Projective generalizations of Lelieuvre's formula

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Abstract

Generalizations of the classical affine Lelieuvre formula to surfaces in projective three-dimensional space and to hypersurfaces in multidimensional projective space are given. A discrete version of the projective Lelieuvre formula is presented too.

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1 Introduction

The classical Lelieuvre formula [1] of affine geometry provides us a way to construct a surface via the affine conormal vector (see *e.g.* [2]-[4]). Namely, it is the relation

$$\boldsymbol{f}_\xi = \sigma \boldsymbol{\nu} \wedge \boldsymbol{\nu}_\xi \quad , \quad \boldsymbol{f}_\eta = -\sigma \boldsymbol{\nu} \wedge \boldsymbol{\nu}_\eta \tag{1.1}$$

between the coordinates $\boldsymbol{f}$ of a surface in R^3 and its conormal $\boldsymbol{\nu}$. The conormal $\boldsymbol{\nu}$ obeys the equation $\boldsymbol{\nu}_{\xi\eta} \parallel \boldsymbol{\nu}$ and the corresponding Blaschke metric is $\Omega = 2det|\boldsymbol{\nu}, \boldsymbol{\nu}_\xi, \boldsymbol{\nu}_\eta| d\xi\, d\eta$. For an indefinite metric $\sigma = 1$ and ξ, η are real-valued asymptotic coordinates while for a positive-definite metric $\sigma = \sqrt{-1}$ and ξ and η are complex conjugate to each other: $\eta = \overline{\xi}$. The Lelieuvre formula (1.1) is an effective tool to study surfaces in affine geometry [2]-[4]. It's generalization to hypersurfaces in R^{n+1} has been given in [5]. The Lelieuvre formula (1.1) provides us also a way to define integrable deformations of affine surface via the Nizhik-Veselov-Novikov (NVN) equation [6].

In this paper we present a projective analog of the Lelieuvre's formula. It is given by

$$f \wedge f_\xi = \sigma \star (\nu \wedge \nu_\xi) \quad , \quad f \wedge f_\eta = -\sigma \star (\nu \wedge \nu_\eta) \tag{1.2}$$

where $f \subset P^3$, $\nu \subset P_3$ and P_3 is a projective space dual to P^3, $\star$ denotes the Hodge star operation and $\sigma^2 = \pm 1$. The relation (1.2) provides an explicit formula (2.29) for f via ν. The projective Lelieuvre map (PLM) (1.2) manifest also a symmetry between f and ν (projective duality). We derive the compatibility condition for (1.2) and prove an invariance of full determinants under the correspondence (1.2). It is shown that (1.2) sets up correspondence between the normalizations of homogeneous coordinates for a surface in P^3 and its dual surface in P_3.

A PLM for hypersurfaces in $(n+1)$-dimensional projective space is also presented. A discrete version of the PLM for discrete surfaces in P^3 is given. We present discrete analogs of the projective Fubini forms. An affine reduction (*i.e.* the corresponding formulae in the gauge $f = (\boldsymbol{f}, -1)$) is considered. For the discrete case we obtain analogs of the Blaschke and affine Fubini cubic forms.

The paper is organized as follows. In section 2 and 3 the PLM (1.2) in asymptotic and conjugate line coordinates is presented and studied. The PLM type formulae for hypersurfaces in P^{n+1} are given in section 4. The discrete analog of the PLM (1.2) is presented in section 5. In section 6 we consider an affine "reduction" of the formulae derived.

2 The projective Lelieuvre map in asymptotic coordinates.

Let P^n and P_n be projective spaces dual to each other with homogeneous coordinates $f = (f_1, f_2, f_3, \ldots, f_{n+1})$ and $\nu = (\nu_1, \nu_2, \nu_3, \ldots, \nu_{n+1})$ respectively. The pairing of P_n and P^n is defined in a standard way:

$$\langle f, \nu \rangle = 0 \quad . \tag{2.1}$$

We denote $\varepsilon_{i_1 \ldots i_{n+1}}$ an alternating tensor in P^n: $\varepsilon_{12\ldots n+1} = 1$. We will denote the wedge product of m vectors as $a_1 \wedge a_2 \wedge \ldots \wedge a_m$. In a fixed basis one has:

$$\begin{aligned}
&(a_1 \wedge a_2 \wedge \ldots \wedge a_n)_i \stackrel{def}{=} [a_1, \ldots, a_n]_i = \varepsilon_{i i_2 \ldots i_{n+1}} a_{1_{i_2}} \ldots a_{n_{i_{n+1}}} \ , \\
&a_1 \wedge a_2 \wedge \ldots \wedge a_{n+1} = det|a_1, a_2, \ldots, a_{n+1}| = \varepsilon_{i_1 i_2 \ldots i_{n+1}} a_{1_{i_1}} a_{i_{i_2}} \ldots a_{i_{i_{n+1}}} \quad .
\end{aligned} \tag{2.2}$$

Note that

$$\langle b, [a_1, \ldots, a_n] \rangle = det|b, a_1, a_2, \ldots, a_n| \quad . \tag{2.3}$$

The Hodge star operation $\star$ on skewsymmetric tensor $T_{i_1 \ldots i_\alpha}$ is defined as usual

$$(\star T)_{i_{k+1} \ldots i_{n+1}} = \frac{1}{k!} \varepsilon_{i_1 i_2 \ldots i_{n+1}} T_{i_1 i_2 \ldots i_k} \tag{2.4}$$

where summation over repeated indices is assumed (here and below). One has $(\star T) = (-1)^{k(n-k)} T$.

In this section we consider the three-dimensional case ($n = 3$). Let $f = f(x, y)$, $\nu = \nu(x, y)$ where x and y are real-valued independent variables.

Definition 2.1 *The projective Lelieuvre map PLM $L : \nu \to f$ is defined by the equations*

$$f \wedge f_x = \star (\nu \wedge \nu_x) \quad , \quad f \wedge f_y = - \star (\nu \wedge \nu_y) \quad . \tag{2.5}$$

The relations (2.5) are manifestly invariant under projective transformations in P^3 and P_3.

Proposition 2.1 *The inverse PLM $L^{-1} : f \to \nu$ is given by equation*

$$\nu \wedge \nu_x = \star (f \wedge f_x) \quad , \quad \nu \wedge \nu_x = - \star (f \wedge f_y) \quad . \tag{2.6}$$

The formula (2.6) is an obvious consequence of (2.4) with $n = 4$, $k = 2$. The formulae (2.5), (2.6) for the PLM apparently manifest the projective duality. It results in the duality $f \leftrightarrow \nu$ of all formulae derived from (2.5) and (2.6).

