Spectral properties of the Dirac operator on pseudo-Riemannian space forms

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Contents

Introduction

This paper is concerned with the spectrum of the Dirac operator on the hyperboloids $X := \{x \in \mathbb{R}^{p+q} \mid -x_1^2 - \cdots - x_p^2 + x_{p+1}^2 + \cdots + x_{p+q}^2 = 1\}$. These are geodesically complete semi-Riemannian manifolds of constant curvature 1, that are connected if $q \neq 1$, and simply connected if $q \neq 2$. Also they are semisimple symmetric spaces as quotients $X = \boldsymbol{SO}(p,q)/\boldsymbol{SO}(p,q-1)$. The spectrum of the Laplace operator and its spectral resolution, which at the same time is the decomposition of $L^2(X)$ into a direct integral of irreducible representations of $\boldsymbol{O}(p,q)$, was determined in a number of papers, e.g., [St] and [Ro]. Both these papers use the asymptotic relation between X and the light cone $\Xi := \{x \in \mathbb{R}^{p+q} \mid -x_1^2 - \cdots - x_p^2 + x_{p+1}^2 + \cdots + x_{p+q}^2 = 0, x \neq 0\}$. [St], for instance, proves that each eigenfunction of the Laplacian on X can be extended to a homogeneous function on $\mathbb{R}^{p+q}$ that lies in the kernel of $\Delta_{R^{p+q}} := \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_p^2} - \frac{\partial^2}{\partial x^1_{p+1}} - \cdots - \frac{\partial^2}{\partial x^2_{p+q}}$ analogously to the classic result that the eigenfunctions of the Laplacian of the sphere are the restrictions of homogeneous harmonic polynomials. Now, $\Delta_{\mathbb{R}^{p+q}}$-harmonic functions are determined either by their value on X or by their value on Ξ and the value of a directional derivative on Ξ. This and the decomposition of $L^2(\Xi)$ into a direct integral of irreducible representations of $\boldsymbol{O}(p,q)$ yield the spectral resolution of Δ_X. The proof of this result, however, makes heavy use of the fact that the Laplace operator is (essentially) self-adjoint, for example, when defining functions of the Laplacian or decomposing $L^2(X)$ into two subspaces on which the Laplace operator has only positive or negative spectrum, so his methods cannot be applied to the Dirac operator, which is *not* self-adjoint with respect to a positive definite scalar product in a Hilbert space. The method of [Ro], however, is the reduction of the Laplace operator to an ordinary differential operator and then describing the resolvent by means of a Green's function. This is done by decomposing X into the orbits of $K' := \boldsymbol{SO}(p)\cdot\boldsymbol{SO}(q) \subset \boldsymbol{SO}(p,q)$ and then restricting everything to the K'-finite elements of $L^2(X)$. Since X has rank one as a semisimple symmetric space, these orbits can be parameterised by the real line; if X had rank r, then one would have to use $\mathbb{R}^r$. The resolvent itself is expressed as a Green's function, and from the resolvent the spectral resolution can be computed. Now this idea can be applied to the Dirac operator where one has to replace $\boldsymbol{SO}(p,q)$ by $\boldsymbol{Spin}(p,q)$ and K' by $K := \boldsymbol{Spin}(p)\cdot\boldsymbol{Spin}(q) \subset \boldsymbol{Spin}(p,q)$.

Let S denote the spinor bundle. Then the main theorem is as follows:

Theorem: *If the dimension or the index of X is even, the closure of the Dirac operator in $L^2(S)$ has no residual spectrum. The point spectrum consists of $\mathbb{Z}\setminus\{0\}$ for even dimension and of $\frac{1}{2}+\mathbb{Z}$ for odd dimension. The imaginary axis belongs to the continuous spectrum. On the K-isotypic components of $L^2(S)$ there is no further spectrum.*

(Theorems 5.1 and 5.2)

Let me describe the contents the paper. In section one the results of [Ro] are cited (without proofs) that are used later in this paper. Part of these results have analogues for the Dirac operator whose proofs are quite similar to those for the Laplacian; others can only be proved using properties of the Laplace operator, for example the self-adjointness.

The second section makes some introductory remarks about the spinor calculus on pseudo-Riemannian manifolds in general and then turns to the case of the hyperboloids. The spinor bundle over the hyperboloids is trivialised and the relevant objects such as the Dirac operator and scalar products are expressed in terms of this trivialisation.

The third section is essentially a rewriting of the results of [Ro]. Representation formulas for eigenspinors are proved and these formulas are used to determine the resolvent by means of a Green's function. The actual computation of Green's function is done in the appendix because it involves rather tedious estimates—it would have been to distracting to place these computations in the body of the text.

The short fourth section sums up the results of the appendix on the Green's function so that one can read the fifth section without going into the details of the appendix.

Section five finally computes the point spectrum and gives some properties of the eigenspaces. The perhaps most interesting result of this section, apart from the point spectrum itself, is the non-degeneracy of the eigenspaces with respect to one of the indefinite scalar products (in case the dimension or the index of X is even).

0 Conventions

Let $\mathbb{N}$ denote the set of natural numbers including zero, i.e., $\mathbb{N} = \{0, 1, 2, \ldots\}$. In contrast let $\mathbb{Z}_+$ denote the set of positive integers: $\mathbb{Z} = \{1, 2, \ldots\}$.

Let K be a compact Lie group and V be a continuous locally convex linear representation of K. Then V is called (K-)isotypic if any two irreducible subrepresentations of V are isomorphic. For a K-invariant subspace $W \subset V$ let $\overline{W}^K$ denote the closure of the sum of all invariant subspaces U of V such that there exists a subrepresentation of W that is isomorphic to U. If V is semisimple (e.g., finite dimensional, or unitary), then it suffices to consider only irreducible U's. A subspace of the form $\overline{W}^K$ is called an isotypic component of V.

An element $v \in V$ is called K-finite if v is contained in a finite dimensional subspace of V which is invariant under K. The space of all K-finite vectors is denoted by V_K. V is called K-finite if $\overline{W}^K$ is finite dimensional for every finite dimensional invariant W . If Y is a smooth (analytic) manifold acted on transitively by K, then every K-finite distribution $T \in \mathscr{D}'(Y)$ is smooth (analytic).

Let A be a continuous K-equivariant map of V into itself. Then the image of each irreducible W is contained in $\overline{W}^K$. For if W' is a subrepresentation of V that is isomorphic to W, than either $A(W')$ is isomorphic to W' (and thus to W) or $A(W) = 0$ because of the equivariance of A and the irreducibility of W. Hence $A(W') \subset \sum_{U \approx W} U$. Then the continuity of A implies $A(\overline{W}^K) \subset \overline{W}^K$. If $\overline{W}^K$ is finite dimensional, the continuity of A is obviously superfluous.

Let E_1 and E_2 be Frchet spaces, $V \subset E_1$ a dense subspace, and A_z a family of not necessarily continuous linear mappings of V into E_2 where $z \in U \subset \mathbb{C}$ is an open subset. A_z is said to depend meromorphically on z at $z = z_0$ if there exist an integer N and (linear) maps a_k, $k \geq N$, such that $A_z e = \sum_{k=N}^{\infty} (z - z_0)^k a_k e$ for all $e \in V$. This is equivalent to $z \mapsto A_z e$ being a meromorphic map at $z = z_0$ of U into E_2 for all $e \in V$ such that the multiplicity of the pole at $z = z_0$ is bounded. A_z is called compact-equicontinuous, if for every compact subset $C \subset U \setminus \{z_0\}$ and every neighbourhood N of 0 in E_2 there is a neighbourhood N' of 0 in V such that $A_z(N') \subset N$ for all $z \in C$. If A_z is compact-equicontinuous, then each a_k is continuous because of

$$a_k e = \int\limits_{|z - z_0| = r} \frac{A_z e}{(z - z_0)^{k+1}} \, dz$$

for small $r > 0$.

This definition of meromorphicity, however, has a serious flaw in that the composition of two meromorphic mappings needn't be meromorphic, i.e., although $E_1 \xrightarrow{A_z} E_2 \xrightarrow{B_z} E_3$ may depend meromorphically on z $B_z \circ A_z$ need not. The same is true for A_z^{-1} even if it exists. Nevertheless, if all spaces are finite dimensional, than both the composition and the inverse are again meromorphic since in this case the meromorphicity of A_z is equivalent to the meromorphicity of the map $U \ni z \mapsto A_z \in \mathrm{Hom}(E_1, E_2)$.

1 The Laplace operator

This section summarises the results and propositions of [Ro] that will be needed in the sequel. Some of these can only be proved using special properties of the Laplacian such as the self-adjointness while others can be easily extended to the Dirac operator.

Let $p \geq 1$ and $q \geq 2$ be natural numbers. Let $\langle x, y\rangle_{p,q}$, or $\langle x, y\rangle$ for short, denote the scalar product $-x_1y_1 - \cdots - x_py_p + x_{p+1}y_{p+1} + \cdots + x_{p+q}y_{p+q}$ ("scalar product" always implies non-degeneracy). Let $G' := \boldsymbol{SO}_o(p,q)$ denote the group of isometries that preserve space and time orientation (which is the connected component of the identity of $\boldsymbol{SO}(p,q)$) and let $K' := \boldsymbol{SO}(p) \cdot \boldsymbol{SO}(q) \subset G'$ be the "standard" maximal compact subgroup. Let $X := \{x \in \mathbb{R}^{p+q} \mid \langle x, x\rangle = 1\}$ be equipped with the pseudo-Riemannian metric of index p that is induced by $\langle .,.\rangle$. Then G' acts transitively on X; this is not true if $q = 1$, here X has two connected components. The isotropy group of $e_{p+q} \in X$ is $H' := \boldsymbol{SO}_o(p, q-1)$. Let $\Xi := \{\xi \in \mathbb{R}^{p+q} \mid \langle \xi, \xi\rangle = 0, \xi \neq 0\}$ denote the light cone, $Y := S^{p-1} \times S^{q-1} \subset \mathbb{R}^p \times \mathbb{R}^q = \mathbb{R}^{p+q}$. G' acts transitively on Ξ and this action induces an action $*$ on Y by $g * y = \alpha(gy)^{-1}gy$ where $\alpha(\xi) > 0$ is the number such that $\alpha(\xi)^{-1}\xi \in Y$ for all $\xi \in \Xi$. Each $x \in \mathbb{R}^{p+q}$ can be written uniquely as $x = x' + x''$ where $x' \in \mathbb{R}^p \times 0$ and $x'' \in 0 \times \mathbb{R}^q$.

Introduce "polar coordinates" on X:

$$\begin{aligned} \mathbb{R}_+ \times Y &\longrightarrow X \\ (\alpha, y) &\longmapsto x(\alpha, y) = \tfrac{1}{2}(\alpha - \alpha^{-1})y' + \tfrac{1}{2}(\alpha + \alpha^{-1})y''. \end{aligned}$$

This is the decomposition of X into the orbits of K', the orbits are the sets $\{\alpha\} \times Y$. Let $X' := \{x(\alpha, y) \mid \alpha > 1\} = \{x \in X \mid x' \neq 0\}$. Then the polar coordinates are analytical coordinates on X'. The metric $g = \langle .,.\rangle|_X$, the volume element dx, and the Laplace operator Δ_X are expressed in terms of the polar coordinates as follows:

$$g = -\alpha^{-2}d\alpha^2 - \frac{1}{4}(\alpha - \alpha^{-1})^2 g_{S^{p-1}} + \frac{1}{4}(\alpha + \alpha^{-1})^2 g_{S^{q-1}}$$

$$dx = \mu(\alpha)\, d\alpha\, dx \text{ where } \mu(\alpha) := \alpha^{-1}\big(\tfrac{1}{2}(\alpha - \alpha^{-1})\big)^{p-1}\big(\tfrac{1}{2}(\alpha + \alpha^{-1})\big)^{q-1}$$

$$\begin{aligned} \Delta_X = \left(\alpha\frac{\partial}{\partial\alpha}\right)^2 + \left((p-1)\frac{\alpha + \alpha^{-1}}{\alpha - \alpha^{-1}} + (q-1)\frac{\alpha - \alpha^{-1}}{\alpha + \alpha^{-1}}\right)\alpha\frac{\partial}{\partial\alpha} \\ - \big(\tfrac{1}{2}(\alpha - \alpha^{-1})\big)^{-2}\Delta_{S^{p-1}} + \big(\tfrac{1}{2}(\alpha + \alpha^{-1})\big)^{-2}\Delta_{S^{q-1}} \end{aligned}$$

where the Laplace operators are defined as $\Delta = -\operatorname{div}\operatorname{grad}$.

A function $f\colon X \to \mathbb{C}$ can be interpreted by means of the polar coordinates as a mapping $v_f\colon (1,\infty) \to \mathbb{C}^Y$ where $v_f(\alpha) := f(x(\alpha, \cdot))$. The main point of Rossmann's approach is the observation that if a function $f \in L^2_{\mathrm{loc}}(X)$ is K'-finite then there exists a finite dimensional K'-invariant subspace $V \subset L^2(Y)$ such that $v_f(\alpha) \in V$ for (almost) all α. Such a subspace is invariant under $\Delta_{S^{p-1}}$ and $\Delta_{S^{q-1}}$. Therefore the question of determining K'-finite eigenfunctions f of the Laplacian reduces to an *ordinary* differential equation for v_f.

Now the Laplace operator satisfies the simple asymptotics $\Delta \sim (\alpha\frac{\partial}{\partial\alpha})^2 + 2\varrho\alpha\frac{\partial}{\partial\alpha}$ for $\alpha \to \infty$ where $\varrho := \frac{1}{2}(p + q - 2)$. The solutions of $\alpha(\alpha u')' + 2\varrho\alpha u' = \lambda u$ ($u\colon (1,\infty) \to V$) are $u(\alpha) = c_+\alpha^{-\varrho+\nu} + c_-\alpha^{-\varrho-\nu}$ where $\lambda = -\varrho^2 + \nu^2$. These functions indeed approximate eigenfunctions of Δ:

Lemma 1 (a) *If $\nu \notin \frac{1}{2}\mathbb{N}$, then for each K'-finite function $\varphi \in \mathscr{E}(Y)$ there exists a unique K'-finite function $\Phi_\nu\varphi \in \mathscr{E}(X')$ such that*

$$\Delta \Phi_\nu \varphi = (-\varrho^2 + \nu^2)\Phi_\nu\varphi$$

and

$$\Phi_\nu\varphi(x(\alpha, y)) = \alpha^{-\varrho+\nu}\big(\varphi(y) + O(\varphi, \alpha, y)\big)$$

where $\alpha \mapsto O(\varphi, \alpha, y)$ is analytic at $\alpha = \infty$ and $O(\varphi, \infty, y) = 0$. Φ_ν depends meromorphically on ν with at most simple poles at $\nu \in \frac{1}{2}\mathbb{N} \setminus \mathbb{N}$.

(b) *If $\nu \notin \mathbb{Z}$, then for each K'-finite eigenfunction $f \in \mathscr{E}(X')$ with $\Delta f = (-\varrho^2+\nu^2)f$ there are unique K'-finite $\varphi_\pm \in \mathscr{E}(Y)$ such that*

$$f = \Phi_\nu\varphi_+ + \Phi_{-\nu}\varphi_-.$$

Φ_ν is injective for $\nu \notin \mathbb{Z}$.

Proof. [Ro] Lemma 4. ⊠

It should be emphasised that $\Phi_\nu\varphi$ is only defined on X', not on the whole of X. To obtain such eigenfunctions one can use the Poisson transformation. For this purpose set

$$\mathscr{E}_{\varepsilon,\nu}(\Xi) := \{f \in \mathscr{E}(\Xi) \mid f(\alpha\xi) = |\alpha|^{-\varrho+\nu}\,\mathrm{sign}(\alpha)^\varepsilon f(\xi) \text{ for all } \alpha \in \mathbb{R}^*\}$$

for $\varepsilon = 0, 1$ and $\nu \in \mathbb{C}$. This space is invariant under G'. Using the identification

$$\begin{aligned} \mathscr{E}_{\varepsilon,\nu}(\Xi) &\longrightarrow \mathscr{E}_\varepsilon(Y) := \{f \in \mathscr{E}(Y) \mid f(-y) = (-1)^\varepsilon f(y)\} \\ f &\longmapsto f|_Y \end{aligned}$$

one obtains a representation $\pi_{\varepsilon,\nu}$ on $\mathscr{E}_\varepsilon(Y)$. Then one defines the Poisson transformation as follows:

$$\begin{aligned} &F_{\varepsilon,\nu} : \mathscr{E}_{\varepsilon,\nu}(\Xi) \longrightarrow \mathscr{E}(X) \\ &F_{\varepsilon,\nu}\varphi(x) = \int_Y |\langle x, y\rangle|^{-\varrho-\nu}\,\mathrm{sign}(\langle x, y\rangle)^\varepsilon\, dy. \end{aligned}$$

This integral is convergent for $\mathrm{Re}(\varrho + \nu) < 1$ and allows a meromorphic continuation:

Lemma 2 (a) *$F_{\varepsilon,\nu}$ is holomorphic as a map of $\mathscr{E}_\varepsilon(Y)$ into $\mathscr{E}(X)$ except for simple Poles at ν with $\varrho + \nu \in \mathbb{Z}_+$ and $\varrho + \nu \equiv \varepsilon + 1 \bmod 2$. $F_{\varepsilon,\nu}$ is compact-equicontinuous (with respect to the C^∞-topology).*

(b) *$F_{\varepsilon,\nu} : \mathscr{E}_{\varepsilon,\nu}(\Xi) \to \mathscr{E}(X)$ is G'-equivariant.*

(c) *$F_{\varepsilon,\nu}$ is injective for $\varrho + \nu \notin \mathbb{Z}$.*

(d) *$\Delta F_{\varepsilon,\nu}\varphi = (-\varrho^2 + \nu^2)F_{\varepsilon,\nu}\varphi$ for $\varphi \in \mathscr{E}_\varepsilon(Y)$.*

Proof. [Ro] Lemma 5 and proof of Theorem 6. ⊠

With these preparations one can show

Theorem 1 *If* $\varrho+\nu\notin\mathbb{Z}$, *each* K'*-finite distribution* $f\in\mathscr{D}'(X)$ *with*

$$\Delta f=(-\varrho^2+\nu^2)f$$

can be written as

$$f=F_{0,\nu}\varphi_0+F_{1,\nu}\varphi_1$$

with uniquely determined K'*-finite* $\varphi_\varepsilon\in\mathscr{E}_\varepsilon(Y)$.

The next lemma describes the connection between the eigenfunctions $\Phi_\nu\varphi$ and $F_{\varepsilon,\nu}\varphi$:

Lemma 3 *There are linear* K'*-equivariant maps depending meromorphically on* ν

$$C_\pm(\varepsilon,\nu)\colon\mathscr{E}_\varepsilon(Y)_{K'}\longrightarrow\mathscr{E}_\varepsilon(Y)_{K'}$$

such that

$$F_{\varepsilon,\nu}\varphi=\Phi_\nu C_+(\varepsilon,\nu)\varphi+\Phi_{-\nu}C_-(\varepsilon,\nu)\varphi$$

for K'*-finite* φ. $C_\pm(\varepsilon,\nu)$ *intertwines* $\pi_{\varepsilon,\nu}$ *and* $\pi_{\varepsilon,\pm\nu}$. $C_+(\varepsilon,\nu)$ *is scalar,* $C_+(\varepsilon,\nu)=c(\varepsilon,\nu)id$ *where*

$$c(\varepsilon,\nu)=2^{2\varrho+1}\pi^\varrho\frac{\Gamma(\nu)}{\Gamma(\varrho+\nu)}\cdot\begin{cases}(-1)^{1+q/2}\tan\left(\frac12\pi(\varrho+\nu+\varepsilon)\right), & \text{if } 2\mid q;\\ (-1)^{(q-1)/2}, & \text{if } 2\nmid q.\end{cases}$$

Furthermore

$$C_-(\varepsilon,\nu)\varphi(y)=2^{\varrho+\nu}\int\limits_Y|\langle y,\overline{y}\rangle|^{-\varrho-\nu}\operatorname{sign}(\langle y,\overline{y}\rangle)^\varepsilon\varphi(\overline{y})\,d\overline{y}$$

for $\operatorname{Re}(\varrho+\nu)\le 0$.

Proof. [Ro] Lemma 7 (the formulas above differ a bit from the ones in [Ro], in Rossmann's formulas the factors $(-1)^{(q-1)/2}$ and $2^{\varrho+\nu}$ are missing). ⊠

The intertwining operators satisfy the following relations:

Lemma 4 (a) $F_{\varepsilon,\nu}C_-(\varepsilon,-\nu)=F_{\varepsilon,-\nu}C_+(\varepsilon,\nu)$.
(b) $C_+(\varepsilon,\nu)C_+(\varepsilon,-\nu)=C_-(\varepsilon,\nu)C_-(\varepsilon,-\nu)=c(\varepsilon,\nu)c(\varepsilon,-\nu)id$.

Proof. [Ro] Lemma 8. ⊠

This implies in particular that $C_-(\varepsilon,\nu)$ is invertible for $\nu\notin\frac12\mathbb{Z}$ because of $\varrho\in\frac12\mathbb{N}$ and Lemma 3. (a) implies that $F_{\varepsilon,\nu}C_-(\varepsilon,\nu)^{-1}=c(\varepsilon,-\nu)^{-1}F_{\varepsilon,-\nu}$.

The following lemma will be useful later on. It is used in [Ro] to prove the irreducibility of the eigenspaces of the Laplace operator under the action of G'.Let $H^{m,n}\subset\mathscr{E}(Y)$ for $m,n\in\mathbb{N}$ denote the space spanned by all products of spherical harmonics of degree m on S^{p-1} and of degree n on S^{q-1} respectively:

$$H^{m,n}:=\mathscr{L}_{\mathbb{C}}\left\{\varphi_1|_{S^{p-1}}\cdot\varphi_2|_{S^{q-1}}\,\middle|\,\begin{matrix}\Delta_{\mathbb{R}^p}\varphi_1=0,\ \varphi_1\text{ homogeneous of degree } m\\ \Delta_{\mathbb{R}^q}\varphi_2=0,\ \varphi_2\text{ homogeneous of degree } n\end{matrix}\right\}.$$

If $p=1$, then $m=0,1$ and $H^{0,n}\oplus H^{1,n}$ is the eigenspace of $\Delta_{S^{q-1}}$ for eigenvalue $n(n+q-2)$. If $p>1$, then $H^{m,n}$ is the joint eigenspace of $\Delta_{S^{p-1}}$ and $\Delta_{S^{q-1}}$ for eigenvalues $m(m+p-2)$ and $n(n+q-2)$ respectively.

Lemma 5 *If $\varrho + \nu_0 \in \mathbb{Z}_+$, $\varrho + \nu_0 \equiv \varepsilon + q \bmod 2$, then the kernel (= null space) of $(\nu - \nu_0)c(\varepsilon,\nu)F_{\varepsilon,\nu}|_{\nu=\nu_0}$ is the closure (in $\mathscr{E}_\varepsilon(Y)$) of $\sum_{m,n} H^{m,n}$ where the sum is over all m, n such that*

$$-\left(m + \frac{p-2}{2}\right) + \left(n + \frac{q-2}{2}\right) < \nu_0 \quad \textit{and} \quad m + n \equiv \varepsilon \bmod 2$$

in case $p > 1$ and

$$n + \frac{q-2}{2} < \nu_0 \quad \textit{and} \quad m + n \equiv \varepsilon \bmod 2$$

in case $p = 1$.

Proof. [Ro] Lemma 11. ⊠

To determine the spectral decomposition of the Laplace operator [Ro] proceeds as follows. First of all the resolvent $R_\lambda = (\Delta - \lambda)^{-1}$ is determined on the space $C_0^\infty(X)_{K'}$ for $\lambda \notin \mathbb{R}$. For this purpose let $V \subset \mathscr{E}(Y)$ be a K'-finite subspace and $V_\varepsilon := \{v \in V \mid v(-y) = (-1)^\varepsilon v(y)\}$ for $\varepsilon = 0, 1$. There are two families of endomorphisms $(f_{\varepsilon,\nu}(\alpha))_{\alpha\in(1,\infty)}$ and $(\varphi_{\varepsilon,\nu}(\alpha))_{\alpha\in(1,\infty)}$ of V_ε such that

$$\begin{aligned}(F_{\varepsilon,\nu}v)(x(\alpha,y)) &= \big(f_{\varepsilon,\nu}(\alpha)v\big)(y)\\ (\Phi_\nu v)(x(\alpha,y)) &= \big(\varphi_{\varepsilon,\nu}(\alpha)v\big)(y)\end{aligned}$$

for all $v \in V_\varepsilon$. Now, if $f \in C_0^\infty(X)_{K'}$ with $v_f(\alpha) \in V_\varepsilon$ for all α, then

$$\big((\Delta - \lambda)^{-1} f\big)(x(\alpha,\cdot)) = \int_1^\infty k_{\varepsilon,\nu}(\alpha,\overline{\alpha})v_f(\overline{\alpha})\mu(\overline{\alpha})\,d\overline{\alpha}$$

where $\lambda = -\varrho^2 + \nu^2$, $\operatorname{Re}\nu > 0$ and

$$k_{\varepsilon,\nu}(\alpha,\overline{\alpha}) = 2^{2\varrho-1}\nu^{-1}c(\varepsilon,\nu)^{-1} \cdot \begin{cases} f_{\varepsilon,\nu}(\alpha)\varphi_{\varepsilon,-\overline{\nu}}(\overline{\alpha})^*, & \text{if } \alpha < \overline{\alpha};\\ \varphi_{\varepsilon,-\nu}(\alpha)f_{\varepsilon,\overline{\nu}}(\overline{\alpha})^*, & \text{if } \alpha > \overline{\alpha};\end{cases} \tag{1}$$

here $\overline{\nu}$ is the complex conjugate of ν and $\Box^*$ denotes the adjoint in $V_\varepsilon \subset L^2(Y)$. From this one can compute the spectral measure E by means of the formulas

$$\int_{[\lambda_1,\lambda_2]} \chi(\lambda)\,dE(\lambda) = \lim_{\mu\downarrow 0}\frac{1}{2\pi i}\int_{\lambda_1}^{\lambda_2}(R_{\lambda-i\mu} - R_{\lambda+i\mu})\,d\lambda$$

where $\chi(\lambda) = 1/2$ for $\lambda = \lambda_1, \lambda_2$ and $\chi(\lambda) = 1$ elsewhere and therefore

$$\int_{(\lambda_1,\lambda_2)} dE(\lambda) = \lim_{\substack{\lambda_1'\downarrow\lambda_1\\ \lambda_2'\uparrow\lambda_2}}\lim_{\mu\downarrow 0}\frac{1}{2\pi i}\int_{\lambda_1'}^{\lambda_2'}(R_{\lambda-i\mu} - R_{\lambda+i\mu})\,d\lambda.$$

These formulas show that the spectral measure is determined by the singularities of the resolvent at the real line. Now R_λ has two kinds of singularities. First poles according to (1) and Lemmas 2, 3. Second the singularities along $(-\infty, -\varrho^2)$ because of the jump of the function $\lambda \mapsto \nu$ with $\lambda = -\varrho^2 + \nu^2$ and $\operatorname{Re}\nu > 0$. The examination of these singularities yields the following result on the spectrum:

Theorem 2 *The point spectrum of Δ consists of all eigenvalues λ with $\lambda = -\varrho^2 + \nu^2$ where $\varrho + \nu \in \mathbb{N}$, $\nu > 0$. The continuous spectrum consists of the interval $(-\infty, -\varrho^2]$.*

2 The spinor calculus

In this section some elementary properties of the Clifford algebra and the spin group for indefinite inner products are summarised, in particular the construction of spin invariant scalar products on the spin module is described. After that some remarks are made about the spinor calculus on pseudo-Riemannian manifolds, and a positive definite scalar product on the spinor bundle of any pseudo-Riemannian spin manifold is constructed (for a detailed discussion of all this see [B]). Finally this calculus is applied to the hyperboloids X: the spinor bundle is being trivialised and the significant objects (scalar products, Clifford multiplication, Dirac operator, etc.) are expressed in terms of this trivialisation.

Let p and q be natural numbers, $q \geq 1$. Let $\mathbb{R}^{p+q-1}$ be equipped with the inner products $(x, y) = \sum_{i=1}^{p+q-1} x_i y_i$ and $\langle x, y\rangle = -\sum_{i=1}^{p} x_i x_i + \sum_{i=p+1}^{p+q-1} x_i y_i$ (here $q-1$ is used instead of q only to be consistent with subsequent notation). Let $\mathbb{R}^{p,q-1}$ denote $\mathbb{R}^{p+q-1}$ if it is endowed with $\langle\,.\,,.\rangle$. Let $\boldsymbol{Cl}_{p+q-1}$ and $\boldsymbol{Cl}_{p,q-1}$ the Clifford algebras for these inner products, where the conventions $x^2 = -(x,x)$ and $x^2 = -\langle x, x\rangle$, respectively, are used for $x \in \mathbb{R}^{p+q-1}$. Let $\boldsymbol{Cl}^{\mathbb{C}}_{p-q+1}$ be the Clifford algebra for $\mathbb{C}^{p+q-1}$ with quadratic form $z \mapsto z_1^2 + \cdots + z_{p+q-1}^2$. Then the isometric embeddings

$$\begin{array}{ccc} \mathbb{R}^{p+q-1} & \longrightarrow & \mathbb{C}^{p+q-1} \\ e_k & \longmapsto & e_k \end{array}$$

and

$$\begin{array}{ccc} \mathbb{R}^{p,q-1} & \longrightarrow & \mathbb{C}^{p+q-1} \\ e_k & \longmapsto & \begin{cases} ie_k, & \text{if } k \leq p, \\ e_k, & \text{if } k \geq p+1; \end{cases} \end{array}$$

induce embeddings of $\boldsymbol{Cl}_{p+q-1}$ and $\boldsymbol{Cl}_{p,q-1}$ into $\boldsymbol{Cl}^{\mathbb{C}}_{p+q-1}$. Let Δ_{p+q-1} be the spin module and $\kappa\colon \boldsymbol{Cl}^{\mathbb{C}}_{p+q-1} \to \mathrm{End}(\Delta_{p+q-1})$ be a spin representation (just choose one in case $p+q-1$ odd). By restriction one obtains representations κ_{p+q-1} and $\kappa_{p,q-1}$ of $\boldsymbol{Cl}_{p+q-1}$ and $\boldsymbol{Cl}_{p,q-1}$, respectively. If Δ_{p+q-1} is viewed as a $\boldsymbol{Cl}_{p,q-1}$-module, it is denoted by $\Delta_{p,q-1}$. The spin groups are then defined as $\boldsymbol{Spin}(p+q-1) := \{x_1 \cdots x_{2r} \mid r \in \mathbb{N}, x_i \in \mathbb{R}^{p+q-1}, (x,x) = 1\}$ and $\boldsymbol{Spin}(p, q-1) := \{x_1 \cdots x_{2r} \mid r \in \mathbb{N}, x_i \in \mathbb{R}^{p,q-1}, (x,x) = \pm 1\}$. The map

$$\begin{array}{rccll} \lambda\colon & \boldsymbol{Spin}(p,q-1) & \longrightarrow & \boldsymbol{SO}(p,q-1) & \\ & g & \longmapsto & \lambda(g) & \quad \text{where } \lambda(g)x = g \cdot x \cdot g^{-1} \text{ for } x \in \mathbb{R}^{p,q-1} \end{array}$$

is a two-sheeted covering. $\boldsymbol{Spin}(p+q-1)$ is connected in case $p+q-1 \geq 2$ but $\boldsymbol{Spin}(p, q-1)$ in general has two connected components, except when $p = q-1 = 1$ (four connected components) and $p = 0$ or $q-1 = 0$ (connected). Let $\boldsymbol{Spin}_+(p, q-1) := \{x_1 \cdots x_{2r} \mid r \in \mathbb{N}, x_i \in \mathbb{R}^{p,q-1}, \langle x, x\rangle = \pm 1, \langle x, x\rangle = 1 \text{ for an even number of } i\}$. Then $\boldsymbol{Spin}_+(p, q-1) =$

$\boldsymbol{Spin}_o(p,q-1)$ if $(p,q-1)\neq(1,1)$; if $(p,q-1)=(1,1)$, $\boldsymbol{Spin}_+(p,q-1)$ has two connected components. One has $\boldsymbol{Spin}_+(p,q-1)=\lambda^{-1}(\boldsymbol{SO}_o(p,q-1))$.

The Lie algebra of $\boldsymbol{Spin}(p,q-1)$ is $\boldsymbol{spin}(p,q-1)=\mathscr{L}_{\mathbb{R}}\{x\cdot y \mid x,y\in\mathbb{R}^{p,q-1}\}$, and the differential λ_* of λ satisfies $\lambda_*(x\cdot y)z=2(\langle x,z\rangle y-\langle y,z\rangle x)$.

