kappa = 0$ for all sections $k \in \Gamma(\mathcal{K})$ and $\kappa \in \Gamma(\mathcal{L}^0)$.*
3. *A form $\alpha \in \Lambda^* M$ is called adapted if $\iota_k \alpha = 0$ and $\kappa \wedge \alpha = 0$ for all sections $k \in \Gamma(\mathcal{K}),\ \kappa \in \Gamma(\mathcal{L}^0)$. $\mathcal{L}^{i-1}$ denotes the bundle of adapted i-forms.*
4. *Let (M^{2n}, g) be a Lorentzian manifold. An almost optical structure $(\mathcal{K}, \mathcal{L}, J)$ is called orthogonal if $\mathcal{K}$ is a null subbundle, $\mathcal{L} = \mathcal{K}^{\perp_g}$ is the orthogonal complement and $J : \mathcal{L}/\mathcal{K} \to \mathcal{L}/\mathcal{K}$ is orthogonal with respect to the Riemannian metric $\hat{g}$ on $\mathcal{L}/\mathcal{K}$.*
   Conversely, let g be a Lorentzian metric on a C^∞-manifold M with almost optical structure $(\mathcal{K}, \mathcal{L}, J)$ so that $(\mathcal{K}, \mathcal{L}, J)$ is orthogonal in g. Then we say that g is an optical metric.
5. *Let $\mathcal{E}$ denote the set of optical metrics on a C^∞-manifold with almost optical structure. Two metrics $g, \tilde{g} \in \mathcal{E}$ are optical equivalent. Conformally equivalent metrics are always optical equivalent.*

Optical metrics exist on every oriented C^∞-manifold M^{2n} with optical structure. In the case that M isn't orientable this need not to be true.
If $(M^{4,1}, g)$ is an oriented Lorentzian manifold and $\mathcal{K}$ is a null subbundle over M than

$$(\mathcal{K}\ ,\ \mathcal{K}^\perp\ ,\ \hat{*} : \mathcal{K}^\perp/\mathcal{K} \to \mathcal{K}^\perp/\mathcal{K})$$

is a natural orthogonal almost optical structure on M^4.
For M^4 a 4-dimensional oriented C^∞-manifold with almost optical structure $(\mathcal{K}, \mathcal{L}, J)$ and g an optical metric on M, it holds:

$$\hat{*} = J \text{ or } \hat{*} = -J.$$

Proposition 3.3 *Let $(M^{2n,1}, g)$ be a Lorentzian manifold (index $g = 1$) and let*

$$g : \mathbf{C} \otimes TM \times \mathbf{C} \otimes TM \to \mathbf{C}$$

be the $\mathbf{C}$-linear extension of g.

1. *Let $(\mathcal{K}, \mathcal{L}, J)$ be an orthogonal almost optical structure on M^{2n}. Then the corresponding complex almost optical structure $\mathcal{N}_{1,0}$ is a maximal totally null subbundle in $\mathbf{C} \otimes TM$ that means $\mathcal{N}_{1,0}$ is totally null and isn't properly inclused in a totally null subbundle of $\mathbf{C} \otimes TM$.*

2. *Let $\mathcal{N}_{1,0} \subset \mathbf{C} \otimes TM$ be maximal totally null then $\mathcal{N}_{1,0} (\simeq (\mathcal{K}, \mathcal{L}, J))$ is an orthogonal almost optical structure on M.*

Proof: 1. Since J is orthogonal, it follows

$$\hat{g}(X - iJX, Y - iJY) = \hat{g}(X, Y) - \hat{g}(JX, JY) - i(\hat{g}(X, JY) + \hat{g}(JX, Y)) = 0$$

for all $X, Y \in \mathcal{L}/\mathcal{K}$. Therefore, $\mathcal{N}_{1,0}$ is totally null. The maximality follows by a dimension argument.

2. Let $\mathcal{N}_{1,0}$ be maximal totally null, it follows $\dim_{\mathbf{C}} \mathcal{N}_{1,0} = n$. Besides, it exists exactly one 1-dimensional null bundle $\mathcal{K} \subset TM$ such that $\mathbf{C} \otimes \mathcal{K} \subset \mathcal{N}_{1,0}$. Then $\mathcal{N}_{1,0} \cap \overline{\mathcal{N}}_{1,0} = \mathbf{C} \otimes \mathcal{K}$. From $\mathcal{N}_{1,0}$, $\overline{\mathcal{N}}_{1,0} \perp (\mathcal{N}_{1,0} \cap \overline{\mathcal{N}}_{1,0})$, it follows

$$Re(\mathcal{N}_{1,0} + \overline{\mathcal{N}}_{1,0}) \perp Re(\mathcal{N}_{1,0} \cap \overline{\mathcal{N}}_{1,0}).$$

Therefore, it is $\mathcal{L} = Re(\mathcal{N}_{1,0} + \overline{\mathcal{N}}_{1,0}) = \mathcal{K}^{\perp}$ and J is orthogonal:

$$\begin{aligned} \hat{g}(J(l_1 + \mathcal{K}_m), J(l_2 + \mathcal{K}_m)) &= g(J(n_1 + \overline{n}_1), J(n_2 + \overline{n}_2)) \\ &= g(i(n_1 - \overline{n}_1), i(n_2 - \overline{n}_2)) = g(n_1, \overline{n}_2) + g(\overline{n}_1, n_2) \\ &= g(n_1 + \overline{n}_1, n_2 + \overline{n}_2) = \hat{g}(l_1 + \mathcal{K}_m, l_2 + \mathcal{K}_m), \end{aligned}$$

$m \in M,\ l_1, l_2 \in \mathcal{L}_m,\ n_1, n_2 \in \mathcal{N}_{1,0m}$ □

Now, we define step by step the integrability of an almost complex structure $(\mathcal{K}, \mathcal{L}, J)$ on a C^∞-manifold M. Let $X \in \Gamma(TM)$ be a vector field. It denotes ϕ^X the flow, which is generated by the field X.

Definition 3.8 *An almost optical structure $(\mathcal{K}, \mathcal{L}, J)$ is called geodesic if*

$$a) \quad [\Gamma(\mathcal{K}), \Gamma(\mathcal{L})] \subset \Gamma(\mathcal{L}).$$

This is a property of the flag geometry. Condition a) means that $\mathcal{L}$ is invariant to the flow ϕ^k of an arbitrary section in $k \in \Gamma(\mathcal{K})$, $\phi^k_{t*}\mathcal{L} = \mathcal{L}\ \forall t$.

Lemma 3.4 [Tra85] *Let M be a C^∞-manifold with almost optical structure $(\mathcal{K}, \mathcal{L}, J)$ and let g be an adapted metric on M. $(\mathcal{K}, \mathcal{L}, J)$ is geodesic if and only if the integral curve of an arbitrary section $k \in \Gamma(\mathcal{K})$ is a pregeodesic on (M, g).*

Now, let M be a C^∞-manifold with a geodesic almost optical structure $(\mathcal{K}, \mathcal{L}, J)$, which in addition satisfies the condition

$$\text{b)} \quad \phi^{k*}_t J = J\ \forall t,\ k \in \Gamma(\mathcal{K}).$$

The 1-dimensional bundle $\mathcal{K}$ over M generates an equivalence relation on M:
Two points $m, m' \in M$ are defined to be in relation, $m \sim^{\mathcal{K}} m'$, if and only if a section $k \in \Gamma(\mathcal{K})$ exists such that $\phi^k_t(m) = m'$ for a $t \in \mathbf{R}$ in the domain of the flow. In words, two points in M are in relation if and only if both lie on the integral curve of a field $k \in \Gamma(\mathcal{K})$.

The bundle $\mathcal{K}$ is called regular if the quotient space $N = M/{\sim}^{\mathcal{K}}$ admits a natural structure of a $(2n-1)$-dimensional manifold such that the canonical projection $\tilde{\pi} : M \to N$ is a submersion.
To every point $m \in M$ exists an open subset $U_m \subset M$ such that the restriction $\mathcal{K}|_{U_m}$ of the bundle is regular over U_m. Then is $U_m/{\sim}^{\mathcal{K}}$ a $(2n-1)$-dimensional manifold.
The almost optical structure $(\mathcal{K}, \mathcal{L}, J)$ on $U_m/{\sim}^{\mathcal{K}}$ induces an almost CR-structure:
From condition a), it follows that $H' := \tilde{\pi}_*(\mathcal{L}|_{U_m}) \subset T(U_m/{\sim}^{\mathcal{K}})$ is a $(2n-2)$-dimensional bundle over $U_m/{\sim}^{\mathcal{K}}$. Because of the condition $\phi_t^{k*}J = J$, the mapping J projects to an almost complex structure J' on the bundle H'. Then (H', J') is an almost CR-structure on $U_m/{\sim}^{\mathcal{K}}$.
In the spezial case that the bundle $\mathcal{K}$ is regular over M the almost optical structure $(\mathcal{K}, \mathcal{L}, J)$ induces canonically an almost CR-structure on the $(2n-1)$-dimensional manifold $M/{\sim}^{\mathcal{K}}$.

Definition 3.9 *An almost optical structure $(\mathcal{K}, \mathcal{L}, J)$ on M is integrable if*

a) $[\Gamma(\mathcal{K}), \Gamma(\mathcal{L})] \subset \Gamma(\mathcal{L})$,

b) $\phi_t^{k*}J = J \ \forall t,\ k \in \Gamma(\mathcal{K})$,

c) for all $m \in M$ the canonically induced almost CR-structure on $U_m/{\sim}^{\mathcal{K}}$ is integrable.

Then $(\mathcal{K}, \mathcal{L}, J)$ is an optical structure.

Remarks:

1. Let $\mathcal{T} \subset \mathcal{L}$ be a $(2n-2)$-dimensional bundle over M such that $\mathcal{K} \oplus \mathcal{T} = \mathcal{L}$. The bundles $\mathcal{L}/\mathcal{K} \cong \mathcal{T}$ are identified by the projection $\mathrm{pr} : \mathcal{L} \to \mathcal{L}/\mathcal{K}$ and an almost complex structure is induced on $\mathcal{T}$. On $\mathcal{L}$, we have the linear bundle morphism $J^{\mathcal{T}} : \mathcal{L} \to \mathcal{L}$ given by
$$J^{\mathcal{T}}|_{\mathcal{T}} \cong J,\ J^{\mathcal{T}}|_{\mathcal{K}} = 0.$$
The integrability conditions b) and c) can equivalently expressed by
($\pounds$ denotes the Lie derivative)

   b') $\quad (\pounds_k J^{\mathcal{T}})(l) \in \mathcal{K} \quad \forall\, l \in \mathcal{L},\ k \in \Gamma(\mathcal{K})$ and

   c')
$$J^{\mathcal{T}}([J^{\mathcal{T}}l_1, l_2] + [l_1, J^{\mathcal{T}}l_2]) - [J^{\mathcal{T}}l_1, J^{\mathcal{T}}l_2] + [l_1, l_2] \in \Gamma(\mathcal{K}),$$
$$[J^{\mathcal{T}}l_1, l_2] + [l_1, J^{\mathcal{T}}l_2] \in \Gamma(\mathcal{L}), \quad \forall\, l_1, l_2 \in \Gamma(\mathcal{L}).$$

2. The integrability conditions for complex almost optical structures $\mathcal{N}_{1,0}$ are equivalently expressed by
$$[\Gamma(\mathcal{N}_{1,0}), \Gamma(\mathcal{N}_{1,0})] \subset \Gamma(\mathcal{N}_{1,0}).$$
This can be proofed as similar as for CR-structures.

Examples:

1. On a 4-dimensional oriented Lorentzian manifold $(M^{4,1}, g)$ a 1-dimensional null bundle $\mathcal{K}$ generates the almost optical structure $(\mathcal{K}, \mathcal{K}^{\perp}, \hat{*})$.

2. Let N^{2n-1} be a manifold with almost CR-structure (H, J). The $2n$-dimensional manifold $M := \mathbf{R} \times N$ admits the almost optical structure

$$(T\mathbf{R}\ ,\ T\mathbf{R} \times H\ ,\ J : T\mathbf{R} \times H/T\mathbf{R} \cong H \to T\mathbf{R} \times H/T\mathbf{R} \cong H).$$

3. In a later section, we will introduce a canonical almost optical structure on the 6-dimensional twistor space of 4-dimensional Lorentzian manifolds.

Notations: *Let M_1 and M_2 be C^∞-manifolds with almost optical structures $(\mathcal{K}_1, \mathcal{L}_1, J_1)$ resp. $(\mathcal{K}_2, \mathcal{L}_2, J_2)$. A diffeomorphism $\phi : M_1 \to M_2$ is an optical diffeomorphism if*

$$\phi_* \mathcal{K}_1 = \mathcal{K}_2,\ \phi_* \mathcal{L}_1 = \mathcal{L}_2 \text{ and } \phi^* J_2 = J_1.$$

Then we denote these both optical structures on M_1 resp. M_2 as equivalent.
If an optical diffeomorphism exists between neighborhoods of points $m_1 \in M_1$ and $m_2 \in M_2$, then we say that the (almost) optical structures in the points $m_1 \in M_1$ and $m_2 \in M_2$ are locally equivalent.

From the definition of integrability of an almost optical structure, it follows that an optical structure $(\mathcal{K}, \mathcal{L}, J)$ on a manifold M^{2n} is locally in every point $m \in M$ optical diffeomorphic to $\mathbf{R} \times N_m$, where N_m is a CR-manifold and $\mathbf{R} \times N_m$ admits the optical structure of example 2. In particular, optical structures in a point $m \in M$ are locally equivalent if and only if the locally induced CR-structures in $\tilde{\pi}(m)$ are locally equivalent.

Definition 3.10 *Let M^{2n} be a manifold with almost optical structure $(\mathcal{K}, \mathcal{L}, J)$. An optical function on M is a complex valued function $f : M \to \mathbf{C}$ such that*

$$\begin{array}{ll} i) & df(k) = 0 \quad \forall\ k \in \mathcal{K}, \\ ii) & df(l + \mathcal{K}_m + iJ(l + \mathcal{K}_m)) = 0 \quad \forall\ l \in \mathcal{L}_m,\ m \in M. \end{array}$$

(Condition $ii)$ makes sense only if condition $i)$ is already satisfied.)

Let $(\mathcal{K}, \mathcal{L}, J)$ be an optical structure on M and let $\mathcal{K}$ be regular over M. Then $\tilde{\pi} : M \to N = M/{\sim^{\mathcal{K}}}$ is a submersion and an optical function f is constant on the submanifolds $\tilde{\pi}^{-1}(n),\ n \in N$. Hence, $f \circ \tilde{\pi}^{-1} : N \to \mathbf{C}$ is a well defined CR-function on N with the induced CR-structure (H', J'). Optical functions on M are in one-to-one correspondence to CR-functions on $M/{\sim^{\mathcal{K}}}$.

Finally, we give an important result on the existence of Maxwell fields on a 4-dimensional manifold with almost optical structure.
A 2-form F on a 4-dimensional oriented Lorentzian manifold, which satisfies $dF = 0$ and $d{*}F = 0$, is a solution of the Maxwell equations.
Let M^4 be an oriented 4-dimensional manifold. A geodesic almost optical structure $(\mathcal{K}, \mathcal{L}, J)$ on M is also denoted as a geometry of rays and an integrable almost optical structure $(\mathcal{K}, \mathcal{L}, J)$ is a geometry of rays without shear.

Let $g, \tilde{g} \in \mathcal{E}$ be optical metrics on M^4 with almost optical structure $(\mathcal{K}, \mathcal{L}, J)$. Then we have

$$\hat{*}_g = \hat{*}_{\tilde{g}} : \mathcal{L}^1 \to \mathcal{L}^1.$$

If $F \in \Lambda^2 M$ is an adapted solution of the Maxwell equation on (M, g), (F is null), then F is also an adapted solution of the Maxwell equation on $(M, \tilde{g})$. In this sense $F \in \mathcal{L}^1$ is a solution of the Maxwell equation to the optical geometry $(\mathcal{K}, \mathcal{L}, J)$ on M.

Proposition 3.5 (Ivor Robinson) [Rob59], [Tra85] or [RT87] *Let M be a 4-dimensional oriented C^∞-manifold with optical structure $(\mathcal{K}, \mathcal{L}, J)$.*

1. *If the optical geometry on M admits a nowhere vanishing adapted solution $F \in \mathcal{L}^1$ of the Maxwell equations, ($dF = 0$, $dJF = 0$), then $(\mathcal{K}, \mathcal{L}, J)$ is a geometry of rays without shear.*

2. *If $\mathcal{K}$ is regular over M, $M/\sim^{\mathcal{K}}$ is realizable in $\mathbf{C}^2$ and $(\mathcal{K}, \mathcal{L}, J)$ is a geometry of rays without shear then M admits an adapted nowhere vanishing solution $F \in \mathcal{L}^1$ of the Maxwell equations.*

4 The twistor space of a Lorentzian manifold

The twistor space $\mathcal{Z}(M)$ of a 4-dimensional oriented Lorentzian manifold M is the set of all null directions in the tangent space TM of the Lorentzian manifold. $\mathcal{Z}(M) \subset \mathbf{P}(TM)$ is a submanifold of the projective tangent space and is a locally trivial fibre bundle over M. The fibres are diffeomorphic to the sphere $\mathbf{S}^2$. Also, the twistor space can be understand as the bundle of the orthogonal optical structures on the tangent space of M. In this point of view the twistor space is handled in [Nur96], [Nur97] and [MS94].
In this section, we describe the twistor space as projective spinor bundle $\mathbf{P}(S^+)$ (resp. $\mathbf{P}(S^-)$).

4.1 The group $\mathbf{Spin}(4,1)$, spinor representation and spinor bundle

We represent now some basic facts of spin geometry. An detailed treatment of this theme can be find e.g. in [Baum81].

Let $(e_1, \dots, e_4)$ be the standard basis of the Minkowski space. $\mathbf{R}^4_1 := (\mathbf{R}, <,>_1)$ $(<e_1, e_1>_1 = -1)$. $\epsilon_k := <e_k, e_k>_1$. $\mathcal{C}_{4,1} := Cliff(\mathbf{R}^4, -<,>_1)$ denotes the Clifford algebra of the Minkowski space. $\mathcal{C}_{4,1} \cong \Lambda \mathbf{R}^{4*}_1$ is 16-dimensional and is multiplicative generated by $\{1, e_1, \dots, e_4\}$ and the relations

$$\begin{aligned} &e_i^2 := e_i \cdot e_i = -\epsilon_i, \\ &e_i \cdot e_j + e_j \cdot e_i = 0 \;\forall i \neq j. \end{aligned}$$

The complexification $\mathcal{C}^{\mathbf{C}}_{4,1} := \mathcal{C}_{4,1} \otimes_{\mathbf{R}} \mathbf{C}$ of this algebra is isomorphic to the matrix algebra $M(4, \mathbf{C})$ (see [Baum81, p. 51]):

Let

$$g_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \; g_2 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \; T = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \text{ and } E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The isomorphism is explicitly given by

$$\begin{aligned} \Phi_{4,1} : \mathcal{C}^{\mathbf{C}}_{4,1} &\to M(4, \mathbf{C}), \\ e_1 \mapsto iE \otimes g_1, &\;\; e_2 \mapsto E \otimes g_2, \\ e_3 \mapsto g_1 \otimes T, &\quad e_4 \mapsto g_2 \otimes T. \end{aligned}$$

(The sign $\otimes$ denotes the Kronecker product for matrices.)
Let $\rho : \mathbf{Spin}(4,1) \to \mathbf{SO}(4,1)$ be the universal covering of the Lorentz group. The group $\mathbf{Spin}(4,1)$ is realized in $\mathcal{C}_{4,1}$ by

$$\mathbf{Spin}(4,1) = \{x_1 \cdot \ldots \cdot x_{2l} |\; x_i \in \mathbf{R}^4_1, \; < x_i, x_i >_1 = \pm 1\} \subset \mathcal{C}_{4,1}.$$

For $s \in \mathbf{Spin}(4,1)$ and $x \in \mathbf{R}^4_1$, it holds $s \cdot x \cdot s^{-1} \in \mathbf{R}^4_1$. The covering ρ is given by

$$\rho(s)x = s \cdot x \cdot s^{-1}.$$

Let $\mathbf{Spin}^+(4,1)$ and $\mathbf{SO}^+(4,1)$ denote the connected components of the units. Then $\pi_1(\mathbf{SO}^+(4,1)) = \mathbf{Z}_2$ and ρ is a 2-fold covering.
The group $\mathbf{Spin}^+(4,1)$ can also be realized in another way (comp. [Baum81, p. 60-61]): Consider the group $\mathbf{SL}(2,\mathbf{C})$. $\mathbf{SL}(2,\mathbf{C})$ is simple connected. Let

$$\tau_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \; \tau_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \; \tau_3 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \text{ and } \tau_4 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

We can identify $x = (x_1, x_2, x_3, x_4) \in \mathbf{R}^4_1$ with the Hermitian matrix

$$\tilde{x} = x_1\tau_1 + x_2\tau_2 + x_3\tau_3 + x_4\tau_4,$$

and $\det \tilde{x} = - < x, x >_1$ holds. The mapping

$$\lambda : A \in \mathbf{SL}(2,\mathbf{C}) \mapsto \lambda(A) \in \mathbf{SO}^+(4,1), \; \lambda(A)x := A\tilde{x}\overline{A}^{\perp},$$

is a 2-fold covering. Therefore, $\mathbf{SL}(2,\mathbf{C})$ and $\mathbf{Spin}^+(4,1)$ are isomorphic.
By restriction of $\Phi_{4,1}$ to $\mathbf{Spin}(4,1)$, we obtain a complex 4-dimensional representation $\Delta_{4,1}$ of the group $\mathbf{Spin}(4,1)$, the spinor representation. The spinor representation $\Delta_{4,1}$ decomposes into the two irreducible $\mathbf{Spin}(4,1)$-representations $\Delta^+_{4,1}$ and $\Delta^-_{4,1}$ of complex dimension 2. The $\mathbf{Spin}^+(4,1)$-representations $\Delta^+_{4,1}|_{\mathbf{Spin}^+(4,1)}$, $\Delta^-_{4,1}|_{\mathbf{Spin}^+(4,1)}$ are isomorphic to the canonical $\mathbf{SL}(2,\mathbf{C})$-representation $\mathbf{C}^2$.
Let $x \in \mathbf{R}^4_1$ and $u \in \Delta_{4,1}$. The Clifford multiplication is given by

$$\mu : \mathbf{R}^4_1 \otimes \Delta_{4,1} \to \Delta_{4,1}, \; x \otimes u \mapsto \Phi(x)u.$$

If $\mathbf{Spin}(4,1)$ acts via ρ on $\mathbf{R}^4_1$ then the Clifford multiplication $\mu : \mathbf{R}^4_1 \otimes \Delta_{4,1} \to \Delta_{4,1}$ is a homomorphism of the $\mathbf{Spin}(4,1)$-representation, which permutes the decomposition $\Delta_{4,1} = \Delta^+_{4,1} \oplus \Delta^-_{4,1}$:

$$\mu : \mathbf{R}^4_1 \otimes \Delta^+ \to \Delta^-, \ \mu : \mathbf{R}^4_1 \otimes \Delta^- \to \Delta^+.$$

