Straightest Geodesics on Polyhedral Surfaces

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Abstract

Geodesic curves are the fundamental concept in geometry to generalize the idea of straight lines to curved surfaces and arbitrary manifolds. On polyhedral surfaces we introduce the notion of discrete geodesic curvature of curves and define straightest geodesics. This allows a unique solution of the initial value problem for geodesics, and therefore a unique movement in a given tangential direction, a property not available in the well-known concept of locally shortest geodesics.

An immediate application is the definition of parallel translation of vectors and a discrete Runge-Kutta method for the integration of vector fields on polyhedral surfaces. Our definitions only use intrinsic geometric properties of the polyhedral surface without reference to the underlying discrete triangulation of the surface or to an ambient space.

1 Introduction

Geodesics on smooth surfaces are the straightest and locally shortest curves. They generalize the concept of euclidean straight lines and play a fundamental role in the study of smoothly curved manifolds. Two basic properties are responsible for their importance: first, geodesics solve the initial value problem which states, that from any point of a manifold there starts a unique geodesic in any direction. Second, the length minimization property provides a solution of the boundary value problem of connecting two given points on a manifold with a locally shortest curve. On smooth surfaces geodesics possess both properties, in contrast to the situation on polyhedral surfaces.

The aim of this paper is to define *straightest curves* on two-dimensional polyhedral surfaces, as opposed to the concepts of locally shortest and quasi-geodesics. Such straightest geodesics will uniquely solve the initial value problem on polyhedral surfaces, and therefore allow to move uniquely on a polyhedral surface in a given direction along a straightest geodesic until the boundary is reached, a property not available for locally shortest geodesics. An application of straightest geodesics is the definition of parallel translation of vectors and higher order numerical integration methods for tangential vector fields. This allows the extension of Runge Kutta methods to polyhedral surfaces.

We consider polyhedral surfaces as two-dimensional simplicial complexes consisting of triangles. Each triangle has a flat metric and the common edge of two neighbouring triangles has the same length in both triangles. The definition of a metric on the polyhedral surface only requires the specification of edge lengths and does not refer to an immersion of the surface in an ambient space. This intrinsic approach allows the definition of straightest geodesics, discrete geodesic curvature, vector fields, and parallel translation of vectors in terms of the geometric data of the surface, such as edge lengths, triangle angles, and discrete curvature properties.

Geodesics on polyhedral surfaces were intensively studied using different definitions. The Russian school of A.D. Alexandrov [1] defines geodesics on polyhedral surfaces as locally shortest curves which leads to important implications in the study of non-regular and regular differential geometry. But shortest geodesics cannot be extended as shortest curves across a spherical vertex with positive Gauß curvature as, for example, the vertex of a cube. Beyond a hyperbolic vertex with negative Gauß curvature there even exists a continuum of extensions. Therefore, shortest geodesics fail to solve the initial value problem for geodesics at vertices of a polyhedral surface.

A.D. Alexandrov also introduced the concept of quasi-geodesics which are limit curves of geodesics on a family of converging smooth surfaces. They form a wider class than shortest geodesics and were amongst others studied by Pogorelov [7] on convex polyhedral surfaces. A quasi-geodesic through a spherical vertex is a curve with right and left angles both less than π, and therefore an inbound direction has multiple extensions.

Shortest geodesics appear in many practical applications. For example, the optimal movement of a robot should have minimal length in its parameter space. Such discrete minimization problems are studied in computational geometry, see for example Dijkstra [4], Sharir and Schorr [10], and Mitchell et.al. [6] for efficient algorithms on the computation of the shortest path in graphs and in polyhedral spaces.

Our paper starts in section 2 with a review of geodesics on smooth surfaces, especially since some of their properties differ from those of geodesics on polyhedral surfaces. In section 3 we will introduce polyhedral surfaces as metric spaces and recall basic facts. Straightest geodesics are defined in section 4 and discussed as solutions of the initial value problem. In section 5 we imbed the notion of straightest lines into the concept of discrete geodesic curvature of arbitrary curves on polyhedral surfaces. This general setting is more appropriate for our later discussions, and straightest geodesics turn out to be those class of curves with vanishing discrete geodesic curvature. As a validation of the definition we prove the Gauß-Bonnet theorem using our notion of discrete geodesic curvature. In section 6 we apply the concept to the definition of parallel translation of tangential vector fields and in section 7 we generalize Runge Kutta methods to the numerical integration of ordinary differential equations on polyhedral surfaces.

Applications of this paper are given in the video *Geodesics and Waves* [8]. The numerics were developed within the visualization environment OORANGE [5].

2 Review of Geodesics on Smooth Surfaces

Geodesics on smooth surfaces can be characterized by different equivalent properties. The generalized properties on polyhedral surfaces will no longer be equivalent and lead to different classes of discrete geodesics. The following material can be found in any introductory text book on differential geometry, see for example [3].

Let M be a smooth surface and $\gamma : I = [a, b] \to M$ a curve parametrized over an interval I. To avoid accelerations tangential to the curve we assume arc length parametrization, i.e. the tangent vector has constant length $|\gamma'| = 1$. A curve γ is called *locally shortest* if it is a critical point of the length functional $L(\gamma_{|[a,b]}) := length(\gamma_{|[a,b]})$ with respect to variations tangential to M which leave the endpoints fixed. Formally, if $\phi : I \to T_\gamma M$ is a tangential vector field along γ with $\phi(a) = 0$ and $\phi(b) = 0$, then we have $\frac{\partial}{\partial \varepsilon} L(\gamma + \varepsilon\phi)_{|\varepsilon=0} = 0$. A critical point of the length functional is usually not a global minimizer compared to curves with the same endpoints.

