Bethe ansatz solution of a closed spin 1 XXZ Heisenberg chain with quantum algebra symmetry

Jon Links†[*], Angela Foerster‡[†] and Michael Karowski‡[‡]

†Department of Mathematics, University of Queensland, Queensland, 4072, Australia

‡Institut für Theoretische Physik, Freie Universtät Berlin, Arnimmallee 14, Berlin, Germany

Abstract

A quantum algebra invariant integrable closed spin 1 chain is introduced and analysed in detail. The Bethe ansatz equations as well as the energy eigenvalues of the model are obtained. The highest weight property of the Bethe vectors with respect to $U_q(sl(2))$ is proved.

[*]e-mail jrl@maths.uq.oz.au
[†]e-mail angela@if.ufrgs.br
[‡]e-mail karowski@physik.fu-berlin.de

1 Introduction

The Quantum Inverse Scattering Method has proved to be a powerful procedure in the analysis of one-dimensional integrable quantum chains or two-dimensional lattice models of statistical mechanics (e.g. see [1]). Central to this formalism is the Yang-Baxter equation whose solutions are sufficient to guarrantee integrability of the associated model. The advent of quantum algebras [2, 3] provided a systematic treatment to obtaining solutions of the Yang-Baxter equation. However the most common approach to the QISM, which was to impose periodic boundary conditions, was quickly realized to be incompatible with the quantum algebra symmetry of the model. Several authors were able to overcome this problem by working with a model on an open chain [4, 5, 6, 7, 8]. This practice made the Bethe ansatz solutions of such models more difficult and in some instances only postulated solutions are available [9].

More recently, it has been demonstated that it is in fact possible to construct closed chain models with preservation of quantum algebra symmetry [10, 11, 12, 13]. Significantly the $U_q(sl(2))$ invariant closed spin 1/2 XXZ model was shown to be connected with a lattice quantization of the Liouville model [14]. The algebraic Bethe ansatz solutions of such models showed that the closed chain quantum algebra invariant case was not fraught with the same difficulties which were faced in the instances of open chains. The existence of such symmetry makes available results such as the highest weight property of the Bethe states for the fundamental representation of $U_q(sl(n))$, and furthermore a characterization of "good" and "bad" states in terms of q-dimensions when q takes values of roots of unity [15]. Initially, just quantum algebra invariant closed chains of the Hecke algebra type were analysed. It was subsequently shown [16] that a more general formulation existed.

Here we wish to expand on the knowledge of closed chain quantum algebra invariant models by undertaking a detailed study of the $U_q(sl(2))$ invariant spin 1 model. Integrable spin 1 models built from a $U_q(sl(2))$ invariant R-matrix have already been the subject of some analysis [17, 18]. Our study of the closed chain $U_q(sl(2))$ invariant spin 1 model exposes new mathematical aspects not present in the previously studied models [10, 11, 12]; viz. the model is not of Hecke algebra type and it is an example of a higher spin system where the most natural approach to the Bethe ansatz solution is to use a transfer matrix defined on an auxiliary space different from the local quantum space. We then find the eigenvalues of the transfer matrix whose auxiliary space is isomorphic to the local quantum space following the method of Babujian and Tsvelick [19]. The need to use two transfer matrices defined on different auxiliary spaces means working with more than one solution of the Yang-Baxter equation. Throughout we will follow the notation of [26, 19] in distinguishing the spaces on which the various operators act. Specifically we use the symbol σ to denote action on the spin 1/2 space and s for action on the spin 1 space.

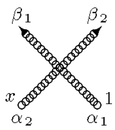
The paper is organized as follows. In section 2 we define some basic quantities, e.g., R matrices, monodromy and transfer matrices. A quantum algebra invariant closed spin-1 chain is introduced and its relation with one of the transfer matrices is presented. In section 3 the system is analysed through a combination of the techniques developed to handle with quantum algebra invariant closed chains [10] and higher-spin chains [18] and the Bethe ansatz equations as well as the energy eigenvalues of the model are obtained. In section 4 we show that the Bethe vectors are highest weight vectors with respect to

$U_q(sl(2))$. We also argue that use of the $U_q(sl(2))$ generators allows use to generate a complete set of states for the model. A summary of our main results is presented in section 5.

2 The model

We begin by recalling the R-matrix for the spin-1/2 chain

$$ {}_{\sigma\sigma}R^{\beta_1\beta_2}_{\alpha_1\alpha_2}(x) = $$



$$ = \frac{1}{{}_{\sigma\sigma}a} \left(\begin{array}{cc|cc} {}_{\sigma\sigma}a & 0 & 0 & 0 \\ 0 & {}_{\sigma\sigma}b & {}_{\sigma\sigma}c_- & 0 \\ \hline 0 & {}_{\sigma\sigma}c_+ & {}_{\sigma\sigma}b & 0 \\ 0 & 0 & 0 & {}_{\sigma\sigma}a \end{array} \right), \tag{1} $$

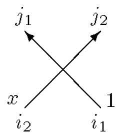
with

$$ {}_{\sigma\sigma}a = xq - \frac{1}{xq}, \qquad {}_{\sigma\sigma}b = x - \frac{1}{x}, \qquad {}_{\sigma\sigma}c_+ = x\left(q - \frac{1}{q}\right), \qquad {}_{\sigma\sigma}c_- = \frac{1}{x}\left(q - \frac{1}{q}\right), \tag{2} $$

which acts in the tensor product of two 2-dimensional auxiliary spaces $\mathbf{C}^2 \otimes \mathbf{C}^2$. Above α_1 , α_2 (β_1 and β_2) are column (row) indices running from 1 to 2.