The Lie algebra $\mathbf{spin}(4,1)$ of the group $\mathbf{Spin}(4,1)$ is realized in $\mathcal{C}_{4,1}$ by

$$Span\{e_i \cdot e_j : 1 \leq i < j \leq 4\}.$$

Let $\{E_{ij} : \ i \leq j\}$ be the standard basis of $\mathbf{so}(4,1)$. The differential $\rho_* : \mathbf{spin}(4,1) \to \mathbf{so}(4,1)$ is given by

$$\rho_*(e_i \cdot e_j) = 2E_{ij}.$$

For the differential of the representation Φ holds: $\Phi_* = \Phi$.
Let $u_1 = \left(\begin{smallmatrix} 1 \\ -i \end{smallmatrix}\right) \in \mathbf{C}^2$, $u_{-1} = \left(\begin{smallmatrix} 1 \\ i \end{smallmatrix}\right) \in \mathbf{C}^2$ and $u(\delta_1, \delta_2) := u_{\delta_1} \otimes u_{\delta_2} \in \mathbf{C}^4$, $\delta_1, \delta_2 = \pm 1$. Then

$$u(1,1) = \begin{pmatrix} 1 \\ -i \\ -i \\ -1 \end{pmatrix}, \ u(1,-1) = \begin{pmatrix} 1 \\ i \\ -i \\ 1 \end{pmatrix}, \ u(-1,1) = \begin{pmatrix} 1 \\ -i \\ i \\ 1 \end{pmatrix}, \ u(-1,-1) = \begin{pmatrix} 1 \\ i \\ i \\ -1 \end{pmatrix},$$

$$\begin{aligned} \Delta^+_{4,1} &= Span_{\mathbf{C}}\{u(1,1), u(-1,-1)\}, \\ \Delta^-_{4,1} &= Span_{\mathbf{C}}\{u(1,-1), u(-1,1)\}. \end{aligned}$$

For the Clifford multiplication, we have

$$\begin{array}{llll} e_1 \cdot u(1,1) = & -u(1,-1), & e_1 \cdot u(-1,-1) = & -u(-1,1), \\ e_2 \cdot u(1,1) = & u(1,-1), & e_2 \cdot u(-1,-1) = & -u(-1,1), \\ e_3 \cdot u(1,1) = & -iu(-1,1), & e_3 \cdot u(-1,-1) = & iu(1,-1), \\ e_4 \cdot u(1,1) = & -u(-1,1), & e_4 \cdot u(-1,-1) = & -u(1,-1), \end{array}$$

$$\begin{array}{llll} e_1 \cdot u(1,-1) = & -u(1,1), & e_1 \cdot u(-1,1) = & -u(-1,-1), \\ e_2 \cdot u(1,-1) = & -u(1,1), & e_2 \cdot u(-1,1) = & u(-1,-1), \\ e_3 \cdot u(1,-1) = & iu(-1,-1), & e_3 \cdot u(-1,1) = & -iu(1,1), \\ e_4 \cdot u(1,-1) = & u(-1,-1), & e_4 \cdot u(-1,1) = & u(1,1). \end{array}$$

On $\Delta_{4,1} = \mathbf{C}^4$, the natural scalar product

$$(v,w)_\Delta := \sum_{i=1}^{4} v_i \cdot \overline{w_i}, \ v, w \in \Delta_{4,1},$$

is given. We have another scalar product

$$< v, w >_\Delta := (e_1 \cdot v, w)_\Delta$$

on $\Delta_{4,1}$. $<,>_\Delta$ is indefinite of index 2 and invariant to the action of $\mathbf{Spin}^+(4,1)$ (comp. [Baum81, p. 68]):

$$< x\cdot v, w >_\Delta = < v, x\cdot w>_\Delta \ \ \forall x \in \mathbf{R}^4_1,\ v,w\in\Delta_{4,1},$$
$$< s\cdot v, s\cdot w >_\Delta = < v, w>_\Delta \ \ \forall s \in \mathbf{Spin}^+(4,1),\ v,w\in\Delta_{4,1}.$$

Let $(M^{4,1}, g)$ be a Lorentzian spin manifold with spin structure (Q, f) (comp. [Baum81, p. 70]).

$$S := Q \times_{\mathbf{Spin}(4,1)} \Delta_{4,1}$$

is the spinor bundle over M. S decomposes into the subbundles $S = S^+ \oplus S^-$, $S^+ := Q \times_{\mathbf{Spin}(4,1)} \Delta^+_{4,1},\ S^- := Q \times_{\mathbf{Spin}(4,1)} \Delta^-_{4,1}$. Let

$$\nabla^S : \Gamma(S) \to \Gamma(T^*M \otimes S)$$

be the spinor derivative.

$$\mu : \Gamma(TM \otimes S) \to \Gamma(S)$$

denotes the Clifford multiplication.
Let $(M^{4,1}, g)$ be in addition space- and time-orientable. The principal fibre bundle Q is reducible to a $\mathbf{Spin}^+(4,1)$-principal fibre bundle $\tilde{Q}$. Then

$$S = \tilde{Q} \times_{\mathbf{Spin}^+(4,1)} \Delta_{4,1}.$$

Further, the bundle S admits a scalar product $<,>_S$, which is induced by the $\mathbf{Spin}^+(4,1)$-invariant scalar product $<,>_\Delta$ on $\Delta_{4,1}$.

4.2 $\mathbf{SO}(4,1)$-action on $\mathbf{P}(\Delta^\pm_{4,1})$

Convention: For simplicity, we will combine in the following analogous assertions in the 'positive' and the 'negative' case to only one by the double index '$^\pm$' (e.g. $\mathbf{P}(\Delta^\pm) = \mathbf{SO}(4,1)/H^\pm \Leftrightarrow \mathbf{P}(\Delta^+) = \mathbf{SO}(4,1)/H^+$ and $\mathbf{P}(\Delta^-) = \mathbf{SO}(4,1)/H^-$).
From the irreducible spinor representation $\Phi^\pm : \mathbf{Spin}(4,1) \to \mathbf{GL}(\Delta^\pm_{4,1})$, we obtain a transitive action of $\mathbf{SO}(4,1)$ on the complex manifold $\mathbf{P}(\Delta^\pm) \cong \mathbf{S}^2$, which leave the complex structure invariant. $\mathbf{P}(\Delta^\pm)$ can be described as homogeneous space $\mathbf{SO}(4,1)/H^\pm$.

$\mathbf{SL}(2,\mathbf{C})$ acts on the complex projective space $\mathbf{P}^1(\mathbf{C})$ transitively via the canonical $\mathbf{SL}(2,\mathbf{C})$-action on $\mathbf{C}^2$:

$$\mathbf{SL}(2,\mathbf{C}) \times \mathbf{P}^1(\mathbf{C}) \to \mathbf{P}^1(\mathbf{C}),\ (A, \left[\begin{pmatrix} a \\ b \end{pmatrix}\right]) \mapsto \left[A\begin{pmatrix} a \\ b \end{pmatrix}\right].$$

$T := \left\{\begin{pmatrix} z & t \\ 0 & z^{-1}\end{pmatrix} : z \in \mathbf{C}^*,\ t \in \mathbf{C}\right\}$ is the isotropy group in the point $\left[\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right] \in \mathbf{P}^1(\mathbf{C})$.
$\underline{T} = \{\begin{pmatrix} a & b \\ 0 & -a\end{pmatrix} :\ a, b \in \mathbf{C}\}$ is the Lie algebra of the isotropy group T.
Then $\mathbf{P}^1(\mathbf{C}) = \mathbf{SL}(2,\mathbf{C})/T$.
Let

$$w : \mathbf{SO}(4,1) \times \mathbf{P}(\Delta^\pm) \to \mathbf{P}(\Delta^\pm),\ (a,[u]) \mapsto [\Phi\rho^{-1}(A)u],\ u \in \Delta^\pm.$$

w is a group action of $\mathbf{SO}(4,1)$ on $\mathbf{P}(\Delta^\pm)$. From the isomorphy of the representations $\Delta^\pm|_{\mathbf{Spin}^+(4,1)}$ and the canonical $\mathbf{SL}(2,\mathbf{C})$-representation $\mathbf{C}^2$, it follows that the $\mathbf{SO}(4,1)$-action w is transitive. Let $H^{\pm}$ be the isotropy group in the point $[\binom{1}{0}^{\pm}] = [u(1,\pm 1)] \in \mathbf{P}(\Delta^{\pm})$. Then

$$\mathbf{P}(\Delta^{\pm}) = \mathbf{SO}(4,1)/H^{\pm}.$$

The representation Φ maps $\mathbf{Spin}(4,1)$ into the complex matrices $\mathbf{GL}(4,\mathbf{C})$. It follows that the complex structure on $\Delta^{\pm}$ is invariant to the $\mathbf{Spin}(4,1)$-action and therefore, the complex canonical structure on $\mathbf{P}(\Delta^{\pm})$ is invariant to the $\mathbf{SO}(4,1)$-action.
Using the formulas to $e_i \cdot u(\delta_1,\delta_2)$ in (4.1), we obtain in the basis $(u(1,1),u(-1,-1))$ resp. $(u(1,-1),u(-1,1))$ for Δ^+ resp. Δ^- the following result for the differential of the spinor representation $\Phi^{\pm}$:

$$\Phi^+_*(\sum_{i<j} a_{ij}e_ie_j) = \begin{pmatrix} -a_{12}+ia_{34} & a_{24}-ia_{13}+a_{14}-ia_{23} \\ ia_{13}-a_{24}+a_{14}-ia_{23} & a_{12}-ia_{34} \end{pmatrix},$$

$$\Phi^-_*(\sum_{i<j} a_{ij}e_ie_j) = \begin{pmatrix} ia_{34}+a_{12} & ia_{13}-ia_{23}+a_{24}-a_{14} \\ -ia_{13}-a_{24}-a_{14}-ia_{23} & -a_{12}-ia_{34} \end{pmatrix}.$$

The Lie algebra of the isotropy group $H^{\pm}$ in $[\binom{1}{0}^{\pm}] \in \mathbf{P}(\Delta^{\pm})$ is

$$\underline{H}^{\pm} = (\Phi^{\pm}_*\rho_*^{-1})^{-1}(\underline{T}).$$

In the standard basis of $\mathbf{so}(4,1)$:

$$\begin{aligned} \underline{H}^{+} &= \{\sum_{i<j} a_{ij}E_{ij} \in \mathbf{so}(4,1) : \ a_{13}=a_{23},\ a_{14}=a_{24}\} \\ &= Span\{E_{12}, E_{13}+E_{23}, E_{14}+E_{24}, E_{34}\}, \end{aligned}$$

$$\begin{aligned} \underline{H}^{-} &= \{\sum_{i<j} a_{ij}E_{ij} \in \mathbf{so}(4,1) : \ a_{13}=-a_{23},\ a_{14}=-a_{24}\} \\ &= Span\{E_{12}, E_{13}-E_{23}, E_{14}-E_{24}, E_{34}\}. \end{aligned}$$

Let $m^{\pm} := Span\{E_{13}\mp E_{23}, E_{14}\mp E_{24}\}$. Then $\mathbf{so}(4,1) = \underline{H}^{\pm}\oplus m^{\pm}$. We don't obtain a reductive decomposition of the Lia algebra $\mathbf{so}(4,1)$.
On the tangent bundle $T\mathbf{SO}(4,1)$, the bundle morphisms $\widehat{J}^+, \widehat{J}^-$ are given by, ($\widetilde{X} \in \Gamma(T\mathbf{SO}(4,1))$ denotes the left-invariant vector field to $X \in \mathbf{so}(4,1)$),

$$\begin{aligned} &\widehat{J}^+(\widetilde{E_{13}}-\widetilde{E_{23}}) = -(\widetilde{E_{14}}-\widetilde{E_{24}}),\ \widehat{J}^+(\widetilde{E_{14}}-\widetilde{E_{24}}) = (\widetilde{E_{13}}-\widetilde{E_{23}}), \\ &\widehat{J}^+(\widetilde{X}) = 0 \quad \forall X \in \underline{H}^+, \end{aligned}$$

$$\begin{aligned} &\widehat{J}^-(\widetilde{E_{13}}+\widetilde{E_{23}}) = -(\widetilde{E_{14}}+\widetilde{E_{24}}),\ \widehat{J}^-(\widetilde{E_{14}}+\widetilde{E_{24}}) = (\widetilde{E_{13}}+\widetilde{E_{23}}), \\ &\widehat{J}^-(\widetilde{X}) = 0 \quad \forall X \in \underline{H}^-. \end{aligned}$$

Lemma 4.1 *Let $\phi^\pm : \mathbf{SO}(4,1) \to \mathbf{P}(\Delta^\pm) \cong \mathbf{SO}(4,1)/H^\pm$ be the natural projection. Then $\phi^\pm_* \circ \widehat{J}^\pm = J^{\mathbf{P}(\Delta^\pm)} \circ \phi^\pm_*$.*

Proof: Let $v_1 := \frac{d}{dt}|_{t=0}[(\begin{smallmatrix}1\\0\end{smallmatrix}) + t(\begin{smallmatrix}0\\1\end{smallmatrix})]$, $v_2 := \frac{d}{dt}|_{t=0}[(\begin{smallmatrix}1\\0\end{smallmatrix}) + t(\begin{smallmatrix}0\\i\end{smallmatrix})]$.
Then $T_{[(\begin{smallmatrix}1\\0\end{smallmatrix})]}\mathbf{P}^1(\mathbf{C}) = Span\{v_1, v_2\}$.
The natural complex structure on $\mathbf{P}^1(\mathbf{C}) \cong \mathbf{P}(\Delta^\pm)$ is in $T_{[(\begin{smallmatrix}1\\0\end{smallmatrix})]}\mathbf{P}^1(\mathbf{C})$ given by $J^{\mathbf{P}^1(\mathbf{C})}v_1 = v_2$ and $J^{\mathbf{P}^1(\mathbf{C})}v_2 = -v_1$.
On $T_e\mathbf{SO}(4,1) \cong \mathbf{so}(4,1)$, we have

$$
\begin{aligned}
\phi^+_*(E_{13} - E_{23}) &= \frac{d}{dt}\Big|_{t=0} \exp(t(E_{13} - E_{23}))H^+ \\
&= \frac{d}{dt}\Big|_{t=0} [\exp \frac{t}{2}(e_1e_3 - e_2e_3)(\begin{smallmatrix}1\\0\end{smallmatrix})^+] \\
&= \frac{d}{dt}\Big|_{t=0} [(1 + \frac{t}{2}(e_1e_3 - e_2e_3))(\begin{smallmatrix}1\\0\end{smallmatrix})^+] \\
&= \frac{d}{dt}\Big|_{t=0} [(\begin{smallmatrix}1\\0\end{smallmatrix})^+ + t(\begin{smallmatrix}0\\i\end{smallmatrix})^+] = v_2, \\
\phi^+_*(E_{14} - E_{24}) &= \frac{d}{dt}\Big|_{t=0} [(\begin{smallmatrix}1\\0\end{smallmatrix})^+ + t(\begin{smallmatrix}0\\1\end{smallmatrix})^+] = v_1, \\
\phi^-_*(E_{13} + E_{23}) &= \frac{d}{dt}\Big|_{t=0} [(\begin{smallmatrix}1\\0\end{smallmatrix})^- - t(\begin{smallmatrix}0\\i\end{smallmatrix})^-] = -v_2, \\
\phi^-_*(E_{14} + E_{24}) &= \frac{d}{dt}\Big|_{t=0} [(\begin{smallmatrix}1\\0\end{smallmatrix})^- - t(\begin{smallmatrix}0\\1\end{smallmatrix})^-] = -v_1.
\end{aligned}
$$

It follows $J^{\mathbf{P}(\Delta^\pm)} \circ \phi^\pm_{*e} = \phi^\pm_{*e} \circ \widehat{J}^\pm$. From the $\mathbf{SO}(4,1)$-invarianz of $J^{\mathbf{P}(\Delta^\pm)}$ follows $\phi^\pm_* \circ \widehat{J}^\pm = J^{\mathbf{P}(\Delta^\pm)} \circ \phi^\pm_*$ □

4.3 A relation between projective spinors and optical structures

We will assign canonically to every projective spinor $[\psi^\pm] \in \mathbf{P}(\Delta^\pm)$ an optical structure on $\mathbf{R}^4_1$. Moreover, we will see that the set $Opt(\mathbf{R}^4_1)$ of the orthogonal optical structures on $\mathbf{R}^4_1$ is naturally identified with the projective spinors $\mathbf{P}(\Delta^+) \cup \mathbf{P}(\Delta^-)$.

Let $Opt(\mathbf{R}^4_1)$ be the set of the orthogonal optical structures on the Minkowski space $\mathbf{R}^4_1$. In the case that $\mathbf{R}^4_1$ has an orientation, $* : \Lambda^2(\mathbf{R}^4)^* \to \Lambda^2(\mathbf{R}^4)^*$ is the Hodge operator, a null direction K generates the optical structure $\sigma = (K, K^\perp, \hat{*})$, where $\hat{*} : K^\perp/K \to K^\perp/K$ is induced by the Hodge operator (comp. 3.2).
Let the Minkowski space $\mathbf{R}^4_1$ have the standard orientation.

$$Opt^+(\mathbf{R}^4_1) := \{\sigma = (K, L, J) \in Opt(\mathbf{R}^4_1) :\ J = \hat{*} : L/K \to L/K\},$$

$$Opt^-(\mathbf{R}^4_1) := \{\sigma = (K, L, J) \in Opt(\mathbf{R}^4_1) :\ J = -\hat{*} : L/K \to L/K\}.$$

Let $P \subset \mathbf{P}(\mathbf{R}^4)$ be the set of null directions in $\mathbf{R}^4_1$. Consider the bijections

$$
\begin{array}{ccccc}
Opt^+(\mathbf{R}^4_1) & \cong & P & \cong & Opt^-(\mathbf{R}^4_1) \\
(K, L, J) & \leftrightarrow & K & \leftrightarrow & (K, L, -J).
\end{array}
$$

Lemma 4.2 *Let $0 \neq \psi^{\pm} \in \Delta^{\pm}$ be a spinor. Then*

$$\hat{\psi}^{\pm} : \mathbf{R}^4_1 \to \Delta^{\mp}_{4,1},\ t \mapsto t\psi^{\pm}$$

is a $\mathbf{R}$-linear mapping such that $\dim\ker\hat{\psi}^{\pm} = 1$ and $\ker\hat{\psi}^{\pm}$ is a null direction in $\mathbf{R}^4_1$.

Proof: Let $0 \neq \psi^+ = \begin{pmatrix} A \\ B \end{pmatrix} \in \Delta^+,\ 0 \neq \psi^- = \begin{pmatrix} C \\ D \end{pmatrix} \in \Delta^-$ and let

$$n^{\psi^+} = \begin{pmatrix} (|A|^2+|B|^2) \\ |A|^2-|B|^2 \\ -2Re(iA\overline{B}) \\ -2Re(A\overline{B}) \end{pmatrix},\ n^{\psi^-} = \begin{pmatrix} |C|^2+|D|^2 \\ |D|^2-|C|^2 \\ 2Re(iC\overline{D}) \\ 2Re(C\overline{D}) \end{pmatrix}.$$

Then

$$\begin{aligned} n^{\psi^+}\cdot\psi^+ &= \begin{pmatrix} -(|A|^2+|B|^2)A+(|A|^2-|B|^2)A-2iRe(iA\overline{B})B+2Re(A\overline{B})B \\ -(|A|^2+|B|^2)B-(|A|^2-|B|^2)B+2iRe(ia\overline{B})A+2Re(A\overline{B})A \end{pmatrix} \\ &= \begin{pmatrix} -2|B|^2A+2A\overline{B}B \\ -2|A|^2B+2\overline{A}BA \end{pmatrix} = 0, \\ n^{\psi^-}\cdot\psi^- &= \begin{pmatrix} -(|C|^2+|D|^2)C-(|D|^2-|C|^2)C-2iRe(iC\overline{D})D+2Re(C\overline{D})D \\ -(|C|^2+|D|^2)D-(|D|^2-|C|^2)D+2iRe(iC\overline{D})C+2Re(C\overline{D})C \end{pmatrix} \\ &= 0 \end{aligned}$$

From $0 = n^{\psi^{\pm}}\cdot n^{\psi^{\pm}}\cdot\psi^{\pm} = -< n^{\psi^{\pm}}, n^{\psi^{\pm}} > \psi^{\pm}$, it follows that $n^{\psi^{\pm}}$ is null.
Let $t \in \mathbf{R}^4_1$ such that $t\psi^{\pm} = 0$. Then $(t+n^{\psi^{\pm}})(t+n^{\psi^{\pm}})\psi^{\pm} = 0 \Rightarrow$
$(tt + tn^{\psi^{\pm}} + n^{\psi^{\pm}}t + n^{\psi^{\pm}}n^{\psi^{\pm}})\psi^{\pm} = -2 < t, n^{\psi^{\pm}} > \psi^{\pm} = 0$.
Since t and $n^{\psi^{\pm}}$ are null, it follows that $t \in \mathbf{R}n^{\psi^{\pm}} \Rightarrow \mathbf{R}n^{\psi^{\pm}} = \ker\hat{\psi}^{\pm}$ is null $\square$

Let

$$K^{\psi^{\pm}} := \ker\hat{\psi}^{\pm},\ L^{\psi^{\pm}} := (K^{\psi^{\pm}})^{\perp},$$

$$J^{\psi^+} := \hat{*} : L^{\psi^+}/K^{\psi^+} \to L^{\psi^+}/K^{\psi^+}, \qquad J^{\psi^-} := -\hat{*} : L^{\psi^-}/K^{\psi^-} \to L^{\psi^-}/K^{\psi^-}.$$

For $0 \neq \psi^{\pm} \in \Delta^{\pm}$, it is $\sigma^{\psi^{\pm}} := (K^{\psi^{\pm}}, L^{\psi^{\pm}}, J^{\psi^{\pm}}) \in Opt^{\pm}(\mathbf{R}^4_1)$ an orthogonal optical structure.
If $0 \neq \psi^{\pm}_2 = c\psi^{\pm}_1 \in \Delta^{\pm}$ for a $c \in \mathbf{C}$ so is $\ker\hat{\psi}^{\pm}_1 = \ker\hat{\psi}^{\pm}_2$. It follows $\sigma^{\psi^{\pm}_1} = \sigma^{\psi^{\pm}_2}$.
Therefore, to each projective spinor an optical structure is assigned,

$$j^{\pm} : \mathbf{P}(\Delta^{\pm}) \to Opt^{\pm}(\mathbf{R}^4_1),\ [\psi^{\pm}] \mapsto \sigma^{\psi^{\pm}}.$$

The mapping $j^{\pm}$ ist bijective:
Let $\Delta^{\pm} = Span_{\mathbf{C}}\{\psi^{\pm}_1, \psi^{\pm}_2\}$ and for $t \in \mathbf{R}^4_1$, it holds: $t\psi^{\pm}_1 = t\psi^{\pm}_2 = 0$. It follows $t = 0$.
Hence, $j^{\pm}$ is injective.
Let $t \in \mathbf{R}^4_1$ be a null vector. Then $tt\psi^{\pm} = 0\ \forall\psi^{\pm} \in \Delta^{\pm}$. There exists $\gamma^{\pm} \in \Delta^{\pm}$ such that $t\gamma^{\pm} \neq 0$. For $t\gamma^{\pm} \in \Delta^{\mp}$, it holds $t(t\gamma^{\pm}) = 0 \Rightarrow j^{\pm}$ is surjective.
The action of $\mathbf{SO}(4,1)$ on $P \cong Opt^{\pm}(\mathbf{R}^4_1)$ and $\mathbf{P}(\Delta^{\pm})$ is identical via $j^{\pm}$:

$$\rho(s)n^{\psi^{\pm}}s\psi^{\pm} = sn^{\psi^{\pm}}s^{-1}s\psi^{\pm} = sn^{\psi^{\pm}}\psi^{\pm} = 0\ \forall s \in \mathbf{Spin}(4,1) \Rightarrow$$
$$j^{\pm}(s\psi^{\pm}) = \rho(s)n^{\psi^{\pm}}.$$

Altogether, we have:

Proposition 4.3 *For $0 \neq \psi^\pm \in \Delta^\pm$, let $\hat{\psi}^\pm : \mathbf{R}^4_1 \to \Delta^\mp$ be the linear mapping $t \mapsto t\psi^\pm$.*

$$\begin{array}{rcrcl} j^+ & : & \mathbf{SO}(4,1)/H^+ \cong \mathbf{P}(\Delta^+) & \to & Opt^+(\mathbf{R}^4_1) \cong P, \\ & & [\psi^+] & \mapsto & \sigma^{\psi^+} = (\ker \hat{\psi}^+, (\ker \hat{\psi}^+)^\perp, \hat{*}), \end{array}$$

$$\begin{array}{rcrcl} j^- & : & \mathbf{SO}(4,1)/H^- \cong \mathbf{P}(\Delta^-) & \to & Opt^-(\mathbf{R}^4_1) \cong P, \\ & & [\psi^-] & \mapsto & \sigma^{\psi^-} = (\ker \hat{\psi}^-, (\ker \hat{\psi}^-)^\perp, -\hat{*}) \end{array}$$

are isomorphisms of the $\mathbf{SO}(4,1)$-action. The natural identification of the complex manifolds $\mathbf{P}(\Delta^+)$ and $\mathbf{P}(\Delta^-)$ via the space of null directions,

$$\mathbf{P}(\Delta^+) \cong P \cong \mathbf{P}(\Delta^-),\ [\left(\begin{smallmatrix} A \\ B \end{smallmatrix}\right)^+] \mapsto n^{[\left(\begin{smallmatrix} A \\ B \end{smallmatrix}\right)^+]} = n^{[\left(\begin{smallmatrix} -\overline{B} \\ \overline{A} \end{smallmatrix}\right)^-]} \mapsto [\left(\begin{smallmatrix} -\overline{B} \\ \overline{A} \end{smallmatrix}\right)^-],$$

is anti-holomorphic.

