Surfaces with flat normal bundle: an explicit construction

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Abstract

An explicit construction of surfaces with flat normal bundle in the Euclidean space E^n (unit hypersphere S^n) in terms of solutions of certain linear system is proposed. In the case of E^3 our formulae can be viewed as the direct Lie sphere analog of the generalized Weierstrass representation of surfaces in conformal geometry or the Lelieuvre representation of surfaces in the affine 3-space.

An explicit parametrization of Ribaucour congruences of spheres by three solutions of the linear system is obtained. In view of the classical Lie correspondence between Ribaucour congruences and surfaces with flat normal bundle in the Lie quadric in P^5 this gives an explicit representation of surfaces with flat normal bundle in the 4-dimensional space form of the Lorentzian signature. Direct projective analog of this construction is the known parametrization of W-congruences by three solutions of the Moutard equation. Under the Plücker embedding W-congruences give rise to surfaces with flat normal bundle in the Plücker quadric.

Integrable evolutions of surfaces with flat normal bundle and parallels with the theory of nonlocal Hamiltonian operators of hydrodynamic type are discussed in the conclusion.

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1 Introduction

Let M^n be a submanifold of the Euclidean space E^{m+n} with the radius-vector $\mathbf{r}$ and m pairwise orthogonal unit normals $\mathbf{n}^\alpha$, the infinitesimal displacements of which are governed by the equations

$$d\mathbf{n}^\alpha = \omega^\alpha_\beta \ \mathbf{n}^\beta \quad mod \ TM^n.$$

The 1-forms ω^α_β define connection in the normal bundle of submanifold M^n [4]. Submanifolds with flat normal bundle are characterized by the existence of the normal frame $\mathbf{n}^\alpha$ for which $\omega^\alpha_\beta = 0$, so that

$$d\mathbf{n}^\alpha \in TM^n \quad \text{for any } \alpha;$$

such normals are called parallel in the normal bundle. Submanifolds with flat normal bundle have been extensively investigated in differential geometry: see e.g. [1], [22], [32] and references therein. Let $u^1, ..., u^n$ be a local coordinate system on M^n. Introducing the metric $(d\mathbf{r}, d\mathbf{r}) = g_{ij}du^i du^j$ and the Weingarten operators $(\overset{\alpha}{w})^i_j$ by the formulae

$$\partial_j \mathbf{n}^\alpha = (\overset{\alpha}{w})^i_j \ \partial_i \mathbf{r},$$

$\partial_j = \partial_{u^j}$, one can write down the Gauss-Codazzi equations of a submanifold with flat normal bundle in the form

$$\begin{gathered} g_{ik}(\overset{\alpha}{w})^k_j = g_{jk}(\overset{\alpha}{w})^k_i, \quad \nabla_k(\overset{\alpha}{w})^i_j = \nabla_j(\overset{\alpha}{w})^i_k, \\ R^{ij}_{kl} = \textstyle\sum_{\alpha=1}^m \{(\overset{\alpha}{w})^i_k(\overset{\alpha}{w})^j_l - (\overset{\alpha}{w})^j_k(\overset{\alpha}{w})^i_l\} \end{gathered} \tag{1}$$

where ∇ is the covariant differentiation generated by g_{ij} and $R^{ij}_{kl} = g^{is}R^j_{skl}$ is the curvature tensor. Moreover, the family of the Weingarten operators is commutative:

$$[\overset{\alpha}{w}, \overset{\beta}{w}] = 0. \tag{2}$$

It was observed recently in [13] that the Gauss-Codazzi equations (1), (2) of submanifolds with flat normal bundle coincide with the skew-symmetry conditions and the Jacobi identities for certain nonlocal Hamiltonian operators of hydrodynamic type (a brief review of this relationship is included in the Appendix).

The class of submanifolds with flat normal bundle is invariant under the following natural transformations:

1. **Conformal transformations** generated by arbitrary translations, rotations and inversions in E^{m+n}. In fact, paper [1] treats submanifolds with flat normal bundle as the objects of conformal geometry referring them to a special conformal frame of spheres.

2. **Normal shifts** which transform the radius-vector $\mathbf{r}$ into $\mathbf{r} + t^\alpha \mathbf{n}^\alpha$; here $t^\alpha = const$ and $\mathbf{n}^\alpha$ are parallel in the normal bundle [32]. Submanifolds related by a normal shift are called parallel.

3. **Gauss map** associating with any submanifold M^n with flat normal bundle in E^{m+n} a submanifold $\tilde{M}^n$ with flat normal bundle spanned by one of it's parallel normals $\mathbf{n}^\alpha$ in the unit hypersphere $S^{m+n-1} \subset E^{m+n}$.

4. **Stereographic projection** mapping submanifold $\tilde{M}^n \subset S^{m+n-1}$ into E^{m+n-1}.

In the case of hypersurfaces (it should be emphasized that any hypersurface of E^{n+1} or S^{n+1} automatically has flat normal bundle) transformations 1, 2 generate a finite-dimensional group of contact transformations known as the Lie sphere group.

Transformations 3, 4 suggest an inductive construction of submanifolds with flat normal bundle: given a submanifold in E^{m+n-1}, we first project it stereographically into the unit hypersphere $S^{m+n-1} \subset E^{m+n}$ and then reconstruct a submanifold M^n in E^{m+n} from the given Gauss image (the last step is essentially nonunique and requires integration of certain linear system). Note that our definition of the Gauss map differs from the standard one, which associates with a given submanifold the family of it's normal subspaces in the Grassmanian.

In the case of surfaces this procedure allows to construct an arbitrary surface with flat normal bundle starting from an orthogonal net on the plane E^2. It requires solving linear equations only. As will be shown in sect.2, this step-by-step construction can be combined into a simple formula representing an arbitrary surface with flat normal bundle in terms of solutions of certain linear system.

The commutativity of the family of Weingarten operators implies the existence of the net of curvature lines. The tangents to the curves of the net are n common eigendirections of $\overset{\alpha}{w}$. This net is not necessarily holonomic (that is, a coordinate net): isoparametric submanifolds are the most important examples of submanifolds with flat normal bundle and nonholonomic (in general) net of curvature distributions [31], [32], [14].

From now on we consider submanifolds with flat normal bundle and holonomic net of curvature lines. Submanifolds of this type naturally arise as the "coordinate" submanifolds of orthogonal coordinate systems in the Euclidean space. Conversely, any submanifold with flat normal bundle and holonomic net of lines of curvature can be included (nonuniquely!) in an orthogonal coordinate system. Particularly interesting examples of submanifolds with flat normal bundle and holonomic net of curvature lines are provided by the embeddings of the Lobachevski space L^n into E^{2n+1}, see e.g. [2], [29], [30]; here the existence of curvature line parametrization was pointed out in [5].

In the coordinates $u^1, ..., u^n$ of the lines of curvature the metric $(d\mathbf{r}, d\mathbf{r})$ and the Weingarten operators $\overset{\alpha}{w}$ become diagonal:

$$(d\mathbf{r}, d\mathbf{r}) = g_{ii}(du^i)^2, \quad (\overset{\alpha}{w})^i_j = (\overset{\alpha}{w})^i \delta^i_j$$

so that the Gauss-Codazzi equations (1) assume the form

$$\begin{gathered} \partial_j (\overset{\alpha}{w})^i = \partial_j \ln \sqrt{g_{ii}}\ ((\overset{\alpha}{w})^j - (\overset{\alpha}{w})^i), \quad i \neq j \\ R^{ij}_{ij} = \textstyle\sum_{\alpha=1}^m (\overset{\alpha}{w})^i (\overset{\alpha}{w})^j; \end{gathered} \tag{3}$$

all other components of the curvature tensor vanish. Introducing the Lame coefficients $H_i = \sqrt{g_{ii}}$ and the rotation coefficients β_{ij} by the formulae

$$\partial_i H_j = \beta_{ij} H_i, \quad i \neq j,$$

one can easily check that $(\overset{\alpha}{w})^i$ are representable in the form