On smooth manifolds the length minimizing property of geodesics can be reformulated as an ordinary differential equation for γ, namely $\gamma''(s)^{\tan \mathrm{M}} = 0$, the Euler-Lagrange equations of the variational problem.

The curvature $\kappa(s) = |\gamma''(s)|$ of a curve measures the infinitesimal turning of the tangent vector at every point $\gamma(s)$. For curves γ on surfaces $M \subset \mathbb{R}^3$, the curvature can be decomposed into the curve's bending in the normal direction n of the surface and its bending in the tangent space in direction of the binormal b. This decomposition leads to the definition of the geodesic curvature κ_g and the normal curvature κ_n of a curve:

$$\begin{aligned} \kappa^2(s) &= |\gamma''(s)|^2 \\ &= \left|\gamma''(s)^{\tan \mathrm{M}}\right|^2 + \left|\gamma''(s)^{\mathrm{nor\ M}}\right|^2 \\ &= \kappa_g^2(s) + \kappa_n^2(s). \end{aligned} \tag{1}$$

The geodesic curvature κ_g of a curve γ measures the tangential acceleration. If $\kappa_g = 0$ then the curve varies up to second order only in direction of the surface normal, therefore it is a *straightest curve* on the surface. The normal curvature κ_n is related with the bending of the surface itself and can be neglected from an intrinsic point of view.

Summarizing, one characterizes smooth geodesics as follows:

Definition 1 *Let M be a smooth two-dimensional surface. A smooth curve $\gamma : I \to M$ with $|\gamma'| = 1$ is a* geodesic *if one of the equivalent properties holds:*

1. *γ is a locally shortest curve.*
2. *γ'' is parallel to the surface normal, i.e.*

$$\gamma''(s)^{tan\ M} = 0. \tag{2}$$

3. *γ has vanishing geodesic curvature $\kappa_g = 0$.*

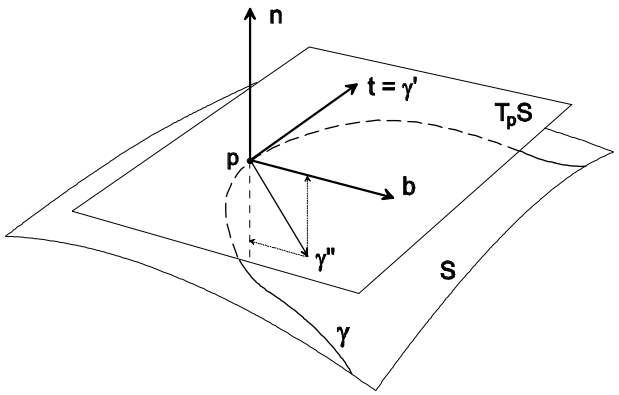

Figure 1: Geodesic and normal curvature of a curve on a smooth surface.

In section 4 we will consider geodesics on polyhedral surfaces and notice that the polygonal equivalents of the above properties lead to different notions of discrete geodesics.

The boundary value problem for geodesics has a solution in every homotopy class and is usually not unique. On the other hand, we have a unique solution for the initial value problem derived from equation (2):

Lemma 2 *Let M be a smooth manifold. Then for any point $p \in \overset{\circ}{M}$ in the interior of M and any tangent direction $v \in T_pM$ the initial value problem*

$$\begin{aligned} \gamma''(s)^{tan\ M} &= 0 \\ \gamma(0) &= p \\ \gamma'(0) &= v \end{aligned} \tag{3}$$

has a unique solution $\gamma : [0, \ell) \to M$, where ℓ is the length of the maximal interval of existence.

3 Curvature of Polyhedral Surfaces

In this section we review some facts on the geometry of polyhedral surfaces. Basic references are for example A.D. Alexandrov and Zalgaller [1] and Reshetnyak [9]. For simplification we restrict ourselves to two-dimensional surfaces consisting of planar triangles. A topological triangle f in a two-dimensional manifold S is a simple domain $f \subset S$ whose boundary is split by three vertices into three edges with no common interior points.

Definition 3 *A* polyhedral surface *S is a two-dimensional manifold (with boundary) consisting of a finite or denumerable set F of topological triangles and an intrinsic metric $\rho(X, Y)$ such that*

1. *Any point $p \in S$ lies in at least one triangle $f \in F$.*

2. *Each point $p \in S$ has a neighbourhood that intersects only finitely many triangles $f \in F$.*

3. *The intersection of any two non-identical triangles g, $h \in F$ is either empty, or consists of a common vertex, or of a simple arc that is an edge of each of the two triangles.*

4. *The intrinsic metric ρ is flat on each triangle, i.e. each triangle is isometric to a triangle in $\mathbb{R}^2$.*

Remark 4 *Most of our considerations apply to a more general class of length spaces. Each face may have an arbitrary metric as long as the metrics of two adjacent faces are compatible, i.e. if the common edge has the same length in both faces, and the triangle inequality holds.*