We calculate concrete the optical structures corresponding to following spinors:

i) $\psi^+ = \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)^+ \in \Delta^+ :$ $\quad K^{\psi^+} = \mathbf{R}(e_1 + e_2),\ L^{\psi^+} = Span\{e_1 + e_2, e_3, e_4\},$
$\quad J^{\psi^+}(e_3 + K^{\psi^+}) = -e_4 + K^{\psi^+},\ J^{\psi^+}(e_4 + K^{\psi^+}) = e_3 + K^{\psi^+}$

ii) $\psi^- = \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)^- \in \Delta^- :$ $\quad K^{\psi^-} = \mathbf{R}(e_1 - e_2),\ L^{\psi^-} = Span\{e_1 - e_2, e_3, e_4\},$
$\quad J^{\psi^-}(e_3 + K^{\psi^-}) = -e_4 + K^{\psi^-},\ J^{\psi^-}(e_4 + K^{\psi^-}) = e_3 + K^{\psi^-}$

iii) $\psi^- = \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)^- \in \Delta^- :$ $\quad K^{\psi^-} = \mathbf{R}(e_1 + e_2),\ L^{\psi^-} = Span\{e_1 + e_2, e_3, e_4\},$
$\quad J^{\psi^-}(e_3 + K^{\psi^-}) = e_4 + K^{\psi^-},\ J^{\psi^-}(e_4 + K^{\psi^-}) = -e_3 + K^{\psi^-}$

iv) $\psi^- = \left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right)^- \in \Delta^- :$ $\quad K^{\psi^-} = \mathbf{R}(e_1 + e_4),\ L^{\psi^-} = Span\{e_1 + e_4, e_2, e_3\},$
$\quad J^{\psi^-}(e_3 + K^{\psi^-}) = -e_2 + K^{\psi^-},\ J^{\psi^-}(e_2 + K^{\psi^-}) = e_3 + K^{\psi^-}$

Lemma 4.4 *Let $\delta : L^{\psi^\pm}/K^{\psi^\pm} \to \Delta^\mp,\ l + K^{\psi^\pm} \mapsto l \cdot \psi^\pm$. Then*

$$\delta(J^{\psi^\pm}(l + K^{\psi^\pm})) = i\delta(l + K^{\psi^\pm}) = il\psi^\pm.$$

Proof: For the optical structure $\sigma^{\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)^\pm} \in Opt^\pm(\mathbf{R}^4_1)$ we have

$$\delta(J^{\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)^\pm}(e_3 + K^{\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)^\pm})) = -e_4 \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)^\pm = \pm \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)^\pm = ie_3 \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)^\pm,$$

$$\delta(J^{\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)^\pm}(e_4 + K^{\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)^\pm})) = e_3 \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)^\pm = ie_4 \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)^\pm.$$

Let $\psi^\pm \in \Delta^\pm$ be arbitrary. It exists $s \in \mathbf{Spin}^+(4,1)$ such that $\psi^\pm = s \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)^\pm$.

$$
\begin{aligned}
\delta(J^{\psi^\pm}(l+K^{\psi^\pm})) &= \delta(\pm\hat{*}(l+K^{\psi^\pm})) = \delta(s\rho(s^{-1})[\pm\hat{*}(l+K^{\psi^\pm})]s^{-1}) \\
&= \delta(\pm s\hat{*}(\rho(s^{-1})(l+K^{\psi^\pm}))s^{-1}) \\
&= \delta(sJ^{s^{-1}\psi^\pm}(\rho(s^{-1})l+K^{s^{-1}\psi^\pm})s^{-1}) \\
&= s\delta(J^{s^{-1}\psi^\pm}(\rho(s^{-1})l+K^{s^{-1}\psi^\pm})) = is\delta(\rho(s^{-1})l+K^{s^{-1}\psi^\pm}) \\
&= i\delta(l+K^{\psi^\pm}) \qquad \square
\end{aligned}
$$

4.4 The twistor spaces $\mathcal{Z}^+(M)$, $\mathcal{Z}^-(M)$

Let $(M^{4,1}, g)$ be a 4-dimensional oriented Lorentzian manifold. $SO(M)$ denotes the principal fibre bundle of the oriented (pseudo)-orthonormal frames of M.

Definition 4.1 *The positive resp. negative twistor space of a 4-dimensional oriented Lorentzian manifold (M, g) is defined as the positive resp. negative projective spinor bundle*

$$
\begin{aligned}
\mathcal{Z}^+(M) &:= \mathbf{P}(S^+) = SO(M) \times_{\mathbf{SO}(4,1)} \mathbf{P}(\Delta^+), \\
\mathcal{Z}^-(M) &:= \mathbf{P}(S^-) = SO(M) \times_{\mathbf{SO}(4,1)} \mathbf{P}(\Delta^-).
\end{aligned}
$$

$\mathcal{Z}^\pm(M)$ is a 6-dimensional oriented C^∞-manifold. We can $\mathcal{Z}^\pm(M)$ also understand as:

$\mathcal{Z}^\pm(M) \cong SO(M) \times_{\mathbf{SO}(4,1)} P$, the bundle of null directions in TM,
$\mathcal{Z}^\pm(M) \cong SO(M) \times_{\mathbf{SO}(4,1)} Opt^\pm(\mathbf{R}^4_1)$, the bundle of optical structures in TM,
$\mathcal{Z}^\pm(M) \cong SO(M) \times_{\mathbf{SO}(4,1)} \mathbf{SO}(4,1)/H^\pm = SO(M)/H^\pm$ associated bundle.

Because $\mathcal{Z}^+(M)$ and $\mathcal{Z}^-(M)$ are naturally identified with the bundle $SO(M) \times_{\mathbf{SO}(4,1)} P$, they are themselves identificable,

$$
\Theta : \mathcal{Z}^+(M) \cong SO(M) \times_{\mathbf{SO}(4,1)} P \cong \mathcal{Z}^-(M).
$$

A global section $\alpha \in \Gamma(M; \mathcal{Z}^\pm(M))$ can be understand as an almost optical structure on M.
Let (M, g) be a spin manifold with spin structure (Q, f) and let $S^\pm\backslash 0$ be the bundle of positive resp. negative spinors without the zero section, then we have the natural projection

$$
q : S^\pm\backslash 0 = Q \times_{\mathbf{Spin}(4,1)} \Delta^\pm_{4,1}\backslash 0 \to Q \times_{\mathbf{Spin}(4,1)} \mathbf{P}(\Delta^\pm_{4,1}) \cong \mathcal{Z}^\pm(M).
$$

Since $\mathcal{Z}^+(M)$ and $\mathcal{Z}^-(M)$ are naturally identified, it is essentially sufficient to investigate only one of the two twistor spaces. Mainly, we will consider in the following the positive twistor space $\mathcal{Z}^+(M)$.
Let (M, g) be space- and time-oriented. Let $T \in \Gamma(TM)$ be a timelike and futur-pointing vector field on M such that $g(T, T) = -1$. The choice of such a field $T \in \Gamma(TM)$ is equivalent to a $\mathbf{SO}(3)$-reduction $SO(3)_T(M)$ of the frame bundle $SO(M)$ over M. Let $\tilde{s} = (s_1, \ldots, s_4)$ be a local orthonormal frame such that $s_1 = T$. We denote $\tilde{s}$ as a T-orthononormal basis. We have the identification

$$
\mathcal{Z}^+(M) \cong SO(3)_T(M) \times_{\mathbf{SO}(3)} \mathbf{P}(\Delta^+),
$$

where $\mathbf{SO}(3)$ acts via the imbedding $i : \mathbf{SO}(3) \hookrightarrow \mathbf{SO}(4,1),\ A \mapsto \left(\begin{smallmatrix} 1 & 0 \\ 0 & A \end{smallmatrix}\right)$. We consider once again the type $\mathbf{P}(\Delta^+) \cong P$ of the fibres of the twistor space. It is

$$i_P : P \cong \mathbf{S}^2,\ \left[\begin{pmatrix} 1 \\ a \\ b \\ c \end{pmatrix}\right] \mapsto \begin{pmatrix} a \\ b \\ c \end{pmatrix},\ a, b, c \in \mathbf{R},$$

a diffeomorphism. The group $\mathbf{SO}(3)$ acts already transitively on P and i_P is an isomorphy of the $\mathbf{SO}(3)$-action on P and the natural $\mathbf{SO}(3)$-action on $\mathbf{S}^2 \cong \mathbf{SO}(3)/\mathbf{SO}(2)$:

$$i_P(\left[\begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix} \begin{pmatrix} 1 \\ a \\ b \\ c \end{pmatrix}\right]) = A \begin{pmatrix} a \\ b \\ c \end{pmatrix}.$$

The twistor space can now be described as an associated bundle to the reduced principal fibre bundle $SO(3)_T(M)$:

$$\mathcal{Z}^+(M) \cong SO(3)_T(M) \times_{\mathbf{SO}(3)} \mathbf{SO}(3)/\mathbf{SO}(2) = SO(3)_T(M)/\mathbf{SO}(2).$$

$$\begin{array}{lccc} i: & (s_2, s_3, s_4)_m \cdot A \in SO(3)_T(M) & \hookrightarrow & (T, s_2, s_3, s_4)_m \cdot \left(\begin{smallmatrix} 1 & 0 \\ 0 & A \end{smallmatrix}\right) \in SO(M) \\ & \downarrow \hat{\pi} & & \downarrow \hat{\pi} \\ & [(s_2, s_3, s_4)_m, A\mathbf{SO}(2)] & \xrightarrow{\cong} & [(T, s_2, s_3, s_4)_m, \left(\begin{smallmatrix} 1 & 0 \\ 0 & A \end{smallmatrix}\right) H^+], \quad m \in M \\ & \in SO(3)_T(M)/\mathbf{SO}(2) & & \in \mathcal{Z}^+(M) \end{array}$$

Let $M = \mathbf{R}^1_1 \times_f N$ be a Lorentz warped product, where (N, h) is a 3-dimensional oriented Riemannian manifold. The Lorentzian metric on M is given by

$$g = -\rho^*(<,>) + (f \circ \rho)^2 \sigma^* h,$$

where $\rho : M \to \mathbf{R}^1_1$ and $\sigma : M \to N$ are the natural projections.
Let $T := \frac{\partial}{\partial t} \in T\mathbf{R}^1_1$ and $SO(N)$ be the frame bundle over N and let

$$Z(N) = SO(N) \times_{\mathbf{SO}(3)} \mathbf{SO}(3)/\mathbf{SO}(2)$$

be the unit sphere bundle over N. We have the diffeomorphisms

$$\Psi : SO(3)_T(M) \cong \mathbf{R} \times SO(N),$$

$$(s_2, s_3, s_4)_{(t,m)} \cdot A \mapsto (t, (\sigma_*(fs_2), \sigma_*(fs_3), \sigma_*(fs_4))_m \cdot A),\ (t, m) \in M,$$

$$\hat{\Psi} : SO(3)_T(M)/\mathbf{SO}(2) \cong \mathbf{R} \times Z(N),$$

$$[(s_2, s_3, s_4), A\mathbf{SO}(2)] \mapsto (t, [(\sigma_*(fs_2), \sigma_*(fs_3), \sigma_*(fs_4))_m, A\mathbf{SO}(2)]).$$

If we think of the twistor space $\mathcal{Z}^+(M) \cong SO(M) \times_{\mathbf{SO}(4,1)} P$ as the bundle of null directions, we obtain the identification

$$\Omega : \mathcal{Z}^+(M) \cong \mathbf{R} \times Z(N),$$

$$[(T, s_2, s_3, s_4)_{(t,m)}, \left[\begin{pmatrix} 1 \\ a \\ b \\ c \end{pmatrix}\right]] \mapsto (t, [(\sigma_*(fs_2), \sigma_*(fs_3), \sigma_*(fs_4))_m, \begin{pmatrix} a \\ b \\ c \end{pmatrix}]),$$

where $\begin{pmatrix} a \\ b \\ c \end{pmatrix} \in \mathbf{S}^2$ and $\left[\begin{pmatrix} 1 \\ a \\ b \\ c \end{pmatrix}\right] \in P$.

In the next section, we will investigate concrete examples of twistor spaces.

The Levi-Civita connection Z in the $\mathbf{SO}(4,1)$-principal fibre bundle $SO(M)$ induces a decomposition of the tangent space of the associated bundle $\mathcal{Z}^+(M)$ into the sum of a horizontal and a vertical bundle

$$T\mathcal{Z}^+(M) = T_V\mathcal{Z}^+(M) \oplus T_H\mathcal{Z}^+(M).$$

But in general, the holonomy group of M isn't included in the group $\mathbf{SO}(3)$. Therefore, the $\mathbf{SO}(4,1)$-connection Z can't necessarily be reduced to a $\mathbf{SO}(3)$-connection in the bundle $SO(3)_T(M)$. This isn't necessarily possible even in the case of a Lorentz warped product $M = \mathbf{R}^1_1 \times_f N$. The horizontal spaces with respect to the Levi-Civita connection of (N,h) in the bundle $TZ(N)$ aren't in relation to the horizontal spaces in $T\mathcal{Z}^+(M) \cong T(\mathbf{R} \times Z(N))$ with respect to the Levi-Civita connection of (M,g).

5 Almost optical structures on the twistor spaces $\mathcal{Z}^+(M)$ and $\mathcal{Z}^-(M)$

On the twistor spaces $\mathcal{Z}^+(M)$ and $\mathcal{Z}^-(M)$ of an oriented Lorentzian 4-manifold exists in both cases a canonical almost complex structure $\mathcal{O}^+_+$ resp. $\mathcal{O}^-_+$. A well known theorem states that these natural almost optical structures on the twistor spaces are integrable if and only if the base manifold M is conformally flat. In [Nur96], the proof to this is given in terms of the Newman-Penrose notation. Here, we will give the proof by directly evaluating the integrability conditions in the frame bundle $SO(M)$. Besides, we will investigate the twistor spaces with optical geometry of Lorentz warped products $M = \mathbf{R}^1_1 \times_f N$, especially the twistor spaces of the three space forms $\mathbf{R}^4_1$, $\mathbf{S}^4_1$ and $\mathbf{H}^4_1$.

5.1 The almost optical structures $\mathcal{O}^+_+$, $\mathcal{O}^+_-$, $\mathcal{O}^-_+$, $\mathcal{O}^-_-$

First, consider the projective spinor bundle

$$\mathcal{Z}^+(M) = SO(M) \times_{\mathbf{SO}(4,1)} \mathbf{P}(\Delta^+_{4,1})$$

of a 4-dimensional oriented Lorentzian manifold $(M^{4,1}, g)$. It denote $\pi : SO(M) \to M$ and $\pi : \mathcal{Z}^+(M) \to M$ the natural projections. Let

$$T\mathcal{Z}^+(M) = T_V\mathcal{Z}^+(M) \oplus T_H\mathcal{Z}^+(M)$$

be the decomposition of the tangent bundle of $\mathcal{Z}^+(M)$ into the horizontal and the vertical part with respect to the Levi-Civita connection Z.

We will now define the canonical almost optical structure $\mathcal{O}_+^+$ on $\mathcal{Z}^+(M)$.
The complex structure on $\mathbf{P}(\Delta^+)$ is invariant to the $\mathbf{SO}(4,1)$-action on $\mathbf{P}(\Delta^+) \cong \mathbf{SO}(4,1)/H^+$. Hence, a natural complex structure J^V on the fibres of $\mathcal{Z}^+(M)$ is given.
Let $p \in \mathcal{Z}^+(M)$ be arbitrary. It is

$$p = (K^p, L^p, J^p) \in Opt^+(T_{\pi(p)}M)$$

an optical structure on $T_{\pi(p)}M$. The vector space $T_{Hp}\mathcal{Z}^+(M)$ admits as optical structure the horizontal lift of $p \in Opt^+(T_{\pi(p)}M)$. Together with the complex structure J_p^V in $T_{Vp}\mathcal{Z}^+(M)$, we obtain for each $p \in \mathcal{Z}^+(M)$ an optical structure on the vector space $T_p\mathcal{Z}^+(M)$ and altogether, we have constructed the almost optical structure $\mathcal{O}_+^+ = (\mathcal{K}^+, \mathcal{L}^+, J_+^+)$ on $\mathcal{Z}^+(M)$.
More exactly, $\mathcal{K}^+ \subset T_H\mathcal{Z}^+(M)$ is given by $\pi_*\mathcal{K}_p^+ = K^p \quad \forall p \in \mathcal{Z}^+(M)$.
For $\mathcal{L}^+ \subset T\mathcal{Z}^+(M)$, it holds $\mathcal{L}_p^+ = \pi_*^{-1}(L^p) \quad \forall p \in \mathcal{Z}^+(M)$.
Then $\mathcal{L}^+ = T_V\mathcal{Z}^+(M) \oplus (T_H\mathcal{Z}^+(M) \cap \mathcal{L}^+)$.
The mapping $J_+^+ : \mathcal{L}^+/\mathcal{K}^+ \to \mathcal{L}^+/\mathcal{K}^+$ is given by

$$J_+^+|_{(T_H\mathcal{Z}^+(M)\cap\mathcal{L}^+)/\mathcal{K}^+} = \pi_{*p}^{-1} J^p \pi_{*p}, \quad J_+^+|_{T_V\mathcal{Z}^+(M)} = J^V.$$

There is another possibility to define an almost optical structure on the screen space $\mathcal{L}^+/\mathcal{K}^+$. Let $J_-^+ : \mathcal{L}^+/\mathcal{K}^+ \to \mathcal{L}^+/\mathcal{K}^+$ be given by

$$J_-^+|_{(T_H\mathcal{Z}^+(M)\cap\mathcal{L}^+)/\mathcal{K}^+} = -\pi_{*p}^{-1} J^p \pi_{*p}, \; J_-^+|_{T_V\mathcal{Z}^+(M)} = J^V.$$

We obtain the almost optical structure $\mathcal{O}_-^+ = (\mathcal{K}^+, \mathcal{L}^+, J_-^+)$ on $\mathcal{Z}^+(M)$.