$$(\overset{\alpha}{w})^i = H^\alpha_i / H_i$$

where H_i^α satisfy the same equations as H_i:

$$\partial_i H_j^\alpha = \beta_{ij} H_i^\alpha.$$

Since

$$R_{ij}^{ij} = -\frac{1}{H_i H_j}(\partial_i \beta_{ij} + \partial_j \beta_{ji} + \sum_{k \neq i,j} \beta_{ki}\beta_{kj}),$$

equations (3) can be rewritten in the form

$$\begin{gathered} \partial_i H_j^\alpha = \beta_{ij} H_i^\alpha \\ \partial_k \beta_{ij} = \beta_{ik}\beta_{kj}, \quad k \neq i,j \\ \partial_i \beta_{ij} + \partial_j \beta_{ji} + \textstyle\sum_{k \neq i,j} \beta_{ki}\beta_{kj} + \sum_{\alpha=1}^m H_i^\alpha H_j^\alpha = 0, \quad i \neq j; \end{gathered} \tag{4}$$

here equations $(4)_2$ are the compatibility conditions of $(4)_1$. In the case of surfaces they simplify to

$$\begin{gathered} \partial_1 H_2^\alpha = \beta_{12} H_1^\alpha, \quad \partial_2 H_1^\alpha = \beta_{21} H_2^\alpha, \\ \partial_1 \beta_{12} + \partial_2 \beta_{21} + \textstyle\sum_{\alpha=1}^m H_1^\alpha H_2^\alpha = 0; \end{gathered} \tag{5}$$

in this form they appear in [9], p. 162.

The formulae for submanifolds with flat normal bundle assume much more symmetric form in the unit hypersphere (the case of the Euclidean space follows by a stereographic projection). The advantage of the unit hypersphere is that unlike the Euclidean case the radius-vector and the unit normals can be treated on equal footing. Let $\mathbf{w}^1$ and $\mathbf{w}^\alpha$, $\alpha = 2, ..., m$ be the radius-vector and the parallel normals of $M^n \subset S^{m+n-1} \subset E^{m+n}$, respectively. We arrange them in an $(m+n) \times m$ matrix W which satisfies the equation

$$W^t W = E_m$$

in view of the orthonormality of $\mathbf{w}^1$, $\mathbf{w}^\alpha$. The requirement that $u^1, ..., u^n$ are coordinates of lines of curvature implies that the matrices $\partial_i W$ are of rank one for any i.

In sect.2 we propose a direct construction of surfaces $M^2 \subset S^{m+1} \subset E^{m+2}$ with flat normal bundle by constructing an $(m+2) \times m$ matrix W satisfying the above properties in terms of m arbitrary solutions $(\kappa^1, s^1), ..., (\kappa^m, s^m)$ of the linear system

$$\begin{aligned} \kappa_x &= \tan\varphi \; s_x \\ \kappa_y &= -\cot\varphi \; s_y \end{aligned} \tag{6}$$

where $\varphi(x, y)$ is an arbitrary function of $x = u^1, y = u^2$. This construction resembles the so-called N-fold Ribaucour transformation known from the theory of integrable systems (see, e.g., [21]) and provides a compact representation of the iterative procedure discussed above. We prove that an arbitrary surface with flat normal bundle can be obtained (locally) within our construction. Particular examples of surfaces with flat normal bundle were discussed in [24] (isothermic surfaces with flat normal bundle), [34] (surfaces flat normal bundle and zero Gaussian curvature); various transformations of surfaces with flat normal bundle were discussed in [9]. The approach of [9] is based on the remark that the radius-vector $\mathbf{r}$ of a surface $M^2 \subset S^{m+1} \subset E^{m+2}$ with flat normal bundle satisfies the Laplace equation

$$\mathbf{r}_{xy} = a\mathbf{r}_x + b\mathbf{r}_y,$$

supplemented by a quadratic constraint $(\mathbf{r},\mathbf{r}) = 1$; thence the name: quadratic conjugate nets.

Construction of sect.2 can be viewed as a direct linearization of system (5) transforming it into the "decoupled" form

$$\partial_1 H_2^\alpha = \beta_{12} H_1^\alpha, \qquad \partial_2 H_1^\alpha = \beta_{21} H_2^\alpha$$

$$\partial_1 \beta_{12} + \partial_2 \beta_{21} = 0.$$

Under the change of variables $\psi^1 = -s_y/\sin\varphi$, $\psi^2 = s_x/\cos\varphi$, system (6) transforms to a more familiar Dirac operator

$$\psi_x^1 = \varphi_y \psi^2, \quad \psi_y^2 = -\varphi_x \psi^1;$$

however, system (6) will be more convenient for our purposes.

In sect.3 we discuss the case of surfaces in E^3 and derive the explicit formulae for the main geometric quantities (such as the radii of principle curvature, fundamental forms and the Lie-invariant density) in terms of linear system (6). The representation of surfaces in E^3 in terms of two solutions of linear system (6) is a direct analog of the generalized Weierstrass representation of surfaces in conformal geometry in terms of the 2-dimensional Dirac operator [17] and the Lelieuvre representation of affine surfaces in terms of the Moutard equation [19].

Sect.4 generalizes construction of surfaces with flat normal bundle to the case of submanifolds of arbitrary dimension carrying holonomic net of curvature lines.

In sect.5 we discuss surfaces with flat normal bundle in the 4-dimensional space of constant curvature $S^{3,1}$ of the Lorentzian signature. These surfaces can be obtained as the images of Ribaucour congruences of spheres under the Lie sphere map [9]. In this particular case our construction allows to "parametrize" Ribaucour congruences (and hence surfaces with flat normal bundle in $S^{3,1}$) by three solutions of system (6).

Construction of sect.5 is the direct Lie sphere analog of the known construction in projective differential geometry parametrizing W-congruences by three solutions of the Moutard equation. Under the Plücker embedding W-congruences correspond to surfaces with flat normal bundle in the Plücker quadric $S^{2,2}$ [9]. We recall this construction in sect.6.

Considering linear system (6) as the Lax operator of the $(2+1)$-dimensional integrable modified Veselov-Novikov (mVN) hierarchy we define integrable evolutions of surfaces with flat normal bundle in the spirit of [17]. Although these evolutions are not completely well-defined, they have a number of interesting properties: in particular, the first local integral of the mVN hierarchy

$$\int\int \varphi_x\varphi_y \ dxdy$$

coincides with the simplest Lie-invariant functional in Lie sphere geometry. This construction is sketched in sect.7.

Parallels between the theory of submanifolds with flat normal bundle and nonlocal Hamiltonian operators of hydrodynamic type are drawn in the Appendix.

2 Construction of surfaces with flat normal bundle in a hypersphere

In this section we propose a construction of surfaces with flat normal bundle in terms of solutions of a linear system

$$\kappa_x = \lambda(x,y) \ s_x$$
$$\kappa_y = \mu(x,y) \ s_y$$

where λ, μ are constrained by $\lambda\mu = -1$. For several reasons it will be convenient to represent this system in the form (6):

$$\kappa_x = \tan\varphi \ s_x$$
$$\kappa_y = -\cot\varphi \ s_y.$$

Let $(\kappa^1, s^1), ..., (\kappa^m, s^m)$ be m arbitrary solutions of (6). First we introduce two matrices: the $2\times m$ matrix

$$U = \begin{pmatrix} \kappa^1 & ... & \kappa^m \\ s^1 & ... & s^m \end{pmatrix}$$

and the $m\times m$ matrix V with the elements $V^{\alpha\beta}$ defined by the formula

$$dV^{\alpha\beta} = \kappa^\alpha d\kappa^\beta + s^\alpha ds^\beta. \tag{7}$$

The right-hand sides in (7) are closed in view of (6) so that $V^{\alpha\beta}$ are correctly defined up to additive constants. We restrict these constants by requiring

