Difference Equations and Highest Weight Modules of $U_q[sl(n)]$

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Abstract

The quantized version of a discrete Knizhnik-Zamolodchikov system is solved by an extension of the generalized Bethe Ansatz. The solutions are constructed to be of highest weight which means they fully reflect the internal quantum group symmetry.

1 Introduction

This article can be considered as an addendum to the article [1] on matrix difference equations and a generalized version of the Bethe ansatz. For an introduction to their rôle in mathematical physics the reader is referred to [1]-[3] and references contained therein.

Though q-deformations of discrete Knizhnik-Zamolodchikov equations have been treated in much detail over the last years [3][4] it became not completely clear how that solutions are related to the underlying symmetry of such problems.

The conventional algebraic formulation of the Bethe ansatz demonstrates the close relation between the eigenvector problem and the representation theory of its connected symmetry group (either classical or q-deformed): Bethe vectors can be constructed to be highest weight vectors of irreducible representations and therefore by simply counting them one makes certain on spanning the whole space of states.

However one has to be careful when moving from classical Lie algebras to a quantum group, as it can be seen when an 1 dimensional periodic XXX-Heisenberg chain is deformed

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to the anisotropic XXZ-model. Deforming the Hamiltonian in a straightforward way won't preserve the (quantum) symmetry. Instead one is forced to change the boundary conditions [5] or to take additional terms (arising from the nontrivial toroidal topology) into account as done in [6].

The behavior of the difference equation

$$Q(\underline{x}; i)\, f(x_1, \ldots, x_i, \ldots, x_N) = f(x_1, \ldots, x_i + \kappa, \ldots, x_N), \quad i = 1, \ldots, N; \tag{1.1}$$

where $f(\underline{x})$ is a vector-valued function on N variables x_i, $Q(\underline{x}; i)$ a family of linear operators and κ an arbitrary shift parameter, indeed resembles this problematic nature: The operators $Q(\underline{x}; i)$ can be regarded as a sort of generalized transfer matrices and therefore the analogy to a quantum spin chain becomes obvious.

In Section 2 we formulate this equation in a way adapted to quantum symmetry and obtain solutions by a generalized Bethe ansatz. In Section 3 they are shown to be highest weight vectors, additionally we calculate their weights. For sake of transparency both sections are fixed to $U_q[sl(2)]$ containing all essential features of a quantum group. Finally, for completeness, we briefly comprehend the aspects of the higher ranked case in Section 4, followed by a summary of the given results.

2 The generalized Bethe ansatz

Consider N vector spaces $V_i \simeq \mathbf{C}^2$, each given as the representation space of the fundamental representation of $U_q[sl(2)]$. The basis vectors will be denoted by $|1\rangle$ resp. $|2\rangle$. The R-matrix then acts as a linear operator on two of such spaces V_i and V_j:

$$R_{ij} : V_i \otimes V_j \to V_j \otimes V_i, \tag{2.1}$$

and is given by the quasitriangular Hopf algebraic structure of $U_q[sl(2)]$ [7]. In the natural basis of tensor products its matrix form reads

$$R = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & q^{-1} & 0 & 0 \\ 0 & (1 - q^{-2}) & q^{-1} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{2.2}$$

If in addition one associates to each space V_i a variable x_i, it is possible to define a 'spectral parameter' dependent R-matrix:

$$R(x) := \frac{q e^{x/2} R - q^{-1} e^{-x/2} P R^{-1} P}{q e^{x/2} - q^{-1} e^{-x/2}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & b(x) & c_-(x) & 0 \\ 0 & c_+(x) & b(x) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{2.3}$$

where P is the permutation operator in the sense of eqn. (2.1)

$$P_{ij}\left(v_i \otimes v_j\right) = v_i \otimes v_j, \quad v_{i,j} \in V_{i,j},$$

and $x = x_i - x_j$. The Boltzmann weights read explicitly

$$b(x) = \frac{e^{x/2} - e^{-x/2}}{qe^{x/2} - q^{-1}e^{-x/2}}, \quad c_{\pm}(x) = \frac{e^{\pm x/2}(q - q^{-1})}{qe^{x/2} - q^{-1}e^{-x/2}}. \tag{2.4}$$

