# The inverse spectral problem for first order systems on the half line

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**Abstract**

On the half line $[0, \infty)$ we study first order differential operators of the form

$$B\frac{1}{i}\frac{d}{dx} + Q(x),$$

where $B := \begin{pmatrix} B_1 & 0 \\ 0 & -B_2 \end{pmatrix}$, $B_1, B_2 \in \mathrm{M}(n, \mathbb{C})$ are self–adjoint positive definite matrices and $Q : \mathbb{R}_+ \to \mathrm{M}(2n, \mathbb{C})$, $\mathbb{R}_+ := [0, \infty)$, is a continuous self–adjoint off–diagonal matrix function.

We determine the self–adjoint boundary conditions for these operators. We prove that for each such boundary value problem there exists a unique matrix spectral function $\sigma$ and a generalized Fourier transform which diagonalizes the corresponding operator in $L^2_\sigma(\mathbb{R}, \mathbb{C})$.

We give necessary and sufficient conditions for a matrix function $\sigma$ to be the spectral measure of a matrix potential $Q$. Moreover we present a procedure based on a Gelfand-Levitan type equation for the determination of $Q$ from $\sigma$.

Our results generalize earlier results of M. Gasymov and B. Levitan.

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# Contents

# 1. Introduction

We consider the differential operator

$$L := B\frac{1}{i}\frac{d}{dx} + Q(x), \tag{1.1}$$

where

$$B := \begin{pmatrix} B_1 & 0 \\ 0 & -B_2 \end{pmatrix},$$

$B_1, B_2 \in \mathrm{M}(n, \mathbb{C})$ are self–adjoint positive definite matrices and $Q : \mathbb{R}_+ \to \mathrm{M}(2n, \mathbb{C})$, $\mathbb{R}_+ := [0, \infty)$, is a continuous self–adjoint matrix function. If $B_1 = B_2 = I_n$ then (1.1) is a Dirac operator.

It turns out that the operator (1.1) subject to the boundary condition

$$f_2(0) = Hf_1(0) \quad \text{with} \quad B_1 = H^* B_2 H \tag{1.2}$$

generates a self-adjoint extension $L_H$ of the minimal operator corresponding to $L$. Here, $f_1(0), f_2(0)$ denote the first resp. last $n$ components of the vector $f(0)$.

Let $Y(x, \lambda)$ be the $2n \times n$ matrix solution of the initial value problem

$$LY = \lambda Y, \quad Y(0, \lambda) = \begin{pmatrix} I \\ H \end{pmatrix}. \tag{1.3}$$

We will prove that there exists a unique increasing right–continuous $n \times n$ matrix function $\sigma(\lambda), \lambda \in \mathbb{R}$, (spectral function or spectral measure) such that we have the symbolic identity

$$\int_{\mathbb{R}} Y(x, \lambda) d\sigma(\lambda) Y(t, \lambda)^* = \delta(x - t) I_{2n}. \tag{1.4}$$

The main purpose of this paper is to investigate the inverse spectral problem for the operator $L_H$. This means to find necessary and sufficient conditions for a $n \times n$ matrix function $\sigma$ to be the spectral function of the boundary value problem (1.1), (1.2).

For a Sturm-Liouville operator this problem has been posed and completely solved by I. Gelfand and B. Levitan in the well–known paper [8] (see also [11], [15], [17]). Later on M. Gasymov and B. Levitan proved similar results for $2 \times 2$ Dirac systems [7], [15, Chap. 12] (see also [6] and [12]).

We note that in [15, Chap. 12] the determination of a potential $Q$ with prescribed spectral function $\sigma$ is incomplete. The self-adjointness of $Q$ is not proved.

The paper is organized as follows. In Section 2 we present some auxiliary results. In particular we prove the self-adjointness of the operator $L_H$.

In Section 3 we introduce the generalized Fourier transform

$$(\mathcal{F}_{H,Q} f)(\lambda) := \int_0^\infty Y(x, \lambda)^* f(x) dx$$

(see (3.19)) for $f \in L^2_{\mathrm{comp}}(\mathbb{R}_+, \mathbb{C}^{2n})$ and establish the existence of an $n \times n$ matrix (spectral) measure $\sigma$ such that the Parseval equality

$$(f, g)_{L^2(\mathbb{R}_+, \mathbb{C}^{2n})} = (\mathcal{F}_{H,Q} f,\ \mathcal{F}_{H,Q} g)_{L^2_\sigma(\mathbb{R})}, \tag{1.4'}$$

which is equivalent to (1.4), holds. In the proof we follow the Levitan–Levinson approximation method [5, Sec. 9.3], [15, Chap. 8]. Moreover we show that $\mathcal{F}_{H,Q}$ is a unitary transformation from $L^2(\mathbb{R}_+, \mathbb{C}^{2n})$ onto $L^2_\sigma(\mathbb{R})$ which diagonalizes the operator $L_H$. Namely, $\mathcal{F}_{H,Q} L_H \mathcal{F}^{-1}_{H,Q} = M_{\mathrm{id}}$ where $M_{\mathrm{id}} : L^2_\sigma(\mathbb{R}) \to L^2_\sigma(\mathbb{R})$ denotes the multiplication operator by the function $\lambda \mapsto \lambda$. Similar results (with similar proofs) hold for Sturm-Liouville operators as well as for higher order differential operators. Our proof is simpler than the corresponding proofs in [5] and [15].

In Section 4 we introduce (under the additional assumption $B_1 = \lambda_1 I_n$)) a triangular transformation operator $I + K$ and present a sketch of proof of the representation $Y(\cdot, \lambda) = ((I + K)e_0)(\cdot, \lambda)$ where $e_0(x, \lambda)$ is the solution of (1.3) with $Q = 0$. Then we derive the linear Gelfand-Levitan equation

$$F(x,t) + K(x,t) + \int_0^x K(x,s)F(s,t)ds = 0, \quad x > t, \tag{1.5}$$

with $F(x,t)$ defined by (4.31). $F$ is the analog of the so-called transition function (cf. [11]). We present two proofs of (1.5). The proof after Theorem 4.7 is close to the proofs in [8] and [15, Chap. 12]. The second one is relatively short. It is based on simple identities for kernels of Volterra operators (see (4.14)–(4.20)). In Proposition 4.5 we derive two representations (4.28) and (4.31) for $F(x,t)$ which easily imply (1.5). In other words, this proof derives the linear equation (1.5) directly from the nonlinear Gelfand-Levitan equation (4.28). This proof seems to be new and is essential in the sequel.

Furthermore, in Section 5 we solve the inverse problem (Theorem 5.2). Namely, starting with the transition matrix function $F(x,t)$ of the form (5.1') we prove the existence of the unique solution $K(x,t)$ of (1.5). Conversely, starting with $K(x,t)$ we determine the matrix potential $Q(x) = iBK(x,x) - iK(x,x)B$ and we prove that $Y(\cdot, \lambda) := ((I + K)e_0)(\cdot, \lambda)$ satisfies the initial value problem (1.3).

Finally, in Section 6 we present some generalizations and improvements of the main result. The degenerate Gelfand-Levitan equation is also considered here.

In conclusion we mention some recent publications close to our work. D. Alpay and I. Gohberg [1],[2] have constructed some explicit formulas for the matrix potential of a Dirac system (1.1) from the rational spectral function. Their approach is based on the results of minimal factorizations and realizations of matrix functions [3].

A new approach to inverse spectral problems for one-dimensional Schrödinger operators with partial information on the potential as well as to different kinds of uniqueness problems on the half-line has been recently proposed by F. Gesztesy and B. Simon (see [9], [10] and references therein).

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# 2. Preliminaries

We consider again the operator (1.1) from the introduction.

In the sequel for a vector $v \in \mathbb{C}^{2n}$ the vectors $v_1, v_2 \in \mathbb{C}^n$ will denote the first resp. last $n$ components of $v$.

In this paper scalar products will be antilinear in the first and linear in the second argument. This is necessary since we will be dealing with vector measures (see (3.18) below).

$L$ is a formally self–adjoint operator acting on $H^1_{\mathrm{comp}}((0,\infty),\mathbb{C}^{2n}) \subset L^2(\mathbb{R}_+,\mathbb{C}^{2n})$. We denote by $L^*$ the adjoint of $L$ in $L^2(\mathbb{R}_+,\mathbb{C}^{2n})$. To obtain self–adjoint extensions we impose boundary conditions of the form

$$H_2 f_2(0) = H_1 f_1(0). \tag{2.1}$$

Here, $H_1, H_2 \in \mathrm{M}(n,\mathbb{C})$ and $f_1(0), f_2(0) \in \mathbb{C}^n$ denote the first $n$ resp. last $n$ components of $f(0)$, where $f \in H^1_{\mathrm{comp}}(\mathbb{R}_+,\mathbb{C}^{2n})$.

**Proposition 2.1** *Let $L_{H_1,H_2}$ be the operator $L^*$ restricted to the domain*

$$\mathcal{D}(L_{H_1,H_2}) := \{f \in \mathcal{D}(L^*) \,|\, H_2 f_2(0) = H_1 f_1(0)\}.$$

*Then the operator $L_{H_1,H_2}$ is self–adjoint iff the matrices $H_1, H_2$ are invertible and $B_1 = H^* B_2 H$, where $H := H_2^{-1} H_1$.*

Consequently we have $L_{H_1,H_2} = L_{H_2^{-1}H_1,I} = L_{H,I}$. From now on we will denote $L_{H,I}$ by $L_H$ and we will write the boundary condition always in the form

$$f_2(0) = H f_1(0). \tag{2.2}$$

**Proof** Since $Q$ is continuous we have $\mathcal{D}(L^*) \subset H^1_{\mathrm{loc}}(\mathbb{R}_+,\mathbb{C}^{2n})$. Now choose a sequence of functions $\chi_m \in C_0^\infty(\mathbb{R})$ with the following properties:

(i) $\chi_m|(-\infty, m] = 1$,

(ii) $0 \le \chi_m \le 1$,

(iii) $|\chi_m'| \le \frac{1}{m}$.

If $f \in \mathcal{D}(L^*)$ then $\chi_m f \to f$ in $L^2(\mathbb{R}_+,\mathbb{C}^{2n})$ and

$$L\chi_m f = B\frac{1}{i}\chi_m' f + \chi_m L f \to Lf \tag{2.3}$$

in $L^2(\mathbb{R}_+,\mathbb{C}^{2n})$. Thus $\chi_m f \to f$ in $\mathcal{D}(L^*)$.

For $f, g \in \mathcal{D}(L^*)$ we then find

$$\begin{aligned} &(L^* f, g) - (f, L^* g) = \lim_{k\to\infty}\lim_{l\to\infty}(L^*\chi_k f, \chi_l g) - (\chi_k f, L^*\chi_l g) \\ = \;& -i\lim_{k\to\infty}\lim_{l\to\infty} < B\chi_k f(0), \chi_l g(0) >_{\mathbb{C}^{2n}} = -i < Bf(0), g(0) >_{\mathbb{C}^{2n}} . \end{aligned} \tag{2.4}$$

Hence, $g \in \mathcal{D}(L^*_{H_1,H_2})$ iff for all $f \in \mathcal{D}(L_{H_1,H_2})$

$$0 = < Bf(0), g(0) >_{\mathbb{C}^{2n}} . \tag{2.5}$$

This shows that any self–adjoint extension of $L$ is given by a Lagrangian subspace $V$ of the symplectic vector space $\mathbb{C}^{2n}$ with symplectic form

$$\omega(v,w) :=< Bv, w >=< B_1v_1, w_1 > - < B_2v_2, w_2 > .$$

Lagrangian means that $\dim V = n$ and $\omega|V = 0$. The domain of such an extension then is

$$\{f \in \mathcal{D}(L^*) \,|\, f(0) \in V\}.$$

Now let $V$ be a Lagrangian subspace of $\mathbb{C}^{2n} = \mathbb{C}^n_+ \oplus \mathbb{C}^n_-$. We denote by $\pi_1, \pi_2$ the orthogonal projections onto the first resp. second factor. Since the symplectic form $\omega$ is positive resp. negative definite on $\ker \pi_1$ resp. $\ker \pi_2$ and since $\dim V = n$ the maps $\pi_1, \pi_2$ restricted to $V$ are isomorphisms

$$\widetilde{\pi}_1 : V \to \mathbb{C}^n_+, \quad \widetilde{\pi}_2 : V \to \mathbb{C}^n_-. \tag{2.6}$$

Hence $V = \{(x, \widetilde{\pi}_2 \circ \widetilde{\pi}_1^{-1} x) \,|\, x \in \mathbb{C}^n_+\}$. Put $H := \widetilde{\pi}_2 \circ \widetilde{\pi}_1^{-1}$. Then $\omega|V = 0$ immediately implies $B_1 = H^* B_2 H$. This proves the proposition. □

**Remark 2.2** 1. The previous proposition shows that the deficiency indices $n_\pm(L)$ are equal to $n$, i.e. $n_\pm(L) = n$. This means that at infinity we do not have to impose a boundary condition. Thus infinity is always in the 'limit point case', which essentially distinguishes first order systems from Sturm–Liouville operators and higher order differential operators ([18, 5]).