Lemma 2.1 *For the PLM (2.5) the relations hold*

$$\langle f_x, \nu \rangle = \langle f_x, \nu_x \rangle = \langle f_{xx}, \nu \rangle = \langle f, \nu_{xx} \rangle = \langle f_{xx}, \nu_{xx} \rangle = 0 \ , \tag{2.7}$$
$$\langle f_y, \nu \rangle = \langle f_y, \nu_y \rangle = \langle f_{yy}, \nu \rangle = \langle f, \nu_{yy} \rangle = \langle f_{yy}, \nu_{yy} \rangle = 0 \ . \tag{2.8}$$

To prove (2.7) and (2.8) we present (2.5) and their differential consequences (2.9)

$$f \wedge f_{xx} = \star (\nu \wedge \nu_{xx}) \quad , \quad f \wedge f_{yy} = - \star (\nu \wedge \nu_{yy}) \tag{2.9}$$

in a component form:

$$f_i f_{kx} - f_k f_{ix} = \varepsilon_{iklm} \nu_l \nu_{mx} \ , \tag{2.10}$$
$$f_i f_{ky} - f_k f_{iy} = -\varepsilon_{iklm} \nu_l \nu_{my} \qquad (i, k = 1, \dots, 4) \tag{2.11}$$

and

$$f_i f_{kxx} - f_k f_{ixx} = \varepsilon_{iklm} \nu_l \nu_{mxx} \ , \tag{2.12}$$
$$f_i f_{kyy} - f_k f_{iyy} = -\varepsilon_{iklm} \nu_l \nu_{myy} \qquad (i, k = 1, \dots, 4) \ . \tag{2.13}$$

The pairing (2.1) is obviously compatible with (2.5) and (2.12), (2.13). Equations (2.10), (2.11) imply

$$\begin{array}{l} f \langle f_x, \nu \rangle - \langle f, \nu \rangle f_x = 0 \ , \\ f \langle f_y, \nu \rangle - \langle f, \nu \rangle f_y = 0 \ , \\ f \langle f_x, \nu_x \rangle - \langle f, \nu_x \rangle f_x = 0 \ , \\ f \langle f_y, \nu_y \rangle - \langle f, \nu_y \rangle f_y = 0 \end{array} \tag{2.14}$$

while (2.12), (2.13) give

$$\begin{array}{l} f \langle f_{xx}, \nu_{xx} \rangle - f_{xx} \langle f, \nu_{xx} \rangle = 0 \ , \\ f \langle f_{yy}, \nu_{yy} \rangle - f_{yy} \langle f, \nu_{yy} \rangle = 0 \ . \end{array} \tag{2.15}$$

For generic f the relations (2.14), (2.15) are equivalent to the relations (2.7) and (2.8). Further from (2.10), (2.11) one gets

$$f\langle f_x,\nu_y\rangle = f\langle f_y,\nu_x\rangle = [\nu,\nu_x,\nu_y] \quad . \tag{2.16}$$

Pairing of both sides of (2.16) with ν_{xy} and use of (2.3) give

$$\langle f_x,\nu_y\rangle\langle f,\nu_{xy}\rangle = -det|\nu,\nu_x,\nu_y,\nu_{xy}| \quad . \tag{2.17}$$

Since $\langle f,\nu_{xy}\rangle = -\langle f_x,\nu_y\rangle$ one gets

$$\langle f_x,\nu_y\rangle^2 = det|\nu,\nu_x,\nu_y,\nu_{xy}| \quad . \tag{2.18}$$

Thus, using (2.16) and (2.18), we prove the

Theorem 2.1 *For the PLM (2.5)* $L:\nu\to f$ *one has*

$$f = \frac{[\nu,\nu_x,\nu_y]}{\sqrt{det|\nu,\nu_x,\nu_y,\nu_{xy}|}} \quad . \tag{2.19}$$

For the inverse PLM $f\to\nu$ *one has*

$$\nu = \frac{[f,f_x,f_y]}{\sqrt{det|f,f_x,f_y,f_{xy}|}} \quad . \tag{2.20}$$

Using now (2.12), one gets

$$\langle f_{xx},\nu_x\rangle f = -\,[\nu,\nu_x,\nu_{xx}] \quad . \tag{2.21}$$

The equality (2.12) implies that

$$\langle f_{xx},\nu_x\rangle\langle f,\nu_{xxx}\rangle = det|\nu,\nu_x,\nu_{xx},\nu_{xxx}| \quad . \tag{2.22}$$

Since $\langle f,\nu_{xxx}\rangle = \langle f_{xx},\nu_x$ one obtains

$$\langle f_{xx},\nu_x\rangle^2 = det|\nu,\nu_x,\nu_{xx},\nu_{xxx}| \quad . \tag{2.23}$$

So

$$f = -\frac{[\nu,\nu_x,\nu_{xx}]}{\sqrt{det|\nu,\nu_x,\nu_{xx},\nu_{xxx}|}} \quad . \tag{2.24}$$

Analogously from (2.13), one gets

$$\langle f_{yy}, \nu_y \rangle^2 = -det|\nu, \nu_y, \nu_{yy}, \nu_{yyy}| \tag{2.25}$$

and

$$f = -\frac{[\nu, \nu_y, \nu_{yy}]}{\sqrt{det|\nu, \nu_y, \nu_{yy}, \nu_{yyy}|}} \ . \tag{2.26}$$

Using the inverse PLM (2.6) (which coincides with the direct one) , one gets

$$\begin{aligned} \langle f_y, \nu_x \rangle^2 &= det|\nu, \nu_x, \nu_y, \nu_{xy}| \ , \\ \langle f_x, \nu_{xx} \rangle^2 &= det|\nu, \nu_x, \nu_{xx}, \nu_{xxx}| \ , \\ \langle f_y, \nu_{yy} \rangle^2 &= -det|\nu, \nu_y, \nu_{yy}, \nu_{yyy}| \ . \end{aligned} \tag{2.27}$$

Comparing (2.18), (2.23), (2.25) with (2.27) and taking into account that $\langle f_y, \nu_x \rangle = \langle f_x, \nu_y \rangle$, $\langle f_{xx}, \nu_x \rangle = -\langle f_x, \nu_{xx} \rangle$, $\langle f_{yy}, \nu_y \rangle = -\langle f_y, \nu_{yy} \rangle$ one gets

Theorem 2.2 *The full determinants are invariant under the PLM (2.5):*

$$\begin{aligned} &det|f, f_x, f_y, f_{xy}| = det|\nu, \nu_x, \nu_y, \nu_{xy}| \ , \\ &det|f, f_x, f_{xx}, f_{xxx}| = det|\nu, \nu_x, \nu_{xx}, \nu_{xxx}| \ , \\ &det|f, f_y, f_{yy}, f_{yyy}| = det|\nu, \nu_y, \nu_{yy}, \nu_{yyy}| \ . \end{aligned} \tag{2.28}$$

The formulae (2.27) and (2.25) provide us the following expressions for projective Fubini forms (which are invariant under unimodular projective transformations):

$$\begin{aligned} F_2 &= 2\langle f_x, \nu_y \rangle dx\, dy = 2\sqrt{det|f, f_x, f_y, f_{xy}|}dxdy = 2\sqrt{det|\nu, \nu_x, \nu_y, \nu_{xy}|}dxdy \ , \\ F_3 &= \langle f_x, \nu_{xx} \rangle dx^3 = \sqrt{det|f, f_x, f_{xx}, f_{xxx}|}dx^3 = \sqrt{det|\nu, \nu_x, \nu_{xx}, \nu_{xxx}|}dx^3 \ , \\ \tilde{F}_3 &= \langle f_y, \nu_y \rangle dy^3 = \sqrt{-det|f, f_y, f_{yy}, f_{yyy}|}dy^3 = \sqrt{-det|\nu, \nu_y, \nu_{yy}, \nu_{yyy}|}dy^3 \ . \end{aligned} \tag{2.29}$$

Further comparing (2.19), (2.24), (2.26) and their dual analogs, one arrives at the following

Theorem 2.3 *The compatibility conditions for the PLM (2.5) are the following*

$$\begin{aligned} \nu_{xx} &= U_1\nu_x + V_1\nu_y + W_1\nu \ , \\ \nu_{yy} &= U_2\nu_x + V_2\nu_y + W_2\nu \end{aligned} \tag{2.30}$$

and

$$\begin{aligned} f_{xx} &= U_1 f_x - V_1 f_y + \tilde{W}_1 f \ , \\ f_{yy} &= -U_2 f_x + V_2 f_y + \tilde{W}_2 f \end{aligned} \tag{2.31}$$

where

$$V_1^2 = \frac{det|\nu, \nu_x, \nu_{xx}, \nu_{xxx}|}{det|\nu, \nu_x, \nu_y, \nu_{xy}|} \quad , \quad U_2^2 = -\frac{det|\nu, \nu_y, \nu_{yy}, \nu_{yyy}|}{det|\nu, \nu_x, \nu_y, \nu_{xy}|} \tag{2.32}$$

and U_1, W_1, V_2, W_2, $\tilde{W}_1$, $\tilde{W}_2$ *are some functions.*

Note that neither ν nor f obey an equation of the form

$$f_{xy} = C f_x + D f_y + E f \quad .$$

Note also that V_1 and U_2 are projective invariants and have the same form in terms of f.