Proposition 5.1 (comp. [Nur96]) *Let $\mathcal{O}_+^+ = (\mathcal{K}^+, \mathcal{L}^+, J_+^+)$ and $\mathcal{O}_-^+ = (\mathcal{K}^-, \mathcal{L}^-, J_-^+)$ be the canonical optical structures on $\mathcal{Z}^+(M)$. Then*

1. *$\mathcal{O}_+^+$, $\mathcal{O}_-^+$ are geodesic,*
2. *$\mathcal{O}_-^+$ is never integrable,*
3. *$\mathcal{O}_+^+$ is integrable if and only if (M, g) is conformally flat,*
4. *the integrability conditions for $\mathcal{O}_+^+$ are satisfied exactly in the points $p = (K^p, L^p, J^p) \in \mathcal{Z}^+(M)$, for those $K^p \subset T_{\pi(p)}M$ is a principal null direction.*

Proof: In the first part of the proof the integrability conditions are translated to the frame bundle $SO(M)$, where they are pointwise evaluated in the 2. part of the proof by a suitable basis.
$i)$ Let a vector bundle $\mathcal{F} \subset \mathcal{L}^+ \cap T_H\mathcal{Z}^+(M)$ over $\mathcal{Z}^+(M)$ be fixed such that

$$\mathcal{L}^+ = \mathcal{K}^+ \oplus \mathcal{F} \oplus T_V\mathcal{Z}^+(M).$$

Let $\mathcal{T} := \mathcal{F} \oplus T_V\mathcal{Z}^+(M)$. By a remark in section (3.2) the integrability conditions from definition 3.9 for $\mathcal{O}_\pm^+$ are equivalent to:
$(J_+^{\mathcal{T}} : \mathcal{L}^+ \to \mathcal{L}^+, \; J_+^{\mathcal{T}}|_{\mathcal{T}} \cong J_+^+, \; J_+^{\mathcal{T}}|_{\mathcal{K}^+} = 0, \quad J_-^{\mathcal{T}} : \mathcal{L}^+ \to \mathcal{L}^+, \; J_-^{\mathcal{T}}|_{\mathcal{T}} \cong J_-^+, \; J_-^{\mathcal{T}}|_{\mathcal{K}^+} = 0)$

a) $[\Gamma(\mathcal{K}^+), \Gamma(\mathcal{L}^+)] \subset \Gamma(\mathcal{L}^+)$,
b') $(\pounds_k J^{\mathcal{T}}_{\pm})(l) \in \mathcal{K}^+ \quad \forall\, l \in \mathcal{L}^+,\ k \in \Gamma(\mathcal{K}^+)$,
c') $[J^{\mathcal{T}}_{\pm} l_1, l_2] + [l_1, J^{\mathcal{T}}_{\pm} l_2] \in \Gamma(\mathcal{L}^+)$ and
$J^{\mathcal{T}}_{\pm}([J^{\mathcal{T}}_{\pm} l_1, l_2] + [l_1, J^{\mathcal{T}}_{\pm} l_2]) - [J^{\mathcal{T}}_{\pm} l_1, J^{\mathcal{T}}_{\pm} l_2] + [l_1, l_2] \in \Gamma(\mathcal{K}^+)$
$\forall\, l_1, l_2 \in \Gamma(\mathcal{L}^+)$.

ii) Now, we construct a flag geometry $(\widehat{\mathcal{K}}^+, \widehat{\mathcal{L}}^+)$ and a bundle morphism $\widehat{J}^{\mathcal{T}}_{\pm} : \widehat{\mathcal{L}}^+ \to \widehat{\mathcal{L}}^+$ on the frame bundle $SO(M)$.
Let $TSO(M) = T_H SO(M) \oplus T_V SO(M)$ be the decomposition into the horizontal and the vertical part with respect to the Levi-Civita connection Z. It is

$$\hat{\pi} : SO(M) \to \mathcal{Z}^+(M) = SO(M) \times_{\mathbf{SO}(4,1)} \mathbf{P}(\Delta^+),\ \tilde{s}_m \mapsto [\tilde{s}_m, [\tbinom{1}{0}^+]],$$

the natural projection. For the horizontal bundle holds: $\hat{\pi}_*(T_H SO(M)) = T_H \mathcal{Z}^+(M)$.
In (4.2), we defined the bundle morphism

$$\widehat{J}^+ : T\mathbf{SO}(4,1) \to T\mathbf{SO}(4,1).$$

For $\phi^+ : T\mathbf{SO}(4,1) \to T\mathbf{SO}(4,1)/H^+$ the canonical projection, we have $\phi^+_* \circ \widehat{J}^+ = J^{\mathbf{P}(\Delta^+)} \circ \phi^+_*$ (Lemma 4.1).
Further, let

$$\widehat{\mathcal{K}}^+ := \hat{\pi}_*^{-1}(\mathcal{K}^+) \cap T_H SO(M) \text{ and } \widehat{\mathcal{L}}^+ := \hat{\pi}_*^{-1}(\mathcal{L}^+) \subset TSO(M).$$

$\widehat{\mathcal{K}}^+$ is a 1-dimensional subbundle in $T_H SO(M)$ and we have for the bundle $\widehat{\mathcal{L}}^+$

$$\widehat{\mathcal{L}}^+ = T_V SO(M) \oplus (\hat{\pi}_*^{-1}(\mathcal{F} \oplus \mathcal{K}^+) \cap T_H SO(M)).$$

The bundle morphism $\widehat{J}^{\mathcal{T}}_{\pm} : \widehat{\mathcal{L}}^+ \to \widehat{\mathcal{L}}^+$ is induced on the fibres $T_V SO(M)$ by the left-invariant mapping $\widehat{J}^+ : T\mathbf{SO}(4,1) \to T\mathbf{SO}(4,1)$ on the tangent space of the fibre type,

$$\widehat{J}^{\mathcal{T}}_{\pm}|_{T_V SO(M)} \cong \widehat{J}^+.$$

On the horizontal part is $\widehat{J}^{\mathcal{T}}_{\pm}$ defined by

$$\widehat{J}^{\mathcal{T}}_{\pm}|_{\hat{\pi}_*^{-1}(\mathcal{F} \oplus \mathcal{K}^+) \cap T_H SO(M)} = \hat{\pi}_*^{-1} \circ J^{\mathcal{T}}_{\pm}|_{\mathcal{F} \oplus \mathcal{K}^+} \circ \hat{\pi}_*.$$

By definition of $\widehat{J}^{\mathcal{T}}_{\pm}$, we have $\hat{\pi}_* \widehat{J}^{\mathcal{T}}_{\pm} = J^{\mathcal{T}}_{\pm} \hat{\pi}_*$.
iii) The conditions a), b') and c') will now be expressed in the frame bundle $SO(M)$ in terms of $\widehat{\mathcal{K}}^+, \widehat{\mathcal{L}}^+$ and $\widehat{J}^{\mathcal{T}}_{\pm}$.
We have $\mathbf{so}(4,1) = m^+ \oplus \underline{H}^+$ (comp. (4.2)). $\widetilde{X} \in \Gamma(TSO(M))$ denotes the fundamental vector field corresponding to $X \in \mathbf{so}(4,1)$. Let

$$T_V^{\underline{H}^+} := Span\{\widetilde{X} \in \Gamma(TSO(M)) :\ X \in \underline{H}^+\},$$

$$T_V^{m^+} := Span\{\widetilde{X} \in \Gamma(TSO(M)) :\ X \in m^+\}$$

be the bundles, which are spanned by the fundamental vector fields of the elements in $\underline{H}^+$ resp. m^+. Then $T_V SO(M) = T_V^{m^+} \oplus T_V^{\underline{H}^+}$ and $T_V^{\underline{H}^+} = \ker \hat{\pi}_*$.
Note that $\hat{\pi}_* \widehat{J}^{\mathcal{T}}_{\pm} = J^{\mathcal{T}}_{\pm} \hat{\pi}_*$ and $T_V^{\underline{H}^+} = \ker \hat{\pi}_*$. The following conditions in the frame bundle $SO(M)$ are equivalent to a), b') and c'):

a') $[\Gamma(\widehat{\mathcal{K}}^+), \Gamma(\widehat{\mathcal{L}}^+)] \subset \Gamma(\widehat{\mathcal{L}}^+)$,

b") $(\pounds_k \widehat{J}^{\mathcal{T}}_{\pm})(l) \in \widehat{\mathcal{K}}^+ \oplus T_V^{\underline{H}^+} \quad \forall\, l \in \widehat{\mathcal{L}}^+,\ k \in \Gamma(\widehat{\mathcal{K}}^+)$,

c") $[\widehat{J}^{\mathcal{T}}_{\pm} l_1, l_2] + [l_1, \widehat{J}^{\mathcal{T}}_{\pm} l_2] \in \Gamma(\widehat{\mathcal{L}}^+)$ and
$\widehat{J}^{\mathcal{T}}_{\pm}([\widehat{J}^{\mathcal{T}}_{\pm} l_1, l_2] + [l_1, \widehat{J}^{\mathcal{T}}_{\pm} l_2]) - [\widehat{J}^{\mathcal{T}}_{\pm} l_1, \widehat{J}^{\mathcal{T}}_{\pm} l_2] + [l_1, l_2] \in \Gamma(\widehat{\mathcal{K}}^+ \oplus T_V^{\underline{H}^+})$
$\forall\, l_1, l_2 \in \Gamma(\widehat{\mathcal{L}}^+)$.

(Also note for this $[fX, gY] = fg[X,Y] + fX(g)Y - gY(f)X$,
$\hat{\pi}_*([X^*, Y^*]) = [X,Y]$ $(X, Y \in \Gamma(T\mathcal{Z}^+(M))$, $\hat{\pi}_*(X^*) = X$, $\hat{\pi}_*(Y^*) = Y)$ and $\widehat{J}^{\mathcal{T}2}_{\pm} l + l \in T_V^{\underline{H}^+} \oplus \widehat{\mathcal{K}}^+ \quad \forall\, l \in \widehat{\mathcal{L}}^+$.)

iv) We choose a suitable local basis in the frame bundle $SO(M)$ to evaluate the integrability conditions.
Let $\tilde{s} = (s_1, \cdots, s_4) : U_m \subset M \to SO(M)$ be a reper in a neighborhood of $m \in M$. We obtain a trivialization of $SO(M)$ over U_m:

$$I : SO(M)|_{U_m} \cong U_m \times \mathbf{SO}(4,1),\ I(q_u) = (u, A),\ q_u = \tilde{s}_u \cdot A.$$

The vector fields $S_i \in \Gamma(TSO(M)|_{U_m}),\ i = 1, \ldots, 4$, are pointwise defined by

$$S_{i(\tilde{s}_u \cdot A)} := I_*^{-1}((\tilde{s}_u \cdot A)_i, 0)_{(u,A)} = I_*^{-1}(\sum_{k=1}^{4} s_{ku} \cdot A_{ki}, 0)_{(u,A)} \quad \forall\, u \in U_m,\ A \in \mathbf{SO}(4,1).$$

Let the fields $Z_i \in \Gamma(TSO(M)|_{U_m})$ be given by

$$\begin{aligned} Z_{i(\tilde{s}_u \cdot A)} &:= S_{i(\tilde{s}_u \cdot A)} - \sum_{k<l} \epsilon_k \epsilon_l \omega_{kl}((\tilde{s}_u \cdot A)_i) \widetilde{[Ad(A^{-1}) E_{kl}]}_{(\tilde{s}_u \cdot A)} \\ &= S_{i(\tilde{s}_u \cdot A)} - \sum_{k<l} \epsilon_k \epsilon_l \omega_{kl}((\tilde{s}_u \cdot A)_i) \left.\frac{d}{dt}\right|_{t=0} \tilde{s}_u \cdot A \exp t Ad(A^{-1}) E_{kl} \\ &\quad \forall\, u \in U_m,\ A \in \mathbf{SO}(4,1), \end{aligned}$$

where $\omega_{kl} := g(\nabla s_k, s_l),\ k, l = 1, \ldots, 4$, denote the local connection forms and E_{kl} are the elements of the standard basis in $\mathbf{so}(4,1)$. Then $\pi_*(Z_{i(\tilde{s}_u \cdot A)}) = \pi_*(S_{i(\tilde{s}_u \cdot A)}) = (\tilde{s}_u \cdot A)_i = \sum_{k=1}^{4} s_{ku} A_{ki}$ and for the Levi-Civita connection Z holds

$$\begin{aligned} Z(dR_{A^{-1}} Z_{i(\tilde{s}_u \cdot A)}) &= Z(I_*^{-1}((\tilde{s}_u \cdot A)_i, 0)_{(u,e)}) - Z(\sum_{k<l} \epsilon_k \epsilon_l \omega_{kl}((\tilde{s}_u \cdot A)_i) (\widetilde{E_{kl}})_{\tilde{s}_u}) \\ &= Z \circ d\tilde{s}((\tilde{s}_u \cdot A)_i) - \sum_{k<l} \epsilon_k \epsilon_l \omega_{kl}((\tilde{s}_u \cdot A)_i) E_{kl} = 0 \end{aligned}$$

for all $A \in \mathbf{SO}(4,1), u \in U_m$. For the last equality note that the local connection forms ω_{kl} are just given by $Z \circ d\tilde{s}$. The fields Z_i are therefore horizontal and

$$T_H SO(M)|_{U_m} = Span\{Z_i : \ i = 1, \ldots, 4\}.$$

The optical structure $(K^p, L^p, J^p) \in Opt^+(T_{\pi(p)}M)$ to $p = [\tilde{s}_u \cdot A, [\binom{1}{0})^+]] \in \mathcal{Z}^+(M)$ is given by (comp. (4.3)):

$$\begin{aligned}
&K^p = \mathbf{R}((\tilde{s}_u \cdot A)_1 + (\tilde{s}_u \cdot A)_2),\\
&L^p = Span\{(\tilde{s}_u \cdot A)_3, (\tilde{s}_u \cdot A)_4\} \oplus K^p \text{ and}\\
&J^p((\tilde{s}_u \cdot A)_4 + K^p) = (\tilde{s}_u \cdot A)_3 + K^p, \ J^P((\tilde{s}_u \cdot A)_3 + K^p) = -(\tilde{s}_u \cdot A)_4 + K^p.
\end{aligned}$$

Hence, the fields Z_i are choosen in such a way that

$$\widehat{\mathcal{K}}^+|_{\pi^{-1}(U_m)} = \mathbf{R}(Z_1 + Z_2), \quad \widehat{\mathcal{L}}^+|_{\pi^{-1}(U_m)} = T_V SO(M) \oplus Span\{Z_1 + Z_2, Z_3, Z_4\}.$$

In the following, let $\mathcal{F}$ be a bundle such that $\mathcal{F}|_{\pi^{-1}(U_m)} = Span\{Z_3, Z_4\}$ for a local frame $\tilde{s} : U_m \subset M \to SO(M)$. Then

$$\mathcal{T}|_{\pi^{-1}(U_m)} = T_V SO(M) \oplus Span\{Z_3, Z_4\},$$

and $\widehat{J}^{\mathcal{T}}_{\pm}$ is on $\pi^{-1}(U_m)$ given by

$$\widehat{J}^{\mathcal{T}}_{\pm}(Z_3) = \mp Z_4, \ \widehat{J}^{\mathcal{T}}_{\pm}(Z_4) = \pm Z_3,$$

$$\widehat{J}^{\mathcal{T}}_{\pm}(\widetilde{E_{14}} - \widetilde{E_{24}}) = \widetilde{E_{13}} - \widetilde{E_{23}}, \ \widehat{J}^{\mathcal{T}}_{\pm}(\widetilde{E_{13}} - \widetilde{E_{23}}) = -(\widetilde{E_{14}} - \widetilde{E_{24}}), \ \widehat{J}^{\mathcal{T}}_{\pm}|_{T^{\mathcal{H}^+}_V} = 0.$$

v) The integrability conditions a'), b") and c") will now be calculated in the point $\tilde{s}_m \in SO(M), \ m \in U_m$ by the use of the basis $\{Z_i : \ i = 1, \ldots, 4\} \cup \{\widetilde{E_{ij}} : \ i < j\}$. For this, let the frame $\tilde{s} : U_m \to SO(M)$ in $m \in U_m$ be parallel, $\nabla s_i(m) = 0$, $i = 1, \ldots, 4$. It follows

$$\begin{aligned}
&\omega_{ij}(m) = 0, \quad [s_i, s_j]_m = \nabla_{s_i} s_j(m) - \nabla_{s_j} s_i(m) = 0 \text{ and}\\
&R_{ijkl}(m) = s_i \omega_{kl}(s_j) - s_j \omega_{kl}(s_i), \ i, j, k, l = 1 \ldots, 4.
\end{aligned}$$

With these equalities the following commutator relations result.

1) First, we obtain that $\mathcal{O}^+_+, \mathcal{O}^+_-$ are geodesic:

$$[S_i, S_j]_{\tilde{s}_m} = 0,$$

$$\begin{aligned}
[Z_i, Z_j]_{\tilde{s}_m} = \quad & [S_i, -\sum_{k<l} \epsilon_k \epsilon_l \omega_{kl}((\tilde{s} \cdot A)_i)(Ad(A^{-1})E_{kl})\widetilde{)}]_{\tilde{s}_m}\\
+ \quad & [-\sum_{k<l} \epsilon_k \epsilon_l \omega_{kl}((\tilde{s} \cdot A)_i)(Ad(A^{-1})E_{kl}\widetilde{)}, S_j]_{\tilde{s}_m}\\
= \quad - & \sum_{k<l} \epsilon_k \epsilon_l R_{ijkl} \widetilde{E_{kl}}_{\tilde{s}_m},
\end{aligned}$$

$$
\begin{aligned}
[Z_1+Z_2,Z_3]_{\tilde s_m} = & \quad (R_{1313}+R_{2313})\widetilde{E_{13}}_{\tilde s_m} + (R_{1314}+R_{2314})\widetilde{E_{14}}_{\tilde s_m} \\
+ & \quad (-R_{1323}-R_{2323})\widetilde{E_{23}}_{\tilde s_m} + (-R_{1324}-R_{2324})\widetilde{E_{24}}_{\tilde s_m} \ + C_1 \\
= & \quad \frac{1}{2}(R_{1313}+R_{2313}+R_{1323}+R_{2323})(\widetilde{E_{13}}_{\tilde s_m} - \widetilde{E_{23}}_{\tilde s_m}) \\
+ & \quad \frac{1}{2}(R_{1314}+R_{2314}+R_{1324}+R_{2324})(\widetilde{E_{14}}_{\tilde s_m} - \widetilde{E_{24}}_{\tilde s_m}) \ + C_2 \\
& \in \widehat{\mathcal{L}}^+, \ C_1, C_2 \in T_V^{H^+},
\end{aligned}
$$

$$
\begin{aligned}
[Z_1+Z_2,Z_4]_{\tilde s_m} = & \quad \frac{1}{2}(R_{1413}+R_{2413}+R_{1423}+R_{2423})(\widetilde{E_{13}}_{\tilde s_m} - \widetilde{E_{23}}_{\tilde s_m}) \\
+ & \quad \frac{1}{2}(R_{1414}+R_{2414}+R_{1424}+R_{2424})(\widetilde{E_{14}}_{\tilde s_m} - \widetilde{E_{24}}_{\tilde s_m}) \ + C \\
& \in \widehat{\mathcal{L}}^+, \ C \in T_V^{H^+},
\end{aligned}
$$

$$
[Z_i, \widetilde{E_{kl}}]_{\tilde s_m} = -\left[\frac{d}{dt}\bigg|_{t=0} (\tilde s_m \cdot \exp tE_{kl})_i \right]^*_{\tilde s_m} = -\left[(\tilde s_m \cdot E_{kl})_i \right]^*_{\tilde s_m}.
$$

($X^* \in TSO(M)$ denotes the horizontal lift of $X \in TM$.)

$$
\begin{aligned}
&[Z_1+Z_2,\widetilde{E_{12}}]_{\tilde s_m} = Z_{2\tilde s_m} + Z_{1\tilde s_m}, &&[Z_1+Z_2,\widetilde{E_{13}}]_{\tilde s_m} = Z_{3\tilde s_m}, \\
&[Z_1+Z_2,\widetilde{E_{14}}]_{\tilde s_m} = Z_{4\tilde s_m}, &&[Z_1+Z_2,\widetilde{E_{23}}]_{\tilde s_m} = -Z_{3\tilde s_m}, \\
&[Z_1+Z_2,\widetilde{E_{24}}]_{\tilde s_m} = -Z_{4\tilde s_m}, &&[Z_1+Z_2,\widetilde{E_{34}}]_{\tilde s_m} = 0
\end{aligned}
$$

It follows $[\Gamma(\widehat{\mathcal{K}}^+), \Gamma(\widehat{\mathcal{L}}^+)] \subset \Gamma(\widehat{\mathcal{L}}^+)$ for $\mathcal{O}_+^+, \mathcal{O}_-^+$ and therefore, $\mathcal{O}_+^+, \mathcal{O}_-^+$ are geodesic.

2) We proof now condition b").

$$
[Z_1+Z_2,\widetilde{E_{13}}-\widetilde{E_{23}}]_{\tilde s_m} = Z_{3\tilde s_m} + Z_{3\tilde s_m}, \quad [Z_1+Z_2,\widetilde{E_{14}}-\widetilde{E_{24}}]_{\tilde s_m} = Z_{4\tilde s_m} + Z_{4\tilde s_m},
$$
$$
[Z_1+Z_2,\widetilde{E_{13}}+\widetilde{E_{23}}]_{\tilde s_m} = 0, \qquad [Z_1+Z_2,\widetilde{E_{14}}+\widetilde{E_{24}}]_{\tilde s_m} = 0.
$$

For $\mathcal{O}_-^+$ we have

$$
[Z_1+Z_2, \widehat{J}_-^{\mathcal{T}}(\widetilde{E_{13}}-\widetilde{E_{23}})]_{\tilde s_m} - \widehat{J}_-^{\mathcal{T}}[Z_1+Z_2,\widetilde{E_{13}}-\widetilde{E_{23}}]_{\tilde s_m} =
$$
$$
-4Z_{4\tilde s_m} \notin \widehat{\mathcal{K}}^+ \oplus T_V^{H^+}
$$

Condition b") isn't satisfied in any point of $SO(M)$ and $\mathcal{O}_-^+$ is never integrable.

For $\mathcal{O}_+^+$ holds:

$$
[Z_1+Z_2, \widehat{J}_+^{\mathcal{T}}(\widetilde{E_{13}}-\widetilde{E_{23}})]_{\tilde s_m} - \widehat{J}_+^{\mathcal{T}}[Z_1+Z_2,\widetilde{E_{13}}-\widetilde{E_{23}}]_{\tilde s_m} = -2Z_{4\tilde s_m} + 2Z_{4\tilde s_m} = 0,
$$
$$
[Z_1+Z_2, \widehat{J}_+^{\mathcal{T}}(\widetilde{E_{14}}-\widetilde{E_{24}})]_{\tilde s_m} - \widehat{J}_+^{\mathcal{T}}[Z_1+Z_2,\widetilde{E_{14}}-\widetilde{E_{24}}]_{\tilde s_m} = \ 2Z_{3\tilde s_m} - 2Z_{3\tilde s_m} = 0,
$$

$$[Z_1+Z_2,\widehat{J}_+^T(\widetilde{E_{13}}+\widetilde{E_{23}})]_{\tilde{s}_m}-\widehat{J}_+^T[Z_1+Z_2,\widetilde{E_{13}}+\widetilde{E_{23}}]_{\tilde{s}_m}=0,$$
$$[Z_1+Z_2,\widehat{J}_+^T(\widetilde{E_{14}}+\widetilde{E_{24}})]_{\tilde{s}_m}-\widehat{J}_+^T[Z_1+Z_2,\widetilde{E_{14}}+\widetilde{E_{24}}]_{\tilde{s}_m}=0,$$
$$[Z_1+Z_2,\widehat{J}_+^T\widetilde{E_{12}}]_{\tilde{s}_m}-\widehat{J}_+^T[Z_1+Z_2,\widetilde{E_{12}}]_{\tilde{s}_m}=Z_{2\tilde{s}_m}+Z_{1\tilde{s}_m}\in\widehat{\mathcal{K}}^+,$$
$$[Z_1+Z_2,\widehat{J}_+^T\widetilde{E_{34}}]_{\tilde{s}_m}-\widehat{J}_+^T[Z_1+Z_2,\widetilde{E_{34}}]_{\tilde{s}_m}=0.$$

We obtain a condition for the integrability:

$$\begin{aligned}&[Z_1+Z_2,\widehat{J}_+^TZ_3]-\widehat{J}_+^T[Z_1+Z_2,Z_3]\in\widehat{\mathcal{K}}^+\oplus T_V^{H^+}\\&[Z_1+Z_2,\widehat{J}_+^TZ_4]-\widehat{J}_+^T[Z_1+Z_2,Z_4]\in\widehat{\mathcal{K}}^+\oplus T_V^{H^+}\end{aligned}\quad\Leftrightarrow$$

$$R_{1413}+R_{2413}+R_{1423}+R_{2423}+R_{1314}+R_{2314}+R_{1324}+R_{2324}=0$$
$$R_{1414}+R_{2414}+R_{1424}+R_{2424}+R_{1313}+R_{2313}+R_{1323}+R_{2323}=0$$

From the conclusion to Lemma 2.4, it follows that this is equivalent to:

$K^{\hat{\pi}(\tilde{s}_m)}=\mathbf{R}(s_{1m}+s_{2m})\subset T_mM$ is principal null direction.