2. For scalar Dirac systems ($n = B_1 = B_2 = 1$) another proof of Proposition 2.1 has been obtained earlier by B.M. Levitan [15, Theorem 8.6.1].

The present proof is adapted from the standard proof of the essential self–adjointness of Dirac operators on complete manifolds (see e.g. [13, Theorem II.5.7]).

3. Another proof of the previous proposition could be given using the uniqueness of the solution of the Goursat problem for the hyperbolic system $\frac{du}{dt} = \pm i L_H^* u$ in $\mathbb{R}^2_+$. This method was also used to prove the essential self–adjointness of all powers of the Dirac operator on a complete manifold (cf. [4]). For the problem considered here we prefered to present an elementary direct proof.

From now on we will assume

$$B_1 = H^* B_2 H. \tag{2.7}$$

Note that this implies that $H$ is invertible.

We first discuss in some detail the case $Q = 0$.

Let $A \in \mathrm{M}(n, \mathbb{C})$ be a positive definite matrix. Then we put for $f \in L^2(\mathbb{R}, \mathbb{C}^n)$

$$\mathcal{F}_A f(\lambda) := \int_{\mathbb{R}} e^{-iA^{-1}x\lambda} f(x) dx. \tag{2.8}$$

Then we have for $f, g \in L^2(\mathbb{R}, \mathbb{C}^n)$ the Parseval equality

$$(f, g) = \frac{1}{2\pi} \int_{\mathbb{R}} (\mathcal{F}_A f)(\lambda)^* A^{-1} (\mathcal{F}_A g)(\lambda) d\lambda. \tag{2.9}$$

To prove (2.9) we may assume $A$ to be diagonal, i.e. $A = \operatorname{diag}(a_1, \dots, a_n)$, because if $A = U\widetilde{A}U^*$ with a unitary matrix $U$ then $(\mathcal{F}_A f)(\lambda) = U(\mathcal{F}_{\widetilde{A}} U^* f)(\lambda)$. Now $\mathcal{F}_A f(\lambda) = (\mathcal{F} f_j(\lambda/a_j))_{j=1,\dots,n}$ and (2.9) follows easily from the Parseval equality for the Fourier transform.

Now let

$$e_0(x,\lambda) := e^{i\lambda B^{-1}x}\begin{pmatrix} I \\ H \end{pmatrix} = \begin{pmatrix} e^{i\lambda B_1^{-1}x} \\ e^{-i\lambda B_2^{-1}x} H \end{pmatrix} \tag{2.10}$$

and put

$$\mathcal{F}_{H,0} f(\lambda) := \int_0^\infty e_0(x,\lambda)^* f(x)dx = \mathcal{F}_{B_1}\widetilde{f}_1(\lambda) + H^*\mathcal{F}_{B_2}\widetilde{f}_2(-\lambda), \tag{2.11}$$

where $\widetilde{f}_j$ denotes the extension by 0 of $f_j$ to $\mathbb{R}$.

If $f, g \in L^2(\mathbb{R}_+, \mathbb{C}^{2n})$ then the integrals

$$\begin{aligned} &\int_{\mathbb{R}} \mathcal{F}_{B_1}\widetilde{f}_1(\lambda)^* B_1^{-1} H^* \mathcal{F}_{B_2}\widetilde{g}_2(-\lambda)d\lambda, \\ &\int_{\mathbb{R}} \mathcal{F}_{B_2}\widetilde{f}_2(-\lambda)^* H B_1^{-1} \mathcal{F}_{B_1}\widetilde{g}_1(\lambda)d\lambda \end{aligned} \tag{2.12}$$

are sums of scalar products of the form

$$\int_{\mathbb{R}} \overline{\mathcal{F}\widetilde{\varphi}(-\lambda)}\mathcal{F}\widetilde{\psi}(\lambda)d\lambda, \tag{2.13}$$

where $\varphi, \psi \in L^2(\mathbb{R}_+)$. These scalar products vanish and hence we end up with the Parseval equality in the case of $Q = 0$

$$\begin{aligned} &\frac{1}{2\pi}\int_{\mathbb{R}} \mathcal{F}_{H,0} f(\lambda)^* B_1^{-1} \mathcal{F}_{H,0} g(\lambda)d\lambda \\ = \; &\frac{1}{2\pi}\int_{\mathbb{R}} \mathcal{F}_{B_1}\widetilde{f}_1(\lambda)^* B_1^{-1} \mathcal{F}_{B_1}\widetilde{g}_1(\lambda)d\lambda + \frac{1}{2\pi}\int_{\mathbb{R}} \mathcal{F}_{B_2}\widetilde{f}_2(-\lambda)^* H B_1^{-1} H^* \mathcal{F}_{B_2}\widetilde{g}_2(-\lambda)d\lambda \\ = \; &(f, g), \end{aligned} \tag{2.14}$$

in view of (2.7) and (2.9)

# 3. The spectral measure

We turn to general $Q$. For future reference we state the boundary value problem for $L$:

$$Lf = \lambda f, \quad f_1(0) = H f_2(0), \quad \text{where} \quad B_1 = H^* B_2 H. \tag{3.1}$$

In this section we prove the existence of a spectral measure function for the self–adjoint operator $L_H$.

Let $Y : \mathbb{R}_+ \times \mathbb{C} \to \mathrm{M}(2n \times n, \mathbb{C})$ be the unique solution of the initial value problem

$$LY = \lambda Y, \quad Y(0,\lambda) = \begin{pmatrix} I \\ H \end{pmatrix}. \tag{3.2}$$

We pick a second matrix $\widetilde{H}$ satisfying (2.7) and denote by $L_b$ the operator $L^*$ restricted to the domain

$$\mathcal{D}(L_b) := \{f \in H^1([0,b],\mathbb{C}^{2n}) \,|\, f_2(0) = Hf_1(0), f_2(b) = \widetilde{H} f_1(b)\}. \tag{3.3}$$

It is clear, that $L_b$ is a discrete self–adjoint operator. We denote its spectrum by

$$\sigma(L_b) := (\lambda_{m,b})_{m=1}^{\infty}. \tag{3.4}$$

$\phi_m$ denotes the normalized eigenfunction corresponding to $\lambda_{m,b}$, i. e.

$$\begin{aligned} &L\phi_m = \lambda_{m,b}\phi_m, \quad \phi_{m,2}(0) = H\phi_{m,1}(0), \\ &\phi_{m,2}(b) = \widetilde{H}\phi_{m,1}(b), \quad \|\phi_m\|_{L^2([0,b],\mathbb{C}^{2n})} = 1. \end{aligned} \tag{3.5}$$

We denote by $Y_i$ the $i$-th column of $Y$. Since the $(Y_i)_{i=1}^m$ form a basis of the solutions of the boundary value problem (3.1), we have

$$\phi_m(x) = \sum_{i=1}^n a_{mi} Y_i(x, \lambda_{m,b}). \tag{3.6}$$

(3.6) and the Parseval equality for the basis $(\phi_m)$ of $L^2([0,b],\mathbb{C}^{2n})$ imply for any $f \in L^2([0,b],\mathbb{C}^{2n})$

$$\begin{aligned} \int_0^b |f(x)|^2 dx &= \sum_{m=1}^{\infty} \left| \int_0^b < \phi_m(x), f(x) > dx \right|^2 \qquad (3.7) \\ = \sum_{m=1}^{\infty} \sum_{i,k=1}^{n} a_{mi}\overline{a_{mk}} &\overline{\int_0^b < Y_i(x,\lambda_{m,b}), f(x) > dx} \int_0^b < Y_k(x,\lambda_{m,b}), f(x) > dx. \end{aligned}$$

Setting

$$\sigma_{ik}(\lambda; b) := \begin{cases} -\displaystyle\sum_{-\lambda<\lambda_{m,b}\le 0} a_{mi}\overline{a_{mk}}, & \lambda < 0, \\ \displaystyle\sum_{0<\lambda_{m,b}\le\lambda} a_{mi}\overline{a_{mk}}, & \lambda > 0, \end{cases} \tag{3.8}$$

this can be rewritten

$$\|f\|^2_{L^2([0,b],\mathbb{C}^{2n})} = \int_{-\infty}^{\infty} \sum_{i,k=1}^{n} \overline{F_i(\lambda)} F_k(\lambda) d\sigma_{ik}(\lambda), \tag{3.9}$$

where $\sigma_{ik}(\lambda) := \sigma_{ik}(\lambda; b)$ and

$$F_i(\lambda) := \int_0^b < Y_i(x,\lambda), f(x) > dx, \quad i = 1,\dots,n. \tag{3.10}$$

**Lemma 3.1** *For each $N > 0$ there exists a constant $A = A(N)$ not depending on $b \in [0,\infty)$ such that*

$$V_{-N}^{N}\sigma_{ik} \le A, \quad i,k = 1,\dots,n, \tag{3.11}$$

*i.e. on compact subintervals of $\mathbb{R}$ the variations of $\sigma_{ik}$ are uniformly bounded in $b$.*

**Proof** In view of (3.2) for each $\varepsilon > 0$ there exists a $h > 0$ such that for $x \in [0,h], \lambda \in [-N,N]$, and $i,j \in \{1,\dots,n\}$ we have

$$|Y_{ij}(x,\lambda) - \delta_{ij}| < \varepsilon. \tag{3.12}$$

Now consider $f_h \in L^2[0,h]$ with

$$f_h(x) \geq 0, \quad \int_0^h f_h(x)dx = 1. \tag{3.13}$$

Denoting by $f_h^j$ the $j$-th column of the $2n \times n$ matrix function $f(x) = f_h(x)\binom{I}{0}$ and setting

$$F_{hi}^j(\lambda) := \int_0^h < Y_i(x,\lambda), f_h^j(x) > dx = \int_0^h \overline{Y_{ji}(x,\lambda)} f_h(x)dx, \tag{3.14}$$

we arrive at the estimates

$$|F_{hi}^j(\lambda)| < \varepsilon,\ i \neq j; \quad 1-\varepsilon < |F_{hi}^j(\lambda)| < 1+\varepsilon. \tag{3.15}$$

(3.15), the Parseval quality (3.9) for $f_h^j$, and the obvious inequality

$$2V_{-N}^N \sigma_{ik} \leq V_{-N}^N \sigma_{ii} + V_{-N}^N \sigma_{kk} \tag{3.16}$$

imply that for each fixed $j \in \{1,\dots,n\}$ and all $b \geq h$

$$\begin{aligned} \int_0^h |f_h^j(x)|^2 dx \ &\geq \ \int_{-N}^N |F_{hj}^j(\lambda)|^2 d\sigma_{jj}(\lambda;b) - 2\sum_{k\neq j} \int_{-N}^N |F_{hj}^j(\lambda)F_{hk}^j(\lambda)||d\sigma_{jk}(\lambda;b)| \\ &\quad - \sum_{i\neq j,k\neq j} \int_{-N}^N |F_{hi}^j(\lambda)F_{hk}^j(\lambda)||d\sigma_{ik}(\lambda;b)| \\ &\geq \ (1-\varepsilon)^2 V_{-N}^N \sigma_{jj} - (n-1)\varepsilon(1+\varepsilon)V_{-N}^N\sigma_{jj} \\ &\quad -(1+\varepsilon)\varepsilon \sum_{k\neq j} V_{-N}^N \sigma_{kk} - \varepsilon^2 \sum_{k\neq j} V_{-N}^N \sigma_{kk}. \end{aligned} \tag{3.17}$$

If $\varepsilon$ is sufficiently small, it follows that for all $b \geq h$

$$\sum_{j=1}^n \|f_h^j\|^2_{L^2([0,h])} \geq C(\varepsilon) \sum_{j=1}^n (\sigma_{jj}(N;b) - \sigma_{jj}(-N;b)),$$

where $C(\varepsilon)$ is independent of $b$. □

Let $\sigma(\lambda) = (\sigma_{ij}(\lambda))_{i,j=1}^n$ be an increasing $n \times n$ matrix function. On the space $C_0(\mathbb{R}, \mathbb{C}^n)$ of continuous $\mathbb{C}^n$–valued functions with compact support we introduce the scalar product

$$(f,g)_{L^2_\sigma} := \int_{\mathbb{R}} f(\lambda)^* d\sigma(\lambda) g(\lambda) := \sum_{i,j=1}^n \int_{\mathbb{R}} \overline{f_i(\lambda)} g_j(\lambda) d\sigma_{ij}(\lambda). \tag{3.18}$$

We denote by $L^2_\sigma(\mathbb{R})$ (cf. [18]) the Hilbert space completion of this space.