For the spin-1 chain the R-matrix is given by [20, 21]

$$ {}_{ss}R^{j_1j_2}_{i_1i_2}(x) = \tag{3} $$



$$ = \frac{1}{{}_{ss}g} \left(\begin{array}{ccc|ccc|ccc} {}_{ss}g & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & {}_{ss}a & 0 & {}_{ss}c_- & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & {}_{ss}b & 0 & {}_{ss}d_- & 0 & {}_{ss}e_- & 0 & 0 \\ \hline 0 & {}_{ss}c_+ & 0 & {}_{ss}a & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & {}_{ss}d_+ & 0 & {}_{ss}f & 0 & {}_{ss}d_- & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & {}_{ss}a & 0 & {}_{ss}c_- & 0 \\ \hline 0 & 0 & {}_{ss}e_+ & 0 & {}_{ss}d_+ & 0 & {}_{ss}b & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & {}_{ss}c_+ & 0 & {}_{ss}a & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & {}_{ss}g \end{array} \right), $$

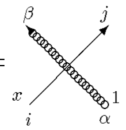
where

$$ \begin{aligned} {}_{ss}g &= xq^2 - \frac{1}{xq^2}, \qquad {}_{ss}a = x - \frac{1}{x}, \qquad {}_{ss}b = \left(x - \frac{1}{x}\right)\left(\frac{x^2 - q^2}{x^2q^2 - 1}\right), \\ {}_{ss}c_- &= \frac{1}{x}\ {}_{ss}c, \qquad {}_{ss}c_+ = x\ {}_{ss}c,, \qquad {}_{ss}c, = \left(q^2 - \frac{1}{q^2}\right), \qquad {}_{ss}f = {}_{ss}a + {}_{ss}e, \\ {}_{ss}d_- &= \frac{1}{x}\ {}_{ss}d, \qquad {}_{ss}d_+ = x\ {}_{ss}d, \qquad {}_{ss}d = \left(\frac{xq}{x^2q^2 - 1}\right)\left(x - \frac{1}{x}\right)\left(q^2 - \frac{1}{q^2}\right), \\ {}_{ss}e_- &= \frac{1}{x^2}\ {}_{ss}e, \qquad {}_{ss}e+ = x^2\ {}_{ss}e, \qquad {}_{ss}e = \left(\frac{xq}{x^2q^2 - 1}\right)\left(q - \frac{1}{q}\right)\left(q^2 - \frac{1}{q^2}\right), \end{aligned} \tag{4} $$

and it acts in $\mathbf{C}^3 \otimes \mathbf{C}^3$, with $\mathbf{C}^3$ a 3-dimensional auxiliary space.

For later convenience we also introduce an R-matrix which acts on $\mathbf{C}^2 \otimes \mathbf{C}^3$ [21]

$$
{}_{\sigma s}R^{\beta j}_{\alpha i}(x) =
$$



$$
= \frac{1}{{}_{\sigma s}a} \left(\begin{array}{ccc|ccc} {}_{\sigma s}a & 0 & 0 & 0 & 0 & 0 \\ 0 & {}_{\sigma s}b & 0 & {}_{\sigma s}d_- & 0 & 0 \\ 0 & 0 & {}_{\sigma s}c & 0 & {}_{\sigma s}d_- & 0 \\ - & - & - & - & - & - \\ 0 & {}_{\sigma s}d_+ & 0 & {}_{\sigma s}c & 0 & 0 \\ 0 & 0 & {}_{\sigma s}d_+ & 0 & {}_{\sigma s}b & 0 \\ 0 & 0 & 0 & 0 & 0 & {}_{\sigma s}a \end{array} \right) , \tag{5}
$$

where

$$
\begin{aligned}
{}_{\sigma s}a &= xq^{3/2} - \frac{1}{xq^{3/2}}, \qquad {}_{\sigma s}b = xq^{1/2} - \frac{1}{xq^{1/2}}, \qquad {}_{\sigma s}c = \frac{x}{q^{1/2}} - \frac{q^{1/2}}{x}, \\
{}_{\sigma s}d_- &= \frac{1}{x}\ {}_{\sigma s}d, \qquad {}_{\sigma s}d_+ = x\ {}_{\sigma s}d, \qquad {}_{\sigma s}d = \sqrt{\left(q - \frac{1}{q}\right)\left(q^2 - \frac{1}{q^2}\right)},
\end{aligned} \tag{6}
$$

These R-matrices satisfy the following properties

- Yang-Baxter equations

$$
R^{\alpha''\beta''}_{\alpha'\beta'}(x/y) R^{\alpha'\gamma''}_{\alpha\ \gamma'}(x) R^{\beta'\gamma'}_{\beta\ \gamma}(y) = R^{\beta''\gamma''}_{\beta'\gamma'}(y) R^{\alpha''\gamma'}_{\alpha'\ \gamma}(x) R^{\alpha'\beta'}_{\alpha\ \beta}(x/y) \ . \tag{7}
$$