Choose a positive definite hermitian product $(.,.)_\Delta$ in $\Delta_{p,q-1}$ ($\mathbb{C}$-linear in the second argument, conjugate-linear in the first) such that $(x\cdot u,v)_\Delta=-(u,x\cdot v)_\Delta$ for all $x\in\mathbb{R}^{p+q-1}$, $u,v\in\Delta_{p+q-1}$; it is unique except for a positive real factor (here $x\cdot u$ means $\kappa_{p+q-1}(x)u$). $(.,.)$ is invariant under $\boldsymbol{Spin}(p+q-1)$. Let $\theta\colon\mathbb{R}^{p,q-1}\to\mathbb{R}^{p,q-1}$ denote the reflection in $0\times\mathbb{R}^{q-1}$, i.e., $\theta e_k=-e_k$ if $k\leq p$ and $\theta e_k=e_k$ if $k\geq p+1$. Then $(x\cdot u,v)=-(u,\theta x\cdot v)$ for $x\in\mathbb{R}^{p,q-1}$ (and here $x\cdot u$ means $\kappa_{p,q-1}(x)u$) because $\kappa_{p,q-1}(x)=i\kappa_{p+q-1}(x)$ if $x\in\mathbb{R}^p\times 0$ and $\kappa_{p,q-1}(x)=\kappa_{p+q-1}(x)$ if $x\in 0\times\mathbb{R}^{q-1}$. One can construct $\boldsymbol{Spin}_+(p,q-1)$-invariant scalar products from $(.,.)$. For this purpose let $a:=i^{p(p-1)/2}e_1\cdots e_p$ and $b:=i^{q(q-1)/2}e_{p+1}\cdots e_{p+q-1}$ in $\boldsymbol{Cl}_{p,q-1}\otimes\mathbb{C}$; instead of $(e_1,\ldots,e_p)$ and $(e_{p+1},\ldots,e_{p+q-1})$ one can take any orthonormal bases of $\mathbb{R}^p\times 0$ and $0\times\mathbb{R}^{q-1}$ with the same orientation as $(e_1,\ldots,e_p)$ and $(e_{p+1},\ldots,e_{p+q-1})$, respectively, because a, say, is except for the factor $i^{p(p-1)/2}$ the image of $e_1\wedge\ldots\wedge e_p$ under the canonical map of $\bigwedge\mathbb{R}^{p,q-1}$ onto $\boldsymbol{Cl}_{p,q-1}$. If $x\in\mathbb{R}^{p,q-1}$, then $x\cdot a=(-1)^p a\cdot\theta x$ and $x\cdot b=(-1)^q b\cdot\theta x$. Furthermore $a^2=b^2=1$. The multiplication with a and b, respectively, is a symmetric isomorphism of $\Delta_{p,q-1}$ with respect to $(.,.)$. Therefore $\langle u,v\rangle_a:=(u,a\cdot v)$ and $\langle u,v\rangle_b:=(u,b\cdot v)$ are non-degenerate hermitian sesquilinear forms on $\Delta_{p,q-1}$. They satisfy $\langle x\cdot u,v\rangle_a=(-1)^p\langle u,x\cdot v\rangle_a$ and $\langle x\cdot u,v\rangle_b=(-1)^q\langle u,x\cdot v\rangle_b$ for $x\in\mathbb{R}^{p,q-1}$. Furthermore these scalar products are invariant under $\boldsymbol{Spin}_+(p,q-1)$, because if $x_1,\ldots,x_{2r}\in\mathbb{R}^{p,q-1}$, $\langle x_i,x_i\rangle=\pm1$, $\langle x_i,x_i\rangle=1$ for an even number of i, then

$$\begin{aligned}\langle x_1\cdots x_{2r}\cdot u\,,\,x_1\cdots x_{2r}\cdot v\rangle_a &= ((-1)^p)^{2r}\langle u\,,\,x_{2r}\cdots x_1\cdot x_1\cdots x_{2r}\cdot u\rangle_a\\ &= x_1^2\cdots x_{2r}^2\langle u,v\rangle_a\\ &= (-1)^{\text{even number}}\cdot\langle u,v\rangle_a\\ &= \langle u,v\rangle_a.\end{aligned}$$

and the same for $\langle .,.\rangle_b$. Since the volume element $e_1\cdots e_{p+q-1}$ acts on $\Delta_{p,q-1}$ by multiplication with $\pm i^{(p+q-1)(p+q-1)/2+(q-1)}$ for odd $p+q-1$ ("$\pm$" depends on which of the two representations of $\boldsymbol{Cl}_{p,q-1}$ on $\Delta_{p,q-1}$ was chosen) in this case $\langle .,.\rangle_b=\pm\langle .,.\rangle_a$, so there is essentially only one spin invariant scalar product. This is also clear from the fact that $\boldsymbol{Spin}_+(p,q-1)$ acts irreducibly on $\Delta_{p,q-1}$.

Consider the well known isomorphism

$$\begin{array}{ccc}\boldsymbol{Cl}_{p,q-1} & \longrightarrow & \boldsymbol{Cl}^0_{p,q}\\ u=u_0+u_1 & \longmapsto & u_0+u_1\cdot e_{p+q}\end{array}\quad(1)$$

where $\boldsymbol{Cl}^0_{p,q}$ denotes the algebra of even elements and $u=u_0+u_1$ is the decomposition of u into its even and its odd part. In this way one obtains a representation $\hat{\kappa}$ of $\boldsymbol{Cl}_{p,q-1}$ in $\Delta_{p,q}$. If $p+q-1$ is even, this representation is isomorphic to $\Delta_{p,q-1}$, if $p+q-1$ is odd, then $\boldsymbol{Cl}_{p,q}$ decomposes into the positive and negative Weyl spinors $\Delta^\pm_{p,q}$, which are invariant under $\boldsymbol{Cl}^0_{p,q}$ and thus also under $\boldsymbol{Cl}_{p,q-1}$. The two representations thus obtained are isomorphic to the two representations that $\boldsymbol{Cl}_{p,q-1}$ has in $\Delta_{p,q-1}$ (see [Tr] Prop. 4; the reason is simply that $\boldsymbol{Cl}_{p,q-1}$ has respectively only one or two irreducible complex representations).

Now, if $u, v \in \Delta_{p,q}$, $x \in \mathbb{R}^{p,q-1}$, then

$$\begin{aligned}(\hat\kappa(x)u, v)_{\Delta_{p,q}} &= (xe_{p+q} \cdot u\ ,\ v)_{\Delta_{p,q}}\\ &= (u\ ,\ e_{p+q}\theta x \cdot v)_{\Delta_{p,q}}\\ &= -(u\ ,\ (\theta x)e_{p+q} \cdot v)_{\Delta_{p,q}}\\ &= -(u, \hat\kappa(\theta x)v)_{\Delta_{p,q}},\end{aligned}$$

so one can introduce $\boldsymbol{Spin}_+(p, q-1)$-invariant scalar products on $\Delta_{p,q}$ with the method mentioned above. In this way $\langle u, v\rangle_a = i^{p(p-1)/2}(u\ ,\ e_1 \cdots e_p \cdot v)_{\Delta_{p,q}}$ for $u, v \in \Delta_{p,q}$ if p is even and $\langle u, v\rangle_b = i^{q(q-1)/2}(u\ ,\ e_{p+1} \cdots e_{p+q} \cdot v)_{\Delta_{p,q}}$ if q is even. One observes that for p even $\langle . , . \rangle_a$ coincides with the scalar product that is constructed analogously using the representation of $\boldsymbol{Cl}_{p,q}$ in $\Delta_{p,q}$, and which is even invariant under $\boldsymbol{Spin}_+(p, q)$. If in addition $p + q$ is even, then $\Delta^+_{p,q}$ and $\Delta^-_{p,q}$ are orthogonal and $\langle . , . \rangle_a$ is non-degenerate on $\Delta^\pm_{p,q}$. The same can be done if q is even. Thus one concludes: If p or q is even, then there exists a (non-degenerate) scalar product on $\Delta^{(\pm)}_{p,q}$ that is invariant under $\boldsymbol{Spin}_+(p, q)$.

Let now (M, g) be a semi-Riemannian manifold of dimension n and index p. Let $O(M, g) := \{(s_1, \ldots, s_n) \mid s_i \in T_xM, x \in M, g(s_i, s_j) = \delta_{ij}\varepsilon_i\}$, where $\varepsilon_i := -1$ if $i \le p$ and $\varepsilon_i := 1$ if $i > p$, denote the bundle of orthonormal frames with structure group $\boldsymbol{O}(p, n-p)$. In addition let (M, g) be space and time oriented, that is, there is fixed a reduction $SO_+(M)$ of $O(M, g)$ to $\boldsymbol{SO}_o(p, n-p) \subset \boldsymbol{O}(p, n-p)$. Since $\boldsymbol{SO}(p) \cdot \boldsymbol{SO}(n-p)$ is maximal compact in $\boldsymbol{SO}_o(p, n-p)$, the bundle $SO_+(M)$ can be reduced to a bundle Q with structure group $\boldsymbol{SO}(p) \cdot \boldsymbol{SO}(n-p)$. Each such reduction Q is equivalent to the choice of a decomposition $TM = T^-M \oplus T^+M$ of the tangent bundle into an g-orthogonal sum of a positive and a negative definite subbundle that both are oriented, as follows. If Q is chosen as above and $x \in M$, take a frame $s = (s_1, \ldots, s_n) \in Q_x$ and let T^-_xM be the space spanned by $s_1, \ldots, s_p$. The orientation in T^-_xM is chosen so that $(s_1, \ldots, s_p)$ is positive time and space oriented. For T^+_xM one proceeds in the same way. The spaces $T^\pm_xM$ and their orientations thus obtained are independent of the choice of $s \in Q_x$. On the other hand Q is determined by $T^\pm M$ because $(s_1, \ldots, s_n) \in Q_x$ if and only if $(s_1, \ldots, s_p)$ is a positive oriented frame of T^-_xM and $(s_{p+1}, \ldots, s_n)$ is a positive oriented frame of T^+_xM.

For each such decomposition there is a g-orthogonal reflection θ in T^+M and an associate Riemannian metric $r(s, t) := g(s, \theta t)$.

Let now P be a spin structure of (M, g), that is, there is a reduction $P \xrightarrow{\Lambda} SO_+(M)$ with respect to $\lambda\colon \boldsymbol{Spin}_+(p, n-p) \to \boldsymbol{SO}_o(p, n-p)$. The spinor bundle S is the associated bundle $S = P \times_{\kappa_{p,n-p}} \Delta_{p,n-p}$. The scalar products $\langle . , . \rangle_{a,b}$ in $\Delta_{p,n-p}$ invariant under $\boldsymbol{Spin}_+(p, n-p)$ induce scalar products $\langle . , . \rangle_{a,b}$ in S, $\langle [p\ ,\ u], [p, v]\rangle_{a,b} := \langle u, v\rangle_{a,b}$. Analogously to the Riemannian case the Clifford multiplication $\mathbb{R}^{p,n-p} \otimes \Delta_{p,n-p} \to \Delta_{p,n-p}$ is equivariant with respect to $\boldsymbol{Spin}(p, n-p)$, so there's a Clifford multiplication $TM \otimes S \to S$. The Levi-Civita connection in $SO_+(M)$ lifts to a connection in P and thus gives rise to a covariant derivation ∇^S in $C^\infty(S)$. The Dirac operator is then the composition of the maps $C^\infty(S) \xrightarrow{\nabla^S} C^\infty(TM^* \otimes S) \xrightarrow{g} C^\infty(TM \otimes S) \to C^\infty(S)$ where the last map is given by the

Clifford multiplication. If $\varphi \in C^\infty(S)$ is a spinor field and $(s_1, \ldots, s_n)$ an orthonormal frame of T_xM, then $D\varphi(x) = \sum_{i=1}^n \varepsilon_i s_i \cdot \nabla^S_{s_i}\varphi(x)$ where $\varepsilon_i = g(s_i, s_i)$.

Since there is no positive definite hermitian product in $\Delta_{p,n-p}$ that is invariant under $\boldsymbol{Spin}_+(p, n-p)$ if $0 < p < n$, *a priori* one cannot define a positive definite hermitian product in S. Nevertheless one can obtain such an inner product as follows. One has to *choose* a decomposition $TM = T^-M \oplus T^+M$ and construct the reduced bundle Q as described above. Then $P' := \Lambda^{-1}(Q) \subset P$ is a reduction of P to the structure group $\boldsymbol{Spin}(p) \cdot \boldsymbol{Spin}(n-p) \subset \boldsymbol{Spin}_+(p, n-p)$. Then $S = P' \times_{\boldsymbol{Spin}(p)\cdot\boldsymbol{Spin}(n-p)} \Delta_{p,n-p}$. Since the positive definite scalar product $(.\,,.)_\Delta$ is invariant under $\boldsymbol{Spin}(p) \cdot \boldsymbol{Spin}(n-p)$ this gives rise to a positive definite scalar product $(.\,,.)_S$. According to the relations between $\langle .\,,.\rangle_{a,b}$ and $(.\,,.)_\Delta$ it follows:

$$
\begin{aligned}
(t \cdot u, v)_S &= -(u, (\theta t) \cdot v)_S, \\
\langle u, v\rangle_q &= i^{p(p-1)/2}(u\ ,\ s_1 \cdots s_p \cdot v)_S \text{ and} \\
\langle u, v\rangle_b &= i^{(n-p)(n-p+1)/2}(u\ ,\ s_{p+1} \cdots s_n \cdot v)_S
\end{aligned}
$$

for all $t \in T_xM$, $u, v \in S_x$, and positive oriented frames $(s_1, \ldots, s_p)$ and $(s_{p+1}, \ldots, s_n)$ of T^-_xM and T^+_xM, respectively.

Recall that a Krein space is a complex Hilbert space H with inner product $(.\,,.)$, together with a self-adjoint involution $J \in \mathrm{Aut}(H)$, also called the fundamental symmetry. This involution J gives rise to an indefinite inner product $\langle x, y\rangle := (x, Jy)$. Since $J \in \mathrm{Aut}(H)$, the mapping $H \to H^*$, $x \mapsto \langle x, \cdot\rangle$ is a conjugate-linear isomorphism of H onto its dual. Therefore for each densely defined operator A in H one has the (J-)adjoint $A^\sharp$ defined by $\mathrm{dom}(A^\sharp) := \{x \in H \mid \mathrm{dom}(A) \ni y \mapsto \langle x, Ay\rangle \in \mathbb{C} \text{ is continuous}\}$ and $\langle A^\sharp x, y\rangle = \langle x, Ay\rangle$ for all $x \in \mathrm{dom}(A^\sharp)$ and $y \in \mathrm{dom}(A)$. A is called (J-)self-adjoint if $A = A^\sharp$, and essentially self-adjoint if $A^\sharp = \overline{A}$, $\overline{A}$ denoting the closure of A.

The adjoint in Krein spaces has some elementary properties with respect to the spectrum:

Lemma 1 *Let $(H, (.\,,.), J)$ be Krein space, A a closed operator, let σ_p, σ_c, and σ_r denote the point, the continuous, and the residual spectrum, respectively. Then:*

(a) *If $\lambda \in \sigma_r(A)$, then $\overline{\lambda} \in \sigma_p(A^\sharp)$.*
(b) *If $\lambda \in \sigma_p(A)$, then $\overline{\lambda} \in \sigma_r(A^\sharp) \cup \sigma_p(A^\sharp)$.*
(c) *If $\lambda_c(A)$, then $\overline{\lambda} \in \sigma_c(A^\sharp)$.*

In particular $\sigma(A^\sharp) = \overline{\sigma(A)}$ (here $\overline{\Box}$ denotes the complex conjugate).

Proof. In the first place $\mathrm{im}(A)^{\perp_{\langle .\,,.\rangle}} = \ker(A^\sharp)$. For $x \perp \mathrm{im}(A)$ if and only if $\langle x, Ay\rangle = 0$ for all $y \in \mathrm{dom}(A)$, which is the same as $x \in \mathrm{dom}(A^\sharp)$ and $A^\sharp x = 0$.

(a) Let $\lambda \in \sigma_r(A)$. Then $\mathrm{im}(A - \lambda)$ is not dense in H. Therefore $\ker(A^\sharp - \overline{\lambda}) = \ker(A - \lambda)^\sharp = \mathrm{im}(A - \lambda)^\perp \neq 0$, hence $\overline{\lambda} \in \sigma_p(A^\sharp)$.

(b) Let $\lambda \in \sigma_p(A)$. Then $\ker(A - \lambda) \neq 0$. Hence $\overline{\mathrm{im}(A^\sharp - \overline{\lambda})} = \ker(A - \lambda)^\perp \neq H$. Thus either $\overline{\lambda} \in \sigma_p(A^\sharp)$ or $\overline{\lambda} \in \sigma_r(A^\sharp)$.

(c) (a) and (b) imply that $\overline{\sigma_p(A) \cup \sigma_r(A)} \subset \sigma_p(A^\sharp) \cup \sigma_r(A^\sharp)$ hence $\overline{\sigma_p(A) \cup \sigma_r(A)} = \sigma_p(A^\sharp) \cup \sigma_r(A^\sharp)$ because of $(A^\sharp)^\sharp = A$. (c) follows immediately. **Q.E.D.**

Consider now a space and time oriented semi-Riemannian spin manifold (M,g) of dimension n and index p with a fixed decomposition $TM = T^-M \oplus T^+M$. On the spinor bundle there are the positive definite scalar product $(.\,,.)_S$ and the indefinite scalar products $\langle .\,,.\rangle_{a,b}$, that are related by $\langle .\,,.\rangle_{a,b} = (\,\cdot\,, J_{a,b}\,\cdot\,)_S$ where $J_{a,b} \in \mathrm{End}(S)$, J_a is the Clifford multiplication with $i^{p(p-1)/1} s_1 \cdots s_p$ and J_b is the Clifford multiplication with $i^{(n-p)(n-p+1)/2} s_{p+1} \cdots s_n$, $(s_1, \ldots, s_n)$ an arbitrary frame that is compatible with both the decomposition $TM = T^-M \oplus T^+M$ and the orientations in $T^\pm M$. Define $L^2(S)$ to be the Hilbert space of square integrable sections of S with respect to $(.\,,.)_S$ and the volume form belonging to g. Then $J_{a,b} \in \mathrm{End}(S)$ induce self-adjoint involutions of $L^2(S)$ that make $L^2(S)$ into a Krein space. Let $\Box^{\sharp_{a,b}}$ denote the adjoint with respect to $\langle .\,,.\rangle_{a,b}$ in $L^2(S)$. Then

Theorem 1 *If the Riemannian metric corresponding to the decomposition $TM = T^-M \oplus T^+M$ complete, then $D^{\sharp_a} = (-1)^p \overline{D}$ and $D^{\sharp_b} = (-1)^{n-p-1} \overline{D}$ where D is the Dirac operator equipped with domain $C_0^\infty(S)$ and $\overline{D}$ is its closure in $L^2(S)$.*

Proof. See [B] Satz 3.19.2. ⊠

Notice that for n even D is essentially self-adjoint with respect to exactly one of $\langle .\,,.\rangle_{a,b}$, while for n odd D is essentially self-adjoint if and only if the index p is even.

Return now to the hyperboloids. Thus let p and q be natural numbers, $p \geq 1$, $q \geq 2$. $\boldsymbol{Spin}_+(p,q)$ acts transitively on $X = \{x \in \mathbb{R}^{p,q} \mid \langle x,x\rangle_{p,q} = 1\}$ via the covering $\lambda\colon \boldsymbol{Spin}_+(p,q) \to \boldsymbol{SO}_o(p,q)$. Let $G := \boldsymbol{Spin}_+(p,q)$, $H := \boldsymbol{Spin}_+(p,q-1) = \lambda^{-1}(\boldsymbol{SO}_o(p,q-1))$. Let $SO_+(X)$ denote the bundle of time and space oriented frames of X where the orientation is chosen so that the frame $(e_1, \ldots, e_{p+q-1})$ is positive oriented. By virtue of

$$\begin{array}{ccc} \boldsymbol{SO}_o(p,q) & \longrightarrow & SO_+(X) \\ A & \longmapsto & (Ae_1, \ldots, Ae_{p+q-1}) \end{array}$$

the bundle $SO_+(X) \to X$ is isomorphic to the bundle $\boldsymbol{SO}_o(p,q) \to X$, $A \mapsto Ae_{p+q}$. Then the covering $\lambda\colon \boldsymbol{Spin}_+(p,q) \to \boldsymbol{SO}_o(p,q)$ is a spin structure; let $\boldsymbol{Spin}_+(p,q) =: Spin(X)$ if it is considered as the spin structure. Then one can trivialise S using (1):

$$\begin{array}{rcl} S = Spin(X) \times_{\boldsymbol{Spin}_+(p,q-1)} \Delta_{p,q}^{(\pm)} & \longrightarrow & X \times \Delta_{p,q}^{(\pm)} \\ {[g,u]} & \longmapsto & (\lambda(g)e_{p+q}\,,\, gu). \end{array} \quad (2)$$

Lemma 2

- *If $t \in T_xX$, $f \in C^\infty(S) = C^\infty(X, \Delta_{p,q}^{(\pm)})$, then*
$$(\nabla_t f)(x) = t(f)(x) + \frac{1}{2} t \cdot x \cdot f(x).$$
- *If $\langle .\,,.\rangle_\Delta$ is a scalar product in $\Delta_{p,q}^{(\pm)}$ that is invariant under $\boldsymbol{Spin}_+(p,q)$, then*
$$\langle u,v\rangle_{S_x} = \langle u,v\rangle_\Delta$$
where $x \in X$, $u,v \in \Delta_{p,q}^{(\pm)} \equiv S_x$, and $\langle .\,,.\rangle_S$ denotes the scalar product in S that is induced by $\langle .\,,.\rangle_\Delta$.

- *Let $t \circledcirc u$ denote the Clifford multiplication of $t \in T_xX \subset \mathbb{R}^{p+q}$ with $u \in S_x \equiv \Delta^{(\pm)}_{p,q}$. Then*
$$t \circledcirc x = t \cdot x \cdot u$$
where $\cdot$ on the right hand side denotes the Clifford multiplication in $\Delta^{(\pm)}_{p,q}$.
- *The natural action of G on S is given by $g \cdot (x,u) = (\lambda(g)x, gu)$.*

Proof. Let $\sigma : G \to G$ denote $\sigma(g) := -e_{p+q} \cdot g \cdot e_{p+q}$. Since $e^2_{p+q} = -1$, σ is an involutive automorphism of G. The group G^σ of fixed points is H. For, $\sigma(g) = g$ is equivalent to $g e_{p+q} g^{-1} = e_{p+q}$, that is, $\lambda(g)e_{p+q} = e_{p+q}$. Therefore $g \in \lambda^{-1}(\boldsymbol{SO}_o(p,q-1)) = H$, as claimed. Hence $X = G/H$ is a symmetric space. Let Z denote the Levi-Civita connection in $Spin(X) = G$, $\mathfrak{g}$ the Lie algebra of G, $\mathfrak{h}$ the Lie algebra of H, $\mathfrak{q} := \{A \in g \mid \sigma_*(A) = -A\}$. Then the theory of symmetric spaces shows that $Z\big((dL_g)_1(A)\big) = pr_{\mathfrak{h}}(A)$ for $A \in \mathfrak{g}$ where $L_g : G \to G$ denotes the multiplication with g from the left and $pr_{\mathfrak{h}}$ is the projection with respect to $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}$. Let $Q := Spin(X) \times_H \boldsymbol{Spin}_+(p,q)$. This is a principal fibre bundle over X with structure group $\boldsymbol{Spin}_+(p,q)$. Then

$$\begin{aligned} Q \quad &\longrightarrow X \times \boldsymbol{Spin}_+(p,q) \\ [g,g'] &\longmapsto (\lambda(g)e_{p+q}\,,\, gg') \quad (3) \end{aligned}$$

is an isomorphism of the principal fibre bundles. Since $S = Spin(X) \times_H \Delta^{(\pm)}_{p,q}$ and G acts on $\Delta^{(\pm)}_{p,q}$, $S \approx Q \times_G \Delta^{(\pm)}_{p,q}$. Then (3) yields a trivialisation of S that is exactly the one of (2). To compute the covariant derivation ∇^S it suffices therefore to see what happens to the Levi-Civita connection when extended to Q and then moved to $X \times \boldsymbol{Spin}_+(p,q)$. For this purpose let $\gamma(t) = g \exp tA$ be a horizontal curve in $Spin(X)$ where $g \in G$ and $A \in \mathfrak{q}$. Then by virtue of (3) this curve turns into

$$t \mapsto \gamma(t) \in Spin(X) \mapsto [\gamma(t), 1] \in Q \mapsto (\lambda(\gamma(t))e_{p+q}\,,\, \gamma(t)) \in X \times \boldsymbol{Spin}_+(p,q).$$

Differentiation yields that

$$(\lambda(g)\lambda_*(A)\,,\, gA) \in T_{(\lambda(g)e_{p+q}\,,\,g)} X \times \boldsymbol{Spin}_+(p,q) \tag{4}$$

is a horizontal vector. Let now $x \in X$, $t \in T_xX$, $x = \lambda(g)e_{p+q}$, $t = \lambda(g)t'$, where $t' \in \mathbb{R}^{p+q-1} \times 0 = T_{e_{p+q}}X$, $f \in C^\infty(X, \Delta^{(\pm)}_{p,q})$, and let $s : X \to X \times \boldsymbol{Spin}_+(p,q)$ denote the "trivial" section $s(x) = (x,1)$. Then

$$\begin{aligned} (\nabla_t f)(x) &= \nabla_t[s,f](x) \\ &= [s\,,\, t(f)(x) + Z(ds_x(t))f(x)] \\ &= [s\,,\, t(f)(x) + Z((t,0))f(x)]. \end{aligned} \tag{5}$$

Therefore one has to compute $Z(t,0)$ for $(t,0) \in T_{(x,1)}X \times \boldsymbol{Spin}_+(p,q)$. Now

$$\lambda_*(-\tfrac{1}{2}t'e_{p+q})e_{p+q} = 2\left(\langle -\tfrac{1}{2}t'\,,\, e_{p+q}\rangle e_{p+q} - \langle -\tfrac{1}{2}e_{p+q}\,,\, e_{p+q}\rangle t'\right) = t',$$

hence

$$T_{(x,1)}X \times \boldsymbol{Spin}_+(p,q) \ni (t,x) = (dr_{g^{-1}})_{(x,g)}(\lambda(g)\lambda_*(-\tfrac{1}{2}te_{p+q})e_{p+q}\ ,\ dL_g(-\tfrac{1}{2}t'e_{p+q})) + (dr_{g^{-1}})_{(x,g)}(0\ ,\ dL_g(\tfrac{1}{2}t'e_{p+q}))$$

where r_g denotes the action of $\boldsymbol{Spin}_+(p,q)$ on $X \times \boldsymbol{Spin}_{(}p,q)$. The first summand is horizontal because of (4), while the second is the fundamental vector field corresponding to $g\frac{1}{2}t'e_{p+q}g^{-1} \in \mathfrak{g}$ at $(x,0)$. Consequently $Z(t,0) = \frac{1}{2}gt'e_{p+q}g^{-1} = \frac{1}{2}t$ because $tx = gt'g^{-1}$ and $x = ge_{p+q}g^{-1}$. Therefore because of (5):

$$\begin{aligned}(\nabla_t f)(x) &= [s\ ,\ t(f)(x) + \tfrac{1}{2}tx \cdot f(x)]\\ &= t(f)(x) + \tfrac{1}{2}tx \cdot f(x),\end{aligned}$$

as claimed.

If $\langle .\, ,.\rangle$ is an inner product in $\Delta_{p,q}^{(\pm)}$ invariant under H, then the associated inner product in S is given by $\langle [g,u],[g,v]\rangle = \langle u,v\rangle$. If then $\Delta_{p,q}^{(\pm)} \ni u \equiv [g,g^{-1}u] \in S_{\lambda(g)e_{p+q}}$, $v \equiv [g,g^{-1}v]$,

$$\begin{aligned}\langle u,v\rangle_{S_{\lambda(g)e_{p+q}}} &= \langle [g\ ,\ g^{-1}u],[g,g^{-1}v]\rangle\\ &= \langle g^{-1}u\ ,\ g^{-1}v\rangle.\end{aligned}$$

Therefore, if $\langle .\, ,.\rangle$ is even invariant under $G = \boldsymbol{Spin}_+(p,q)$, the assertion follows.

Let $x = \lambda(g)e_{p+q}$, $t = \lambda(g)t'$, $u \in \Delta_{p,q}^{(\pm)} \equiv S_x$. Then

$$\begin{aligned}t\circledcirc &= [g,t'] \circledcirc [g,g^{-1}u]\\ &= [g,t'e_{p+q} \cdot g^{-1}u]\\ &= g \cdot t'e_{p+q} \cdot g^{-1}u\\ &= t \cdot x \cdot u\end{aligned}$$

according to (1).

Finally, the action of G on S is given by $g \cdot (\lambda(g)e_{p+q}\ ,\ u) \equiv g' \cdot [g,g^{-1}u] = [g'g,g^{-1}u] \equiv (\lambda(g'g)e_{p+q}\ ,\ g' \cdot u)$. **Q.E.D.**

Lemma 3 *If $x \in X$, $f \in C^\infty(X,\Delta_{p,q}^{(\pm)})$, D the Dirac operator, then*

$$Df(x) = -\frac{n}{2} - x \cdot \sum_{i=1}^{n} \varepsilon_i s_i \cdot s_i(f)(x)$$

where $(s_1,\ldots,s_n)$ is an orthonormal frame of T_xX , $n := \dim X = p+q-1$, and $\varepsilon_i = g(s_i,s_i)$.

Proof. $Df(x)$ is given by $Df(x) = \sum_{i=1}^n \varepsilon_i s_i \circledcirc \nabla_{s_i} f(x)$, hence

$$\begin{aligned}Df(x) &= \sum_{i=1}^{n} \varepsilon_i s_i x \cdot \big(s_i(f)(x) + \tfrac{1}{2}s_i x f(x)\big)\\ &= \sum_{i=1}^{n} \varepsilon_i(-xs_i)s_i(f)(x) + \frac{1}{2}\sum_{i=1}^{n} \varepsilon_i(-s_i x x s_i)f(x)\\ &= -x \cdot \sum_{i=1}^{n} \varepsilon_i s_i \cdot s_i(f)(x) - \frac{1}{2}nf(x).\end{aligned}$$

Q.E.D.

Lemma 4 *The Dirac operator D and the Laplace operator Δ satisfy*

$$\left(D+\frac{1}{2}\right)^2=\left(\frac{n-1}{2}\right)^2+\Delta.$$

Proof. Let $x\in X$, let $(s_1,\dots,s_n)$ by a local x-synchronous frame, that is, $\nabla_t s_i(x)=0$ for all $t\in T_xX$. Then

$$\begin{aligned}s_i(s_j)(s)&=\nabla_{s_i}s_j(x)+\langle s_i(s_j),x\rangle\cdot x\\&=0+\big(s_i(\langle s_j,id\rangle)-\langle s_i,s_j(id)\rangle\big)\cdot x\\&=-\varepsilon_i\delta_{ij}x.\end{aligned}$$

Hence

$$\begin{aligned}\left(D+\frac{1}{2}\right)^2 f&=\left(\frac{n-1}{2}\right)^2 f+(n-1)x\cdot\sum_{i=1}^n\varepsilon_i s_i\cdot s_i(f)+\sum_{i,j=1}^n\varepsilon_i\varepsilon_j x s_i\cdot s_i(x s_j\cdot s_j(f))\\&=\left(\frac{n-1}{2}\right)^2 f+(n-1)x\cdot\sum_{i=1}^n\varepsilon_i s_i\cdot s_i(f)\\&\quad+\sum_{i,j=1}^n\varepsilon_i\varepsilon_j x s_i\cdot\big(s_i s_j\cdot s_j(f)-\varepsilon_i\delta_{ij}xx\cdot s_j(f)+x s_j\cdot s_i(s_j(f))\big).\end{aligned}$$

Now, the first part of the last sum equals $-nx\cdot\sum_{j=1}^n\varepsilon_j s_j\cdot s_j(f)$ because of $s_i^2=-\varepsilon_i$, the second is $x\cdot\sum_{i=1}^n\varepsilon_i s_i\cdot s_i(f)$ because of $x^2=-1$, and the third equals $\sum_{i,j=1}^n\varepsilon_i\varepsilon_j s_i s_j\cdot s_i(s_j(f))$ because $xs_i=-s_ix$. Since $[s_i,s_j](x)=0$ (because the s_i are x-synchronous) and $s_is_j=-s_js_i$ for $i\neq j$, in the last term only the summation for $i=j$ is needed. Since $s_i^2=-\varepsilon_i$ this is just Δf. **Q.E.D.**

Finally a positive definite hermitian product on the spinor bundle has to be defined. This is done as described at the beginning of this section. To find a decomposition $TM=T^-M\oplus T^+M$ one can proceed as follows. Consider the following map:

$$\begin{aligned}\Phi:\quad X\quad&\longrightarrow\quad B^p\times S^{q-1}\\x'+x''&\longmapsto\Big((1+|x''|)^{-1}x',\ |x''|^{-1}x''\Big)\end{aligned}$$

where $x=x'+x''$ such that $x'\in\mathbb{R}^p\times 0$ and $x''\in 0\times\mathbb{R}^q$, and $B^p:=\{a\in\mathbb{R}^p\mid |a|<1\}$. This is a diffeomorphism whose inverse is given by

$$\begin{aligned}\Phi^{-1}:\ B^p\times S^{q-1}&\longrightarrow\quad X\\(a,b)\quad&\longmapsto\left(\frac{2}{1-|a|^2}a\,,\ \frac{1+|a|^2}{1-|a|^2}b\right).\end{aligned}$$

The metric transforms as follows:

$$\big((\Phi^{-1})^*\langle.\,,.\rangle_{p,q}\big)_{(a,b)}=-\frac{4}{(1-|a|^2)^2}g_{B^p}+\left(\frac{2}{1-|a|^2}-1\right)^2 g_{S^{q-1}}.$$

Therefore $T(B^p \times S^{q-1}) = TB^p \oplus TS^{q-1}$ is an orthogonal decomposition into a positive and a negative definite subbundle. Furthermore the Riemannian metric associated to this decomposition is a warped product of two complete Riemannian metrics, namely the hyperbolic space B^p and the sphere S^{q-1}. Hence the associated Riemannian metric is complete. Moving this decomposition to X by virtue of Φ^{-1} one has

$$\begin{aligned} T_x^+X &= \{(0, z'') \mid z'' \in \mathbb{R}^q, \langle x'', z''\rangle = 0\}, \\ T_x^-X &= \left\{(z' \,,\, \tfrac{\langle x', z'\rangle}{|x''|^2} x'') \mid z' \in \mathbb{R}^p\right\}. \end{aligned} \tag{6}$$

At first sight this decomposition seems to be achieved by pure coincidence owing to the diffeomorphism Φ. But consider a semisimple Lie group G with involution σ, and a subgroup H such that $(G^\sigma)_o \subset H \subset G^\sigma$, so G/H is a symmetric space. Choose a Cartan involution θ commuting with σ, set $K := G^\theta$. Let $\mathfrak{p} := E_{-1}(\theta_*)$ and $\mathfrak{q} := E_{-1}(\sigma_*)$ be the eigenspaces. Then G/H is diffeomorphic to a fibre bundle over $K/(K \cap H)$ with fibre $\mathfrak{p} \cap \mathfrak{q}$. If G/H is equipped with the metric induced by the Killing form, then these fibres are maximal positive definite submanifolds. Hence the tangent spaces at these fibres constitute a maximal positive definite subbundle of $T\,G/H$. In case $G = \boldsymbol{Spin}_+(p,q)$, $H = \boldsymbol{Spin}_+(p, q-1)$, and $K = \boldsymbol{Spin}(p) \cdot \boldsymbol{Spin}(q)$ this results in the decomposition (6). The fibres (i.e. B^p) are here negative definite because $\langle .\,, .\rangle_{p,q}|_X$ is not induced by the Killing form but a negative multiple.