$R(x)$ satisfies the Yang Baxter equation

$$R_{12}(x_1 - x_2)R_{13}(x_1 - x_3)R_{23}(x_2 - x_3) = R_{23}(x_2 - x_3)R_{13}(x_1 - x_3)R_{12}(x_1 - x_2). \tag{2.5}$$

One defines a monodromy matrix $T_0(\underline{x}, x_0)$ acting on the tensor product space $V = \bigotimes_{i=1}^N V_i$ and an additional auxiliary space $V_0 \simeq \mathbf{C}^2$

$$T_0(\underline{x}, x_0) := R_{10}(x_1 - x_0)R_{20}(x_2 - x_0)\ldots R_{N0}(x_N - x_0). \tag{2.6}$$

However as we will see later it is more useful to work with the doubled monodromy matrix as proposed in [8] some years ago as an application of the 'reflection' equation introduced in [9]. We will use in the following the special type that has been introduced in [6] and is given by

$$\mathcal{T}_0(\underline{x}, x_0) := R_{01}R_{02}\ldots R_{0N}R_{10}(x_1 - x_0)R_{20}(x_2 - x_0)\ldots R_{N0}(x_N - x_0). \tag{2.7}$$

Its dependency on V_0 becomes obvious if $\mathcal{T}_0$ is written as a matrix w.r.t. the auxiliary space:

$$\mathcal{T}_0 = \begin{pmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{pmatrix}. \tag{2.8}$$

Equation (2.5) implies the Yang-Baxter equation for $\mathcal{T}$

$$R_{ab}(v - u)\mathcal{T}_a(\underline{x}; u)R_{ba}\mathcal{T}_b(\underline{x}; v) = \mathcal{T}_b(\underline{x}; v)R_{ba}\mathcal{T}_a(\underline{x}; u)R_{ba}(v - u) \tag{2.9}$$

giving the following commutation relations for the operators $\mathcal{A}, \mathcal{B}$ and $\mathcal{D}$:

$$\begin{aligned}
[\mathcal{B}(\underline{x}; u), \mathcal{B}(\underline{x}; v)] &= 0, \\
\mathcal{A}(\underline{x}; u)\mathcal{B}(\underline{x}; v) &= q^{-1}b^{-1}(u - v)\mathcal{B}(\underline{x}; v)\mathcal{A}(\underline{x}; u) \\
&\quad - \mathcal{B}(\underline{x}; u)\left[q^{-1}\frac{c_-(u - v)}{b(u - v)}\mathcal{A}(\underline{x}; v) + (1 - q^{-2})\mathcal{D}(\underline{x}; v)\right], \\
\mathcal{D}(\underline{x}; u)\mathcal{B}(\underline{x}; v) &= qb^{-1}(v - u)\mathcal{B}(\underline{x}; v)\mathcal{A}(\underline{x}; u) - q\frac{c_-(v - u)}{b(v - u)}\mathcal{B}(\underline{x}; u)\mathcal{D}(\underline{x}; v),
\end{aligned} \tag{2.10}$$

(for a detailed proof see [6]).

Analogous to the definition (2.7) consider a further set of monodromy type matrices defined by

$$\mathcal{T}^Q(\underline{x}; i) := R_{01}R_{02}\dots R_{0N}R_{10}(x_1 - x_0)\dots P_{i0}\dots R_{N0}(x_N - (x_0 + \kappa)), \quad (i = 1, \dots, N). \tag{2.11}$$

where κ is the arbitrary shift parameter having already appeared in eqn. (1.1). They still have the block structure of (2.8) but do no longer depend on a parameter x_0 of the auxiliary space. The monodromy matrices $\mathcal{T}$ and $\mathcal{T}^Q$ now fulfill another Yang-Baxter equation

$$R_{ba}(x_i - u)\,\mathcal{T}_b(\underline{x}'; u)\,R_{ab}\,\mathcal{T}_a^Q(\underline{x}; i) = \mathcal{T}_a^Q(\underline{x}; i)\,R_{ba}\,\mathcal{T}_b(\underline{x}; u)\,R_{ab}(x_i + \kappa - u). \tag{2.12}$$