- generalized Cherednik reflection property [22]

$$
R^{\alpha\ \beta}_{\alpha'\beta'}(x)(R^{-1})^{\alpha'\beta'}_{\gamma\ \delta}(y^{-1}) = R^{\alpha\ \beta}_{\alpha'\beta'}(y)(R^{-1})^{\alpha'\beta'}_{\gamma\ \delta}(x^{-1}) \tag{8}
$$

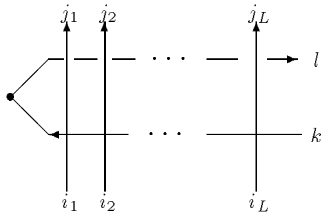
- crossing unitarity [23]

$$
(R^{t_1})^{\alpha\beta}_{\alpha'\beta'}(x\eta) K^{\alpha'}_{\alpha''}((R^{-1})^{t_1})^{\alpha''\beta'}_{\gamma'\delta}(x)(K^{-1})^{\gamma'}_{\gamma} = \delta^{\alpha}_{\gamma}\delta^{\beta}_{\delta}. \tag{9}
$$

where t_1 denotes matrix transposition in the first space, η is a crossing parameter and $K = K^t$ is the crossing matrix. Explicit forms for K are given below. We remark that eq. (8) is the natural generalization of Cherednik's reflection property to the case where the R-matrix acts on two non-isomorphic spaces.

Let us now introduce the "doubled" monodromy matrix ${}_{ss}\mathcal{U}$

$$
{}_{ss}\mathcal{U}^{l\{j\}}_{k\{i\}}(x) =
$$



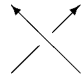
$$
\tag{10}
$$

$$
= \quad {}_{ss}R_{-\,l'\,j_1'}^{\ \ l_2\,j_1}\ {}_{ss}R_{-\,l_2\,j_2'}^{\ \ l_3\,j_2} \ldots\ {}_{ss}R_{-\,l_L\,j_L'}^{\ \ l\,j_L}\ {}_{ss}R^{l'\,j_1'}_{k_2\,i_1}(1/x)\ {}_{ss}R^{k_2\,j_2'}_{k_3\,i_2}(1/x) \ldots\ {}_{ss}R^{k_L\,j_L'}_{k\,i_L}(1/x) \ ,
$$

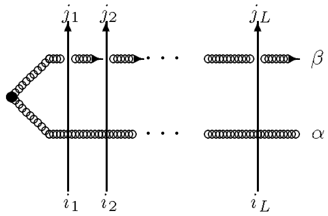
which acts in the tensor product of a three-dimensional auxiliary space and a quantum space $\mathbf{C}^3 \otimes \mathbf{C}^{3L}$ and can be regarded as a 3×3 matrix of matrices acting in the quantum space

$$
{}_{ss}\mathcal{U}^l_k(x) = \left(\begin{array}{ccc} {}_{ss}\mathcal{U}^1_1 & {}_{ss}\mathcal{U}^1_2 & {}_{ss}\mathcal{U}^1_3 \\ {}_{ss}\mathcal{U}^2_1 & {}_{ss}\mathcal{U}^2_2 & {}_{ss}\mathcal{U}^2_3 \\ {}_{ss}\mathcal{U}^3_1 & {}_{ss}\mathcal{U}^3_2 & {}_{ss}\mathcal{U}^3_3 \end{array} \right) \ , \tag{11}
$$

Above the constant ${}_{ss}R$-matrix is defined as

$$ {}_{ss}R_{-} = -\lim_{x\to 0}\ {}_{ss}R^{-1}(x) = $$  $$, \tag{12} $$

For later convenience we also define the auxiliary "doubled" monodromy matrix

$$ {}_{\sigma s}\mathcal{U}^{\beta\{j\}}_{\alpha\{i\}}(x) = $$  $$ \tag{13} $$

$$ = {}_{\sigma s}R_{-}{}^{\beta_2\, j_1}_{\alpha'\, j_1'}\ {}_{\sigma s}R_{-}{}^{\beta_3\, j_2}_{\beta_2\, j_2'} \dots {}_{\sigma s}R_{-}{}^{\beta\, j_L}_{\beta_L\, j_L'}\ {}_{\sigma s}R^{\alpha'\, j_1'}_{\alpha_2\, i_1}(1/x)\ {}_{\sigma s}R^{\alpha_2\, j_2'}_{\alpha_3\, i_2}(1/x) \dots {}_{\sigma s}R^{\alpha_L\, j_L'}_{\alpha\, i_L}(1/x) , $$

where ${}_{\sigma s}R_{-}$ corresponds to the leading term in the limit of the matrix ${}_{\sigma s}R^{-1}(x)$ for $x \to 0$, analogously to ${}_{ss}R_{-}$ (see eq. (12)) It acts on $\mathbf{C}^2 \otimes \mathbf{C}^{3L}$ and can be represented as a 2×2 matrix in the auxiliary space whose entries are matrices acting in the quantum space

$$ {}_{\sigma s}\mathcal{U}^{\beta}_{\alpha}(x) = \begin{pmatrix} A & B \\ C & D \end{pmatrix} , \tag{14} $$