$$V^{\alpha\alpha} = \frac{(\kappa^\alpha)^2 + (s^\alpha)^2 + 1}{2}, \tag{8}$$

$$V^{\alpha\beta} + V^{\beta\alpha} = \kappa^\alpha\kappa^\beta + s^\alpha s^\beta, \qquad \alpha \neq \beta; \tag{9}$$

both restrictions are compatible with (7). In the matrix form conditions (7), (8) and (9) can be rewritten as follows:

$$dV = U^t dU \tag{10}$$

$$V + V^t = U^t U + E_m \tag{11}$$

where E_m denotes the $m \times m$ identity matrix. Let us also point out that

$$U_x = \begin{pmatrix} \tan\varphi \\ 1 \end{pmatrix} \left(s_x^1 \; ... \; s_x^m\right)$$
$$U_y = \begin{pmatrix} -\cot\varphi \\ 1 \end{pmatrix} \left(s_y^1 \; ... \; s_y^m\right) \tag{12}$$

where the right-hand sides are undestood as the products of the 2×1 and $1 \times m$ matrices. Consequently, U_x and U_y are matrices of rank 1. Let us introduce finally the $(m+2) \times m$ matrix

$$W = \begin{pmatrix} UV^{-1} \\ \dots\dots\dots\dots \\ V^{-1} - E_m \end{pmatrix} \tag{13}$$

which consists of the $2 \times m$ upper submatrix UV^{-1} (V is assumed to be invertible) and the $m \times m$ lower submatrix $V^{-1} - E_m$.

Lemma 1. *Matrix W satisfies the equation $W^t W = E_m$.*

Proof:

As far as

$$W^t = \left((V^t)^{-1}U^t \;\vdots\; (V^t)^{-1} - E_m\right)$$

we have

$$W^t W = (V^t)^{-1}U^t U V^{-1} + ((V^t)^{-1} - E_m)(V^{-1} - E_m) =$$

$$(V^t)^{-1}(V + V^t - E_m)V^{-1} + ((V^t)^{-1} - E_m)(V^{-1} - E_m) = E_m$$

In this calculation we made use of (11). q.e.d.

Thus, m columns $\mathbf{w}^1, ..., \mathbf{w}^m$ of the matrix W are pairwise orthogonal unit vectors in E^{m+2}. Let us choose, for instance, $\mathbf{w}^1$ as the radius-vector of a surface M^2 which by a construction lies in the unit hypersphere $S^{m+1} \subset E^{m+2}$. We are going to show that the remaining vectors $\mathbf{w}^\alpha$, $\alpha = 2, ..., m$ are orthogonal to M^2 and can be interpreted as it's unit normals. The proof is based on the following

Lemma 2. *Matrices W_x and W_y are of rank one.*

Proof:

Let us demonstrate this for the matrix W_x:

$$W_x = \begin{pmatrix} U_x V^{-1} - UV^{-1}V_x V^{-1} \\ \dots\dots\dots\dots\dots\dots \\ -V^{-1}V_x V^{-1} \end{pmatrix} = \begin{pmatrix} U_x V^{-1} - UV^{-1}U^t U_x V^{-1} \\ \dots\dots\dots\dots\dots\dots \\ -V^{-1}U^t U_x V^{-1} \end{pmatrix} =$$

$$\begin{pmatrix} E_2 - UV^{-1}U^t \\ \dots\dots\dots\dots \\ -V^{-1}U^t \end{pmatrix} U_x V^{-1}.$$

Since U_x is of rank one, the rank of W_x cannot exeed one as well. q.e.d.

Hence, $\mathbf{w}_x^\alpha$ are proportional to $\mathbf{w}_x^1$. Similarly, $\mathbf{w}_y^\alpha$ are proportional to $\mathbf{w}_y^1$. As far as $TM^2 = span\ \{\mathbf{w}_x^1, \mathbf{w}_y^1\}$ and $\mathbf{w}^\alpha$ are orthogonal to $\mathbf{w}_x^\alpha$ and $\mathbf{w}_y^\alpha$ (indeed, $\mathbf{w}^\alpha$ are unit vectors), the orthogonality of $\mathbf{w}^\alpha$ and TM^2 follows directly. In fact we have proved a stronger result: since $d\mathbf{w}^\alpha \subset TM^2$, the surface M^2 automatically has flat normal bundle. Moreover, x and y are coordinates of the lines of curvature. The metric $(d\mathbf{w}^1, d\mathbf{w}^1)$ and the second fundamental forms $(d\mathbf{w}^1, d\mathbf{w}^\alpha)$ of the surface M^2 are the elements of the matrix

$$dW^t dW = (X^t)^{-1} dU^t dU X^{-1} \tag{14}$$

(we skip this straightforward calculation). Since the (α, β)-element of $dU^t dU$ is of the form

$$d\kappa^\alpha d\kappa^\beta + ds^\alpha ds^\beta = s_x^\alpha s_x^\beta \ \frac{dx^2}{\cos^2 \varphi} + s_y^\alpha s_y^\beta \ \frac{dy^2}{\sin^2 \varphi},$$

the matrix $dW^t dW$ contains no mixed terms $dxdy$. This gives another proof of the fact that in the coordinates x, y all fundamental forms of M^2 are diagonal. We can formulate the main result of this paper:

Theorem 1. *For any number m of solutions of linear system (6) each column of the matrix W can be considered as the radius-vector of a surface $M^2 \subset S^{m+1} \subset E^{m+2}$ with flat normal bundle parametrized by coordinates x, y of the lines of curvature. The remaining columns play the role of $m-1$ normals which are parallel in the normal bundle. The metric and the second fundamental forms of the surface M^2 are given by (14).*

Conversely, an arbitrary surface M^2 with flat normal bundle can be obtained (locally) within this construction.

Proof:

It remains to prove the last statement. This proof is constructive and provides an explicit representation of the surface M^2 in terms of solutions of system (6). Let p_0 be a nongeneric point of $M^2 \subset S^{m+1} \subset E^{m+2}$ (that is, the net of curvature lines is a coordinate net in a neighbourhood of p_0). Let $\mathbf{w}^1$ be the radius-vector of M^2 and $\mathbf{w}^\alpha$, $\alpha = 2, ..., m$, the set of parallel normals. We arrange the columns $\mathbf{w}^1, ..., \mathbf{w}^m$ in an $(m+2) \times m$ matrix W which satisfies the equation

$$W^t W = E_m \tag{15}$$

in view of the orthonormality of $\mathbf{w}^1, ..., \mathbf{w}^m$. Moreover, one can always choose coordinates in the ambient space in such a way that in the point p_0 the matrix W would be of the form

$$W = \begin{pmatrix} 0 \\ \dots \\ E_m \end{pmatrix}.$$

In a neighbourhood of p_0 the matrix W can be represented as follows:

$$W = \begin{pmatrix} UV^{-1} \\ \dots\dots\dots\dots \\ V^{-1} - E_m \end{pmatrix}$$

where U and V are the $2 \times m$ and $m \times m$ matrices, respectively. In the point p_0 we have $U = 0, \quad V^{-1} = 2E_m$ so that V is indeed invertible in a neighbourhood of p_0. Condition (15) implies (11):

$$V + V^t = U^t U + E_m.$$

Let us utilise the fact that W_x and W_y are of rank one (we recall that x, y are coordinates of the lines of curvature). Differentiation of W with respect to x results in

$$W_x = \begin{pmatrix} U_x - UV^{-1}V_x \\ \cdots\cdots\cdots\cdots \\ -V^{-1}V_x \end{pmatrix} V^{-1}$$

implying that V_x is of rank one and hence can be represented in the form

$$V_x = \begin{pmatrix} a_1 \\ \vdots \\ a_m \end{pmatrix} \begin{pmatrix} \xi^1 & ... & \xi^m \end{pmatrix}$$

viewed as the product of $m \times 1$ and $1 \times m$ matrices. For the same reasons,

$$U_x = \begin{pmatrix} \lambda \\ \beta \end{pmatrix} \begin{pmatrix} \xi^1 & ... & \xi^m \end{pmatrix}.$$