**Theorem 3.2** *There exists an increasing $n \times n$ matrix function $\sigma(\lambda), \lambda \in \mathbb{R}$, (spectral function) such that the map*

$$\mathcal{F}_{H,Q} : L^2_{\text{comp}}(\mathbb{R}_+, \mathbb{C}^{2n}) \ni f \mapsto (\mathcal{F}_{H,Q} f)(\lambda) := F(\lambda) := \int_0^\infty Y(x,\lambda)^* f(x) dx \qquad (3.19)$$

*extends by continuity to a unitary transformation from $L^2(\mathbb{R}_+, \mathbb{C}^{2n})$ onto the space $L^2_\sigma(\mathbb{R})$, i.e. for $f, g \in L^2(\mathbb{R}_+, \mathbb{C}^{2n})$ we have the Parseval equation*

$$\int_{\mathbb{R}_+} f^*(t) g(t) dt = \int_{\mathbb{R}} F^*(\lambda) d\sigma(\lambda) G(\lambda) \qquad (3.20)$$

*with $F, G$ being the $\mathcal{F}_{H,Q}$–transforms of $f, g$.*

*If $\sigma$ is normalized by requiring it to be right–continuous then it is unique.*

**Remark 3.3** 1. The Parseval identity may be symbolically rewritten as

$$\int_{\mathbb{R}} Y(x,\lambda) d\sigma(\lambda) Y(t,\lambda)^* = \delta(x-t) I_{2n}. \qquad (3.20')$$

To obtain (3.20') from (3.20) it suffices to set in (3.20) $f(\xi) = \delta_x(\xi) \otimes e_i,\ g(\xi) = \delta_t(\xi) \otimes e_j (1 \le i, j \le 2n)$ and to note that $(F_{H,Q} f)(\lambda) = Y(x,\lambda)^* e_i,\ (F_{H,Q} g)(\lambda) = Y(t,\lambda)^* e_j$.

2. From now on we will consider – without saying this explicitly – only right–continuous $n \times n$ matrix functions. Such a function $\sigma$ is determined by its corresponding matrix measure $d\sigma$.

**Proof** i) Let $\sigma_b(\lambda) = (\sigma_{ik}(\lambda; b))$ be the increasing $n \times n$ matrix function defined by (3.8). In view of Lemma 3.1 we can apply Helly's theorem and find a sequence $b_p \to \infty$, such that the limit

$$\lim_{b_p \to \infty} \sigma_{b_p}(\lambda) = \sigma(\lambda)$$

exists at each point of continuity of $\sigma(\lambda)$.

To obtain the Parseval equality (3.20) it is natural to pass to the limit in (3.9) as $b_p \to \infty$. We start with a function $f \in C_0^\infty((0,a), \mathbb{C}^{2n})$.

Similar to (3.9) one obtains from (3.7), (3.8) the equality

$$\begin{aligned} \sum_{|\lambda_m| \le N} \left| \int_0^a < \phi_m(x), f(x) > dx \right|^2 &= \int_{-N}^N \sum_{i,k} \overline{F_i(\lambda)} F_k(\lambda) d\sigma_{ik}(\lambda; b) \\ &= \int_{-N}^N F(\lambda)^* d\sigma(\lambda; b) F(\lambda) \qquad (3.21) \end{aligned}$$

for $b > a$. Furthermore, from (3.7) and (3.21) we infer

$$\begin{aligned} &\left| \int_0^a |f(x)|^2 dx - \int_{-N}^N F(\lambda)^* d\sigma(\lambda; b) F(\lambda) \right| \\ = &\sum_{|\lambda_m| > N} \left| \int_0^b < \phi_m(x), f(x) > dx \right|^2 \le \frac{1}{N^2} \|L_b f\|^2. \end{aligned}$$

Passing to the limit as $b = b_p \to \infty$ we obtain the Parseval equality for $f$. Since $C_0^\infty(\mathbb{R}_+, \mathbb{C}^{2n})$ is dense in $L^2(\mathbb{R}_+, \mathbb{C}^{2n})$ (3.20) is proved for any $f \in L^2(\mathbb{R}_+, \mathbb{C}^{2n})$.

ii) So far we have proved that $\mathcal{F}_{H,Q} : L^2(\mathbb{R}_+, \mathbb{C}^{2n}) \longrightarrow L^2_\sigma(\mathbb{R})$ is an isometry. To prove surjectivity we mimick the proof of [5, Sec. 9.3] for second order operators.

Note first that for $f \in \mathcal{D}(L_H)$ we have

$$(\mathcal{F}_{H,Q} L_H f)(\lambda) = -Y(0,\lambda)^* \frac{1}{i} B f(0) + \lambda(\mathcal{F}_{H,Q} f)(\lambda) = \lambda(\mathcal{F}_{H,Q} f)(\lambda) \tag{3.22}$$

since in view of (3.1) and (3.2) $Y(0,\lambda)^* B f(0) = 0$.

For $f \in L^2_{\text{comp}}(\mathbb{R}_+, \mathbb{C}^{2n}) \cap \mathcal{D}(L_H)$ formula (3.22) follows from integration by parts. For arbitrary $f \in \mathcal{D}(L_H)$ it follows from the fact that $L^2_{\text{comp}}(\mathbb{R}_+, \mathbb{C}^{2n}) \cap \mathcal{D}(L_H)$ is a core for $L_H$. The latter follows from the proof of Proposition 2.1.

Next we construct the adjoint of $\mathcal{F}_{H,Q}$: we put for $g \in L^2_{\sigma,\text{comp}}(\mathbb{R})$

$$(\mathcal{G}_H g)(x) := \int_{\mathbb{R}} Y(x,\lambda) d\sigma(\lambda) g(\lambda). \tag{3.23}$$

Then for $f \in L^2_{\text{comp}}(\mathbb{R}_+, \mathbb{C}^{2n})$

$$\begin{aligned} (\mathcal{G}_H g, f)_{L^2(\mathbb{R}_+, \mathbb{C}^{2n})} &= \int_0^\infty (\mathcal{G}_H g)(x)^* f(x) dx \\ &= \int_0^\infty \int_{\mathbb{R}} g(\lambda)^* d\sigma(\lambda) Y(x,\lambda)^* f(x) dx \\ &= (g, \mathcal{F}_{H,Q} f)_{L^2_\sigma(\mathbb{R})}. \end{aligned} \tag{3.24}$$

From the estimate

$$|(\mathcal{G}_H g, f)_{L^2(\mathbb{R}_+, \mathbb{C}^{2n})}| = |(g, \mathcal{F}_{H,Q} f)_{L^2_\sigma(\mathbb{R})}| \le \|g\|_{L^2_\sigma(\mathbb{R})} \|f\|_{L^2(\mathbb{R}_+, \mathbb{C}^{2n})} \tag{3.25}$$

we infer that $\mathcal{G}_H$ extends by continuity for $L^2_\sigma(\mathbb{R})$. Moreover, it equals the adjoint of $\mathcal{F}_{H,Q}$, i.e.

$$\mathcal{G}_H = \mathcal{F}^*_{H,Q}. \tag{3.26}$$

Since $\mathcal{F}_{H,Q}$ is an isometry it remains to prove injectivity of $\mathcal{F}^*_{H,Q}$.

It follows from (3.22) and Proposition 2.1

$$\mathcal{F}_{H,Q}(L_H - \zeta)^{-1} = (M_{\text{id}} - \zeta)^{-1} \mathcal{F}_{H,Q} \tag{3.27}$$

for $\zeta = \nu + i\varepsilon \in \mathbb{C} \setminus \mathbb{R}$. Here $M_{\text{id}} : L^2_\sigma(\mathbb{R}) \to L^2_\sigma(\mathbb{R}), (M_{\text{id}} g)(\lambda) := \lambda g(\lambda)$ denotes the operator of multiplication by $\lambda$. Therefore $(L_H - \bar{\zeta})^{-1} \mathcal{F}^*_{H,Q} = \mathcal{F}^*_{H,Q}(M_{\text{id}} - \bar{\zeta})^{-1}$ and hence we have the implication:

$$\Phi \in \ker \mathcal{F}^*_{H,Q} \Longrightarrow (M_{\text{id}} - \zeta)^{-1} \Phi \in \ker \mathcal{F}^*_{H,Q} \quad \text{for all } \zeta \in \mathbb{C} \setminus \mathbb{R}. \tag{3.28}$$

We put

$$\widetilde{Y}(x,\lambda) := \int_0^x Y(t,\lambda) dt. \tag{3.29}$$

Note that the $i$–th row $\widetilde{Y}_i(x,\lambda)$ of $\widetilde{Y}$ satisfies

$$\widetilde{Y}_i(x,\lambda)^* = \int_0^x Y(t,\lambda)^* e_i dt = (\mathcal{F}_{H,Q}(1_{[0,x]} \otimes e_i))(\lambda), \tag{3.30}$$

where $e_i$ denotes the $i$–th unit vector in $\mathbb{C}^{2n}$.

Now let $\Phi \in \ker \mathcal{F}_{H,Q}^*$.

In particular $\widetilde{Y}_i(x,\cdot)^* \in L^2_\sigma(\mathbb{R})$ and thus in view of (3.24) and (3.28) we have for $x \geq 0, \varepsilon > 0, \nu \in \mathbb{R}$

$$0 = (1_{[0,x]} \otimes e_i, \mathcal{F}_{H,Q}^* \frac{\varepsilon}{(M_{\mathrm{id}} - \nu)^2 + \varepsilon^2} \Phi) = (\widetilde{Y}_i^*, \frac{\varepsilon}{(M_{\mathrm{id}} - \nu)^2 + \varepsilon^2} \Phi), \tag{3.31}$$

thus

$$0 = \int_{\mathbb{R}} \widetilde{Y}(x,\lambda) \frac{\varepsilon}{(\lambda - \nu)^2 + \varepsilon^2} d\sigma(\lambda)\Phi(\lambda). \tag{3.32}$$

Since $\widetilde{Y}(x,\cdot)d\sigma(\cdot)\Phi(\cdot)$ is $L^1$ the dominated convergence theorem implies for $\alpha, \beta \in \mathbb{R}$

$$\begin{aligned} 0 &= \lim_{\varepsilon \to 0} \int_\alpha^\beta \int_{\mathbb{R}} \widetilde{Y}(x,\lambda) \frac{\varepsilon}{(\lambda - \nu)^2 + \varepsilon^2} d\sigma(\lambda)\Phi(\lambda) d\nu \\ &= \int_{\mathbb{R}} \widetilde{Y}(x,\lambda) \lim_{\varepsilon \to 0} \int_\alpha^\beta \frac{\varepsilon}{(\lambda - \nu)^2 + \varepsilon^2} d\nu d\sigma(\lambda)\Phi(\lambda) \\ &= \pi \int_\alpha^\beta \widetilde{Y}(x,\lambda) d\sigma(\lambda)\Phi(\lambda). \end{aligned} \tag{3.33}$$

Differentiating by $x$ and putting $x = 0$ yields for $\alpha, \beta \in \mathbb{R}$

$$0 = \int_\alpha^\beta Y(0,\lambda) d\sigma(\lambda)\Phi(\lambda).$$

Since $Y(0,\lambda) = \binom{I}{H}$ and $H$ is invertible we have

$$\int_\alpha^\beta d\sigma(\lambda)\Phi(\lambda) = 0$$

for all $\alpha, \beta \in \mathbb{R}$. This implies $\Phi = 0$ in $L^2_\sigma(\mathbb{R})$.