Using equations (7,8) we can prove that the following Yang-Baxter equations are fulfilled

$$ {}_{\sigma\sigma}R^{\alpha\ \beta}_{\alpha'\beta'}(y/x)\ {}_{\sigma s}\mathcal{U}^{\beta'}_{\gamma'}(x)\ {}_{\sigma\sigma}R_{-}{}^{\alpha'\gamma'}_{\delta'\ \alpha}\ {}_{\sigma s}\mathcal{U}^{\delta'}_{\delta}(y) = {}_{\sigma s}\mathcal{U}^{\alpha}_{\alpha'}(y)\ {}_{\sigma\sigma}R^{+}{}^{\alpha'\ \beta}_{\delta'\beta'}\ {}_{\sigma s}\mathcal{U}^{\beta'}_{\gamma'}(x)\ {}_{\sigma\sigma}(R^{-1})^{\delta'\gamma'}_{\delta\ \gamma}(x/y) \tag{15} $$

$$ {}_{\sigma s}R^{\alpha\ i}_{\alpha' i'}(y/x)\ {}_{ss}\mathcal{U}^{i'}_{j'}(x)\ {}_{\sigma s}R_{-}{}^{\alpha' j'}_{\beta'\ j}\ {}_{\sigma s}\mathcal{U}^{\beta'}_{\beta}(y) = {}_{\sigma s}\mathcal{U}^{\alpha}_{\alpha'}(y)\ {}_{\sigma s}R^{+}{}^{\alpha'\ i}_{\beta' i'}\ {}_{ss}\mathcal{U}^{i'}_{j'}(x)\ {}_{\sigma s}(R^{-1})^{\beta' j'}_{\beta\ j}(x/y) \tag{16} $$

$$ {}_{ss}R^{\alpha\ i}_{\alpha' i'}(y/x)\ {}_{ss}\mathcal{U}^{i'}_{j'}(x)\ {}_{ss}R_{-}{}^{\alpha' j'}_{\beta'\ j}\ {}_{ss}\mathcal{U}^{\beta'}_{\beta}(y) = {}_{ss}\mathcal{U}^{\alpha}_{\alpha'}(y)\ {}_{ss}R^{+}{}^{\alpha'\ i}_{\beta' i'}\ {}_{ss}\mathcal{U}^{i'}_{j'}(x)\ {}_{ss}(R^{-1})^{\beta' j'}_{\beta\ j}(x/y) . \tag{17} $$

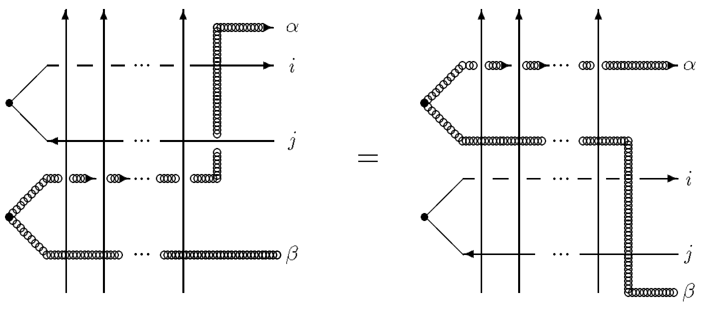
Above we have defined (for R-matrices acting on any two spaces)

$$ R^{+} = \lim_{x\to\infty} R(x). $$

For later use we also define

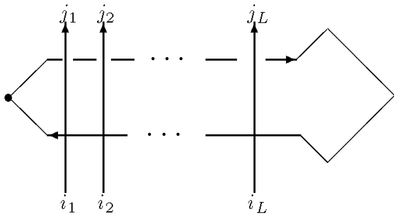
$$ R^{-} = \lim_{x\to 0} R(x). $$

Equation (16) is depicted graphically below. Similar graphical representations apply for eqs. (15,17) but will not be presented.



Finally, the spin-1 transfer matrix is constructed by taking the spin-1 Markov trace of the monodromy matrix (10) in the auxiliary space

$$ {}_{ss}\mathcal{T}^{\{j\}}_{\{i\}}(x) = \sum_{\alpha} \ {}_{s}K^{\alpha}_{\alpha} \ {}_{ss}\mathcal{U}^{\alpha\{j\}}_{\alpha\{i\}} \ = \tag{18} $$

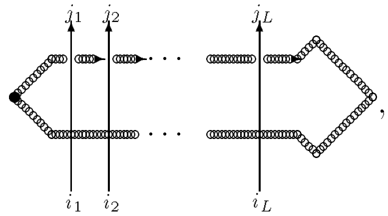


where

$$ {}_{s}K \ = \begin{pmatrix} q^2 & & \\ & 1 & \\ & & q^{-2} \end{pmatrix} \ , \tag{19} $$

By using equations (9,17) it can be shown that this transfer matrix forms a commuting family, i.e., it commutes for different spectral parameters. A quantum algebra invariant spin-1 XXZ Hamiltonian with closed boundary conditions will be obtained later from it. However, in order to diagonalize it, the usual algebraic Bethe ansatz scheme which applies to monodromy matrices whose auxiliary space is the fundamental representation can not be adopted. As in other higher-spin chains [18, 24, 25, 26, 27] this problem can be solved by introducing an auxiliary spin-1/2 transfer matrix ${}_{\sigma s}\mathcal{T}$ which commutes with the spin-1 transfer matrix ${}_{ss}\mathcal{T}$. This spin-1/2 auxiliary transfer matrix is constructed using the auxiliary ${}_{\sigma s}R(x)$ (5) and doubled monodromy ${}_{\sigma s}\mathcal{U}(x)$ (13) matrices and is given by

$$ {}_{\sigma s}\mathcal{T}^{\{j\}}_{\{i\}}(x) = \sum_{\alpha} \ {}_{\sigma}K^{\alpha}_{\alpha} \ {}_{\sigma s}\mathcal{U}^{\alpha\{j\}}_{\alpha\{i\}} \ = \tag{20} $$



with

$$ {}_{\sigma}K \ = \begin{pmatrix} q & \\ & q^{-1} \end{pmatrix} \ , \tag{21} $$