Differentiation of (11) with respect to x gives

$$(V_x - U^t U_x) + (V_x - U^t U_x)^t = 0,$$

implying that $V_x - U^t U_x$ is skew-symmetric. On the other hand, it is of rank one. Since the rank of skew-symmetric matrix is necessarily even, we conclude that $V_x = U^t U_x$. Similarly, $V_y = U^t U_y$, so that

$$dV = U^t dU$$

which coincides with (10). By normalizing ξ we can always represent U_x in the form

$$U_x = \begin{pmatrix} \lambda \\ 1 \end{pmatrix} \begin{pmatrix} \xi^1 & ... & \xi^m \end{pmatrix}.$$

Similarly,

$$U_y = \begin{pmatrix} \mu \\ 1 \end{pmatrix} \begin{pmatrix} \eta^1 & ... & \eta^m \end{pmatrix}.$$

The compatibility conditions $U_{xy} = U_{yx}$ imply $\xi^i = s^i_x, \ \eta^i = s^i_y$ so that

$$U = \begin{pmatrix} \kappa^1 & ... & \kappa^m \\ s^1 & ... & s^m \end{pmatrix}$$

where $\kappa^i_x = \lambda s^i_x, \ \kappa^i_y = \mu s^i_y$. The condition $\lambda\mu = -1$ follows from the compatibility of the equations (10) for V. q.e.d.

3 Particular case: surfaces in E^3

Here we give a direct construction of surfaces $M^2 \subset E^3$ in terms of two solutions of linear system (6). Under stereographic projection $S^3 \to E^3$ this construction reduces to that of the preceeding section. We treat this simplest case separately in order to bring together the necessary formulae.

Let (κ^1, s^1) and (κ^2, s^2) be two solutions of system (6). Defining the functions A and B by the formulae

$$dA = \kappa^1 d\kappa^2 + s^1 ds^2, \qquad B = \frac{(\kappa^1)^2 + (s^1)^2 + 1}{2}$$

we introduce a surface $M^2 \subset E^3$ with the radius-vector $\mathbf{r}$ and the unit normal $\mathbf{n}$:

$$\mathbf{r} = \begin{pmatrix} \kappa^2 - \kappa^1 A/B \\ s^2 - s^1 A/B \\ -A/B \end{pmatrix}, \qquad \mathbf{n} = \begin{pmatrix} \kappa^1/B \\ s^1/B \\ 1/B - 1 \end{pmatrix} \tag{16}$$

(in a somewhat different context these formulae were proposed in [11], [12]). A direct calculation results in the Weingarten equations

$$\begin{array}{c} \mathbf{r}_x = \rho^1 \mathbf{n}_x \\ \mathbf{r}_y = \rho^2 \mathbf{n}_y \end{array} \tag{17}$$

where ρ^1, ρ^2 are the radii of principle curvature of the surface M^2:

$$\rho^1 = \lambda^1 B - A, \quad \rho^2 = \lambda^2 B - A;$$

here $\lambda^1 = s_x^2/s_x^1$, $\lambda^2 = s_y^2/s_y^1$. Formulae (17) imply that x, y are coordinates of the lines of curvature. By a construction the normal $\mathbf{n}$ and the third fundamental form

$$(d\mathbf{n}, d\mathbf{n}) = \frac{(d\kappa^1)^2 + (ds^1)^2}{B^2} = \left(\frac{s_x^1}{B}\right)^2 \frac{dx^2}{\cos^2\varphi} + \left(\frac{s_y^1}{B}\right)^2 \frac{dy^2}{\sin^2\varphi}$$

depend only on the first solution (κ^1, s^1) of system (6). Varying (κ^2, s^2), we reconstruct the full class of Combescure-equivalent surfaces. All of them have parallel normals and parallel tangents to the lines of curvature in the corresponding points x, y; hence they have one and the same spherical image of the lines of curvature. The results of sect.2 imply that any surface can be represented (locally) by formulae (16) in a neighbourhood of the nonumbilic point.

In the discussion of Lie sphere geometry of surfaces in 3-space Blaschke [3] introduced the Lie-invariant functional which assumes the following form in terms of the radii of principle curvature ρ^1, ρ^2 [10]:

$$\int\int \frac{\rho_x^1 \rho_y^2}{(\rho^1 - \rho^2)^2}\, dxdy \tag{18}$$

As far as $A_x = \lambda^1 B_x,\ A_y = \lambda^2 B_y$ we have

$$\frac{\rho^1_x \rho^2_y}{(\rho^1 - \rho^2)^2} = \frac{\lambda^1_x \lambda^2_y}{(\lambda^1 - \lambda^2)^2}.$$

Moreover,

$$\frac{\lambda^1_x \lambda^2_y}{(\lambda^1 - \lambda^2)^2}\ dx \wedge dy = \frac{\lambda^1_y \lambda^2_x}{(\lambda^1 - \lambda^2)^2}\ dx \wedge dy - d\left(\frac{d\lambda^2}{\lambda^1 - \lambda^2}\right) = \varphi_x \varphi_y\ dx \wedge dy - d\left(\frac{d\lambda^2}{\lambda^1 - \lambda^2}\right)$$

(in this calculation we have used the formula $s_{xy} = -\varphi_x \kappa_y - \varphi_y \kappa_x$ which follows from (6)). Thus for compact surfaces without umbilic points (for instance, immersed tori) functional (18) coincides with

$$\int\int \varphi_x \varphi_y\ dxdy.$$

As we will see in sect.7, this functional can be interpreted as the first conservation law of the (2+1)-dimensional integrable mVN hierarchy associated with linear system (6).

4 Construction of submanifolds with flat normal bundle carrying coordinate net of curvature lines

The case of submanifolds M^n of dimension n greater than two requires certain modifications in the construction of sect.2. We start with a Dirac operator

$$\partial_i H_j = \beta_{ij} H_i, \quad i, j = 1, ..., n, \quad i \neq j \tag{19}$$

where the Lame coefficients H_i and the rotation coefficients β_{ij} are functions of n independent variables u^i, $\partial_i = \partial_{u^i}$. We require that the rotation coefficients satisfy the zero curvature conditions

$$\begin{gathered} \partial_k \beta_{ij} = \beta_{ik} \beta_{kj}, \quad i \neq j \neq k \\ \partial_i \beta_{ij} + \partial_j \beta_{ji} + \textstyle\sum_{k \neq i,j}^{n} \beta_{ki} \beta_{kj} = 0, \quad i \neq j. \end{gathered} \tag{20}$$

Equations (19), (20) are well-known in the theory of n-orthogonal curvilinear coordinate systems [7]. Let us introduce the so-called direction-cosines: n pairwise orthogonal unit vectors $\mathbf{X}_i = (X_{1i}, ..., X_{ni})$ satisfying the equations

$$\begin{gathered} \partial_j \mathbf{X}_i = \beta_{ij} \mathbf{X}_j \\ \partial_i \mathbf{X}_i = -\textstyle\sum_{k \neq i}^{n} \beta_{ki} \mathbf{X}_k \end{gathered} \tag{21}$$

which are compatible in view of (20) and define the $n \times n$ orthogonal matrix X_{ji}. In order to construct a submanifold M^n with flat normal bundle in the unit hypersphere $S^{m+n-1} \subset E^{m+n}$ we choose m arbitrary solutions $\mathbf{H}^\alpha = (H^\alpha_1, ..., H^\alpha_n),\ \alpha = 1, ..., m$ of the linear system (19). Defining m vector-functions $\mathbf{s}^\alpha = (s^\alpha_1, ..., s^\alpha_n)$ by the formulae

$$ds^\alpha_i = \sum_{k=1}^{n} X_{ik} H^\alpha_k du^k \tag{22}$$

which are compatible in view of (20) and (21) and taking into account the orthogonality of X_{ji}, one can easily check the identity

$$\sum_{i=1}^{n}(ds_i^\alpha)^2 = \sum_{i=1}^{n}(H_i^\alpha)^2(du^i)^2$$

implying that s_i^α are flat coordinates of the flat diagonal metric

$$\sum_{i=1}^{n}(H_i^\alpha)^2(du^i)^2.$$

Let us introduce two matrices: the $n\times m$ matrix

$$U = \begin{pmatrix} s_1^1 & ... & s_1^m \\ \multicolumn{3}{c}{\dots\dots\dots} \\ s_n^1 & ... & s_n^m \end{pmatrix}$$

and the $m\times m$ matrix V with the elements $V^{\alpha\beta}$ defined by the formula

$$dV^{\alpha\beta} = (\mathbf{s}^\alpha, d\mathbf{s}^\beta) = \sum_{i=1}^{n} s_i^\alpha ds_i^\beta \tag{23}$$