Equations (2.30) and (2.31) are known one. They define surfaces in the three- dimensional projective spaces dual to each other (see *e.g.* [7]). The relations of the form (2.19),(2.20) and (2.28) derived in a different situation also can be found in [7].

3 The PLM for elliptic surfaces

Similar to the standard affine Lelieuvre formula for elliptic surfaces (see *e.g.* [4]) there is an elliptic version of the PLM (2.5).

Definition 3.1 *An elliptic version of the PLM is given by the relation*

$$f \wedge df = \star (\nu \wedge \star d\nu) \tag{3.1}$$

where $f \subset P^3$,

$$\langle f, \nu \rangle = 0 \tag{3.2}$$

and $\star d\nu$ *is a dual* $1-$*form.*

In local coordinates (x, y), $f = f(x, y)$, $\nu = \nu(x, y)$ and (3.1) is

$$f \wedge f_x = - \star (\nu \wedge \nu_y) \quad , \quad f \wedge f_y = \star (\nu \wedge \nu_x) \quad . \tag{3.3}$$

The inverse PLM $L^{-1} : f \to \nu$ is given by $\nu \wedge \star d\nu = \star(f \wedge df)$. The differential consequences of (3.3) are of the form

$$\begin{array}{l} f \wedge f_{xx} = -\star(\nu_x \wedge \nu_y) - \star(\nu \wedge \nu_{xy}) \quad , \\ f \wedge f_{yy} = \star(\nu_y \wedge \nu_x) + \star(\nu \wedge \nu_{xy}) \end{array} \tag{3.4}$$

and

$$\begin{array}{l} f_y \wedge f_x + f \wedge f_{xy} = -\star(\nu \wedge \nu_{yy}) \ , \\ f_x \wedge f_y + f \wedge f_{xy} = \star(\nu \wedge \nu_{xx}) \quad . \end{array} \tag{3.5}$$

Using (3.2), (3.3), one gets

Lemma 3.1 *For the PLM (3.1) one has*

$$\begin{array}{l} \langle f, \nu_x \rangle = \langle f_x, \nu \rangle = \langle f, \nu_y \rangle = \langle f_y, \nu \rangle = 0 \ , \\ \langle f_x, \nu_y \rangle = \langle f_y, \nu_x \rangle = 0 \ , \\ \langle f_{xy}, \nu \rangle = \langle f, \nu_{xy} \rangle = 0 \end{array} \tag{3.6}$$

and

$$\langle f_x, \nu_x \rangle = \langle f_y, \nu_y \rangle \quad . \tag{3.7}$$

Equations (3.3) imply that

$$f \langle f_x, \nu_x \rangle = [\nu, \nu_x, \nu_y] \tag{3.8}$$

and

$$f \langle f_y, \nu_y \rangle = [\nu, \nu_x, \nu_y] \tag{3.9}$$

From (3.8) one gets

$$\langle f, \nu_{xx} \rangle \langle f_x, \nu_x \rangle = -det|\nu, \nu_x, \nu_{xy}, \nu_{xx}| \quad . \tag{3.10}$$

Since $\langle f, \nu_{xx} \rangle = -\langle f_x, \nu_x \rangle$, one obtains

$$\langle f_x, \nu_x \rangle^2 = det|\nu, \nu_x, \nu_y, \nu_{xx}| \quad . \tag{3.11}$$

Analogously (3.9) gives

$$\langle f_y, \nu_y \rangle^2 = det|\nu, \nu_x, \nu_y, \nu_{yy}| \quad . \tag{3.12}$$

Thus as a consequence of (3.8), (3.9), (3.11), (3.12) one has

Theorem 3.1 *For the PLM map (3.1)*

$$f = \frac{[\nu, \nu_x, \nu_y]}{\sqrt{det|\nu, \nu_x, \nu_y, \nu_{xx}|}} = \frac{[\nu, \nu_x, \nu_y]}{\sqrt{det|\nu, \nu_x, \nu_y, \nu_{yy}|}} \quad . \tag{3.13}$$

In a similar manner one can show that for the inverse PLM

$$\nu = \frac{[f, f_x, f_y]}{\sqrt{det|f, f_x, f_y, f_{xx}|}} = \frac{[f, f_x, f_y]}{\sqrt{det|f, f_x, f_y, f_{yy}|}} \tag{3.14}$$

and

$$\begin{aligned} \langle f_x, \nu_x \rangle^2 &= -det|f, f_x, f_y, f_{xx}| \ , \\ \langle f_y, \nu_y \rangle^2 &= -det|f, f_x, f_y, f_{yy}| \quad . \end{aligned} \tag{3.15}$$

Comparison of (3.11), (3.12) with (3.15) leads to

Theorem 3.2 *Full determinants change signs under the PLM (3.1):*

$$\begin{aligned} det|f, f_x, f_y, f_{xx}| &= -det|\nu, \nu_x, \nu_y, \nu_{xx}| \ , \\ det|f, f_x, f_y, f_{yy}| &= -det|\nu, \nu_x, \nu_y, \nu_{yy}| \quad . \end{aligned} \tag{3.16}$$

Further from (3.4) one gets

$$f \langle f_{xx}, \nu_x \rangle = [\nu, \nu_x, \nu_{xy}] \quad . \tag{3.17}$$

Since $\langle f, \nu_y \rangle = 0$ (3.17) implies

$$det|\nu, \nu_x, \nu_y, \nu_{xy}| = 0 \quad . \tag{3.18}$$

Then equations (3.16) and (3.11), (3.12) ($\langle f_x, \nu_x \rangle = \langle f_y, \nu_y \rangle = 0$) imply

$$det|\nu, \nu_x, \nu_y, \nu_{yy}| = -det|\nu, \nu_x, \nu_y, \nu_{xx}| \quad . \tag{3.19}$$

Using (3.16), (3.17) and relations

$$\begin{aligned} f \wedge (f_{yy} - f_{xx}) &= 2 \star (\nu \wedge \nu_{xy}) \ , \\ 2f \wedge f_{xy} &= - \star (\nu \wedge [\nu_{yy} - \nu_{xx}]) \end{aligned} \tag{3.20}$$

one gets

Theorem 3.3 *The compatibility conditions for the PLM (3.1) are of the form*

$$\nu_{xy} = U\nu_x + V\nu_y + W\nu \ , \tag{3.21}$$
$$\nu_{yy} - \nu_{xx} = -2\tilde{V}\nu_x + 2\tilde{U}\nu_y + C\nu \tag{3.22}$$

and

$$f_{xy} = \tilde{U}f_x + \tilde{V}f_y + \tilde{W}f \ , \tag{3.23}$$
$$f_{yy} - f_{xx} = -2Vf_x + 2Uf_y + \tilde{C}f \tag{3.24}$$

where U, V, W, C, $\tilde{U}$, $\tilde{V}$, $\tilde{W}$, $\tilde{C}$ are some functions.