Using eqs. (9,16) we can also show that the above transfer matrices commute

$$ [\ {}_{\sigma s}\mathcal{T}(x) , \ \ {}_{ss}\mathcal{T}(y)] \ = \ 0 \ \ . \tag{22} $$

Therefore we can simultaneously diagonalize ${}_{\sigma s}\mathcal{T}$ and ${}_{ss}\mathcal{T}$ which will be presented in the next section.

A quantum algebra invariant closed spin-1 Hamiltonian can be defined through

$$ \mathcal{H} \ = \ \ {}_{ss}\mathcal{T}'(x) \ \ {}_{ss}\mathcal{T}^{-1}(x) \ |_{x=1} \ \ , \tag{23} $$

where the prime indicates differentiation with respect to the variable x. This yields (see [16] for details about this general construction)

$$ \mathcal{H} \ = \sum_{n=1}^{L-1} h_n + h_0 \ \ , \tag{24} $$

where

$$h_n \;\propto\; \vec{J}_n\,.\,\vec{J}_{n+1} - (\vec{J}_n\,.\,\vec{J}_{n+1})^2 + \frac{(q-q^{-1})^2}{2}\Big[J^z_n J^z_{n+1} + (J^z_n)^2 + (J^z_{n+1})^2 - (J^z_n J^z_{n+1})^2\Big]$$
$$- \;(q^{1/2}-q^{-1/2})^2\Big[(J^x_n J^x_{n+1} + J^y_n J^y_{n+1})J^z_n J^z_{n+1} + J^z_n J^z_{n+1}(J^x_n J^x_{n+1} + J^y_n J^y_{n+1})\Big] \qquad (25)$$

and $\vec{J}_n$ are spin-1 generators of $sl(2)$. The boundary term h_0 is given by

$$h_0 = \underbrace{{}_{ss}\hat{R}^-_1\ {}_{ss}\hat{R}^-_2 \ldots {}_{ss}\hat{R}^-_{L-1}}_{\mathrm{G}}\ h_{L-1}\ \underbrace{{}_{ss}\hat{R}^+_{L-1}\ldots {}_{ss}\hat{R}^+_2\ {}_{ss}\hat{R}^+_1}_{G^{-1}}\ , \qquad (26)$$

with

$${}_{ss}\hat{R}^{\pm\,\{\gamma\}}_{n\ \{\beta\}} = \mathbf{1}^{\gamma_1}_{\beta_1}\otimes\mathbf{1}^{\gamma_2}_{\beta_2}\otimes\ldots\otimes\ {}_{ss}R^{\pm\ \gamma_n\gamma_{n+1}}_{\ \ \beta_{n+1}\beta_n}\otimes\ldots\mathbf{1}^{\gamma_L}_{\beta_L} \qquad n=1,2,\ldots L-1\ . \qquad (27)$$

In eq. (24) L is the number of lattice sites. The operators H, h_n and $\hat{R}^{\pm}_n$ $(n=1,2,\ldots L-1)$ act on the "quantum space" $\mathbf{C}^{3L}$ (for simplicity, we omit the quantum space indices and write them only whenever necessary). The model is periodic in the sense that the operator G^{-1} maps h_n into h_{n-1}

$$G^{-1}h_nG = h_{n-1} \qquad n=2,\ldots L-1\ , \qquad (28)$$

and h_1 into h_0

$$G^{-1}h_1G = Gh_{L-1}G^{-1}\ . \qquad (29)$$

The quantum algebra invariance of such a construction is discussed in [16].

3 Bethe ansatz method

In this section we solve the eigenvalue problem of the transfer matrix

$${}_{ss}\mathcal{T}\Psi = (q^2\ {}_{ss}\mathcal{U}^1_1 + \ {}_{ss}\mathcal{U}^2_2 + q^{-2}\ {}_{ss}\mathcal{U}^3_3)\Psi = \ {}_{ss}\Lambda\Psi\ , \qquad (30)$$

(and consequently that of the Hamiltonian (24)) through a combination of the techniques developed to handle with quantum group invariant closed chains [10] and higher-spin chains [18]. First, from the fact that eq. (22) is satisfied, ${}_{ss}\mathcal{T}$ and ${}_{\sigma s}\mathcal{T}$ have a common set of eigenvectors, which can be determined by applying the algebraic Bethe ansatz method to ${}_{\sigma s}\mathcal{T}$ (20). Following Babujian [26], the vector Ψ can be written as

$$\Psi = B(x_1)\,B(x_2)\ldots B(x_M)\Phi \qquad (31)$$

where Φ is the reference state defined by the equation

$$C\Phi = 0$$

whose solution is $\Phi = \otimes^L_{i=1}|1>_i$. It is an eigenstate of A and D

$$A(x)\Phi = q^{\frac{3L}{2}}\Phi\ ,$$
$$D(x)\Phi = q^{-\frac{L}{2}}\frac{{}_{\sigma s}c(1/x)^L}{{}_{\sigma s}a(1/x)^L}\Phi\ . \qquad (32)$$