3) It will be proofed now that condition c") is satisfied in a point of $SO(M)$ if and only if condition b") is there satisfied. Hence, we obtain no more conditions for the integrability.

The fibres of $\mathcal{Z}^+(M)$ are complex submanifolds, because of this conditions c') and c") restricted to the fibres $T_V\mathcal{Z}^+(M)$ resp. $T_VSO(M)$ are certainly satisfied.

$$[Z_3,\widetilde{E_{13}}-\widetilde{E_{23}}]_{\tilde{s}_m}=Z_{1\tilde{s}_m}-Z_{2\tilde{s}_m},\quad[Z_4,\widetilde{E_{13}}-\widetilde{E_{23}}]_{\tilde{s}_m}=0,$$
$$[Z_3,\widetilde{E_{14}}-\widetilde{E_{24}}]_{\tilde{s}_m}=0,\quad[Z_4,\widetilde{E_{14}}-\widetilde{E_{24}}]_{\tilde{s}_m}=Z_{1\tilde{s}_m}-Z_{2\tilde{s}_m},$$
$$[Z_3,\widetilde{E_{12}}]_{\tilde{s}_m}=0,\quad[Z_4,\widetilde{E_{12}}]_{\tilde{s}_m}=0,$$
$$[Z_3,\widetilde{E_{34}}]_{\tilde{s}_m}=-Z_{4\tilde{s}_m},\quad[Z_4,\widetilde{E_{34}}]_{\tilde{s}_m}=Z_{3\tilde{s}_m},$$
$$[Z_3,\widetilde{E_{13}}+\widetilde{E_{23}}]_{\tilde{s}_m}=Z_{1\tilde{s}_m}+Z_{2\tilde{s}_m},\quad[Z_4,\widetilde{E_{13}}+\widetilde{E_{23}}]_{\tilde{s}_m}=0,$$
$$[Z_3,\widetilde{E_{14}}+\widetilde{E_{24}}]_{\tilde{s}_m}=0,\quad[Z_4,\widetilde{E_{14}}+\widetilde{E_{24}}]_{\tilde{s}_m}=Z_{1\tilde{s}_m}+Z_{2\tilde{s}_m},$$
$$[Z_i,Z_j]_{\tilde{s}_m}\in\widehat{\mathcal{L}}^+.$$

It follows $[\widehat{J}_+^Tl_1,l_2]_{\tilde{s}_m}+[l_1,\widehat{J}_+^Tl_2]_{\tilde{s}_m}\in\widehat{\mathcal{L}}^+\quad\forall\, l_1,l_2\in\Gamma(\widehat{\mathcal{L}}^+)$.
Let N be the Nijenhuis tensor.

$$N(Z_3,Z_4)_{\tilde{s}_m}=\widehat{J}_+^T([\widehat{J}_+^TZ_3,Z_4]_{\tilde{s}_m}+[Z_3,\widehat{J}_+^TZ_4]_{\tilde{s}_m})-[\widehat{J}_+^TZ_3,\widehat{J}_+^TZ_4]_{\tilde{s}_m}+[Z_3,Z_4]_{\tilde{s}_m}=0,$$
$$N(Z_3,Z_3)_{\tilde{s}_m}=N(Z_4,Z_4)_{\tilde{s}_m}=0,$$
$$N(Z_3,\widetilde{E_{13}}+\widetilde{E_{23}})_{\tilde{s}_m},\ N(Z_3,\widetilde{E_{14}}+\widetilde{E_{24}})_{\tilde{s}_m},\ N(Z_4,\widetilde{E_{13}}+\widetilde{E_{23}})_{\tilde{s}_m},\ N(Z_4,\widetilde{E_{14}}+\widetilde{E_{24}})_{\tilde{s}_m}\in\widehat{\mathcal{K}}^+,$$
$$N(Z_3,\widetilde{E_{12}})_{\tilde{s}_m}=N(Z_4,\widetilde{E_{12}})_{\tilde{s}_m}=0,$$
$$N(Z_3,\widetilde{E_{34}})_{\tilde{s}_m}=\widehat{J}_+^T(-Z_{3\tilde{s}_m})-Z_{4\tilde{s}_m}=0,\quad N(Z_4,\widetilde{E_{34}})_{\tilde{s}_m}=\widehat{J}_+^T(-Z_{4\tilde{s}_m})+Z_{3\tilde{s}_m}=0.$$

$$N(Z_3, \widetilde{E_{13}} - \widetilde{E_{23}})_{\tilde{s}_m} = -N(Z_4, \widetilde{E_{14}} - \widetilde{E_{24}})_{\tilde{s}_m} =$$
$$\widehat{J}_+^{\mathcal{T}}([-Z_4, \widetilde{E_{13}} - \widetilde{E_{23}}]_{\tilde{s}_m} + [Z_3, -\widetilde{E_{14}} + \widetilde{E_{24}}]_{\tilde{s}_m}) - [Z_4, \widetilde{E_{14}} - \widetilde{E_{24}}]_{\tilde{s}_m} + [Z_3, \widetilde{E_{13}} - \widetilde{E_{23}}]_{\tilde{s}_m} = 0,$$
$$N(Z_3, \widetilde{E_{14}} - \widetilde{E_{24}})_{\tilde{s}_m} = -N(Z_4, \widetilde{E_{13}} - \widetilde{E_{23}})_{\tilde{s}_m} =$$
$$\widehat{J}_+^{\mathcal{T}}([-Z_4, \widetilde{E_{14}} - \widetilde{E_{24}}]_{\tilde{s}_m} + [Z_3, -\widetilde{E_{13}} + \widetilde{E_{23}}]_{\tilde{s}_m}) - [-Z_4, \widetilde{E_{13}} - \widetilde{E_{23}}]_{\tilde{s}_m} + [Z_3, \widetilde{E_{14}} - \widetilde{E_{24}}]_{\tilde{s}_m} = 0,$$
$$N(Z_1 + Z_2, \widetilde{E_{13}} - \widetilde{E_{23}})_{\tilde{s}_m} = \widehat{J}_+^{\mathcal{T}}(-2Z_{4\tilde{s}_m}) + 2Z_{3\tilde{s}_m} = 0,$$
$$N(Z_1 + Z_2, \widetilde{E_{14}} - \widetilde{E_{24}})_{\tilde{s}_m} = \widehat{J}_+^{\mathcal{T}}(2Z_{3\tilde{s}_m}) + 2Z_{4\tilde{s}_m} = 0,$$
$$N(Z_1 + Z_2, \widetilde{E_{13}} + \widetilde{E_{23}})_{\tilde{s}_m} = N(Z_1 + Z_2, \widetilde{E_{14}} + \widetilde{E_{24}})_{\tilde{s}_m} = N(Z_1 + Z_2, \widetilde{E_{34}})_{\tilde{s}_m} = 0,$$
$$N(Z_1 + Z_2, \widetilde{E_{34}})_{\tilde{s}_m} = Z_{1\tilde{s}_m} + Z_{2\tilde{s}_m} \in \widehat{\mathcal{K}}^+.$$

We obtain again the same conditions for the integrability as in 2):

$$N(Z_1 + Z_2, Z_3)_{\tilde{s}_m} = -\widehat{J}_+^{\mathcal{T}}[Z_1 + Z_2, Z_4]_{\tilde{s}_m} + [Z_1 + Z_2, Z_3]_{\tilde{s}_m} \in \widehat{\mathcal{K}}^+ \oplus T_{\overline{V}}^{\underline{H}^+} \text{ and}$$

$$N(Z_1 + Z_2, Z_4)_{\tilde{s}_m} = -\widehat{J}_+^{\mathcal{T}}[Z_1 + Z_2, -Z_3]_{\tilde{s}_m} + [Z_1 + Z_2, Z_4]_{\tilde{s}_m} \in \widehat{\mathcal{K}}^+ \oplus T_{\overline{V}}^{\underline{H}^+}.$$

4) Until now we have that the conditions a'), b") and c") for $\widehat{\mathcal{K}}^+, \widehat{\mathcal{L}}^+, \widehat{J}_+^+$ are satisfied in $\tilde{s}_m \in SO(M)$ if and only if $K^{\hat{\pi}(\tilde{s}_m)} \subset T_m M$ is a principal null direction. This proofs the 4. assertion of the theorem. But in the tangent space $T_m M$ exist at most 4 principal null directions or the Weyl tensor must vanish in $m \in M$, $W_m = 0$. Hence, the integrability conditions for $\mathcal{O}_+^+$ are satisfied everywhere on $\mathcal{Z}^+(M)$ if and only if (M, g) is conformally flat. □

In analogy to the case of the positive twistor space $\mathcal{Z}^+(M)$, the almost optical structure

$$\mathcal{O}_+^- = (\mathcal{K}^-, \mathcal{L}^-, J_+^-)$$

is defined on the twistor space $\mathcal{Z}^-(M)$:
$\mathcal{K}^- \subset T_H \mathcal{Z}^-(M)$ is a 1-dimensional horizontal bundle such that

$$\pi_*(\mathcal{K}_p^-) = K^p \subset T_{\pi(p)} M \quad \forall\, p = (K^p, L^p, J^p) \in Opt^-(T_{\pi(p)}M) \subset \mathcal{Z}^-(M),$$

$\mathcal{L}^- \subset T\mathcal{Z}^-(M)$ is a bundle such that $\mathcal{L}_p^- = \pi_*^{-1}(L^p)$ $\forall\, p \in \mathcal{Z}^-(M)$ and for the bundle morphism $J_+^- : \mathcal{L}^-/\mathcal{K}^- \to \mathcal{L}^-/\mathcal{K}^-$ holds

$$J_+^-|_{(T_H \mathcal{Z}^-(M) \cap \mathcal{L}^-)/\mathcal{K}^-} = \pi_{*p}^{-1} J^p \pi_{*p}, \quad J_+^-|_{T_V \mathcal{Z}^-(M)} = J^V.$$

(J^V is induced on the fibres by $J^{\mathbf{P}(\Delta^-)}$.)
Moreover, let $J_-^- : \mathcal{L}^-/\mathcal{K}^- \to \mathcal{L}^-/\mathcal{K}^-$,

$$J_-^-|_{(T_H \mathcal{Z}^-(M) \cap \mathcal{L}^-)/\mathcal{K}^-} = -\pi_{*p}^{-1} J^p \pi_{*p}, \quad J_-^-|_{T_V \mathcal{Z}^-(M)} = J^V$$

and we obtain the almost optical structure $(\mathcal{K}^-, \mathcal{L}^-, J_-^-)$.

The two twistor spaces with canonical almost optical structures $(\mathcal{Z}^+(M), \mathcal{O}_+^+)$ and $(\mathcal{Z}^-(M), \mathcal{O}_+^-)$ aren't in general optical diffeomorphic. In any case, a bundle morphism $\Xi : \mathcal{Z}^+(M) \to \mathcal{Z}^-(M)$, whose induced mapping on M is the identity, $\pi \circ \Xi = \pi$, $(\pi : \mathcal{Z}^{\pm}(M) \to M)$, isn't an optical diffeomorphism.

Proof: Suppose, it exists such a bundle morphism

$$
\begin{array}{ccc}
\Xi : \mathcal{Z}^+(M) & \cong & \mathcal{Z}^-(M), \\
\searrow & & \swarrow \\
& M &
\end{array}
$$

which is also an optical diffeomorphism. Then $\pi_* \Xi_* = \pi_*$ and it follows

$$K^p = \pi_* \Xi_* (\mathcal{K}_p^+) = \pi_* (\mathcal{K}_{\Xi(p)}^-) = K^{\Xi(p)} \quad \forall\, p = (K^p, L^p, J^p) \in \mathcal{Z}^+(M).$$

Hence, the mapping Ξ has to be the identification via the bundle of null directions

$$\Xi = \Theta : \mathcal{Z}^+(M) \cong SO(M) \times_{\mathbf{SO}(4,1)} P \cong \mathcal{Z}^-(M).$$

But we have

$$\Theta^*(J_+^-|_{(T_H \mathcal{Z}^-(M) \cap \mathcal{L}^-)/\mathcal{K}^-}) = -J_+^+|_{(T_H \mathcal{Z}^+(M) \cap \mathcal{L}^+)/\mathcal{K}^+}, \ \Theta^*(J_+^-|_{T_V \mathcal{Z}^-(M)}) = -J_+^+|_{T_V \mathcal{Z}^+(M)}.$$

(The identification $\mathbf{P}(\Delta^+) \cong P \cong \mathbf{P}(\Delta^-)$ is anti-holomorphic, comp. (4.3).)
The assumption was false □

But the proof shows that $(\mathcal{K}^+, \mathcal{L}^+, J_+^+) \cong (\mathcal{K}^-, \mathcal{L}^-, -J_+^-)$ via the identification $\Theta : \mathcal{Z}^+(M) \cong \mathcal{Z}^-(M)$. The canonical almost optical structures $\mathcal{O}_+^+$ and $\mathcal{O}_+^-$ don't essentially differ. This makes clear that the analogous assertions to proposition 5.1 holds:

$\mathcal{O}_+^-$ is integrable if and only if M is conformally flat and $\mathcal{O}_-^-$ is never integrable.

5.2 The twistor spaces of $\mathbf{R}_1^4$, $\mathbf{S}_1^4$ and $\mathbf{H}_1^4$

A. The twistor space $\mathcal{Z}^+(M)$ of a Lorentz warped product $M = \mathbf{R}_1^1 \times_f N$ with optical structure $\mathcal{O}_+^+$

Let $M = \mathbf{R}_1^1 \times_f N$ be a Lorentz warped product, where (N, h) is an oriented 3-dimensional Riemannian manifold. In the last section, we suggest that the twistor space

$$\Omega : \mathcal{Z}^+(M) = SO(M)/H^+ \cong \mathbf{R} \times Z(N)$$

is related to the bundle of unit spheres over the Riemannian manifold (N, h). On the bundle $Z(N)$, a canonical almost CR-structure $(H^{Z(N)}, J^{Z(N)})$ was defined in section (3.1). The product $\mathbf{R} \times Z(N)$ admits the natural optical structure (comp. (3.2))

$$
\begin{aligned}
\mathcal{O}^{Z(N)} &= (\mathcal{K}^{Z(N)}, \mathcal{L}^{Z(N)}, J^{Z(N)}) \\
:&= (T\mathbf{R}, T\mathbf{R} \times H^{Z(N)}, J^{Z(N)} : T\mathbf{R} \times H^{Z(N)}/T\mathbf{R} \to T\mathbf{R} \times H^{Z(N)}/T\mathbf{R}) \\
&\quad (T\mathbf{R} \times H^{Z(N)}/T\mathbf{R} \cong H^{Z(N)}).
\end{aligned}
$$

Proposition 5.2 *Let $M := \mathbf{R}^1_1 \times_f N$ be a Lorentz warped product, $M_1 := \mathbf{R}^1_1 \times_1 N$ denotes the corresponding Lorentz warped product with warping function $\tilde{f} = 1$.*

1. *The twistor spaces with canonical almost optical structures $(\mathcal{Z}^+(M), \mathcal{O}^+_+)$ and $(\mathcal{Z}^+(M_1), \mathcal{O}^+_+)$ are optical diffeomorphic. ($\mathcal{Z}^+(\mathbf{R}^1_1 \times_f N)$ with almost optical structure $\mathcal{O}^+_+$ is 'independent' of the warping function.)*

2. *Let M be in addition conformally flat. The optical structures $\mathcal{O}^+_+$ and $\mathcal{O}^{Z(N)}$ on the twistor space $\mathcal{Z}^+(M) \cong \mathbf{R} \times Z(N)$ are in every point of the twistor space locally equivalent. In particular, the CR-structure on the manifold $U_{(t_0,n_0)}/\sim^{\mathcal{K}^+}$, $U_{(t_0,n_0)} \ni (t_0, n_0)$, that is induced by $(\mathcal{Z}^+(M), \mathcal{O}^+_+)$ in a point $(t_0, n_0) \in \mathcal{Z}^+(M)$, is locally equivalent to the CR-structure in the point $n_0 \in Z(N)$.*

3. *Let M be conformally flat and in addition, let the distribution $\mathcal{K}^+$ over the twistor space $\mathcal{Z}^+(M)$ be regular. Moreover, let exist a complete vector field $K \in \Gamma(T\mathcal{Z}^+(M))$ such that $\mathcal{K}^+ = \mathbf{R} \cdot K$. Then $\mathcal{Z}^+(M)$ is a trivial $\mathbf{R}$-fibration over the manifold $\mathcal{Z}^+(M)/\sim^{\mathcal{K}^+}$. The twistor space with optical structure $(\mathcal{Z}^+(M), \mathcal{O}^+_+)$ is optical diffeomorphic to $(\ \mathbf{R} \times Z(N), \mathcal{O}^{Z(N)}\ )$ and the CR-manifolds $\mathcal{Z}^+(M)/\sim^{\mathcal{K}^+} \cong Z(N)$ are CR-diffeomorphic.*

Proof: First, we will describe the optical structure $\mathcal{O}^+_+$ in a point of a suitable local trivialization of the twistor space $\mathcal{Z}^+(M)$.
i) The fibre type $\mathbf{SO}(4,1)/H^+ \cong P$ of the twistor space $\mathcal{Z}^+(M)$ is identified with $\mathbf{SO}(3)/\mathbf{SO}(2) \cong \mathbf{S}^2$ (comp. (4.4)):

$$i_P : \mathbf{SO}(4,1)/H^+ \cong \mathbf{SO}(3)/\mathbf{SO}(2),\ \begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix} H^+ \leftrightarrow A\mathbf{SO}(2).$$

Let E_{ij} denote the elements of the standard basis of $\mathbf{so}(4,1)$ and F_{ij} those of $\mathbf{so}(3)$. For the differential i_{P*} in $T_{eH^+}(\mathbf{SO}(4,1)/H^+)$ holds:

$$\frac{d}{dt}\Big|_{t=0} \exp tE_{13}H^+ = \frac{d}{dt}\Big|_{t=0} \exp tE_{13} \cdot \exp -t(E_{13}+E_{23})H^+ = \frac{d}{dt}\Big|_{t=0} \exp -tE_{23}H^+ \mapsto$$

$$\begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} = \frac{d}{dt}\Big|_{t=0} \exp -tF_{12}\mathbf{SO}(2) \in T_{(1,0,0)}\mathbf{S}^2 = T_{e\mathbf{SO}(2)}(\mathbf{SO}(3)/\mathbf{SO}(2)),$$

$$\frac{d}{dt}\Big|_{t=0} \exp tE_{14}H^+ \mapsto \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} = \frac{d}{dt}\Big|_{t=0} \exp -tF_{13}\mathbf{SO}(2),$$

$$\frac{d}{dt}\Big|_{t=0} \exp tE_{23}H^+ \mapsto \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \frac{d}{dt}\Big|_{t=0} \exp tF_{12}\mathbf{SO}(2),$$

$$\frac{d}{dt}\Big|_{t=0} \exp tE_{24}H^+ \mapsto \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \frac{d}{dt}\Big|_{t=0} \exp tF_{13}\mathbf{SO}(2),$$

$$\Big(\frac{d}{dt}\Big|_{t=0} \exp tE_{12}H^+ = \frac{d}{dt}\Big|_{t=0} \exp tE_{34}H^+ = 0\Big).$$

ii) Now, let $(\hat{s}_2, \hat{s}_3, \hat{s}_4) : U_{n_0} \to SO(N)$ be a local orthonormal frame in a neighborhood of $n_0 \in N$. Let the local frame $(\hat{s}_2, \hat{s}_3, \hat{s}_4)$ be in addition parallel in the point n_0,

$\nabla \hat{s}_i(n_0) = 0,\ i = 2,3,4.\ T := \frac{\partial}{\partial t} \in \Gamma(TM)$ is a timelike vector field, $g(T,T) = -1$. Then

$$\tilde{s} = (T, s_2, s_3, s_4) = (T, \frac{1}{f}\hat{s}_2, \frac{1}{f}\hat{s}_3, \frac{1}{f}\hat{s}_4) : \mathbf{R} \times U_{n_0} \to SO(M)$$

is a local orthonormal frame in M. In the points $(t, n_0) \in \mathbf{R} \times U_{n_0}$, it holds

$$[T, s_i] = -\frac{f'}{f} s_i,\ [s_i, s_j] = 0,\ i,j = 2,3,4\ ,$$

$$\nabla_T T = 0,\ \nabla_T s_i = 0,\ \nabla_{s_i} T = [s_i, T] = \frac{f'}{f} s_i,\ \nabla_{s_i} s_j = \delta_{ij} \frac{f'}{f} T,\quad i,j = 2,3,4.$$

The local frame $\tilde{s}$ gives a local trivialization of the frame bundle

$$SO(M)|_{\mathbf{R} \times U_{n_0}} \cong \mathbf{R} \times U_{n_0} \times \mathbf{SO}(4,1).$$

In the points $(t, n_0, e) \in \mathbf{R} \times U_{n_0} \times \mathbf{SO}(4,1)$ of the trivialization, the horizontal vectors

$$Z_i = (s_i, -\sum_{k<l} \epsilon_k \epsilon_l \omega_{kl}(s_i) \widetilde{E_{kl}}\ ),\quad i = 1, \ldots, 4,\ (s_1 = T),$$

are given. We have for the differential of the natural projection $\hat{\pi} : SO(M) \to \mathcal{Z}^+(M)$ in the points $(t, n_0, e) \in \mathbf{R} \times U_{n_0} \times \mathbf{SO}(4,1)$

$$\hat{\pi}_*(Z_1) = \hat{\pi}_*((T, 0)) = (T, 0) \in T_{(t, n_0, eH^+)} \mathbf{R} \times U_{n_0} \times \mathbf{SO}(4,1)/H^+,$$

$$\hat{\pi}_*(Z_2) = \hat{\pi}_*((s_2, \omega_{12}(s_2)\widetilde{E_{12}})) = (s_2, \left.\frac{d}{dr}\right|_{r=0} \exp(r\omega_{12}(s_2)E_{12})H^+) = (s_2, 0),$$

$$\hat{\pi}_*(Z_3) = \hat{\pi}_*((s_3, \omega_{13}(s_3)\widetilde{E_{13}})) = (s_3, \left.\frac{d}{dr}\right|_{r=0} \exp(r\omega_{13}(s_3)E_{13})H^+),$$

$$\hat{\pi}_*(Z_4) = \hat{\pi}_*((s_4, \omega_{14}(s_4)\widetilde{E_{14}})) = (s_4, \left.\frac{d}{dr}\right|_{r=0} \exp(r\omega_{14}(s_4)E_{14})H^+).$$

For the identification $\Omega : \mathcal{Z}^+(M) \cong \mathbf{R} \times Z(N)$ locally holds

$$\Omega = id \times i_P : \mathbf{R} \times U_{n_0} \times \mathbf{SO}(4,1)/H^+ \to \mathbf{R} \times U_{n_0} \times \mathbf{SO}(3)/\mathbf{SO}(2).$$