Again we give some commutation rules relating their matrix elements:

$$\begin{aligned}
\mathcal{A}^Q(\underline{x}; i)\mathcal{B}(\underline{x}; u) &= q^{-1}b^{-1}(x_i + \kappa - u)\mathcal{B}(\underline{x}'; u)\mathcal{A}^Q(\underline{x}; i) \\
&\quad -\mathcal{B}^Q(\underline{x}; i)\left[q^{-1}\frac{c_-(x_i + \kappa - u)}{b(x_i + \kappa - u)}\mathcal{A}(\underline{x}; u) + (1 - q^{-2})\mathcal{D}(\underline{x}; u)\right], \\
\mathcal{D}^Q(\underline{x}; i)\mathcal{B}(\underline{x}; u) &= qb^{-1}(x_i - u)\mathcal{B}(\underline{x}'; u)\mathcal{D}^Q(\underline{x}; i) - q\frac{c_-(u - x_i)}{b(u - x_i)}\mathcal{B}^Q(\underline{x}; i)\mathcal{D}(\underline{x}; u).
\end{aligned} \tag{2.13}$$

The first terms in eqn. (2.10) and (2.13) are called 'wanted' resp. all others 'unwanted' ones. Now taking the Markov trace[1] over $\mathcal{T}^Q$ gives the operator on the l.h.s. of the difference equation (1.1)

$$Q(\underline{x}; i) := \mathrm{tr}_q \mathcal{T}^Q(\underline{x}; i) = \mathcal{A}^Q(\underline{x}; i) + q^{-2}\mathcal{D}^Q(\underline{x}; i). \tag{2.14}$$

Denote by Ω the usual reference state ($\Omega = |1\rangle^{\otimes N}$) and apply an arbitrary number m of $\mathcal{B}$ operators thereto defining the following

Bethe ansatz vector:

$$f(\underline{x}) = \sum_{\underline{u}} B(\underline{x}; u_m)\dots B(\underline{x}; u_1)\,\Omega\, g(\underline{x}; \underline{u}), \tag{2.15}$$

where the summation over $\underline{u}$ is specified by

$$\sum_{\underline{u}} = \sum_{\substack{l_1 \in \mathbf{Z} \\ u_1 = \tilde{u}_1 + l_1\kappa}} \dots \sum_{\substack{l_m \in \mathbf{Z} \\ u_m = \tilde{u}_m + l_m\kappa}} \quad (\underline{\tilde{u}} \text{ arbitrary set of complex numbers}) \tag{2.16}$$

and the the function $g(\underline{x}; \underline{u})$ is defined by

$$g(\underline{x}; \underline{u}) = \prod_{i,j} \psi(x_i - u_j) \prod_{k<l} \tau(u_k - u_l). \tag{2.17}$$

[1]It's asymmetric form results from the choice of normalization in eqn. (2.2)

Theorem: The difference equation (1.1) defined by eqn. (2.14) is solved by the Bethe vectors (2.15) if the functions $\psi(x)$ and $\tau(x)$ satisfy the difference equations:

$$q^{-1}b(x+\kappa)\,\psi(x+\kappa)=\psi(x),\quad q^2\frac{\tau(x)}{b(x)}=\frac{\tau(x-\kappa)}{b(-x+\kappa)}. \tag{2.18}$$