Next we apply $A(x)$ and $D(x)$ to Ψ (31), push them through all the $B's$ using the following commutation rules derived from the Yang-Baxter relation (15)

$$
\begin{aligned}
A(x)B(y) &= \frac{1}{q}\frac{{}_{\sigma\sigma}a(x/y)}{{}_{\sigma\sigma}b(x/y)}B(y)A(x) - \frac{1}{q}\frac{{}_{\sigma\sigma}c_-(x/y)}{{}_{\sigma\sigma}b(x/y)}B(x)A(y) - \frac{q-1/q}{q}B(x)D(y) \\
D(x)B(y) &= q\frac{{}_{\sigma\sigma}a(y/x)}{{}_{\sigma\sigma}b(y/x)}B(y)D(x) - q\frac{{}_{\sigma\sigma}c_-(y/x)}{{}_{\sigma\sigma}b(y/x)}B(x)D(y)\,,
\end{aligned} \tag{33}
$$

and apply them to the reference state Φ using eqs. (32). ¿From the first terms of the r.h.s. of eqs. (32) we get the "wanted" contributions, while the other terms originate the "unwanted" terms, since they can never give a vector proportional to Ψ

$$
\begin{aligned}
A(x)\Psi &= q^{\frac{3L}{2}-M}\prod_{i=1}^{M}\frac{{}_{\sigma\sigma}a(x/x_i)}{{}_{\sigma\sigma}b(x/x_i)}\Psi + \text{u. t.} \\
D(x)\Psi &= q^{-\frac{L}{2}+M}\frac{{}_{\sigma s}c(1/x)^L}{{}_{\sigma s}a(1/x)^L}\prod_{i=1}^{M}\frac{{}_{\sigma\sigma}a(x_i/x)}{{}_{\sigma\sigma}b(x_i/x)}\Psi + \text{u. t.}
\end{aligned} \tag{34}
$$

The cancellation of all unwanted terms ensure that Ψ, as given by (31) is an eigenstate of the transfer matrix ${}_{\sigma s}\mathcal{T}(x)$ (20) and this indeed happens if the Bethe ansatz equations (BAE) hold

$$
q^{2(1+L-M)}\left(\frac{{}_{\sigma s}a(1/x_k)}{{}_{\sigma s}c(1/x_k)}\right)^L\prod_{i=1}^{M}\frac{{}_{\sigma\sigma}a(x_k/x_i)}{{}_{\sigma\sigma}b(x_k/x_i)}\frac{{}_{\sigma\sigma}b(x_i/x_k)}{{}_{\sigma\sigma}a(x_i/x_k)} = -1\ , \qquad k=1,\dots M\ , \tag{35}
$$

Note that these equations are much simpler than those obtained for the quantum group invariant spin 1 chain with open boundary conditions (see [18]). Also in the limit $q\to 1$ we recover the BAE for the usual periodic case [26].

Let us now find the eigenvalues of ${}_{ss}\mathcal{T}(x)$ by acting with this transfer matrix on Ψ (31), according to (30). For this purpose we need the commutation relations between ${}_{ss}\mathcal{U}_1^1(x)$, ${}_{ss}\mathcal{U}_2^2(x)$, ${}_{ss}\mathcal{U}_3^3(x)$ and ${}_{ss}\mathcal{U}_2^1(y)$ and their action on the reference state Φ. Rewriting the Yang-Baxter equation (16) we can find the following relations

$$
{}_{\sigma s}\mathcal{U}^{\alpha}_{\alpha'}(y)\ {}_{\sigma s}R^{+\alpha'\ i}_{\ \beta\ i'}\ {}_{ss}\mathcal{U}^{i'}_{j}(x) = {}_{\sigma s}R^{\alpha\ i}_{\alpha' i'}(y/x)\ {}_{ss}\mathcal{U}^{i'}_{j'}(x)\ {}_{\sigma s}R_{-\beta' j''}^{\ \alpha'\ j'}\ {}_{\sigma s}\mathcal{U}^{\beta'}_{\beta''}(y)\ {}_{\sigma s}R^{\beta'' j''}_{\beta\ j}(x/y) \tag{36}
$$