To prove the uniqueness statement we assume we had another increasing right continuous $n \times n$ matrix function $\varrho$ such that $\mathcal{F}_{H,Q}$ is a unitary transformation from $L^2(\mathbb{R}_+, \mathbb{C}^{2n})$ onto $L^2_\varrho(\mathbb{R})$. Then in view of (3.20) we have for all $F, G \in L^2_{\mathrm{comp}}(\mathbb{R}, \mathbb{C}^n)$

$$\int_{\mathbb{R}} F(\lambda)^* d\sigma(\lambda) G(\lambda) = \int_{\mathbb{R}} F(\lambda)^* d\varrho(\lambda) G(\lambda),$$

and hence the two Radon vector measures $d\sigma$ and $d\varrho$ coincide. By the right–continuity this implies $\sigma = \varrho$. □

**Example 3.4** (2.14) shows that in the case $Q = 0$ we can choose for $\sigma$ the function $\sigma_0(\lambda) := \frac{1}{2\pi} B_1^{-1} \lambda$.

**Remark 3.5** The proof of Theorem 3.2 is based on the approximation method proposed independently by B.M. Levitan [15, Chap. 8] and N. Levinson [5, Chap. 9].

# 4. Transformation operator and Gelfand–Levitan equation

**1.** We present a special case of [16, Theorem 7.1]. In the sequel we assume $B$ to be a diagonal matrix, which can be achieved by conjugating $L$ by an appropriate unitary matrix. We assume in addition that $B_1$ is scalar, i.e.

$$B_1 = \lambda_1 I_n, \quad B_2 = \operatorname{diag}(\lambda_2 I_{n_2}, \ldots, \lambda_r I_{n_r}), \quad n_2 + \ldots + n_r = n. \tag{4.1}$$

**Remark 4.1** Let

$$H = \operatorname{col}(H_2, \ldots, H_r)$$

be the block–matrix representation of $H$ with $n_j \times n$ blocks $H_j$. Then (2.7) can be rewritten as

$$B_1 = \sum_{j=2}^{r} \lambda_j H_j^* H_j.$$

Furthermore, since $H$ is invertible we have $\operatorname{rank} H_j = n_j$ for $j = 2, \ldots, r$.

We define

$$\Omega := \{(x,t) \in \mathbb{R}^2 \,|\, 0 \le t \le x\}. \tag{4.2}$$

**Theorem 4.2** *Let $Q = (Q_{ij})_{i,j=1}^r : \mathbb{R}_+ \longrightarrow \mathrm{M}(2n, \mathbb{C})$ continuous, where $Q_{ij}$ denotes the block–matrix decomposition with respect to the orthogonal decompostion $\mathbb{C}^{2n} = \bigoplus_{i=1}^{r} \mathbb{C}^{n_i}$. Moreover, we assume that $Q$ is off–diagonal, i.e.*

$$Q_{ii} = 0, \quad i = 1, \ldots, r. \tag{4.3}$$

*Let again $Y$ be the solution of the initial value problem* (3.2)*. Then there exists a unique continuous function $K : \Omega \longrightarrow \mathrm{M}(2n, \mathbb{C})$ such that we have*

$$Y(x,\lambda) = e_0(x,\lambda) + \int_0^x K(x,t) e_0(t,\lambda) dt. \tag{4.4}$$

*If $Q \in C^1(\mathbb{R}_+, \mathrm{M}(2n,\mathbb{C}))$ then $K \in C^1(\Omega, \mathrm{M}(2n,\mathbb{C}))$ and it satisfies*

$$B\partial_x K(x,t) + \partial_t K(x,t) B + iQ(x) K(x,t) = 0, \tag{4.5a}$$

$$BK(x,x) - K(x,x)B + iQ(x) = 0, \tag{4.5b}$$

$$K(x,0) B \begin{pmatrix} I \\ H \end{pmatrix} = 0. \tag{4.5c}$$

*If $Q \in C(\mathbb{R}_+, \mathrm{M}(n,\mathbb{C})$ then $K$ is the generalized continuous solution of* (4.5)*.*

*Conversely, if $K$ is a (generalized) solution of* (4.5) *then $Y(x,\lambda)$ defined by* (4.4) *is the (generalized) solution of the initial value problem* (3.2)*.*

**Sketch of proof** i) Suppose that $K \in C^1(\mathbb{R}_+, \mathrm{M}(2n,\mathbb{C}))$ and that formula (4.4) holds. Substituting (4.4) into (3.2) and integrating by parts one obtains

$$[BK(x,x) - K(x,x)B + iQ(x)] e_0(x,\lambda) + K(x,0) B e_0(0,\lambda)$$
$$+ \int_0^x [B\partial_x K(x,t) + \partial_t K(x,t) B + iQ(x) K(x,t)] e_0(t,\lambda) dt = 0. \tag{4.6}$$

Since $e_0(0,\lambda) = \binom{I}{H}$ does not depend on $\lambda$ one concludes from (4.6) and the Riemann–Lebesgue Lemma that (4.6) is equivalent to (4.5). Thus in this case the representation (4.4) is equivalent to the solvability of the problem (4.5).

ii) Next we prove the existence of a (not unique) solution of the problem (4.5a)–(4.5b). Let R(x,t) be one of them. Using the block–matrix representation $R(x,t) = (R_{ij}(x,t))_{i,j=1}^r$ we rewrite the problem (4.5a)–(4.5b) as

$$\lambda_i \partial_x R_{ij}(x,t) + \lambda_j \partial_t R_{ij}(x,t) = -i \sum_{p=1}^{r} Q_{ip}(x) R_{pj}(x,t), \quad 1 \le i,j \le r \tag{4.7}$$

$$R_{ij}(x,x) = -i(\lambda_i - \lambda_j)^{-1}\, Q_{ij}(x), \quad 1 \le i \ne j \le r \tag{4.8}$$

It is clear that the system (4.7) is hyperbolic with real characteristics $l_{ij} : x = k_{ij}t + c(k_{ij} = \lambda_j \lambda_i^{-1})$. Thus, in $\Omega = \{0 \le t \le x < \infty\}$ we have the incomplete characteristic Cauchy problem (4.7), (4.8) with $(2n)^2 - n_1^2 - \ldots - n_r^2$ scalar conditions (4.8). Fixing $x_0 \in \mathbb{R}_+$ and setting

$$k_{\min} = \max\{k_{ij} : \; k_{ij} \in (0,1), \; 1 \le i,j \le r\}, \; k_{\max} = k_{\min}^{-1}$$

we consider the triangle $\triangle_{ABC}$ confined by the lines $AB : \; x = t, \; AC : \; x - x_0 = k_{\min} t, \; BC : \; x - x_0 = k_{\max} t$. We preserve the notation $Q(x)$ for a continuous extension to $\mathbb{R}$ of the function $Q(x)$ with the same norm. Furthermore, we denote by $a$ and $b$ the abscissas of the points $A$ and $B$ respectively. Now we impose the following $n_1^2 + \ldots + n_r^2$ conditions on the characteristic line $AC$ :

$$R_{jj}(x, (x-1)k_{\min}) = 0, \quad \text{for} \quad x \in [a, x_0], \quad j \in 1, ..., r. \tag{4.9}$$

Thus, we arrive at the Goursat problem (4.7)–(4.9) for the hyperbolic system (4.7) in the triangle $\triangle_{ABC}$. Integrating the system (4.7) along the characteristics and using (4.8), (4.9) one deduces the system of integral equations

$$\lambda_i R_{ij}(x,t) = \frac{\lambda_i Q_{ij}(\xi_{ij}(x,t))}{\lambda_i - \lambda_j} + \int_{\xi_{ij}(x,t)}^{x} \sum_{p=1}^{r} Q_{ip}(\xi) R_{pj}(\xi, (\xi - x)k_{ij} + t) d\xi \tag{4.10}$$

where for brevity it is set $\lambda_i(\lambda_i - \lambda_j)^{-1} Q_{ij}(\xi_{ij}(x,t)) = 0$ for $i = j$ and

$$\xi_{ij}(x,t) = \begin{cases} (\lambda_j x - \lambda_i t)(\lambda_j - \lambda_i)^{-1}, & i \ne j, \\ a + (1-a)(x-t), & i = j. \end{cases}$$

For $Q \in C^1(\mathbb{R}, \mathrm{M}(n,\mathbb{C})$ the system (4.10) is equivalent to the Goursat problem (4.7)–(4.9). The solvability (and uniqueness) of the solution of (4.10) is proven by the method of successive approximations.

For $Q \in C(\mathbb{R}, \mathrm{M}(n,\mathbb{C})) \setminus C^1(\mathbb{R}, \mathrm{M}(n,\mathbb{C})$ we understand the solution of (4.7)–(4.9) as a solution of (4.10).

iii) To finish the proof, starting with the solution $R(x,t)$ of the Goursat problem (4.7)–(4.9) we introduce a convolution operator

$$\Phi : f \to \int_0^x \Phi(x-t) f(t) dt$$

with $\Phi(x) = \operatorname{diag}(\Phi_1(x), \ldots, \Phi_r(x))$ being a block - diagonal $2n \times 2n$ matrix function, consisting of $n_j \times n_j$ blocks $\Phi_j$ and define the operator $K$ by the equality $I + K = (I + R)(I + \Phi)$. It is clear that $K$ is a Volterra operator with the kernel

$$K(x,t) = R(x,t) + \Phi(x-t) + \int_0^x R(x,s)\Phi(s-t)ds \tag{4.11}$$

Since the operator $I+R$ intertwines the restrictions $L_0$ and $-iB \otimes D_0$ of the operators $L$ and $-iB \otimes D$ onto $W^1_{2,0}[0,1] \otimes \mathbb{C}^{2n} = \{f \in W^1_2 \otimes \mathbb{C}^{2n} : \ f(0) = 0\}$, that is $L_0(I+R) = (I+R)(-iB \otimes D_0)$, so is $I + K$. This fact amounts to saying that $K(x,t)$ satisfies the problem (4.5a)–(4.5b). To satisfy the condition (4.5c) it suffices (in view of (4.11)) to choose $\Phi(x)$ as the solution of the equation

$$\Phi(x)B\begin{pmatrix} I \\ H \end{pmatrix} + \int_0^x R(x,s)\Phi(s)B\begin{pmatrix} I \\ H \end{pmatrix} ds = -R(x,0)B\begin{pmatrix} I \\ H \end{pmatrix}. \tag{4.12}$$

Let $H = col(H_2, \ldots, H_r)$ be the block matrix representation of $H$ corresponding to the representation (4.1). Since $\operatorname{rank} H_j = n_j$ $(2 \le j \le r)$ (see Remark 4.1) the Volterra equation (4.12) is of the second kind and therefore has the unique solution $\Phi \in C[0,\infty) \otimes \mathbb{C}^{2n \times 2n}$. Thus $K(x,t)$ is the required solution of (4.5a)–(4.5c). □

**2.** We continue with some general remarks about Volterra operators:

For any continuous matrix function $K : \Omega \longrightarrow \mathrm{M}(2n, \mathbb{C})$ we obtain a Volterra operator

$$Kf(x) := \int_0^x K(x,t)f(t)dt \tag{4.13}$$

acting on $C(\mathbb{R}_+, \mathbb{C}^{2n})$ or $L^2([0,a], \mathbb{C}^{2n})$ for any $a > 0$. By slight abuse of notation we will use the same symbol for the operator and its kernel. The set of operators $I + K$ with $K$ being a Volterra operator forms a group. The operator

$$R := (I+K)^{-1} - I \tag{4.14}$$

is again a Volterra operator with continuous kernel $R(x,t), t \le x$. From the equation

$$I = (I+R)(I+K) = (I+K)(I+R) \tag{4.15}$$

we deduce

$$RK = KR = -R - K. \tag{4.16}$$

Put

$$F := R + R^* + RR^*. \tag{4.17}$$

The kernel of $F$ obviously is

$$F(x,t) = \begin{cases} R(x,t) + \displaystyle\int_0^t R(x,s)R(t,s)^* ds, & x > t, \\ R(t,x)^* + \displaystyle\int_0^x R(x,s)R(t,s)^* ds, & x < t. \end{cases} \tag{4.18}$$