Equations (3.23), (3.24) characterize a surface in P^3 parameterized by conjugate lines (see [7]).

Thus, the PLM (3.1) is the map between surfaces in dual spaces P_3 and P^3 parameterized by conjugate lines.

The PLM's (2.5) and (3.1) and the corresponding formulae can be written in a unique common form. For this purpose we introduce the variables ξ and η defined as $\xi = x$, $\eta = y$ in the case (2.5) and as $\xi = x+iy = z$, $\eta = x-iy = \overline{z}$ in the case (3.1). Then the formulae (2.5) and (3.1) take the form

$$f \wedge f_\xi = \sigma \star (\nu \wedge \nu_\xi) \quad , \quad f \wedge f_\eta = -\sigma \star (\nu \wedge \nu_\eta) \tag{3.25}$$

where $\sigma = 1$ in the real case and $\sigma = -\sqrt{-1}$, $\xi = z$, $\eta = \overline{z}$ for the case considered in this section.

Note that there are other compatibility conditions for the formulae (3.25) different from those given by (2.29), (2.30) or (3.21)-(3.24). Indeed, written in coordinates formulae (3.25) are equivalent to the following

$$\left(\frac{f_k}{f_i}\right)_\xi = \sigma f_i^{-2}\varepsilon_{iklm}\nu_l\nu_{m\xi} \quad , \quad \left(\frac{f_k}{f_i}\right)_\eta = -\sigma f_i^{-2}\varepsilon_{iklm}\nu_l\nu_{m\eta} \quad . \tag{3.26}$$

An obvious compatibility condition for (3.26) is equivalent to the system

$$\nu_{k\xi\eta} = (\log f_i)_\eta \, \nu_{k\xi} + (\log f_i)_\xi \, \nu_{k\eta} + u_i\nu_k \tag{3.27}$$

$(i \neq k,\ i,k = 1,\ldots,4)$ where u_i are some function.

From the inverse formulae (3.25) one gets

$$f_{k\xi\eta} = (\log \nu_i)_\eta f_{k\xi} + (\log \nu_i)_\xi f_{k\eta} + \tilde{u}_i f_k \quad (i \neq k) \tag{3.28}$$

where $\tilde{u}_i$ are some function. However, equations (3.27) and (3.28) are not form-invariant under projective transformations. So in contrast to equations (2.29), (2.30) or (3.21)-(3.24), they do not characterize projective properties of surfaces.

All the results derived until this section for the real projective space RP^3 are apparently extendable to the space CP^3. In this case $f \subset CP^3$, $\nu \subset CP_3$ and ξ, η are complex valued variables. An intermediate case $f \subset CP^3$, $\nu \subset CP_3$ and $\xi = z$, $\eta = \overline{z}$, $z \in C$, $\sigma = -\sqrt{-1}$ which provides the PLM for surfaces with positive-defined metric in CP^3 could be of particular interest to the theory of Riemann surfaces.

The PLM discussed above was formulated in asymptotic coordinates or in conjugate line coordinates. The PLM for surfaces in RP^3 can be formulated in general coordinates on the surfaces. We will get these formulae in the next section as a particular case of the PLM for hypersurfaces.

4 The projective Lelieuvre map for hypersurfaces.

Let $f = (f_1, \ldots, f_{n+2})$ and $\nu = (\nu_1, \ldots, \nu_{n+2})$ be homogeneous coordinates in dual projective spaces P^{n+1} and P_{n+1} paired by (2.1). Consider hypersurfaces $M : f(x_1, \ldots, x_n) \subset P^{n+1}$ and $M^* : \nu(x_1, \ldots, x_n) \subset P_{n+1}$ where $x_1, \ldots, x_n$ are any local coordinates on surfaces.

Definition 4.1 *The PLM $L : \nu \to f$ for hypersurfaces in P^{n+1} is defined by the system of equation*

$$f \wedge f_{x_\alpha} = \sum_{\beta=1}^{n} A_{\alpha\beta} \star \left(\nu_{x_1} \wedge \ldots \wedge \nu_{x_{\beta-1}} \wedge \nu \wedge \nu_{x_{\beta+1}} \wedge \ldots \wedge \nu_{x_n} \right) \quad , \quad \alpha = 1, \ldots, n \tag{4.1}$$

where $A_{\alpha\beta}$ $(\alpha,\ \beta = 1, \ldots, n)$ are functions on $x_1, \ldots, x_n$.

The inverse PLM $L^{-1} : f \to \nu$ is given by

$$\nu_{x_1} \wedge \ldots \wedge \nu_{x_{\beta-1}} \wedge \nu \wedge \nu_{x_{\beta+1}} \wedge \ldots \wedge \nu_{x_n} = \star \sum_{\gamma=1}^{n} \left(A^{-1}\right)_{\beta\gamma} f \wedge f_{x_\gamma} \quad , \quad \beta = 1, \ldots, n \tag{4.2}$$

where A^{-1} is the matrix inverse to the matrix A ($\det A \neq 0$).

In local coordinates in P_{n+1} the formulae (3.1) look like

$$\begin{gathered} f_i f_{kx_\alpha} - f_k f_{ix_\alpha} = \textstyle\sum_{\beta=1}^{n} A_{\alpha\beta} \varepsilon_{ikl_1 \ldots l_n} \nu_{l_1 x_1} \ldots \nu_{l_{\beta-1} x_{\beta-1}} \nu_{l_\beta} \nu_{l_{\beta+1} x_{\beta+1}} \ldots \nu_{l_n x_n} \ , \\ (i, k = 1, \ldots, n+2 \quad , \quad \alpha = 1, \ldots, n) \ . \end{gathered} \tag{4.3}$$

Lemma 4.1 *For the PLM (4.1) one has*

$$\langle f_{x_\alpha}, \nu \rangle = \langle f, \nu_{x_\alpha} \rangle = 0 \quad , \quad \alpha = 1, \ldots, n \ . \tag{4.4}$$

Equations (4.3) imply that

$$\langle f, \nu \rangle f_{x_\alpha} - \langle f_{x_\alpha}, \nu \rangle f = 0 \quad , \quad \alpha = 1, \ldots, n \ . \tag{4.5}$$

Due to (2.1), one gets (4.4).