Let $\gamma \subset S$ be a curve whose segments on each face are rectifiable. Then the *length of* γ is well-defined and given by

$$Length(\gamma) = \sum_{f \in F} Length(\gamma_{|f}). \tag{4}$$

The neighbourhood of a vertex is isometric to a cone and is characterized by the total vertex angle:

Definition 5 *Let S be a polyhedral surface and $v \in S$ a vertex. Let $F = \{f_1, ..., f_m\}$ be the set of faces containing p as a vertex, and θ_i be the interior angle of the face f_i at the vertex p, compare figure 2. Then the* total vertex angle $\theta(p)$ *is given by*

$$\theta(p) = \sum_{i=1}^{m} \theta_i(p). \tag{5}$$

Interior points p of a face or of an open edge have a neighbourhood which is isometric to a planar euclidean domain and we define $\theta(p) = 2\pi$.

All points of a polyhedral surface can be classified according to the sign of the *vertex angle excess* $2\pi - \theta(p)$:

Definition 6 *A vertex p of a polyhedral surface S with total vertex angle $\theta(p)$ is called* euclidean, spherical, *or* hyperbolic *if its angle excess $2\pi - \theta(p)$ is $= 0$, > 0, or < 0. Respectively, interior points of a face or of an open edge are euclidean.*

The neighbourhood of a vertex can be isometrically unfolded to a (partial or multiple) covering of a part of the euclidean plane. There exist three situations as shown in figure 2 which metrically characterize the vertex. For example, the tip of a convex cone is a spherical vertex and a saddle point is hyperbolic. On the other hand, a spherical vertex need not be the tip of a convex cone. The isometric unfolding of sets of a polyhedral surface is a common procedure to study the geometry.

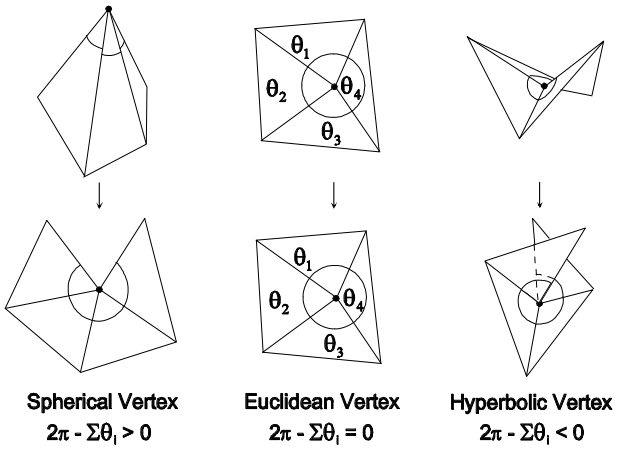

Figure 2: Classification of vertices on a polyhedral surface according to the excess of the vertex angle, and their unfolding in a planar domain.

The Gauß curvature of a general manifold is a central intrinsic property of the geometry and can be computed in terms of the metric. It influences, for example, the parallel translation of vectors along curves. The Gauß curvature of a piecewise linear surface is concentrated at the isolated vertices since all other points on the surface have a neighbourhood isometric to a planar euclidean domain with zero curvature. It is therefore more appropriate to work with the concept of total Gauß curvature.

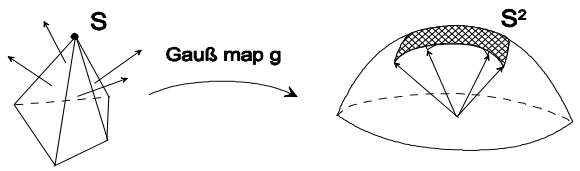

Figure 3: The Gauß map assigns to each point $p \in S$ of a surface its normal vector $n(p) \in \mathbb{S}^2$. At edges and vertices of a polyhedral surface the image of the Gauß map is the spherical convex hull of the normal vectors of adjacent faces.

Using following definition the curvature can be measured directly in metrical terms of the surface S.

Definition 7 *The* (total) Gauß curvature $K(p)$ *of a vertex* p *on a polyhedral surface* S *is defined as the vertex angle excess*

$$\begin{aligned} K(p) &= 2\pi - \theta(p) \\ &= 2\pi - \sum_{i=1}^{m} \theta_i(p). \end{aligned} \tag{6}$$

An immediate consequence is that euclidean vertices have curvature $K = 0$, spherical vertices have $K > 0$, and hyperbolic vertices have $K < 0$. For example, the vertices of a cube each have Gauß curvature $\frac{\pi}{2}$.

For a smooth surface S imbedded into $\mathbb{R}^3$ the curvature measures the infinitesimal turn of the normal vector of the surface and can be defined via the *Gauß map* $g : S \to \mathbb{S}^2$ which assigns to each point p on a surface S its normal vector $n(p)$, see figure 3. The total Gauß curvature $K(\Omega)$ of a domain $\Omega \subset S$ is given by the area of its spherical image: $K(\Omega) = \text{area } g(\Omega)$. It is an easy calculation to show that this relation also holds for the Gauß curvature of a vertex on a polyhedral surface.