The right-hand sides in (23) are closed in view of (22) so that $V^{\alpha\beta}$ are correctly defined up to additive constants. We restrict these constants by requiring

$$V^{\alpha\alpha} = \frac{(\mathbf{s}^\alpha, \mathbf{s}^\alpha)+1}{2} = \frac{\sum_{i=1}^{n}(s_i^\alpha)^2+1}{2}, \tag{24}$$

$$V^{\alpha\beta}+V^{\beta\alpha} = (\mathbf{s}^\alpha, \mathbf{s}^\beta) = \sum_{i=1}^{n} s_i^\alpha s_i^\beta, \qquad \alpha\neq\beta; \tag{25}$$

both restrictions are compatible with (23). In the matrix form conditions (23)–(25) can be rewritten as follows:

$$dV = U^t dU$$

$$V+V^t = U^tU + E_m.$$

We also point out that as a consequence of (22) all matrices $\partial_i U$ are of rank one. Let us introduce finally the $(m+n)\times m$ matrix W

$$W = \begin{pmatrix} UV^{-1} \\ \dots\dots\dots \\ V^{-1}-E_m \end{pmatrix}$$

which consists of the $n\times m$ upper submatrix UV^{-1} (V is assumed invertible) and the $m\times m$ lower submatrix $V^{-1}-E_m$. It is straightforward to check that

$$W^tW = E_m.$$

Thus, m columns $\mathbf{w}^1, ..., \mathbf{w}^m$ of the matrix W are pairwise orthogonal unit vectors in E^{m+n}. Choosing any one of them (say, $\mathbf{w}^1$) as the radius-vector of a submanifold $M^n \subset$

$S^{m+n-1} \subset E^{m+n}$ one can show that the remaining vectors $\mathbf{w}^{\alpha}$, $\alpha = 2, ..., m$ are orthogonal to M^n and parallel in the normal bundle. Moreover, u^i are coordinates of the lines of curvature. The proofs repeat those from sect.2. It is important to emphasise that for $n > 2$ this construction requires the solution of the nonlinear n-wave system (20) which does not depend on the codimension of the embedding.

In the case of surfaces equations (19) are of the form

$$\partial_1 H_2 = -\partial_2 \varphi \ H_1, \quad \partial_2 H_1 = \partial_1 \varphi \ H_2$$

where $\beta_{12} = -\partial_2 \varphi, \ \ \beta_{21} = \partial_1 \varphi$ as a consequence of (20). It follows from (21) that

$$\mathbf{X}_1 = (\cos\varphi, \ - \sin\varphi), \quad \mathbf{X}_2 = (\sin\varphi, \ \cos\varphi),$$

so that equations (22) assume the form

$$ds_1 = \cos\varphi \ H_1 du^1 + \sin\varphi \ H_2 du^2$$

$$ds_2 = -\sin\varphi \ H_1 du^1 + \cos\varphi \ H_2 du^2$$

implying for s_1, s_2 the linear system

$$\partial_2 s_1 = \tan\varphi \ \partial_2 s_2$$

$$\partial_1 s_1 = -\cot\varphi \ \partial_1 s_2$$

which coincides with (6). Thus in the case $n = 2$ our construction reduces to that of sect.2.

5 Ribaucour congruences and surfaces with flat normal bundle in $S^{3,1}$

In this section we describe the construction of surfaces with flat normal bundle in a 4-dimensional space of constant curvature $S^{3,1}$ of the Lorentzian signature $(+++-)$. This construction is based on the notion of Ribaucour congruence of spheres which we briefly recall here for a convenience of the reader.

Let $M^2 \subset E^3$ be a surface with the radius-vector $\mathbf{r}$ and the unit normal $\mathbf{n}$ satisfying the Weingarten equations

$$\mathbf{r}_x = \rho^1 \mathbf{n}_x$$
$$\mathbf{r}_y = \rho^2 \mathbf{n}_y$$

where ρ^1, ρ^2 are the radii of principle curvature. A sphere $S^2(R)$ of radius R and center $\mathbf{r} - R\mathbf{n}$ touches M^2 at the point $\mathbf{r}$. Specifying R as a function of x, y we obtain an arbitrary congruence (that is, two-parameter family) of spheres tangent to our surface. Besides the surface M^2 itself, this congruence has another envelope $\tilde{M}^2$.

Definition. *A congruence of spheres is called the Ribaucour congruence if the lines of curvature on M^2 and $\tilde{M}^2$ correspond to each other.*

It is known (see e.g. [9], p. 194-196) that the function $R(x,y)$ giving rise to a congruence of Ribaucour is expressible in the form

$$R = \frac{P}{Q} \tag{26}$$

where P and Q are solutions of the linear system

$$\begin{aligned} P_x &= \rho^1 Q_x \\ P_y &= \rho^2 Q_y. \end{aligned} \tag{27}$$

Thus the construction of Ribaucour congruences with the given envelope M^2 reduces to the solution of linear system (27). One can show by a direct calculation that the function R defined by (26), (27) satisfies the nonlinear equation

$$R_{xy} = \left(a + \ln(\rho^1 - R)_y\right) R_x + \left(b + \ln(\rho^2 - R)_x\right) R_y \tag{28}$$

where

$$a = \frac{\rho^1_y}{\rho^2 - \rho^1}, \qquad b = \frac{\rho^2_x}{\rho^1 - \rho^2}.$$

Substitution (26) linearises the nonlinear equation (28).

With any sphere of radius R and center $\xi = (\xi^1, \xi^2, \xi^3)$ we associate a point

$$\mathbf{Z} = (z^1 : z^2 : z^3 : z^4 : z^5 : z^6)$$

in projective space P^5 with six homogeneous coordinates

$$\begin{gathered} z^1 = \xi^1, \quad z^2 = \xi^2, \quad z^3 = \xi^3, \\ z^4 = \tfrac{1-(\xi,\xi)+R^2}{2}, \quad z^5 = \tfrac{1+(\xi,\xi)-R^2}{2}, \quad z^6 = R \end{gathered} \tag{29}$$

which satisfy the quadratic relation

$$(z^1)^2 + (z^2)^2 + (z^3)^2 + (z^4)^2 - (z^5)^2 - (z^6)^2 = 0. \tag{30}$$

This construction is Lie's famous correspondence between spheres in E^3 and points on the Lie quadric (30) in P^5, see e.g. [20], [3], [6]; coordinates z^i are known as hexaspherical coordinates. Introducing inhomogeneous coordinates $\tilde{z}^i = z^i/z^6$, $i = 1, ..., 5$, we rewrite (30) in the form

$$(\tilde{z}^1)^2 + (\tilde{z}^2)^2 + (\tilde{z}^3)^2 + (\tilde{z}^4)^2 - (\tilde{z}^5)^2 = 1$$

which represents the 4-dimensional quadric $S^{3,1}$ of constant curvature 1 and the signature $(+++-)$ in the Lorentzian space with the metric $(d\tilde{z}^1)^2 + (d\tilde{z}^2)^2 + (d\tilde{z}^3)^2 + (\tilde{z}^4)^2 - (d\tilde{z}^5)^2$. With any congruence of spheres we thus associate a surface in $S^{3,1}$.

Proposition. ([9], p. 253) *Ribaucour congruences of spheres correspond to surfaces with flat normal bundle in* $S^{3,1}$.