We obtain $\quad (\hat{Z}_i := (id \times i_P)_* \circ \hat{\pi}_*(Z_i)\ )$

$$\hat{Z}_1 = (T, 0),\ \hat{Z}_2 = (s_2, 0) \in T_{(t, n_0, e\mathbf{SO}(2))} \mathbf{R} \times U_{n_0} \times \mathbf{SO}(3)/\mathbf{SO}(2),$$

$$\hat{Z}_3 = (s_3, -\left.\frac{d}{dr}\right|_{r=0} \exp(r\frac{f'}{f}F_{12})\mathbf{SO}(2)),$$

$$\hat{Z}_4 = (s_4, -\left.\frac{d}{dr}\right|_{r=0} \exp(r\frac{f'}{f}F_{13})\mathbf{SO}(2)).$$

iii) For the optical structure $\mathcal{O}_+^+ = (\mathcal{K}^+, \mathcal{L}^+, J_+^+)$ in the tangent space of a point $p_0 = (t, n_0, e\mathbf{SO}(2)) \in \mathbf{R} \times U_{n_0} \times \mathbf{SO}(3)/\mathbf{SO}(2)$ holds

$$(V_1 := (0, \frac{d}{dr}\Big|_{r=0} \exp tF_{12}\mathbf{SO}(2)),\ V_2 := (0, \frac{d}{dr}\Big|_{r=0} \exp tF_{13}\mathbf{SO}(2))\ ),$$

$$\mathcal{K}_{p_0}^+ = \mathbf{R}(\hat{Z}_1 + \hat{Z}_2),\ \mathcal{L}_{p_0}^+ = Span\{\hat{Z}_1 + \hat{Z}_2, \hat{Z}_3, \hat{Z}_4, V_1, V_2\},$$
$$J_+^+(\hat{Z}_3 + \mathcal{K}_{p_0}^+) = -\hat{Z}_4 + \mathcal{K}_{p_0}^+,\ J_+^+(\hat{Z}_4 + \mathcal{K}_{p_0}^+) = \hat{Z}_3 + \mathcal{K}_{p_0}^+,$$
$$J_+^+(V_1 + \mathcal{K}_{p_0}^+) = -V_2 + \mathcal{K}_{p_0}^+,\ J_+^+(V_2 + \mathcal{K}_{p_0}^+) = V_1 + \mathcal{K}_{p_0}^+.$$

It follows

$$J_+^+((s_3, 0) + \mathcal{K}_{p_0}^+) = (-s_4, 0) + \mathcal{K}_{p_0}^+,\ J_+^+((s_4, 0) + \mathcal{K}_{p_0}^+) = (s_3, 0) + \mathcal{K}_{p_0}^+,$$

$$J_+^+(V_1 + \mathcal{K}_{p_0}^+) = -V_2 + \mathcal{K}_{p_0}^+,\ J_+^+(V_2 + \mathcal{K}_{p_0}^+) = V_1 + \mathcal{K}_{p_0}^+.$$

1. Consider the diffeomorphism

$$\Xi : \mathcal{Z}^+(M) \cong \mathbf{R} \times Z(N) \to \mathcal{Z}^+(M_1) \cong \mathbf{R} \times Z(N),\ (t, n) \mapsto (F(t), n),$$

where $F : \mathbf{R} \to \mathbf{R}$ is a diffeomorphism such that $F' = \frac{1}{f}$. Then $\Xi_*(\mathcal{K}_f^+) = (\mathcal{K}_1^+)$ and the mapping $\Xi : (\mathcal{Z}^+(M), \mathcal{O}_+^+) \to (\mathcal{Z}^+(M_1), \mathcal{O}_+^+)$ is an optical diffeomorphism.

2. On the trivialization $U_{n_0} \times \mathbf{SO}(3)/\mathbf{SO}(2)$ of the bundle of unit spheres $Z(N)$, the canonical CR-structure is given in the point $(n_0, e\mathbf{SO}(2))$ by

$$H^{Z(N)} = Span\{(\hat{s}_3, 0), (\hat{s}_4, 0), V_1, V_2\},$$

$$J^{Z(N)}((\hat{s}_3, 0)) = (-\hat{s}_4, 0),\ J^{Z(N)}((\hat{s}_4, 0)) = (\hat{s}_3, 0),$$
$$J^{Z(N)}(V_1) = -V_2,\ J^{Z(N)}(V_2) = V_1.$$

Let $\mathcal{O}_+^+$ be integrable, hence M conformally flat. Let the neighborhood U_{n_0} be in addition such that the bundle $\mathcal{K}^+$ over $\mathcal{Z}^+(M)|_{\mathbf{R}\times U_{n_0}}$ is regular. $(\mathcal{Z}^+(M)|_{\mathbf{R}\times U_{n_0}})/\sim^{\mathcal{K}^+}$ and $\{t\} \times Z(N)|_{U_{n_0}} \cong Z(N)|_{U_{n_0}}$, $t \in \mathbf{R}$, are naturally identified and the induced CR-structure of $\mathcal{O}_+^+$ on $(\mathcal{Z}^+(M)|_{\mathbf{R}\times U_{n_0}})/\sim^{\mathcal{K}^+} \cong Z(N)|_{U_{n_0}}$ (if M isn't conformally flat $(\mathcal{Z}^+(M)|_{\mathbf{R}\times U_{n_0}})/\sim^{\mathcal{K}^+}$ don't admit a natural induced CR-structure, comp. proof of proposition 5.1) is identical to the canonical CR-structure $(H^{Z(N)}, J^{Z(N)})$. This proofs the 2. assertion of the proposition.

3. Now, let the bundle $\mathcal{K}^+ = \mathbf{R} \cdot K$ over $\mathcal{Z}^+(M)$ be regular. Then, it exists a nowhere vanishing vector field S on $\mathcal{Z}^+(M) \cong \mathbf{R} \times Z(N)$ tangent to $Z(N)$ such that $\mathcal{K}^+ = \mathbf{R}(T + S)$. The flow $\phi_t^S : \mathbf{R} \times M \to M$ is by assumption a 1-parameter group on M.

$$\Xi : (\mathcal{Z}^+(M), \mathcal{O}_+^+) \to (\mathbf{R} \times Z(N), \mathcal{O}^{Z(N)}),\ (t, n) \mapsto (t, \phi_{-t}^S(n))$$

is a diffeomorphism. $\mathcal{Z}^+(M)/\sim^{\mathcal{K}}$ and $\{0\} \times Z(N) \cong Z(N)$ are naturally identified and $\mathcal{Z}^+(M)$ is a trivial $\mathbf{R}$-fibration over $\mathcal{Z}^+(M)/\sim^{\mathcal{K}^+} \cong Z(N)$. The induced CR-structure

(H', J') on $\mathcal{Z}^+(M)/\sim^{\mathcal{K}^+}$ is equivalent to the CR-structure $(H^{Z(N)}, J^{Z(N)})$ on $Z(N)$, and Ξ is an optical diffeomorphism $\square$

The assumptions of the 2. and 3. assertion in the proposition are very spezial, because the assumption that the Lorentz warped product is conformally flat implies that the Riemannian 3-manifold (N, h) has constant sectional curvature and is therefore in every point local isometric to one of the space forms $\mathbf{R}^3$, $\mathbf{S}^3$ or $\mathbf{H}^3$.

Now, we study the twistor spaces $\mathcal{Z}^+(\mathbf{R}^4_1)$, $\mathcal{Z}^+(\mathbf{S}^4_1)$ and $\mathcal{Z}^+(\mathbf{H}^4_1)$ with almost optical structure $\mathcal{O}^+_+$. On these twistor spaces, we can just apply proposition 5.2.

B. The twistor space $\mathcal{Z}^+(\mathbf{R}^4_1)$ of the Minkowski space with optical structure $\mathcal{O}^+_+$

Let $\mathbf{R}^4_1 = (\mathbf{R}^4, <,>_1)$ be the flat Minkowski space. The twistor space $\mathcal{Z}^+(\mathbf{R}^4_1)$ can be understood as

$$\begin{array}{ccccccc} \mathbf{R}^4_1 \times \mathbf{P}(\Delta^+) & \cong & \mathbf{R}^4_1 \times P & \cong & \mathbf{R}^4_1 \times \mathbf{S}^2 & \cong & \mathbf{R}^4_1 \times \mathbf{SO}(3)/\mathbf{SO}(2). \\ (x, [\psi^+]) & \mapsto & (x, \mathbf{R}n^{\psi^+}) & \mapsto & (x, \begin{pmatrix} a \\ b \\ c \end{pmatrix}) & \mapsto & (x, A\mathbf{SO}(2)) \\ & & = (x, \left[\begin{pmatrix} 1 \\ a \\ b \\ c \end{pmatrix}\right]) & & = (x, A\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}) & & \end{array}$$

First, the optical structure $\mathcal{O}^+_+$ on $\mathcal{Z}^+(M)$ can locally be described as follows:
Let $A : U \subset \mathbf{S}^2 \to \mathbf{SO}(3)$ be a local section in the frame bundle $SO(\mathbf{S}^2)$,
let $\{\frac{\partial}{\partial x_i} : \ i = 1, \dots, 4\}$ be the standard coordinates on $\mathbf{R}^4_1$ and
$\{E_{ij} : \ 1 \le i < j \le 3\}$ the standard basis of the Lie algebra $\mathbf{so}(3)$, $\widetilde{E_{ij}} \in \Gamma(\mathbf{SO}(3))$ denotes the corresponding left-invariant vector fields. Let

$$Z_i := (\ [(\frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_4}) \cdot A]_i, 0) \in \Gamma(T\mathbf{R}^4_1 \times TU), \quad i = 1, \dots, 4.$$

On $\mathbf{R}^4_1 \times U \subset \mathbf{R}^4_1 \times \mathbf{S}^2 \cong \mathcal{Z}^+(\mathbf{R}^4_1)$ the optical structure $\mathcal{O}^+_+$ is given in the point $u = (x, \begin{pmatrix} a \\ b \\ c \end{pmatrix}) = (x, A\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}) \in \mathbf{R}^4_1 \times U$ by

$$\mathcal{K}^+_u = \mathbf{R}(Z_{1u} + Z_{2u}) = \mathbf{R}(\frac{\partial}{\partial x_1} + a\frac{\partial}{\partial x_2} + b\frac{\partial}{\partial x_3} + c\frac{\partial}{\partial x_4}, 0) \subset T\mathbf{R}^4_1 \times TU,$$

$$\mathcal{L}^+_u = \mathcal{K}^+_u \oplus Span\{Z_{3u}, Z_{4u}\} \oplus Span\{\widetilde{E_{12u}}, \widetilde{E_{13u}}\}.$$

For $Z_i, Z_j \in \Gamma(T\mathbf{R}^4_1 \times TU)$ holds: $[Z_i, Z_j] = 0$. This proofs that $\mathcal{L}^+$ is an involutive distribution. Then J^+_+ on $\mathcal{L}^+|_{T\mathbf{R}^4_1 \times U}$ is given by

$$J^+_+(\widetilde{E_{12}} + \mathcal{K}^+) = -\widetilde{E_{13}} + \mathcal{K}^+, \ J^+_+(\widetilde{E_{13}} + \mathcal{K}^+) = \widetilde{E_{12}} + \mathcal{K}^+,$$

$$J^+_+(Z_3 + \mathcal{K}^+) = -Z_4 + \mathcal{K}^+, \ J^+_+(Z_4 + \mathcal{K}^+) = Z_3 + \mathcal{K}^+.$$

The twistor space $\mathcal{Z}^+(\mathbf{R}^4_1)$ with optical structure $\mathcal{O}^+_+$ can be globally described: consider the Cartesian product $\mathbf{R}\times\mathbf{R}\times T\mathbf{S}^2$ with almost optical structure $\widehat{\mathcal{O}} := (\widehat{\mathcal{K}},\widehat{\mathcal{L}},\widehat{J})$,

$$\widehat{\mathcal{K}} := \mathbf{R}(\frac{\partial}{\partial t_1}) = T\mathbf{R}, \quad \widehat{\mathcal{L}} := T\mathbf{R}\times T(T\mathbf{S}^2),$$

$$\widehat{\mathcal{L}}/\widehat{\mathcal{K}} \cong T(T\mathbf{S}^2), \quad \widehat{J} \cong -J^{T\mathbf{S}^2} : T(T\mathbf{S}^2) \to T(T\mathbf{S}^2).$$

($J^{T\mathbf{S}^2}$ denotes the natural complex structure on $T\mathbf{S}^2$ that is induced by the natural complex structure $J^{\mathbf{S}^2}$ of the sphere $\mathbf{S}^2$.)

$(\mathcal{Z}^+(\mathbf{R}^4_1),\mathcal{O}^+_+)$ and $(\mathbf{R}\times\mathbf{R}\times T\mathbf{S}^2,\widehat{\mathcal{O}})$ are optical diffeomorphic. In particular, the bundle $\mathcal{K}^+$ over $\mathcal{Z}^+(\mathbf{R}^4_1)$ is regular and $\mathcal{Z}^+(\mathbf{R}^4_1)$ is a $\mathbf{R}-$fibration over the quotient manifold $\mathcal{Z}^+(\mathbf{R}^4_1)/\!\sim^{\mathcal{K}^+}$.

$\mathcal{Z}^+(\mathbf{R}^4_1)/\!\sim^{\mathcal{K}^+}$ with canonical induced CR-structure (H',J') is as CR-manifold equivalent to the Cartesian product $\mathbf{R}\times T\mathbf{S}^2$ ($\cong N\mathbf{S}^2\oplus T\mathbf{S}^2$, $N\mathbf{S}^2$ the normal bundle of the sphere $\mathbf{S}^2$) with CR-structure ($T(T\mathbf{S}^2), -J^{T\mathbf{S}^2}$). From proposition 5.2 3., it follows that this is the bundle of unit spheres $Z(\mathbf{R}^3)$ with canonical CR-structure.

Proof: We have $T\mathbf{S}^2 = \mathbf{SO}(3)\times_{Ad\mathbf{SO}(2)}\underline{S}$, $\underline{S} = Span\{E_{12},E_{13}\}$. Consider the product $\mathbf{R}^3\times\mathbf{S}^2$.

$$\begin{array}{rccl}\rho_1 : & \mathbf{R}^3\times\mathbf{S}^2 & \to & \mathbf{R}\times T\mathbf{S}^2 \cong N\mathbf{S}^2\oplus T\mathbf{S}^2 \\ & (x_2,x_3,x_4,\begin{pmatrix} a\\ b\\ c\end{pmatrix}) & \mapsto & (t_2,[A,t_3E_{12}+t_4E_{13}]) \\ & = (A\begin{pmatrix} t_2\\ t_3\\ t_4\end{pmatrix},A\begin{pmatrix} 1\\ 0\\ 0\end{pmatrix}) & & = (t_2,\frac{d}{dr}\Big|_{r=0} A\exp r(t_3E_{12}+t_4E_{13})\mathbf{SO}(2))\end{array}$$

is a diffeomorphism. For the twistor space $\mathcal{Z}^+(\mathbf{R}^4_1)$, we have the identification

$$\begin{array}{rccl}\rho_2 : & \mathcal{Z}^+(\mathbf{R}^4_1) & \to & \mathbf{R}\times\mathbf{R}\times T\mathbf{S}^2 \cong \mathbf{R}\times(N\mathbf{S}^2\oplus T\mathbf{S}^2). \\ & (\begin{pmatrix} 1 & 0\\ 0 & A\end{pmatrix}\begin{pmatrix} t_1\\ t_2\\ t_3\\ t_4\end{pmatrix},A\begin{pmatrix} 1\\ 0\\ 0\end{pmatrix}) & \to & (t_1,t_2,[A,t_3E_{12}+t_4E_{13}])\end{array}$$

Let $T := \frac{\partial}{\partial t_1}$, $N := \frac{\partial}{\partial t_2} \in \Gamma(T(\mathbf{R}\times\mathbf{R}\times T\mathbf{S}^2))$ be the normal unit vector. We have

$$\rho_{2*}\left((\frac{\partial}{\partial x_1},0)\right) = T,\ \rho_{2*}(Z_2) = N,\ \rho_{2*}(Z_3),\rho_{2*}(Z_4) \in T(T\mathbf{S}^2) \subset T(\mathbf{R}\times\mathbf{R}\times T\mathbf{S}^2).$$

Then $\mathcal{O}^+_+$ on $\mathbf{R}\times\mathbf{R}\times T\mathbf{S}^2$ is via the identification ρ_2 given by

$$\mathcal{K}^+ = \mathbf{R}(T+N),\ \mathcal{L}^+ = \mathcal{K}^+\oplus T(T\mathbf{S}^2) \text{ and}$$
$$\mathcal{L}^+/\mathcal{K}^+ \cong T(T\mathbf{S}^2),\ J^+_+ \cong -J^{T\mathbf{S}^2} : T(T\mathbf{S}^2) \to T(T\mathbf{S}^2).$$

Consider the transformation

$$\begin{array}{rcl}\rho_3 : (\mathcal{Z}^+(\mathbf{R}^4_1),\mathcal{O}^+_+) \cong (\mathbf{R}\times\mathbf{R}\times T\mathbf{S}^2,\mathcal{O}^+_+) & \to & (\mathbf{R}\times\mathbf{R}\times T\mathbf{S}^2,\widehat{\mathcal{O}}) \\ (t_1,t_2,v) & \mapsto & (t_1+t_2,t_1-t_2,v)\end{array}.$$

It is obvious that ρ_3 is an optical diffeomorphism. $\square$

C. The twistor space $\mathcal{Z}^+(\mathbf{S}^4_1)$ of the pseudosphere $\mathbf{S}^4_1$ with optical structure $\mathcal{O}^+_+$

i) The pseudosphere $\mathbf{S}^4_1$

Let

$$\mathbf{S}^4_1 := \{x \in \mathbf{R}^5_1 : < x, x >_1 = 1\} \subset \mathbf{R}^5_1$$

be the pseudosphere (comp. [O'N83, chap. 4]). $\mathbf{S}^4_1$ is with the induced metric g of $\mathbf{R}^5_1$ a 4-dimensional Lorentzian manifold with constant sectional curvature $k = 1$. Hence, $\mathbf{S}^4_1$ is conformally flat. Let the orientation on $\mathbf{S}^4_1$ be such that

$$\tilde{e}_0 = (\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_4}, \frac{\partial}{\partial x_5}) \text{ in } T_{x_0}\mathbf{S}^4_1,\ x_0 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \in \mathbf{S}^4_1,$$

is positive oriented.

$\mathbf{S}^4_1$ can be described as homogeneous space:

The isometry group $\mathbf{SO}(5,1)$ of $\mathbf{R}^5_1$ acts transitively on $\mathbf{S}^4_1 \subset \mathbf{R}^5_1$. Let this action be denoted by τ. The isotropy group in the point $x_0 \in \mathbf{S}^4_1$ is the group $\mathbf{SO}(4,1)$, that is imbedded in $\mathbf{SO}(5,1)$ by

$$i : \mathbf{SO}(4,1) \hookrightarrow \mathbf{SO}(5,1),\ (A_{ij}) \mapsto \begin{pmatrix} A_{11} & 0 & A_{1i} \\ 0 & 1 & 0 \\ A_{i1} & 0 & A_{ij} \end{pmatrix}.$$

We obtain

$$\mathbf{S}^4_1 \cong \mathbf{SO}(5,1)/\mathbf{SO}(4,1),\ Ax_0 \mapsto A\mathbf{SO}(4,1).$$

Let $\{E_{ij} :\ 1 \leq i < j \leq 5\}$ be the standard basis of $\mathbf{so}(5,1)$ and

$$b := Span\{E_{12}, E_{23}, E_{24}, E_{25}\} \subset \mathbf{so}(5,1).$$

Then

$$\mathbf{so}(5,1) = \mathbf{so}(4,1) \oplus b,\ [b, \mathbf{so}(4,1)] \subset b,\ [b,b] \subset \mathbf{so}(4,1).$$

Hence, $\mathbf{S}^4_1 \cong \mathbf{SO}(5,1)/\mathbf{SO}(4,1)$ is a symmetric space.