4 Discrete Straightest Geodesics

Our approach to discrete geodesics on polyhedral surfaces concentrates on the property of a curve to be straightest rather than locally shortest. Both properties are equivalent for geodesics on smooth surfaces, as mentioned in section 2, but locally shortest curves on polygonal surfaces do not allow a unique extension, for example, beyond spherical vertices of the surface. The original motivation for our study was to define a unique way to move straight ahead in a given direction on a polyhedral surface. Applications are, for example, the tracing of moving particles restricted to flow along a polyhedral surface, the solution of initial value problems on polyhedral surfaces related with given tangential vector fields, and the intrinsic generalization of numerical algorithms for ordinary differential equations to polygonal surfaces.

The concept of shortest geodesics in graphs, polyhedral manifolds, and more general length spaces has been studied by a number of authors in different fields, see for example [4][6][1][2]. For our applications this concept has a central missing property, namely, the initial value problem for geodesics is not uniquely solvable and in some cases has no solution: first, no shortest geodesics can be extended through a spherical vertex since it could be shortened by moving off the corner, and second, there exists a family of possible extensions of a geodesic as a shortest curve through a hyperbolic vertex: every extension with curve angles $\theta_l, \theta_r \in [\pi, \theta - \pi]$ is locally shortest where θ is the total vertex angle. See lemma 11 and figure 4.

Quasi-geodesics are a different approach which was introduced by A.D. Alexandrov (see the references to the original Russian literature in [1]) and investigated on convex surfaces by Pogorelov [7] and others. They appear as limit sets of smooth geodesics when smooth surfaces approximate, for example, a polyhedral surface. On polyhedral surfaces quasi-geodesics are characterized by their fulfillment of the inequation $|\pi - \theta_l| + |\pi - \theta_r| - |2\pi - \theta_l - \theta_r| \geq 0$ at each point, where θ_l and θ_r are the two angles of the curve, and $\theta_l + \theta_r = \theta$ is the total vertex angle of the point. Compare figure 4 for the notation. At hyperbolic vertices with $\theta > 2\pi$ the definition is identical to that for shortest geodesics, while at spherical vertices with $\theta < 2\pi$ curves with $\pi - \theta_l \geq 0$ and $\pi - \theta_r \geq 0$ are quasi-geodesics.

In the following definition we introduce straightest geodesics which are a new class of discrete geodesics on polyhedral surfaces. This class has a non-empty intersection with the set of shortest geodesics and is a subset of quasi-geodesics.

Definition 8 *Let S be a polyhedral surface and $\gamma \subset S$ a curve. Then γ is a* straightest geodesic *on S if for each point $p \in \gamma$ the left and right curve angles θ_l and θ_r at p are equal, see figure 4.*

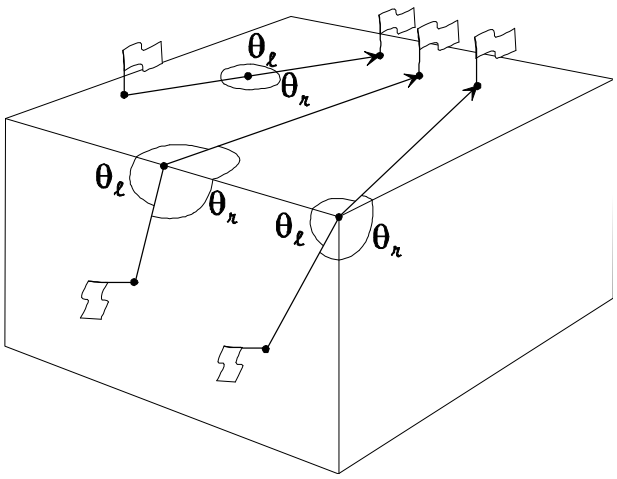

Figure 4: Notion of left and right curve angles θ_l and θ_r with $\theta_l + \theta_r = \theta$.

A straightest geodesic in the interior of a face is locally a straight line, and across an edge it has equal angles on opposite sides as shown in color plate ???. The definition of straightest geodesics on faces and through edges is identical to the concept of shortest geodesics but at vertices the concepts differ. Our definition fits into the more general discussion of discrete geodesic curvature of curves on a polyhedral surface, this will be discussed in detail in section 5.

The following theorem proves the unique solvability of the initial value problem for straightest geodesics. To state the problem we start with the notion of a tangent vector on a polyhedral surface:

Definition 9 *Let S be a polyhedral surface and $p \in S$ a point. A* polyhedral tangent vector *v with base point p lies in the plane of an adjacent face and locally points into the face. The polyhedral tangent space T_pS consists of all polyhedral tangent vectors at p.*

We remark, that the polyhedral tangent bundle TS can be equipped with the structure of a topological vector bundle by introducing normalized angles as in definition 17, but do not pursue this property. Instead, we use the fact that polyhedral tangent vectors are characterized solely by intrinsic properties of the geometry rather than by reference to an ambient space.

Theorem 10 (Discrete Initial Value Problem) *Let S be a polyhedral surface and $p \in S$ a point with polyhedral tangent vector $v \in T_pS$. Then there exists a unique straightest geodesic γ with*

$$\begin{aligned} \gamma(0) &= p \\ \gamma'(0) &= v, \end{aligned} \tag{7}$$

and the geodesic extends to the boundary of S.