Remark: As a variation of the solutions given in [4] the following functions fulfill this conditions:

$$\psi(x)=\frac{(q^2e^x;e^{-\kappa})_\infty}{(e^x;e^{-\kappa})_\infty},\quad \tau(x)=(1-e^x)\frac{(q^{-2}e^{x-\kappa};e^{-\kappa})_\infty}{(q^2e^x;e^{-\kappa})_\infty}, \tag{2.19}$$

where

$$(z;p)_\infty:=\prod_{n=0}^{\infty}(1-zp^n).$$

Proof: We apply the operator $Q(\underline{x};i)$ in its decomposition (2.14) to $f(\underline{x})$. Using the relations (2.11) and (2.14) one commutes the operators $\mathcal{A}^Q$ and $\mathcal{D}^Q$ to the right, where they act on the reference state Ω according to

$$\mathcal{A}^Q(\underline{x};i)\Omega=\Omega,\quad \mathcal{D}^Q(\underline{x},i)\Omega=0,$$

respectively

$$\mathcal{A}(\underline{x};u)\Omega=\Omega,\quad \mathcal{D}(\underline{x},u)\Omega=\prod_{j=1}^{N}b(x_j-u)\,\Omega.$$

The wanted term contribution of $\mathcal{A}^Q$ reads

$$\begin{aligned}&\mathcal{A}(\underline{x};i)\sum_{\underline{u}}\mathcal{B}(\underline{x};u_m)\dots\mathcal{B}(\underline{x};u_1)\,\Omega\,g(\underline{x},\underline{u})=\\ &=\sum_{\underline{u}}\mathcal{B}(\underline{x}';u_m)\dots\mathcal{B}(\underline{x}';u_1)\,\Omega\,\prod_{j=1}^{m}q^{-1}b^{-1}(x_i+\kappa-u_j)g(\underline{x},\underline{u})=f(\underline{x}'),\end{aligned}$$

where in the last step the quasi periodic property of ψ (eqn.(2.18)) has been used. The $q^{-2}\mathcal{D}^Q$ wanted contribution vanishes due to the fact that $\mathcal{D}^Q(\underline{x};i)\,\Omega=0$.

In a second step one has to verify that all other terms cancel each other under the sum (2.16). Denote the unwanted terms obtained from $\mathcal{A}^Q$ respectively $q^{-2}\mathcal{D}^Q$ that are proportional to $\mathcal{B}^Q(\underline{x};i)\mathcal{B}(\underline{x};u_{m-1})\dots\mathcal{B}(\underline{x};u_1)\,\Omega$ by $\mathrm{uw}_{A,D}^{(i,j)}$. (They result when one commutes first 'unwanted' due to (2.13) and then always wanted due to (2.10).)

$$\begin{aligned}&\mathrm{uw}_A^{(i,m)}=\\ &\left[-q^{-1}\frac{c_-(x_i+\kappa-u_m)}{b(x_i+\kappa-u_m)}\prod_{k<m}q^{-1}b^{-1}(u_m-u_k)-(1-q^{-2})\prod_{k<m}qb^{-1}(u_k-u_m)\prod_{j=1}^{N}q^{-1}b(x_j-u_m)\right]\\ &\qquad\mathcal{B}^Q(\underline{x};i)\mathcal{B}(\underline{x};u_{m-1})\dots\mathcal{B}(\underline{x};u_1)\,\Omega g(\underline{x};\underline{u})\end{aligned} \tag{2.20}$$

$$\mathrm{uw}_D^{(i,m)} =$$

$$-q^{-1}\frac{c_-(u_m-x_i)}{b(u_m-x_i)}\prod_{k<m} q\,b^{-1}(u_k-u_m)\prod_{j=1}^{N} q^{-1}b(x_i-u_m)\mathcal{B}^Q(\underline{x};i)\mathcal{B}(\underline{x};u_{m-1})\dots\mathcal{B}(\underline{x};u_1)\,\Omega g(\underline{x};\underline{u}). \tag{2.21}$$

Using the symmetry property $\frac{c_-}{b}(-x) = -\frac{c_-}{b}(x) - (1-q^{-2})$ combine (2.21) and the second term of (2.20). Then both eqns. of (2.18) are applied to this term and obviously this term cancels with the first one of (2.20) under the sum (2.16) which completes the proof.