Choosing the decomposition (6) one obtains

Lemma 5 *For $x \in X$ and $u \in \Delta_{p,q}^{(\pm)}$ the norm $\|u\|_x := \|(x,u)\|_{S_x}$ is given by*

$$\|u\|_x^2 = |x''| \cdot \|u\|^2_{\Delta_{p,q}^{(\pm)}} + |x''|^{-1} \cdot (u \,,\, x'x'' \cdot u)_{\Delta_{p,q}^{(\pm)}}$$

where again $x = x' + x''$, $x' \in \mathbb{R}^p \times 0$ and $x'' \in 0 \times \mathbb{R}^q$. $\|\cdot\|$ is invariant under $K := \boldsymbol{Spin}(p) \cdot \boldsymbol{Spin}(q)$.

Proof. One has $\|u\|_x = \|[g, g^{-1}u]\| = \|g^{-1}u\|_{\Delta_{p,q}^{(\pm)}}$ if $x = \lambda(g)e_{p+q}$ and $\lambda(g)e_i \in T_x^-X$ if $i = 1, \ldots, p$ and $\lambda(g)e_i \in T_x^+M$ if $i = p+1, \ldots, p+q-1$ (because this is equivalent to g lying in the reduction of $Spin(X)$ determined by (6)). Thus if $k \in K$,

$$\|ku\|_{\lambda(k)x} = \|[kg, g^{-1}]\| = \|g^{-1}u\| = \|u\|_x$$

because kg lies in the reduction defined by (6), too ((6) is obviously K-invariant), thus proving the K-invariance. Now, let x be of the form $x = \pm\sqrt{a^2-1}\,e_p + a\,e_{p+q}$ where $a \geq 1$. Let $g := \mp\sqrt{\frac{a+1}{2}} + \sqrt{\frac{a-1}{2}} e_p e_{p+q}$. Then

$$g = \left(\mp\sqrt{\frac{a+1}{2}} e_p + \sqrt{\frac{a-1}{2}} e_{p+q}\right)\left(a\,e_p \mp \sqrt{a^2-1} e_{p+q}\right)$$

is a product of two negative unit vectors of $\mathbb{R}^{p,q}$. Hence $g \in \boldsymbol{Spin}_+(p,q)$. Furthermore

$g^{-1} = \mp\sqrt{\frac{a+1}{2}} - \sqrt{\frac{a-1}{2}} e_p e_{p+q}$ and

$$\lambda(g) = \begin{pmatrix} 1 & & & 0 & & & 0 \\ & \ddots & & \vdots & & & \vdots \\ & & 1 & 0 & & & 0 \\ & & & a & & & \pm\sqrt{a^2-1} \\ & & & 0 & 1 & & 0 \\ & & & \vdots & & \ddots & \vdots \\ & & & 0 & & 1 & 0 \\ & & & \pm\sqrt{a^2-1} & & & a \end{pmatrix} \in \boldsymbol{SO}_o(p,q).$$

This shows that g lies in the reduction of $Spin(X)$ determined by (6). Hence for $u \in \Delta^{(\pm)}_{p,q}$:

$$\begin{aligned} \|u\|^2_x &= \|g^{-1}u\|^2_{\Delta^{(\pm)}_{p,q}} \\ &= \left\| \left(\mp\sqrt{\frac{a+1}{2}} - \sqrt{\frac{a-1}{2}} e_p e_{p+q} \right) u \right\|^2 \\ &= \frac{a+1}{2}\|u\|^2 \pm \frac{\sqrt{a^2-1}}{2}(u\ ,\ e_p e_{p+q}\cdot u) \pm \frac{\sqrt{a^2-1}}{2}(e_p e_{p+q}\cdot u\ ,\ u) + \frac{a-1}{2}\|e_p e_{p+q}\cdot u\|^2 \\ &= \frac{a+1}{2}\|u\|^2 \pm \sqrt{a^2-1}(u\ ,\ e_p e_{p+q}\cdot u) + \frac{a-1}{2}\|e_p e_{p+q}\cdot u\|^2 \\ &= a\,\|u\|^2 \pm \sqrt{a^2-1}\,(u\ ,\ e_p e_{p+q} u) \\ &= |x''|\cdot\|u\|^2 + \|x''\|^{-1}\cdot(u\ ,\ x'x''\cdot u) \end{aligned}$$

because $x' = \pm\sqrt{a^2-1}e_p$ and $x'' = ae_{p+q}$. If $x \in X$ is arbitrary, there exists a $k \in K$ such that $x_0 := \lambda(k) = \sqrt{a^2-1}e_p + ae_{p+q}$ if $p \neq 1$, and $x_0 := \lambda(k) = \pm\sqrt{a^2-1}e_p + ae_{p+q}$ if $p = 1$ (thus "$\pm$" is only necessary for $p = 1$), where $a = |x''|$. Then

$$\begin{aligned} \|u\|^2_x = \|u\|^2_{\lambda(k^{-1})x_0} &= \|ku\|^2_{x_0} \\ &= |x''_0|\cdot\|ku\|^2 + |x''_0|^{-1}\cdot(ku\ ,\ x'_0 x''_0\cdot ku) \\ &= |x''|\cdot\|u\|^2 + |x''|^{-1}(u\ ,\ x'x''\cdot u) \end{aligned}$$

because $\|\cdot\|_{\Delta^{(\pm)}_{p,q}}$ is invariant under K and $x' = kx'_0k^{-1}$ and $x'' = kx''_0k^{-1}$. **Q.E.D.**

3 K-finite eigenspinors

This section follows quite closely the way paved by [Ro]. Thus the smooth K-finite eigenspinors of the Dirac operator for almost all eigenvalues are determined by reducing the Dirac operator an ordinary differential operator. After that conditions for the square integrability of these eigenspinors are formulated that allow some partial results on the point and the residual spectrum.

Fix one and for all $\Delta := \Delta_{p,q}^{(\pm)}$ (in case $p+q$ even). Consider the polar coordinates of section 1, $(0,\infty)\times Y \ni (\alpha, y) \mapsto x(\alpha, y) \in X$. To each $f \in \Delta^X$ a $v_f \in (\Delta^Y)^{(0,\infty)}$ is associated by virtue of $v_f(\alpha)(\,\cdot\,) = f(x(\alpha,\,\cdot\,))$. K acts on Δ^Y as $(k\cdot\varphi)(y) := k(\varphi(\lambda^{-1}y))$ and also on Δ^X (same definition). Then obviously $v_{k\cdot f}(\alpha) = (k\cdot v_f)(\alpha)$ because $\lambda(k)x(\alpha,y) = x(\alpha, \lambda(k)y)$. If f_1 and f_2 are maps of X into Δ that are equal almost everywhere (= except on a set of measure zero), then $v_{f_1}(\alpha) = v_{f_2}(\alpha)$ almost everywhere for almost all α. A map $f \in \Delta^X$ is called of type V where $V \subset \Delta^Y$ is a subspace if $v_f(\alpha) \in V$ for almost all α. The K-invariant positive definite scalar product and the standard metric on Y define $L^2(Y, \Delta)$.

Lemma 1 *For each K-finite $f \in L^2_{\mathrm{loc}}(S)$ there exists a finite dimensional subspace $V \subset L^2(Y,\Delta)$ invariant under K such that f is of type V. On the other hand, if V is finite dimensional K-invariant and $f \in L^2_{\mathrm{loc}}(S)$ is of type V, then f is K-finite.*

Proof. First let f be K-finite. This means that there is a finite dimensional subspace $V' \subset L^2_{\mathrm{loc}}(S)$ that is K-invariant such that $f \in V'$. Take a basis $f_1, \ldots, f_r$ of V'. Choose functions $\tilde f_1, \ldots, \tilde f_r$ that represent $f_1, \ldots, f_r$. Then one can define a family of mappings $\mathrm{res}_\alpha : v' \to L^2(Y, \Delta)$ as

$$\mathrm{res}_\alpha\left(\sum_{i=1}^r a_i\, f_i\right) := \sum_{i=1}^r a_i\, v_{\tilde f_i}(\alpha)$$

for $\alpha \in (0,\infty)$. These res_α are K-equivariant. Since V' is finite dimensional and $L^2(Y,\Delta) \approx L^2(Y) \otimes_{\mathbb{C}} \Delta$ is K-finite as a tensor product of a K-finite and a finite dimensional representation, there's a finite dimensional K-invariant subspace $V \subset L^2_{\mathrm{loc}}(Y,\Delta)$ such that $\mathrm{res}_\alpha(V') \subset V$ for all α. To see this decompose $V' = V_1' \oplus \cdots V_s'$ into a direct sum of irreducible subspaces. Then $\mathrm{res}_\alpha(V_i')$ is either isomorphic to V_i' or 0. Since the space spanned by all subrepresentations of $L^2(Y,\Delta)$ isomorphic to V_i' is finite dimensional, the assertion follows.

Now let $v_f(\alpha) \in V$ for almost all α. Then $f = \sum_i a_i\varphi_i$ where $\varphi_1, \ldots, \varphi_t$ is a basis of V and $a_i \in L^2_{\mathrm{loc}}(\mathbb{R}_+)$. Thus $k\cdot f = \sum_i a_i k\cdot\varphi_i = \sum_{i,j} a_i b_{ij}(k)\varphi_j$ for $k \in K$ where $b_{ij}(k)$ are some coefficients. Hence f is contained in the K-invariant finite dimensional subspace of $L^2_{\mathrm{loc}}(S)$ spanned by $a_i\varphi_j$, $i,j = 1,\ldots,s$. **Q.E.D.**

Let $S' := S|_{X'}$ denote the restriction of the spinor bundle to X'. Then $L^2(S) = L^2(S')$. Let $\mathscr{I}$ denote the endomorphism of the trivial vector bundle $Y\times\Delta$, defined by $\mathscr{I}u = -y'y''\cdot u$ for $u \in \Delta \equiv \{y\}\times\Delta$ ($y = y' + y''$ with $y' \in \mathbb{R}^p \times 0$ and $y'' \in 0\times\mathbb{R}^q$, as usual). $\mathscr{I}$ is K-equivariant (K acts on $Y\times\Delta$ as $k\cdot(y,u) = (\lambda(k)y, ku)$). Likewise $\mathscr{I}$ denotes the induced endomorphisms of $L^2(Y,\Delta)$ and $L^2(S')$ ($= L^2(S)$): $(\mathscr{I}f)(y) := -y'y''\cdot f(y)$ and $(\mathscr{I}f)(x(\alpha,y)) := -y'y''\cdot f(x(\alpha,y))$. With these conventions Lemma 2.5 says

$$(u,v)_{S_{x(\alpha,y)}} = \tfrac{1}{2}(\alpha+\alpha^{-1})(u,v)_\Delta - \tfrac{1}{2}(\alpha-\alpha^{-1})(u,\mathscr{I}v)_\Delta. \tag{1}$$

$\mathscr{I}$ is involutive and isometric, therefore a unitary self-adjoint isomorphism (wherever defined). Hence $\alpha^{-1}\|u\|^2_{S_{x(\alpha,y)}} \le \|u\|^2_\Delta \le \alpha\,\|u\|^2_{S_{x(\alpha,y)}}$ because $\mathscr{I}$ has only eigenvalues ±1. Let $L^2(S)_V$ denote the space of K-finite spinors of type V.

Lemma 2 *$L^2(S)_V$ is a closed subspace of $L^2(S)$. As a representation of K one has $L^2(S)_V \approx \sum_{i=0}^{\infty} V$ (infinite Hilbert sum). In particular $L^2(S)_V$ is isotypic if V is irreducible and $L^2(S)_V$ is an isotypic component of $L^2(S)$ if $V = \overline{V'}^K$ for some irreducible V'. If W is some other finite dimensional subspace of $L^2(Y,\Delta)$ invariant under K such that $\overline{V}^K \cap \overline{W}^K = 0$, then $L^2(S)_V$ and $L^2(S)_W$ are orthogonal with respect to every (maybe indefinite) K-invariant L^2-continuous scalar product in $L^2(S)$.*

Proof. Let $(f_n)_{n\in\mathbb{N}} \subset L^2(S)_V$ be a sequence converging to $f \in L^2(S)$. Choose a subsequence $(f_{n_k})_{k\in\mathbb{N}}$ that converges almost everywhere to f. Then $v_{n_k}(\alpha)$ converges to $v_f(\alpha)$ almost everywhere for almost all α. Since V is finite dimensional, it is closed against pointwise convergence. Hence $v_f(\alpha) \in V$ for almost all α, that is, f is of type V.

The map $(\alpha, y) \mapsto x(\alpha, y)$ of $(1,\infty) \times Y$ into X' induces an isometry of $L^2(S)_V$ onto $L^2((1,\infty), V)$ where $(1,\infty)$ is endowed with measure $\mu(\alpha)\,d\alpha$ and V is equipped with a variable scalar product:

$$(\varphi,\psi)_\alpha := \tfrac{1}{2}(\alpha+\alpha^{-1})(\varphi,\psi)_{L^2(Y,\Delta)} - \tfrac{1}{2}(\alpha-\alpha^{-1})(\varphi,\mathscr{I}\psi)_{L^2(Y,\Delta)}$$

(cf. (1)). Let $A(\alpha)\varphi := \frac{1}{2}(\alpha+\alpha^{-1})\varphi - \frac{1}{2}(\alpha-\alpha^{-1})\mathscr{I}\varphi$ for $\varphi \in V$. Choose orthonormal bases $(a_n)_{n\in\mathbb{N}}$ of $L^2((1,\infty),\mathbb{C};\mu(\alpha)\,d\alpha)$ and $(v_i)_i$ of $(V,(.\,,.)_{L^2})$. Then $(a_n(\alpha)\cdot A(\alpha)^{\frac{1}{2}}v_i)_{n\in\mathbb{N},i}$ is an orthonormal basis of $L^2((1,\infty),V)$. Now the second assertion follows because as a K-module $V \approx \sum_i \mathbb{C}\cdot a_n(\alpha)\cdot A(\alpha)v_i$.

If $\overline{V}^K \cap \overline{W}^K = 0$, then V and W have no irreducible subrepresentations in common (in the sense of isomorphism: there are no K-invariant subspaces $0 \neq V' \subset V$ and $0 \neq W' \subset W$ such that V' and W' are isomorphic). Since $L^2(S)_V$ and V have the same irreducible components (and likewise for $L^2(S)_W$ and W), $L^2(S)_V$ and $L^2(S)_W$ have no irreducible components in common. Hence each K-invariant continuous pairing between $L^2(S)_V$ and $L^2(S)_W$ is zero. **Q.E.D.**

Lemma 3 *The Dirac operator is expressed in terms of the polar coordinates as follows*

$$D = -\frac{n}{2} + \alpha\mathscr{I}\cdot\frac{\partial}{\partial\alpha} + \left(1+\frac{\alpha+\alpha^{-1}}{\alpha-\alpha^{-1}}\mathscr{I}\right)D' - \left(1+\frac{\alpha-\alpha^{-1}}{\alpha+\alpha^{-1}}\mathscr{I}\right)D''$$

where

$$D'\varphi := y'\cdot\sum_{i=1}^{p-1} s_i'\cdot s_i'(\varphi) \quad \text{and} \quad D''\varphi := y''\cdot\sum_{i=1}^{q-1} s_i''\cdot s_i''(\varphi).$$

Here $x = x(\alpha,y)$ and $(s_1',\dots,s_{p-1}')$ is an orthonormal frame of $T_{y'}S^{p-1}$ and $(s_1'',\dots,s_{q-1}'')$ is an orthonormal frame of $T_{y''}S^{q-1}$. D' and D'' are differential operators in $\mathscr{E}(Y,\Delta)$. They satisfy the following relations:

$$D'^2 = (p-1)D' + \Delta_{S^{p-1}}, \qquad D''^2 = -(q-2)D'' + \Delta_{S^{q-1}}, \qquad D'D'' = D''D',$$
$$\mathscr{I}D' + D'\mathscr{I} = (p-1)\mathscr{I}, \qquad \mathscr{I}D'' + D''\mathscr{I} = -(q-1)\mathscr{I}, \qquad k\cdot(D'^{(\prime)}\varphi) = D'^{(\prime)}(k\cdot\varphi).$$

Proof. According to Lemma 2.3 the Dirac operator is given by

$$Df(x) = -\frac{n}{2}f(x) - x\cdot\sum_{i=1}^{n}\varepsilon_i s_i\cdot s_i(f)(x)$$

where $(s_1, \dots, s_n)$ is an arbitrary orthonormal basis of T_xX. If the $x = x(\alpha, y)$ and the s_i' and s_i'' are chosen as above, then $(s_1, \dots, s_n) := (\alpha\partial_\alpha, s_1', \dots, s_{p-1}', s_1'', \dots, s_{q-1}'')$ is an orthonormal basis of T_xX where $\partial_\alpha = \frac{1}{2}(1+\alpha^{-2})y' + \frac{1}{2}(1-\alpha^{-2})y'' \in T_xX$. Let for clearness $\hat{f}$ denote $\hat{f}(\alpha, y) := f(x(\alpha, y))$; $\hat{f}$ and f will be identified otherwise. Then

$$\begin{aligned}
\varepsilon_1 s_1 \cdot s_1(f) &= -\alpha^2\partial_\alpha \cdot \partial_\alpha(f) = -\alpha\left(\frac{1}{2}(\alpha+\alpha^{-1})y' + \frac{1}{2}(\alpha-\alpha^{-1})y''\right) \cdot \frac{\partial \hat{f}}{\partial\alpha}, \\
\varepsilon_i s_i \cdot s_i(f) &= -s_{i-1}' \cdot 2(\alpha-\alpha^{-1})^{-1} s_{i-1}'(\hat{f}) \qquad \text{if } i = 2, \dots, p, \\
\varepsilon_i s_i \cdot s_i(f) &= s_{i-p}'' \cdot 2(\alpha+\alpha^{-1})^{-1} s_{i-p}(\hat{f}) \qquad \text{if } i = p+1, \dots, p+q-1.
\end{aligned}$$

Since $x = \frac{1}{2}(\alpha-\alpha^{-1})y' + \frac{1}{2}(\alpha+\alpha^{-1})y''$, $y'^2 = -y''^2 = 1$, and $y'y'' = -y''y'$, one has $x \cdot \alpha\partial_\alpha = y'y''$, and therefore $x \cdot \varepsilon_1 s_1 \cdot s_1(f) = \alpha\mathscr{I}\frac{\partial}{\partial\alpha}\hat{f}$. Furthermore

$$\begin{aligned}
-x \cdot \sum_{i=2}^{p} s_i \cdot s_i(f) &= \sum_{i=1}^{p-1}\big((\alpha-\alpha^{-1})y' + (\alpha+\alpha^{-1})y''\big)(\alpha-\alpha^{-1})^{-1} s_i' \cdot s_i'(\hat{f}) \\
&= D'f + \frac{\alpha+\alpha^{-1}}{\alpha-\alpha^{-1}}\mathscr{I}D'f.
\end{aligned}$$

In the same way one proves

$$-x \cdot \sum_{i=p+1}^{n} s_i \cdot s_i(f) = -D''f - \frac{\alpha-\alpha^{-1}}{\alpha+\alpha^{-1}}\mathscr{I}D''f.$$

The assertions about D'^2 and D''^2 are proved analogously to Lemma 2.4.

Let now $(s_1', \dots, s_{p-1}')$ and $(s_1'', \dots, s_{q-1}'')$ *local* orthonormal frames. Then:

$$\begin{aligned}
D'D''\varphi &= \sum_{i,j} y's_i' \cdot s_i'\big(y''s_j'' \cdot s_j''(\varphi)\big) \\
&= \sum_{i,j} y's_i'y''s_j'' \cdot s_i's_j''(\varphi) \\
&= \sum_{i,j} y''s_j''y's_i' \cdot s_j''s_i'(\varphi) \\
&= D''D'\varphi
\end{aligned}$$

Also:

$$\begin{aligned}
(\mathscr{I}D' + D'\mathscr{I})\varphi &= -\sum_i (y'y''y's_i' \cdot s_i'(\varphi) + y's_i' \cdot s_i'(y'y'' \cdot \varphi)) \\
&= -\sum_i (-y''s_i' \cdot s_i'(\varphi) + y's_i's_i'y'' \cdot \varphi + y's_i'y'y'' \cdot s_i'(\varphi)) \\
&= -\sum (-y''s_i' \cdot s_i'(\varphi) + y'y'' \cdot \varphi - s_i'y'' \cdot s_i'(\varphi)) \\
&= (p-1)\mathscr{I}\varphi.
\end{aligned}$$

Let $(s_1', \dots, s_{p-1}')$ be an orthonormal basis of $T_{\lambda(k^{-1})y'} S^{p-1}$. Then:

$$\begin{aligned}
\big(k \cdot (D'\varphi)\big)(y) &= k \cdot \big((D'\varphi)(\lambda(k^{-1})y)\big) \\
&= k \cdot \left(\lambda(k^{-1})y' \cdot \sum_i s_i' \cdot s_i'(\varphi)(\lambda(k^{-1})y)\right) \\
&= y \cdot \sum_i \lambda(k)s_i' \cdot k \cdot \big((\lambda(k)s_i')(\varphi \circ \lambda(k^{-1}))\big)(y) \\
&= y \cdot \sum_i \lambda(k)s_i' \cdot (\lambda(k)s_i')\big(k \cdot (\varphi \circ \lambda(k^{-1}))\big)(y) \\
&= (D'(k \cdot \varphi))(y)
\end{aligned}$$

because $(\lambda(k)s_1', \dots \lambda(k)s_{p-1}')$ is an orthonormal basis of $T_{y'} S^{p-1}$. **Q.E.D.**

Lemma 4 *For each finite dimensional K-invariant subspace $V \subset L^2(Y, \Delta)$ its "K-closure" $\overline{V}^K$ is invariant under D', D'', and $\mathscr{I}$. One has a decomposition $\overline{V}^K = \overline{V}_0^K \oplus \overline{V}_1^K$ that too is invariant under K, D', D'', and $\mathscr{I}$, where W_ε for $\varepsilon = 0, 1$ denotes the subspace of respectively even and odd functions in W.*

Proof. Consider the parity operator P: $(P\varphi)(y) := \varphi(-y)$. P is obviously a K-invariant operator. Hence $\overline{V}^K$ is mapped by P into itself and so decomposes into the eigenspaces of P, which are just $\overline{V}_\varepsilon^K$, $\varepsilon = 0, 1$. Since D', D'', and $\mathscr{I}$ are K-invariant, too, $\overline{V}^K$ is stable under these operators. The invariance of $\overline{V}_\varepsilon^K$ under D', D'', and $\mathscr{I}$ follows then immediately from the definitions of these operators. **Q.E.D.**

Just as for the Laplacian these lemmas allow the reduction of the Dirac operator on K-finite functions to an ordinary differential operator. Thus let f be a (smooth) K-finite spinor. According to Lemma 1 there's a finite dimensional subspace $V \subset \mathscr{E}(Y, \Delta)$ invariant under K such that f is of type V. Using Lemma 3 the equation $Df = \lambda f$, $\lambda \in \mathbb{C}$, becomes an ordinary differential equation for v_f, which is a function on $(1, \infty)$ with values in $V \subset \overline{V}^K$. Thus analogously to Lemma 1.1 one has the following

Lemma 5 (a) *If $\nu \in \mathbb{C}$, $\nu \notin \frac{1}{2}\mathbb{Z}_+$, $\mathscr{I}\varphi = \varphi$, φ K-finite, there exists exactly one K-finite $\Psi_\nu^+ \varphi \in \mathscr{E}(X', \Delta)$ such that*

$$(D + \tfrac{1}{2})\, \Psi_\nu^+ \varphi = \nu\, \Psi_\nu^+ \varphi \quad \textit{und} \quad \Psi_\nu^+ \varphi(x(\alpha, y)) = \alpha^{-\varrho+\nu}\big(\varphi(y) + \alpha^{-1} O(\varphi, \alpha, y)\big)$$

where $\alpha \mapsto O(\varphi, \alpha, y)$ is analytic at $\alpha = \infty$ and $O(\varphi, \infty, y) = 0$. Ψ_ν^+ depends meromorphically on ν with at most simple poles at $\nu \in \mathbb{Z}_+$. In particular Ψ_ν^+ allows a holomorphic continuation at $\nu \in \frac{1}{2} + \mathbb{N}$.

(b) *If $\nu \in \mathbb{C}$, $\nu \notin -\frac{1}{2}\mathbb{N}$, $\mathscr{I}\varphi = -\varphi$, φ K-finite, there exists exactly one K-finite $\Psi_\nu^- \varphi \in \mathscr{E}(X', \Delta)$ such that*

$$(D + \tfrac{1}{2})\, \Psi_\nu^- \varphi = \nu\, \Psi_\nu^- \varphi \quad \textit{und} \quad \Psi_\nu^- \varphi(x(\alpha, y)) = \alpha^{-\varrho-\nu}\big(\varphi(y) + \alpha^{-1} O(\varphi, \alpha, y)\big)$$

where $\alpha \mapsto O(\varphi, \alpha, y)$ is analytic at $\alpha = \infty$ and $O(\varphi, \infty, y) = 0$. Ψ_ν^- depends meromorphically on ν with at most simple poles at $\nu \in -\mathbb{N}$. In particular Ψ_ν^- allows a holomorphic continuation at $\nu \in -(\frac{1}{2} + \mathbb{N})$.

(c) *If* $\nu \notin \mathbb{Z}$, *then for each* K*-finite* $f \in \mathscr{E}(X', \Delta)$ *with* $(D + \frac{1}{2})f = \nu f$, *there exist unique* K*-finite* $\varphi_\pm$ *such that*

$$f = \Psi_\nu^+ \varphi_+ + \Psi_\nu^- \varphi_-.$$

$\Psi_\nu^\pm$ *is* K*-equivariant. If* $\varphi \in V$, V *finite dimensional and* K*-invariant, then* $\Psi_\nu^\pm \varphi$ *is of type* $\overline{V}^K$. *One has*

$$\Psi_\nu^+ = \Phi_\nu \quad \textit{and} \quad \Psi_\nu^- = \Phi_{-\nu} \circ (1 + \nu{-1}(D' - D'' - \varrho)).$$

$\Psi_\nu^{\mathfrak{p}}$ *preserves parity.*

Proof. Suppose that $\Psi_\nu^\pm \varphi$ exists as claimed. Then

$$\Psi_\nu^\pm \varphi(x(\alpha, y)) = \alpha^\sigma \sum_{k=0}^\infty \alpha^{-k} v_k(y) \tag{2}$$

where $\sigma = -\varrho \pm \nu$. This expression is meant in the sense that for each $y \in Y$ there exists a number $C_y > 0$ such that this series is convergent for $\alpha > C_y$, $y =$ const. Since $\Psi_\nu^\pm \varphi$ is K-finite, there's a finite dimensional subspace $V \subset \mathscr{E}(Y, \Delta)$ such that $\Psi_\nu^\pm \varphi$ is of type V; without loss of generality one can assume that $\overline{V}^K = V$ and $\varphi \in V$. Since respectively $v_0(y) = \lim_{\alpha \to \infty} \alpha^{\varrho - \nu} \Psi_\nu^+ \varphi(x(\alpha, y)) = \varphi(y)$ and $v_0(y) = \lim_{\alpha \to \infty} \alpha^{\varrho + \nu} \Psi_\nu^- \varphi(x(\alpha, y)) = \varphi(y) + \nu^{-1}(D' - D'' - \varrho)\varphi(y)$, one has $v_0 \in V$. Hence $\alpha\big(\Psi_\nu^\pm \varphi(x(\alpha, \cdot\,)) - \alpha^\sigma v_0(\,\cdot\,)\big) \in V$ for all $\alpha > 1$. This expression converges to v_1 for $\alpha \to \infty$. Since $V \subset \mathscr{E}(Y, \Delta)$ is finite dimensional, it is closed against pointwise convergence (since V is finite dimensional, there's exactly one Hausdorff locally convex topologies on V; both the topology of pointwise convergence and the C^∞-topology are such, hence they're the same; finally finite dimensions imply closedness of $V \subset \mathscr{E}(Y, \Delta)$). Therefore $v_1 \in V$. This shows by induction that $v_k \in V$ for all $k \in \mathbb{N}$. Now, if all $v_k \in V$ and (2) converges with respect to the topology of pointwise convergence, then (2) converges with respect to the C^∞-topology, too, because both topologies are Hausdorff locally convex on the finite dimensional V. In particular it is permissible to exchange differentiation and summation. Since $\Psi_\nu^\pm \varphi$ satisfies $(D + \frac{1}{2} - \nu)\Psi_\nu^\pm \varphi = 0$, using Lemma 3 one obtains:

$$\begin{aligned}(\nu + \varrho)\alpha^\sigma \sum_{k=0}^\infty \alpha^{-k} v_k = \alpha^\sigma \sum_{k=0}^\infty (\sigma - k)\alpha^{-k} \mathscr{I} v_k + \alpha^\sigma \sum_{k=0}^\infty \alpha^{-k}(D' - D'')v_k \\ + \frac{\alpha + \alpha^{-1}}{\alpha - \alpha^{-1}} \alpha^\sigma \sum_{k=0}^\infty \alpha^{-k} \mathscr{I} D' v_k - \frac{\alpha - \alpha^{-1}}{\alpha + \alpha^{-1}} \alpha^\sigma \sum_{k=0}^\infty \alpha^{-k} \mathscr{I} D'' v_k.\end{aligned}$$

Multiplying this equation with $\alpha^{-\sigma}(\alpha + \alpha^{-1})(\alpha - \alpha^{-1})$ and comparing coefficients this yields:

$$\begin{aligned}&\big(\nu - (\varrho + \sigma - k) - (1 + \mathscr{I})(D' - D'' - \varrho)v_k\big) = \\ &= 2\mathscr{I}(D' + D'')v_{k-2} + \big(\nu + \varrho - (\sigma - k + 4)\mathscr{I} + (1 + \mathscr{I})(D' - D'')\big)v_{k-4}.\end{aligned} \tag{3}$$

Consider first the case $\sigma = -\varrho + \nu$. From the relations

$$\mathscr{I} D' + D' \mathscr{I} = (p-1)\mathscr{I} \quad \text{und} \quad \mathscr{I} D'' + D'' \mathscr{I} = -(q-1)\mathscr{I}$$

of Lemma 3 it follows that $D' - (p-1)/2$, $D'' + (q-1)/2$, and $D' - D'' - \varrho$ anti-commute with $\mathscr{I}$. Using this one has

$$\big(\nu - k + \nu\mathscr{I} + (1+\mathscr{I})(D' - D'' - \varrho)\big)\big(\nu - (\varrho + \sigma - k)\mathscr{I} - (1+\mathscr{I})(D' - D'' - \varrho)\big) = -k(k-2\nu)\mathscr{I}\,.$$

Hence equation (3) determines v_k, $k \geq 1$, uniquely if $\nu \notin \frac{1}{2}\mathbb{Z}_+$ and $\varphi = v_0$ is chosen such that

$$\big(\nu - (D' - D'' - \varrho)\big)(1-\mathscr{I})\varphi = 0.$$

It follows that

$$\begin{aligned} 0 &= \mathscr{I}\big(\nu - (D' - D'' - \varrho)\big)(1-\mathscr{I})\varphi \\ &= \big(\nu + (D' - D'' - \varrho)\big)\mathscr{I}(1-\mathscr{I})\varphi \\ &= -\big(\nu + (D' - D'' - \varrho)\big)(1-\mathscr{I})\varphi, \end{aligned}$$

hence $2\nu(1-\mathscr{I})\varphi = 0$. Therefore the recursion (3) can be carried out if $\mathscr{I}\varphi = \varphi$ (and if $\nu \neq 0$, this is also a necessary condition on φ). By multiplying the right hand side of (3) with $-k^{-1}(k-2\nu)^{-1}\mathscr{I}\big(\nu - k + \nu\mathscr{I} + (1+\mathscr{I})(D' - D'' - \varrho)\big)$ one obtains an explicit formula for v_k in terms of v_{k-2} and v_{k-4}. Obviously the map $\varphi = v_0 \mapsto v_k$ is K-equivariant. Furthermore $v_k = 0$ if $2 \nmid k$, so one can define the v_k also if $\nu \in \frac{1}{2}\mathbb{N} \setminus \mathbb{N}$; just set $v_k = 0$ for $2 \nmid k$ and compute v_k for $2 \mid k$ by (3). Now, D' and D'' are bounded operators in the finite dimensional vector space V, hence for each compact set $B \subset \mathbb{C} \setminus \mathbb{Z}_+$ and each $\varepsilon > 0$ there is a constant $C_{B,\varepsilon}$ such that $\|v_k\| \leq C_{B,\varepsilon}(1+\varepsilon)^k\|v_0\|$ for all $k \in \mathbb{N}$, $\nu \in B$, and any norm $\|\cdot\|$ in V (which induces the C^∞-topology on V). This shows the convergence of the series (2) for $\alpha \in (1,\infty)$, that $\Psi_\nu^+\varphi$ is an eigenspinor, and that Ψ_ν^+ depends meromorphically on ν, so (a) is proved.

(b) is proved in the same way. Here $\sigma = -\varrho - \nu$, and

$$\big(-k + (1+\mathscr{I})(D' - D'' - \varrho)\big)\big(\nu - (\varrho + \nu - k)\mathscr{I} - (1+\mathscr{I})(D' - D'' - \varrho)\big) = -k\big(\nu + (k+\nu)\mathscr{I}\big).$$

Since $\mathscr{I}^2 = 1$, the map on the right hand side of this equation is invertible if and only if $k \neq 0$ and $k \neq 2\nu$. If $\nu \notin -\frac{1}{2}\mathbb{N}$, (3) determines v_k, $k \geq 1$, uniquely if $v_0 = \varphi$ is chosen so that

$$\nu(1+\mathscr{I})v_0 = (D' - D'' - \varrho)(1-\mathscr{I})v_0.$$

Let $v_0 = v_0^+ + v_0^-$ be the decomposition of v_0 into eigenvectors of $\mathscr{I}$. Then this equation is equivalent to

$$\nu\, v_0^+ = (D' - D'' - \varrho)v_0^-\,.$$

Now (b) follows immediately.