Furthermore, using (4.16) we conclude

$$F + K + KF = R + K + R^* + RR^* + KR + KR^* + KRR^* = R^* \tag{4.19}$$

thus we have the "Gelfand–Levitan equation"

$$F + K - R^* + KF = 0. \tag{4.20}$$

**Proposition 4.3** *Let $K : \Omega \longrightarrow \mathrm{M}(2n, \mathbb{C})$ be continuous and let $R : \Omega \longrightarrow \mathrm{M}(2n, \mathbb{C})$ be the continuous kernel of the Volterra operator $(I + K)^{-1} - I$. Then the function $F : \mathbb{R}_+^2 \longrightarrow \mathrm{M}(2n, \mathbb{C})$ defined by (4.18) satisfies the "Gelfand–Levitan equation"*

$$F(x,t) + K(x,t) + \int_0^x K(x,s)F(s,t)ds = 0, \quad x > t, \tag{4.21}$$

$$F(x,t) - R(t,x)^* + \int_0^x K(x,s)F(s,t)ds = 0, \quad x < t. \tag{4.22}$$

*Conversely, if $F_1 : \Omega \longrightarrow \mathrm{M}(2n, \mathbb{C})$ is continuous and satisfies (4.21) then $F_1 = F|\Omega$.*

**Proof** It only remains to prove the assertion about $F_1$. The difference $F(x,t) - F_1(x,t)$ satisfies the equation

$$F(x,t) - F_1(x,t) + \int_0^x K(x,s)[F(s,t) - F_1(s,t)]ds = 0, \quad 0 \le t \le x.$$

For each fixed $t \in [0, x]$ this is a homogeneous Volterra equation of the second kind and consequently has only the trivial solution $F(x,t) - F_1(x,t) = 0$. □

From now on $K$ denotes the unique Volterra operator with continuous kernel satisfying (4.4). As before $R$ denotes the Volterra operator defined by $R := (I + K)^{-1} - I$.

In particular we have in view of (4.4)

$$e_0(x, \lambda) = ((I + R)Y(\cdot, \lambda))(x) = Y(x, \lambda) + \int_0^x R(x,t)Y(t, \lambda)dt. \tag{4.23}$$

**Lemma 4.4** 1. *Let $\sigma$ be the spectral function of the boundary value problem (3.1) and let $g \in L^2_{\mathrm{comp}}(\mathbb{R}_+, \mathbb{C}^{2n})$. Put*

$$G_0(\lambda) := (\mathcal{F}_{H,0}g)(\lambda) = \int_0^\infty e_0(x, \lambda)^* g(x)dx. \tag{4.24}$$

*Then $G_0 \in L^2_\sigma(\mathbb{R})$ and if*

$$\int_{\mathbb{R}} G_0(\lambda)^* d\sigma(\lambda) G_0(\lambda) = 0 \tag{4.25}$$

*then $g = 0$.*

2. *We have $\mathcal{F}_{H,0}(L^2_{\mathrm{comp}}(\mathbb{R}_+, \mathbb{C}^{2n})) = \mathcal{F}_{H,Q}(L^2_{\mathrm{comp}}(\mathbb{R}_+, \mathbb{C}^{2n}))$.*

**Proof** In view of (4.23) we have

$$\begin{aligned} G_0(\lambda) &= \int_0^\infty [Y(x,\lambda)^* + \int_0^x Y(t,\lambda)^* R(x,t)^* dt] g(x)dx \\ &= \int_0^\infty Y(x,\lambda)^* [g(x) + \int_x^\infty R(t,x)^* g(t)dt]dx, \end{aligned} \tag{4.26}$$

hence $G_0(\lambda)$ is also the $\mathcal{F}_{H,Q}$–transform of the function

$$\widetilde{g}(x) := ((I+R^*)g)(x) = g(x) + \int_x^\infty R(t,x)^* g(t)dt. \tag{4.27}$$

Since $g \in L^2_{\text{comp}}(\mathbb{R}_+, \mathbb{C}^{2n})$ we also have $\widetilde{g} \in L^2_{\text{comp}}(\mathbb{R}_+, \mathbb{C}^{2n})$. This shows the inclusion $\mathcal{F}_{H,0}(L^2_{\text{comp}}(\mathbb{R}_+, \mathbb{C}^{2n})) \subset \mathcal{F}_{H,Q}(L^2_{\text{comp}}(\mathbb{R}_+, \mathbb{C}^{2n}))$. The converse inclusion is proved analogously using (4.4) instead of (4.23).

In view of the Parseval equality (Theorem 3.2) we find

$$\int_{\mathbb{R}} G_0(\lambda)^* d\sigma(\lambda) G_0(\lambda) = \int_0^\infty \widetilde{g}(x)^* \widetilde{g}(x)dx,$$

which by assumption (4.25) implies $\widetilde{g} = 0$. Since $g$ has compact support (4.27) is a Volterra equation and thus $g = 0$. □

**Proposition 4.5** *Let $\sigma$ be the spectral function of the boundary value problem* (3.1) *and let $\sigma_0 = \frac{1}{2\pi}\lambda\lambda_1^{-1} I_n$ be the corresponding spectral function for $Q = 0$. We abbreviate $\Sigma := \sigma - \sigma_0$.*

1. *Let $I + R$ be the transformation operator of the form* (4.23) *and $F$ be the $2n \times 2n$ matrix function defined by* (4.18), *i.e.*

$$F(x,t) := \begin{cases} R(x,t) + \displaystyle\int_0^t R(x,s)R(t,s)^* ds, & x > t > 0, \\ R(t,x)^* + \displaystyle\int_0^x R(x,s)R(t,s)^* ds, & 0 < x < t. \end{cases} \tag{4.28}$$

*Then we have for all $f, g \in L^2_{\text{comp}}(\mathbb{R}_+, \mathbb{C}^{2n})$*

$$\int_{\mathbb{R}} F_0(\lambda)^* d\Sigma(\lambda) G_0(\lambda) = \int_0^\infty \int_0^\infty f(x)^* F(x,t) g(t) dx dt, \tag{4.29}$$

*where $F_0, G_0$ denote the $\mathcal{F}_{H,0}$–transforms of $f, g$.*

2. *Putting*

$$\widetilde{e}_0(x,\lambda) := \int_0^x e_0(t,\lambda)dt \tag{4.30}$$

*the function*

$$\widetilde{F}(x,t) := \int_{\mathbb{R}} \widetilde{e}_0(x,\lambda) d\Sigma(\lambda) \widetilde{e}_0(t,\lambda)^* \tag{4.31}$$

*exists and has a continuous mixed second derivative which coincides with $F(x,t)$, i.e. $\frac{\partial^2}{\partial x \partial t}\widetilde{F}(x,t) = F(x,t)$.*

3. *Conversely, given any increasing $n \times n$ matrix function $\sigma$ put $\Sigma := \sigma - \sigma_0$. If the integral* (4.31) *exists and has a continuous mixed second derivative $F_1(x,t) := \frac{\partial^2}{\partial x \partial t}\widetilde{F}(x,t)$ then* (4.29) *holds for all $f, g \in L^2_{\text{comp}}(\mathbb{R}_+, \mathbb{C}^{2n})$ with $F_1$ instead of $F$.*

**Remark 4.6** We note that the identity (4.29) characterizes the spectral function of the problem (3.1). More precisely, if $\varrho$ is an increasing $n \times n$ matrix function such that(4.29) holds with $\Sigma_\varrho := \varrho - \sigma_0$ then $\varrho = \sigma$.

Indeed from (4.29) we infer

$$\begin{aligned} &\int_{\mathbb{R}} F_0(\lambda)^* d\sigma(\lambda) G_0(\lambda) = \int_{\mathbb{R}} F_0(\lambda)^* d\varrho(\lambda) G_0(\lambda), \\ &F_0 := \mathcal{F}_{H,0} f, \quad G_0 := \mathcal{F}_{H,0} g, \end{aligned} \tag{4.32}$$

for all $f, g \in L^2_{\text{comp}}(\mathbb{R}_+, \mathbb{C}^{2n})$. By Theorem 2.2 and Lemma 4.4, 2. this implies that (4.32) holds for all $F_0, G_0 \in L^2_\varrho(\mathbb{R})$, in particular it holds for all $F_0, G_0 \in C(\mathbb{R}, \mathbb{C}^n)$ with compact support. Thus the vector measures $d\sigma, d\varrho$ and hence the right–continuous functions $\varrho, \sigma$ coincide.

**Proof** 1. In view of (4.26) $F_0$ is the $\mathcal{F}_{H,Q}$–transform of

$$\widetilde{f}(x) = f(x) + \int_x^\infty R(t,x)^* f(t) dt, \tag{4.27'}$$

thus the Parseval equality (3.20) gives

$$\begin{aligned} &\int_{\mathbb{R}} F_0(\lambda)^* d\Sigma(\lambda) G_0(\lambda) = \int_{\mathbb{R}} F_0(\lambda)^* d\sigma(\lambda) G_0(\lambda) - (f,g) = (\widetilde{f}, \widetilde{g}) - (f,g) \\ = &\int_0^\infty (f(x) + \int_x^\infty R(t,x)^* f(t) dt)^* (g(x) + \int_x^\infty R(t,x)^* g(t) dt) dx - (f,g) \\ = &\int_0^\infty \int_0^\infty f(x)^* F(x,t) g(t) dx dt \end{aligned}$$

by a straightforward calculation.

2. For $x, t \geq 0$ and $f_0, g_0 \in \mathbb{C}^{2n}$ we apply 1. with $f(u) := 1_{[0,x]}(u) f_0, g(v) := 1_{[0,t]}(v) g_0$ and find

$$f_0^* \int_R \widetilde{e}_0(x,\lambda) d\Sigma(\lambda) \widetilde{e}_0(t,\lambda)^* g_0 = f_0^* \int_0^x \int_0^y F(u,v) du dv g_0, \tag{4.33}$$

which implies the first assertion.

3. To prove the converse statement we note that now we have (4.33) with $F_1(x,t) = \frac{\partial^2}{\partial x \partial t} \widetilde{F}(x,t)$. This identity implies (4.29) with $F_1$ instead of $F$ for step functions

$$f = \sum_{j=1}^n f_j 1_{[a_j, b_j[}, \quad g = \sum_{j=1}^n g_j 1_{[c_j, d_j[}, \quad f_j, g_j \in \mathbb{C}^{2n}. \tag{4.34}$$

There is a slight subtlety since $\Sigma$ is not necessarily increasing. However, we conclude from (4.29) and the Parseval equality that for all step functions $f, g$

$$(F_0, G_0)_{L^2_\sigma(\mathbb{R})} = (f,g)_{L^2(\mathbb{R}_+, \mathbb{C}^{2n})} + \int_0^\infty \int_0^\infty f(x)^* F(x,t) g(t) dx dt. \tag{4.35}$$

Since $\sigma$ is increasing the assertion now follows from the denseness of the step functions in $L^2_{\text{comp}}(\mathbb{R}_+, \mathbb{C}^{2n})$. To complete the proof it remains to note that the equality $F(x,t) = F_1(x,t)$ is a consequence of (4.29) and (4.35)

□

Combining Propositions 4.3 and 4.5 one immediately obtains the following theorem.

**Theorem 4.7** *Let $\sigma$ be the spectral measure of the problem* (3.1) *and* $\sigma_0(\lambda) = \frac{1}{2\pi}\lambda_1^{-1}\lambda I_n$. *Then with $F$ defined by* (4.31) *we have the Gelfand–Levitan equation*

$$F(x,t) + K(x,t) + \int_0^x K(x,s)F(s,t)ds = 0, \quad t < x. \tag{4.36}$$

**Remark 4.8** Note that by Proposition 4.5 2. the function $F$ is continuous also on the diagonal. In view of (4.18) the continuity of $F$ at the diagonal implies $R(x,x) = R(x,x)^*$.

**Proof** We present a second proof of the Gelfand–Levitan equation based on the formula (4.31) for $F$, which is similar to [8] and [15, Chap. 12].