Further (4.1) implies

$$\langle f_{x_\alpha}, \nu_{x_\gamma} \rangle f = \sum_{\beta=1}^{n} A_{\alpha\beta} \left[\nu_{x_\gamma}, \nu_{x_1}, \ldots, \nu_{x_{\beta-1}}, \nu, \nu_{x_{\beta+1}}, \ldots, \nu_{x_n}\right] \quad , \quad \alpha, \gamma = 1, \ldots, n \ . \tag{4.6}$$

Since

$$\left[\nu_{x_\gamma}, \nu_{x_1}, \ldots, \nu_{x_{\beta-1}}, \nu, \nu_{x_{\beta+1}}, \ldots, \nu_{x_n}\right] = -\delta_{\gamma\beta} \left[\nu, \nu_{x_1}, \ldots, \nu_{x_n}\right] \tag{4.7}$$

where $\delta_{\alpha\beta}$ is the Kroneker symbol, one has

$$\langle f_{x_\alpha}, \nu_{x_\gamma} \rangle f = -A_{\alpha\gamma} \left[\nu, \nu_{x_1}, \ldots, \nu_{x_n}\right] \quad , \quad \alpha, \gamma = 1, \ldots, n \ . \tag{4.8}$$

It follows from (4.8) that

$$\langle f_{x_\alpha}, \nu_{x_\gamma} \rangle \langle f_{x_\beta}, \nu_{x_\delta} \rangle = -A_{\alpha\gamma} det|\nu_{x_\beta x_\delta}, \nu, \nu_{x_1}, \ldots, \nu_{x_n}| \quad , \quad \alpha, \beta, \delta, \gamma = 1, \ldots, n \ . \tag{4.9}$$

Exchanging indices $(\alpha,\beta)\leftrightarrow(\beta,\alpha)$ in (4.9), one also gets

$$\langle f_{x_\beta},\nu_{x_\delta}\rangle\langle f_{x_\alpha},\nu_{x_\gamma}\rangle = -A_{\beta\delta} det|\nu_{x_\alpha x_\gamma},\nu,\nu_{x_1},\dots,\nu_{x_n}| \;\; , \;\; \alpha,\beta,\delta,\gamma = 1,\dots,n \; . \tag{4.10}$$

From (4.9) or (4.10) it follows at $\alpha=\beta$, $\gamma=\delta$ that

$$\langle f_{x_\alpha},\nu_{x_\gamma}\rangle^2 = A_{\alpha\gamma} det|\nu_{x_\alpha x_\gamma},\nu,\nu_{x_1},\dots,\nu_{x_n}| \; . \tag{4.11}$$

Taking into account (4.8) and (4.11), one gets the following

Theorem 4.1 *For the PLM (4.1) for hypersurfaces one has*

$$f = -\left(\frac{A_{\alpha\gamma}}{det|\nu_{x_\alpha x_\gamma},\nu,\nu_{x_1},\dots,\nu_{x_n}|}\right)^{\frac{1}{2}}[\nu,\nu_{x_1},\dots,\nu_{x_n}] \; . \tag{4.12}$$

Further comparing (4.9) and (4.10), one obtains the equation

$$det|A_{\alpha\gamma}\nu_{x_\beta x_\delta} - A_{\beta\delta}\nu_{x_\alpha x_\gamma},\nu,\nu_{x_1},\dots,\nu_{x_n}| = 0 \;\; , \;\; \alpha,\beta,\delta,\gamma = 1,\dots,n \; . \tag{4.13}$$

This equation implies the following

Theorem 4.2 *The compatibility conditions for the PLM (4.1) are given by the system of equations*

$$A_{\alpha\gamma}\nu_{x_\beta x_\delta} - A_{\beta\delta}\nu_{x_\alpha x_\gamma} + \sum_{\rho=1}^{n} U^{(\rho)}_{\alpha,\beta,\gamma,\delta}\nu_{x_\rho} + W_{\alpha,\beta,\gamma,\delta}\nu = 0 \;\; , \;\; \alpha,\beta,\gamma,\delta = 1,\dots,n \tag{4.14}$$

$U^{(\rho)}_{\alpha,\beta,\gamma,\delta}$ *and* $W_{\alpha\beta\gamma\delta}$ *are some functions.*

These functions vanish when simultaneously $\alpha=\beta$ and $\gamma=\delta$ and for those $\alpha,\beta,\gamma,\delta$ for which both $A_{\alpha\gamma}=0$ and $A_{\beta\delta}=0$.

Corollary 4.1 *In virtue of the compatibility condition (4.14) the factor in the formula (4.12) is independent on choice of indices* α, β.

In the particular case $n=2$ the formula (4.1), (4.12) (4.14) give the PLM for surface in P_3 (P^3) in general coordinate system. At $A_{11}=A_{22}=0$, $A_{12}=A_{21}=-2$ one reproduces the results of section 3 while at the case $A_{12}=A_{21}=0$, $A_{11}=A_{22}=-2$ one gets the formulae of section 3.

5 Projective Lelieuvre map for discrete surfaces

Discrete surfaces (maps $\boldsymbol{Z}_2 \to R^N$) are the subject of intensive study now (see *e.g.* [8]-[9]). A discrete analog of the Lelieuvre formula for discrete affine spheres has been found recently in [10].

Here we present the projective Lelieuvre formulae for discrete surfaces in P^3. So let $f : \boldsymbol{Z}_2 \to P^3$ and $\nu : \boldsymbol{Z}_2 \to P_3$ with the pairing (2.1). Thus $f = f(n_1, n_2)$ and $\nu = \nu(n_1, n_2)$ where n_1, n_2 are integers. We denote the shift operators as T_1 and T_2: $T_1 f(n_1, n_2) = f(n_1 + 1, n_2)$, $T_2 f(n_1, n_2) = f(n_1, n_2 + 1)$. For compactness we will denote $f_1 = T_1 f$, $f_{11} = T_1^2 f$, $f_2 = T_2 f$, $f_{22} = T_2^2 f$, $f_{-1} = T_1^{-1} f$ etc. and will omit arguments of f and ν.

Definition 5.1 *Discrete PLM $L : \nu \to f$ is given by relations*

$$f \wedge f_1 = \star(\nu \wedge \nu_1) \quad , \quad f \wedge f_2 = -\star(\nu \wedge \nu_2) \quad . \tag{5.1}$$

The inverse map $f \to \nu$ is of the same form

$$\nu \wedge \nu_1 = \star(f \wedge f_1) \quad , \quad \nu \wedge \nu_2 = -\star(f \wedge f_2) \quad . \tag{5.2}$$

In coordinates (5.1) looks like

$$\begin{array}{l} f_i f_{1k} - f_k f_{1i} = \frac{1}{2}\varepsilon_{iklm}\left(\nu_l \nu_{1m} - \nu_m \nu_{1l}\right) , \\ f_i f_{2k} - f_k f_{2i} = -\frac{1}{2}\varepsilon_{iklm}\left(\nu_l \nu_{2m} - \nu_m \nu_{2l}\right) \end{array} \quad i, k = 1, \ldots, 4 \quad . \tag{5.3}$$

So the PLM (5.1) is, in fact, the identification: *1)* of the polar Plücker coordinates of discrete surface in P_3 in direction $T_1^n \nu$ with the corresponding Plücker coordinates in P^3 and *2)* of the anti-polar Plücker coordinates in P_3 in direction $T_2^n \nu$ with the corresponding Plücker coordinates in P^3.

Lemma 5.1 *For discrete PLM (5.1) one has*

$$\langle f_\alpha, \nu \rangle = \langle f, \nu_\alpha \rangle = 0 \quad , \quad \alpha = 1, 2 \quad . \tag{5.4}$$

From (5.3) one gets

$$\begin{array}{l} f\langle f_\alpha, \nu \rangle - \langle f, \nu \rangle f_\alpha = 0 \, , \\ \langle f_\alpha, \nu_\alpha \rangle - \langle f, \nu_\alpha \rangle f_\alpha = 0 \quad , \quad \alpha = 1, 2 \quad . \end{array} \tag{5.5}$$

Since $\langle f,\nu\rangle = \langle f_\alpha,\nu_\alpha\rangle = 0$ one gets (5.4).