Indeed, with $\xi = \mathbf{r} - R\mathbf{n}$ and $R = P/Q$ the 6-vector $\mathbf{Z}$ defined as in (29) satisfies the Laplace equation

$$\mathbf{Z}_{xy} = \left(a + \ln(\rho^1 - R)_y\right)\mathbf{Z}_x + \left(b + \ln(\rho^2 - R)_x\right)\mathbf{Z}_y.$$

Hence

$$\tilde{\mathbf{Z}}_{xy} = \left(a + \left(\ln\frac{\rho^1 - R}{z^6}\right)_y\right)\tilde{\mathbf{Z}}_x + \left(b + \left(\ln\frac{\rho^2 - R}{z^6}\right)_x\right)\tilde{\mathbf{Z}}_y$$

so that all components of the 5-vector $\tilde{\mathbf{Z}}$ lying on the quadric $S^{3,1}$ satisfy one and the same Laplace equation. This implies that $\tilde{\mathbf{Z}}$ is the radius-vector of a surface with flat normal bundle in $S^{3,1}$.

Now we are ready to give a general formula for surfaces with flat normal bundle in $S^{3,1}$: for that purpose we construct Ribaucour congruence of spheres in E^3 choosing three solutions (κ^1, s^1), (κ^2, s^2), (κ^3, s^3) of linear system (6). First we introduce the functions A, B, V^{31}, V^{32} as follows:

$$dA = \kappa^1 d\kappa^2 + s^1 ds^2, \qquad B = \tfrac{(\kappa^1)^2 + (s^1)^2 + 1}{2},$$

$$dV^{31} = \kappa^3 d\kappa^1 + s^3 ds^1, \qquad dV^{32} = \kappa^3 d\kappa^2 + s^3 ds^2$$

and define a surface $M^2 \subset E^3$ as in sect.3. It's radius-vector $\mathbf{r}$ and the unit normal $\mathbf{n}$ are the following:

$$\mathbf{r} = \begin{pmatrix} \kappa^2 - \kappa^1 A/B \\ s^2 - s^1 A/B \\ -A/B \end{pmatrix}, \qquad \mathbf{n} = \begin{pmatrix} \kappa^1/B \\ s^1/B \\ 1/B - 1 \end{pmatrix}.$$

The radii of principal curvature ρ^1, ρ^2 are given by

$$\rho^1 = \lambda^1 B - A, \quad \rho^2 = \lambda^2 B - A$$

where $\lambda^1 = s^2_x/s^1_x$, $\lambda^2 = s^2_y/s^1_y$. In order to define a Ribaucour congruence of spheres with the given envelope M^2 we have to specify a pair of functions P, Q satisfying (27). As one can verify directly, these can be choosen as follows:

$$P = V^{32} - V^{31}A/B, \qquad Q = V^{31}/B$$

so that

$$R = \frac{P}{Q} = \frac{V^{32}}{V^{31}}B - A.$$

The corresponding congruence of spheres with centers $\xi = \mathbf{r} - R\mathbf{n}$ and radii R generate a surface with flat normal bundle in $S^{3,1}$ with homogeneous hexaspherical coordinates

$$z^1 = \kappa^2 - \kappa^1 V^{32}/V^{31},$$

$$z^2 = s^2 - s^1 V^{32}/V^{31},$$

$$z^3 = (B-1)V^{32}/V^{31} - A,$$

$$z^4 = 1 - {\kappa^2}^2 - {s^2}^2 + 2(\kappa^1\kappa^2 + s^1 s^2 - A)V^{32}/V^{31},$$

$$z^5 = 1 + {\kappa^2}^2 + {s^2}^2 + 2(A - \kappa^1\kappa^2 - s^1 s^2)V^{32}/V^{31},$$

$$z^6 = BV^{32}/V^{31} - A.$$

As in the case of surfaces with flat normal bundle in a hypersphere, this construction is entirely expressed in terms of solutions of linear system (6).

6 W-congruences and surfaces with flat normal bundle in $S^{2,2}$

Construction of surfaces with flat normal bundle in a 4-dimensional space of constant curvature $S^{2,2}$ of the signature $(++--)$ is based on the notion of W-congruences of lines which are the direct projective analogs of Ribaucour congruences of spheres. Under the Plücker embedding W-congruences correspond to surfaces with flat normal bundle in the Plücker quadric in P^5. We recall this construction following [15], p.139-142.

Let us consider four arbitrary solutions $\xi^1, \xi^2, \xi^3, \xi^4$ of the Moutard equation

$$\xi_{xy} = Q(x,y)\xi \tag{31}$$

and introduce the functions S^{ij} by the formulae

$$dS^{ij} = (\xi^i_x\xi^j - \xi^j_x\xi^i)dx + (\xi^j_y\xi^i - \xi^i_y\xi^j)dy, \quad i \neq j, \quad i,j = 1,...,4,$$

the right-hand sides of which are closed one-forms in view of (31). Let M^2 be a surface in 3-space with the radius-vector

$$\mathbf{r} = (S^{23}, S^{31}, S^{12}). \tag{32}$$

Formula (32) is known as the Lelieuvre representation of surfaces in 3-space in the asymptotic parametrization x, y [19]. By a construction

$$\mathbf{r}_x = \xi_x \times \xi, \quad \mathbf{r}_y = \xi \times \xi_y$$

where $\xi = (\xi^1, \xi^2, \xi^3)$. In order to construct a W-congruence with the given focal surface M^2 we introduce another surface $\tilde{M}^2$ with the radius-vector

$$\tilde{\mathbf{r}} = \mathbf{r} + \tilde{\xi} \times \xi$$

where

$$\tilde{\xi} = \frac{1}{\xi^4}(S^{41}, S^{42}, S^{43});$$

vector $\tilde{\xi}$ can be interpreted as the Moutard transformation of ξ with the help of the fourth solution ξ^4. As pointed out in [15], the lines passing through $\mathbf{r}(x,y)$ and $\tilde{\mathbf{r}}(x,y)$ are tangent to both surfaces M^2 and $\tilde{M}^2$. Since x, y are asymptotic coordinates on both surfaces, this congruence is a W-congruence with the two focal surfaces M^2 and $\tilde{M}^2$. The homogeneous coordinates of $\mathbf{r}$ and $\tilde{\mathbf{r}}$ are of the form

$$
\begin{gathered}
\mathbf{r} = (S^{23},\ S^{31},\ S^{12},\ 1), \\
\tilde{\mathbf{r}} = (S^{23}\xi^4 + S^{42}\xi^3 - S^{43}\xi^2,\ S^{31}\xi^4 + S^{43}\xi^1 - S^{41}\xi^3,\ S^{12}\xi^4 + S^{41}\xi^2 - S^{42}\xi^1,\ 1).
\end{gathered} \tag{33}
$$

For any two points $\mathbf{a}$ and $\mathbf{b}$ in P^3 with homogeneous coordinates $\mathbf{a} = (a^1 : a^2 : a^3 : a^4)$, $\mathbf{b} = (b^1 : b^2 : b^3 : b^4)$ the Plücker coordinates of the line $(\mathbf{a}, \mathbf{b})$ are the six numbers

$$(p^{12} : p^{13} : p^{14} : p^{23} : p^{42} : p^{34})$$

($p^{ij} = a^i b^j - a^j b^i$) viewed as homogeneous coordinates of a point in P^5 lying on the Plücker quadric

$$p^{12}p^{34} + p^{13}p^{42} + p^{14}p^{23} = 0. \tag{34}$$

In the coordinates

$$
\begin{aligned}
z^1 &= \tfrac{p^{12}+p^{34}}{2}, \quad z^4 = \tfrac{p^{12}-p^{34}}{2}, \\
z^2 &= \tfrac{p^{13}+p^{42}}{2}, \quad z^5 = \tfrac{p^{13}-p^{42}}{2}, \\
z^3 &= \tfrac{p^{14}+p^{23}}{2}, \quad z^6 = \tfrac{p^{14}-p^{23}}{2}
\end{aligned}
$$

equation (34) assumes the form

$$(z^1)^2 + (z^2)^2 + (z^3)^2 - (z^4)^2 - (z^5)^2 - (z^6)^2 = 0.$$

Introducing $\tilde{z}^i = z^i/z^6,\ i = 1, ..., 5$ we can rewrite this equation in the form

$$(\tilde{z}^1)^2 + (\tilde{z}^2)^2 + (\tilde{z}^3)^2 - (\tilde{z}^4)^2 - (\tilde{z}^5)^2 = 1$$

which defines the 4-dimensional quadric $S^{2,2}$ of constant curvature 1 and the signature $(++--)$ in the Lorentzian space with the metric $(d\tilde{z}^1)^2 + (d\tilde{z}^2)^2 + (d\tilde{z}^3)^2 - (\tilde{z}^4)^2 - (d\tilde{z}^5)^2$. With any congruence of lines we thus associate a surface in $S^{2,2}$.