To verify (c) one only has to note that

$$\overline{V}^K \ni \varphi \mapsto \Psi_\nu^+\varphi^+ + \Psi_\nu^-\varphi^-$$

($\varphi = \varphi^+ + \varphi^-$, $\mathscr{I}\varphi^\pm = \pm\varphi^\pm$) is an injective mapping of $\overline{V}^K$ into the space of eigenspinors of type $\overline{V}^K$. Since the dimension of this space of eigenvectors is precisely the dimension of $\overline{V}^K$ (because the eigenvalue equation is an elliptic first order differential equation), this mapping is also surjective, the assertion follows. **Q.E.D.**

Remark: The proof shows that $(\nu - k)\,\Psi_\nu^+\varphi\big|_{\nu=k} = \alpha^{-\varrho-k}P_k\varphi$ for $k \in \mathbb{Z}_+$ for some linear map $P_k \in \mathrm{End}(\overline{V}^K)$ and analogously $(\nu - k)\,\Psi_\nu^-\varphi\big|_{\nu=k} = \alpha^{-\varrho+k}P_k\varphi$ for $k \in \mathbb{Z}_-$. With more effort one can show that $P_k \neq 0$ (for some $\overline{V}^K$) so that $\Psi_\nu^\pm$ has indeed poles at $\nu \in \mathbb{Z}_\pm$.

The following lemma will be used to find those of the eigenspinors just constructed which are square integrable at infinity, that is, square integrable on sets of the form $\{x(\alpha, y) \mid \alpha > C, y \in Y\}$ for some $C > 1$.

Lemma 6 *Let $\nu \in \mathbb{C}$; then:*

(a) *If* $\mathrm{Re}\,\nu \geq \frac{1}{2}$, $\varphi \in L^2(Y,\Delta)$, $\alpha^{-\varrho+\nu}\varphi \in L^2_\infty(S)$, *then* $\varphi = 0$.
(b) *If* $\mathrm{Re}\,\nu < -\frac{1}{2}$, $\varphi \in L^2_{\mathrm{loc}}(S')$ *such that* $\|\varphi(x(\alpha, \cdot))\|_{L^2(Y,\Delta)} \leq \mathrm{const}$ *for all (sufficiently large)* α , *then* $\alpha^{-\varrho+\nu}\varphi(x(\alpha, y)) \in L^2_\infty(S)$.
(c) *If* $\mathrm{Re}\,\nu \in [-\frac{1}{2}, \frac{1}{2})$, $\varphi \in L^2(Y,\Delta)$, *then* $\alpha^{-\varrho+\nu}\varphi \in L^2_\infty(S)$ *if and only if* $\mathscr{I}\varphi = 0$.

Here $L^2_\infty(S)$ *denotes the space of all spinors square integrable at infinity.*

Proof. Let

$$I := \int_C^\infty |\alpha^{-\varrho+\nu}|^2 \int_Y \|\varphi(x(\alpha, y))\|^2_{x(\alpha,y)}\mu(\alpha)\,dy\,d\alpha$$

for $C > 1$. Since $\mu(\alpha) = 2^{-2\varrho}\alpha^{-1}(\alpha - \alpha^{-1})^{p-1}(\alpha + \alpha^{-1})^{q-1}$, obviously $\mu(\alpha) \sim 2^{-2\varrho}\alpha^{2\varrho-1}$ as $\alpha \to \infty$. Hence the convergence of the integral I is equivalent to the convergence of

$$I' := \int_C^\infty \alpha^{2\,\mathrm{Re}\,\nu - 1} \int_Y \|\varphi(x(\alpha, y))\|^2_{x(\alpha,y)}\,dy\,d\alpha.$$

Let $\varphi = \varphi_+ + \varphi_-$ where $\mathscr{I}\varphi_\pm = \pm\varphi_\pm$. Then (cf. (1))

$$\|\varphi(x(\alpha, y))\|^2_{x(\alpha,y)} = \alpha\|\varphi_-(x(\alpha, y))\|^2_\Delta + \alpha^{-1}\|\varphi_+(x(\alpha, y))\|^2_\Delta.$$

Therefore

$$I' = \int_C^\infty \alpha^{2\,\mathrm{Re}\,\nu}\|\varphi_-(x(\alpha, \cdot))\|^2_{L^2(Y,\Delta)}\,d\alpha + \int_C^\infty \alpha^{2\,Re\nu - 2}\|\varphi_+(x(\alpha, \cdot))\|^2_{L^2(Y,\Delta)}\,d\alpha.$$

For (a) and (c) this reduces to

$$I' = \|\varphi_-\|^2_{L^2(Y,\Delta)}\int_C^\infty \alpha^{2\,\mathrm{Re}\,\nu}\,d\alpha + \|\varphi_+\|^2_{L^2(Y,\Delta)}\int_C^\infty \alpha^{2\,\mathrm{Re}\,\nu - 2}\,d\alpha,$$

these two cases are now obvious. For case (b) one only has to notice that together with $\|\varphi(x(\alpha, \cdot))\|_{L^2(Y,\Delta)}$ also $\|\varphi_\pm(x(\alpha, \cdot))\|_{L^2(Y,\Delta)}$ is bounded because the decomposition of $L^2(Y,\Delta)$ into the eigenspaces of $\mathscr{I}$ is orthogonal with respect to $(.\,,.)_{L^2(Y,\Delta)}$. **Q.E.D.**

Lemma 7 *The smooth K-finite L^2_∞-spinors on X' that are eigenspinors of $D + \frac{1}{2}$ corresponding to the eigenvalue $\nu \in \mathbb{C} \setminus \mathbb{Z}$ are exactly:*

$$\begin{aligned}
\operatorname{Re}\nu > \tfrac{1}{2} &: \ f = \Psi_\nu^- \varphi, \ \mathscr{I}\varphi = -\varphi; \\
\operatorname{Re}\nu = \tfrac{1}{2} &: \ f = 0; \\
\operatorname{Re}\nu < \tfrac{1}{2} &: \ f = \Psi_\nu^+ \varphi, \ \mathscr{I}\varphi = \varphi.
\end{aligned}$$

Proof. Lemma 5 shows that $f = \Psi_\nu^+ + \varphi_+ + \Psi_\nu^- \varphi_-$ with suitable $\varphi_\pm$.

• $\operatorname{Re}\nu > \frac{1}{2}$: From Lemma 5 and Lemma 6 it is clear that $\Psi_\nu^- \varphi \in L^2_\infty(S)$. Consider $U := \{y \in Y \mid \varphi_+(y) \neq 0\}$. If $f \in L^2_\infty(S)$, then $\hat{f} \in L^2_\infty(S)$, too, where

$$\hat{f}(x(\alpha, y)) := \begin{cases} \Psi_\nu^+ \varphi_+(x(\alpha, y)) & \text{if } y \in U; \\ 0 & \text{elsewhere.} \end{cases}$$

For $y \in U$ one has:

$$\begin{aligned}
\|f(x(\alpha, y))\|_{x(\alpha,y)} &= \|\alpha^{-\varrho+\nu}\varphi_+(y) + \alpha^{-\varrho+\nu-2}R(\varphi, \alpha, y)\|_{x(\alpha,y)} \\
&\geq |\alpha^{-\varrho+\nu}| \left(\|\varphi_+(y)\|_{x(\alpha,y)} - \alpha^{-2}\|R(\varphi, \alpha, y)\|_{x(\alpha,y)}\right) \\
&\geq |\alpha^{-\varrho+\nu-\frac{1}{2}}| \left(\|\varphi_+(y)\|_\Delta - \alpha^{-1}\|R(\varphi, \alpha, y)\|_\Delta\right)
\end{aligned}$$

where $\|R(\varphi, \alpha, y)\| \leq \text{const}$ as $\alpha \to \infty$, $y \in Y$. Hence for $\alpha \to \infty$: $\|\hat{f}(x(\alpha, y))\|_{x(\alpha,y)} \sim |\alpha^{-\varrho+\nu-\frac{1}{2}}| \cdot \|\mathbb{1}_U(y) \cdot \varphi(y)\| = |\alpha^{-\varrho+\nu}| \cdot \|\mathbb{1}_U(y) \cdot \varphi_+(y)\|_{x(\alpha,y)}$ where $\mathbb{1}_U$ denotes the characteristic function of U. Therefore $\hat{f} \in L^2_\infty(S)$ if and only if $\alpha^{-\varrho+\nu}\mathbb{1}_U \cdot \varphi_+ \in L^2_\infty(S')$, which implies $\varphi_+ = 0$ (cf. Lemma 6).

• $\operatorname{Re}\nu = \frac{1}{2}$: From Lemma 5 and Lemma 6 it follows that

$$f \equiv \alpha^{-\varrho+\nu}\varphi_+ + \alpha^{-\varrho-\nu}\varphi_- + \alpha^{-\varrho-\nu}(D' - D'' - \varrho)\varphi_- \bmod L^2_\infty(S').$$

The third summand is in $L^2_\infty(S')$ because $\mathscr{I}(D' - D'' - \varrho)\varphi_- = -(D' - D'' - \varrho)\mathscr{I}\varphi_- = (D' - D'' - \varrho)\varphi_-$. Hence $f \in L^2_\infty(S)$ if and only if $\alpha^{-\varrho+\nu}\varphi_+ + \alpha^{-\varrho-\nu}\varphi_- \in L^2_\infty(S)$. Now

$$\|\alpha^{-\varrho+\nu}\varphi_+ + \alpha^{-\varrho-\nu}\varphi_-\|^2_{x(\alpha,y)} = \alpha^{2\varrho}\left(\|\varphi_+\|^2_\Delta + \|\varphi_-\|^2_\Delta\right).$$

Integrating this with respect to the weight $\mu(\alpha) \sim 2^{2\varrho}\alpha^{2\varrho-1}$ the integral is finite only if $\varphi_+ = \varphi_- = 0$.

• $\operatorname{Re}\nu < \frac{1}{2}$: Here by Lemma 5 and Lemma 6 $\Psi_\nu^+ \varphi_+ \equiv \alpha^{-\varrho+\nu}\varphi_+ \bmod L^2_\infty(S')$, hence again by Lemma 6 $\Psi_\nu^+ \varphi_+ \in L^2_\infty(S')$. Just as in the case $\operatorname{Re}\nu > \frac{1}{2}$ one shows that $\Psi_\nu^- \varphi_- \in L^2_\infty(S')$ implies $\alpha^{-\varrho-\nu}\varphi_- + \alpha^{-\varrho-\nu}\nu^{-1}(D' - D'' - \varrho)\varphi_- \in L^2_\infty(S)$. Since

$$\begin{aligned}
\|\alpha^{-\varrho-\nu}\varphi_- + \alpha^{-\varrho-\nu}\nu^{-1}(D' - D'' - \varrho)\varphi_-\|^2_{x(\alpha,y)} &= \alpha^{-2\varrho-2\operatorname{Re}\nu+1}\|\varphi_-\|^2_\Delta \\
&\quad + \alpha^{-2\varrho-2\operatorname{Re}\nu-1}\|\nu^{-1}(D' - D'' - \varrho)\varphi_-\|^2_\Delta,
\end{aligned}$$

this is not integrable with respect to the weight $\mu(\alpha)$ for $\operatorname{Re}\nu < \frac{1}{2}$ if $\varphi_- \neq 0$. **Q.E.D.**

Just as for the Laplace operator on functions one can define a Poisson transformation to obtain eigenspinors defined on X (and not only on X'). Thus let

$$\mathscr{E}_{\varepsilon,\nu}(\Xi,\Delta) := \{f \in \mathscr{E}(\Xi,\Delta) \mid f(\alpha\xi) = |\alpha|^{\varrho+\nu}\,\mathrm{sign}(\alpha)^\varepsilon f(\xi)\}$$
$$\mathscr{E}_\varepsilon(Y,\Delta) := \{\varphi \in \mathscr{E}(Y,\Delta) \mid \varphi(-y) = (-1)^\varepsilon\varphi(y)\}$$

for $\varepsilon = 0, 1$, $\nu \in \mathbb{C}$. $G = \boldsymbol{Spin}_+(p,q)$ acts on $\mathscr{E}_{\varepsilon,\nu}(\Xi,\Delta)$ by

$$(\pi_{\varepsilon,\nu}(g)f)(\xi) = g \cdot f(\lambda(g)^{-1}\xi);$$

as in the case of functions one can identify $\mathscr{E}_{\varepsilon,\nu}(\Xi,\Delta)$ with $\mathscr{E}_\varepsilon(Y,\Delta)$. Define the Poisson transformation as

$$F_{\varepsilon,\nu} : \mathscr{E}_{\varepsilon,\nu}(\Xi,\Delta) \longrightarrow \mathscr{E}(X,\Delta)$$
$$F_{\varepsilon,\nu}\varphi(x) = \int_Y |\langle x,y\rangle|^{-\varrho-\nu}\,\mathrm{sign}(\langle x,y\rangle)^\varepsilon\varphi(y)\,dy$$

One has

Lemma 8 (a) *$F_{\varepsilon,\nu} : \mathscr{E}_\varepsilon(Y,\Delta) \to \mathscr{E}(X,\Delta)$ depends meromorphically on ν with simple poles at those ν that satisfy $\varrho+\nu \in \mathbb{Z}_+$ and $\varrho+\nu \equiv \varepsilon+1 \bmod 2$. $F_{\varepsilon,\nu}$ is continuous with respect to the C^∞-topology.*

(b) *$f_{\varepsilon,\nu} : \mathscr{E}_{\varepsilon,\nu}(\Xi,\Delta) \to \mathscr{E}(X,\Delta)$ is a G-map.*

(c) *$F_{\varepsilon,\nu}$ is injective if $\varrho+\nu \notin \mathbb{Z}$.*

(d) *If $\varrho+\nu \notin \mathbb{Z}$, then $F_{\varepsilon,\nu}$ is an eigenspinor of $D+\frac{1}{2}$ for eigenvalue ν if and only if $\mathscr{I}\varphi = \varphi$.*

(e) *If $V \subset \mathscr{E}_\varepsilon(Y,\Delta)$ is finite dimensional and invariant under K, $\varphi \in V$, then $F_{\varepsilon,\nu}\varphi$ is of type $\overline{V}^K$.*

Proof. Assertions (a)–(d) follow immediately from Lemma 1.2. (e) follows from the K-equivariance of $F_{\varepsilon,\nu}$. To prove (d) show first

$$(D+\tfrac{1}{2})\big(|\langle x,y\rangle|^\sigma \cdot v\big) = -(\varrho+\sigma)\,|\langle x,y\rangle|^\sigma \cdot v - \sigma|\langle x,y\rangle|^\sigma\,\mathrm{sign}(\langle x,y\rangle)\,xy\cdot v$$

for $v \in \Delta$, $y \in Y$, $\sigma > 1$. For this purpose choose an orthonormal basis $(s_1,\dots,s_n)$ of T_xX. Then by Lemma 2.3

$$(D+\tfrac{1}{2})\big(|\langle x,y\rangle|^\sigma \cdot v\big) = -\varrho|\langle x,y\rangle|^\sigma \cdot v - x\cdot\Big(\sum_{i=1}^n \varepsilon_i s_i(|\langle x,y\rangle|^\sigma)\,s_i\Big)xy\cdot v.$$

For $t \in T_xX$ one has $t(|\langle x,y\rangle|^\sigma) = \sigma|\langle x,y\rangle|^{\sigma-1}\,\mathrm{sign}(\langle x,y\rangle)\langle t,x\rangle$. Since $(s_1,\dots,s_n,x)$ is an orthonormal basis of $\mathbb{R}^{p,q}$ it follows that

$$(D+\tfrac{1}{2})\big(|\langle x,y\rangle|^\sigma \cdot v\big) = -\varrho|\langle x,y\rangle|^\sigma - \sigma|\langle x,y\rangle|^{\sigma-1}\,\mathrm{sign}(\langle x,y\rangle)x\cdot(y - \langle x,y\rangle\,x)\cdot v,$$

so the assertion follows from the fact that $x^2 = -1$. Thus by differentiating under the integral sign for $\mathrm{Re}(\varrho+\nu) < 0$ one obtains

$$(D * \tfrac{1}{2})F_{\varepsilon,\nu}\varphi(x) = \nu F_{\varepsilon,\nu}\varphi(x) + (\varrho+\nu)xe_{p+q}\cdot F_{\varepsilon,\nu}\hat{\varphi}(x)$$

where $\hat{\varphi}(y) := e_{p+q}y\cdot\varphi(y)$ (e_{p+q} was introduced here only because $\Delta = \Delta^{(\pm)}_{p,q}$ is only a $\boldsymbol{Cl}^0_{p,q}$-module if $2 \mid p+q$ therefore one needs an even number of vectors; one could have taken any other vector instead of e_{p+q}). Hence $F_{\varepsilon,\nu}\varphi$ is an eigenspinor for eigenvalue ν if and only if $(\varrho+\nu)xe_{p+q}\cdot F_{\varepsilon,\nu}\hat{\varphi}(x) = 0$, hence (because of (b) and $x^2 = e_{p+q}^2 = -1$) if and only if $y\cdot\varphi(y) = 0$ for all $y \in Y$. Now,

$$y\cdot\varphi(y) = (y'+y'')\cdot\varphi(y) = y''(-y'y''-1)\cdot\varphi(y) = y''\cdot(\mathscr{I}-1)\varphi(y).$$

Since $y''^2 = -1$, this proves (d) if $\mathrm{Re}(\varrho+\nu) < 0$. Thus meromorphicity proves (d) for all ν such that $\varrho+\nu \notin \mathbb{Z}$. **Q.E.D.**

Lemma 9 (a) *Each K-finite distribution $f \in \mathscr{D}'(S) = \mathscr{D}(X)'\otimes\Delta$ that is an eigendistribution for eigenvalue $\lambda = \nu - \frac{1}{2}$, $\lambda + \varrho \notin \frac{1}{2}+\mathbb{Z}$ can be written as*

$$f = F_{0,\nu}\varphi_0 + F_{1,\nu}\varphi_1$$

for unique K-finite $\varphi_\varepsilon \in \mathscr{E}_\varepsilon(Y,\Delta)$, $\mathscr{I}\varphi_\varepsilon = \varphi_\varepsilon$.

(b) *One has $\sigma_p(\overline{D}) \subset \frac{1}{2}\mathbb{Z}\setminus\{0\}$. $\sigma_p(\overline{D}) \subset \frac{1}{2}+\mathbb{Z}$ if the dimension of X is odd.*

(c) *$\overline{D}$ has empty residual spectrum if the dimension or the index of X is even. In the remaining case $\sigma_r(\overline{D}) \subset \frac{1}{2}+\mathbb{Z}$.*

Proof. (a) is clear from Lemma 2.4, Theorem 1.1 and Lemma 8 (notice that $\sum_\varepsilon F_{\varepsilon,\nu}\varphi_\varepsilon$ can only be an eigenspinor if $F_{\varepsilon,\nu}\varphi_\varepsilon$ are because both $F_{\varepsilon,\nu}$ and D preserve parity).

(b) Let $E_\lambda(\overline{D})$ denote the eigenspace of $\overline{D}$ corresponding to eigenvalue λ; this is a closed subspace of $L^2(S)$. Since both D and $\|\cdot\|$ are K-invariant (D is even G-invariant), so is $\overline{D}$. Hence $E_\lambda(\overline{D})$ is invariant under K, thus a (continuous) unitary representation of K. Therefore the k-finite are dense in $E_\lambda(\overline{D})$. Lemma 7 implies that $E_\lambda(\overline{D}) = 0$ if $\mathrm{Re}\,\nu = \frac{1}{2}$, that is, if $\mathrm{Re}\,\lambda = 0$. Let now $\mathrm{Re}\,\lambda > 0$. The K-finite eigenspinors f are known: $f = \sum_\varepsilon F_{\varepsilon,\nu}\varphi_\varepsilon$ where $\mathscr{I}\varphi_\varepsilon = \varphi_\varepsilon$ and $f = \Psi_\nu^-\varphi' = \Phi_{-\nu}(\varphi' + \nu^{-1}(D'-D''-\varrho)\varphi')$ where $\mathscr{I}\varphi' = -\varphi'$ (Lemma 7). From Lemma 1.3 it follows

$$f = \Phi_\nu(c(0,\nu)\varphi_0 + c(1,\nu)\varphi_1) + \Phi_{-\nu}(C_-(0,\nu)\varphi_0 + C_-(1,\nu)\varphi_1);$$

everything works fine since $\nu \notin \mathbb{Z}$ and $\varrho+\nu \notin \mathbb{Z}$ because $\lambda \notin \frac{1}{2}\mathbb{Z}$. Therefore

$$\begin{aligned} c(0,\nu)\varphi_0 + c(1,\nu)\varphi_1 &= 0,\\ C_-(0,\nu)\varphi_0 + C_-(1,\nu)\varphi_1 &= \varphi' + \nu^{-1}(D'-D''-\varrho)\varphi'. \end{aligned}$$

Since $c(\varepsilon,\nu) \neq 0$ for $\nu \notin \frac{1}{2}\mathbb{Z}$ and $\psi_\varepsilon(-y) = (-1)^\varepsilon\varphi_\varepsilon(y)$, $\varphi_\varepsilon = 0$ for $\varepsilon = 0,1$, hence $f = 0$. The case $\mathrm{Re}\,\lambda < 0$ is treated similarly.

Since both $\Psi_\nu^\pm$ and $F_{\varepsilon,\nu}$ are defined for $\nu \in \frac{1}{2}+\mathbb{Z}$ (that is, $\lambda \in \mathbb{Z}$) if the dimension is odd, the same reasoning shows that $\sigma_p(\overline{D}) \subset \frac{1}{2}+\mathbb{Z}$.

(c) Suppose, that $\lambda \in \sigma_r(\overline{D})$. Then $\overline{\lambda} \in \sigma_p(D^\sharp)$ where $D^\sharp$ is the adjoint with respect to one of the two indefinite scalar products in S (cf. Lemma 2.1). Since $\overline{D} = D^\sharp$ for even dimension or even index for one of these scalar products (Theorem 2.1), it follows that $\overline{\lambda} \in \sigma_p(\overline{D}) \subset \mathbb{R}$. Then $\lambda = \overline{\lambda}$ and so $\lambda \in \sigma_r(\overline{D}) \cap \sigma_p(\overline{D}) = \emptyset$. Hence $\sigma_r(\overline{D}) = \emptyset$ in case of even dimension or even index. For odd dimension and odd index one has $D^\sharp = -\overline{D}$, thus $\lambda \in \sigma_r(\overline{D})$ implies only $-\lambda \in \sigma_p(\overline{D}) \subset \frac{1}{2} + \mathbb{Z}$, thus $\sigma_r(\overline{D}) \subset \frac{1}{2} + \mathbb{Z}$. **Q.E.D.**

4 The resolvent on the isotypic components

General Assumption: From now on let p and q be chosen so that there's a scalar product in $\Delta = \Delta^{(\pm)}_{p,q}$ invariant under G; therefore let p or q be even.

Lemma 1 *Let D denote the Dirac operator equipped with domain $C_0^\infty(S)$, and let D' denote the Dirac operator with $C_0^\infty(S)_K$ as domain, that is, the space of smooth K-finite spinors with compact support. Then the closures $\overline{D}$ and $\overline{D'}$ in $L^2(S)$ coincide, i.e., have the same domain.*

Proof. The inclusion $\overline{D'} \subset \overline{D}$ is trivial because $C_0^\infty(S)_K \subset C_0^\infty(S)$. The inclusion $\overline{D'} \supset \overline{D}$ follows from the density of $C_0^\infty(S)_K$ in $C_0^\infty(S)$ with respect to the C_0^∞-topology (see for instance [He] Lemma IV.1.9). **Q.E.D.**

Lemma 2 *Let $(H, (.\,,.), J)$ be a Krein space with indefinite inner product $\langle .\,,. \rangle = (.\,, J)$, let $U \subset H$ be a closed subspace that is non-degenerate with respect to $\langle .\,,. \rangle$, and let A be a (densely) defined operator $A\colon \mathrm{dom}(A) \subset H \to H$ that is essentially self-adjoint with respect to $\langle .\,,. \rangle$. Assume that $pr_U(\mathrm{dom}(A)) = \mathrm{dom}(A) \cap U$ where pr_U is the orthogonal projection with respect to $\langle .\,,. \rangle$ onto U and that $A(\mathrm{dom}(A) \cap U) \subset U$ (that is, U is invariant under A). Define $\mathrm{dom}(A|_U) := \mathrm{dom}(A) \cap U$ and $A|_U\colon \mathrm{dom}(A|_U) \to U$. Then $A|_U$ is a densely defined operator in U, that is essentially self-adjoint with respect to $\langle .\,,. \rangle|_U$.*

Proof. $\mathrm{dom}(A|_U) = \mathrm{dom}(A) \cap U = pr_U(\mathrm{dom}(A)) \subset U$ is dense because $\mathrm{dom}(A) \subset H$ is dense and $pr_U : H \to U$ is continuous.

Let $x, y \in \mathrm{dom}(A|_U) = \mathrm{dom}(A) \cap U$. Then $\langle A|_U x, y \rangle = \langle Ax, y \rangle = \langle x, Ay \rangle = \langle x, A|_U y \rangle$ because $A \subset A^\sharp$ (the adjoint with respect to $\langle .\,,. \rangle$). Hence $A|_U \subset (A|_U)^\sharp$.

Since U is non-degenerate, $H = U \oplus U^\perp$ where $U^\perp$ denotes the orthogonal complement with respect to $\langle .\,,. \rangle$. If now $z \in \mathrm{dom}(A)$, let $z = z_1 + z_2$ according to this decomposition. Then $z_1 \in pr_U(\mathrm{dom}(A)) = \mathrm{dom}(A) \cap U \subset \mathrm{dom}(A)$ and therefore $z_2 \in \mathrm{dom}(A)$, too. One has $\langle Az_2\,, \mathrm{dom}(A) \cap U \rangle = \langle z_2\,, A(\mathrm{dom}(A) \cap U) \rangle = 0$, hence $Az_2 \in U^\perp$ because $\mathrm{dom}(A) \cap U$ is dense in U. This means $pr_U(Az) = A\, pr_U(z)$, that is, A commutes with pr_U.

Let $x \in \mathrm{dom}(A|_U)^\sharp$. Then $|\langle x, Ay \rangle| \le \mathrm{const}\, \|y\|$ for all $y \in \mathrm{dom}(A|_U)$. If $z \in \mathrm{dom}(A)$, then $|\langle x, Az \rangle| = |\langle x\,,\ pr_U\, Az \rangle| = |\langle x\,,\ A\, pr_U\, z \rangle| \le \mathrm{const}\, \|pr_U\, z\| \le \mathrm{const}\, \|z\|$, hence $x \in \mathrm{dom}(A^\sharp) = \mathrm{dom}(\overline{A})$. By the very definition of $\overline{A}$ this means that there's a sequence $x_n \in \mathrm{dom}(A)$ such that $x_n \to x$ and $Ax_n \to \overline{A}x$. But then $\mathrm{dom}(A) \cap U = pr_U(\mathrm{dom}(A)) \ni pr_U\, x_n \to pr_U\, x$ and $A\, pr_U\, x_n = pr_U\, Ax_n \to pr_U\, \overline{A}x$. Thus $A\, pr_U\, x_n$ converges to $\overline{A}x$, hence $x \in \mathrm{dom}(\overline{A|_U})$. **Q.E.D.**

Let now $V \subset L^2(Y,\Delta)$ be K-finite subspace with $V = \overline{V}^K$. By Lemma 3.2 $L^2(S)_V \subset L^2(S)$ is closed. Let $\langle .\,,.\rangle$ be a scalar product that is induced by a G-invariant scalar product on Δ. Then $L^2(S)_V$ is non-degenerate with respect to $\langle .\,,.\rangle$. In fact, $\sum_W L^2(S)_W \subset L^2(S)$ is dense (sum over all K-finite subspaces $W \subset L^2(Y,\Delta)$) and $L^2(S)_V \perp_{\langle .\,,.\rangle} L^2(S)_W$ if $\overline{V}^K \cap \overline{W}^K = 0$. The Dirac operator $(D, C_0^\infty(S)$ is essentially self-adjoint with respect to $\langle .\,,.\rangle$ in $L^2(S)$. Instead of $C_0^\infty(S)$ one can choose $C_0^\infty(S)_K$ as the domain of D without changing its closure $\overline{D}$. The orthogonal projection onto $L^2(S)_V$ with respect to $\langle .\,,.\rangle$ maps $C_0^\infty(S)_K$ into itself if $V = \overline{V}^K$. In fact, let $f \in C_0^\infty(S)$ be a K-finite spinor. Then there's a K-invariant finite dimensional subspace $W' \subset C_0\infty(S)$ with $f \in W'$. Let $W' = W_1' + \cdots + W_r'$ be a decomposition into irreducible subspaces, let $f = f_1 + \cdots + f_r$ accordingly. Since again $f_i \in C_0^\infty(S)$, without loss of generality one can assume that W' is irreducible. Then $W \subset L^2(S)_V$ if there's a $W \subset V$ that's isomorphic to W' as a K-representation, and $W \perp L^2(S)_V$ otherwise (cf. Lemma 3.2). Thus the image of f under the orthogonal projection onto $L^2(S)_V$ is either f or 0, and thus again in $C_0^\infty(S)_K$.

As a K-invariant operator, D leaves $C_0^\infty(S) \cap L^2(S)_V$ invariant (Lemma 3.2 again). One has $x(\alpha,-y) = -x(\alpha,y)$. Hence a spinor f of type V that satisfies $f(-x) = (-1)^\varepsilon f(x)$ is even of type $V_\varepsilon(= \{\varphi \in V \mid \varphi(-y) = (-1)^\varepsilon \varphi(y)\}$). Furthermore for a spinor f with $f(-x) = (-1)^\varepsilon f(x)$ $\;Df$ satisfies the same relation. Hence Lemma 2 yields the following

Lemma 3 *The restrictions $D\big|_{L^2(S)_V}$ and $D\big|_{L^2(S)_{V_\varepsilon}}$ of the Dirac operator for $V = \overline{V}^K$ are essentially self-adjoint with respect to $\langle .\,,.\rangle$ in $L^2(S)$ and $L^2(S)_{V_\varepsilon}$, respectively.*

Now identify $L^2(S)_V$ with $L^2((1,\infty), V; \mu(\alpha)\,d\alpha, \|\cdot\|_\alpha)$. Here

$$\|v\|_\alpha^2 = \tfrac{1}{2}(\alpha + \alpha^{-1})\,\|v\|^2_{L^2(Y,\Delta)} - \tfrac{1}{2}(\alpha - \alpha^{-1})(v, \mathscr{I} v)_{L^2(Y,\Delta)}$$

where $v \in V$. Let $V^{\pm\mathscr{I}} := \{\varphi \in V \mid \mathscr{I}\varphi = \pm\varphi\}$. These subspaces of V are totally isotropic and dual with respect to $\langle .\,,.\rangle$ (see the appendix, page 46), and so are $V_\varepsilon^{\mathscr{I}}$ and $V_\varepsilon^{-\mathscr{I}}$. If $A \in \mathrm{Hom}(V_\varepsilon^{\pm\mathscr{I}}, V_\varepsilon)$, then one can think of the adjoint $A^\sharp$ with respect to $\langle .\,,.\rangle$ as an element of $\mathrm{Hom}(V_\varepsilon, V_\varepsilon^{\mp\mathscr{I}})$.

The maps $F_{\varepsilon,\nu}$, Φ_ν, and $\Psi_\nu^\pm$ correspond to families of homomorphisms $f_{\varepsilon,\nu}(\alpha) \in \mathrm{Hom}(V_\varepsilon^{\mathscr{I}}, V_\varepsilon)$, $\varphi_\nu(\alpha) \in \mathrm{End}(V_\varepsilon)$, and $\psi_\nu^\pm(\alpha) \in \mathrm{Hom}(V_\varepsilon^{\pm\mathscr{I}}, V_\varepsilon)$ for $\alpha > 1$ by

$$\begin{aligned}(F_{\varepsilon,\nu}v)(x(\alpha,y)) &= (f_{\varepsilon,\nu}(\alpha)v)(y),\\ (\Phi_\nu v)(x(\alpha,y)) &= (\varphi_\nu(\alpha)v)(y),\\ (\Psi_\nu^\pm v)(x(\alpha,y)) &= (\psi_\nu^\pm(\alpha)v)(y).\end{aligned}$$

In the appendix the following is proved:

Theorem 1 *Let $V \subset L^2(Y,\Delta)$ be a finite dimensional K-invariant subspace with $\overline{V}^K = V$. Let $\overline{D}_{V_\varepsilon} := \overline{D}\big|_{L^2(S)_{V_\varepsilon}}$. Then:*

(a) *If $\lambda \in i\mathbb{R}$, then $\lambda \in \sigma_c(\overline{D}_{V_\varepsilon})$ (even if $p \equiv q \equiv 1 \bmod 2$).*

(b) *If $\operatorname{Re}\lambda \notin \frac{1}{2}\mathbb{Z}$, $\nu := \lambda + \frac{1}{2}$, then $\lambda \in \varrho(\overline{D}_{V_\varepsilon})$, and the resolvent is given by*

$$\big((\overline{D}_{V_\varepsilon} - \lambda)^{-1} f\big)(\alpha) = \int_1^\infty k_\nu(\alpha,\overline{\alpha}) f(\overline{\alpha}) \mu(\overline{\alpha})\, d\overline{\alpha},$$

where

$$k_\nu(\alpha,\overline{\alpha}) = 2^{2\varrho} c(\varepsilon,\nu)^{-1} \cdot \begin{cases} f_{\varepsilon,\nu}(\alpha)\psi_{\overline{\nu}}^-(\overline{\alpha})^\sharp & \text{if } \alpha < \overline{\alpha}, \\ \psi_\nu^-(\alpha) f_{\varepsilon,\overline{\nu}}(\overline{\alpha})^\sharp & \text{if } \alpha > \overline{\alpha}; \end{cases}$$

in case $\operatorname{Re}\lambda > 0$, *and*

$$k_\nu(\alpha,\overline{\alpha}) = 2^{2\varrho} \cdot \begin{cases} f_{\varepsilon,\nu}(\alpha) C_-(\varepsilon,\nu)^{-1}\big(1+\nu^{-1}(D'-D''-\varrho)\big)\psi_{\overline{\nu}}^+(\overline{\alpha})^\sharp & \text{if } \alpha < \overline{\alpha}, \\ \psi_\nu^+(\alpha)\Big(f_{\varepsilon,\overline{\nu}}(\overline{\alpha}) C_-(\varepsilon,\overline{\nu})^{-1}\big(1+\overline{\nu}^{-1}(D'-D''-\varrho)\big)\Big)^\sharp & \text{if } \alpha > \overline{\alpha}; \end{cases}$$

in case $\operatorname{Re}\lambda < 0$.

Proof. See Lemma 3.9 and Lemma 4 of the appendix for (a). **Q.E.D.**

5 The point spectrum

This section determines the point spectrum of the Dirac operator by studying the poles of the resolvent. In addition some properties of the eigenspaces are proved.

The **General Assumption** of section 4 is still in force, i.e., p or q is even.