The relations (5.3) and their shifted versions give

$$f\langle f_1,\nu_2\rangle = [\nu,\nu_1,\nu_2] \quad , \quad f\langle f_2,\nu_1\rangle = [\nu,\nu_1,\nu_2] \ , \tag{5.6}$$

$$f_1\langle f_{11},\nu\rangle = [\nu,\nu_1,\nu_{11}] \quad , \quad f_1\langle f_{12},\nu\rangle = [\nu,\nu_1,\nu_{12}] \ , \tag{5.7}$$

$$f_2\langle f_{22},\nu\rangle = -\,[\nu,\nu_2,\nu_{22}] \quad , \quad f_2\langle f_{12},\nu\rangle = [\nu,\nu_2,\nu_{12}] \ , \tag{5.8}$$

and

$$f_{12}\langle f_2,\nu_1\rangle = -\,[\nu,\nu_2,\nu_{12}] \quad . \tag{5.9}$$

From (5.6)-(5.9) it follows

Lemma 5.2 *For the discrete PLM (5.1) one has*

$$\langle f_1,\nu_2\rangle = \langle f_2,\nu_1\rangle \ , \tag{5.10}$$

$$\langle f,\nu_{12}\rangle = \langle f_{12},\nu\rangle \tag{5.11}$$

and

$$\begin{aligned} \langle f_{12},\nu\rangle\langle f_1,\nu_2\rangle &= -det|\nu,\nu_1,\nu_2,\nu_{12}| \ , \\ \langle f_{11},\nu\rangle\langle f_1,\nu_2\rangle &= det|\nu,\nu_1,\nu_2,\nu_{11}| \ , \\ \langle f_{22},\nu\rangle\langle f_1,\nu_2\rangle &= det|\nu,\nu_1,\nu_2,\nu_{22}| \ . \end{aligned} \tag{5.12}$$

Further equations (5.6)-(5.8) give rise to the following

Theorem 5.1 *The compatibility conditions for the PLM (5.1) have the form*

$$\begin{aligned} \nu_{11} &= A_1\nu_{12} + B_1\nu_1 + C_1\nu \ , \\ \nu_{22} &= A_2\nu_{12} + B_2\nu_2 + C_2\nu \end{aligned} \tag{5.13}$$

where

$$A_1 = -\frac{\langle f_{11},\nu\rangle}{\langle f_{12},\nu\rangle} \ , \ C_1 = \frac{\langle f_{11},\nu_{12}\rangle}{\langle f_{12},\nu\rangle} \quad , \ A_2 = -\frac{\langle f_{22},\nu\rangle}{\langle f_{12},\nu\rangle} \quad , \ C_2 = -\frac{\langle f_{22},\nu_{12}\rangle}{\langle f_{12},\nu\rangle} \quad . \tag{5.14}$$

and correspondingly

$$\begin{aligned} f_{11} &= -A_1 f_{12} + \tilde{B}_1 f_1 + C_1 f \ , \\ f_{22} &= -A_2 f_{12} + \tilde{B}_2 f_2 + C_2 f \end{aligned} \tag{5.15}$$

where B_1, B_2, $\tilde{B}_1$, $\tilde{B}_2$ are some functions.

For the inverse PLM one gets the formulae (5.10)-(5.12) with the substitution $f \leftrightarrow \nu$, in particular,

$$\nu\langle f_2, \nu_1\rangle = [f, f_1, f_2] \tag{5.16}$$

and

$$\langle f, \nu_{12}\rangle\langle f_2, \nu_1\rangle = -det|f, f_1, f_2, f_{12}| \ . \tag{5.17}$$

Comparing (5.12) and (5.17), one gets

Theorem 5.2 *For the PLM (5.1) $\nu \to f$ one has*

$$det|f, f_1, f_2, f_{12}| = det|\nu, \nu_1, \nu_2, \nu_{12}| \ . \tag{5.18}$$

So the volume of simplex with with vertices at the origin and the points ν, ν_1, ν_2, ν_{12} is preserved by the PLM (5.1).

The formulae of this section are apparently reduced to those of section 2 in the continuous limit $T_\alpha \nu = \nu + \frac{\partial \nu}{\partial x^\alpha} dx^\alpha$, $\alpha = 1, 2$, $dx^\alpha \to 0$. In particular

$$\begin{array}{l} det|f, f_1, f_2, f_{12}| \to det|f, f_x, f_y, f_{xy}|\,(dx\,dy)^2 \ , \\ det|f, f_1, f_{11}, f_{111}| \to det|f, f_x, f_{xx}, f_{xxx}|\,(dx)^6 \ , \\ det|f, f_2, f_{22}, f_{222}| \to det|f, f_y, f_{yy}, f_{yyy}|\,(dy)^6 \end{array} \tag{5.19}$$

and similar for determinants with ν.

Comparing (5.19) with (2.29) and using (5.18), one gets

Proposition 5.1 *The quantities*

$$\begin{array}{l} F_2^d = 2\sqrt{\langle f_{12}, \nu\rangle\langle f_1, \nu_2\rangle} = \sqrt{det|f, f_1, f_2, f_{12}|} = \sqrt{det|\nu, \nu_1, \nu_2, \nu_{12}|} \ , \\ F_3^d = \sqrt{-\langle f_1 \nu_{111}\rangle\langle f_{11}, \nu\rangle} = \sqrt{det|f, f_1, f_{11}, f_{111}|} = \sqrt{det|\nu, \nu_1, \nu_{11}, \nu_{111}|} \ , \\ \tilde{F}_3^d = \sqrt{\langle f_2, \nu_{222}\rangle\langle f_{22}, \nu\rangle} = \sqrt{det|f, f_2, f_{22}, f_{222}|} = \sqrt{det|\nu, \nu_2, \nu_{22}, \nu_{222}|} \end{array} \tag{5.20}$$

are the discrete analogs of the Fubini's form (2.29).

6 Affine and dual affine gauges

Let us consider an affine "reduction" of the formulae derived. To get to affine geometry relations one should pass to inhomogeneous coordinates (say $\left(\frac{f_1}{f_4}, \frac{f_2}{f_4}, \frac{f_3}{f_4}\right)$) or choose the "gauge" $f_4 = -1$. There is also a possibility to do the same in the dual space P_3.

Let us consider first the gauge $f_4 = -1$. In this case $\nu_4 = \langle \boldsymbol{f}, \boldsymbol{\nu} \rangle$, where we denote $\boldsymbol{f} = (f_1, f_2, f_3)$, $\boldsymbol{\nu} = (\nu_1, \nu_2, \nu_3)$. The formulae (2.5) are reduced obviously to (1.1). Equations (3.27) with $i = 4$ are affine form-invariant. So in the gauge $f_4 = -1$ the affine conormal $\boldsymbol{\nu}$ in addition to equations (2.29) obeys also the equation

$$\boldsymbol{\nu}_{xy} = U_4 \boldsymbol{\nu} \quad . \tag{6.1}$$

Taking into account this equation, one can show that

$$det|\nu, \nu_x, \nu_y, \nu_{xy}| = \langle \boldsymbol{f}_x, \boldsymbol{\nu}_y \rangle det|\boldsymbol{\nu}, \boldsymbol{\nu}_x, \boldsymbol{\nu}_y| \quad . \tag{6.2}$$

Using this relation, one obtains from (2.18) that

$$\langle \boldsymbol{f}_x, \boldsymbol{\nu}_y \rangle = det|\boldsymbol{\nu}, \boldsymbol{\nu}_x, \boldsymbol{\nu}_y| \tag{6.3}$$

and hence

$$det|\nu, \nu_x, \nu_y, \nu_{xy}| = \left(det|\boldsymbol{\nu}, \boldsymbol{\nu}_x, \boldsymbol{\nu}_y|\right)^2 \quad . \tag{6.4}$$