Proof. There exists a face f of S which contains the initial point p and, for a small number $\varepsilon > 0$, the straight line $\gamma(t) := p + tv$ with $t \in [0, \varepsilon)$. γ is a straightest geodesic and a solution of equation 7. If we extend γ beyond the small interval and γ reaches an edge or a vertex of S for larger values of t then definition 8 of straightest geodesics uniquely defines how to extend γ beyond the edge or vertex. That is to proceed in that direction for which the left and right curve angles of γ at the vertex are equal. ■

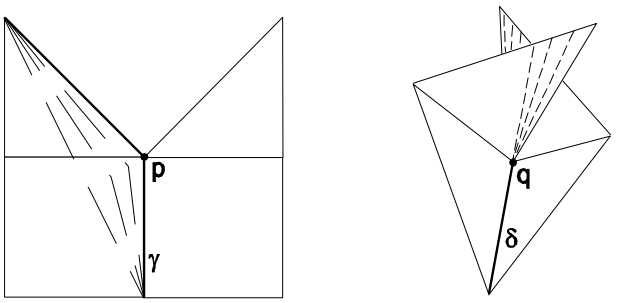

Figure 5: Locally shortest geodesics cannot be extended through a spherical vertex p and there exist multiple continuations at a hyperbolic vertex q.

The concepts of straightest and shortest geodesics differ on polyhedral surfaces. For example, as shown in the following lemma, the theorem above does not hold for locally shortest geodesics approaching a spherical or hyperbolic vertex. As long as a geodesic γ does not meet a vertex of a polyhedral surface both concepts are equal and γ is both, straightest and locally shortest. The following lemma comprehends the differences:

Lemma 11 *On a polyhedral surface S the concepts of straightest and locally shortest geodesics differ in the following way (see figure 5):*

1. *A geodesic γ containing no surface vertex is both straightest and locally shortest.*
2. *A straightest geodesic γ through a spherical vertex is not locally shortest.*
3. *There exists a family of shortest geodesics γ_θ through a hyperbolic vertex with the same inbound direction. Only one of the shortest geodesics extends the inbound direction as straightest geodesic.*
4. *Straightest geodesics do not solve the boundary value problem for geodesics since there exist shadow regions in the neighbourhood of a hyperbolic vertex where two points cannot be joined by a straightest geodesic.*

Proof. Ad 1.) We unfold the faces met by the geodesic to an isometric strip of faces in the euclidean plane. The geodesic γ is unfolded to a euclidean straight line in the interior of the strip which is locally shortest and fulfills the angle condition of definition 8.

Ad 2.) Let γ be a straightest geodesic through a spherical vertex with curvature $K > 0$. We unfold the adjacent faces to a planar domain by cutting along the outbound direction of γ. The image of γ in the plane has a corner at the vertex with curve angle $\frac{\theta}{2} = \pi - \frac{K}{2} < \pi$ at both sides. Therefore, γ is not locally shortest since it can be shortened by smoothing the corner in either direction as shown on the left in figure 5.

Ad 3.) A hyperbolic vertex has curvature $K < 0$. Let γ_0 be the unique straightest geodesic though the vertex which extends the inbound direction. We unfold the adjacent faces to a planar domain by cutting along the outbound direction of γ_0, then γ_0 has a curve angle $\frac{\theta}{2} = \pi - \frac{K}{2} > \pi$ at both sides of the corner. Assume a curve with the same inbound but a different outbound direction. Whenever both angles between the inbound and outbound direction are bigger than or equal to π, we cannot locally shorten the curve. Therefore all such curves are locally shortest. ■

5 Discrete Geodesic Curvature

We define the notion of geodesic curvature of curves on piecewise linear surfaces with the later aim of defining parallel translation of vectors along arbitrary curves. Additionally, vanishing geodesic curvature should characterize straightest geodesics. The definition should comply with the known (total) curvature of polygons in the euclidean plane, and the Gauß-Bonnet equation should hold. In the following, we assume curves to be smooth on faces and to have well-defined polyhedral tangent directions at the edges and vertices of the surface. Similar to the discrete Gauß curvature for surfaces, the discrete geodesic curvature is the equivalent of the total geodesic curvature of smooth surfaces.

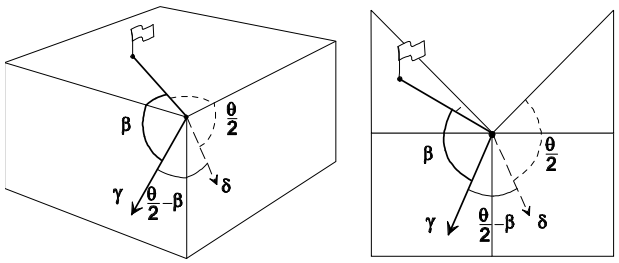

Figure 6: The discrete geodesic curvature of a curve γ is the normalized angle between γ and a discrete straightest geodesic δ.

Definition 12 *Let γ be a curve on a polyhedral surface S. Let θ be the total vertex angle and β one of the two curve angles of γ at p. Then the* discrete geodesic curvature *κ_g of γ at p is given by*

$$\kappa_g = \frac{2\pi}{\theta}\left(\frac{\theta}{2} - \beta\right). \tag{8}$$

Choosing the other curve angle $\beta' = \theta - \beta$ changes the sign of κ_g.