For $f, g \in L^2_{\mathrm{comp}}(\mathbb{R}_+, \mathbb{C}^{2n})$ we consider

$$\mathrm{I}(f,g) := \int_{\mathbb{R}} \int_0^\infty dx \int_0^\infty dt\, f(x)^* Y(x,\lambda) d\sigma(\lambda) e_0(t,\lambda)^* g(t). \tag{4.37}$$

Substituting (4.4) for $Y$ we find using the Parseval equality and Lemma 4.5

$$\begin{aligned} \mathrm{I}(f,g) &= (f,g) + \int_0^\infty \int_0^\infty f(x)^* F(x,t) g(t) dx dt \\ &\quad + \underbrace{\int_{\mathbb{R}} \int_0^\infty dx \int_0^\infty dt\, f(x)^* \int_0^\infty K(x,s) e_0(s,\lambda) ds d\sigma(\lambda) e_0(t,\lambda)^* g(t)}_{\mathrm{II}(f,g)} \qquad (4.38) \\ \mathrm{II}(f,g) &= \int_{\mathbb{R}} \int_0^\infty dx \int_0^\infty ds \int_0^\infty f(x)^* K(x,s) dx e_0(s,\lambda) d\sigma(\lambda) e_0(t,\lambda)^* g(t). \end{aligned}$$

Writing $d\sigma = d\Sigma + d\sigma_0$ and using Lemma 4.5 we find

$$\begin{aligned} \mathrm{II}(f,g) &= \int_0^\infty \int_0^\infty f(x)^* K(x,t) g(t) dx dt \\ &\quad + \int_0^\infty \int_0^\infty \int_0^\infty f(x)^* K(x,s) F(s,t) g(t) dx dt ds, \end{aligned} \tag{4.39}$$

hence

$$\mathrm{I}(f,g) = (f,g) + \int_0^\infty \int_0^\infty f(x)^* [F(x,y) + K(x,y) + \int_0^x K(x,t)F(t,y)dt] g(y) dx dy. \tag{4.40}$$

Now if $\operatorname{supp} f \subset [b,\infty)$, $\operatorname{supp} g \subset [0,a]$, $a < b$, then $(f,g) = 0$ and

$$\int_0^\infty e_0(x,\lambda)^* g(x) dx$$

is the $\mathcal{F}_{H,Q}$–transform of

$$g(x) + \int_x^\infty R(t,x)^* g(t) dt$$

which also has support in $[0,a]$, hence by the Parseval equality $\mathrm{I}(f,g) = 0$. This implies the assertion. □

# 5. The inverse problem

**Proposition 5.1** *Let $\sigma(\lambda)$ be a $n \times n$ matrix function satisfying:*

1. *If $g \in L^2_{\mathrm{comp}}(\mathbb{R}_+, \mathbb{C}^{2n})$ and if*
$$\int_{\mathbb{R}} G_0(\lambda)^* d\sigma(\lambda) G_0(\lambda) = 0,$$
*where $G_0$ is the $\mathcal{F}_{H,0}$–transform of $g$, then $g = 0$.*

2. *The function*
$$\widetilde{F}(x,t) := \int_{\mathbb{R}} \widetilde{e}_0(x,\lambda) d\Sigma(\lambda) \widetilde{e}_0(t,\lambda)^* \tag{5.1}$$
*with $\Sigma = \sigma - \sigma_0$ exists, and has a continuous mixed second derivative*
$$F(x,t) := \frac{\partial^2}{\partial x \partial t} \widetilde{F}(x,t). \tag{5.1'}$$

*Then the Gelfand–Levitan equation* (4.36) *has a unique continuous solution $K : \Omega \longrightarrow$ $\mathrm{M}(2n, \mathbb{C})$.*

*Moreover, if $F(x,t)$ is continuously differentiable, then so is $K(x,t)$.*

**Proof** Since for fixed $x$ equation (4.36) is a Fredholm equation it suffices to show that the dual equation
$$k(t) + \int_0^x k(s) F(t,s)^* ds = 0, \tag{5.2}$$
where $k : [0,x] \to M(2n, \mathbb{C})$ is square integrable, has only the zero solution. Looking at the individual columns in (5.2) it suffices to show that
$$g(t)^* + \int_0^x g(s)^* F(t,s)^* ds = 0, \quad g \in L^2([0,x], \mathbb{C}^{2n}) \tag{5.3}$$
implies $g = 0$. Extending $g$ by 0 to $\mathbb{R}_+$ we may consider $g$ as an element of $L^2_{\mathrm{comp}}(\mathbb{R}_+, \mathbb{C}^{2n})$ and (5.3) implies in view of 2. and Proposition 4.5, 3.
$$\begin{aligned} 0 &= \|g\|^2 + \int_0^\infty \int_0^\infty g(s)^* F(s,t) g(t) ds dt \\ &= \|g\|^2 + \int_{\mathbb{R}} G_0(\lambda)^* d\Sigma(\lambda) G_0(\lambda) = \int_{\mathbb{R}} G_0(\lambda)^* d\sigma(\lambda) G_0(\lambda) \end{aligned}$$
and thus $g = 0$ by 1.

The proof of $C^1$–smoothness of $K(x,t)$ is similar to that used in [15] and [8] and is omitted. □

Next we prove the main result of this paper:

**Theorem 5.2** *For an increasing $n \times n$ matrix function $\sigma(\lambda)$ to be the spectral measure function of the boundary value problem* (3.1) *with continuous $2n \times 2n$ matrix potential $Q$ satisfying* (4.3) *it is necessary and sufficient that the conditions* 1. *and* 2. *of Proposition* 5.1 *hold.*

**Proof** The necessity was proved in Lemma 4.4 and Proposition 4.5.

To prove the sufficiency we assume that the conditions 1. and 2. of Proposition 5.1 hold:

i) Starting with $\sigma(\lambda)$ we define $\widetilde{F}, F$ by (5.1) and (5.1'). Then we consider the Gelfand–Levitan equation (4.36)

$$\Phi(x,t) := F(x,t) + K(x,t) + \int_0^x K(x,s)F(s,t)ds = 0, \quad x > t. \tag{5.4}$$

By Proposition 5.1 this equation has a unique continuous solution $K : \Omega \to \mathrm{M}(2n, \mathbb{C})$.

Then $F$ also equals the right hand side of (4.18): namely, starting with $K$ we consider the operator $R$ of the form (4.14) and introduce $F_1$ by (4.18). According to Proposition 4.3 $F_1$ and $K$ are connected by equation (4.21). Thus $F$ defined by (5.1) and $F_1$ defined by (4.18) satisfy the equation (5.4) and therefore we infer from Proposition 4.3 that $F = F_1$.

We collect further properties of $F$: in view of (4.18) we have

$$F(x,t) = F(t,x)^*. \tag{5.5a}$$

By continuity, the equation (4.18) also holds for $x = t$ and consequently $R(x,x)$ is self–adjoint. Therefore, so is $K(x,x) = -R(x,x)$. Furthermore,

$$\partial_t F(x,t)B = -B\partial_x F(x,t), \tag{5.5b}$$

where this equality holds in the distributional sense if $F$ is only continuous. To see this let $f, g \in C_0^\infty((0,\infty), \mathbb{C}^{2n})$. In view of (4.29) and (3.22) applied with $Q = 0$ we calculate

$$\begin{aligned}
\int_0^\infty \int_0^\infty f(x)^* \partial_t F(x,t) B g(t) dx dt &= -i \int_0^\infty \int_0^\infty f(x)^* F(x,t) \frac{1}{i} B \partial_t g(t) dx dt \\
= -i \int_{\mathbb{R}} (\mathcal{F}_{H,0} f)(\lambda)^* d\Sigma(\lambda) \lambda (\mathcal{F}_{H,0} g)(\lambda) &= -i \int_{\mathbb{R}} (\mathcal{F}_{H,0} \frac{1}{i} B f')(\lambda)^* d\Sigma(\lambda) (\mathcal{F}_{H,0} g)(\lambda) \\
= -\int_0^\infty \int_0^\infty f(x)^* B \partial_x F(x,t) g(t) dx dt. &
\end{aligned}$$

Moreover, it follows from (5.1) and (2.10) that with some matrix function $T(t)$ we have

$$F(0,t) = \begin{pmatrix} I \\ H \end{pmatrix} T(t). \tag{5.5c}$$

We now define (cf. (4.4))

$$Y(x,\lambda) = e_0(x,\lambda) + \int_0^x K(x,t) e_0(t,\lambda) dt \tag{5.6}$$

and we will show that the properties (5.5a-c) imply that $Y(x,\lambda)$ satisfies the initial value problem

$$B \frac{1}{i} \frac{dY(x,\lambda)}{dx} + Q(x) Y(x,\lambda) = \lambda Y(x,\lambda), \quad Y(0,\lambda) = e_0(0,\lambda) = \begin{pmatrix} I \\ H \end{pmatrix}, \tag{5.7}$$

where

$$Q(x) := iBK(x,x) - iK(x,x)B. \tag{5.8}$$

Note that since $K(x,x)$ is self–adjoint $Q(x)$ is self–adjoint, too. Moreover, from (5.8) we also conclude that $Q(x)$ is off–diagonal, i.e. $Q_{ii} = 0$.

It follows from (5.5c) that

$$F(x,0)BF(0,t) = T(x)^*[B_1 - H^*B_2H]T(t) = 0. \tag{5.9}$$

Plugging (5.9) into the Gelfand–Levitan equation (5.4) gives

$$K(x,0)B\begin{pmatrix} I \\ H \end{pmatrix} = 0 \quad \text{for} \quad x \in [0,\infty). \tag{5.10}$$

ii) For the moment we assume in addition that $F$ is continuously differentiable. Then by Proposition 5.1 $K$ also is continuously differentiable. Differentiating (5.4) we obtain

$$\begin{aligned} B\partial_x\Phi(x,t) &= B\partial_x F(x,t) + B\partial_x K(x,t) + BK(x,x)F(x,t) \\ &\quad + \int_0^x B\partial_x K(x,s)F(s,t)ds = 0, \end{aligned} \tag{5.11}$$

$$\begin{aligned} \partial_t\Phi(x,t)B &= \partial_t F(x,t)B + \partial_t K(x,t)B \\ &\quad + \int_0^x K(x,s)\partial_t F(s,t)Bds = 0. \end{aligned} \tag{5.12}$$

Integrating by parts and using (5.5b) and (5.10) we obtain

$$\begin{aligned} &\int_0^x K(x,s)\partial_t F(s,t)Bds = -\int_0^x K(x,s)B\partial_s F(s,t)ds \\ = &\int_0^x \partial_s K(x,s)BF(s,t)ds - K(x,x)BF(x,t). \end{aligned} \tag{5.13}$$

Adding up (5.11) and (5.12) and using (5.13) and the Gelfand–Levitan equation (5.4) we obtain

$$\begin{aligned} &B\partial_x K(x,t) + \partial_t K(x,t)B + iQ(x)K(x,t) \\ &+ \int_0^x [B\partial_x K(x,s) + \partial_s K(x,s)B + iQ(x)K(x,s)]F(s,t)ds = 0. \end{aligned} \tag{5.14}$$

Since the homogeneous integral equation corresponding to the Gelfand–Levitan equation (5.4) has only the trivial solution (see the proof of Proposition 5.1) we infer from (5.14) that

$$B\partial_x K(x,t) + \partial_t K(x,t)B + iQ(x)K(x,t) = 0. \tag{5.15}$$

Since $K$ satisfies the relations (5.10), (5.8) and (5.15) it follows from Theorem 4.2 that $Y(x,\lambda)$ (cf. (5.6)) satisfies the initial value problem (5.7).

iii) We now assume that $F$ is just continuous. Assume for the moment that for $\delta > 0$ we have a continuously differentiable matrix function $F^\delta : \mathbb{R}_+^2 \to \mathrm{M}(2n,\mathbb{C})$ with the properties:

$$F^\delta \text{ converges to } F \text{ as } \delta \to 0 \text{ uniformly on compact subsets of } \mathbb{R}^2_+. \tag{5.16a}$$

$$F^\delta \text{ satisfies (5.5a-c).} \tag{5.16b}$$

We fix $x_0 > 0$. For $0 < x \le x_0$ let $T_F$ be the integral operator in $C([0,x],\mathbb{C}^{2n})$ defined by $(T_F f)(t) := \int_0^x f(s)F(s,t)ds$. The proof of Proposition 5.1 shows that $-1 \notin \operatorname{spec} T_F$. Thus for $\delta \le \delta_0(x_0)$ we have $-1 \notin \operatorname{spec} T_{F^\delta}$ and the Gelfand–Levitan equations

$$((I + T_{F^\delta})K_\delta(x,.))(t) = K_\delta(x,t) + \int_0^x K_\delta(x,s)F^\delta(s,t)ds = -F^\delta(x,t) \tag{5.17}$$

have (for each fixed $x \in (0,x_0]$) unique solutions $K_\delta(x,t)$, $(x,t) \in [0,x_0]^2$, which converge to $K$ as $\delta \to 0$ uniformly on $[0,x_0]^2$. Since $F^\delta$ is $C^1$ it can be shown (cf. the proof of Proposition 5.1) that $K_\delta$ is $C^1$, too.