Then the fourth component of the relation (2.19) is satisfied identically while for $\boldsymbol{f}$ one gets a standard expansion of $\boldsymbol{f}$ in normal and tangent components. The relation (2.20) is reduced to the known expression of $\boldsymbol{\nu}$ in terms of $\boldsymbol{f}$. The formulae (2.22) and (2.25) give rise to

$$\langle \boldsymbol{f}_{xx}, \boldsymbol{\nu}_x \rangle = det|\boldsymbol{\nu}, \boldsymbol{\nu}_x, \boldsymbol{\nu}_{xx}| \quad , \quad \langle \boldsymbol{f}_{yy}, \boldsymbol{\nu}_y \rangle = -det|\boldsymbol{\nu}, \boldsymbol{\nu}_y, \boldsymbol{\nu}_{yy}| \quad . \tag{6.5}$$

Finally the formulae (2.28) become

$$\begin{aligned} det|\boldsymbol{f}_x, \boldsymbol{f}_y, \boldsymbol{f}_{xy}| &= \left(det|\boldsymbol{\nu}, \boldsymbol{\nu}_x, \boldsymbol{\nu}_{xy}|\right)^2 = F^2 \ , \\ det|\boldsymbol{f}_x, \boldsymbol{f}_{xx}, \boldsymbol{f}_{xxx}| &= \left(det|\boldsymbol{\nu}, \boldsymbol{\nu}_x, \boldsymbol{\nu}_{xx}|\right)^2 = A^2 \ , \\ det|\boldsymbol{f}_y, \boldsymbol{f}_{yy}, \boldsymbol{f}_{yyy}| &= -\left(det|\boldsymbol{\nu}, \boldsymbol{\nu}_y, \boldsymbol{\nu}_{yy}|\right)^2 = -B^2 \quad . \end{aligned} \tag{6.6}$$

As a result, the Fubini's forms (2.29) are reduced to

$$F_2 \to F\ dx\, dy \quad , \quad F_3 \to A\ dx^3 \quad , \quad \tilde{F}_3 \to B\ dy^3 \tag{6.7}$$

i.e. to the well-known affine Blaschke and cubic Fubini's forms in terms of coordinates $\boldsymbol{f}$ and affine conormal $\boldsymbol{\nu}$ (see *e.g.* [2]-[3]).

So, in the gauge $f_4 = -1$ one reproduces the relations for the affine Lelieuvre map for surfaces with indefinite metric.

Now let $\nu_4 = 1$. So $f_4 = -\langle \boldsymbol{f}, \boldsymbol{\nu} \rangle$. The inverse map (2.5) $f \to \nu$ is of the form (1.1) while the direct map $\nu \to f$ is given by

$$\left(\frac{\boldsymbol{f}}{\langle \boldsymbol{f}, \boldsymbol{\nu} \rangle}\right)_x = \frac{[\boldsymbol{\nu}, \boldsymbol{\nu}_x]}{(\langle \boldsymbol{f}, \boldsymbol{\nu} \rangle)^2} \ , \quad \left(\frac{\boldsymbol{f}}{\langle \boldsymbol{f}, \boldsymbol{\nu} \rangle}\right)_y = -\frac{|\boldsymbol{\nu}, \boldsymbol{\nu}_y]}{(\langle \boldsymbol{f} \boldsymbol{\nu} \rangle)^2} \ . \tag{6.8}$$

The compatibility condition for (6.8) are given by (2.29), (2.30) plus equations

$$\begin{aligned} \boldsymbol{\nu}_{xy} &= (\log a)_x \boldsymbol{\nu}_y + (\log a)_y \boldsymbol{\nu}_x + u \boldsymbol{\nu} \ , \\ \boldsymbol{f}_{xy} &= \tilde{u} \boldsymbol{f} \end{aligned} \tag{6.9}$$

where $a = -\langle \boldsymbol{f}, \boldsymbol{\nu} \rangle$ and u, $\tilde{u}$ are some functions. The vector $\boldsymbol{\nu}$ is not the standard affine conormal but $\frac{\boldsymbol{\nu}}{\langle \boldsymbol{f}, \boldsymbol{\nu} \rangle}$ is. Further the formula (2.19) gives

$$\boldsymbol{f} = \frac{[\boldsymbol{\nu}_x, \boldsymbol{\nu}_y]}{\sqrt{det|\boldsymbol{\nu}_y, \boldsymbol{\nu}_x, \boldsymbol{\nu}_{xy}|}} \tag{6.10}$$

and the relations (2.28) are reduced to those of (6.6) with substitution $f \leftrightarrow \nu$, in particular, to

$$\left(det|\boldsymbol{f}, \boldsymbol{f}_x, \boldsymbol{f}_y|\right)^2 = -det|\boldsymbol{\nu}_x, \boldsymbol{\nu}_y, \boldsymbol{\nu}_{xy}| \tag{6.11}$$

Thus in the dual affine gauge $\nu_4 = 1$ the PLM generates a class of surfaces in P_3.

The choice of the gauge $f_4 = -1$ imposes no constraints on surfaces. The choice of particular gauge only destroys the projective covariance of formulae and symmetry between dual spaces P_3 and P^3.

If one now demands that $f_4 = -1$ and $\nu_4 = 1$ simultaneously then one constraints surfaces by the condition

$$\langle \boldsymbol{f}, \boldsymbol{\nu} \rangle = 1 \tag{6.12}$$

Such a class consists of affine spheres. For affine spheres all formulae are symmetric under substitution $\boldsymbol{f} \leftrightarrow \boldsymbol{\nu}$ (see also [10]). So the affine spheres form the particular class of surfaces for which the general projective duality via the PLM (2.5) is restored as the duality on the affine level.

Similar results are valid for affine "reduction" of the PLM (3.1). For the PLM (4.1) for hypersurfaces the choice of the gauge $f_{n+2} = -1$ reduces (4.1) to the formula

$$\boldsymbol{f}_{x\alpha} = -\sum_{\beta=1}^{n} A_{\alpha\beta}\left[\boldsymbol{\nu}_{x_1},\ldots,\boldsymbol{\nu}_{x_{\beta-1}},\boldsymbol{\nu},\boldsymbol{\nu}_{x_{\beta+1}},\ldots,\boldsymbol{\nu}_{x_n}\right] \ . \tag{6.13}$$

derived in [5] ($\boldsymbol{\nu}=(\nu_1,\ldots,\nu_{n+1})$). From the formula (4.8) for the $n+2-$th component one gets

$$A_{\alpha\gamma} = -\frac{\langle \boldsymbol{f}_{x\alpha},\boldsymbol{\nu}_{x\gamma}\rangle}{det|\boldsymbol{\nu},\boldsymbol{\nu}_{x_1},\ldots,\boldsymbol{\nu}_{x_n}|} \tag{6.14}$$

that coincides with that of [5]. The compatibility conditions for the map (6.13) are given by the equations (4.14) for $\boldsymbol{\nu}$.

At last, let us consider the affine gauges for the discrete PLM (5.1). At the gauge $f_4=-1$ the formulae (5.1) take the form

$$\boldsymbol{f}_1-\boldsymbol{f} = [\boldsymbol{\nu},\boldsymbol{\nu}_1] \quad , \boldsymbol{f}_2-\boldsymbol{f} = -\left[\boldsymbol{\nu},\boldsymbol{\nu}_2\right] \quad . \tag{6.15}$$

The conditions (5.4) and (5.10) become ($\nu_4=\langle \boldsymbol{f},\boldsymbol{\nu}\rangle$)

$$\langle(\boldsymbol{f}-\boldsymbol{f}_\alpha),\boldsymbol{\nu}_\alpha\rangle = 0 \ \ , \ \ \langle(\boldsymbol{f}_\alpha-\boldsymbol{f}),(\boldsymbol{\nu}_\alpha-\boldsymbol{\nu})\rangle = 0 \ \ , \ \ \alpha=1,2 \tag{6.16}$$

and

$$\langle(\boldsymbol{f}_1-\boldsymbol{f}_2),(\boldsymbol{\nu}_1+\boldsymbol{\nu}_2)\rangle = 0 \ . \tag{6.17}$$

which are obviously satisfied due to (6.15).