Using the notion of discrete geodesic curvature we obtain a new characterization of straightest geodesics since they bisect the total vertex angle θ, i.e. $\beta = \frac{\theta}{2}$:

Lemma 13 *Let S be a polyhedral surface and $\gamma \subset S$ a curve. Then γ is a straightest geodesic if and only if γ has vanishing discrete geodesic curvature.*

Remark 14 *1.) Let γ be a polygon in the euclidean plane S and $p \in \gamma$ be a vertex with curve angle β. Then the discrete geodesic curvature equals the total curvature of γ at p defined by the spherical image of its normal vectors.*
2.) Let S be a polyhedral surface and let γ touch a vertex $p \in S$, i.e. $\beta = 0$. Then the geodesic curvature of γ at p is $\kappa_g = \pi$, i.e. it can be measured in the euclidean face and without influence of the vertex angle θ at p.
3.) Shortest geodesics through a hyperbolic vertex with vertex angle $\theta > 2\pi$ have geodesic curvatures κ_g in the interval $\left[-\pi(1-\frac{2\pi}{\theta}), \pi(1-\frac{2\pi}{\theta})\right]$.

Straightest geodesics are natural generalizations of straight lines in euclidean space. For example, geodesic triangles on surfaces can be defined as simply connected regions bounded by three straightest segments, and geodesic polygons as piecewise straightest curves.

The Gauß-Bonnet theorem relates the topology and geometry of surfaces. It is a remarkable consequence of the definition of discrete geodesic curvature that this fundamental theorem still holds. In fact, one can even reverse the arguments and derive our formula for geodesic curvature from the requirement that the equation of Gauß-Bonnet should hold.

There have been different formulations of the Gauß-Bonnet theorem on polyhedral surfaces, each expressing the Euler characteristic $\chi(\Omega)$ of a domain Ω using different curvature terms. For example, Reshetnyak [9] only uses the Gauß curvature of interior vertices and defines the curvature of the boundary curve by $\kappa = \pi - \beta$, where β is the inner curve angle of the boundary. We refine this approach and split his definition of boundary curvature in two components, a geodesic curvature of the boundary curve and a partial Gauß curvature, where the vertices $p \in \partial\Omega$ contribute to the total Gauß curvature of Ω. The following natural definition determines the contribution of boundary vertices to the total Gauß curvature of Ω. The contribution is proportional to the curve angle β:

Definition 15 *Let $\Omega \subset S$ be a domain on a polyhedral surface with boundary $\Gamma = \partial\Omega$. If $\theta(p)$ is the total vertex angle and $\beta(p)$ the inner curve angle at a vertex $p \in \Gamma$, then the* partial Gauß curvature $K_{|\Omega}$ *of Ω at p is proportional to β:*

$$K_{|\Omega}(p) = \frac{\beta}{\theta} K(p). \tag{9}$$

If $\beta = 0$ then the vertex has no partial Gauß curvature, and $\beta = \theta$ leads to a full contribution of the total Gauß curvature $K = 2\pi - \theta$ to Ω. In the following we simplify the notation by omitting the subindex $_{|\Omega}$.

Theorem 16 (Discrete Gauss-Bonnet) *Let S be a polyhedral surface and $\Omega \subset S$ a domain with boundary curve Γ and Euler characteristic $\chi(\Omega)$. Then the equation*

$$\sum_{p\in\Omega} K(p) + \kappa_g(\Gamma) = 2\pi\chi(\Omega) \tag{10}$$

holds where the total Gauß curvature of Ω includes the partial Gauß curvature at boundary points. If Γ is piecewise straightest then the total geodesic curvature is the sum of the geodesic curvature at the vertices of Γ.

Proof. For the proof we use the version

$$\sum_{p\in\overset{\circ}{\Omega}} K(p) + \sum_{p\in\Gamma} (\pi - \beta(p)) = 2\pi\chi(\Omega)$$

proved by Reshetnyak [9] where only interior vertices of Ω contribute to the total Gauß curvature. Let $p \in \Gamma$ be a boundary vertex, then we have the splitting

$$K_{|\Omega}(p) - \kappa_g(p) = \pi - \beta(p)$$

which proves the assumption. ■

6 Parallel Translation of Vectors

Numerical methods for the integration of ordinary differential equations rely on the possibility for parallel translation of vectors in the euclidean plane. For example, higher order Runge-Kutta methods do several trial shots in a single integration step to compute the final shooting direction and translate direction vectors to their current positions. When transferring such integration methods to surfaces, which are not described by local charts, it is necessary to compare vectors with different base points on the curved surface.

We use the notion of polyhedral tangent vectors formulated in definition 9 and define an intrinsic version of parallel translation of vectors which uses no ambient space as reference. We start with two definitions of angles:

Definition 17 *Let S be a polyhedral surface and $p \in S$ a point with total vertex angle θ. The* euclidean angle $\measuredangle(v, w)$ *between tangent vectors $v, w \in T_pS$ is the angle between corresponding vectors in the unfolded neighbourhood of p measured in $\mathbb{R}^2$, i.e. $\measuredangle(v, w) \in \left[-\frac{\theta}{2}, \frac{\theta}{2}\right]$. The* normalized angle $\alpha(v, w)$ *is obtained by scaling:*

$$\alpha(v, w) := \frac{2\pi}{\theta}\measuredangle(v, w). \tag{11}$$

The normalized and euclidean angles are identical at points which are not vertices of the surface. In practical applications one measures the euclidean angle at first, and then uses the normalized angle to avoid case distinctions at vertices of the surface as seen, for example, in the following lemma:

Lemma 18 *Let Δ be a geodesic triangle on a polyhedral surface S whose edges are straightest segments. If α_1, α_2, and α_3 are the normalized angles of Δ then we have*

$$\alpha_1 + \alpha_2 + \alpha_3 - \pi = \int_\Delta K. \tag{12}$$

Proof. Denote the euclidean angles of Δ with β_i and the vertex angles with θ_i . Then the geodesic curvature of the boundary of Δ at one of its vertices is given by

$$\kappa_g = \frac{2\pi}{\theta}\left(\frac{\theta}{2} - \beta\right) = \pi - \alpha \tag{13}$$

and the assumption follows directly from the discrete Gauß-Bonnet equation (10). ■

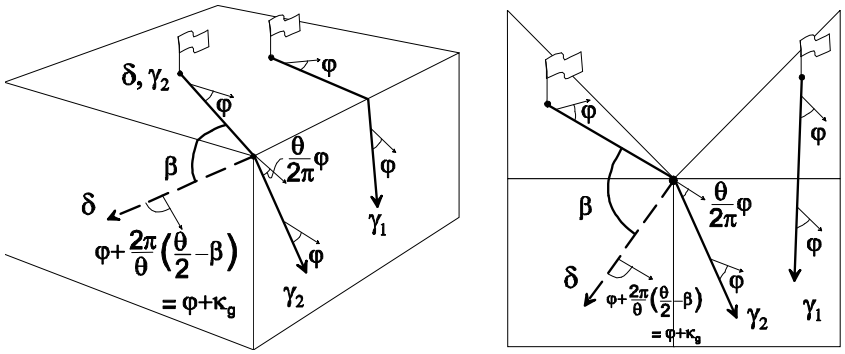

Figure 7: Parallel translation of vectors along straightest geodesics γ_1,γ_2 and an arbitrary curve δ.

On polyhedral surfaces we can use the concept of straightest geodesics and normalized angles to define parallel translation along geodesics and arbitrary curves similar to the smooth case:

Definition 19 *Let $\gamma : I \to S$ be a parametrized straightest geodesic on a polyhedral surface S. A tangential vector field $v : I \to TS$ with $v(s) \in T_{\gamma(s)}S$ is a* parallel vector field *along γ if the normalized angle $\alpha(v(s), \gamma'(s))$ is constant.*

Definition 20 *Let κ_g be the geodesic curvature of a curve $\gamma : I \to S$ with $\gamma(0) = p$ and let $v_0 \in T_pS$ be a tangent vector with normalized angle $\alpha(0) := \frac{2\pi}{\theta(p)}\measuredangle(v_0, \gamma'(0))$. Then v_0 uniquely extends to a parallel vector field v with $v(s) \in T_{\gamma(s)}S$ along γ with $v(0) = v_0$. $v(s)$ is defined by the normalized angle $\alpha(s)$ it encloses with $\gamma'(s)$:*

$$\alpha(s) = \alpha(0) + \int_0^s \kappa_g(t)\, dt. \tag{14}$$

The formula is well-known for curves on smooth surfaces. In the discrete situation we have made direct use of the definition of discrete geodesic curvature and the notion of normalized angles at vertices.

7 Runge Kutta on Discrete Surfaces

The tracing of particles on a surface by integrating a given vector field with Euler or Runge Kutta methods requires an additional effort to keep the trace on the surface. For example, one may use local coordinate charts of a surface to transform the integration to the planar euclidean domain. Here the metrical distortion between surface and euclidean domain must be respected and a preprocessing step to generate the charts and transitions between neighbouring charts is required.

If the vector field is given on a curved surface in an ambient space, say $\mathbb{R}^3$, then a usual tangent vector "points into the ambient space", leading the numerical particle trace off the surface without additional projection methods.

The concepts of straightest geodesics and polyhedral tangent vectors offer an intrinsic tool to solve these problems. In euclidean methods, the vector $v|_{\gamma(s)}$ is interpreted as tangent vector to the particle trace $\gamma(s)$, and the straight line through $\gamma(s)$ with direction $v|_{\gamma(s)}$ is the first order approximation of γ. The idea on surfaces is to use polyhedral tangent vectors defined in definition 9 and to replace the straight line with a straightest geodesic through $\gamma(s)$ with initial direction $v|_{\gamma(s)}$:

Definition 21 (Geodesic Euler Method) *Let S be a polyhedral surface with a polyhedral tangential vector field v on S, let $y_0 \in S$ be an initial point, and let $h > 0$ a (possibly varying) stepsize. For each point $p \in S$ let $\delta(t, p, v(p))$ denote the unique straightest geodesic through p with initial direction $v(p)$ and evaluated at the parameter value t. A single iteration step of the* geodesic Euler method *is given by*

$$y_{i+1} := \delta(h, y_{i,} v(y_i)). \tag{15}$$

This produces a sequence of points $\{y_0, y_1, ...\}$ on S which are connected by straightest geodesic segments of length h. For each $i \in \{0, 1, ...\}$ we define

$$\gamma(ih + t) := \delta(t, y_{i,} v(y_i)), \;\; t \in [0, h] \tag{16}$$

and obtain a piecewise straightest, continuous curve $\gamma : [0, \ell) \to S$ of some length ℓ such that each segment $\gamma_{|[ih,\ (i+1)h]}$ is a straightest geodesics.