which yield the commutation rules

$$
\begin{aligned}
{}_{ss}\mathcal{U}_1^1(x)B(y) &= \frac{1}{q^2}\frac{{}_{\sigma s}a(x/y)}{{}_{\sigma s}c(x/y)}B(y)\ {}_{ss}\mathcal{U}_1^1(x) - \frac{1}{q}\frac{{}_{\sigma s}d_-(x/y)}{{}_{\sigma s}c(x/y)}\ {}_{ss}\mathcal{U}_2^1(x)A(y) \\
&- \frac{1}{q}\sqrt{(1-\frac{1}{q^2})(q^2-\frac{1}{q^2})}\Big(\ {}_{ss}\mathcal{U}_2^1(x)D(y) + \frac{{}_{\sigma s}d_-(x/y)}{{}_{\sigma s}c(x/y)}\ {}_{ss}\mathcal{U}_3^1(x)C(y)\Big) \\
{}_{ss}\mathcal{U}_2^2(x)B(y) &= \frac{{}_{\sigma s}a(y/x)\ {}_{\sigma s}a(x/y)}{{}_{\sigma s}b(y/x)\ {}_{\sigma s}b(x/y)}B(y)\ {}_{ss}\mathcal{U}_2^2(x) - \frac{{}_{\sigma s}d_-(y/x)}{{}_{\sigma s}b(y/x)}\ {}_{ss}\mathcal{U}_2^1(x)D(y) \\
&- \frac{{}_{\sigma s}d_-(x/y)}{{}_{\sigma s}b(x/y)}\Big(\frac{1}{q}\ {}_{ss}\mathcal{U}_3^2(x)A(y) + q\frac{{}_{\sigma s}d_-(y/x)}{{}_{\sigma s}b(y/x)}\ {}_{ss}\mathcal{U}_3^1(x)C(y)\Big) \\
&- \sqrt{(1-\frac{1}{q^2})(q^2-\frac{1}{q^2})}\ {}_{ss}\mathcal{U}_3^2(x)D(y) \\
{}_{ss}\mathcal{U}_3^3(x)B(y) &= q^2\frac{{}_{\sigma s}a(y/x)}{{}_{\sigma s}c(y/x)}B(y)\ {}_{ss}\mathcal{U}_3^3(x) - q^2\frac{{}_{\sigma s}d_-(y/x)}{{}_{\sigma s}c(y/x)}\ {}_{ss}\mathcal{U}_3^2(x)D(y)
\end{aligned} \tag{37}
$$

We also observe that

$$ {}_{ss}\mathcal{U}_1^1(x)\Phi = \Phi, \qquad {}_{ss}\mathcal{U}_2^2(x)\Phi = q^{-2L}\ {}_{ss}a(1/x)^L\Phi, \qquad {}_{ss}\mathcal{U}_3^3(x)\Phi = q^{-4L}\ {}_{ss}b(1/x)^L\Phi \quad (38) $$

Then applying the transfer matrix ${}_{ss}\mathcal{T}$ on the vector Ψ (31) and using eqs. (36) and (38) we get the eigenvalues of ${}_{ss}\mathcal{T}$

$$ \begin{aligned} {}_{ss}\Lambda(x) &= q^{2-2M}\prod_{i=1}^{M}\frac{{}_{\sigma s}a(x/x_i)}{{}_{\sigma s}c(x/x_i)} + q^{-2L}\ {}_{ss}a(1/x)^L\prod_{i=1}^{M}\frac{{}_{\sigma s}a(x/x_i)\ {}_{\sigma s}a(x_i/x)}{{}_{\sigma s}b(x/x_i)\ {}_{\sigma s}b(x_i/x)} \\ &+ q^{2(M-2L-1)}\ {}_{ss}b(1/x)^L\prod_{i=1}^{M}\frac{{}_{\sigma s}a(x_i/x)}{{}_{\sigma s}c(x_i/x)} \end{aligned} \quad (39) $$

We have obtained (39) by taking into account only the first terms on the r.h.s. of eqs. (36). All other terms generate "unwanted" contributions and the condition of their equality to zero yields the BAE (35). A simpler way to recover the BAE from (39) is by demanding that the eigenvalue ${}_{ss}\Lambda(x)$ (39) has no poles at $x = q^{\pm 1/2}x_i$, since ${}_{ss}\mathcal{T}$ is an analytic function in x. Finally, we obtain the eigenvalues of the Hamiltonian (24) from (23) and (39)

$$ E = \sum_{i=1}^{M}\frac{-2(q^2-q^{-2})}{(x_i^{-1}q^{-1/2}-x_iq^{1/2})(x_i^{-1}q^{3/2}-x_iq^{-3/2})} \quad (40) $$

In the rational limit $q\to 1$, this expression reduces to that obtained by Babujian [26] for the usual periodic case (with appropriate rescaling).

4 Highest weight property

In this section we show that the Bethe vectors are highest weight vectors with respect to $U_q(sl(2))$. We begin by writing

$$ \begin{aligned} {}_{\sigma s}R^{+} &= \begin{pmatrix} q^{\frac{1}{2}h} & 0 \\ q^{\frac{-1}{2}}(q-q^{-1})e & q^{\frac{-1}{2}h} \end{pmatrix}, \\ {}_{\sigma s}R_{-} &= \begin{pmatrix} q^{\frac{-1}{2}h} & -q^{\frac{1}{2}}(q-q^{-1})f \\ 0 & q^{\frac{1}{2}h} \end{pmatrix}, \end{aligned} \quad (41) $$

where h, e, f are the $sl(2)$ generators in the spin 1 representation. Next, defining the constant auxiliary monodromy matrix as

$$ {}_{\sigma s}\mathcal{U}^{-\beta}_{\ \alpha} = \lim_{x\to 0}\ {}_{\sigma s}\mathcal{U}_{\alpha}^{\beta}(x) = (\ {}_{\sigma s}R_{-})^{\beta\, j}_{\alpha'\, j'}\,(\ {}_{\sigma s}R^{+})^{\alpha'\, j'}_{\alpha\, i} \quad (42) $$

we have from (41)

$$ C^{-} = q^{-1/2}(q-q^{-1})q^{\frac{-1}{2}h}e. \quad (43) $$

The Bethe vectors (31) are highest weight vectors if

$$ C^{-}\Psi = 0 \quad (44) $$

This can be proven by observing that from the Yang-Baxter algebra (15) we can obtain the following relation