The compatibility conditions for (6.15) are given by equations (5.13) for $\boldsymbol{\nu}$ and also by the equation

$$\boldsymbol{\nu}_{12}+\boldsymbol{\nu} = H\left(\boldsymbol{\nu}_1+\boldsymbol{\nu}_2\right) \tag{6.18}$$

where H is some function. Note that equation (6.18) is not an affine reduction of some projectively covariant compatibility conditions. At the gauge $f_4 = -1$, using (5.4) and (5.6)-(5.8), one obtains

$$\begin{array}{rcl}\langle f_2,\nu_1\rangle &=& \langle(\boldsymbol{f}_1-\boldsymbol{f}),(\boldsymbol{\nu}_1-\boldsymbol{\nu})\rangle = det|\boldsymbol{\nu},\boldsymbol{\nu}_1,\boldsymbol{\nu}_2| \ , \\ \langle f_{11},\nu\rangle &=& -\langle(\boldsymbol{f}_{11}-\boldsymbol{f}),(\boldsymbol{\nu}_1-\boldsymbol{\nu})\rangle = det|\boldsymbol{\nu},\boldsymbol{\nu}_1,\boldsymbol{\nu}_{11}| \ , \\ \langle f_{22},\nu\rangle &=& -\langle(\boldsymbol{f}_{22}-\boldsymbol{f}),(\boldsymbol{\nu}_2-\boldsymbol{\nu})\rangle = -det|\boldsymbol{\nu},\boldsymbol{\nu}_2,\boldsymbol{\nu}_{22}| \ , \end{array} \tag{6.19}$$

and

$$\langle f_{22},\nu\rangle = \langle(\boldsymbol{f}_{12}-\boldsymbol{f}),\boldsymbol{\nu}\rangle\langle(\boldsymbol{f}_{12}-\boldsymbol{f}),\boldsymbol{\nu}_{12}\rangle = -det|\boldsymbol{\nu},\boldsymbol{\nu}_2,\boldsymbol{\nu}_{12}| \ . \tag{6.20}$$

Further, using the properties of determinants and (6.16), (6.20) at $f_4 = -1$ one gets

$$det|f, f_1, f_2, f_{12}| = det|\boldsymbol{f}_1 - \boldsymbol{f}, \boldsymbol{f}_2 - \boldsymbol{f}, \boldsymbol{f}_{12} - \boldsymbol{f}| \tag{6.21}$$

and

$$det|\nu, \nu_1, \nu_2, \nu_{12}| = \langle(\boldsymbol{f}_1 - \boldsymbol{f}), \boldsymbol{\nu}_{12}\rangle det|\boldsymbol{\nu}, \boldsymbol{\nu}_1, \boldsymbol{\nu}_2| = det|\boldsymbol{\nu}, \boldsymbol{\nu}_1, \boldsymbol{\nu}_{12}| det|\boldsymbol{\nu}, \boldsymbol{\nu}_1, \boldsymbol{\nu}_2| \ . \tag{6.22}$$

In virtue of (5.18) and (6.21), (6.22) we have

Theorem 6.1 *For the discrete affine Lelieuvre map (6.15)*

$$det|\boldsymbol{f}_1 - \boldsymbol{f}, \boldsymbol{f}_2 - \boldsymbol{f}, \boldsymbol{f}_{12} - \boldsymbol{f}| = det|\boldsymbol{\nu}, \boldsymbol{\nu}_1, \boldsymbol{\nu}_{12}| det|\boldsymbol{\nu}, \boldsymbol{\nu}_1, \boldsymbol{\nu}_2| \ . \tag{6.23}$$

In continuous limit $\boldsymbol{\nu}_1 = \boldsymbol{\nu} + \boldsymbol{\nu}_x\, dx$, $\boldsymbol{\nu}_2 = \boldsymbol{\nu} + \boldsymbol{\nu}_y\, dy$ the formulae (6.15) convert into the classical Lelieuvre formula, the relation (6.23) is reduced to the first equation (2.28) and

$$\begin{aligned} det|\boldsymbol{\nu}, \boldsymbol{\nu}_1, \boldsymbol{\nu}_2| &\to det|\boldsymbol{\nu}, \boldsymbol{\nu}_x, \boldsymbol{\nu}_y|\, dx\, dy \ , \\ det|\boldsymbol{\nu}, \boldsymbol{\nu}_1, \boldsymbol{\nu}_{11}| &\to det|\boldsymbol{\nu}, \boldsymbol{\nu}_x, \boldsymbol{\nu}_{xx}|\, dx^3 \ , \\ det|\boldsymbol{\nu}, \boldsymbol{\nu}_2, \boldsymbol{\nu}_{22}| &\to det|\boldsymbol{\nu}, \boldsymbol{\nu}_y, \boldsymbol{\nu}_{yy}|\, dy^3 \ . \end{aligned} \tag{6.24}$$

So we have

Proposition 6.1 *The* l.h.s. *of (6.24) and (6.19) or better*

$$\begin{aligned} \Omega_2 &= \langle(\boldsymbol{f}_2 - \boldsymbol{f}), (\boldsymbol{\nu}_1 - \boldsymbol{\nu})\rangle = det|\boldsymbol{\nu}, \boldsymbol{\nu}_1, \boldsymbol{\nu}_2| \ , \\ \Omega_3 &= \langle(\boldsymbol{f}_1 - \boldsymbol{f}_{-1}), (\boldsymbol{\nu} - \boldsymbol{\nu}_{-1})\rangle = -det|\boldsymbol{\nu}_{-1}, \boldsymbol{\nu}, \boldsymbol{\nu}_2| \ , \\ \tilde{\Omega}_3 &= \langle(\boldsymbol{f}_2 - \boldsymbol{f}_{-2}), (\boldsymbol{\nu} - \boldsymbol{\nu}_{-2})\rangle = -det|\boldsymbol{\nu}_{-2}, \boldsymbol{\nu}, \boldsymbol{\nu}_2| \ . \end{aligned} \tag{6.25}$$

are the discrete analogs of the Blaschke and Fubini cubic forms of affine surfaces.

Finally, we consider the dual affine gauge $\nu_4 = 1$. In this case the inverse map (5.1) has a simple form

$$\boldsymbol{\nu}_1 - \boldsymbol{\nu} = [\boldsymbol{f}, \boldsymbol{f}_1] \quad , \quad \boldsymbol{\nu}_2 - \boldsymbol{\nu} = -[\boldsymbol{f}, \boldsymbol{f}_2] \tag{6.26}$$

and instead of formulae (6.16)-(6.25) one has those with the substitutions $\boldsymbol{\nu} \leftrightarrow \boldsymbol{f}$.

A class of discrete surfaces for which both $f_4 = -1$ and $\nu_4 = 1$, *i.e* $\langle \boldsymbol{f}, \boldsymbol{\nu} \rangle = 1$ is given by the discrete affine spheres. They have been studied recently in [10] where the formulae (6.15)-(6.18), (6.26) have been derived in this particular case.

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