The definition of the geodesic Euler method is intrinsic and no projection of the tangent vectors or tangent directions onto the surface are required during integration. If the original vector field is not a polyhedral tangential field then an initial generation of a polyhedral tangential vector field is required in a preprocessing step, however, this step is part of the formulation of the numerical problem and not of the integration method.

Using the concept of parallel translation it is straight forward to define higher order integration methods in a similar intrinsic way. For simplicity, we restrict to a 4-th order geodesic Runge Kutta method:

Definition 22 (Geodesic Runge-Kutta Method) *Let S be a polyhedral surface with a polyhedral tangential vector field v on S, let $y_0 \in S$ be an initial point, and let $h > 0$ a (possibly varying) stepsize. For each point $p \in S$ let $\delta(t, p, v(p))$ denote*

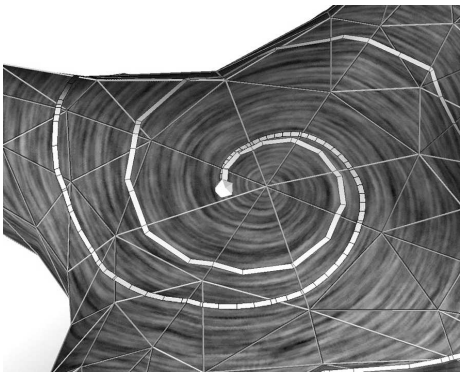

Figure 8: The two piecewise straightest geodesics are solutions computed with the geodesic Euler method (outer curve, stepsize h) and 4th order Runge Kutta method (inner curve, stepsize $4h$). Note, that the geodesic segments extend across triangle edges and vertices. Also, a comparison with the underlying flow shows the expected better approximation quality of the geodesic Runge-Kutta method.

the unique straightest geodesic through p with initial direction $v(p)$ and evaluated at the parameter value t. A single iteration step of the geodesic Runge Kutta method *is given by*

$$y_{i+1} := \delta(h, y_{i,} v_i) \tag{17}$$

where the direction v_i is a polyhedral tangent vector at y_i obtained as follows: we denote the parallel translation of vectors along a geodesic δ to $\delta(0)$ by $\pi_{|\delta}$ and iteratively define

$$\begin{aligned} v_i^1 &: = v(y_i) \\ v_i^2 &: = \pi_{|\delta_1} \circ v(\delta_1(\frac{h}{2}, y_i, v_i^1)) \\ v_i^3 &: = \pi_{|\delta_2} \circ v(\delta_2(\frac{h}{2}, y_i, v_i^2)) \\ v_i^4 &: = \pi_{|\delta_3} \circ v(\delta_3(h, y_i, v_i^3)) \end{aligned} \tag{18}$$

and

$$v_i := \frac{1}{6}(v_i^1 + 2v_i^2 + 2v_i^3 + v_i^4) \tag{19}$$

where the curves δ_i are straightest geodesics through y_i with initial direction v_i^j for $j \in \{1, 2, 3\}$.

8 Conclusion

On polyhedral surfaces we introduced the concept of straightest geodesics and discrete geodesic curvature of curves. We applied the concept to define the parallel translation of tangential vectors and generalized Runge Kutta methods to polyhedral surfaces. These concepts allow a uniform and intrinsic description of geometric and numerical properties on polyhedral surfaces.

References

[1] A. D. Aleksandrov and V. A. Zalgaller. *Intrinsic Geometry of Surfaces*, volume 15 of *Translation of Mathematical Monographs*. AMS, 1967.

[2] S. B. Alexander and R. L. Bishop. Comparison theorems for curves of bounded geodesic curvature in metric spaces of curvature bounded above. *Diff. Geom. Appl.*, 6(1):67–86, 1996.

[3] M. P. D. Carmo. *Differential Geometry of Curves and Surfaces*. Prentice-Hall, 1976.

[4] E. Dijkstra. A note on two problems in connection with graphs. *Numer. Math.*, 1:269–271, 1959.

[5] C. Gunn, A. Ortmann, U. Pinkall, K. Polthier, and U. Schwarz. Oorange - a visualization environment for mathematical experiments. In H.-C. Hege and K. Polthier, editors, *Visualization and Mathematics*, pages 249–265. Springer Verlag, Heidelberg, 1997.

[6] J. S. B. Mitchell, D. M. Mount, and C. H. Papadimitriou. The discrete geodesic problem. *SIAM J. Comput.*, 16(4):647–668, 1987.

[7] A. V. Pogorelov. Quasigeodesic lines on a convex surface. *Amer. Math. Soc. Transl.*, I. Ser. 6(72):430–473, 1952.

[8] K. Polthier, M. Schmies, M. Steffens, and C. Teitzel. Video on geodesics and waves. Siggraph'97 Video Review, 1997.

[9] Y. G. Reshetnyak. *Geometry IV*, volume 70 of *Encyclopaedia of Mathematical Sciences*, chapter 1. Two-Dimensional Manifolds of Bounded Curvature, pages 3–164. Springer Verlag, 1993.

[10] M. Sharir and A. Schorr. On shortest paths in polyhedral space. *SIAM J. Comput.*, 15(1):193–215, 1986.