The group $\mathbf{SO}(5,1)$ acts not only transitively on $\mathbf{S}^4_1 \cong \mathbf{SO}(5,1)/\mathbf{SO}(4,1)$, but also simple transitively on the frame bundle $SO(\mathbf{S}^4_1)$ (see [O'N83, chap.4]). $\mathbf{SO}(5,1)$ is identified with the frame bundle,

$$\mathbf{SO}(5,1) \cong SO(\mathbf{S}^4_1),\ A \mapsto d\tau_A(\tilde{e}_0).$$

The subspace $b \subset \mathbf{so}(5,1)$ is $Ad(\mathbf{SO}(4,1))$-invariant and the bundle of the horizontal spaces in $T\mathbf{SO}(5,1)$ with respect to the Levi-Civita connection of $(\mathbf{S}^4_1, g)$ is given by the left-invariant vector fields to the elements of b,

$$T_H\mathbf{SO}(5,1) = \mathbf{SO}(5,1) \times b \subset T\mathbf{SO}(5,1).$$

$\mathbf{S}^4_1$ is isometric to the Lorentz warped product $\mathbf{R}^1_1 \times_f \mathbf{S}^3$ with warping function $f = \cosh t$:

$$\nu_1 : \mathbf{R}^1_1 \times_f \mathbf{S}^3 \cong \mathbf{S}^4_1 \subset \mathbf{R}^5_1,\ (t,p) \mapsto (\sinh t, (1 + \sinh^2 t)^{\frac{1}{2}} p).$$

If $\mathbf{S}^3$ and $\mathbf{S}^4_1$ are described as homogeneous spaces so is ν_1 given by

$$\begin{array}{rrcl} \nu_1 \ : & \mathbf{R}^1_1 \times_f \mathbf{SO}(4)/\mathbf{SO}(3) & \cong & \mathbf{SO}(5,1)/\mathbf{SO}(4,1), \\ & (s, A\mathbf{SO}(3)) & \mapsto & A\exp(-sE_{12})\mathbf{SO}(4,1), \end{array}$$

where $i : \mathbf{SO}(4) \hookrightarrow \mathbf{SO}(5,1),\ A \mapsto \begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix}$, is the imbedding of $\mathbf{SO}(4)$ in $\mathbf{SO}(5,1)$.

$$(\quad A\exp(-sE_{12})\mathbf{SO}(4,1) \mapsto \begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix} \begin{pmatrix} \sinh s \\ (1+\sinh^2 s)^{\frac{1}{2}} \\ 0 \\ 0 \\ 0 \end{pmatrix} \overset{\nu_1^{-1}}{\mapsto} (s, A\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}) \mapsto (s, A\mathbf{SO}(3)). \quad)$$

ii) The twistor space $\mathcal{Z}^+(\mathbf{S}^4_1)$
We have

$$\mathcal{Z}^+(\mathbf{S}^4_1) = SO(\mathbf{S}^4_1) \times_{\mathbf{SO}(4,1)} \mathbf{SO}(4,1)/H^+ = \mathbf{SO}(5,1)/H^+.$$

In (4.4), we identified the twistor space $\mathcal{Z}^+(M)$ of the Lorentz warped product $M = \mathbf{R}^1_1 \times_f N$ with $\mathbf{R} \times Z(N)$. We have (comp. (3.1))

$$Z(\mathbf{S}^3) = \mathbf{SO}(4) \times_{\mathbf{SO}(3)} \mathbf{SO}(3)/\mathbf{SO}(2) = \mathbf{SO}(4)/\mathbf{SO}(2).$$

The identification Ω is given by

$$\begin{array}{rcl} \Omega : \mathcal{Z}^+(\mathbf{S}^4_1) \cong \mathbf{SO}(5,1)/H^+ & \to & \mathbf{R} \times \mathbf{SO}(4)/\mathbf{SO}(2) \cong \mathbf{R} \times Z(\mathbf{S}^3). \\ B\exp -sE_{12}H^+ & \mapsto & (s, B\mathbf{SO}(2)) \end{array}$$

$$(\quad B\exp -sE_{12}H^+ \in \mathbf{SO}(5,1)/H^+ \mapsto [B\exp -sE_{12}, \left[\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}\right]] \in \mathcal{Z}^+(\mathbf{S}^4_1) \overset{\Omega}{\mapsto}$$

$$(s, [B, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}]) \in \mathbf{R} \times Z(\mathbf{S}^3) \mapsto (s, B\mathbf{SO}(2)) \in \mathbf{R} \times \mathbf{SO}(4)/\mathbf{SO}(2). \quad)$$

The frame bundle $\mathbf{SO}(4)$ over $\mathbf{S}^3$ is trivial. The twistor space $\mathcal{Z}^+(\mathbf{S}^4_1)$ is diffeomorphic to $\mathbf{R} \times \mathbf{S}^3 \times \mathbf{S}^2$.

iii) The optical structure $\mathcal{O}^+_+$ on $\mathcal{Z}^+(\mathbf{S}^4_1)$
On $\mathbf{SO}(5,1)/H^+$, we have a canonical $\mathbf{SO}(5,1)$-invariant almost optical structure. For this, consider the Lie algebra $\mathbf{so}(5,1)$:

$$\mathbf{so}(5,1) = m^+ \oplus \underline{H}^+ \oplus b, \qquad (\mathbf{so}(4,1) = m^+ \oplus \underline{H}^+, \text{ comp. } (4.2)).$$

The subspace b is $Ad(\mathbf{SO}(4,1))$-invariant and therefore also $Ad(H^+)$-invariant, $[b, \underline{H}^+] \subset b$. On the subspace b exists an $Ad(H^+)$-invariant optical structure $(\underline{K}, \underline{L}, \underline{J})$: Let

$$\underline{K} := \mathbf{R}(-E_{12} + E_{23}) \subset b, \quad \underline{L} := Span\{E_{24}, E_{25}\} \oplus \underline{K} \subset b\ .$$

$\underline{K}$ and $\underline{L}$ are $Ad(H^+)$-invariant: $[\underline{K}, \underline{H}^+] \subset \underline{K}, \quad [\underline{L}, \underline{H}^+] \subset \underline{L}$.
On $\underline{L}/\underline{K}$, the $Ad(H^+)$-equivariant complex structure is given by

$$\underline{J} : \underline{L}/\underline{K} \to \underline{L}/\underline{K}, \quad \underline{J}(E_{24} + \underline{K}) = -E_{25} + \underline{K},\ \underline{J}(E_{25} + \underline{K}) = E_{24} + \underline{K}.$$

Now, consider the tangent bundle

$$T\mathbf{SO}(5,1)/H^+ = T_H\mathbf{SO}(5,1)/H^+ \oplus T_V\mathbf{SO}(5,1)/H^+.$$

The horizontal bundle in $T\mathbf{SO}(5,1)/H^+$ is given by

$$T_H\mathbf{SO}(5,1)/H^+ = \mathbf{SO}(5,1) \times_{AdH^+} b.$$

Let

$$\widehat{\mathcal{K}} := \mathbf{SO}(5,1) \times_{AdH^+} \underline{K}, \quad \widehat{\mathcal{L}} := (\mathbf{SO}(5,1) \times_{AdH^+} \underline{L}) \;\oplus\; T_V\mathbf{SO}(5,1)/H^+$$

and let $\widehat{J} : \widehat{\mathcal{L}}/\widehat{\mathcal{K}} \to \widehat{\mathcal{L}}/\widehat{\mathcal{K}}$ be given by

$$\widehat{J}|_{(\mathbf{SO}(5,1)\times_{AdH^+}\underline{L})/\widehat{\mathcal{K}}}[A, l + \underline{K}] = [A, \underline{J}(l + \underline{K})], \quad \widehat{J}|_{T_V\mathbf{SO}(5,1)/H^+} = J^V \cong J^{\mathbf{P}(\Delta^+)}.$$

$\widehat{\mathcal{O}} = (\widehat{\mathcal{K}}, \widehat{\mathcal{L}}, \widehat{J})$ is an almost optical structure on $\mathbf{SO}(5,1)/H^+$ and is invariantly defined to the $\mathbf{SO}(5,1)$-action on $\mathbf{SO}(5,1)/H^+$.
Consider now that $\mathcal{O}_+^+$ on $\mathcal{Z}^+(\mathbf{S}_1^4)$ is also $\mathbf{SO}(5,1)$-invariantly defined. Also note that in the point $x_0 \in \mathbf{S}_1^4$ holds

$$\left.\frac{d}{dr}\right|_{r=0} \exp r(-E_{12} + E_{23})\mathbf{SO}(4,1) = \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_3},$$

$$\left.\frac{d}{dr}\right|_{r=0} \exp rE_{24}\mathbf{SO}(4,1) = \frac{\partial}{\partial x_4}, \quad \left.\frac{d}{dr}\right|_{r=0} \exp rE_{25}\mathbf{SO}(4,1) = \frac{\partial}{\partial x_5}$$

and it follows that $\mathcal{O}_+^+$ and $\widehat{\mathcal{O}}$ in $eH^+ \in \mathbf{SO}(5,1)/H^+$ are equal. Via the identification $\mathcal{Z}^+(\mathbf{S}_1^4) \cong \mathbf{SO}(5,1)/H^+$, the optical structures $\mathcal{O}_+^+$ and $\widehat{\mathcal{O}}$ are identical.

iv) From proposition 5.2 1., it follows that $(\mathcal{Z}^+(\mathbf{R}_1^1 \times_f \mathbf{S}^3), \mathcal{O}_+^+)$ and $(\mathcal{Z}^+(\mathbf{R}_1^1 \times_1 \mathbf{S}^3), \mathcal{O}_+^+)$ are optical diffeomorphic. The bundle $\mathcal{K}^+$ on $\mathcal{Z}^+(\mathbf{R}_1^1 \times_1 \mathbf{S}^3) \cong \mathbf{R} \times \mathbf{SO}(4)/\mathbf{SO}(2)$ is given by

$$\mathcal{K}^+ = \mathbf{R}(T + S), \; S = \left.\frac{d}{dr}\right|_{r=0} A \exp rF_{12}\mathbf{SO}(2).$$

(F_{ij} are the standard basis elements of $\mathbf{so}(4)$.)
To $(\mathcal{Z}^+(\mathbf{S}_1^4), \mathcal{O}_+^+) \cong (\mathbf{R} \times \mathbf{SO}(4)/\mathbf{SO}(2), \mathcal{O}_+^+)$, we can apply proposition 5.2 3., since $\mathbf{SO}(4)/\mathbf{SO}(2)$ is compact.

$$\begin{array}{rcl} (\mathcal{Z}^+(\mathbf{S}_1^4), \mathcal{O}_+^+) \cong (\mathbf{R} \times \mathbf{SO}(4)/\mathbf{SO}(2), \mathcal{O}_+^+) & \cong & (\mathbf{R} \times \mathbf{SO}(4)/\mathbf{SO}(2), \mathcal{O}^{Z(\mathbf{S}^3)}) \\ (t, A\mathbf{SO}(2)) & \mapsto & (t, A \exp -tF_{12}\mathbf{SO}(2)) \end{array}$$

is the optical diffeomorphism. The underlying CR-manifold is $(Z(\mathbf{S}^3), H^{Z(\mathbf{S}^3)}, J^{Z(\mathbf{S}^3)})$ (see (3.1)).

D. The twistor space $\mathcal{Z}^+(\mathbf{H}^4_1)$ of the pseudohyperbolic space $\mathbf{H}^4_1$ with optical structure $\mathcal{O}^+_+$

i) The pseudohyperbolic space $\mathbf{H}^4_1$

Let

$$\mathbf{H}^4_1 := \{x \in \mathbf{R}^5_2 :< x, x >_2= -1\} \subset \mathbf{R}^5_2$$

be the pseudohyperbolic space (comp. [O'N83]). $\mathbf{H}^4_1$ is with the induced metric of $\mathbf{R}^5_2$ a Lorentzian 4-manifold with constant sectional curvature $k = -1$ and is conformally flat. Let the basis $(\frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_4}, \frac{\partial}{\partial x_5}) \in T_{x_0}\mathbf{H}^4_1,\ x_0 = (1,0,0,0,0)$, be positive oriented. $\mathbf{H}^4_1$ is a symmetric space:

The isometry group $\mathbf{SO}(5,2)$ of $\mathbf{R}^5_2$ acts transitively on $\mathbf{H}^4_1$. The isotropy group in the point $x_0 \in \mathbf{H}^4_1$ is the group $\mathbf{SO}(4,1)$, imbedded in $\mathbf{SO}(5,1)$ by

$$i : \mathbf{SO}(4,1) \hookrightarrow \mathbf{SO}(5,2),\ A \mapsto \begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix}.$$

We have $\mathbf{H}^4_1 \cong \mathbf{SO}(5,2)/\mathbf{SO}(4,1)$. Let $\{E_{ij} : 1 \le i < j \le 5\}$ be the standard basis of $\mathbf{so}(5,2)$ and

$$b := Span\{E_{12}, E_{13}, E_{14}, E_{15}\} \subset \mathbf{so}(5,2).$$

It holds:

$$\mathbf{so}(5,2) = \mathbf{so}(4,1) \oplus b,\ [b, \mathbf{so}(4,1)] \subset b,\ [b,b] \subset \mathbf{so}(4,1).$$

The frame bundle $SO(\mathbf{H}^4_1)$ is naturally identified with the group $\mathbf{SO}(5,2)$. The bundle of the horizontal spaces with respect to the Levi-Civita connection is in $T\mathbf{SO}(5,2)$ given by

$$T_H\mathbf{SO}(5,2) = \mathbf{SO}(5,2) \times b \subset T\mathbf{SO}(5,2).$$

The pseudohyperbolic space $\mathbf{H}^4_1$ is isometric to a warped product with warping function $f = x_1 = \cosh r$ on $\mathbf{H}^3$, r is the radius of the polar coordinates:

$$\begin{array}{rcl} \mathbf{H}^4_1 \cong \mathbf{SO}(5,2)/\mathbf{SO}(4,1) & \leftrightarrow & \mathbf{SO}^+(4,1)/\mathbf{SO}(3) \times_f \mathbf{S}^1_1, \\ \exp -sE_{12}A^+\mathbf{SO}(4,1) & \mapsto & (A^+\mathbf{SO}(3), e^{is}) \end{array}$$

where $i_2 : \mathbf{SO}^+(4,1) \hookrightarrow \mathbf{SO}(5,2),\ A^+ \mapsto \begin{pmatrix} A_{11} & 0 & A_{1j} \\ 0 & 1 & 0 \\ A_{i1} & 0 & A_{ij} \end{pmatrix}$ is the imbedding. H^4_1 is locally isometric to a Lorentz warped product of the form $]a,b[\times_{\sin t}\mathbf{H}^3$.

ii) The twistor space $\mathcal{Z}^+(\mathbf{H}^4_1)$

Let

$$\mathcal{Z}^+(\mathbf{H}^4_1) = SO(\mathbf{H}^4_1) \times_{\mathbf{SO}(4,1)} \mathbf{SO}(4,1)/H^+ = \mathbf{SO}(5,2)/H^+.$$

The bundle of unit spheres over $\mathbf{H}^3$ is given by

$$Z(\mathbf{H}^3) = \mathbf{SO}^+(4,1)/\mathbf{SO}(2)$$

(comp. (3.1)) and it exists the identification

$$\begin{array}{rcl} \widehat{\Omega} : \mathcal{Z}^+(\mathbf{H}^4_1) \cong \mathbf{SO}(5,2)/H^+ & \to & \mathbf{SO}^+(4,1)/\mathbf{SO}(2) \times \mathbf{S}^1 \cong Z(\mathbf{H}^3) \times \mathbf{S}^1. \\ \exp -sE_{12}A^+H^+ & \mapsto & (A^+\mathbf{SO}(2), e^{is}) \end{array}$$

The twistor space $\mathcal{Z}^+(\mathbf{H}^4_1)$ is diffeomorphic to $\mathbf{R}^3 \times \mathbf{S}^1 \times \mathbf{S}^2$.

iii) The optical structure $\mathcal{O}^+_+$ on $\mathcal{Z}^+(\mathbf{H}^4_1)$

On $\mathcal{Z}^+(\mathbf{H}^4_1) \cong \mathbf{SO}(5,2)/H^+$ the canonical $\mathbf{SO}(5,2)$-invariant optical structure $\mathcal{O}^+_+$ is defined as follows:

The Lie algebra $\mathbf{so}(5,2)$ decomposes into:

$$\mathbf{so}(5,2) = m^+ \oplus \underline{H}^+ \oplus b.$$

The subspace b is $Ad(H^+)$-invariant and it exists on it an $Ad(H^+)$-invariant optical structure $(\underline{K}, \underline{L}, \underline{J})$:

$$\underline{K} := \mathbf{R}(E_{12} + E_{13}) \subset b, \ \underline{L} := Span\{E_{12} + E_{13}, E_{14}, E_{15}\} \subset b,$$

$$\underline{J} : \underline{L}/\underline{K} \to \underline{L}/\underline{K}, \ \underline{J}(E_{14} + \underline{K}) = -E_{15} + \underline{K}, \ \underline{J}(E_{15} + \underline{K}) = E_{14} + \underline{K}.$$

Let $T\mathbf{SO}(5,2)/H^+ = (\mathbf{SO}(5,2) \times_{AdH^+} b) \oplus T_V\mathbf{SO}(5,2)/H^+$.

The $\mathbf{SO}(5,2)$-invariant optical structure $\mathcal{O}^+_+$ is

$$\mathcal{K}^+ := \mathbf{SO}(5,2) \times_{AdH^+} \underline{K}, \ \mathcal{L}^+ := (\mathbf{SO}(5,2) \times_{AdH^+} \underline{L}) \oplus T_V\mathbf{SO}(5,2)/H^+,$$

$$J^+_+ : \mathcal{L}^+/\mathcal{K}^+ \to \mathcal{L}^+/\mathcal{K}^+, \quad \begin{array}{c} J^+_+|_{(\mathbf{SO}(5,2)\times_{AdH^+}\underline{L})/\mathcal{K}^+}[A, l + \underline{K}] = [A, \underline{J}(l + \underline{K})], \\ J^+_+|_{T_V\mathbf{SO}(5,2)/H^+} = J^V \cong J^{\mathbf{P}(\Delta^+)} \end{array} .$$

Proposition 5.2 2. can locally be applied to the twistor space $\mathcal{Z}^+(\mathbf{H}^4_1)$, since $\mathbf{H}^4_1$ is locally isometric to a Lorentz warped product. Locally in every point, $(\mathcal{Z}^+(\mathbf{H}^4_1), \mathcal{O}^+_+)$ and $(\mathbf{R} \times Z(\mathbf{H}^3), \mathcal{O}^{Z(\mathbf{H}^3)})$ are optical equivalent. The locally induced CR-structures of $(\mathcal{Z}^+(\mathbf{H}^4_1), \mathcal{O}^+_+)$ are locally equivalent to $Z(\mathbf{H}^3)$ with the canonical CR-structure $(H^{Z(\mathbf{H}^3)}, J^{Z(\mathbf{H}^3)})$.

Remark: Let g_1 and g_2 be conformally equivalent Lorentzian metrics on the oriented 4-dimensional manifold M. The twistor spaces $\mathcal{Z}^+(M, g_1)$ and $\mathcal{Z}^+(M, g_2)$ are canonically identified. But in general, the almost optical structure $\mathcal{O}^+_+$ on $\mathcal{Z}^+(M)$ isn't invariant to conformal change of the metric. The discussed examples show this fact. Locally in any point, $\mathcal{O}^+_+$ on $\mathcal{Z}^+(\mathbf{R}^4_1)$ and on $\mathcal{Z}^+(\mathbf{S}^4_1)$ aren't optical equivalent. Considering the induced CR-structures, it becomes clear that $H^{Z(\mathbf{R}^3)}$ on $Z(\mathbf{R}^3)$ is an involutive distribution, but $H^{Z(\mathbf{S}^3)}$ on $Z(\mathbf{S}^3)$ isn't involutive (comp. (3.1)). It follows that the CR-structures aren't equivalent and moreover, they have very different properties. Hence, the almost optical structures $\mathcal{O}^+_+$ on $\mathcal{Z}^+(\mathbf{R}^4_1)$ and $\mathcal{Z}^+(\mathbf{S}^4_1)$ aren't equivalent.

In the Riemannian case the construction of the twistor space $\mathcal{A}(M)$ with the natural almost complex structure is invariant to conformal change of the metric of the base manifold M. Hence, to a 4-manifold with conformal geometry corresponds uniquely an almost complex manifold. In the Lorentzian case such a correspondence to the optical geometry doesn't exist.

6 Optical-geometric interpretation of the twistor equation

A twistor spinor on a 4-dimensional oriented Riemannian manifold (M^4, g) with spin structure can be interpreted as a holomorphic section in the 1-dimensional holomorphic vector bundle $[(S^-)^* \backslash 0] \times_{\mathbf{C}^*} \mathbf{C}$ over the twistor space $\mathcal{A}(M)$ of the Riemannian manifold M (comp. [BFGK91, chap. 3]). In the case of a 4-dimensional Lorentzian manifold $(M^{4,1}, g)$ with spin structure, we can interpret a twistor spinor in terms of optical geometry on the twistor space $\mathcal{Z}^+(M)$ as an 'optical' section in the bundle $\mathcal{H} := [(S^+)^* \backslash 0] \times_{\mathbf{C}^*} \mathbf{C}$.
Moreover, a twistor spinor on M without zeros admits another interpretation:
To a twistor spinor without zeros uniquely corresponds an almost optical structure on M.

6.1 Twistor spinors

Let $(M^{4,1}, g)$ be a space- and time-oriented Lorentzian manifold of dimension 4 with spin structure (Q, f), where Q is a $\mathbf{Spin}^+(4,1)$-principal fibre bundle over M.

$$S := Q \times_{\mathbf{Spin}^+(4,1)} \Delta_{4,1}$$

is the spinor bundle. The spinor bundle admits the scalar product $<,>_S$. ∇^S denotes the spinor derivative, μ is the Clifford multiplication (see (4.1)). Let

$$p : TM \otimes S \to \ker \mu$$

be the orthogonal projection of $TM \otimes S$ to the kernel of the Clifford multiplication. This is locally given by

$$p(X \otimes \phi) = X \otimes \phi + \frac{1}{4} \sum_{i=1}^{4} \epsilon_i \cdot s_i \otimes s_i X \cdot \phi, \quad X \in TM, \ \phi \in S,$$

where (s_1, s_2, s_3, s_4) is a local orthonormal frame. The twistor operator is given by

$$\mathcal{D} := p \circ \nabla^S : \Gamma(S) \to \Gamma(\ker \mu).$$

Locally,

$$\mathcal{D}\phi = \sum_{k=1}^{4} \epsilon_k s_k \otimes (\nabla^S_{s_k} \phi + \frac{1}{4} s_k D\phi),$$

where $D : \Gamma(S) \xrightarrow{\nabla^S} \Gamma(T^*M \otimes S) \xrightarrow{g} \Gamma(TM \otimes S) \xrightarrow{\mu} \Gamma(S)$ is the Dirac operator. A spinor $\phi \in \Gamma(S)$, which satisfies the equation $\mathcal{D}\phi = 0$, is called a twistor spinor.

Proposition 6.1 [BFGK91, chap. 1] *Let $\psi \in \Gamma(S)$ be a spinor. The following conditions are equivalent:*

1. *ψ is a twistor spinor.*
2. *The so-called twistor equation $\nabla^S_X \psi + \frac{1}{4} X \cdot D\psi = 0$ is satisfied for all vector fields $X \in \Gamma(TM)$.*
3. *$X \cdot \nabla^S_Y \psi + Y \cdot \nabla^S_X \psi = \frac{1}{2} g(X,Y) D\psi \quad \forall\, X, Y \in \Gamma(TM)$.*
4. *The spinor field $X \cdot \nabla^S_X \psi$ is independent of the unit vector field X.*

To every spinor $\phi \in \Gamma(S)$ a vector field $V_\phi \in \Gamma(TM)$ is associated (comp. [Spi95, chap. 2]):

$$V_\phi := -\sum_{i=1}^{4} \epsilon_i < s_i \cdot \phi, \phi >_S s_i.$$

Let $S = S^+ \oplus S^-$ be the decomposition of the spinor bundle into its positive and its negative part. Then $\phi = \phi^+ + \phi^-$, $\phi^+ \in \Gamma(S^+)$ and $\phi^- \in \Gamma(S^-)$.

Lemma 6.2 [Spi95, chap. 2] *For the associated field $V_\phi \in \Gamma(TM)$ of a spinor $\phi \in \Gamma(S)$ holds:*

1. *$V_\phi = V_{\phi^+} + V_{\phi^-}$.*
2. *If $\phi \in \Gamma(S)$ is a twistor spinor so is V_ϕ a conformal vector field.*
3. *ϕ, V_ϕ have the same zero set.*
4. *V_{ϕ^+}, V_{ϕ^-} are null vector fields.*
5. *If ϕ is a twistor spinor then $\iota_{V_\phi} W = 0$. (W is the Weyl tensor.)*

From 5. in Lemma 6.2 follows that if $W \neq 0$ on M and ϕ is a twistor spinor without zeros on M then the Lorentzian manifold $(M^{4,1}, g)$ has Petrov type N. In particular, V_{ϕ^+} and V_{ϕ^-} are parallel.

6.2 The interpretation of the twistor equation in the bundle $\mathcal{H}$

Let $S^* := Q \times_{\mathbf{Spin}^+(4,1)} \Delta^*_{4,1}$ be the dual spinor bundle, where

$$\mathbf{Spin}^+(4,1) \times \Delta^*_{4,1} \to \Delta^*_{4,1},\ (s, \psi(\cdot)) \mapsto \psi(s^{-1} \cdot),$$

is the action on the dual spinors. Let $(S^+)^*$ and $(S^-)^*$ be the bundles of the complex linear forms on S^+ resp. S^-. We have $S^* = (S^+)^* \oplus (S^-)^*$. $S^+ \backslash 0$ resp. $(S^+)^* \backslash 0$ denote the bundles without the zero section in each case.