Moreover, $K_\delta$ satisfies (5.10) for $0 < x \le x_0$ which follows from (5.16b) and (5.17). Now part ii) of this proof shows that $K_\delta$ also satisfies (5.15) with $Q_\delta(x) := iBK_\delta(x,x) - iK_\delta(x,x)B, x \in [0,x_0]$. Hence, $K_\delta$ satisfies (4.5a-c) (on $[0,x_0]^2$) and therefore,

$$Y_\delta(x,\lambda) := e_0(x,\lambda) + \int_0^x K_\delta(x,t)e_0(t,\lambda)dt, \quad 0 \le x \le x_0,$$

satisfies the initial value problem (5.7) with $Q_\delta$ instead of $Q$.

Since $F^\delta(x,t)^* = F^\delta(t,x)$ one concludes as in part i) of this proof that $Q_\delta(x)^* = Q_\delta(x)$.

Since $K_\delta$ converges to $K$ as $\delta \to 0$ uniformly on $[0,x_0]^2$, $Q_\delta$ converges to $Q$ uniformly on $[0,x_0]$. Thus $Y(x,\lambda)$ satisfies the initial value problem (5.7) on $[0,x_0]$. Since $x_0$ was arbitrary $Y(x,\lambda)$ satisfies (5.7) on $\mathbb{R}_+$.

It remains to prove the existence of the sequence $F^\delta$:

Let $F(x,t) := (F_{ij}(x,t))_{i,j=1}^r$ be the block matrix representation with respect to the orthogonal decomposition $\mathbb{C}^{2n} = \oplus_{i=1}^r \mathbb{C}^{n_i}$.

It follows from (5.1) and (5.1') that

$$F_{ij}(x,t) = f_{ij}(\mu_i x - \mu_j t), \qquad f_{ij}(\xi) = H_i g(\xi) H_j^*, \tag{5.18}$$

with $\mu_i = \lambda_i^{-1}$, $1 \le i \le r$, and $H_1 := I_{n_1} = I_n$. Here the map $g : \mathbb{R} \to \mathrm{M}(n \times n, \mathbb{C})$ is continuous and satisfies $g(\xi)^* = g(-\xi)$. Therefore the maps $f_{ij} : \mathbb{R} \to M(n_i \times n_j, \mathbb{C})$ are continuous and satisfy $f_{ij}(\xi)^* = f_{ji}(-\xi)$. We note that if the measure $\Sigma(\lambda)$ is finite, that is $\int_{\mathbb{R}} |d\Sigma(\lambda)| \in M(n,\mathbb{C})$, then $g(\xi) = \int_{\mathbb{R}} e^{i\lambda\xi} d\Sigma(\lambda)$.

We put

$$g^\delta(\xi) := \frac{1}{2\delta}\int_{\xi-\delta}^{\xi+\delta} g(s)ds, \quad f_{ij}^\delta(\xi) := H_i g^\delta(\xi) H_j^*, \quad F_{ij}^\delta(x,t) := f_{ij}^\delta(\mu_i x - \mu_j t), \tag{5.19}$$

and $F^\delta(x,t) := (F_{ij}^\delta(x,t))_{i,j=1}^r$.

Obviously, $F^\delta$ is continuously differentiable and satisfies (5.16a).

It is clear from (5.18) that

$$g^\delta(\xi)^* = \frac{1}{2\delta}\int_{\xi-\delta}^{\xi+\delta} g^\delta(s)^* ds = \frac{1}{2\delta}\int_{\xi-\delta}^{\xi+\delta} g^\delta(-s)ds = \frac{1}{2\delta}\int_{-\xi-\delta}^{-\xi+\delta} g^\delta(s)ds = g^\delta(-\xi), \tag{5.20}$$

and thus $f_{ij}^{\delta}(\xi)^* = f_{ji}^{\delta}(-\xi)$.

In view of (5.19) and (5.20) $F^{\delta}$ satisfies (5.5a,b). To prove the property (5.5c) for $F^{\delta}$ we note that in view of (5.18) and (5.19) $F_{ij}^{\delta}(0,t) = f_{ij}^{\delta}(-\mu_j t) = H_i g^{\delta}(-\mu_j t) H_j^*$ and consequently

$$F^{\delta}(0,t) = (F_{ij}^{\delta}(0,t))_{i,j=1}^{r} = (H_i g^{\delta}(-\mu_j t) H_j^*)_{i,j=1}^{r} =: \begin{pmatrix} I \\ H \end{pmatrix} T^{\delta}(t), \tag{5.21}$$

where $T^{\delta}(t) = (g^{\delta}(-\mu_1 t)H_1^*, g^{\delta}(-\mu_2 t)H_2^*, \ldots, g^{\delta}(-\mu_r t)H_r^*)$.

This proves that $F^{\delta}$ satisfies (5.5c). Summing up, we have proved that $F^{\delta}$ satisfies (5.16a,b).

iv) Starting with an increasing $n \times n$ matrix function $\sigma(\lambda)$ satisfying the conditions 1. and 2. of Proposition 5.1 we have constructed the boundary value problem (3.1) resp. (5.7). To complete the proof it remains to show that $\sigma(\lambda)$ is, in fact, the spectral function for the problem (5.7).

Let $\varrho(\lambda)$ be the spectral function of the problem (5.7). Starting with $\Sigma_{\varrho} := \varrho - \sigma_0$ we define $F_{\varrho}$ by (5.1'). Then by Theorem 4.7 $K$ satisfies the Gelfand-Levitan equation (4.36) with $F_{\varrho}$. On the other hand, in view of (5.4) $K$ satisfies the Gelfand–Levitan equation with $F$ instead of $F_{\varrho}$. From Proposition 4.3 we infer $F = F_{\varrho}$.

By Remark 4.6 this implies $\varrho = \sigma$. □

**Remark 5.3** The case $n = 1$ and $B_1 = B_2 = 1$, i.e. the case of a $2 \times 2$ Dirac system, is due to M. Gasymov and B. Levitan [7], [15, Chap. 12]. We note, however, that the proof in [15, Chap. 12] is incomplete, since the self-adjointness of $Q$ is not proved.

# 6. Some generalizations, comments, examples

## 6.1 Generalization of the main result Theorem 5.2

Before we have investigated an operator $L$ of the form (3.1) starting with the operator $L_0$ (with $Q = 0$). This has an obvious generalization. Namely, we may investigate two operators $L_1 := L_{1,H}, L_2 := L_{2,H}$ and consider $L_2$ as a perturbation of $L_1$. More precisely, let

$$L_j = \frac{1}{i} B \frac{d}{dx} + Q_j, \tag{6.1}$$

and

$$\mathcal{D}(L_j) = \{f \in \mathcal{D}(L_j^*) \mid f_1(0) = H f_2(0)\}, \quad B_1 = H^* B_2 H. \tag{6.2}$$

Furthermore, let $Y_j$ be the $2n \times n$ matrix solution of the initial value problem (3.2) (with $L_j$ instead of $L$). By Theorem 3.2 $Y_j$ admits the representation $Y_j(., \lambda) = (I + K_j) e_0(., \lambda)$, where $K_j$ is a Volterra operator with kernel $K_j(x,t)$. Therefore

$$Y_2(x,\lambda) = ((I+K)Y_1(.,\lambda))(x) = Y_1(x,\lambda) + \int_0^x K(x,t) Y_1(t,\lambda)dt, \tag{6.3}$$

where

$$I + K = (1 + K_2)(I + K_1)^{-1}. \tag{6.4}$$

Repeating the arguments used in the proof of Theorem 4.2 one concludes that if $Q_1, Q_2 \in C^1(\mathbb{R}_+, \mathrm{M}(2n, \mathbb{C}))$ then $K \in C^1(\Omega, \mathrm{M}(2n, \mathbb{C}))$ and, moreover, $K$ satisfies the following Goursat problem

$$B\partial_x K(x,t) + \partial_t K(x,t)B + iQ_2(x)K(x,t) - iK(x,t)Q_1(t) = 0, \tag{6.5a}$$
$$BK(x,x) - K(x,x)B = i(Q_1(x) - Q_2(x)), \tag{6.5b}$$
$$K(x,0)B\begin{pmatrix} I \\ H \end{pmatrix} = 0. \tag{6.5c}$$

We also note that (6.5a)–(6.5c) may be deduced directly from (6.4) and (4.5) for $K_1, K_2$. For example (6.5b) follows from (4.5b) and the identity $K(x,x) = K_2(x,x) - K_1(x,x)$.

Putting $R := (I + K)^{-1} - I$ we obtain from (6.3)

$$Y_1(x,\lambda) = ((I+R)Y_2(.,\lambda))(x) = Y_2(x,\lambda) + \int_0^x R(x,t)Y_2(t,\lambda)dt. \tag{6.6}$$

Since Proposition 4.3 remains valid in the case under consideration, the following result, being a complete analog of Proposition 4.5, may be obtained in the same way as Proposition 4.5.

**Proposition 6.1** *Let $\sigma_j(\lambda)$ be the $n \times n$ spectral function (cf. Theorem 3.2) of the operator $L_j, j = 1, 2$, and $\Sigma := \sigma_2 - \sigma_1$.*

1. *Let $F(x,t)$ be defined by (4.28) with $R(x,t)$ being the kernel of the transformation operator (6.6). Then we have for all $f, g \in L^2_{\mathrm{comp}}(\mathbb{R}_+, \mathbb{C}^{2n})$*

$$\int_{\mathbb{R}} F_1(\lambda)^* d\Sigma(\lambda) G_1(\lambda) = \int_0^\infty \int_0^\infty f(x)^* F(x,t) g(t) dx dt, \tag{6.7}$$

*where $F_1$ and $G_1$ are the $\mathcal{F}_{H,Q_1}$-transforms of $f$ and $g$ respectively.*

2. *Putting*

$$\widetilde{Y}_1(x,\lambda) := \int_0^x Y_1(t,\lambda)dt$$

*the function*

$$\widetilde{F}(x,t) := \int_{\mathbb{R}} \widetilde{Y}_1(x,\lambda) d\Sigma(\lambda) \widetilde{Y}_1(t,\lambda)^* \tag{6.8}$$

*exists and has a continuous mixed second derivative which coincides with $F(x,t)$, i.e. $\frac{\partial^2}{\partial x \partial t}\widetilde{F}(x,t) = F(x,t)$.*

3. *Conversely, given any increasing $n \times n$ matrix function $\sigma_2$ put $\Sigma := \sigma_2 - \sigma_1$. If the integral (6.8) exists and has a continuous mixed second derivative $F_1(x,t) := \frac{\partial^2}{\partial x \partial t}\widetilde{F}(x,t)$ then (6.7) holds for all $f, g \in L^2_{\mathrm{comp}}(\mathbb{R}_+, \mathbb{C}^{2n})$ with $F_1$ instead of $F$.*

Combining Propositions 6.1 and 4.3 we arrive at the Gelfand-Levitan equation:

**Proposition 6.2** *Let $\sigma_j$ be the spectral function of the problem (3.1) with $Q_j = Q_j^*$, $j = 1, 2$, instead of $Q$. Then with $F$ defined by (6.8) we have the Gelfand-Levitan equation (4.36).*

Now we are ready to present a generalization of the main result (Theorem 5.2).