3 Bethe vectors and highest weight modules

The generators of $U_q[sl(2)]$ can be derived from the monodromy matrix $T_0(\underline{x};u)$ (2.7) in the limits $u \to \pm\infty$:

$$T = \begin{pmatrix} T^{11} & T^{12} \\ T^{21} & T^{22} \end{pmatrix} := \lim_{u\to-\infty} T_0(\underline{x};u) = q^{-N} \begin{pmatrix} 1 & 0 \\ (q-q^{-1})J_+ & 1 \end{pmatrix} q^{\mathbf{W}};$$

$$\tilde{T} := \lim_{u\to+\infty} T_0(\underline{x};u) = q^N q^{-\mathbf{W}} \begin{pmatrix} 1 & (q-q^{-1})J_- \\ 0 & 1 \end{pmatrix}, \tag{3.1}$$

where $\mathbf{W} = \mathrm{diag}\{W_1, W_2\}$ contains the Cartan elements. In order to prove the highest weight property of $f(\underline{x})$ i.e. the statement

$$T^{21} f(\underline{x}) \propto J_+ f(\underline{x}) = 0, \tag{3.2}$$

we introduce analogous to (3.1) as a limit of $\mathcal{T}(\underline{x};u)$

$$\mathcal{T} := \tilde{T}^{-1} T. \tag{3.3}$$

First we show that $\mathcal{T}^{21} f(\underline{x}) = 0$. The Yang-Baxter equation (2.12) implies

$$\left[\mathcal{T}^{21}, \mathcal{B}(u)\right] = (1-q^{-2})\left[\mathcal{A}(u)\mathcal{T}^{22} - \mathcal{T}^{22}\mathcal{D}(u)\right]. \tag{3.4}$$

Again due to the commutativity of the $\mathcal{B}$-operators it is sufficient to consider the term proportional to $\mathcal{B}(u_m)\dots\mathcal{B}(u_2)$. Because $\mathcal{A}(u), \mathcal{D}(u)$ and $\mathcal{T}^{22}$ act diagonal on Ω it remains to show that

$$\sum_{\underline{u}} \left[\mathcal{A}(u_1) - \mathcal{D}(u_1)\right] \Omega\, g(\underline{x};\underline{u}) = 0,$$

which follows directly from eqn. (2.18). Since $\tilde{T}^{-1}$ is an invertible operator eqn. (3.3) implies the statement (3.2).

The weights ω of the Bethe vectors $f(\underline{x})$ are defined by

$$q^{\mathbf{W}} f(\underline{x}) = q^{\omega} f(\underline{x}), \quad \omega = (\omega_1, \omega_2).$$

The commutation relations

$$\mathcal{A}\mathcal{B}(u) = q^{-2}\mathcal{B}(u)\mathcal{A}; \quad \mathcal{D}\mathcal{B}(u) = q^{2}\mathcal{B}(u)\mathcal{D}$$

and eqn. (3.3) therefore imply

$$T^{11}f(\underline{x}) = q^{N-m}f(\underline{x}) \quad \text{and} \quad T^{22}f(\underline{x}) = q^{m}f(\underline{x}),$$

giving the weight vector $\omega = (N-m, m)$ as expected.

4 The higher ranked case $U_q[sl(n)]$

In this last section we briefly discuss the case of an $U_q[sl(n)]$ difference equation. (For a more detailed description of the nested Bethe ansatz method in general we refer the reader to [1] and [6])

Denote by E_{ij} the unit matrices in $M_{n,n}(\mathbf{C})$. The $U_q[sl(n)]$ R-matrix is then given by

$$R = \sum_i E_{ii}\otimes E_{ii} + q^{-1}\sum_{i\neq j} E_{ii}\otimes E_{jj} + (1-q^{-2})\sum_{i>j} E_{ij}\otimes E_{ji}, \tag{4.1}$$

wheras the definitions for $R(x)$, $T_0(\underline{x};u)$ and $\mathcal{T}_0(\underline{x};u)$ can directely be overtaken from the equations (2.3), (2.6) and (2.7). The latter two operators are considered now as $n\times n$ matrices; the commutation relations of their elements read in analogy to eqns. (2.10) and (2.13).