Lemma 1 *Let V be finite dimensional vector space with positive definite hermitian product $(.,.)$, let $U \subset \mathbb{C}$ be an open subset, $\lambda_0 \in U$ a fixed point, let $\mathscr{L}(L^2(I,V))$ be the Banach space of continuous linear operators in $L^2(I,V)$ for some interval $I \subset \mathbb{R}$, and let $n \in \mathbb{N}$. Let $N \subset I \times I$ be a closed set of measure zero, $k:(U\setminus\{\lambda_0\})\times(I^2\setminus N) \to \operatorname{End}(V)$ a continuous, locally bounded map, such that for all $(s,t) \in (I^2\setminus N)$ the map $U \ni \lambda \mapsto k_\lambda(s,t)$ is meromorphic with a pole at λ_0 of at most nth order, such that the integral operator K_λ, defined by $(K_\lambda f)(s) := \int_I k_\lambda(s,t) f(t)\,dt$, is a bounded operator in $L^2(I,V)$ for $\lambda \neq \lambda_0$, and such that the map $\lambda \mapsto K_\lambda \in \mathscr{L}(L^2(I,V))$ is holomorphic except at $\lambda = \lambda_0$. Then λ_0 is a pole of K_λ of at most nth order. The coefficients of the Laurent series of K_λ at $\lambda = \lambda_0$ are integral operators whose kernels are the Laurent coefficients of $k_\lambda(s,t)$.*

If for one $(s,t) \in (I^2 \setminus N)$ the map $\lambda \mapsto k_\lambda(s,t)$ has a pole of order equal to n, then the pole of K_λ at $\lambda = \lambda_0$ is of order equal to n.

Proof. Since K_λ has an isolated singularity at $\lambda = \lambda_0$, one has a Laurent expansion:

$$K_\lambda = \sum_{k\in\mathbb{Z}} (\lambda-\lambda_0)^k a_k$$

(convergence in $\mathscr{L}(L^2(I,V))$). Then

$$a_k = \frac{1}{2\pi i} \int\limits_{|\lambda-\lambda_0|=r} \frac{K_\lambda}{(\lambda-\lambda_0)^{k+1}}\,d\lambda$$

if $r > 0$ is sufficiently small. Let $f, g \in C_0(I, V)$. Then

$$2\pi i(a_k\, f, g)_{L^2} = \int\limits_{|\lambda-\lambda_0|=r} \frac{(K_\lambda f, g)_{L^2}}{(\lambda-\lambda_0)^{k+1}}\, d\lambda$$

$$= \int\limits_{|\lambda-\lambda_0|=r} \int\limits_I \frac{\big((K_\lambda f)(s)\ ,\ g(s)\big)_V}{(\lambda-\lambda_0)^{k+1}}\, ds\, d\lambda$$

$$= \int\limits_{|\lambda-\lambda_0|=r} \int\limits_I \int\limits_I \frac{1}{(\lambda-\lambda_0)^{k+1}} \big(k_\lambda(s,t) f(t)\ ,\ g(s)\big)_V\, dt\, ds\, d\lambda.$$

Since k is locally bounded and f and g are compactly supported, one may interchange the integrals:

$$2\pi i(a_k\, f, g)_{L^2} = \int\limits_I \int\limits_I \Big(\int\limits_{|\lambda-\lambda_0|=r} \frac{k_\lambda(s,t) f(t)}{(\lambda-\lambda_0)^{k+1}}\, d\lambda\ ,\ g(s)\Big)_V\, dt\, ds$$

$$= \int\limits_I \int\limits_I \Bigg(\Big(\int\limits_{|\lambda-\lambda_0|=r} \frac{k_\lambda(s,t)}{(\lambda-\lambda_0)^{k+1}}\, d\lambda\Big) f(t),\ g(s)\Bigg)_V\, dt\, ds,$$

hence

$$(a_k\, f)(s) = \int\limits_I \frac{1}{2\pi i} \int\limits_{|\lambda-\lambda_0|=r} \frac{k_\lambda(s,t)}{(\lambda-\lambda_0)^{k+1}}\, d\lambda f(t)\, dt.$$

Since $k_\lambda(s,t)$ has for fixed (s,t) a pole of at most nth order at $\lambda = \lambda_0$, $a_k = 0$ if $k < -n$. Therefore K_λ has a pole of at most nth order at $\lambda = \lambda_0$. If now $k_\lambda(s,t)$ has for some (s,t) a pole of order n, than

$$\int\limits_{|\lambda-\lambda_0|=r} \frac{k_\lambda(s,t)}{(\lambda-\lambda_0)^{-n+1}}\, d\lambda \neq 0,$$

hence $a_{-n} \neq 0$. **Q.E.D.**

Remark: The proof applies to the more general situation where a measure $\mu(t)\, dt$ is used instead of the Lebesgue measure dt, μ some positive and (for simplicity) continuous function, and when the constant hermitian product $(.\,,.)$ is replaced by a smooth variable product $t \mapsto (.\,,.)_t$. In order not to overload the formulation of the Lemma 1 I decided to restrict to the simplest case.

Lemma 2 (a) *Let E be a Banach space, $A\colon \mathrm{dom}(A) \subset E \to E$ a densely defined closed operator. Let $R_\lambda := (A-\lambda)^{-1}$ denote the resolvent for $\lambda \in \varrho(A)$. As is well known, the mapping $\lambda \mapsto R_\lambda$ is holomorphic for $\lambda \in \varrho(A)$. Assume that R_λ has a simple pole at $\lambda = \lambda_0$ and set $P := -(\lambda-\lambda_0)R_\lambda|_{\lambda=\lambda_0}$. Then $\lambda \in \sigma_p(A)$ and P is a projection with kernel $\mathrm{im}(A-\lambda_0)$ and image $\ker(A-\lambda_0)$. In particular $\mathrm{im}(A-\lambda) \subset E$ is closed and $E = \ker(A-\lambda_0) \oplus \mathrm{im}(A-\lambda_0)$. If $x \in \mathrm{dom}(A)$, $Ax \in \mathrm{dom}(A)$, and $(A-\lambda_0)^2 x = 0$, then already $(A-\lambda_0)x = 0$.*

(b) *If in addition E is a Krein space with indefinite product $\langle .\,,. \rangle$, $A^\sharp = A$, and $\lambda_0 \in \mathbb{R}$, then P is the orthogonal projection with respect to $\langle .\,,. \rangle$ onto the eigenspace; in particular this space is non-degenerate.*

Proof. (a) One has $(\lambda-\lambda_0)(\lambda'-\lambda_0)R_\lambda R_{\lambda'} \to P^2$ where $\lambda,\lambda' \to \lambda_0$. Now $R_\lambda R_{\lambda'} = (\lambda'-\lambda)^{-1}(R_{\lambda'}-R_\lambda)$. Set $\lambda := \lambda_0+\delta$ and $\lambda' := \lambda_0+2\delta$ where $\delta \to 0$. Then $P^2 = \lim_{\delta\to 0}(-2\delta)(R_{\lambda_0+\delta}-R_{\lambda_0+2\delta}) = 2P-P = P$. Thus P is a projection.

Let now $x \in E$, $y := P(-x)$. Then $y = \lim_{\lambda\to 0}(\lambda-\lambda_0)R_\lambda x$. By the definition of the resolvent it follows that $(\lambda-\lambda_0)R_\lambda x \in \operatorname{dom}(A)$ and

$$A(\lambda-\lambda_0)R_\lambda x = (A-\lambda+\lambda)(\lambda-\lambda_0)R_\lambda x = (\lambda-\lambda_0)x + \lambda(\lambda-\lambda_0)R_\lambda$$

if $\lambda \neq \lambda_0$. If λ goes to λ_0, then the hand right side of this expression converges to $\lambda_0 y$. Since A is closed, $y \in \operatorname{dom}(A)$ and $Ay = \lambda_0 y$.

Let now $y \in \ker(A-\lambda_0)$. Then $y = R_\lambda(A-\lambda)y = R_\lambda(A-\lambda_0+\lambda_0-\lambda)y = (\lambda_0-\lambda)R_\lambda y$ and therefore $(\lambda-\lambda_0)R_\lambda y = -y$ for all $\lambda \in \varrho(A)$. Hence $y = Py$. This eventually shows $\operatorname{im} P = \ker(A-\lambda_0)$.

Now let $x \in \operatorname{dom}(A)$, $y := (A-\lambda_0)x$. Then $Py = -\lim_{\lambda\to\lambda_0}(\lambda-\lambda_0)R_\lambda(A-\lambda_0)x = -\lim_{\lambda\to\lambda_0}(\lambda-\lambda_0)(x+(\lambda-\lambda_0)x) = 0$.

On the other hand $\lim_{\lambda\to\lambda_0}(\lambda-\lambda_0)R_\lambda|_{\ker P} = 0$, so $R_\lambda|_{\ker P}$ is holomorphic at $\lambda = \lambda_0$. If then $y \in \ker P$, set $x := \lim_{\lambda\to\lambda_0} \mathbb{R}_\lambda y$. Then $(A-\lambda_0)x = \lim_{\lambda\to\lambda_0} R_\lambda y = y$. This eventually proves $\operatorname{im}(A-\lambda_0) = \ker P$.

Finally let $x \in \operatorname{dom}(A)$, such that $Ax \in \operatorname{dom}(A)$ and $(A-\lambda_0)^2 x = 0$. Then $(A-\lambda_0)x \in \ker(A-\lambda) = \operatorname{im} P$, hence $(A-\lambda_0)x = P(A-\lambda_0)x$. Therefore:

$$\begin{aligned}(A-\lambda_0)x &= \lim_{\lambda\to\lambda_0}(\lambda-\lambda_0)R_\lambda(A-\lambda+\lambda-\lambda_0)x\\ &= \lim_{\lambda\to\lambda_0}(\lambda-\lambda_0)(1+(\lambda-\lambda_0)R_\lambda)x\\ &= 0.\end{aligned}$$

(b) It suffices to show that $P^\sharp = P$. By the very definition of P one has $P^\sharp = -\lim_{\lambda\to\lambda_0}(\overline{\lambda}-\lambda_0)(R_\lambda)^\sharp$ because $\lambda_0 \in \mathbb{R}$. Since $A^\sharp = A$, $(R_\lambda)^\sharp = R_{\overline{\lambda}}$, hence $P^\sharp = P$. **Q.E.D.**

Remark: If the order n of the pole of R_λ at $\lambda = \lambda_0$ is greater than 1, one obtains analogously $\ker(A-\lambda_0) = \big((\lambda-\lambda_0)^n R_\lambda|_{\lambda=\lambda_0}\big)$. But the last statement of (a) is no longer true.

These two lemmas are now used to determine the point spectrum and the K-finite eigenspinors of the Dirac operator in case the space form has even dimension or even index.

Theorem 1 *If the dimension of $X = \boldsymbol{Spin}_+(p,q)/\boldsymbol{Spin}_+(p,q-1)$, $p \geq 1$ and $q \geq 2$, is even, the point spectrum of the Dirac operator consist of all nonzero integers. The K-finite eigenspinors f corresponding to the eigenvalue $\lambda = \nu - \frac{1}{2} \in \mathbb{Z}_+$ are exactly*

$$f = \operatorname*{res}_{\nu'=\nu} c(\varepsilon,\nu')^{-1}F_{\varepsilon,\nu'}\varphi_\varepsilon = (\nu'-\nu)c(\varepsilon,\nu')^{-1}F_{\varepsilon,\nu'}\varphi_\varepsilon|_{\nu'=\nu}$$

for K-finite $\varphi_\varepsilon \in \mathscr{E}_\varepsilon(Y,\Delta)$, $\mathscr{I}\varphi_\varepsilon = \varphi_\varepsilon$ where $\varrho+\nu \equiv \varepsilon+1 \bmod 2$. If $V \subset L^2(Y,\Delta)$ is a finite dimensional subspace with $\overline{V}^K = V$, then $\sigma_p(\overline{D}_V) \subset \mathbb{Z}\setminus\{0\}$, $\sigma_c(\overline{D}_V) = i\mathbb{R}$, and $\sigma_R(\overline{D}_V) = \emptyset$. The eigenspaces of both $\overline{D}$ and $\overline{D}_V$ are non-degenerate with respect to $\langle .\,,.\rangle$.

Proof. According to Theorem 4.1 the resolvent $(\overline{D}_{V_\varepsilon}-\lambda)^{-1}$ of $\overline{D}$ on $L^2(S)_{V_\varepsilon} \approx L^2((1,\infty), V_\varepsilon; \|\cdot\|_\alpha, \mu(\alpha)\,d\alpha)$ for $\overline{V}^K = V$ is given by an integral operator with kernel

$$k_\nu(\alpha,\overline{\alpha}) = 2^{2\varrho}c(\varepsilon,\nu)^{-1}\cdot\begin{cases} f_{\varepsilon,\nu}(\alpha)\psi^-_{\overline{\nu}}(\overline{\alpha})^\sharp & \text{if } \alpha<\overline{\alpha};\\ \psi^-_\nu(\alpha)f_{\varepsilon,\overline{\nu}}(\overline{\alpha})^\sharp & \text{if } \alpha>\overline{\alpha};\end{cases}$$

if $\lambda\in\mathbb{C}\setminus\frac{1}{2}\mathbb{Z}$, $\operatorname{Re}\lambda>0$. Thus by the two lemmas above one has to determine the poles with respect to ν of this kernel. Since ψ^-_ν doesn't have poles for $\operatorname{Re}\nu>\frac{1}{2}$, one has to find the poles of $c(\varepsilon,\nu)f_{\varepsilon,\nu}$.

By Lemma 1.2 $F_{\varepsilon,\nu}$, equipped with domain $C^\infty(Y)$, has simple poles at those ν that satisfy $\varrho+\nu\in\mathbb{Z}_+$ and $\varrho+\nu\equiv\varepsilon+1 \bmod 2$. Therefore $F_{\varepsilon,\nu}|_{L^2(S)_{V_\varepsilon}}$ has at most simple poles at these ν's. By definition of $f_{\varepsilon,\nu}$ one has $f_{\varepsilon,\nu}(\alpha)v = (F_{\varepsilon,\nu}v)(x(\alpha,\cdot))$ for $v\in V$. Hence $f_{\varepsilon,\nu}(\alpha)$ is meromorphic with at most simple poles at the above mentioned ν's. By Lemma 1.3 $c(\varepsilon,\nu)^{-1}$ has no poles or zeros in the right half plane (if $2\nmid q$) or simple zeros at those ν that satisfy $\varrho+\nu+\varepsilon\in 1+2\mathbb{N}$ and simple poles at those ν that satisfy $\varrho+\nu+\varepsilon\in 2\mathbb{N}$ (if $2\mid q$). This means that $c(\varepsilon,\nu)^{-1}f_{\varepsilon,\nu}(\alpha)$ has at most simple poles at ν with $\varrho+\nu+\varepsilon+q\in 2\mathbb{N}$. By Lemma 1 $(\overline{D}_{V_\varepsilon}-\lambda)^{-1}$ has at most simple poles at $\lambda\in\mathbb{Z}_+$ (and is holomorphic elsewhere).

It remains to show that $(\nu'-\nu)c(\varepsilon,\nu')^{-1}F_{\varepsilon,\nu'}|_{\nu'=\nu}$ does not vanish on $\{\varphi\in\mathscr{E}_\varepsilon(Y,\Delta)_K \mid \mathscr{I}\varphi=\varphi\}$. For this purpose let $\varphi\in\mathscr{E}_\varepsilon(Y)_K$ be a complex-valued function such that $(\nu'-\nu)c(\varepsilon,\nu')^{-1}F_{\varepsilon,\nu'}|_{\nu'=\nu}\varphi\neq 0$; such a φ exists because $c(\varepsilon,\nu')^{-1}F_{\varepsilon,\nu'}$ has a pole at $\nu'=\nu$ with $\operatorname{Re}\nu>\frac{1}{2}$ and $\varrho+\nu+\varepsilon+q\in 2\mathbb{N}$, $\mathscr{E}_\varepsilon(Y)_K$ is dense in $\mathscr{E}_\varepsilon(Y)$, and $(\nu'-\nu)c(\varepsilon,\nu')^{-1}F_{\varepsilon,\nu'}|_{\nu'=\nu}$ is continuous with respect to the C^∞-topology. Then $(\nu'-\nu)c(\varepsilon,\nu')^{-1}F_{\varepsilon,\nu'}|_{\nu'=\nu}(\varphi\cdot u)\neq 0$ for an arbitrary $u\in\Delta\setminus\{0\}$. Let $\varphi\cdot u=\varphi_+ + \varphi_-$ with $\mathscr{I}\varphi_\pm=\pm\varphi_\pm$. Then for at least one of $\varphi_\pm$: $(\nu'-\nu)c(\varepsilon,\nu')^{-1}F_{\varepsilon,\nu'}|_{\nu'=\nu}\varphi_\pm\neq 0$. If this is the case for φ_+, then we're done. If not, then do the following. Since $\dim X=p+q-1$ is even, (exactly) one of p and q is even; first assume that p is even. Then

$$\begin{aligned}(\nu'-\nu)c(\varepsilon,\nu')^{-1}F_{\varepsilon,\nu'}|_{\nu'=\nu}(e_1\cdots e_p)\cdot\varphi_- &= (\nu'-\nu)c(\varepsilon,\nu')^{-1}(e_1\cdots e_p)\cdot F_{\varepsilon,\nu'}|_{\nu'=\nu}\varphi_-\\ &= (e_1\cdots e_p)\cdot(\nu'-\nu)c(\varepsilon,\nu)^{-1}F_{\varepsilon,\nu}|_{\nu'=\nu}\varphi_-\\ &\neq 0\end{aligned}$$

($F_{\varepsilon,\nu}$ commutes with the Clifford multiplication as can be easily seen for $\operatorname{Re}(\nu+\nu)<0$ from the definition of $F_{\varepsilon,\nu}$ as an integral operator, and for all other ν this follows from uniqueness of meromorphic continuation). Since $\mathscr{I}(e_1\cdots e_p)\cdot\varphi_- = -(e_1\cdots e_p)\cdot\mathscr{I}\varphi_- = (e_1\cdots e_p)\cdot\varphi_-$, one has an element of $\{\varphi\in\mathscr{E}_\varepsilon(Y,\Delta)_K\mid\mathscr{I}\varphi=\varphi\}$ that is not annihilated by $(\nu'-\nu)c(\varepsilon,\nu')^{-1}F_{\varepsilon,\nu'}|_{\nu'=\nu}$. In the same way one proceeds if q is even.

Putting this together it follows that $\sigma_p(\overline{D})\cap\{\operatorname{Re}\lambda>0\}=\mathbb{Z}_+$. Since the spinor bundle decomposes into even and odd Weyl spinors, that are intertwined by the Dirac operator, the spectrum of the Dirac operator is symmetric with respect to 0. By Theorem 4.1 one has $i\mathbb{R}\subset\sigma_c(\overline{D}_V)$, which in particular means that $0\notin\sigma_p(\overline{D})$. Thus $\sigma_p(\overline{D})=\mathbb{Z}\setminus\{0\}$ and $\sigma_c(\overline{D}_V)=i\mathbb{R}$.

By Lemma 2 $P:=-(\nu'-\nu)(\overline{D}_{V_\varepsilon}+\frac{1}{2}-\nu')^{-1}|_{\nu'=\nu}$ is the orthogonal (with respect to $\langle.,.\rangle$) projection of $L^2(S)_{V_\varepsilon}$ onto the eigenspace of $\overline{D}_{V_\varepsilon}$ corresponding to the eigenvalue λ.

According to Lemma 1 P is an integral operator with kernel

$$p_\nu(\alpha,\overline{\alpha}) := 2^{2\varrho}\cdot\begin{cases}(\nu'-\nu)\,c(\varepsilon,\nu')^{-1}f_{\varepsilon,\nu'}(\alpha)\big|_{\nu'=\nu}\psi^-_{\overline{\nu}}(\overline{\alpha})^\sharp & \text{if } \alpha<\overline{\alpha};\\ \psi^-_\nu(\alpha)\big((\nu'-\nu)\,c(\varepsilon,\nu')^{-1}f_{\varepsilon,\nu'}(\overline{\alpha})\big|_{\nu'=\overline{\nu}}\big)^\sharp & \text{if } \alpha>\overline{\alpha}.\end{cases}$$

Now $p_\nu\colon(1,\infty)^2\to\mathrm{End}(V_\varepsilon)$ is analytical being the kernel of P. In fact, let $(v_1,\dots,v_r)$ be an orthonormal basis of the L^2-eigenspace, whose dimension is at most the dimension of V_ε because the $Dv=\lambda v$ is an ordinary first order differential equation; such an orthonormal basis exists because the eigenspace is non-degenerate by Lemma 2. Then $p_\nu(\alpha,\overline{\alpha})=\sum_i\langle v_i(\overline{\alpha}),\cdot\rangle v_i(\alpha)$. This mapping is analytical because the v_i's are such being solutions of an elliptic ordinary differential equation with analytic coefficients. Then the uniqueness of analytical continuation implies that

$$\begin{aligned}p_\nu(\alpha,\overline{\alpha}) &= 2^{2\varrho}(\nu'-\nu)c(\varepsilon,\nu')^{-1}f_{\varepsilon,\nu'}(\alpha)\big|_{\nu'=\nu}\psi_{\overline{\nu}}(\overline{\alpha})^\sharp\\ &= 2^{2\varrho}\,\psi^-_\nu(\alpha)\,\big((\nu'-\nu)c(\varepsilon,\nu')^{-1}f_{\varepsilon,\nu'}(\overline{\alpha})\big|_{\nu'=\overline{\nu}}\big)^\sharp\end{aligned}$$

for *all* $\alpha,\overline{\alpha}\in(1,\infty)$. Therefore the eigenspinors f of $\overline{D}_{V_\varepsilon}$ are given by

$$\begin{aligned}f(\alpha) &= \int_1^\infty p_\nu(\alpha,\overline{\alpha})v(\overline{\alpha})\mu(\overline{\alpha})\,d\overline{\alpha}\\ &= \int_1^\infty 2^{2\varrho}(\nu'-\nu)c(\varepsilon,\nu')^{-1}f_{\varepsilon,\nu'}(\alpha)\big|_{\nu'=\nu}\psi^-_{\overline{\nu}}(\overline{\alpha})^\sharp v(\overline{\alpha})\mu(\overline{\alpha})\,d\overline{\alpha}\\ &= (\nu'-\nu)c(\varepsilon,\nu')^{-1}f_{\varepsilon,\nu'}(\alpha)\big|_{\nu'=\nu}\int_1^\infty 2^{2\varrho}\psi^-_{\overline{\nu}}(\overline{\alpha})^\sharp v(\overline{\alpha})\mu(\overline{\alpha})\,d\overline{\alpha}\end{aligned}$$

for some $v\in V_\varepsilon$. Set $\varphi:=\int_1^\infty 2^{2\varrho}\psi^-_{\overline{\nu}}(\overline{\alpha})^\sharp v(\overline{\alpha})\mu(\overline{\alpha})\,d\overline{\alpha}\in V_\varepsilon^{\mathscr{I}}$, then

$$f(\alpha)=(\nu'-\nu)c(\varepsilon,\nu')^{-1}f_{\varepsilon,\nu'}(\alpha)\big|_{\nu'=\nu}\varphi,$$

that is,

$$f=(\nu'-\nu)c(\varepsilon,\nu')^{-1}F_{\varepsilon,\nu'}\big|_{\nu'=\nu}\varphi.$$

Q.E.D.

Theorem 2 *If the dimension of* $X=\boldsymbol{Spin}_+(p,q)/\boldsymbol{Spin}_+(p,q-1)$, $p\geq 1$ *and* $q\geq 2$, *is odd, and the index even, the point spectrum of the Dirac operator is given by* $\sigma_p(\overline{D})=\frac12+\mathbb{Z}$. *For a finite dimensional* $V\subset L^2(Y,\Delta)$ *with* $\overline{V}^K=V$ *one has* $\sigma_p(\overline{D}_V)\subset\frac12+\mathbb{Z}$, $\sigma_c(\overline{D}_V)=i\mathbb{R}$, *and* $\sigma_r(\overline{D}_V)=\emptyset$. *The eigenspaces are non-degenerate.*

Proof. The proof runs analogously to the proof of Theorem 1 by determining the poles of the resolvent. Unfortunately there's no symmetry argument as in the even dimensional case.

Consider again the right half plane $\operatorname{Re}\lambda > 0$. Again one has ($\nu = \lambda + \frac{1}{2}$):

$$k_\nu(\alpha,\overline{\alpha}) = 2^{2\varrho}\cdot\begin{cases} c(\varepsilon,\nu)^{-1}f_{\varepsilon,\nu}(\alpha)\psi^-_{\overline{\nu}}(\overline{\alpha})^\sharp & \text{if } \alpha<\overline{\alpha};\\ \psi^-_\nu(\alpha)\big(c(\varepsilon,\overline{\nu})^{-1}f_{\varepsilon,\overline{\nu}}(\overline{\alpha})\big)^\sharp & \text{if } \alpha>\overline{\alpha}.\end{cases}$$

As in the even dimensional case one finds that this kernel has simple poles, but now for $\nu\in\mathbb{Z}_+$, that is $\lambda\in\frac{1}{2}+\mathbb{Z}$.

On the right half plane $\operatorname{Re}\lambda<0$ one has

$$k_\nu(\alpha,\overline{\alpha}) = 2^{2\varrho}\cdot\begin{cases} f_{\varepsilon,\nu}(\alpha)C_-(\varepsilon,\nu)^{-1}\big(1+\nu^{-1}(D'-D''-\varrho)\big)\psi^+_{\overline{\nu}}(\overline{\alpha})^\sharp & \text{if } \alpha<\overline{\alpha};\\ \psi^+_\nu(\alpha)\Big(f_{\varepsilon,\overline{\nu}}(\overline{\alpha})C_-(\varepsilon,\overline{\nu})^{-1}\big(1+\overline{\nu}^{-1}(D'-D''-\varrho)\big)\Big)^\sharp & \text{if } \alpha>\overline{\alpha}.\end{cases}$$

Now ψ^+_ν has no poles for $\operatorname{Re}\nu<\frac{1}{2}$, so one has to find the poles and their multiplicities of $F_{\varepsilon,\nu}C_-(\varepsilon,\nu)^{-1}\big(1+\nu^{-1}(D'-D''-\varrho)\big)$, equipped with domain $\{\varphi\in\mathscr{E}_\varepsilon(Y,\Delta)\mid\mathscr{I}\varphi=-\varphi\}$. According to Lemma 1.4 one has $F_{\varepsilon,\nu}C_-(\varepsilon,\nu)^{-1}=c(\varepsilon,ny)^{-1}F_{\varepsilon,-\nu}$. Lemma 1.3 shows that $c(\varepsilon,\nu)^{-1}F_{\varepsilon,-\nu}$ has simple poles at $\nu\in\mathbb{Z}_-$ that satisfies $\varrho+\nu\equiv\varepsilon+1 \bmod 2$; if $\nu=0$, then $c(\varepsilon,-\nu)^{-1}F_{\varepsilon,-\nu}$ has a simple zero if $\varrho\equiv\varepsilon+1\bmod 2$, and if $\varrho\equiv\varepsilon\bmod 2$, $c(\varepsilon,-\nu)$ and $F_{\varepsilon,-\nu}$ are defined and $\neq 0$.

Consider now $H^{m,n}\otimes\Delta\subset\mathscr{E}(Y,\Delta)$. By section 1 this is the joint eigenspace of $\varDelta_{S^{p-1}}$ and $\varDelta_{S^{q-1}}$ corresponding to eigenvalues $m(m+p-2)$ and $n(n+q-2)$, respectively if $p>1$. Since $\varDelta_{S^{p-1}}=D'^2-(p-2)D'$ and $\varDelta_{S^{q-1}}=D''^2+(q-2)D''$ by Lemma 3.3, $H^{m,n}\otimes\Delta$ is invariant under D' and D''. Furthermore $H^{m,n}\otimes\Delta$ splits into at most two eigenspaces of D' for eigenvalues $m+p-2$ and $-m$, and analogously into at most two eigenspaces of D'' for eigenvalues n and $-(n+q-")$. Hence $H^{m,n}\otimes\Delta$ splits into at most four eigenspaces of $1+\nu^{-1}(D'-D''-\varrho)$ for eigenvalues $1+\nu^{-1}(m-n+p-2-\varrho)$, $1+\nu^{-1}(m+n+p+q-4-\varrho)$, $1+\nu^{-1}(-m-n-\varrho)$, and $1+\nu^{-1}(-m+n+q-2-\varrho)$. Therefor $1+\nu^{-1}(D'-D''-\varrho)$ is an isomorphism of $H^{m,n}\otimes\Delta$ onto itself if these four number are different from 0. By Lemma 1.5 $c(\varepsilon,-\nu)^{-1}F_{\varepsilon,-\nu}|_{\nu=-k}$ is injective on $H^{m,n}\otimes\Delta$, where $m+n\equiv\varepsilon\bmod 2$, if $-(m+p-2)+(n+q-2)\geq k$ ($k\in\mathbb{Z}_+$ arbitrary). Hence there's obviously a pair (m,n) of natural numbers such that $c(\varepsilon,-\nu)^{-1}F_{\varepsilon,-\nu}(1+\nu^{-1}(D'-D''-\varrho))\big|_{\nu=-k}$ does not vanish on $H^{m,n}\otimes\Delta$. Therefore $c(\varepsilon,-\nu)^{-1}F_{\varepsilon,-\nu}(1+\nu^{-1}(D'-D''-\varrho))\big|_{\nu=-k}$ is not identically zero on $\mathscr{E}_\varepsilon(Y,\Delta)_K=\mathscr{E}_\varepsilon(Y)_K\otimes\Delta$. The same argument as in the proof of Theorem 1 shows now that there's a $\varphi\in\mathscr{E}_\varepsilon(Y,\Delta)_K$ such that $c(\varepsilon,-\nu)^{-1}F_{\varepsilon,-\nu}(1+\nu^{-1}(D'-D''-\varrho))\big|_{\nu=-k}\varphi\neq 0$ and $\mathscr{I}\varphi=-\varphi$. If $p=1$, one can argue analogously. Thus one obtains that the resolvent of $\overline{D}_{V_\varepsilon}$ has for $\nu\in\mathbb{Z}_-$ a pole of at most first order for every V with $\overline{V}^K=V$, and that there's a V such that the resolvent actually has a pole.

The case $\nu=0$ is treated similarly, except that the pole of the resolvent is due to $1+\nu^{-1}(D'-D''-\varrho)$ and not to $c(\varepsilon,-\nu)F_{\varepsilon,-\nu}$. **Q.E.D.**

Remarks: 1) One can of course also in the even dimensional case determine the point spectrum smaller than zero in this way, but by using the symmetry of the spectrum one doesn't need the sophisticated Lemma 1.5.

2) Using Lemma 1.3 one can show even for p and q odd, that $\sigma_p(\overline{D})=\frac{1}{2}+\mathbb{Z}$ and then analogously to the proof of Lemma 3.9 (c) one can show that $\sigma_r(\overline{D})=\emptyset$. But this method

doesn't yield the representation of K-finite eigenspinors of Theorem 1 (or an analogous representation that could be obtained for Theorem 2), and it doesn't say anything about the continuous spectrum of $\overline{D}$ on the isotypic components or the non-degeneracy of the eigenspaces.

Finally some remarks seem appropriate:

- The question arises if the continuous spectrum of $\overline{D}$ on the whole of $L^2(S)$ (and not only on the K-isotypic components) coincides with $i\mathbb{R}$. Experience has shown that this will most likely not be the case. For expample, one can easily show, using a Fourier transformation, that the continous spectrum of the Dirac operator on the flat $\mathbb{R}^{p,q}$ with $p, q \geq 1$ is all of $\mathbb{C}$; analogously using Fourier series instead of Fourier transformation the continuous spectrum on the flat torus $\mathbb{R}^{p,q}/\mathbb{Z}^{p+q}$ is $\mathbb{C} \setminus \sigma_p(\overline{D})$.
- There are a number of properties of the Dirac operator on the K-isotypic components, however, that make it reasonable to believe that this operator is not "to far" from a self-adjoint operator. More precisely, I think that the Dirac operator restricted to the K-isotypic components is a spectral operator (see [DS2] for the definition and properties of spectral operators).
- By Theorem 1 for the even dimensional case (or an analogue for the odd dimensional case but even index) every K-finite L^2-eigenspinor for eigenvalue $\lambda = \nu - \frac{1}{2}$ lies in the image of $\mathscr{E}_\varepsilon(Y, \Delta)_K^{\mathscr{I}}$ under the mapping $(\nu' - \nu)c(\varepsilon, \nu')^{-1} F_{\varepsilon,\nu'}|_{\nu'=\nu}$ with $\varepsilon \equiv \varrho + \nu + 1 \bmod 2$. Since this mapping depends meromorphically on ν' with no pole at $\nu' = \nu$, it is continuous in ν' in a neighbourhood of ν, and because $F_{\varepsilon,\nu'}$ is G-equivariant, so is $(\nu' - \nu)c(\varepsilon, \nu')^{-1} F_{\varepsilon,\nu'}|_{\nu'=\nu}$. Since the space of all K-finite (smooth) eigenspinors for each eigenvalue is $\mathfrak{g}$-invariant (with $\mathfrak{g} = \boldsymbol{spin}(p, q)$, the Lie algebra of G) and $F_{\varepsilon,\nu}$ is G-equivariant and bijective (see Lemma 3.9) from $\mathscr{E}_\varepsilon(Y, \Delta)_K^{\mathscr{I}}$ onto this eigenspace if $\nu \notin \frac{1}{2}\mathbb{Z}$, $\mathscr{E}_\varepsilon(Y, \Delta)_K^{\mathscr{I}}$ is $\mathfrak{g}$-invariant (in fact, it can be proved directly without recourse to any $F_{\varepsilon,\nu}$ that $\mathscr{E}_\varepsilon(Y, \Delta)^{\mathscr{I}}$ is invariant under G). It follows that the space of square integrable K-finite eigenspinors is a $(\boldsymbol{spin}(p, q), K)$-module. The surprising fact is not that the K-finite eigenspinors are $\mathfrak{g}$-invariant, but that the *square integrable* K-finite eigenspinors are $\mathfrak{g}$-invariant.
- During the whole discussion the case $q = 1$, that is, X is the (Riemannian) hyperbolic space of negative metric, was excluded. This was done not because this case differs essentially from $q \geq 2$. The reason is that X was defined as $X = \boldsymbol{Spin}_+(p, q)/\boldsymbol{Spin}_+(p, q-1)$ to have an invariant scalar product, accordingly for $q = 1$ this would mean $X = \boldsymbol{Spin}_+(p, 1)/\boldsymbol{Spin}(p)$. But then multiplication with -1 would not leave X invariant, therefore the decompostion of the C^∞ and L^2 spaces into even and odd elements would be meaningless. The alternative is to define $X = \boldsymbol{Spin}(p, 1)/\boldsymbol{Spin}(p)$; most of the results for $q \geq 2$ are just the same with this setting. Regardless wich alternative is chosen, it would by tiresome to make the distinction every time. The only difference between $q = 1$ and $q \geq 2$ is that for $q = 1$ the Poisson transformation has *no* poles ([Ro] Lemma 5). Therefore the Dirac operator has no point spectrum. Since $\overline{D}^* = -\overline{D}$ (adjoint with respect to the positive definite scalar product on S), here $\sigma(\overline{D}) = \sigma_c(\overline{D}) = i\mathbb{R}$. One could even use the method of Rossmann to find the spectral resolution of the Dirac operator. The only paper that examines the Dirac operator on

the hyperbolic space that I know of is [Bu]; there “only” the spectrum of its square is computed.