$$C^{-}B(x) = B(x)C^{-} + (1-q^{-2})\Big(A(x)D^{-} - D^{-}D(x)\Big) \tag{45}$$

which, using the fact that $C^{-}\Phi = 0$, allows us to write

$$C^{-}\Psi = \sum_{i=1}^{M} Y_i W_i \Phi \tag{46}$$

where

$$W_i = B(x_1)B(x_2)\dots B(x_{i-1})B(x_{i+1})\dots B(x_M). \tag{47}$$

The Y_i can be computed using eqs. (32) and (33) which yields

$$\begin{aligned} Y_i &= (1-q^{-2})\, q^{3/2L} \;\; {}_{\sigma s}a(1/x_i)^L \prod_{j\neq i}^{M} q^{-1} \frac{{}_{\sigma\sigma}a(x_i/x_j)}{{}_{\sigma\sigma}b(x_i/x_j)} \\ &- (1-q^{-2}) q^{-L/2} \;\; {}_{\sigma s}c(1/x_i)^L \prod_{j\neq i}^{M} q \frac{{}_{\sigma\sigma}a(x_j/x_i)}{{}_{\sigma\sigma}b(x_j/x_i)} \end{aligned} \tag{48}$$

Because of the BAE (35), each of the co-efficients Y_i vanishes which implies

$$C^{-}\Psi = 0.$$

It immediately follows that each of the Bethe states are highest weight states. By using the $U_q(sl(2))$ lowering operator f we obtain additional states which will also be eigenstates of the transfer matrix because of the quantum symmetry of the model.

For generic values of the deformation parameter q it is well known that the dimensions and weight spectrum of the finite dimensional irreducible representations of $U_q(sl(2))$ are in 1-1 correspondence with those of $sl(2)$. Since it is known [25, 28] in the $q = 1$ case that the Bethe states combined with the $sl(2)$ symmetry give a complete set of states for the model, it should be possible to prove using methods developed in [29] that this is also true for the model described above.

5 Conclusions

We have solved a quantum algebra invariant integrable closed spin-1 chain by an algebraic Bethe ansatz approach. Particularly eigenstates of the model were constructed and their energy eigenvalues evaluated. A proof of the highest weight property of the Bethe vectors with respect to $U_q(sl(2))$ was also presented. A natural extension of this work would be to generalize the results of the spin-1 chain to corresponding chains of arbitrary spin s.

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References

[1] Korepin V E, Izergin A G and Bogoliubov N M 1993 "Quantum Inverse Scattering Method, Correlation Functions and Algebraic Bethe Ansatz" (Cambridge University Press, Cambridge)

[2] Jimbo M 1985 *Lett. Math. Phys.* **10** 63

[3] Drinfeld V G 1986 *Proc. Int. Cong. Math.* (Berkeley) 798

[4] Destri C and De Vega H J 1992 Nucl. Phys **B361**; 361; 1992 Nucl. Phys. B **374** 692

[5] González-Ruiz A and De Vega H J 1994 Nucl. Phys. **B417** 553

[6] Foerster A and Karowski M 1993 Nucl. Phys. **B408**, 512

[7] González-Ruiz A 1994 Nucl. Phys.**B424** 468

[8] Yue R, Fan H and Hou B 1996 Nucl. Phys.**B462** 167

[9] Artz S, Mezincescu L and Nepomechie R I 1995 Int. J. Mod. Phys. **A10** 1937; 1995 J. Phys. **A28** 5131

[10] Karowski M and Zapletal A 1994 *Nucl. Phys. B* **419** 567

[11] Grosse H, Pallua S, Prester P and Raschhofer E 1994 *J. Phys. A* **27** 4761

[12] Foerster A 1996 *J. Phys.* **A29** 7625

[13] Lima-Santos A and Ghiotto R C T 1998 *J. Phys.* **A31** 505

[14] Faddeev L D and Tirkkonen O 1995 *Nucl. Phys.* **B453** 647

[15] Karowski M and Zapletal A 1994 *J. Phys. A* **27** 7419

[16] Links J and Foerster A 1996 *J. Phys. A* **30** 2483

[17] Batchelor M T, Mezincescu L, Nepomechie R I and Rittenberg V 1990 *J. Phys. A* **23** L141

[18] Mezincescu L, Nepomechie R I and Rittenberg V 1990 *Phys. Lett.* **A147** 70

[19] Babujian H M and Tsvelick A M 1986 *Nucl. Phys.* **B265** [FS15] 24

[20] Zamolodchikov A B and Fateev V A 1980 *Sov. J. Nucl. Phys.* **32** 298

[21] Kulish P P and Reshetikhin N Yu 1983 *J. Sov. Math.* **23** 2435

[22] Cherednik I 1984 *Theor. Mat. Fiz.* **61** 35

[23] Links J R and Gould M D 1996 *Int. J. Mod. Phys.* **B10** 3461

[24] Kulish P P, Reshetikhin N Yu and Sklyanin E K, 1981 *Lett. Math. Phys.* **5** 393

[25] Takhtajan L 1982 *Phys. Lett.* **A87** 479

[26] Babujian H M 1983 *Nucl. Phys.* **B215** [FS7] 317

[27] Kirillov A N and Reshetikhin 1987 *J. Phys.* **A20** 1565

[28] Kirillov A N 1985 *J. Sov. Math.* **30** 2298

[29] Kirillov A N, Liskova N A 1997 *J. Phys.***A 30** 1209
Kirillov A N, Liskova N A preprint hep-th/9607012