The Clifford multiplication of a vector $X \in T_mM$ with a dual spinor $\xi \in S_m^*$ is given by

$$(X \cdot \xi)(\cdot) = \xi(X \cdot).$$

For each $\xi^+ \in (S^+)_m^*$ exactly one spinor $\phi^- \in S_m^-$ exists such that

$$\xi^+(\psi^+) = <\psi^+, \phi^- >_S \quad \forall\, \psi^+ \in S_m^+.$$

Note for this that $<,>_S$ is totally null on S^+ and S^-.
In (4.3), to each spinor $0 \neq \alpha^+ \in \Delta_{4,1}^+$ an optical structure $\sigma^{\alpha^+} \in Opt^+(\mathbf{R}_1^4)$ was assigned. We take the same notation for a spinor $0 \neq \psi^+ \in S^+$. Then $\sigma^{\psi^+} = (K^{\psi^+}, L^{\psi^+}, J^{\psi^+}) \in Opt^+(T_{\pi(\psi^+)}M)$. Analogous to the definition of σ^{ψ^+} to a spinor $\psi^+ \in S^+$, we assign to the dual spinor $\xi^+ \in (S^+)_m^*$ the optical structure

$$\sigma^{\xi^+} = (K^{\xi^+} := \ker \hat{\xi}^+,\ L^{\xi^+} := (K^{\xi^+})^\perp,\ J^{\xi^+} := \hat{*}) \in Opt^+(T_mM),$$

where $\hat{\xi}^+ : T_mM \to (S^-)_m^*,\ t \mapsto t\xi^+$.
For $\xi^+ = <\cdot, \phi^- >_S$, we have

$$\sigma^{\xi^+} = (K^{\phi^-}, L^{\phi^-}, -J^{\phi^-}).$$

Proof: $X \in \ker \hat{\xi}^+ \Leftrightarrow X\xi^+ = 0 \Leftrightarrow \xi^+(X\cdot) = 0 \Leftrightarrow < X\cdot, \phi^- >_S = 0 \Leftrightarrow$
$<\cdot, X\phi^- >_S = 0 \Leftrightarrow X\phi^- = 0 \Leftrightarrow X \in K^{\phi^-}$, since $\sigma^{\phi^-} \in Opt^-(T_mM)$ is negative, it follows $J^{\xi^+} = -J^{\phi^-} : L^{\xi^+}/K^{\xi^+} \to L^{\xi^+}/K^{\xi^+}$ $\square$

On the bundle $S^+\backslash 0$ and $(S^+)^*\backslash 0$, the natural almost optical structures

$$\mathcal{O}^S = (\mathcal{K}^S, \mathcal{L}^S, J^S) \text{ and } \mathcal{O}^{S^*} = (\mathcal{K}^{S^*}, \mathcal{L}^{S^*}, J^{S^*})$$

are given: let

$$TS^+\backslash 0 = T_VS^+\backslash 0 \oplus T_HS^+\backslash 0 \text{ and } T(S^+)^*\backslash 0 = T_V(S^+)^*\backslash 0 \oplus T_H(S^+)^*\backslash 0$$

be the decompositions into the horizontal and the vertical bundles with respect to the spinor connection. $T_VS^+\backslash 0$ and $T_V(S^+)^*\backslash 0$ admit natural complex structures. $T_{H\psi^+}S^+\backslash 0,\ \psi^+ \in S^+\backslash 0$, admits as optical structure the horizontal lift of $\sigma^{\psi^+} \in Opt^+(T_{\pi(\psi^+)}M)$. $T_{H\xi^+}(S^+)^*\backslash 0,\ \xi^+ \in (S^+)^*\backslash 0$, admits as optical structure the horizontal lift of $\sigma^{\xi^+} \in Opt^+(T_{\pi(\xi^+)}M)$.

$$\begin{array}{ccccc}
(S^+)^*\backslash 0 & \xrightarrow{q_1} & \mathcal{Z}^+(M) = SO(M) \times_{\mathbf{SO}(4,1)} \mathbf{P}(\Delta^+) & \xleftarrow{q_2} & S^+\backslash 0 \\
& \pi \searrow & \downarrow \pi & \swarrow \pi & \\
& & M & &
\end{array}$$

$q_2 : S^+\backslash 0 \to \mathcal{Z}^+(M)$ is the natural projection of a spinor to a projective spinor. $q_1 : (S^+)^*\backslash 0 \to \mathcal{Z}^+(M)$ is given by $q_1(\xi^+) = \ker \xi^+ \in \mathcal{Z}^+(M)$.
The almost optical structure $\mathcal{O}^S$ on $S^+\backslash 0$ is the lift of $\mathcal{O}_+^+$ on $\mathcal{Z}^+(M)$:

$$q_{2*}(\mathcal{K}^S) = \mathcal{K}^+,\ q_{2*}(\mathcal{L}^S) = \mathcal{L}^+,\ q_{2*}J^S = J_+^+ q_{2*}.$$

Also, $\mathcal{O}^{S^*}$ on $(S^+)^*\backslash 0$ is the lift of $\mathcal{O}^+_+$ on $\mathcal{Z}^+(M)$:

Proof: We have to show that $\sigma^{q_1(\xi^+)} = \sigma^{\xi^+}\ \forall \xi^+ \in (S^+)^*\backslash 0$.
For $\binom{1}{0}^{+*} \in \Delta^{+*}$ is $\ker \binom{1}{0}^{+*} = \mathbf{C}\cdot\binom{0}{1}^{+}$ and we have

$$\sigma^{\binom{1}{0}^{+*}} = -\sigma^{\binom{1}{0}^{-}} = \sigma^{\binom{0}{1}^{+}} = \sigma^{\ker\binom{1}{0}^{+*}}.$$

In a suitable oriented orthonormal basis (s_1, s_2, s_3, s_4) of $T_{\pi(\xi^+)}M,\ \xi^+ \in (S^+)^*\backslash 0$, holds:
$\sigma^{\xi^+} \in Opt^+(T_{\pi(\xi^+)}M) \cong \sigma^{\binom{1}{0}^{+*}} \in Opt^+(\mathbf{R}^4_1)$.
It follows $\sigma^{\xi^+} = \sigma^{q_1(\xi^+)}\ \forall \xi^+ \in (S^+)^*\backslash 0$ $\square$

The almost optical structures $\mathcal{O}^{S^*}$ and $\mathcal{O}^S$ are integrable if and only if $\mathcal{O}^+_+$ on $\mathcal{Z}^+(M)$ is integrable, hence if M is conformally flat.

$$(S^+)^*\backslash 0 \times \mathbf{C}^* \to (S^+)^*\backslash 0,\ ([q,\xi^+],c) \mapsto [q, c\cdot\xi^+],\quad q \in Q,\ \xi^+ \in \Delta^{+*},\ c \in \mathbf{C}^*,$$

is a $\mathbf{C}^*$-right action on $(S^+)^*\backslash 0$ and

$$(\ (S^+)^*\backslash 0\ ,\ q_1\ ,\ \mathcal{Z}^+(M)\ ,\ \mathbf{C}^*\ )$$

is a $\mathbf{C}^*$-principal fibre bundle. The associated bundle

$$\mathcal{H} := [(S^+)^*\backslash 0] \times_{\mathbf{C}^*} \mathbf{C}$$

is a 1-dimensional complex vector bundle over $\mathcal{Z}^+(M)$.
Let $\psi^+ \in \Gamma(S^+)$ and $\hat{\psi}^+ : (S^+)^*\backslash 0 \to \mathbf{C},\ \xi^+ \mapsto \xi^+(\psi^+)$. Then

$$\hat{\psi}^+(A\cdot\xi^+) = A\cdot\hat{\psi}^+(\xi^+) \text{ for } A \in \mathbf{C}^*$$

and it follows $\hat{\psi}^+ \in \Gamma(\mathcal{Z}^+(M)\ ;\mathcal{H})$.
Conversely, let $F : (S^+)^*\backslash 0 \to \mathbf{C}$ be a function such that $F(A\xi^+) = AF(\xi^+)$ and $F(\xi_1^+ + \xi_2^+) = F(\xi_1^+) + F(\xi_2^+)$. We say that F is a linear section in the bundle $\mathcal{H}$, $F \in \Gamma_{lin}(\mathcal{Z}^+(M)\ ;\mathcal{H})$. Also, we can F understand as an element in $\Gamma((S^+)^{**})$, $F \in \Gamma((S^+)^{**}) \cong \Gamma(S^+)$. Hence, we have the following one-to-one correspondence:

$$\Gamma(S^+) \cong \Gamma_{lin}(\mathcal{Z}^+(M)\ ;\mathcal{H}),\ \psi^+ \leftrightarrow \hat{\psi}^+.$$

A twistor spinor on M, interpreted as a linear section in the bundle $\mathcal{H}$, can be characterized on $\mathcal{Z}^+(M)$ in terms of optical geometry.

First, we need some preparations:

Lemma 6.3 *Let $A : \mathbf{R}^4_1 \to \Delta^+_{4,1}$ be a mapping with the properties*

1. *A is $\mathbf{R}$-linear,*
2. *$\xi^+(A(n)) = 0 \quad \forall\ \xi^+ \in \Delta^{+*},\ n \in K^{\xi^+}$,*

3. $\xi^+(A(J^{\xi^+}(l+K^{\xi^+}))) = i\xi^+(A(l+K^{\xi^+})) \quad \forall\, \xi^+ \in \Delta^{+*},\ l \in L^{\xi^+}$
   (this condition makes only sense if 2. is already satisfied),

4. $\sum_i \epsilon_i e_i A(e_i) = 0 \qquad ((e_1, e_2, e_3, e_4)$ *standard basis in* $\mathbf{R}^4_1$*).*

Then $A = 0$.

Proof: *i*) Denote $\xi^+ = < \cdot, \phi^- >_S$ and $\sigma^{\xi^+} = -\sigma^{\phi^-}$. The properties 2. and 3. are equivalent to

$$< A(n), \phi^- >_S = 0 \quad \forall\, \phi^- \in \Delta^-,\ n \in K^{\phi^-},$$

$$< A(J^{\phi^-}(l+K^{\phi^-})), \phi^- >_S = -i < A(l+K^{\phi^-}), \phi^- >_S \quad \forall\, \phi^- \in \Delta^-,\ l \in L^{\phi^-}.$$

ii) If A satisfies the conditions 1. - 4. so do $\lambda \cdot \Phi(s) \circ A\ \forall s \in \mathbf{Spin}^+(4,1),\ \lambda \in \mathbf{R}$.
iii) Suppose the mapping $A \neq 0$ has the properties as in the Lemma stated. Without lost of generality, we can suppose $A(e_1) = \binom{1}{0}^+$. Further, let

$$A(e_2) = \binom{a_2}{b_2}^+,\ A(e_3) = \binom{a_3}{b_3}^+,\ A(e_4) = \binom{a_4}{b_4}^+,\ a_i,\ b_i \in \mathbf{C}.$$

From the optical structures $\sigma^{\binom{1}{0}^-}$, $\sigma^{\binom{0}{1}^-}$ and $\sigma^{\binom{1}{1}^-}$ (see (4.3)), we obtain

$$< A(e_1 - e_2), \binom{1}{0}^- >_\Delta = 0,$$

$$< A(J^{\binom{1}{0}^-}(e_3 + K^{\binom{1}{0}^-})), \binom{1}{0}^- >_\Delta = -i < A(e_3 + K^{\binom{1}{0}^-}), \binom{1}{0}^- >_\Delta,$$

$$< A(e_1 + e_2), \binom{0}{1}^- >_\Delta = 0,$$

$$< A(J^{\binom{0}{1}^-}(e_3 + K^{\binom{0}{1}^-})), \binom{0}{1}^- >_\Delta = -i < A(e_3 + K^{\binom{0}{1}^-}), \binom{0}{1}^- >_\Delta,$$

$$< A(e_1 + e_4), \binom{1}{1}^- >_\Delta = 0,$$

$$< A(J^{\binom{1}{1}^-}(e_3 + K^{\binom{1}{1}^-})), \binom{1}{1}^- >_\Delta = -i < A(e_3 + K^{\binom{1}{1}^-}), \binom{1}{1}^- >_\Delta$$

$$\Leftrightarrow$$

$$a_2 = 1,\ b_2 = 0,\ 1 + a_4 + b_4 = 0,\ a_4 = ia_3,\ -b_4 = ib_3,\ a_2 + b_2 = i(a_3 + b_3)$$

$$\Leftrightarrow$$

$$A(e_2) = \binom{1}{0}^-,\ A(e_3) = \binom{0}{-i}^-,\ A(e_4) = \binom{0}{-1}^-.$$

It follows $\sum \epsilon_i e_i \cdot A(e_i) = 4\binom{1}{0}^- \neq 0$. This is a contradiction to the assumption. $\square$

Lemma 6.4 *Let $A:\mathbf{R}^4_1\to\Delta^+_{4,1}$ be a mapping with the properties*

1. *A is $\mathbf{R}$-linear,*
2. *$\xi^+(A(n))=0 \quad \forall\,\xi^+\in\Delta^{+*},\ n\in K^{\xi^+}$,*
3. *$\xi^+(A(J^{\xi^+}(l+K^{\xi^+})))=i\xi^+(A(l+K^{\xi^+})) \quad \forall\,\xi^+\in\Delta^{+*},\ l\in L^{\xi^+}$.*

Then exists a spinor $\psi^-\in\Delta^-_{4,1}$ such that $A(x)=x\cdot\psi^-\ \forall x\in\mathbf{R}^4_1$.

Proof: Let

$$\psi^-:=-\frac{1}{4}\sum_{i=1}^{4}\epsilon_i\cdot e_iA(e_i)\text{ and }A^*:\mathbf{R}^4_1\to\Delta^+,\ A^*(x)=A(x)-x\cdot\psi^-.$$

Then

$$\sum_{i=1}^{4}\epsilon_i\cdot e_iA^*(e_i)=\sum\epsilon_i\cdot e_iA(e_i)-\sum\epsilon_i\cdot e_i\cdot e_i\psi^-=-4\psi^-+4\psi^-=0.$$

A^* satisfies also the rest of the conditions in Lemma 6.3:

1. $x\mapsto x\psi^-$ is $\mathbf{R}$-linear,
2. $< n\psi^-,\phi^- >_\Delta=< \psi^-,n\phi^- >_\Delta=0 \quad \forall\ \phi^-\in\Delta^-,\ n\in K^{\phi^-}$,
3. $< J^{\phi^-}(l+K^{\phi^-})\psi^-,\phi^- >_\Delta=< \psi^-,J^{\phi^-}(l+K^{\phi^-})\phi^- >_\Delta$ $=< \psi^-,i(l+K^{\phi^-})\phi^- >_\Delta=-i< (l+K^{\phi^-})\psi^-,\phi^- >_\Delta$.

From Lemma 6.3 follows $A^*=0$, hence $A(x)=x\cdot\psi^-$ $\square$

Proposition 6.5 *A section $\psi^+\in\Gamma(S^+)$ is a twistor spinor, $\mathcal{D}\psi^+=0$, if and only if $\hat{\psi}^+\in\Gamma_{lin}(\mathcal{Z}^+(M)\ ;\mathcal{H})$ is an optical function on $(S^+)^*\backslash 0$, that means $\hat{\psi}^+$ has the properties:*

1. *$\hat{\psi}^+:(S^+)^*\backslash 0\to\mathbf{C}$ is constant on the integral curves to $\mathcal{K}^{S^*}$,*
   $d\hat{\psi}^+(k)=0\ \forall k\in\mathcal{K}^{S^*}$,
2. *$d\hat{\psi}^+((l+\mathcal{K}^{S^*})+iJ^{S^*}(l+\mathcal{K}^{S^*}))=0 \quad \forall\ l\in\mathcal{L}^{S^*}$.*

Proof: $\hat{\psi}^+$ is $\mathbf{C}$-linear on the fibres $(S^+)^*_m\backslash 0,\ m\in M$. Hence, condition 2. is satisfied if and only if

$$d\hat{\psi}^+((l+\mathcal{K}^{S^*})+iJ^{S^*}(l+\mathcal{K}^{S^*}))=0 \quad \forall\ l\in\mathcal{L}^{S^*}\cap T_H(S^+)^*\backslash 0.$$

Let (s_1,s_2,s_3,s_4) be a local orthonormal frame in $U\subset M$ such that $\nabla s_i(m_0)=0$, $i=1,\ldots,4$, for a $m_0\in U$. ψ^+ is locally given by a function
$\psi^+:U\to\Delta^+_{4,1}$ and

$$\hat{\psi}^+:U\times(\Delta^{+*}\backslash 0)\to\mathbf{C},\ (m,\xi^+)\mapsto\xi^+(\psi^+(m)).$$

The conditions 1. and 2. in a point $m \in U$ are equivalent to

$$\xi^+(d\psi^+(k)) \quad \forall\, k \in K^{\xi^+} \subset T_mU,\ \xi^+ \in (S^+)^*_m\backslash 0,$$

$$\xi^+(d\psi^+(J^{\xi^+}(l+K^{\xi^+}))) = i\xi^+(d\psi^+(l+K^{\xi^+})) \quad \forall\, \xi^+ \in (S^+)^*_m\backslash 0,\ l \in L^{\xi^+} \subset T_mU.$$

From $\nabla s_i(m_0) = 0$, it follows that $\hat{\psi}^+$ is an optical function in $m_0 \in M$ if and only if

$$\xi^+(\nabla^S_k\psi^+) = 0,\ \xi^+(\nabla^S_{J(l+K^{\xi^+})}\psi^+) = i\xi^+(\nabla^S_{l+K^{\xi^+}}\psi^+)$$

for all $\xi^+ \in (S^+)^*_{m_0}\backslash 0,\ k \in K^{\xi^+} \subset T_{m_0}M,\ l \in L^{\xi^+} \subset T_{m_0}M$.
If ψ^+ is a twistor spinor, from proposition 6.1, it follows that

$$\xi^+(\nabla^S_k\psi^+) = \xi^+(-\frac{1}{4}kD\psi^+) = -\frac{1}{4}k\xi^+(D\psi^+) = 0,$$

$$\begin{aligned}
\xi^+(\nabla^S_{J(l+K^{\xi^+})}\psi^+) &= -\xi^+(\frac{1}{4}J^{\xi^+}(l+K^{\xi^+})\cdot D\psi^+) = -J^{\xi^+}(l+K^{\xi^+})\xi^+(\frac{1}{4}D\psi^+) \\
&\overset{Lemma 4.4}{=} -i(l+K^{\xi^+})\xi^+(\frac{1}{4}D\psi^+) = i\xi^+(-\frac{1}{4}(l+K^{\xi^+})\cdot D\psi^+) \\
&= i\xi^+(\nabla^S_{l+K^{\xi^+}}\psi^+).
\end{aligned}$$

Hence, ψ^+ is an optical function on $(S^+)^*\backslash 0$.
Converselly, let ψ^+ be an optical function on $(S^+)^*\backslash 0$. It follows that the mapping

$$X \in T_mM \mapsto \nabla^S_X\psi^+$$

satisfies the conditions of Lemma 6.4. We have $\nabla^S_X\psi^+ = X\cdot\phi^-$ for a $\phi^- \in S^-$ and it follows that ψ^+ is a twistor spinor. □

From the existence of a twistor spinor on M it follows that the Weyl tensor in a point, in which the twistor spinor has no zero, is of Petrov type N or it vanishes. Therefore, the almost optical structure $\mathcal{O}^+_+$ on $\mathcal{Z}^+(M)$ isn't necessarily integrable if a twistor spinor exists.
From the proposition follows that with respect to a local trivialization a twistor spinor corresponds locally to optical functions on $\mathcal{Z}^+(M)$. Moreover, in the case that $(M^{4,1}, g)$ is conformally flat, these local optical functions are related to a CR-function on the induced CR-manifold.

Corollar 6.6 *Let $(M^{4,1}, g)$ be conformally flat. A twistor spinor $\psi^+ \in \Gamma(M\,;S^+)$ corresponds locally to a CR-function on a 5-dimensional manifold.*

6.3 The associated optical structure to a twistor spinor

Let $(M^{4,1}, g)$ be an oriented 4-dimensional Lorentzian manifold. A section $\alpha^+ \in \Gamma(M\,;\mathcal{Z}^+(M))$ is an orthogonal almost optical structure on M. Let $M^{4,1}$ be in addition a time-oriented spin manifold with spin structure (Q, f). A spinor field $\phi^+ \in \Gamma(S^+)$ without zeros projects to a section $[\phi^+] \in \Gamma(M\ ; \mathcal{Z}^+(M))$ in the twistor space. Hence, a spinor field ϕ^+ without zeros can be understand as an orthogonal optical structure $\mathcal{O}^{\phi^+} = (\mathcal{K}^{\phi^+}, \mathcal{L}^{\phi^+}, J^{\phi^+}) := [\phi^+]$ on M. The associated vector field to a spinor field has the following further property.

Lemma 6.7 *Let $\phi^+ \in \Gamma(S^+)$ be a spinor and $V_{\phi^+} \in \Gamma(TM)$ the associated vector field. Then $V_{\phi^+} \cdot \phi^+ = 0$, $(V_{\phi^-} \cdot \phi^- = 0)$.*

The proof is an easy calculation. Moreover, we have $V_{\phi^+} \in K^{\phi^+} = \ker \hat{\phi}^+$ (comp. (4.3)). Hence, it follows from the Lemma that the almost optical structure $\mathcal{O}^{\phi^+}$ to $0 \neq \phi^+ \in \Gamma(S^+)$ is just given by $0 \neq V_{\phi^+} \in \Gamma(TM)$:

$$\mathcal{O}^{\phi^+} = (\mathbf{R} \cdot V_{\phi^+}, (\mathbf{R} \cdot V_{\phi^+})^{\perp}, J = \hat{*}).$$

Proposition 6.8 *Let $\phi^+ \in \Gamma(S^+)$ be a twistor spinor without zeros. Then $\mathcal{O}^{\phi^+}$ is an integrable almost optical structure.*

Proof: Lemma 6.2 states that V_{ϕ^+} is a conformal vector field, which has no zeros. Let $\kappa = g(V_{\phi^+}, \cdot)$ be a 1-form. For a suitable function F on M holds

$$F\kappa = (\pounds_{V_{\phi^+}} g)(V_{\phi^+}, \cdot) = V_{\phi^+}(\kappa(\cdot)) - g(\pounds_{V_{\phi^+}} V_{\phi^+}, \cdot) - g(V_{\phi^+}, \pounds_{V_{\phi^+}} \cdot) = \pounds_{V_{\phi^+}} \kappa.$$

It follows that $\mathcal{L}^{\phi^+} = \ker \kappa$ is invariant to the flow of V_{ϕ^+}. Besides, we have $\phi_t^{V_{\phi^+}} * = * \ \forall t$, the Hodge operator is invariant to the flow of a conformal vector field. By a dimension argument the last integrability condition is trivially satisfied. $\square$

We obtain from proposition 3.5:

Conclusion: Let $\phi^+ \in \Gamma(S^+)$ be a twistor spinor without zeros. Let $(M^{4,1}, \mathcal{K}^{\phi^+})$ be regular and $M^{4,1}/\sim^{\mathcal{K}^{\phi^+}}$ realizable in $\mathbf{C}^2$. Then M admits a nowhere vanishing Maxwell field.

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