Appendix: The Green's function

The purpose of this appendix is first to prove some theorems about Green's functions and then to apply these theorems to the Dirac operator. This will result in a complete description of its spectrum on the K-isotypic components.

Let $I=(\alpha_0,\alpha_1)$ be an open interval, bounded or not, let $\mu\in C^\infty(I,\mathbb{R}_+)$ be a positive function; this introduces a measure $d\mu(\alpha):=\mu(\alpha)\,d\alpha$ where $d\alpha$ is the Lebesgue measure on I. Let V be (finite dimensional) complex Hilbert space with inner product $(.\,,.)$. Let $a,b\in C^\infty(I,\mathrm{End}(V))$ such that $a(\alpha)\in \mathrm{Gl}(V)$ for all $\alpha\in I$.*) Let $Du:=a\,u'+b\,u$ denote the corresponding ordinary differential operator. Let $U\subset C^\infty(I,V)$ be a subspace of $L^2(I,V;d\mu(\alpha))$ that contains $C_0^\infty(I,V)$. Assume that D maps U into $L^2(I,V;d\mu(\alpha))$. Let $\overline{D}$ denote the closure of (D,U) in L^2 and D^* be the adjoint; is easily seen that $D^*u=-\mu(\mu a^*u)'+b^*u$ where $\Box^*$ denotes the adjoint with respect to $(.\,,.)$.

Theorem 1 (a) *Let $0\in\varrho(\overline{D})$. Then there exists a function $k(\alpha,\overline{\alpha})$ (the Green's function) with values in $\mathrm{End}(V)$, that is square integrable with respect to $\overline{\alpha}$, such that*

$$(\overline{D}^{-1}v)(\alpha)=\int\limits_J k(\alpha,\overline{\alpha})v(\overline{\alpha})\,d\mu(\overline{\alpha}) \tag{1}$$

for all $v\in L^2(I,V)$. k has the following properties:

(i) *k is smooth on $I\times I\setminus\Delta_I$.*

(ii) *One has $(Dk(\,\cdot\,,\overline{\alpha}))(\alpha)=0$ and $-\frac{\partial}{\partial\overline{\alpha}}\big(\mu(\overline{\alpha})k(\alpha,\overline{\alpha})\circ a(\overline{\alpha})\big)+\mu(\overline{\alpha})k(\alpha,\overline{\alpha})\circ b(\overline{\alpha})$ for $\alpha\neq\overline{\alpha}$.*

(iii) *One has $k(\overline{\alpha}+0,\overline{\alpha})-k(\overline{\alpha}-0,\overline{\alpha})=\mu(\overline{\alpha})^{-1}a(\overline{\alpha})^{-1}$ and $k(\alpha,\alpha+0)-k(\alpha,\alpha-0)=-\mu(\alpha)^{-1}a(\alpha)^{-1}$.*

(iv) *If $\overline{\alpha}\in I$ is fixed and $\chi\in C^\infty(I)$ such that $1-\chi\in C_0^\infty(I)$ and $\chi=0$ in a neighbourhood of $\overline{\alpha}$, then $\chi(\,\cdot\,)k(\,\cdot\,,\overline{\alpha})v\in\mathrm{dom}(\overline{D})$ for all $v\in V$. If on the other hand $\alpha\in I$ is fixed and $\chi\in C^\infty(I)$ such that $1-\chi\in C_0^\infty(I)$ and $\chi=0$ in a neighbourhood of α, then $\chi(\,\cdot\,)k(\alpha,\,\cdot\,)^*v\in\mathrm{dom}(D^*)$. In other words: k satisfies the "boundary conditions" of $\overline{D}$ in the first variable, and the "boundary conditions" of D^* in the second variable.*

(b) *If a function k has the properties* (i)–(iv) *such that $\int_I k(\,\cdot\,,\overline{\alpha})u(\overline{\alpha})\,d\mu(\overline{\alpha})$ is in L^2 for all $u\in L^2$, then $0\in\varrho(\overline{D})$ and k is the Green's function.*

Proof. The proof follows quite closely [DS1] Lemma XIII.3.4.

(a) One has $\mathrm{dom}(\overline{D})\subset H^1_{\mathrm{loc}}(I,V)$ and $\overline{D}^{-1}:L^2(I,V)\to H^1_{\mathrm{loc}}(I,V)$ is continuous. For, let $u\in\mathrm{dom}(\overline{D})$. Then there is a sequence $u_n\in U=\mathrm{dom}(D)$ such that $u_n\to u$ and $Du_n\to\overline{D}u$ in $L^2(I,V)$. Then $u_n'=a^{-1}(Du_n-b\,u_n)\to a^{-1}(\overline{D}u-b\,u)$ in $L^2_{\mathrm{loc}}(I,V)$ because a^{-1} and b are smooth, hence bounded on compact subsets of I. This shows $u\in H^1_{\mathrm{loc}}(I,V)$. Also one has $(\overline{D}^{-1}v)'=a^{-1}(v-b\,\overline{D}v)$ for $v\in L^2(I,V)$, which shows the continuity of

*) Everything remains true even if V is infinite dimensional provided that $a(\alpha)$, $a(\alpha)^{-1}$, and $b(\alpha)$ are continuous for all $\alpha\in I$.

$\overline{D}^{-1}: L^2(I,V) \to H^1_{\mathrm{loc}}(I,V)$. Therefore the map $L^2(I,V) \ni u \mapsto (\overline{D}^{-1}u)(\alpha)$ is continuous for each fixed α. Hence there exists a uniquely determined function $k(\alpha,\,\cdot\,) \in L^2(I,\mathrm{End}(V))$ such that $(\overline{D}^{-1}u)(\alpha) = \int_I k(\alpha,\overline{\alpha})u(\overline{\alpha})\,d\mu(\overline{\alpha})$ for all $u \in L^2(I,V)$.

Show now that $k \in L^2_{\mathrm{loc}}(I\times I,\mathrm{End}(V))$ (note that until now it's not even clear that k is measurable). Let $u \in C_0^\infty(\Omega\times\Omega,V)$, $\Omega \subset I$ compact. Then there's a constant const such that

$$\|\big(\overline{D}^{-1}u(\alpha,\,\cdot\,)\big)(\overline{\alpha})\| \le \mathrm{const}\|u(\alpha,\,\cdot\,)\|_{L^2}$$

for all $\alpha \in I$ and $\overline{\alpha} \in \Omega$ because of the continuity of $\overline{D}^{-1}: L^2(I,V) \to H^1_{\mathrm{loc}}(I,V) \subset L^\infty_{\mathrm{loc}}(I,V)$. Furthermore the mapping $(\alpha,\overline{\alpha}) \mapsto \big(\overline{D}^{-1}u(\alpha,\,\cdot\,)\big)(\overline{\alpha})$ is continuous. To see this let $\alpha,\alpha',\overline{\alpha},\overline{\alpha}' \in \Omega$. Then

$$\begin{aligned}
&\|\big(\overline{D}^{-1}u(\alpha,\,\cdot\,)\big)(\overline{\alpha}) - \big(\overline{D}^{-1}u(\alpha',\,\cdot\,)\big)(\overline{\alpha}')\| \le \\
&\quad\le \|\big(\overline{D}^{-1}u(\alpha,\,\cdot\,)\big)(\overline{\alpha}) - \big(\overline{D}^{-1}u(\alpha',\,\cdot\,)\big)(\overline{\alpha})\| + \|\big(\overline{D}^{-1}u(\alpha',\,\cdot\,)\big)(\overline{\alpha}) - \big(\overline{D}^{-1}u(\alpha',\,\cdot\,)\big)(\overline{\alpha}')\| \\
&\quad\le \|\overline{D}^{-1}u(\alpha,\,\cdot\,) - \overline{D}^{-1}u(\alpha',\,\cdot\,)\|_{L^\infty(\Omega,V)} + \|\frac{d}{d\overline{\alpha}}\overline{D}^{-1}u(\alpha',\,\cdot\,)\|_{L^\infty(\Omega,V)}\cdot|\overline{\alpha}'-\overline{\alpha}| \\
&\quad\le \mathrm{const}\cdot\|u(\alpha,\,\cdot\,)-u(\alpha',\,\cdot\,)\|_{L^2} + \|a^{-1}u(\alpha',\,\cdot\,) - a^{-1}b\overline{D}^{-1}u(\alpha',\,\cdot\,)\|_{L^\infty(\Omega,V)}\cdot|\overline{\alpha}-\overline{\alpha}'| \\
&\quad\le \mathrm{const}\cdot\mu(\Omega)^{\frac12}\left\|\frac{\partial u}{\partial\alpha}\right\|_{L^\infty(\Omega\times\Omega,V)}\cdot|\alpha-\alpha'| + \mathrm{const}\cdot|\overline{\alpha}-\overline{\alpha}'|
\end{aligned}$$

where "const" is a "constant" that depends on $\overline{D}^{-1}$, u, and Ω, but not on $\alpha,\alpha',\overline{\alpha},\overline{\alpha}' \in \Omega$, and where $\mu(\Omega) := \int_\Omega d\mu(\alpha)$. The last line then shows the continuity of $(\alpha,\overline{\alpha}) \mapsto \big(\overline{D}^{-1}u(\alpha,\,\cdot\,)\big)(\overline{\alpha})$. Hence the following integral on the left hand side is reasonable and can be estimated as follows:

$$\begin{aligned}
\int_\Omega \|\big(\overline{D}^{-1}u(\alpha,\,\cdot\,)\big)(\alpha)\|\,d\mu(\alpha) &\le \mathrm{const}\int_\Omega \|u(\alpha,\,\cdot\,)\|_{L^2}\,d\mu(\alpha) \\
&\le \mathrm{const}\,\mu(\Omega)^{\frac12}\Big(\int_\Omega \|u(\alpha,\,\cdot\,)\|^2_{L^2}\,d\mu(\alpha)\Big)^{\frac12} \\
&= \mathrm{const}\,\mu(\Omega)^{\frac12}\|u\|_{L^2}.
\end{aligned}$$

So the map $u \mapsto \int_\Omega \big(\overline{D}^{-1}u(\alpha,\,\cdot\,)\big)(\alpha)\,d\mu(\alpha)$ is L^2-continuous of $C_0^\infty(\Omega\times\Omega,V)$ to V, hence there is a $\hat{k}_\Omega \in L^2(\Omega\times\Omega,\mathrm{End}(V))$ such that

$$\int_\Omega \big(\overline{D}^{-1}u(\alpha,\,\cdot\,)\big)(\alpha)\,d\mu(\alpha) = \int_\Omega\int_\Omega \hat{k}_\Omega(\alpha,\overline{\alpha})u(\alpha,\overline{\alpha})\,d\mu(\alpha)\,d\mu(\overline{\alpha})$$

that is,

$$\int_\Omega\Big(\int_\Omega k_\Omega(\alpha,\overline{\alpha})u(\alpha,\overline{\alpha})\,d\mu(\overline{\alpha})\Big)\,d\mu(\alpha) = \int_\Omega\int_\Omega \hat{k}_\Omega(\alpha,\overline{\alpha})u(\alpha,\overline{\alpha})\,d\mu(\alpha)\,d\mu(\overline{\alpha}).$$

Choose now $u(\alpha,\overline{\alpha}) = f(\alpha)\cdot v(\overline{\alpha})$ where $f \in C_0^\infty(\Omega)$, $v \in C_0^\infty(\Omega, V)$. Then

$$\int_\Omega f(\alpha)\cdot\Big(\int_\Omega k_\Omega(\alpha,\overline{\alpha})v(\overline{\alpha})\,d\mu(\overline{\alpha})\Big)\,d\mu(\alpha) = \int_\Omega f(\alpha)\cdot\Big(\int_\Omega \hat{k}_\Omega(\alpha,\overline{\alpha})v(\overline{\alpha})\,d\mu(\overline{\alpha})\Big)\,d\mu(\alpha).$$

This implies

$$\int k(\alpha,\overline{\alpha})v(\overline{\alpha})\,d\mu(\overline{\alpha}) = \int_\Omega \hat{k}_\Omega(\alpha,\overline{\alpha})v(\overline{\alpha})\,d\mu(\overline{\alpha})$$

for all $\alpha \in \Omega$, which in turn implies that $k|_{\Omega\times\Omega} = \hat{k}_\Omega$ because k is uniquely determined by (1). Thus k is indeed in L^2_{loc}.

Let now $u = Dv$, $v \in C_0^\infty(I,V)$. Then

$$v(\alpha) = \int_I k(\alpha,\overline{\alpha})\big(a(\overline{\alpha})\,v'(\overline{\alpha}) + b(\overline{\alpha})\,v(\overline{\alpha})\big)\,d\mu(\overline{\alpha}). \tag{2}$$

Partial integration yields

$$-\frac{\partial}{\partial\overline{\alpha}}\big(\mu(\overline{\alpha})k(\alpha,\overline{\alpha})\circ a(\overline{\alpha})\big) + \mu(\overline{\alpha})k(\alpha,\overline{\alpha})\circ b(\overline{\alpha}) = \delta_\alpha(\overline{\alpha})\,id_V \tag{3}$$

where δ_α is the delta distribution located at α. On the other hand

$$(v,u)_{L^2} = (\overline{D}(\overline{D}^{-1}v)\,,\,u)_{L^2} = (\overline{D}^{-1}v\,,\,D^*u)_{L^2} \tag{4}$$

for $u,v \in C_0^\infty(I,V)$, that is,

$$\begin{aligned}\int_I \big(v(\overline{\alpha})\,,\,u(\overline{\alpha})\big)\,d\mu(\overline{\alpha}) &= \int_I\int_I \big(k(\alpha,\overline{\alpha})v(\overline{\alpha})\,,\,(D^*u)(\alpha)\big)\,d\mu(\alpha)\,d\mu(\overline{\alpha})\\ &= \int_I\int_I \big(v(\overline{\alpha})\,,\,k(\alpha,\overline{\alpha})^*(D^*u)(\alpha)\big)\,d\mu(\alpha)\,d\mu(\overline{\alpha}),\end{aligned}$$

hence

$$u(\overline{\alpha}) = \int_I k(\alpha,\overline{\alpha})^*(D^*u)(\alpha)\mu(\alpha)\,d\mu(\overline{\alpha}). \tag{5}$$

Again partial integration yields

$$\frac{\partial}{\partial\alpha}\big(k(\alpha,\overline{\alpha})^*\big)a(\alpha)^* + k(\alpha,\overline{\alpha})^*b(\alpha)^* = \mu(\overline{\alpha})^{-1}\delta_{\overline{\alpha}}(\alpha)\,id_V,$$

hence

$$a(\alpha)\,\frac{\partial k}{\partial\alpha}(\alpha,\overline{\alpha}) + b(\alpha)k(\alpha,\overline{\alpha}) = \mu(\overline{\alpha})^{-1}\delta_{\overline{\alpha}}(\alpha)\,id_V. \tag{6}$$

Equations (3) and (6) imply (ii).

To prove (iii) note that (3) can be written as

$$-\frac{\partial}{\partial\overline{\alpha}}\big(\mu(\overline{\alpha})k(\alpha,\overline{\alpha})\circ a(\overline{\alpha})\big)+\mu(\overline{\alpha})k(\alpha,\overline{\alpha})\circ b(\overline{\alpha})=\frac{d\theta_\alpha}{d\overline{\alpha}}(\overline{\alpha})\,id_V$$

where $\theta_\alpha(\overline{\alpha})=0$ if $\overline{\alpha}<\alpha$, and $\theta_\alpha(\overline{\alpha})=1$ if $\overline{\alpha}>\alpha$. This is equivalent to

$$\begin{aligned}\frac{\partial}{\partial\overline{\alpha}}\Big(\mu(\overline{\alpha})k(\alpha,\overline{\alpha})a(\overline{\alpha})+\theta_\alpha(\overline{\alpha})\,id_V\Big)+\Big(\mu(\overline{\alpha})k(\alpha,\overline{\alpha})a(\overline{\alpha})+\theta_\alpha(\overline{\alpha})\,id_V\Big)a(\overline{\alpha})^{-1}b(\overline{\alpha})&=\\=\theta_\alpha(\overline{\alpha})a(\overline{\alpha})^{-1}b(\overline{\alpha}).&\end{aligned}$$

Since the right hand side of this expression is locally integrable, $\mu(\overline{\alpha})k(\alpha,\overline{\alpha})a(\overline{\alpha})+\theta_\alpha(\overline{\alpha})\,id_V$ is continuous in $\overline{\alpha}$. Since $\theta_{\overline{\alpha}}$ has a jump at $\overline{\alpha}=\alpha$ of height one, it follows that $\mu(\,\cdot\,)k(\alpha,\,\cdot\,)a(\,\cdot\,)$ has a jump of height $-id_V$. This is the second assertion of (iii); the first is proved analogously using (6).

From (3) it follows that $k(\alpha,\overline{\alpha})$ depends smoothly on $\overline{\alpha}$ except for $\overline{\alpha}=\alpha$, and (6) shows that k depends smoothly on α except for $\alpha=\overline{\alpha}$. Since the solution of an ordinary differential equation depends smoothly on the initial condition, this shows that k is smooth on $(I\times I)\setminus\Delta_I$.

Now (4) is valid for $v\in L^2(I,V)$ and $u\in\mathrm{dom}(D^*)$. Hence (5) holds for $u\in\mathrm{dom}(D^*)$. Since $\mathrm{dom}(D^*)\subset H^1_{\mathrm{loc}}(I,V)$ and $D^*u=-\mu(\mu a^*u)'+b^*u$, by partial integration from (ii) and (iii) follows

$$\lim_{\substack{\alpha_1'\uparrow\alpha_1\\ \alpha_0'\downarrow\alpha_0}}\mu(\alpha)k(\alpha,\overline{\alpha})^*a(\alpha)^*u(\alpha)\Big|_{\alpha=\alpha_0'}^{\alpha=\alpha_1'}=0. \tag{7}$$

Let $u\in\mathrm{dom}(D^*)$ and $v\in V$. Then for any χ like in (iv) one obtains by partial integration using (7), (ii), and (iii)

$$\begin{aligned}(\chi(\,\cdot\,)k(\,\cdot\,,\overline{\alpha})v\,,\,D^*u)_{L^2}&=\int_I\big(\chi(\alpha)k(\alpha,\overline{\alpha})v\,,\,D^*u(\alpha)\big)\,d\mu(\alpha)\\&=\lim_{\substack{\alpha_1'\uparrow\alpha_1\\ \alpha_0'\downarrow\alpha_0}}\int_{\alpha_0'}^{\alpha_1'}\cdots\\&=\int_I\big(\chi'(\alpha)a(\alpha)k(\alpha,\overline{\alpha})v\,,\,u(\alpha)\big)\,d\mu(\alpha)\end{aligned}$$

because $\chi'=0$ near the boundary of I (note that $\chi k(\,\cdot\,,\alpha)$ is smooth). Since the right hand side is L^2-continuous in u, one has $\chi k(\,\cdot\,,\overline{\alpha})v\in\mathrm{dom}((D^*)^*)=\mathrm{dom}(\overline{D})$. In the same vein (2) yields the second part of (iv).

(b) Let k be a function satisfying (i)–(iv). Let

$$(Kv)(\alpha):=\int_I k(\alpha,\overline{\alpha})v(\overline{\alpha})\,d\mu(\overline{\alpha})$$

for $v \in L^2(I,V)$. By assumption $Kv \in L^2(I,V)$. To prove $K = \overline{D}^{-1}$ one has to show $\overline{D}Kv = v$ and $K\overline{D}u = u$ for $v \in L^2(I,V)$ and $u \in \mathrm{dom}(\overline{D})$.

• Let $v \in L^2(I,V)$, $w \in \mathrm{dom}(D^*)$. Choose $\chi \in C^\infty(I \times I)$ such that $\chi(\,\cdot\,,\overline{\alpha})$ satisfies the first hypothesis of (iv). Then

$$\begin{aligned}(Kv, D^*u) &= \int\limits_I\int\limits_I \big(k(\alpha,\overline{\alpha})v(\overline{\alpha})\,,\ D^*w(\alpha)\big)\,d\mu(\overline{\alpha})\,d\mu(\alpha)\\ &= \int\limits_I\int\limits_I \big(\chi(\alpha,\overline{\alpha})k(\alpha,\overline{\alpha})v(\overline{\alpha})\,,\ D^*w(\overline{\alpha})\big)\,d\mu(\alpha)\,d\mu(\overline{\alpha})\\ &\qquad + \int\limits_I\int\limits_I \big((1-\chi(\alpha,\overline{\alpha}))k(\alpha,\overline{\alpha})v(\overline{\alpha})\,,\ D^*w(\alpha)\big)\,d\mu(\alpha)\,d\mu(\overline{\alpha}).\end{aligned}$$

Then (iv) implies

$$(Kv, D^*w) = \int\limits_I\int\limits_I \Big(D_\alpha\big(\chi(\alpha,\overline{\alpha})k(\alpha,\overline{\alpha})\big)v(\overline{\alpha})\,,\ w(\alpha)\Big)\,d\mu(\alpha)\,d\mu(\overline{\alpha}) + \text{second term};$$

here D_α should indicate that D has to be applied with fixed $\overline{\alpha}$ and variable α. Hence

$$(Kv, D^*w) = \int\limits_I\int\limits_I \Big(\frac{\partial\chi}{\partial\alpha}(\alpha,\overline{\alpha})a(\alpha)k(\alpha,\overline{\alpha})v(\overline{\alpha})\,,\ w(\alpha)\Big)\,d\mu(\alpha)\,d\mu(\overline{\alpha}) + \text{second term}$$

because of (ii). The inner integral of the second term can be split as

$$\int\limits_I \big((1-\chi(\alpha,\overline{\alpha}))k(\alpha,\overline{\alpha})v(\overline{\alpha})\,,\ D^*w(\alpha)\big)\,d\mu(\alpha) = \int\limits_{\alpha_0}^{\overline{\alpha}} \cdots + \int\limits_{\overline{\alpha}}^{\alpha_1} \cdots\ .$$

Using $D^*w = -\mu^{-1}(\mu a^* w)' + b^* w$ one obtains by partial integration (keeping (ii) in mind):

inner integral of the second term =

$$\begin{aligned}&= -\mu(\overline{\alpha})\big(a(\overline{\alpha})k(\overline{\alpha}-0,\overline{\alpha})v(\overline{\alpha})\,,\ w(\overline{\alpha})\big) + \int\limits_{\alpha_0}^{\overline{\alpha}} \Big(-\frac{\partial\chi}{\partial\alpha}(\alpha,\overline{\alpha})a(\alpha)k(\alpha,\overline{\alpha})v(\overline{\alpha})\,,\ w(\alpha)\Big)\,d\mu(\alpha)\\ &\quad + \mu(\overline{\alpha})\big(a(\overline{\alpha})k(\alpha,\overline{\alpha})v(\overline{\alpha})\,,\ w(\overline{\alpha})\big) + \int\limits_{\overline{\alpha}}^{\alpha_1} \Big(-\frac{\partial\chi}{\partial\alpha}(\alpha,\overline{\alpha})a(\alpha)k(\alpha,\overline{\alpha})v(\overline{\alpha})\,,\ w(\alpha)\Big)\,d\mu(\alpha),\end{aligned}$$

hence because of (ii):

$$= (v(\overline{\alpha})\,,\ u(\overline{\alpha})) + \int\limits_I \Big(-\frac{\partial\chi}{\partial\alpha}(\alpha,\overline{\alpha})a(\alpha)k(\alpha,\overline{\alpha})v(\alpha)\,,\ w(\alpha)\Big)\,d\mu(\alpha).$$

This implies

$$\text{second term} = (v,w) - \int\limits_I\int\limits_I \Big(\frac{\partial\chi}{\partial\alpha}(\alpha,\overline{\alpha})a(\alpha)k(\alpha,\overline{\alpha})v(\alpha)\,,\ w(\alpha)\Big)\,d\mu(\alpha)\,d\mu(\overline{\alpha}).$$

Putting all together yields $(Kv, D^*w) = (v, w)$. Hence $Kv \in \operatorname{dom}((D^*)^*) = \operatorname{dom}(\overline{D})$ and $\overline{D}Kv = v$ which proves the first assertion.

- Let $u \in \operatorname{dom}(\overline{D})$, $w \in C_0^\infty(I, V)$. Then

$$(w, KDu) = \int\limits_I \Big(\int\limits_I \big(w(\alpha)\ ,\ k(\alpha,\overline{\alpha})Du(\overline{\alpha})\big)\, d\mu(\overline{\alpha})\Big)\, d\mu(\alpha).$$

Just as above one proves $(w, KDu) = (w, u)$ by inserting χ and splitting the integral into two integrals over (α_0, α) and (α, α_1); since $u \in \operatorname{dom}(\overline{D}) \subset H^1_{\mathrm{loc}}(I, V)$, partial integration is permitted. **Q.E.D.**

Theorem 2 *Change the assumptions of Theorem 1 in that the constant scalar product* $(.\,,.)$ *in* V *is replaced by a variable product* $\alpha \mapsto (.\,,.)_\alpha$ *depending smoothly on* $\alpha \in I$. *Then* (a) *and* (b) *of Theorem 1 remain true except that* (iv) *has to be replaced by* (iv'):

(iv') *The first part of* (iv) *remains unchanged. The second has to be replaced by: If* $\alpha \in I$ *is fixed and* $\chi \in C^\infty(I)$ *such that* $1-\chi \in C_0^\infty(I)$ *and* $\chi = 0$ *in a neighbourhood of* α, *then* $\overline{\alpha} \mapsto \chi(\overline{\alpha})k(\alpha,\overline{\alpha})^{*_{\overline{\alpha}}}u(\overline{\alpha})$ *is in* $\operatorname{dom}(D^*)$ *for all* $u \in C^\infty(I, V)$ *that satisfy* $(u(\,\cdot\,), v) = \mathrm{const}$ *for all* $v \in V$; *here* $\Box^{*_{\overline{\alpha}}}$ *denotes the adjoint with respect to* $(.\,,.)_{\overline{\alpha}}$.

Proof. Let $(.\,,.)_c$ be a constant positive definite hermitian product on V. Then there exists a uniquely determined $A \in C^\infty(I, \operatorname{End}(V))$ such that $A(\alpha)$ is symmetric and positive definite with respect to $(.\,,.)_c$ and $(x, y)_\alpha = (A(\alpha)x\ ,\ A(\alpha)y)_c$ for all $\alpha \in I$. Let $\Box^{*_c}$ denote the adjoint with respect to $(.\,,.)_c$, $\Box^{*_\alpha}$ the adjoint with respect to $(.\,,.)_\alpha$, and $\Box^*$ the adjoint in $L^2(I, V; \|\cdot\|_\alpha, d\mu(\alpha))$. If $f: I \to V$ is a mapping, let $(Af)(\alpha) := A(\alpha)f(\alpha)$. Then A is an isometry of $L^2(I, V; \|\cdot\|_\alpha, d\mu(\alpha))$ onto $L^2(I, V; (.\,,.)_c, d\mu(\alpha))$; for simplicity let these spaces be denoted by L^2 and L^2_c. Let $D_c := A \circ D \circ A^{-1}$ be the corresponding differential operator in L^2_c. Then $\overline{D}_c = A\overline{D}A^{-1}$, $\overline{D}_c^{-1} = A\overline{D}^{-1}A^{-1}$, and $D_c^{*_c} = AD^*A^{-1}$ because A is unitary. In particular $\varrho(\overline{D}) = \varrho(\overline{D}_c)$.

(a) Let $0 \in \varrho(\overline{D}) = \varrho(\overline{D}_c)$, let k_c be the Green's function of $\overline{D}_c$ according to Theorem 1. Then

$$\begin{aligned}(\overline{D}^{-1}v)(\alpha) &= (A^{-1}\overline{D}_c^{-1}A)v(\alpha)\\ &= A(\alpha)^{-1}(\overline{D}_c^{-1}Av)(\alpha)\\ &= A(\alpha)^{-1}\int\limits_I k_c(\alpha,\overline{\alpha})A(\overline{\alpha})v(\overline{\alpha})\, d\mu(\overline{\alpha}).\end{aligned}$$

Hence the Green's function of D is $k(\alpha,\overline{\alpha}) = A(\alpha)^{-1}k_c(\alpha,\overline{\alpha})A(\overline{\alpha})$. k therefore satisfies both (i) and the first part of (ii) trivially. Now, $D_cu = ADA^{-1}u = A(aA^{-1}u)' + AbA^{-1}u = AaA^{-1}u' + A\big((aA^{-1})' + bA^{-1}\big)u$. Hence the second part of (ii) is clear, too, and so is (iii). Since $\operatorname{dom}(\overline{D}) = \operatorname{dom}(A^{-1}\overline{D}_cA) = A^{-1}\operatorname{dom}(\overline{D}_c)$, the first part of (iv) is clear. Now comes the modification. Let $\alpha \in I$ be fixed, choose $\chi \in C^\infty(I)$ such that $\chi = 0$ in a neighbourhood of α and $\chi = 1$ near the boundary of I, and $u \in V$. Then by Theorem 1 $\chi(\,\cdot\,)k_c(\alpha,\,\cdot\,)^{*_c}u \in \operatorname{dom}(\overline{D}_c^{*_c})$. Since $\operatorname{dom}(\overline{D}_c^{*_c}) = \operatorname{dom}(AD^*A^{-1}) = A\operatorname{dom}(D^*)$,

$k_c(\alpha,\overline{\alpha}) = A(\alpha)k(\alpha,\overline{\alpha})A(\overline{\alpha})^{-1}$, and by the self-adjointness of $A(\alpha)$ and $A(\overline{\alpha})$ one has $\chi(\cdot)A(\cdot)^{-1}k(\alpha,\cdot)^{*_c}A(\alpha)u \in A\,\mathrm{dom}(D^*)$. Now, $\Box^{*_c} = A(\overline{\alpha})^2\Box^{*_{\overline{\alpha}}}A(\overline{\alpha})^{-1}$. Consequently,

$$\overline{\alpha} \mapsto \chi(\overline{\alpha})k(\alpha,\overline{\alpha})^{*_{\overline{\alpha}}}A(\overline{\alpha})^{-1}u$$

is an element of $\mathrm{dom}(D^*)$ for all $u \in V$ and $\alpha \in I$. Let $u(\alpha) := A(\alpha)u$. These functions $\alpha \mapsto u(\alpha)$ are characterised by the condition that $(u(\alpha),v)_\alpha = \mathrm{const}$ for all $v \in V$. In fact, one has $(A(\alpha)^{-2}u,v)_\alpha = (A(\alpha)^{-1}u, A(\alpha)^{-1}v)_\alpha = (u,v)_c = \mathrm{const}$, and if $(u(\alpha),v)_\alpha = \mathrm{const}$, then $u := A(\alpha)^2u(\alpha) \in V$ is independent of α. This proves (a).

(b) Let k be a function satisfying (i)–(iii) and (iv'). According to the proof of (a) define $k_c(\alpha,\overline{\alpha}) := A(\alpha)k(\alpha,\overline{\alpha})A(\overline{\alpha})^{-1}$. Then by reverting the steps of (a) one shows that k_c is the Green's function of D_c using Theorem 1. Hence k is the Green's function of D.

Q.E.D.

Theorem 3 *Let I, V, U, μ, a, and b be as in Theorem 1, let $(.\,,.)_\alpha$ be a positive definite hermitian product in V, depending smoothly on α. Let $\langle .\,,.\rangle$ be a possibly indefinite hermitian product such that the endomorphisms $J(\alpha)$, defined by $\langle x,y\rangle = (J(\alpha)x,y)_\alpha$, satisfies $J(\alpha)^2 = id_V$. Let the inner product in $L^2(I,V;\|\cdot\|_\alpha, d\mu(\alpha))$, defined by integrating $\langle .\,,.\rangle$, be denoted by $\langle .\,,.\rangle$, too. Let $(Jf)(\alpha) := J(\alpha)f(\alpha)$ for $f \in L^2$. Then $\langle f,g\rangle_{L^2} = (Jf,g)_{L^2}$, $J^2 = id_{L^2}$. Let $\Box^\sharp$ denote the adjoint with respect to $\langle .\,,.\rangle$. Then:*

(a) *If $0 \in \varrho(\overline{D})$, then $0 \in \varrho(D^\sharp)$, too, and the Green's functions k and l of D and $D^\sharp$, respectively, satisfy $l(\alpha,\overline{\alpha}) = k(\overline{\alpha},\alpha)^\sharp$.*

(b) *If $\overline{D} = D^\sharp$ and $k \in L^2_{\mathrm{loc}}(I\times I,\mathrm{End}(V))$ is a function such that for all $u \in L^2$ the map $\alpha \mapsto \int_I k(\alpha,\overline{\alpha})u(\overline{\alpha})\,d\mu(\overline{\alpha})$ is again L^2 and*

- (i) *k is smooth on $(I\times I)\setminus\Delta_I$;*
- (ii) *$\big(Dk(\cdot,\overline{\alpha})\big)(\alpha) = 0$ for $\alpha \neq \overline{\alpha}$;*
- (iii) *$k(\overline{\alpha}+0,\overline{\alpha}) - k(\overline{\alpha}-0,\overline{\alpha}) = \mu(\overline{\alpha})^{-1}a(\overline{\alpha})^{-1}$ for $\overline{\alpha} \in I$;*
- (iv) *If $\overline{\alpha} \in I$ is fixed and $\chi \in C^\infty(I)$ such that $1-\chi \in C_0^\infty(I)$ and $\chi = 0$ in a neighbourhood of $\overline{\alpha}$, then $\chi(\cdot)k(\cdot,\overline{\alpha})v \in \mathrm{dom}(\overline{D})$ for all $v \in V$;*
- (v) *$k(\overline{\alpha},\alpha) = k(\alpha,\alpha)^\sharp$;*

then $0 \in \varrho(\overline{D})$ and k is the Green's function of D.