**Theorem 6.3** *Let $\sigma_1(\lambda)$ be the spectral function of the operator $L_1$ of the form* (6.1). *For an increasing $n \times n$ matrix function $\sigma(\lambda)$ to be the spectral function of the boundary value problem* (3.1) *with (unique) continuous $2n \times 2n$ matrix potential $Q$ satisfying* (4.3) *it is necessary and sufficient that the following conditions hold:*

1. *If $g \in L^2_{\text{comp}}(\mathbb{R}_+, \mathbb{C}^{2n})$ and if*

$$\int_{\mathbb{R}} G_1(\lambda)^* d\sigma(\lambda) G_1(\lambda) = 0,$$

*where $G_1$ is the $\mathcal{F}_{H,Q_1}$–transform of $g$, then $g = 0$.*

2. *The function*

$$\widetilde{F}(x,t) := \int_{\mathbb{R}} \widetilde{Y}_1(x,\lambda) d\Sigma(\lambda) \widetilde{Y}_1(t,\lambda)^*, \qquad \widetilde{Y}_1(x,\lambda) := \int_0^x Y_1(t,\lambda)dt, \tag{6.9}$$

*with $\Sigma = \sigma - \sigma_1$ exists and has a continuous mixed second derivative*

$$F(x,t) := \frac{\partial^2}{\partial x \partial t} \widetilde{F}(x,t). \tag{6.9'}$$

*Moreover $Q$ has $m$ continuous derivatives iff $F(x,t)$ has $m$ continuous partial derivatives with respect to $x$, that is $D_x^m F(x,t)$ is continuous.*

**Remark 6.4** 1. If $\Sigma(\lambda)$ is increasing then condition 1. of Theorem 6.3 is trivially fulfilled.

For let $g \in L^2_{\text{comp}}(\mathbb{R}_+, \mathbb{C}^{2n})$ with

$$\begin{aligned} 0 &= \int_{\mathbb{R}} G_1(\lambda)^* d\sigma(\lambda) G_1(\lambda) \\ &= \int_{\mathbb{R}} G_1(\lambda) d\sigma_1(\lambda) G_1(\lambda) + \int_{\mathbb{R}} G_1(\lambda) d\Sigma(\lambda) G_1(\lambda), \end{aligned} \tag{6.10}$$

where $G_1$ is the $\mathcal{F}_{H,Q_1}$–transform of $g$. Since $\Sigma(\lambda)$ is assumed to be increasing both summands on the right hand side of (6.10) are nonnegative and hence 0. Then Theorem 3.2 implies $g = 0$.

2. Assume that the matrix measure $\Sigma(\lambda)$ is finite, i.e. $\int_{\mathbb{R}} |d\Sigma(\lambda)| \in \mathbb{C}^{n\times n}$. Then the condition 2 of Theorem 6.3 is fulfilled. This implies that the spectral function can be prescribed on an arbitrary finite interval. More precisely, there exists a boundary value problem (3.1) with continuous $Q$ satisfying (3.3) and such that its spectral function $\sigma(\lambda)$ coincides on an arbitrary finite interval with a prescribed increasing $n \times n$ spectral measure.

3. We mention another situation in which condition 1. of Theorem 6.3 is obsolete. Namely, let $n = 1$ and assume that the (scalar) spectral function $\sigma(\lambda)$ of the boundary value problem (3.1) is not discrete, i.e. its support contains at least one finite limit point. Then the condition 1. in Theorem 6.3 is automatically satisfied. Indeed, let $g \in L^2_{\text{comp}}(\mathbb{R}_+, \mathbb{C}^{2n})$ and assume that

$$0 = \int_{\mathbb{R}} G_1(\lambda)^* d\sigma(\lambda) G_1(\lambda) = \int_{\mathbb{R}} |G_1(\lambda)|^2 d\sigma(\lambda).$$

Thus $G_1$ vanishes on the support of $\sigma$.

Since $\operatorname{supp} g$ is compact $G_1(\lambda)$ is an entire function of exponential type. In view of our assumption that the support of $\sigma$ contains at least one finite limit point we conclude $G_1 = 0$ and using Theorem 3.2 we find $g = 0$.

**Sketch of Proof** The necessity is proved in just the same way as Lemma 4.4 and Proposition 4.5.
*Sufficiency:* Starting with $\sigma(\lambda)$ we define $\widetilde{F}, F$ by (6.8) with $\Sigma(\lambda) := \sigma(\lambda) - \sigma_1(\lambda)$. Then we consider the Gelfand-Levitan equation

$$F(x,t) + K(x,t) + \int_0^x K(x,s)F(s,t)ds = 0, \quad t < x. \tag{6.11}$$

with $F$ defined by (6.8). Following the proof of Proposition 5.1 one concludes that (6.11) has a unique continuous solution $K : \Omega \to \mathrm{M}(2n, \mathbb{C})$. Next we define $Y(x,\lambda)$ setting $Y(.,\lambda) = (I+K)Y_1(.,\lambda)$ and show that $Y(x,\lambda)$ satisfies the initial value problem (5.7) with

$$Q(x) = Q_1(x) + iBK(x,x) - iK(x,x)B. \tag{6.12}$$

Since $Q_1$ satisfies (4.3) we infer from (6.12) that $Q$ also satisfies (4.3). Moreover the self-adjointness of $Q_1$ may be proved as in the proof of Theorem 5.1.

Furthermore, we note that if $F$ is continuously differentiable it satisfies the equality

$$BD_xF(x,t) + D_tF(x,t)B = -iQ_1(x)F(x,t) + iF(x,t)Q_1(t) \tag{6.13}$$

and according to Proposition 5.1 $K$ is continuously differentiable, too. If $F$ is just continuous then (6.13) still holds in the distributional sense. This is shown similar to (5.5b).

Since $Y_1(0,\lambda) = \binom{I}{H}$ we may argue exactly as in (5.5c), (5.9), (5.10) to obtain

$$K(x,0)B\begin{pmatrix} I \\ H \end{pmatrix} = 0, \quad \text{for} \quad x \in [0,\infty). \tag{6.14}$$

In view of (6.11)-(6.14) the relation (6.5a) for $K$ is proved along the same lines as part ii) of the proof of Theorem 5.2.

Thus $K$ satisfies the initial value problem (6.5a)–(6.5c). Therefore $Y(x,\lambda)$ satisfies the initial value problem (5.7) with $Q$ defined by (6.12).

If now $F$ is just continuous then one proceeds as in part iii) of the proof of Theorem 5.2.

That $\sigma$ is indeed the spectral function of the problem (5.7) with $Q$ from (6.12) is shown as part iv) of Theorem 5.2. Instead of (5.1), Theorem 4.7, (5.4), and Proposition 4.5 one uses (6.8), Proposition 6.2, (6.11), and Proposition 6.1. □

## 6.2 The degenerate Gelfand–Levitan equation

We discuss solutions of the Gelfand–Levitan equation in the special case where $\Sigma(\lambda)$ is a step function:

We consider the situation of Theorem 6.3 and fix an operator $L_1$ of the form (6.1) with spectral function $\sigma_1(\lambda)$.

Let $A \in \mathrm{M}(n, \mathbb{C})$ be a hermitian nonnegative matrix and

$$\Sigma(\lambda) := A\ 1_{[a,\infty)}(\lambda) \tag{6.15}$$

an increasing step function with one jump of "height" $A$.

We show that

$$\sigma := \sigma_1 + \Sigma \tag{6.16}$$

is the spectral function of the boundary value problem (3.1) for some (unique) continuous self–adjoint $2n \times 2n$–matrix potential $Q$ satisfying (4.3).

Since jumps of the spectral function correspond to eigenvalues this shows in particular that for a given potential $Q_1$ and given real number $a$ there is a potential $Q$ such that

$$\operatorname{spec}(L_1 + Q - Q_1) = \operatorname{spec}(L_1) \cup \{a\}. \tag{6.17}$$

For the proof we have to verify the conditions 1. and 2. of Theorem 5.3. By Remark 6.4 condition 1. is fulfilled since $A$ is nonnegative. To verify 2. we calculate

$$\begin{aligned} \widetilde{F}(x,t) &= \int_{\mathbb{R}} \widetilde{Y}_1(x,\lambda) d\Sigma(\lambda) \widetilde{Y}_1(t,\lambda)^* \\ &= \widetilde{Y}_1(x,a) A \widetilde{Y}_1(t,a)^*. \end{aligned}$$

Obviously, this has a continuous mixed second derivative, namely

$$F(x,t) := \frac{\partial^2}{\partial x \partial t} \widetilde{F}(x,t) = Y_1(x,a) A Y_1(t,a)^*. \tag{6.18}$$

In this case we can solve the Gelfand–Levitan equation explicitly. First we introduce for $x > 0$

$$T(x) := \int_0^x Y_1(s,a)^* Y_1(s,a) ds. \tag{6.19}$$

From

$$Y_1(0,a)^* Y_1(0,a) = I + H^* H \geq I$$

we infer that $T(x) > 0$ is positive definite for $x \geq 0$.

We put for $t \leq x$

$$\begin{aligned} K(x,t) &:= -Y_1(x,a) A Y_1(t,a)^* + Y_1(x,a) A T(x) (A^{-1} + T(x))^{-1} Y_1(t,a)^* \\ &= -Y_1(x,a)(A^{-1} + T(x))^{-1} Y_1(t,a)^*. \end{aligned} \tag{6.20}$$

Note that $A^{-1} + T(x) \geq T(x)$ is positive definite, thus invertible.

Then one immediately checks that $K(x,t)$ solves the Gelfand–Levitan equation (6.11) corresponding to $F$ and consequently determines $Q$ by means of of (6.12).

Summarizing the previous considerations we arrive at the following proposition.

**Proposition 6.5** *Let $L_1$ be an operator of the form* (6.1), (6.2) *with the spectral function $\sigma_1(\lambda)$ and let $\Sigma(\lambda)$ be of the form* (6.15). *Then $\sigma = \sigma_1 + \Sigma$ is the spectral function of the boundary value problem* (3.1) *with $2n \times 2n$ matrix potential*

$$Q(x) = Q_1(x) + i\{Y_1(x,a)(A^{-1} + T(x))^{-1} Y_1^*(x,a) B - B Y_1(x,a)(A^{-1} + T(x))^{-1} Y_1^*(x,a)\}.$$

**Corollary 6.6** *Under the assumptions of the previous Proposition* 6.5 *let* $Q_1 = 0$ *(i.e.* $L_1 = -iB\frac{d}{dx}$*). Then the* $2n \times 2n$ *matrix potential corresponding to the spectral function* $\sigma(\lambda) = \frac{1}{2\pi}\lambda I_n + \Sigma(\lambda)$ *with one jump of "height"* $A$ *is given by*

$$Q(x) = ie^{iaB^{-1}x}\{S(x)B - BS(x)\}e^{-iaB^{-1}x},$$

*where* $S(x) = \binom{I}{H}(A^{-1} + x(I + H^*H))^{-1}(I, H^*)$.

If $\Sigma$ is a general increasing step function then the Gelfand–Levitan equation is still solvable. However, we do not have such an explicit formula:

**Proposition 6.7** *Let* $L_1$ *be an operator of the form* (6.1), (6.2) *with spectral function* $\sigma_1(\lambda)$. *Furthermore, let* $-\infty < a_1 < \ldots < a_r < \infty$ *be real numbers and* $A_j \in \mathrm{M}(n, \mathbb{C})$ *nonnegative matrices.*

*Then for the increasing step function*

$$\Sigma(\lambda) := \sum_{j=1}^{r} A_j 1_{[a_j,\infty)}(\lambda)$$

*there exists a unique continuous matrix potential* $Q$ *satisfying* (4.3) *such that* $\sigma_1 + \Sigma$ *is the spectral function of* $L_1 + Q - Q_1$.

*Namely,* $Q$ *is uniquely determined by* (6.12) *with* $K(x,t)$ *being the solution of the Gelfand–Levitan equation* (6.11).

**Proof** This follows by induction from Theorem 6.3 and the preceding discussion.

The conditions 1. and 2. of Theorem 6.3 can also immediately be checked directly: namely, condition 1. is fulfilled in view of Remark 6.4 since $\Sigma$ is increasing. Condition 2. follows immediately from

$$\widetilde{F}(x,t) = \sum_{j=1}^{r} \widetilde{Y}_1(x, a_j)A_j\widetilde{Y}_1(t, a_j)^* \quad \text{and} \quad F(x,t) = \sum_{j=1}^{r} Y_1(x, a_j)A_jY_1(t, a_j)^*. \qquad \square$$

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