$$\begin{aligned}
\mathcal{A}(\underline{x};u)\mathcal{B}_\gamma(\underline{x};v) &= q^{-1}b^{-1}(u-v)\mathcal{B}_\gamma(\underline{x};v)\mathcal{A}(\underline{x};u) \\
&\quad -q^{-1}\left[\frac{c_-(u-v)}{b(u-v)}\mathcal{B}_\gamma(\underline{x};u)\mathcal{A}(\underline{x};v) + (q-q^{-1})\mathcal{B}_\alpha(\underline{x};u)\mathcal{D}_{\alpha\gamma}(\underline{x};v)\right], \\
\mathcal{D}_{\beta\gamma}(\underline{x};u)\mathcal{B}_\delta(\underline{x};v) &= q\,b^{-1}(v-u)[\mathcal{B}_{\gamma''}(\underline{x};v)\mathcal{D}_{\beta'\delta'}(\underline{x};u)R^{\delta'\gamma'}_{\delta\gamma}(v-u)R^{\gamma''\beta}_{\beta'\gamma'} \\
&\quad -c_-(v-u)R^{\gamma'\beta}_{\beta'\gamma}\mathcal{B}_{\gamma'}(\underline{x};u)\mathcal{D}_{\beta'\delta}(\underline{x};v)] \\
\mathcal{A}^Q(\underline{x};i)\mathcal{B}_\gamma(\underline{x};u) &= q^{-1}b^{-1}(x_i+\kappa-u)\mathcal{B}_\gamma(\underline{x}';u)\mathcal{A}^Q(\underline{x};i) \\
&\quad -q^{-1}\left[\frac{c_-(x_i+\kappa-u)}{b(x_i+\kappa-u)}\mathcal{B}^Q_\gamma(\underline{x};i)\mathcal{A}(\underline{x};u) + (q-q^{-1})\mathcal{B}^Q_\alpha(\underline{x};i)\mathcal{D}_{\alpha\gamma}(\underline{x};u)\right], \\
\mathcal{D}^Q_{\beta\gamma}(\underline{x};i)\mathcal{B}_\delta(\underline{x};u) &= q\,b^{-1}(u-x_i)[\mathcal{B}_{\gamma''}(\underline{x}';u)\mathcal{D}^Q_{\beta'\delta'}(\underline{x};i)R^{\delta'\gamma'}_{\delta\gamma}(u-x_i)R^{\gamma''\beta}_{\beta'\gamma'} \\
&\quad -c_-(u-x_i)R^{\gamma'\beta}_{\beta'\gamma}\mathcal{B}^Q_{\gamma'}(\underline{x};i)\mathcal{D}_{\beta'\delta}(\underline{x};u)\big],
\end{aligned}$$

where the greek indices run from 2 to n. The operators $Q(\underline{x};i)$, which define eqn. (1.1) are given by the $U_q[sl(n)]$ Markov trace

$$Q(\underline{x};i) := \mathrm{tr}_q\mathcal{T}^Q(\underline{x};i) = \mathcal{A}^Q(\underline{x};i) + \sum_{\alpha=2}^{n} q^{-2(\alpha-1)}\mathcal{D}^Q_{\alpha\alpha}(\underline{x};i), \tag{4.2}$$