Proposition. ([9], p. 254) *W-congruences of lines correspond to surfaces with flat normal bundle in $S^{2,2}$.*

Applying Plücker construction to the congruence $(\mathbf{r}, \tilde{\mathbf{r}})$ we obtain six homogeneous Plükker coordinates

$$
\begin{gathered}
p^{12} = S^{23}(S^{43}\xi^1 - S^{41}\xi^3) - S^{31}(S^{42}\xi^3 - S^{43}\xi^2) \\
p^{13} = S^{23}(S^{41}\xi^2 - S^{42}\xi^1) - S^{12}(S^{42}\xi^3 - S^{43}\xi^2) \\
p^{14} = S^{43}\xi^2 - S^{42}\xi^3
\end{gathered}
$$

$$p^{23} = S^{31}(S^{41}\xi^2 - S^{42}\xi^1) - S^{12}(S^{43}\xi^1 - S^{41}\xi^3)$$
$$p^{42} = S^{43}\xi^1 - S^{41}\xi^3$$
$$p^{34} = S^{42}\xi^1 - S^{41}\xi^2$$

which define a surface with flat normal bundle in $S^{2,2}$ in terms of four solutions of the Moutard equation.

7 Integrable evolutions of surfaces with flat normal bundle

Integrable evolutions of surfaces governed by (2 + 1)-dimensional integrable equations have been introduced in [17]. The most interesting examples include evolution of surfaces in conformal geometry based on the generalized Weierstrass representation [17], [25], [27], [28], and evolution in projective geometry based on the Lelieuvre representation of surfaces in 3-space [17], [18]. The main idea is that linear systems used to construct a surface (the two-dimensional Dirac operator in the case of Weierstrass representation and the Moutard equation in the Lelieuvre case) are viewed as the Lax operators of the integrable $(2+1)$-dimensional hierarchies so that the corresponding t-evolutions act on the induced surfaces. Here we sketch the construction of the third integrable evolution based on the linear system (6) which is relevant to Lie sphere geometry.

Linear system (6)

$$\kappa_x = \tan\varphi \; s_x$$
$$\kappa_y = -\cot\varphi \; s_y$$

can be supplemented with the following t-evolution of κ and s

$$\begin{aligned} \kappa_t &= \kappa_{xxx} - 3\cot\varphi \; \varphi_x(\kappa_{xx} - \varphi_x s_x) + 3p\kappa_x \\ s_t &= s_{xxx} + 3\tan\varphi \; \varphi_x(s_{xx} + \varphi_x\kappa_x) + 3ps_x \end{aligned} \tag{35}$$

which implies the integrable (2+1)-dimensional equation for φ:

$$\begin{aligned} \varphi_t &= \varphi_{xxx} - \varphi_x^3 + 3p\varphi_x \\ p_y &= (\varphi_x\varphi_y)_x. \end{aligned} \tag{36}$$

Similarly the τ-evolution

$$\begin{aligned} \kappa_\tau &= \kappa_{yyy} + 3\tan\varphi \; \varphi_y(\kappa_{yy} - \varphi_y s_y) + 3q\kappa_y \\ s_\tau &= s_{yyy} - 3\cot\varphi \; \varphi_y(s_{yy} + \varphi_y\kappa_y) + 3qs_y \end{aligned} \tag{37}$$

implies the nonlinear equation

$$\begin{aligned} \varphi_\tau &= \varphi_{yyy} - \varphi_y^3 + 3q\varphi_y \\ q_x &= (\varphi_x\varphi_y)_y. \end{aligned} \tag{38}$$

Both these t- and τ-evolutions are compatible. Their linear combination

$$\varphi' = \varphi_{xxx} + \varphi_{yyy} - \varphi_x^3 - \varphi_y^3 + 3p\varphi_x + 3q\varphi_y$$

$$p_y = (\varphi_x \varphi_y)_x \tag{39}$$

$$q_x = (\varphi_x \varphi_y)_y$$

is known as the $(2+1)$-dimensional potential mKdV equation, or the modified Veselov-Novikov (mVN) equation [16]. Evolution of surfaces in E^3 governed by (39) was also discussed in [26].

Under the change of variables

$$\psi^1 = -\frac{s_y}{\sin\varphi}, \quad \psi^2 = \frac{s_x}{\cos\varphi}$$

linear system (6) transforms to a more familiar Dirac operator

$$\psi_x^1 = \varphi_y \psi^2$$

$$\psi_y^2 = -\varphi_x \psi^1$$

while evolutions (35) and (37) assume the forms

$$\psi_t^1 = \psi_{xxx}^1 - 3\varphi_{xy}\psi_x^2 + 3\varphi_y p \psi^2$$

$$\psi_t^2 = \psi_{xxx}^2 + 3(p\psi^2)_x - 3\varphi_{xx}\varphi_x\psi^2 \tag{40}$$

$$p_y = (\varphi_x \varphi_y)_x$$

and

$$\psi_\tau^1 = \psi_{yyy}^1 + 3(q\psi^1)_y - 3\varphi_{yy}\varphi_y\psi^1$$

$$\psi_\tau^2 = \psi_{yyy}^2 + 3\varphi_{xy}\psi_y^1 - 3\varphi_x q \psi^1 \tag{41}$$

$$q_x = (\varphi_x \varphi_y)_y,$$

respectively. All these evolutions preserve the integral

$$\int\int \varphi_x \varphi_y \, dxdy \tag{42}$$

which is the first conservation law in the mVN hierarchy. Geometric meaning of this functional in the context of Lie sphere geometry was clarified in sect.3. Evolutions (35) and (37) induce integrable evolutions of surfaces with flat normal bundle. Restricting to the case of surfaces in E^3 we see that these evolutions preserve the Lie-invariant functional (18) playing a role similar to that of the Willmore functional in conformal geometry and the projective area functional in projective geometry which are invariants of the evolutions discussed in [17], [18], [25], [27], [28]. Thus evolutions of surfaces introduced in this section are essentially Lie-geometric.

Remark. Strictly speaking, evolutions of surfaces introduced above are not completely well-defined: they depend on the particular parametrization of the surface M^2 by coordinates x, y of the lines of curvature as well as on the nonlocalities p, q entering the equations (35) and (37). The only objects which indeed have an invariant geometric meaning are the integrals of these evolutions (corresponding to certain Lie-invariant functionals, (42) being the simplest of them) and their stationary points. It is probably more correct to speak about foliations of 3-space by one-parameter families of surfaces which include the given surface M^2. This point of view was in fact suggested in [26]. The investigation of Lie-geometric properties of these foliations is beyond the scope of this paper.