Proof. (a) If $0 \in \varrho(\overline{D})$, then $0 \in \varrho(D^\sharp)$, too, by Lemma 2.1. Let $u,v \in C_0^\infty(I,V)$. Then:

$$\begin{aligned}\langle (D^\sharp)^{-1}u\,,\,v\rangle &= \langle (\overline{D}^{-1})^\sharp u\,,\,v\rangle = \langle u, \overline{D}^{-1}v\rangle\\ &= \int_I\Big(\int_I \langle u(\alpha)\,,\,k(\alpha,\overline{\alpha})v(\overline{\alpha})\rangle\,d\mu(\overline{\alpha})\Big)\,d\mu(\alpha)\\ &= \int_I\Big(\int_I \langle k(\alpha,\overline{\alpha})^\sharp u(\alpha)\,,\,v(\overline{\alpha})\rangle\,d\mu(\overline{\alpha})\Big)\,d\mu(\alpha)\end{aligned}$$

and

$$\langle (D^\sharp)^{-1}u\,,\,v\rangle = \int_I\Big(\int_I \langle l(\alpha,\overline{\alpha})u(\overline{\alpha})\,,\,v(\alpha)\rangle\,d\mu(\overline{\alpha})\Big)\,d\mu(\alpha).$$

Now $l(\alpha,\overline{\alpha}) = k(\overline{\alpha},\alpha)^\sharp$ follows.

(b) One has to verify (ii), (iii), and (iv') of Theorem 2. By assumption $D^\sharp = \overline{D}$. Since $D^\sharp u = -a^\sharp u' + (b^\sharp - \mu^-1(\mu a^\sharp)')u$, in particular $a^\sharp = -a$ and $b^\sharp = b - \mu^{-1}(\mu a)'$.

(ii) One has (for arguments $\neq \alpha$):

$$\begin{aligned}
-(\mu k(\alpha,\,\cdot\,)a)' + \mu k(\alpha,\,\cdot\,)b &= -k(\,\cdot\,,\alpha)(\mu a)' - \mu k(\alpha,\,\cdot\,)'a + \mu k(\alpha,\,\cdot\,)(b^\sharp + \mu^{-1}(\mu a)') \\
&= -\mu k(\alpha,\,\cdot\,)'a + \mu k(\alpha,\,\cdot\,)b^\sharp \\
&= -\mu k(\alpha,\,\cdot\,)'a + \mu(bk(\,\cdot\,,\alpha))^\sharp \quad \text{because of (v)} \\
&= -\mu k(\alpha,\,\cdot\,)'a - \mu(ak(\,\cdot\,,\alpha)')^\sharp \quad \text{because of (ii)} \\
&= -\mu\Big(k(\alpha,\,\cdot\,)' - (k(\,\cdot\,,\alpha)')^\sharp\Big)a \quad \text{because } a^\sharp = -a \\
&= 0 \quad \text{because of (v).}
\end{aligned}$$

(iii) One has

$$\begin{aligned}
k(\alpha,\alpha+0) - k(\alpha,\alpha-0) &= \big(k(\alpha+0,\alpha) - k(\alpha-0,\alpha)\big)^\sharp \\
&= \big(\mu(\alpha)a(\alpha)^{-1}\big)^\sharp \\
&= -\mu(\alpha)a(\alpha)^{-1}.
\end{aligned}$$

(iv') Consider the situation of Theorem 2, so let $\alpha \in I$, $\chi \in C^\infty(I)$ such that $\chi = 0$ in a neighbourhood of α and $\chi = 1$ near the boundary of I, $u \in C^\infty(I,V)$ such that $(u(\,\cdot\,),v) = \text{const}$ for all $v \in V$. Since $(x,y)_{\overline{\alpha}} = \langle J(\overline{\alpha})^{-1}x\,,\,y\rangle$, this means $u(\overline{\alpha}) = J(\overline{\alpha})u_0$ for some $u_0 \in V$. Then

$$\begin{aligned}
\chi k(\alpha,\,\cdot\,)^* u &= \chi J k(\alpha,\,\cdot\,)^\sharp J^{-1} u \\
&= \chi J k(\,\cdot\,,\alpha) J^{-1} u \\
&= \chi J k(\,\cdot\,,\alpha) u_0.
\end{aligned}$$

It follows by assumption (iv) that $\chi k(\alpha,\,\cdot\,)^* u \in J\,\mathrm{dom}(\overline{D})$. Now, $\mathrm{dom}(\overline{D}) = J^{-1}\,\mathrm{dom}(D^*)$ because of $\overline{D} = D^\sharp = J^{-1}D^*J$, hence $\chi k(\alpha,\,\cdot\,)^* u \in \mathrm{dom}(D^*)$. **Q.E.D.**

Remark: If the product $\langle\,.\,,.\rangle$ is replaced by a variable product $\langle\,.\,,.\rangle_\alpha$, then the Green's function satisfies $k(\alpha,\overline{\alpha})^{\sharp_\alpha} = k(\alpha,\overline{\alpha})^{\sharp_{\overline{\alpha}}} = k(\overline{\alpha},\alpha)$. But this condition doesn't seem to be sufficient for k being the Green's function.

* * *

General assumption: Let p or q be even so that there's a scalar product $\langle\,.\,,.\rangle$ on $\Delta = \Delta_{p,q}^{(\pm)}$ invariant under $\boldsymbol{Spin}_+(p,q)$. Let $(\,.\,,.)$ denote the usual $\boldsymbol{Spin}(p+q)$-invariant scalar product, let $J \in \mathrm{End}(\Delta_{p,q}^{(\pm)})$ be defined by $\langle u,v\rangle = (Ju,v)$. The scalar product $\langle\,.\,,.\rangle$ on Δ induces by integration a scalar product on $L^2(S)$, again denoted by $\langle\,.\,,.\rangle$. Then $D^\sharp = \overline{D}$ with respect to this product in $L^2(S)$ (Theorem 2.1).

Returning to the determination of the resolvent of the Dirac operator consider a finite dimensional K-invariant subspace $V \subset L^2(Y,\Delta)$ such that $\overline{V}^K = V$. Let $V_\varepsilon := \{v \in V \mid$

$\forall y \in Y : v(-y) = (-1)^\varepsilon v(y)\}$, as usual. The polar coordinates $x = x(\alpha, y)$ induce an isometry of $L^2(S)_V$ onto $L^2(I, V; \|\cdot\|_\alpha, d\mu(\alpha))$ where $I = (1, \infty)$, $\mu(\alpha) = \alpha^{-1}\bigl(\frac{1}{2}(\alpha - \alpha^{-1})\bigr)^{p-1} \times \bigl(\frac{1}{2}(\alpha + \alpha^{-1})\bigr)^{q-1}$, and $\|v\|_\alpha^2 = \frac{1}{2}(\alpha + \alpha^{-1})\|v\|_{L^2}^2 - \frac{1}{2}(\alpha - \alpha^{-1})(v, \mathscr{I}v)_{L^2}$. Let U denote the subspace of $L^2(I, V)$ that corresponds to $C_0^\infty(S) \cap L^2(S)_V$ under this identification. The Dirac operator D transforms into an ordinary differential operator, again denoted by D, $Dv = av' + bv$ with $a(\alpha) = \alpha\mathscr{I}$, $b(\alpha) = -\frac{1}{2}n + (1 + \frac{\alpha+\alpha^{-1}}{\alpha-\alpha^{-1}}\mathscr{I})D' - (1 + \frac{\alpha-\alpha^{-1}}{\alpha+\alpha^{-1}}\mathscr{I})D''$ (cf. Lemma 3.3). Let $V_\varepsilon^{\pm\mathscr{I}} := \{v \in V_\varepsilon \mid \mathscr{I}v = \pm v\}$. One has $V_\varepsilon = V_\varepsilon^{\mathscr{I}} \oplus V_\varepsilon^{-\mathscr{I}}$ because $\mathscr{I}^2 = 1$. Now, $\mathscr{I}$ is skew with respect to $\langle .\,,.\rangle$ because $\langle y'y'' \cdot u\,,\, v\rangle = -\langle u\,,\, y'y'' \cdot v\rangle$ for all $y = y' + y'' \in Y$. Hence the eigenspaces of $\mathscr{I}$ in V_ε are totally isotropic. Since $\langle .\,,.\rangle$ is non-degenerate on V_ε (because $V = V_0 \oplus V_{\mathbb{1}}$ and $V_0 \perp_{\langle .\,,.\rangle} V_1$), $\langle .\,,.\rangle$ is a non-degenerate pairing between $V_\varepsilon^{\mathscr{I}}$ and $V_\varepsilon^{-\mathscr{I}}$. In particular, $\dim V_\varepsilon^{\mathscr{I}} = \dim V_\varepsilon^{-\mathscr{I}} = \frac{1}{2}\dim V_\varepsilon$. Let $\overline{D}_{V_\varepsilon} := \overline{D}\big|_{L^2(S)_{V_\varepsilon}}$.

Let $\nu \in \mathbb{C} \setminus \frac{1}{2}\mathbb{Z}$. For such a ν there are the mappings $\Phi_\nu|_{V_\varepsilon}$, $\Psi_\nu^\pm\big|_{V_\varepsilon^{\pm\mathscr{I}}}$, and $F_{\varepsilon,\nu}\big|_{V_\varepsilon^{\mathscr{I}}}$ to $C^\infty(S')_K$ (remember: $S' := S\big|_{X'}$). These maps correspond to families of homomorphisms of V_ε, $V_\varepsilon^{\pm\mathscr{I}}$, and $V_\varepsilon^{\mathscr{I}}$, respectively, to V_ε:

$$\begin{aligned}(\Phi_\nu v)(x(\alpha, y)) &= (\varphi_\nu(\alpha)v)(y)\\ (\Psi_\nu^\pm v)(x(\alpha, y)) &= (\psi_\nu^\pm(\alpha)v)(y)\\ (F_{\varepsilon,\nu}v)(x(\alpha, y)) &= (f_{\varepsilon,\nu}(\alpha)v)(y)\end{aligned}$$

for $\alpha \in I$ and $y \in Y$. The Lemmas 3.7 and 3.9 can now be formulated as follows:

Lemma 1 (a) *The eigenspinors $f \in C^\infty(I, V_\varepsilon)$ of $D + \frac{1}{2}$ that are square integrable at infinity are exactly:*

$\operatorname{Re}\nu > \frac{1}{2}\ :\ f(\alpha) = \psi_\nu^-(\alpha)v,\ \mathscr{I}v = -v,\ v \in V_\varepsilon;$

$\operatorname{Re}\nu = \frac{1}{2}\ :\ f = 0;$

$\operatorname{Re}\nu < \frac{1}{2}\ :\ f(\alpha) = \psi_\nu^+(\alpha)v,\ \mathscr{I}\varphi = \varphi,\ v \in V_\varepsilon.$

(b) *The eigenspinors $f \in C^\infty(I, V_\varepsilon)$ of $D + \frac{1}{2}$ for eigenvalue $\nu \in \mathbb{C} \setminus \frac{1}{2}\mathbb{Z}$ whose pull-backs to X are smooth are exactly*

$$f(\alpha) = f_{\varepsilon,\nu}(\alpha)v, \quad v \in V_\varepsilon.$$

Let $f_1^\nu := f_{\varepsilon,\nu}$ and $f_2^\nu := \psi_\nu^\pm$ (according to $\operatorname{Re}\nu \lessgtr \frac{1}{2}$). Then $f_1^\nu(\alpha) \in \operatorname{Hom}(V_\varepsilon^{\mathscr{I}}, V_\varepsilon)$, $f_2^\nu(\alpha) \in \operatorname{Hom}(V_\varepsilon^{\pm\mathscr{I}}, V_\varepsilon)$. Since $V_\varepsilon^{\mathscr{I}}$ and $V_\varepsilon^{-\mathscr{I}}$ are dual under $\langle .\,,.\rangle$, one can define the adjoint homomorphisms $f_1^\nu(\alpha)^\sharp \in \operatorname{Hom}(V_\varepsilon, V_\varepsilon^{-\mathscr{I}})$ and $f_2^\nu(\alpha)^\sharp \in \operatorname{Hom}(V_\varepsilon, V_\varepsilon^{\mp\mathscr{I}})$. Let $V_{\varepsilon,1} := V_\varepsilon^{\mathscr{I}}$ and $V_{\varepsilon,2} := V_\varepsilon^{\pm\mathscr{I}}$, and also $V_{\varepsilon,1}^\sharp := V_\varepsilon^{-\mathscr{I}}$ and $V_{\varepsilon,2}^\sharp := V_\varepsilon^{\pm\mathscr{I}}$ (that is, $V_{\dots}^\sharp$ is the dual $V_{\dots}$ of with respect to $\langle .\,,.\rangle$).

Lemma 2 *The homomorphism $a_{ij}(\nu) := \mu(\alpha)f_j^{\overline{\nu}}(\alpha)^\sharp a(\alpha)f_i^\nu(\alpha) \in \operatorname{Hom}(V_{\varepsilon,i}, V_{\varepsilon,j}^\sharp)$ ($\overline{\nu}$ being the complex conjugate of ν) is independent of α. One has $a_{ij}(\nu)^\sharp = -a_{ji}(\overline{\nu})$.*

Proof. $D = a\frac{d}{d\alpha} + b$ is (formally) self-adjoint with respect to $\langle .\,,.\rangle$, hence $a^\sharp = -a$ and $b^\sharp = b - \mu^{-1}(\mu a)'$. Then for arbitrary $u \in V_{\varepsilon,i}$ and $v \in V_{\varepsilon,j}$ and $\alpha_1, \alpha_2 \in I$ one has:

$$\begin{aligned}
\int_{\alpha_1}^{\alpha_2} \langle f_i^\nu(\alpha)u\,,\, f_j^{\overline{\nu}}(\alpha)v\rangle\, d\mu(\alpha) &= \frac{1}{\nu}\int_{\alpha_1}^{\alpha_2} \langle (a(f_i^\nu)' + bf_i^\nu)u\,,\, f_j^{\overline{\nu}}v\rangle(\alpha)\, d\mu(\alpha)\\
&= \frac{1}{\nu}\int_{\alpha_1}^{\alpha_2} \big(\mu\langle af_i^\nu u\,,\, f_j^{\overline{\nu}}v\rangle\big)'\, d\alpha\\
&\quad + \frac{1}{\nu}\int_{\alpha_1}^{\alpha_2} \big(\langle (b - \mu^{-1}(\mu a)')f_i^\nu u\,,\, f_j^{\overline{\nu}}v\rangle - \langle af_i^\nu u\,,\, (f_j^{\overline{\nu}})'v\rangle\big)\, d\mu(\alpha)\\
&= \frac{1}{\nu}\mu\langle af_i^\nu u\,,\, f_j^{\overline{\nu}}v\rangle\Big|_{\alpha_1}^{\alpha_2} + \frac{1}{\nu}\int_{\alpha_1}^{\alpha_2}\langle f_i^\nu u\,,\, a(f_j^{\overline{\nu}})'v + bf_j^{\overline{\nu}}v\rangle\mu\, d\alpha\\
&= \frac{1}{\nu}\,\mu\langle af_i^\nu u\,,\, f_j^{\overline{\nu}}v\rangle\Big|_{\alpha_1}^{\alpha_2} + \frac{1}{\nu}\int_{\alpha_1}^{\alpha_2}\langle f_i^\nu u\,,\, D(f_j^{\overline{\nu}}v)\rangle\mu\, d\alpha\\
&= \frac{1}{\nu}\,\mu\langle af_i^\nu u\,,\, f_j^{\overline{\nu}}v\rangle\Big|_{\alpha_1}^{\alpha_2} + \int_{\alpha_1}^{\alpha_2}\langle f_i^\nu u, f_j^{\overline{\nu}}v\rangle\mu\, d\alpha,
\end{aligned}$$

which proves the first assertion. The second follows immediately from the definition using $a^\sharp = -a$. **Q.E.D.**

Now one can construct the Green's function k_ν of $D + \frac{1}{2} - \nu$ for $\nu \in \mathbb{C} \setminus \frac{1}{2}\mathbb{Z}$, $\operatorname{Re}\nu \neq \frac{1}{2}$, following the lines of Theorem 3. Assume that k_ν exists, that is, $\nu - \frac{1}{2} \in \varrho(\overline{D})$. By Theorem 3 (ii) and (iv) it is reasonable to assume that

$$k_\nu(\alpha, \overline{\alpha}) = \begin{cases} f_{\varepsilon,\nu}\, b_1(\overline{\alpha}, \nu) & \text{if } \alpha < \overline{\alpha};\\ \psi_\nu^\pm(\alpha)\, b_2(\overline{\alpha}, \nu) & \text{if } \alpha > \overline{\alpha},\end{cases}$$

for properly chosen $b_1(\overline{\alpha}, \nu) \in \operatorname{Hom}(V_\varepsilon, V_\varepsilon^{\mathscr{I}})$ and $b_2(\overline{\alpha}, \nu) \in \operatorname{Hom}(V_\varepsilon, V_\varepsilon^{\pm\mathscr{I}})$. In fact, for $\alpha > \overline{\alpha}$ this is the only possibility according to Theorem 3 (ii) and Lemma 1. Now, by Theorem 3 one has

$$b_1(\overline{\alpha}, \nu)^\sharp f_{\varepsilon,\nu}(\alpha)^\sharp = \psi_{\overline{\nu}}^\pm(\overline{\alpha}) b_2(\alpha, \overline{\nu}). \tag{8}$$

$f_{\varepsilon,\nu}(\alpha)$ and $\psi_\nu^\pm(\alpha)$ are injective for all $\alpha \in I$. Indeed, if $f_{\varepsilon,\nu}(\alpha)v = 0$ for some α, then $f_{\varepsilon,\nu}(\alpha)v = 0$ for *all* α because $\alpha \mapsto f_{\varepsilon,\nu}(\alpha)v$ is the solution of an ordinary elliptic first order differential equation. The same argument applies for $\psi_\nu^\pm$. Therefore $f_{\varepsilon,\nu}(\alpha)^\sharp$ and $\psi_\nu^\pm(\alpha)^\sharp$ are surjective. Because of (8) $b_1(\overline{\alpha}, \nu)$ takes its values in the range of $\psi_\nu^\pm(\overline{\alpha})$. Hence there's a (unique) $b(\alpha, \nu) \in \operatorname{Hom}(V_\varepsilon^{\mp\mathscr{I}}, V_\varepsilon)$ such that $b_1(\alpha, \nu)^\sharp = \psi_\nu^\pm(\alpha) b(\alpha, \nu)$. Then (8) is equivalent to

$$\psi_{\overline{\nu}} \pm (\overline{\alpha}) b(\overline{\alpha}, \nu)^\sharp f_{\varepsilon,\nu}(\alpha)^\sharp = \psi_{\overline{\nu}}^\pm(\overline{\alpha}) b_2(\alpha, \overline{\nu}),$$

hence

$$b(\overline{\alpha}, \nu)^\sharp f_{\varepsilon,\nu}(\alpha)^\sharp = b_2(\alpha, \overline{\nu}).$$

Since the right side is independent of $\overline{\alpha}$ and since $f_{\varepsilon,\nu}(\alpha)^\sharp$ is surjective, it follows that $b(\overline{\alpha},\nu) =: b(\nu)$ is independent of $\overline{\alpha}$ and $b_2(\alpha,\nu) = b(\nu)^\sharp f_{\varepsilon,\nu}(\alpha)^\sharp$. Thus one has to find a $b(\nu) \in \mathrm{Hom}(V_\varepsilon, V_\varepsilon^{\mp\mathscr{I}}$ such that

$$k_\nu(\alpha,\overline{\alpha}) = \begin{cases} f_{\varepsilon,\nu}(\alpha)\, b(\nu)\, \psi_{\overline{\nu}}^{\pm}(\overline{\alpha})^\sharp & \text{if } \alpha < \overline{\alpha}; \\ \psi_\nu^{\pm}(\alpha)\, b(\overline{\nu})^\sharp\, f_{\varepsilon,\overline{\nu}}(\overline{\alpha})^\sharp & \text{if } \alpha > \overline{\alpha}; \end{cases} \tag{9}$$

is the Green's function. By Theorem 3 two more conditions are to be satisfied namely the "jump condition" (iii) of this theorem and a square integrability.

The "jump condition" can be expressed using (9) as follows:

$$\psi_\nu^{\pm}(\alpha)\, b(\overline{\nu})^\sharp f_{\varepsilon,\overline{\nu}}(\alpha)^\sharp - f_{\varepsilon,\nu}(\alpha)\, b(\nu)\, \psi_{\overline{\nu}}^{\pm}(\alpha)^\sharp = \mu(\alpha)^{-1} a(\alpha)^{-1}. \tag{10}$$

Multiplying this equation by $\mu(\alpha) f_i^{\overline{\nu}}(\alpha)^\sharp a(\alpha)$ $(i = 1,2)$ from the left one obtains

$$a_{2i}(\nu)\, b(\overline{\nu})^\sharp f_{\varepsilon,\overline{\nu}}(\alpha)^\sharp - a_{1i}(\nu)\, b(\nu)\, \psi_{\overline{\nu}}^{\pm}(\alpha)^\sharp = f_i^{\overline{\nu}}(\alpha)^\sharp. \tag{11}$$

These two equations $(i = 1,2)$ are equivalent to (10). To see this it suffices to show that $\ker f_1^{\overline{\nu}}(\alpha)^\sharp \cap f_2^{\overline{\nu}}(\alpha)^\sharp = \big(\mathrm{im}\, f_1^{\overline{\nu}}(\alpha) + \mathrm{im}\, f_2^{\overline{\nu}}(\alpha)\big)^{\perp\langle\cdot,\cdot\rangle} = 0$. For this it suffices to show that $\mathrm{im}\, f_1^{\overline{\nu}}(\alpha) \cap \mathrm{im}\, f_2^{\overline{\nu}}(\alpha) = 0$ since then $\dim\big(\mathrm{im}\, f_1^{\overline{\nu}}(\alpha) + \mathrm{im}\, f_2^{\overline{\nu}}(\alpha)\big) = \dim V_\varepsilon^{\mathscr{I}} + \dim V_\varepsilon^{\pm\mathscr{I}} = \dim V_\varepsilon$. Suppose therefore that $x \in \mathrm{im}\, f_1^{\overline{\nu}}(\alpha) \cap \mathrm{im}\, f_2^{\overline{\nu}}(\alpha)$. Then there exist $y,z \in V$ such that $x = f_1^{\overline{\nu}}(\alpha)y = f_2^{\overline{\nu}}(\alpha)z$. But then $f_1^{\overline{\nu}}(\alpha)y = f_2^{\overline{\nu}}(\alpha)z$ for *all* $\alpha \in I$ because both sides of this equation are solutions of an ordinary differential equation (that is first order and elliptic). Now, the left hand side of this equation can be lifted to a smooth eigenspinor of D on X, while the right hand side is square integrable at infinity. Therefore the lift of this function to X is a smooth L^2-eigenspinor of D for eigenvalue $\overline{\nu} - \frac{1}{2} \notin \frac{1}{2}\mathbb{Z}$. Consequently, $f_1^{\overline{\nu}}(\alpha)y = f_2^{\overline{\nu}}(\alpha)z = 0$ for all α; in particular $x = 0$.

Multiplying (11) by $\mu(\alpha)a(\alpha)f_j^{\nu}(\alpha)$ from the right one obtains

$$a_{2i}(\nu)\, b(\overline{\nu})^\sharp a_{ji}(\nu) - a_{1i}(\nu)\, b(\nu)\, a_{j2}(\nu) = a_{ji}(\nu). \tag{12}$$

In the same vein one can show that these four equations $(i,j = 1,2)$ are equivalent to the single equation (10).

Now one has to distinguish the two cases $\mathrm{Re}\,\nu > \frac{1}{2}$ and $\mathrm{Re}\,\nu < \frac{1}{2}$.

- Let $\mathrm{Re}\,\nu > \frac{1}{2}$. Then $f_2^\nu = \psi_\nu^-$. From Lemma 3.5 it follows that for $\alpha \to \infty$

$$f_2^\nu(\alpha) \sim \alpha^{-\varrho-\nu}\big(1 + \nu^{-1}(D' - D'' - \varrho)\big)\; i_{V_\varepsilon^{-\mathscr{I}}} \in \mathrm{Hom}(V_\varepsilon^{-\mathscr{I}}, V_\varepsilon)$$

where $i_{V_\varepsilon^{-\mathscr{I}}}$ is the inclusion map $V_\varepsilon^{-\mathscr{I}} \hookrightarrow V_\varepsilon$. Hence

$$f_2^{\overline{\nu}}(\alpha)^\sharp \sim \alpha^{-\varrho-\nu}\, pr_{V_\varepsilon^{\mathscr{I}}} \circ \big(1 + \nu^{-1}(D' - D'' - \varrho)^\sharp\big) \in \mathrm{Hom}(V_\varepsilon, V_\varepsilon^{\mathscr{I}})$$

($pr_{V_\varepsilon^{\mathscr{I}}} = (i_{V_\varepsilon^{-\mathscr{I}}})^\sharp$ is the projection onto $V_\varepsilon^{\mathscr{I}}$ with respect to $V_\varepsilon = V_\varepsilon^{\mathscr{I}} \oplus V_\varepsilon^{-\mathscr{I}}$). According to Lemma 1.3 one has

$$f_{\varepsilon,\nu}(\alpha) = c(\varepsilon,\nu)\, \varphi_\nu(\alpha) + \varphi_{-\nu}(\alpha)\, C_-(\varepsilon,\nu), \tag{13}$$

and from Lemma 1.1 it follows that

$$f_1^{\nu}(\alpha) = f_{e_{p+q},s\nu}(\alpha) \sim c(\varepsilon,\nu)\,\alpha^{-\varrho+\nu}\, i_{V_\varepsilon^{\mathscr{I}}} \in \mathrm{Hom}(V_\varepsilon^{\mathscr{I}}, V_\varepsilon)$$

and therefore

$$f_1^{\overline{\nu}}(\alpha)^\sharp \sim c(\varepsilon,\nu)\,\alpha^{-\varrho+\nu}\, pr_{V_\varepsilon^{-\mathscr{I}}} \in \mathrm{Hom}(V_\varepsilon, V_\varepsilon^{-\mathscr{I}})$$

(note that $c(\varepsilon,\overline{\nu}) = \overline{c(\varepsilon,\nu)}$ because of Lemma 1.4). Furthermore $\mu(\alpha) \sim 2^{-2\varrho}\alpha^{2\varrho-1}$ and $a(\alpha) = \alpha\mathscr{I}$. Hence:

$$a_{11}(\nu) = \lim_{\alpha\to\infty} 2^{-2\varrho}\alpha^{2\varrho} f_{\varepsilon,\overline{\nu}}(\alpha)^\sharp \mathscr{I} f_{\varepsilon,\nu}(\alpha). \tag{14}$$

To analyse this expression, note that $f_{\varepsilon,\nu} = c(\varepsilon,\nu)\varphi_\nu + \varphi_{-\nu}C_-(\varepsilon,\nu)$ and that $\varphi_{\pm\nu}$ can be developed in a series of powers $\alpha^{-\varrho\pm\nu-k}$, $k \in \mathbb{N}$. Since the limit (14) exists, one needs to consider only those powers of α whose product yields a constant; all powers of higher order are automatically zero, the lower orders are irrelevant. Now the powers that occur on the right hand side of (14) are exactly $\alpha^{2\nu-k}$, α^{-k}, and $\alpha^{-2\nu-k}$, $k \in \mathbb{N}$. Since $2\nu \notin \mathbb{Z}$, one needs to consider only the powers $\alpha^{-\varrho\pm\nu}$ in $f_{\varepsilon,\nu}$ and $f_{\varepsilon,\overline{\nu}}^\sharp$. Eventually:

$$\begin{aligned} a_{11}(\nu) &= 2^{-2\varrho}\, pr_{V_\varepsilon^{-\mathscr{I}}} \left(c(\varepsilon,\nu)\mathscr{I} C_-(\varepsilon,\nu) + C_-(\varepsilon,\overline{\nu})^\sharp \mathscr{I} c(\varepsilon,\nu)\right) i_{V_\varepsilon^{\mathscr{I}}} \\ &= 2^{-2\varrho} c(\varepsilon,\nu)\, pr_{V_\varepsilon^{-\mathscr{I}}} \left(-C_-(\varepsilon,\nu) + C_-(\varepsilon,\overline{\nu})^\sharp\right) i_{V_\varepsilon^{\mathscr{I}}}\,. \end{aligned}$$

Now, $C_-(\varepsilon,\overline{\nu})^\sharp = C_-(\varepsilon,\nu)$, for

$$C_-(\varepsilon,\nu)u(y) = 2^{\varrho+\nu} \int\limits_Y |\langle y,\overline{y}\rangle|^{-\varrho-\nu} \,\mathrm{sign}(\langle y,\overline{y}\rangle)^\varepsilon u(\overline{y})\, d\overline{y},$$

according to Lemma 1.3 if $\mathrm{Re}\,\nu < \mathrm{Re}\,\varrho$. Hence

$$\begin{aligned} \langle C_-(\varepsilon\,,\ \nu)u, v\rangle &= 2^{\varrho+\nu} \int\limits_Y\int\limits_Y |\langle y,\overline{y}\rangle|^{-\varrho-\nu}\,\mathrm{sign}(\langle y,\overline{y}\rangle)^\varepsilon \langle u(\overline{y})\,,\ v(y)\rangle\, d\overline{y}\, dy \\ &= \langle u\,,\ C_-(\varepsilon,\overline{\nu})v\rangle. \end{aligned}$$

For all other ν (in particular those with $\mathrm{Re}\,\nu > \frac{1}{2}$) the assertion follows from the meromorphicity of $C_-(\varepsilon,\nu)$. Thus $a_{11}(\nu) = 0$. The computation of the other a_{ij} is simpler. From the asymptotics it follows immediately that $a_{21}(\nu) = 2^{-2\varrho}c(\varepsilon,\nu)\, i_{V_\varepsilon^{-\mathscr{I}}}$ since $D' - D'' - \varrho$ anticommutes with $\mathscr{I}$, and therefore $a_{12}(\nu) = -a_{21}(\overline{\nu})^\sharp = -2^{-2\varrho}c(\varepsilon,\nu)\, i_{V_\varepsilon^{\mathscr{I}}}$. Furthermore $a_{22}(\nu) = 0$. Putting this into (12) one obtains

$$b(\nu) = 2^{2\varrho}c(\varepsilon,\nu)^{-1},$$

hence the Green's function is given by

$$k_\nu(\alpha,\overline{\alpha}) = 2^{2\varrho}c(\varepsilon,\nu)^{-1}\cdot \begin{cases} f_{\varepsilon,\nu}(\alpha)\,\psi_{\overline{\nu}}^-(\overline{\alpha})^\sharp & \text{if } \alpha < \overline{\alpha}, \\ \psi_\nu^-(\alpha)\, f_{\varepsilon,\overline{\nu}}(\overline{\alpha})^\sharp & \text{if } \alpha > \overline{\alpha}, \end{cases}$$

provided that the integral operator with kernel k_ν maps $L^2(I, V_\varepsilon; \|\cdot\|_\alpha, d\mu(\alpha))$ into itself.

• Let $\operatorname{Re}\nu < \frac{1}{2}$. Then $f_2^\nu(\alpha) = \psi_\nu^+(\alpha) \in \operatorname{Hom}(V_\varepsilon^{\mathscr{I}}, V_\varepsilon)$, $f_2^\nu(\alpha) \sim \alpha^{-\varrho+\nu}\, i_{V_\varepsilon^{\mathscr{I}}}$ as $\alpha \to \infty$, and $f_2^{\overline{\nu}}(\alpha)^\sharp \sim \alpha^{-\varrho+\nu}\, pr_{V_\varepsilon^{-\mathscr{I}}}$. The computation of a_{11} is the same as for $\operatorname{Re}\nu > \frac{1}{2}$, again one has $a_{11} = 0$. Also $a_{22} = 0$ because the highest order term is $\alpha^{-2\varrho}\, pr_{V_\varepsilon^{-\mathscr{I}}} \circ i_{V_\varepsilon^{\mathscr{I}}} = 0$. The asymptotics imply $a_{12}(\nu) = -2^{-2\varrho}\, pr_{V_\varepsilon^{-\mathscr{I}}}\, C_-(\varepsilon,\nu)\, i_{V_\varepsilon^{\mathscr{I}}} = -a_{21}(\nu)$. Hence the only condition on $b(\nu)$ according to (12) is

$$2^{-2\varrho}\, pr_{V_\varepsilon^{-\mathscr{I}}}\, C_-(\varepsilon,\nu)\, i_{V_\varepsilon^{\mathscr{I}}}\, b(\nu)\, pr_{V_\varepsilon^{-\mathscr{I}}}\, C_-(\varepsilon,\nu)\, i_{V_\varepsilon^{\mathscr{I}}} = pr_{\varepsilon^{-\mathscr{I}}}\, C_-(\varepsilon,\nu)\, i_{V_\varepsilon^{\mathscr{I}}},$$

which means that

$$b(\nu) = 2^{2\varrho}\bigl(pr_{V_\varepsilon^{-\mathscr{I}}}\, C_-(\varepsilon,\nu)\, i_{V_\varepsilon^{\mathscr{I}}}\bigr)^{-1} \in \operatorname{Hom}(V_\varepsilon^{-\mathscr{I}}, V_\varepsilon^{\mathscr{I}}),$$

provided that the right hand side were defined. To see this consider a $v \in V_\varepsilon^{\mathscr{I}}$. Then $F_{\varepsilon,\nu}v \in C^\infty(S)_K$ is an eigenspinor of $D + \frac{1}{2}$ for eigenvalue ν. Since $F_{\varepsilon,\nu}v = \Phi_\nu c(\varepsilon,\nu)v + \Phi_{-\nu}C(\varepsilon,\nu)v$ and $\Phi_\nu = \Psi_\nu^+$, $\Phi_{-\nu}C(\varepsilon,\nu)v$ is an eigenspinor, too (although only defined on S'). Then (the proof of) Lemma 3.5 shows that $C_-(\varepsilon,\nu)v = v' + \nu^{-1}(D' - D'' - \varrho)v'$ for some $v' \in V_\varepsilon^{-\mathscr{I}}$. Then $(D' - D'' - \varrho)v' \in V_\varepsilon^{\mathscr{I}}$ implies $v' = pr_{V_\varepsilon^{-\mathscr{I}}}\, C_-(\varepsilon,\nu)v$. If then $pr_{V_\varepsilon^{-\mathscr{I}}}\, C_-(\varepsilon,\nu)v = 0$, this would imply that $C_-(\varepsilon,\nu)v = 0$, and therefore $v = 0$ since $C_-(\varepsilon,\nu)$ is injective by Lemma 1.4. Hence $pr_{V_\varepsilon^{-\mathscr{I}}}\, C_-(\varepsilon,\nu)$ is an injective mapping of $V_\varepsilon^{\mathscr{I}}$ into $V_\varepsilon^{-\mathscr{I}}$. Since these spaces have the same dimension, $pr_{V_\varepsilon^{-\mathscr{I}}}\, C_-(\varepsilon,\nu)$ is bijective. Furthermore,

$$\bigl(pr_{V_\varepsilon^{-\mathscr{I}}}\, C_-(\varepsilon,\nu)\, i_{V_\varepsilon^{\mathscr{I}}}\bigr)^{-1} = C_-(\varepsilon,\nu)^{-1} \circ \bigl(1 + \nu^{-1}(D' - D'' - \varrho)\bigr) \in \operatorname{Hom}(V_\varepsilon^{-\mathscr{I}, V_\varepsilon^{\mathscr{I}}}).$$

Now, $C_-(\varepsilon,\overline{\nu})^\sharp = C_-(\varepsilon,\nu)$, hence

$$\begin{aligned} k_\nu(\alpha,\overline{\alpha}) &= 2^{2\varrho}\cdot\begin{cases} f_{\varepsilon,\nu}(\alpha)\,\bigl(pr_{V_\varepsilon^{-\mathscr{I}}}\, C_-(\varepsilon,\nu)\, i_{V_\varepsilon^{\mathscr{I}}}\bigr)^{-1}\psi_{\overline{\nu}}^+(\overline{\alpha})^\sharp & \text{if } \alpha < \overline{\alpha}, \\ \psi_\nu^+(\alpha)\,\bigl(pr_{V_\varepsilon^{-\mathscr{I}}}\, C_-(\varepsilon,\nu)\, i_{V_\varepsilon^{\mathscr{I}}}\bigr)^{-1} f_{\varepsilon,\overline{\nu}}(\overline{\alpha})^\sharp & \text{if } \alpha > \overline{\alpha}; \end{cases} \\ &= 2^{2\varrho}\cdot\begin{cases} f_{\varepsilon,\nu}(\alpha)\, C_-(\varepsilon,\nu)^{-1}\bigl(1 + \nu^{-1}(D' - D'' - \varrho)\bigr)\,\psi_{\overline{\nu}}^+(\overline{\alpha})^\sharp & \text{if } \alpha < \overline{\alpha}, \\ \psi_\nu^+(\alpha)\,\Bigl(f_{\varepsilon,\overline{\nu}}(\overline{\alpha})\, C_-(\varepsilon,\overline{\nu})^{-1}\bigl(1 + \nu^{-1}(D' - D'' - \varrho)\bigr)\Bigr)^\sharp & \text{if } \alpha > \overline{\alpha}; \end{cases} \end{aligned}$$

provided that the corresponding integral operator maps L^2 into itself.