Furtheron we denote the number of particles by N_n. The Bethe vectors solving (1.1) are created by the action of N_{n-1} $\mathcal{B}$-Operators and read

$$f(\underline{x}) = \sum_{\underline{u}} \mathcal{B}_{\beta_{N_{n-1}}}(\underline{x}; u_{N_{n-1}}) \ldots \mathcal{B}_{\beta_1}(\underline{x}; u_1)\, \Omega\, g^{\underline{\beta}}(\underline{x}, \underline{u}), \tag{4.3}$$

where, in contrast to section 2, $g(\underline{x}; \underline{u})$ is a function with values in $V^{(n-1)} = \otimes^{N_{n-1}} \mathbf{C}^{(n-1)}$ given by the ansatz

$$g(\underline{x}; \underline{u}) = \prod_{i,j} \psi(x_i - u_j) \prod_{k<l} \tau(u_k - u_l)\, f^{(n-1)}(\underline{u}) \tag{4.4}$$

with functions $\psi(x)$ and $\tau(x)$ as given by (2.19) and a (yet undetermined) function $f^{(n-1)}$ with values in $V^{(n-1)}$. To prove eqn. (1.1) one has to apply $Q(\underline{x}; i)$ to $f(\underline{x})$; the 'wanted' contribution of $\mathcal{A}^Q$ again produces the r.h.s. of eqn. (1.1). On the other hand the 'unwanted' terms cancel exactly if $f^{(n-1)}$ satisfies the $n-1$ dimensional analogue of eqn. (1.1). Therefore we repeat the ansatz (4.3) for $f^{(n-1)}$ and all the resulting subsequent Bethe ansatz levels, where consequently the number of $\mathcal{B}$-operators used at the kth level is denoted by N_{n-k}. Finally after $n-2$ steps the problem has been reduced to the $U_q[sl(2)]$ problem already solved in Section 2.

The highest weight property of the Bethe vectors (4.3) is proved in a way parallel to Section 3. At some stages the higher ranked case is a little more involved, but those aspects have been already treated carefully in [12].

The resulting weight vector ω then reads

$$\omega = (\omega_1, \ldots, \omega_n) = (N_n - N_{n-1}, \ldots, N_2 - N_1, N_1), \tag{4.5}$$

again fulfilling the maximal weight condition

$$\omega_1 \geq \ldots \geq \omega_n \geq 0. \tag{4.6}$$

Summary

Starting from the $U_q[sl(2)]$ R-matrix we derived a family of q-deformed discrete Knizhnik-Zamolodchikov equations and constructed solutions via the generalization of the algebraic Bethe ansatz as developed in [1]. These solutions have been shown to be of highest weight w.r.t. the underlying quantum group structure. Using the variant of the nested Bethe ansatz method we extended the results to the higher ranked symmetry of $U_q[sl(n)]$. An application of these results can be found in [13].

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References

[1] H. Babujian, M. Karowski and A. Zapletal, J. Phys. A Math. Gen. **30** (1997) 6425.

[2] V. Tarasov and A. Varchenko, Invent. math. **128** (1997) 501.

[3] I. Frenkel and N.Yu. Reshetikhin, Commun. Math. Phys. **146** (1992) 1.

[4] N.Yu. Reshetikhin, Lett. Math. Phys. **26** (1992) 153.

[5] L. Mezincescu and R.I. Nepomechie, Mod. Phys. Lett. **A6** (1991) 2497;
C. Destri and H.J. de Vega, Nucl. Phys. **B361** (1992) 361.

[6] M. Karowski and A. Zapletal, Nucl. Phys. **B419** (1994) 567.

[7] M. Jimbo, Lett. Math. Phys. **10** (1985) 63.

[8] E.K. Sklyanin, J. Phys. **A21** (1988) 2375.

[9] I. Cherednik, Theor. Mat. Fiz. **61** (1984) 35.

[10] H. Bethe, Z. Phys. **71** (1931) 205.

[11] L.A. Takhtadzhan and L.D. Faddeev, Russian Math. Surveys **34** (1979) 186.

[12] M. Karowski and A. Zapletal, J. Phys. A Math. Gen. **27** (1994) 7419.

[13] H. Babujian, A. Fring, M. Karowski and A. Zapletal, Exact Form Factors in Integrable Field Theories: the Sine-Gordon Model, hep-th/9805185.