8 Appendix. Nonlocal Hamiltonian operators and submanifolds with flat normal bundle

Let us consider an infinite-dimensional phase space of vector functions $u = \{u^i(x), i = 1, \ldots, n\}$, where the Poisson bracket of two functionals $F = \int f(u, u_x, \ldots)\ dx$ and $G = \int g(u, u_x, \ldots)\ dx$ is given by the formula

$$\{I, J\} = \int \frac{\delta F}{\delta u^i}\ A^{ij}\ \frac{\delta G}{\delta u^j}\ dx; \tag{43}$$

here A^{ij} is an operator of hydrodynamic type

$$A^{ij} = g^{ij}(u)\ d + b^{ij}_k(u)\ u^k_x, \quad d = \frac{d}{dx}. \tag{44}$$

The theory of such brackets was developed by Dubrovin and Novikov in [8]. A fundamental observation was that this theory is essentially differential-geometric. Indeed, if we take $det\ g^{ij} \neq 0$ (such Poisson brackets are called nondegenerate) and represent b^{ij}_k in the form $b^{ij}_k = -g^{is}\Gamma^j_{sk}$, it is not difficult to show that under point transformations $\tilde{u}^i = \tilde{u}^i(u^1, \ldots, u^n)$ the coefficients g^{ij} transform as components of a type $(2, 0)$ tensor, while Γ^j_{sk} transform as Christoffel symbols of an affine connection. The condition for the operator (44) to be Hamiltonian (i.e., to define a bracket that is skew-symmetric and satisfies the Jacobi identity) imposes strict constraints on g^{ij} and Γ^j_{sk}.

Theorem 2. [8] *1. The bracket defined by (43) and (44) is skew-symmetric if and only if the tensor g^{ij} is symmetric (i.e., defines a pseudo-Riemannian metric) and the connection Γ^j_{sk} is compatible with the metric: $\nabla_k g^{ij} = 0$.*

2. The bracket defined by (43) and (44) satisfies the Jacobi identity if and only if the connection Γ^j_{sk} is symmetric and its curvature tensor vanishes.

In other words, the metric $g_{ij}\ du^i du^j$ is flat (here $g_{ik}g^{kj} = \delta^j_i$), and Γ^j_{sk} are the coefficients of the corresponding Levi-Civita connection. It follows from this that for Hamiltonian operators of the form (44) we have an infinite-dimensional analog of the Darboux theorem: in the flat coordinates $g^{ij} = \epsilon^i\delta^{ij}$ $(\epsilon^i = \pm 1)$, $\Gamma^j_{sk} = 0$, and we obtain a particularly simple expression for A^{ij} with constant coefficients: $A^{ij} = \epsilon^i\delta^{ij}d$.

If for Hamiltonian we select the hydrodynamic functional $H = \int h(u)dx$, where density does not explicitely depend on the derivatives $u_x, u_{xx}, \ldots$, we obtain a Hamiltonian system of hydrodynamic type

$$u^i_t = A^{ij}\frac{\delta H}{\delta u^j} = v^i_j(u)\ u^j_x,$$

where the matrix v^i_j is given by the formula $v^i_j = \nabla^i\nabla_j h$. One may consult the survey [33] for the necessary information concerning differential geometry, integrability and applications of Hamiltonian systems of hydrodynamic type.

The first nonlocal generalization of Hamiltonian operators of hydrodynamic type was proposed by Mokhov and the author in [23]:

$$A^{ij} = g^{ij}d - g^{is}\Gamma^j_{sk}u^k_x + c\ u^i_x d^{-1}u^j_x, \quad c = const. \tag{45}$$

Formally, A^{ij} can be treated as a linear combination of the local operator (44) (henceforth, we assume $det\ g^{ij} \neq 0$) and the nonlocal term $u^i_x d^{-1}u^j_x$.

The conditions required for the operator (45) to be Hamiltonian depend on the constant c in a nontrivial way.

Theorem 3. [23] *1. The bracket defined by (43) and (45) is skew-symmetric if and only if the tensor g^{ij} is symmetric and the connection Γ^j_{sk} is compatible with the metric: $\nabla_k g^{ij} = 0$.*

2. The bracket defined by (43) and (45) satisfies the Jacobi identity if and only if the metric $g_{ij}du^i du^j$ has constant curvature c, i.e., $R^{ij}_{kl} = c(\delta^i_k\delta^j_l - \delta^j_k\delta^i_l)$.

Further generalizations of nonlocal Hamiltonian operators (45) lean in the direction of modifying the nonlocal "tail":

$$A^{ij} = g^{ij}d - g^{is}\Gamma^j_{sk}u^k_x + w^i_k u^k_x d^{-1} w^j_l u^l_x. \tag{46}$$

The conditions required for the operator (46) to be Hamiltonian impose certain restrictions on the metric $g^{ij}(u)$, the connection $\Gamma^j_{sk}(u)$ and the operator $w^i_j(u)$:

Theorem 4. [13] *1. The bracket defined by (43) and (46) is skew-symmetric if and only if the tensor g^{ij} is symmetric and the connection Γ^j_{sk} is compatible with the metric: $\nabla_k g^{ij} = 0$.*

2. The bracket defined by (43) and (46) satisfies the Jacobi identity if and only if the connection Γ^j_{sk} is symmetric, and the metric g_{ij} (with lower indices) and the operator w^i_j satisfy the Gauss-Codazzi equations:

$$g_{ik}w^k_j = g_{jk}w^k_i, \qquad \nabla_k w^i_j = \nabla_j w^i_k,$$

$$R^{ij}_{kl} = w^i_k w^j_l - w^j_k w^i_l, \qquad (R^{ij}_{kl} \equiv g^{is}R^j_{skl}).$$

In other words, the classical Gauss-Codazzi equations of hypersurfaces M^n in a pseudo-Euclidean space E^{n+1} are nothing but the Jacobi identity for the Poisson bracket (43), (46)! Here the metric g_{ij} plays the role of the first quadratic form of M^n, w^i_j, the role of the Weingarten operator (shape-operator). If M^n is a hyperplane in E^{n+1}, it's Weingarten operator vanishes and we obtain the Hamiltonian operator (44). If M^n is a unit hypersphere, then it's Weingarten operator $w^i_j = \delta^i_j$ yields the operator (45) with $c = 1$.

Further generalizations involve "lengthening" the nonlocal tail of the Hamiltonian operator:

$$A^{ij} = g^{ij}d - g^{is}\Gamma^j_{sk}u^k_x + \sum_{\alpha=1}^{m} (\overset{\alpha}{w})^i_k u^k_x d^{-1} (\overset{\alpha}{w})^j_l u^l_x. \tag{47}$$

Theorem 5. [13] *1. The bracket defined by (43) and (47) is skew-symmetric if and only if the tensor g^{ij} is symmetric and the connection Γ^j_{sk} is compatible with the metric: $\nabla_k g^{ij} = 0$.*

2. The bracket defined by (43) and (47) satisfies the Jacobi identity if and only if the connection Γ^j_{sk} is symmetric, and the metric g_{ij} (with lower indices), and the set of operators $\overset{\alpha}{w}$ satisfy the Gauss-Codazzi equations of submanifolds $M^n \subset E^{n+m}$ with flat normal bundle:

$$g_{ik}(\overset{\alpha}{w})^k_j = g_{jk}(\overset{\alpha}{w})^k_i, \qquad \nabla_k(\overset{\alpha}{w})^i_j = \nabla_j(\overset{\alpha}{w})^i_k,$$

$$R^{ij}_{kl} = \sum_{\alpha=1}^{N} \{(\overset{\alpha}{w})^i_k(\overset{\alpha}{w})^j_l - (\overset{\alpha}{w})^j_k(\overset{\alpha}{w})^i_l\},$$

$$[\overset{\alpha}{w}, \overset{\beta}{w}] = 0.$$

This remarkable correspondence between nonlocal Hamiltonian operators and submanifolds $M^n \subset E^{n+m}$ with flat normal bundle was clarified in [13], where it was demonstrated that the operator (47) arises as the result of Dirac reduction of the flat operator $\delta^{IJ}\frac{d}{dx}$, defined in the ambient space E^{n+m} (here $I, J = 1, \ldots, n+m$), to a submanifold M^n.

Thus, the classification of Hamiltonian operators of the type (47) is equivalent to the classification of submanifolds with flat normal bundle. The relevance of nonlocal Hamiltonian operators to the theory of integrable systems of hydrodynamic type as well as their further properties and examples are discussed in [13].

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