Now one has to show that the integral operators map L^2 into itself. Following a familiar pattern, the first step is a simple estimate:

Lemma 3 *One has*

$$\|\psi_\nu^\pm(\alpha)\| \le \text{const}\,(\alpha - 1)^{-1}$$

as $\alpha \downarrow 1$ *;* $\|\psi_\nu^\pm(\alpha)\|$ *denotes the operator norm of* $\psi_\nu^\pm(\alpha) \in \operatorname{Hom}(V_\varepsilon^{\pm\mathscr{I}}, V_\varepsilon)$.

Proof. One has $(D + \frac{1}{2} - \nu)\psi_\nu^\pm(\,\cdot\,)v = 0$ for all $v \in V_\varepsilon^{\pm\mathscr{I}}$; let $f(\alpha) := \psi_\nu^\pm(\alpha)v$. Then $f' = -a^{-1}(b - \nu)f$ (where $D = a(\,\cdot\,)' + b$). Hence $(\|f\|^2)' = 2\operatorname{Re}(f, f') \ge -2\|f\|\cdot\|f'\| \ge$

$-2\|a^{-1}(b-\nu)\|\cdot\|f\|^2$. Consequently $(\ln\|f\|)' \geq -\|a^{-1}(b-\nu)\|$. By integrating this inequality and applying the exponential function one gets for $\alpha' < \alpha''$:

$$\|f(\alpha')\| \leq \|f(\alpha'')\| \cdot \exp \int_{\alpha'}^{\alpha''} \|a^{-1}(b-\nu)\|.$$

It is easily seen for the Dirac operator that $\|a(\alpha)^{-1}(b(\alpha)-\nu)\| \leq \mathrm{const}\,(\alpha'')(\alpha-1)^{-1}$ for $\alpha \leq \alpha''$. Putting this into the inequality above one obtains

$$\|f(\alpha)'\| \leq \mathrm{const}(\alpha'') \cdot \|f(\alpha'')\|\,(\alpha'-1)^{-1},$$

hence

$$\|\psi_\nu^\pm(\alpha')v\| \leq \mathrm{const}(\alpha'') \cdot \|\psi_\nu^\pm(\alpha'')v\|\,(\alpha'-1)^{-1},$$

and therefore

$$\|\psi_\nu^\pm(\alpha')\| \leq \mathrm{const}(\alpha'') \cdot \|\psi_\nu^\pm(\alpha'')\|\,(\alpha'-1)^{-1}$$

for $\alpha' \leq \alpha''$. **Q.E.D.**

Define now two integral operators:

$$K_{\mathrm{I}}u(\alpha) := \psi_\nu^\pm(\alpha)\,b(\overline{\nu})^\sharp \int_1^\alpha f_{\varepsilon,\overline{\nu}}(\overline{\alpha})^\sharp u(\overline{\alpha})\,d\mu(\overline{\alpha}),$$

$$K_{\mathrm{II}}u(\alpha) := f_{\varepsilon,\nu}(\alpha)\,b(\nu) \int_\alpha^\infty \psi_{\overline{\nu}}^\pm(\overline{\alpha})^\sharp u(\overline{\alpha})\,d\mu(\overline{\alpha}).$$

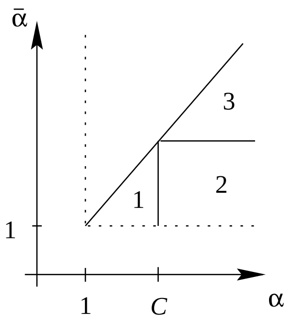
Then $K_{\mathrm{I}}u + K_{\mathrm{II}}u = Ku := \int_I k_\nu(\,\cdot\,,\overline{\alpha})u(\overline{\alpha})\,d\mu(\overline{\alpha})$. It suffices to show that K_{I} maps L^2 into itself because the kernel of K_{II} corresponding to ν is the adjoint (with respect to $\langle.\,,.\rangle$) of the kernel of K_{I} corresponding to $\overline{\nu}$, hence $K_{\mathrm{II},\nu} = K_{\mathrm{I},\overline{\nu}}^\sharp$.

Let $C > 1$ be arbitrary. Then K_{I} is split, too, now into three parts K_1, K_2, and K_3. Let

$$K_1u(\alpha) := \begin{cases} \psi_\nu^\pm(\alpha)\,b(\overline{\nu})^\sharp \displaystyle\int_1^\alpha f_{\varepsilon,\overline{\nu}}(\overline{\alpha})^\sharp u(\overline{\alpha})\,d\mu(\overline{\alpha}) & \text{if } \alpha < C, \\ 0 & \text{if } \alpha > C; \end{cases}$$

$$K_2u(\alpha) := \begin{cases} 0 & \text{if } \alpha < C, \\ \psi_\nu^\pm(\alpha)\,b(\overline{\nu})^\sharp \displaystyle\int_1^C f_{\psi,\overline{\nu}}(\overline{\alpha})^\sharp u(\overline{\alpha})\,d\mu(\overline{\alpha}) & \text{if } \alpha > C; \end{cases}$$

$$K_3u(\alpha) := \begin{cases} 0 & \text{if } \alpha < C, \\ \psi_\nu^\pm(\alpha)\,b(\overline{\nu})^\sharp \displaystyle\int_1^C f_{\varepsilon,\overline{\nu}}(\overline{\alpha})^\sharp u(\overline{\alpha})\,d\mu(\overline{\alpha}) & \text{if } \alpha > C. \end{cases}$$



Then $K_{\mathrm{I}} = K_1 + K_2 + K_3$. (The reason for this splitting of K_{I} is the decomposition of $\{(\alpha,\overline{\alpha}) \mid \alpha > \overline{\alpha}\}$ into three parts—see the right picture.)

• For $\alpha < C$ one has:

$$\begin{aligned}
\|K_{\mathrm{I}}u(\alpha)\|_\alpha &\leq \|\psi_\nu^\pm(\alpha)\|_\alpha \|b(\overline{\nu})^\sharp\|_\alpha \int_1^\alpha \|f_{\varepsilon,\overline{\nu}}(\overline{\alpha})^\sharp\|_{\overline{\alpha}} \|u(\overline{\alpha})\|_{\overline{\alpha}}\, d\mu(\overline{\alpha}) \\
&\leq \|\psi_\nu^\pm(\alpha)\|_\alpha \|b(\overline{\nu})^\sharp\|_\alpha \|u\|_{L^2} \left(\int_1^\alpha \|f_{\varepsilon,\overline{\nu}}(\overline{\alpha})^\sharp\|_{\overline{\alpha}}^2\, d\mu(\overline{\alpha}) \right)^{\frac{1}{2}} \\
&\leq \|\psi_\nu^\pm(\alpha)\|_\alpha \|b(\overline{\nu})^\sharp\|_\alpha \|u\|_{L^2} \cdot (\alpha - 1) \max_{1 \leq \overline{\alpha} \leq C} \|f_{\varepsilon,\overline{\nu}}(\overline{\alpha})^\sharp \mu(\overline{\alpha})\|_\alpha \\
&\leq \mathrm{const}(C)\, \|u\|_{L^2}
\end{aligned}$$

because of Lemma 3 and since $\|\cdot\|_\alpha$ is continuous on $\bar{I}$.
• Obviously K_2 is continuous ($f_{\varepsilon,\nu}^\sharp$ is continuous on $[1, C]$ and $\psi_\nu^\pm$ is square integrable at infinity).
• To show the continuity of K_3 consider first the differential operator $i\frac{d}{dt}$ with domain $H^1(\mathbb{R}) \subset L^2(\mathbb{R})$. As is well known this operator is self-adjoint, hence $\sigma(\frac{d}{dt}) \subset i\mathbb{R}$ (in fact, one has equality but this is not important here). By Theorem 1 there exists the Green's function for $(\frac{d}{dt} - \lambda)^{-1}$ if $\lambda \notin i\mathbb{R}$. Using the conditions for a Green's function of this theorem it is easily seen that it is given by

$$\big((\tfrac{d}{dt} + \lambda)^{-1} f\big)(t) = -e^{-\lambda t} \int_{-\infty}^t e^{\lambda s} f(s)\, ds$$

if $\lambda \in \mathbb{R}$, $\lambda > 0$. Hence this integral operator is continuous, so there is a constant "const" such that

$$\int_{\mathbb{R}} \left(e^{-\lambda t} \int_{-\infty}^t e^{\lambda s} f(s)\, ds \right)^2 dt \leq \mathrm{const} \int_{\mathbb{R}} f(t)^2\, dt$$

for all $f\colon \mathbb{R} \to \mathbb{R}_+$. Substituting $t = \ln\alpha$ and $s = \ln\overline{\alpha}$ ($\alpha, \overline{\alpha} \in \mathbb{R}_+$) yields

$$\int_0^\infty \left(\alpha^{-\lambda - \frac{1}{2}} \int_0^\alpha \overline{\alpha}^{\lambda - 1} f(\ln\overline{\alpha})\, d\overline{\alpha} \right)^2 d\alpha \leq \mathrm{const} \int_0^\infty \alpha^{-1} f(\ln\alpha)^2\, d\alpha.$$

Setting finally $g(\alpha) := \alpha^{-\frac{1}{2}} f(\ln\alpha)$ and $\nu := \lambda + \frac{1}{2}$ results in

$$\int_0^\infty \left(\alpha^{-\nu} \int_0^\alpha \overline{\alpha}^{\nu - 1} g(\overline{\alpha}) \right)^2 d\alpha \leq \mathrm{const} \int_0^\infty g(\alpha)^2\, d\alpha. \tag{$*$}$$

The cases $\operatorname{Re}\nu > \frac{1}{2}$ and $\operatorname{Re}\nu < \frac{1}{2}$ must be considered separately.

•• $\operatorname{Re}\nu > \frac{1}{2}$: Here $K_3 u(\alpha) = 2^{-2\varrho} \overline{c(\varepsilon,\nu)}^{-1} \psi_\nu^-(\alpha) \int_C^\alpha f_{\varepsilon,\overline{\nu}}(\overline{\alpha})^\sharp u(\overline{\alpha})\, d\mu(\overline{\alpha})$. Lemmas 1.1 and 1.3 imply that

$$c(\varepsilon,\overline{\nu})^{-1} f_{\varepsilon,\overline{\nu}}(\alpha) = \alpha^{-\varrho + \overline{\nu}} i_{V_\varepsilon^{\mathscr{I}}} + \alpha^{-\varrho + \overline{\nu} - 1} A_\nu(\alpha)$$

where $A_\nu(\alpha) \in \mathrm{Hom}(V_\varepsilon^{\mathscr{I}}, V_\varepsilon)$, $A_\nu(\alpha) \to A_\nu(\infty)$ as $\alpha \to \infty$. Since $c(\varepsilon, \overline{\nu}) = \overline{c(\varepsilon, \nu)}$,

$$c(\varepsilon,\nu)^{-1} f_{\varepsilon,\overline{\nu}}(\alpha)^\sharp = \alpha^{-\varrho+\nu}\, pr_{V_\varepsilon^{-\mathscr{I}}} + \alpha^{-\varrho+\nu-1} A_\nu(\alpha)^\sharp.$$

Putting this into the definition of K_3 one obtains

$$2^{2\varrho} K_3 u(\alpha) = \psi_\nu^-(\alpha) \int_C^\alpha \overline{\alpha}^{-\varrho+\nu} (pr_{V_\varepsilon^{-\mathscr{I}}}\, u)(\overline{\alpha})\, d\mu(\overline{\alpha}) + \psi_\nu^-(\alpha) \int_C^\alpha \overline{\alpha}^{-\varrho+\nu-1} A_\nu(\overline{\alpha})^\sharp u(\overline{\alpha})\, d\mu(\overline{\alpha}).$$

Let $u_1(\alpha) := \psi_\nu^-(\alpha) \int_C^\alpha \overline{\alpha}^{-\varrho+\nu} (pr_{V_\varepsilon^{-\mathscr{I}}}\, u)(\overline{\alpha})\, d\mu(\overline{\alpha})$ for $\alpha \geq C$. Then:

$$\begin{aligned}
\|u_1(\alpha)\|_\alpha &\leq \alpha^{\frac{1}{2}} \|u_1(\alpha)\|_{L^2} \\
&\leq \alpha^{\frac{1}{2}} \|\psi_\nu^-(\alpha)\|_{L^2} \int_C^\alpha \overline{\alpha}^{-\varrho+\mathrm{Re}\,\nu} \|pr_{V_\varepsilon^{-\mathscr{I}}}\, u(\overline{\alpha})\|_{L^2}\, d\mu(\overline{\alpha}) \\
&= \alpha^{\frac{1}{2}} \|\psi_\nu^-(\alpha)\|_{L^2} \int_C^\alpha \overline{\alpha}^{-\varrho+\mathrm{Re}\,\nu-\frac{1}{2}} \|pr_{V_\varepsilon^{-\mathscr{I}}}\, u(\overline{\alpha})\|_{\overline{\alpha}}\, d\mu(\overline{\alpha}) \\
&\leq \mathrm{const}\ \alpha^{-\varrho-\mathrm{Re}\,\nu+\frac{1}{2}} \int_C^\alpha \overline{\alpha}^{-\varrho+\mathrm{Re}\,\nu-\frac{1}{2}} \|pr_{V_\varepsilon^{-\mathscr{I}}}\, u(\overline{\alpha})\|_{\overline{\alpha}}\, d\mu(\overline{\alpha}).
\end{aligned} \tag{15}$$

From $\mu(\alpha) \sim 2^{2\varrho}\alpha^{2\varrho-1}$ for $\alpha \to \infty$ it follows that

$$\|u_1(\alpha)\|_\alpha \mu(\alpha)^{\frac{1}{2}} \leq \mathrm{const}\ \alpha^{-\mathrm{Re}\,\nu} \int_C^\alpha \overline{\alpha}^{\mathrm{Re}\,\nu-1} \big(\|pr_{V_\varepsilon^{-\mathscr{I}}}\, u(\overline{\alpha})\|_{\overline{\alpha}} \mu(\overline{\alpha})^{\frac{1}{2}}\big)\, d\overline{\alpha}.$$

Then $(*)$ implies that the L^2-norm (with respect to Lebesgue measure) of $\|u_1(\alpha)\|_\alpha \mu(\alpha)^{\frac{1}{2}}$ can be estimated from above by the L^2-norm of $\|pr_{V_\varepsilon^{-\mathscr{I}}}\, u(\overline{\alpha})\|_{\overline{\alpha}}^2 \mu(\overline{\alpha})^{\frac{1}{2}}$, hence by the L^2-norm of $\|u(\overline{\alpha})\|_{\overline{\alpha}} \mu(\overline{\alpha})^{\frac{1}{2}}$ because $\|u(\overline{\alpha})\|_{\overline{\alpha}}^2 = \|pr_{V_\varepsilon^{\mathscr{I}}}\, u(\overline{\alpha})\|_{\overline{\alpha}}^2 + \|pr_{V_\varepsilon^{-\mathscr{I}}}\, u(\overline{\alpha})\|_{\overline{\alpha}}^2 \geq \|pr_{V_\varepsilon^{-\mathscr{I}}}\, u(\overline{\alpha})\|_{\overline{\alpha}}^2$. This means that the map $u \mapsto u_1$ is continuous in $L^2(I, V_\varepsilon; \|\cdot\|_\alpha, d\mu(\alpha))$.

Now to the second term. Let $u_2(\alpha) := \psi_\nu^-(\alpha) \int_C^\alpha \overline{\alpha}^{-\varrho+\nu-1} A_\nu(\overline{\alpha})^\sharp u(\overline{\alpha})\, d\mu(\overline{\alpha})$. Then:

$$\begin{aligned}
\|u_2(\alpha)\|_\alpha &\leq \alpha^{\frac{1}{2}} \|u_2(\alpha)\|_{L^2} \\
&\leq \mathrm{const}\ \alpha^{-\varrho-\mathrm{Re}\,\nu+\frac{1}{2}} \int_C^\alpha \overline{\alpha}^{-\varrho+\mathrm{Re}\,\nu-1} \|A_\nu(\overline{\alpha})^\sharp\|_{L^2} \cdot \|u(\overline{\alpha})\|_{L^2}\, d\mu(\overline{\alpha}) \\
&\leq \mathrm{const}\ \alpha^{-\varrho-\mathrm{Re}\,\nu+\frac{1}{2}} \int_C^\alpha \overline{\alpha}^{-\varrho+\mathrm{Re}\,\nu-1} \|u(\overline{\alpha})\|_{L^2}\, d\mu(\overline{\alpha}) \\
&\leq \mathrm{const}\ \alpha^{-\varrho-\mathrm{Re}\,\nu+\frac{1}{2}} \int_C^\alpha \overline{\alpha}^{-\varrho+\mathrm{Re}\,\nu-\frac{1}{2}} \|u(\overline{\alpha})\|_{\overline{\alpha}}\, d\mu(\overline{\alpha}).
\end{aligned}$$

This is the same as for u_1 (cf. (15)).
•• Let $\operatorname{Re}\nu < \frac{1}{2}$. Then $K_3u(\alpha) = \psi_\nu^+(\alpha)b(\overline{\nu})^\sharp \int_C^\alpha f_{\varepsilon,\overline{\nu}}(\overline{\alpha})^\sharp u(\overline{\alpha})\, d\mu(\overline{\alpha})$. Considering only the highest order terms of $f_{\varepsilon,\overline{\nu}}$ one sees

$$f_{\varepsilon,\overline{\nu}}(\alpha) = \alpha^{-\varrho+\overline{\nu}}c(\varepsilon,\overline{\nu})\, i_{V_\varepsilon^{\mathscr{I}}} + \alpha^{-\varrho-\overline{\nu}}C_-(\varepsilon,\overline{\nu})\, i_{V_\varepsilon^{\mathscr{I}}} + \alpha^{-\varrho+|\operatorname{Re}\nu|-1}A_\nu(\alpha)$$

where $A_\nu(\alpha) \in \operatorname{Hom}(V_\varepsilon^{\mathscr{I}}, V_\varepsilon)$, $\|A_\nu(\alpha)\|_{L^2}$ as $\alpha \to \infty$. Hence

$$f_{\varepsilon,\overline{\nu}}(\alpha)^\sharp = \alpha^{-\varrho+\nu}c(\varepsilon,\nu)\, pr_{V_\varepsilon^{-\mathscr{I}}} + \alpha^{-\varrho-\nu}\, pr_{V_\varepsilon^{-\mathscr{I}}}\, C_-(\varepsilon,\nu) + \alpha^{-\varrho+|\operatorname{Re}\nu|-1}A_\nu(\alpha)^\sharp.$$

Also $\psi_\nu^+(\alpha) \sim \alpha^{-\varrho+\nu} + O(\alpha^{-\varrho+\nu-1}$. Putting these estimates into the definition of K_3 this splits K_3u into six summands (three for $f_{\varepsilon,\nu}$ times two for ψ_ν^+). So let

$$u_1(\alpha) := \alpha^{-\varrho+\nu}b(\overline{\nu})^\sharp \int_C^\alpha \overline{\alpha}^{\varrho+\nu}\, pr_{V_\varepsilon^{-\mathscr{I}}}\, u(\overline{\alpha})\, d\mu(\overline{\alpha})$$

for $\alpha > C$ (and $u_1(\alpha) := 0$ for $\alpha < C$). Since $b(\overline{\nu})^\sharp = b(\nu) \in \operatorname{Hom}(V_\varepsilon^{-\mathscr{I}}, V_\varepsilon^{\mathscr{I}})$, $\|u_1(\alpha)\|_\alpha = \alpha^{-\frac{1}{2}}\|u_1(\alpha)\|_{L^2}$. Hence

$$\begin{aligned}\|u_1(\alpha)\|_\alpha &\leq \text{const}\ \alpha^{-\varrho+\operatorname{Re}\nu-\frac{1}{2}} \int_C^\alpha \overline{\alpha}^{-\varrho+\operatorname{Re}\nu}\|pr_{V_\varepsilon^{-\mathscr{I}}}\, u(\overline{\alpha})\|_{L^2}\, d\mu(\overline{\alpha})\\ &= \text{const}\ \alpha^{-\varrho+\operatorname{Re}\nu-\frac{1}{2}} \int_C^\alpha \overline{\alpha}^{-\varrho+\operatorname{Re}\nu-\frac{1}{2}}\|pr_{V_\varepsilon^{-\mathscr{I}}}\, u(\overline{\alpha})\|_{\overline{\alpha}}\, d\mu(\overline{\alpha}).\end{aligned} \tag{16}$$

Therefore

$$\|u_1(\alpha)\|_\alpha \mu(\alpha)^{\frac{1}{2}} \leq \text{const}\ \alpha^{\operatorname{Re}\nu-1} \int_\alpha^C \overline{\alpha}^{\operatorname{Re}\nu-1}\big(\|pr_{V_\varepsilon^{-\mathscr{I}}}\, u(\overline{\alpha})\|_{\overline{\alpha}}\mu(\overline{\alpha})^{\frac{1}{2}}\big)\, d\overline{\alpha}. \tag{17}$$

Now, $\operatorname{Re}\nu - 1 < -\frac{1}{2}$, hence $\overline{\alpha} \mapsto \overline{\alpha}^{\operatorname{Re}\nu-1}$ is square integrable at infinity (with respect to Lebesgue measure). Since also $\|pr_{V_\varepsilon^{-\mathscr{I}}}(\overline{\alpha})\|_{\overline{\alpha}}\mu(\overline{\alpha})^{\frac{1}{2}}$ is square integrable (with respect to Lebesgue measure), the integral on the right hand side is bounded for all α by the Cauchy Schwarz inequality, so the left hand side can be estimated by the L^2-function $\alpha \mapsto \alpha^{\operatorname{Re}\nu-1}$.

Let

$$u_2(\alpha) := (\psi_\nu^+(\alpha) - \alpha^{-\varrho+\nu})b(\overline{\nu})^\sharp \int_C^\alpha \overline{\alpha}^{-\varrho+\nu}\, pr_{V_\varepsilon^{-\mathscr{I}}}\, u(\overline{\alpha})\, d\mu(\overline{\alpha}).$$

Then:

$$\begin{aligned}\|u_2(\alpha)\|_\alpha \leq \alpha^{\frac{1}{2}}\|u_2(\alpha)\|_{L^2} &\leq \text{const}\ \alpha^{-\varrho+\operatorname{Re}\nu-\frac{1}{2}} \int_C^\alpha \overline{\alpha}^{-\varrho+\operatorname{Re}\nu}\|pr_{V_\varepsilon^{-\mathscr{I}}}\, u(\overline{\alpha})\|_{L^2}\, d\mu(\overline{\alpha})\\ &= \text{const}\ \alpha^{-\varrho+\operatorname{Re}\nu-\frac{1}{2}} \int_C^\alpha \overline{\alpha}^{-\varrho+\operatorname{Re}\nu-\frac{1}{2}}\|pr_{V_\varepsilon^{-\mathscr{I}}}\, u(\overline{\alpha})\|_{\overline{\alpha}}\, d\mu(\overline{\alpha}).\end{aligned}$$

This expression occurred already in the discussion of u_1 (cf. (17)).

Let

$$u_3(\alpha) := \alpha^{-\varrho+\nu}(\alpha) b(\overline{\nu})^\sharp \int_C^\alpha \overline{\alpha}^{-\varrho-\nu}\, pr_{V_\varepsilon^- \mathscr{I}}\, C_-(\varepsilon,\nu) u(\overline{\alpha})\, d\mu(\overline{\alpha}).$$

As before:

$$\begin{aligned}\|u_3(\alpha)\|_\alpha = \alpha^{-\frac12}\|u_3(\alpha)\|_{L^2} &\le \text{const } \alpha^{-\varrho+\operatorname{Re}\nu-\frac12}\int_C^\alpha \overline{\alpha}^{-\varrho-\operatorname{Re}\nu}\|pr_{V_\varepsilon^- \mathscr{I}}\, C_-(\varepsilon,\nu)u(\overline{\alpha})\|_{L^2}\, d\mu(\overline{\alpha})\\ &\le \text{const } \alpha^{-\varrho+\operatorname{Re}\nu-\frac12}\int_C^\alpha \overline{\alpha}^{-\varrho-\operatorname{Re}\nu}\|u(\overline{\alpha})\|_{L^2}\, d\mu(\overline{\alpha}) \qquad (18)\\ &\le \text{const } \alpha^{-\varrho+\operatorname{Re}\nu-\frac12}\int_C^\alpha \overline{\alpha}^{-\varrho-\operatorname{Re}\nu+\frac12}\|u(\overline{\alpha})\|_{\overline{\alpha}}\, d\mu(\overline{\alpha}).\end{aligned}$$

Hence

$$\|u_3(\alpha)\|_\alpha \mu(\alpha)^{\frac12} \le \text{const } \alpha^{\operatorname{Re}\nu-1}\int_C^\alpha \overline{\alpha}^{-\operatorname{Re}\nu}\big(\|u(\overline{\alpha})\|_{\overline{\alpha}}\mu(\overline{\alpha})^{\frac12}\big)\, d\overline{\alpha}.$$

Since $1-\operatorname{Re}\nu > \frac12$, $(*)$ can be used to estimate the l^2-norm of u_3.

Let

$$u_4(\alpha) := (\psi_\nu^+(\alpha) - \alpha^{-\varrho+\nu}) b(\overline{\nu})^\sharp \int_C^\alpha \overline{\alpha}^{-\varrho-\nu}\, pr_{V_\varepsilon^- \mathscr{I}}\, u(\overline{\alpha})\, d\mu(\overline{\alpha}).$$

Then:

$$\|u_4(\alpha)\|_\alpha \le \alpha^{\frac12}\|u_4(\alpha)\|_{L^2} \le \text{const } \alpha^{-\varrho+\operatorname{Re}\nu-\frac12}\int_C^\alpha \overline{\alpha}^{-\varrho-\operatorname{Re}\nu}\|u(\alpha)\|_{L^2}\, d\mu(\overline{\alpha}),$$

This expression occurred already in the discussion of u_3 (cf. (18)).

Let

$$u_5(\alpha) := \alpha^{-\varrho+\nu} b(\overline{\nu})^\sharp \int_C^\alpha \overline{\alpha}^{-\varrho+|\operatorname{Re}\nu|-1} A_\nu(\overline{\alpha})^\sharp u(\overline{\alpha})\, d\mu(\overline{\alpha}).$$

Then:

$$\begin{aligned}\|u_5(\alpha)\|_\alpha = \alpha^{-\frac12}\|u_5(\alpha)\|_{L^2} &\le \text{const } \alpha^{-\varrho+\operatorname{Re}\nu-\frac12}\int_C^\alpha \overline{\alpha}^{-\varrho+|\operatorname{Re}\nu|-1}\|A_\nu(\overline{\alpha})^\sharp u(\overline{\alpha})\|_{L^2}\, d\mu(\overline{\alpha})\\ &\le \text{const } \alpha^{-\varrho+\operatorname{Re}\nu-\frac12}\int_C^\alpha \overline{\alpha}^{-\varrho+|\operatorname{Re}\nu|-\frac12}\|u(\overline{\alpha})\|_{\overline{\alpha}}\, d\mu(\overline{\alpha}),\end{aligned}$$

and therefore

$$\|u_5(\alpha)\|_\alpha \mu(\alpha)^{\frac{1}{2}} \leq \text{const } \alpha^{\operatorname{Re}\nu-1} \int\limits_C^\alpha \overline{\alpha}^{|\operatorname{Re}\nu|-1}\big(\|u(\overline{\alpha})\|_{\overline{\alpha}}\mu(\overline{\alpha})^{\frac{1}{2}}\big)\, d\overline{\alpha}. \tag{19}$$

Again $(*)$ applies.

And finally let

$$u_6(\alpha) := (\psi_\nu^+(\alpha) - \alpha^{-\varrho+\nu}) b(\overline{\nu})^\sharp \int\limits_C^\alpha \overline{\alpha}^{-\varrho+|\operatorname{Re}\nu|} A_\nu(\overline{\alpha})^\sharp u(\overline{\alpha})\, d\mu(\overline{\alpha}).$$

As done before:

$$\begin{aligned}
\|u_6(\alpha)\|_\alpha \mu(\alpha)^{\frac{1}{2}} &\leq \alpha^{\frac{1}{2}} \|u_6(\alpha)\|_{L^2} \mu(\alpha)^{\frac{1}{2}} \\
&\leq \text{const } \alpha^{\operatorname{Re}\nu-1} \int\limits_C^\alpha \overline{\alpha}^{-\varrho+|\operatorname{Re}\nu|-1} \|A_\nu(\overline{\alpha})^\sharp u(\overline{\alpha})\|_{L^2}\, d\mu(\overline{\alpha}) \\
&\leq \text{const } \alpha^{\operatorname{Re}\nu-1} \int\limits_C^\alpha \overline{\alpha}^{-\varrho+|\operatorname{Re}\nu|-\frac{1}{2}} \|u(\overline{\alpha})\|_{\overline{\alpha}}\, d\mu(\overline{\alpha}) \\
&\leq \text{const } \alpha^{\operatorname{Re}\nu-1} \int\limits_C^\alpha \overline{\alpha}^{|\operatorname{Re}\nu|-1}\big(\|u(\overline{\alpha})\|_{\overline{\alpha}}\mu(\overline{\alpha})^{\frac{1}{2}}\big)\, d\overline{\alpha}.
\end{aligned}$$

This is again (19).

Sum up: The k_ν are actually the Green's functions.

Finally some remarks on the case that was excluded from consideration, namely the case that p and q are both odd. In this case there exists no scalar product on $\Delta_{p,q}^{(\pm)}$ that is invariant under $\boldsymbol{Spin}_+(p,q)$. Consequently there's no scalar product that's independent of $\alpha \in I$. Therefore Theorem 3 cannot be applied, so one has to fall back to Theorem 2. But in proving the continuity of the integral operators one meets the difficulty of estimating expressions like $\|A^{\sharp_\alpha}\|_\alpha$ as $\alpha \to \infty$, where A is an endomorphism of V_ε. Furthermore the evenness of p or q was needed to show that $\dim V_\varepsilon^{\mathscr{I}} = \dim V_\varepsilon^{-\mathscr{I}} = \frac{1}{2}\dim V_\varepsilon$ for the proofs of the Theorems 5.1 and 5.2. Nevertheless some result can be formulated for arbitrary p and q:

Lemma 4 *Irrespectively of the parity of p and q, $i\mathbb{R} \subset \sigma_c(\overline{D})$.*

Proof. If not so, by Theorem 1 a Green's function $k_{\nu+\frac{1}{2}}(\alpha,\overline{\alpha})$ for $\overline{D}+\frac{1}{2}-(\nu+\frac{1}{2})$ had to exist if $\nu \in i\mathbb{R}$, $\nu \in \varrho(\overline{D})$. Then for all $\overline{\alpha}$ the mapping $\alpha \mapsto k_{\nu+\frac{1}{2}}(\alpha,\overline{\alpha})$ were an eigenfunction of $D+\frac{1}{2}$ for eigenvalue $\nu+\frac{1}{2}$ square integrable at infinity. According to Lemma 1 this would imply that $k_{\nu+\frac{1}{2}}(\alpha,\overline{\alpha}) = 0$ for $\alpha > \overline{\alpha}$. Hence $k_{\nu+\frac{1}{2}} = 0$ by Theorem 3 and the subsequent remark. Thus the resolvent $\big(\overline{D}+\frac{1}{2}-(\nu+\frac{1}{2})\big)^{-1}$ were zero, which is obviously a contradiction.

Q.E.D.

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