Axioms for Weak Bialgebras

FLORIAN NILL
Inst. theor. Physik, FU-Berlin, Arnimallee 14, D-14195 Berlin, Germany

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Abstract

Let $(\mathcal{A}, \mathbf{1})$ be a finite dimensional unital associative algebra over a field K, which is also equipped with a coassociative counital coalgebra structure (Δ, ε). $\mathcal{A}$ is called a *weak bialgebra* if the coproduct $\Delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ satisfies $\Delta(ab) = \Delta(a)\Delta(b)$. We do **not** require $\Delta(\mathbf{1}) = \mathbf{1} \otimes \mathbf{1}$ **nor** multiplicativity of the counit $\varepsilon : \mathcal{A} \to K$. Instead, we propose a new set of counit axioms, which are modelled so as to guarantee that $\mathrm{Rep}\,\mathcal{A}$ becomes a monoidal category with unit object given by the cyclic left $\mathcal{A}$-module $\mathcal{E} := (\mathcal{A} \rightharpoonup \hat{\mathbf{1}}) \subset \hat{\mathcal{A}}$, where $\hat{\mathbf{1}} \equiv \varepsilon$ is the unit in the dual weak bialgebra $\hat{\mathcal{A}}$. Under these *monoidality axioms* $\mathcal{E}$ and $\bar{\mathcal{E}} := (\hat{\mathbf{1}} \leftharpoonup \mathcal{A})$ become commuting unital subalgebras of $\hat{\mathcal{A}}$, which are trivial if and only if ε is multiplicative. We also propose axioms for an antipode $S : \mathcal{A} \to \mathcal{A}$, such that the category $\mathrm{Rep}\,\mathcal{A}$ becomes rigid. S is uniquely determined, provided it exists. If a monoidal weak bialgebra $\mathcal{A}$ has an antipode S, then its dual $\hat{\mathcal{A}}$ is monoidal if and only if S is a bialgebra anti-homomorphism, in which case S is also invertible. In this way we obtain a definition of weak Hopf algebras which in Appendix A will be shown to be equivalent to the one given independently by G. Böhm and K. Szlachányi. Special examples are given by the face algebras of T. Hayashi and the generalised Kac algebras of T. Yamanouchi.

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* Present address. e-mail: nill@physik.fu-berlin.de florian.nill@physik.uni-muenchen.de
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1 Introduction

In [MS,S] G. Mack and V. Schomerus have introduced the notion of *weak* coproducts $\Delta : \mathcal{G} \to \mathcal{G} \otimes \mathcal{G}$ on quasi-Hopf algebras $\mathcal{G}$ by allowing Δ to be non-unital, $\Delta(\mathbf{1}) \neq \mathbf{1} \otimes \mathbf{1}$. Examples are semisimple quotients of quantum groups at $q =$ roots of unity. The underlying motivation was to obtain symmetry candidates $\mathcal{G}$ in low dimensional quantum field theories. Technically this may be understood as a Tannaka-Krein like reconstruction program [Ma,Hä], starting from the rigid monoidal category of Doplicher-Haag-Roberts (DHR) endomorphisms on a local observable algebra $\mathcal{M}$. In this way one may successfully match the quantum field theoretic fusion rules with *non-integer* ("statistical" or "categorical" or "q-") dimensions with those of $\mathrm{Rep}\,\mathcal{G}$.

The price to pay in this setting is the quasi-coassociativity of the coproduct Δ. Thus, the dual $\hat{\mathcal{G}}$ of $\mathcal{G}$ is not an object of the same kind. In particular $\hat{\mathcal{G}}$ is not even an associative algebra, which makes it impossible to define an analogue of the DHR-field algebra $\mathcal{F} = \mathcal{M} \rtimes \hat{\mathcal{G}}$.[1] On the other hand, in Ocneanu's approach of recovering "quantum symmetries" from (depth 2) Jones inclusions [Oc,Da,Lo,Szy,EN,Ya,NW] one always expects a concept of symmetry algebras $\mathcal{A}$, such that the dual $\hat{\mathcal{A}}$ is of the same type (due to the two-step periodicity in any Jones tower).

In this work I propose a new axiomatic approach to weak (Hopf) bialgebras $(\mathcal{A}, \mathbf{1}, \Delta, \varepsilon)$, which strictly meets this duality principle. In particular, I start from the observation that dualizing the property $\Delta(\mathbf{1}) \neq \mathbf{1} \otimes \mathbf{1}$ suggests to allow non-multiplicative counits as well, i.e. $\varepsilon(ab) \neq \varepsilon(a)\varepsilon(b)$. On the other hand, I don't give up coassociativity of the coproduct Δ, such that the dual $(\hat{\mathcal{A}}, \hat{\mathbf{1}}, \hat{\Delta}, \hat{\varepsilon})$ is of the same kind. For simplicity - and to make this duality strategy manifest - throughout I will restrict myself to finite dimensional algebras $\mathcal{A}$ over a field K. A generalization to infinite dimensional settings together with appropriate topological (like C^*- or von-Neumann algebraic) structures should be a future goal.

A first announcement of the present work has been given in 1994 [N2]. Subsequent discussions with H.-W. Wiesbrock and K.-H. Rehren have soon lead to first applications in Jones theory and quantum field theory [W,Re]. In 1996 G. Böhm and K. Szlachányi [BSz,Sz] independently came up with very similar ideas. The main progress of the present paper in comparison with the BSz-approach is that here I propose so-called *(co)monoidality axioms*, the necessity and consequences of which are discussed individually and without referring to antipode structures. Also, the antipode axioms presented here are simpler than those of Böhm-Szlachányi and are motivated by a more general analysis of *rigidity structures* on weak bialgebras. In this way I will end up with a set of axioms for weak Hopf algebras, which will be shown in Appendix A to be equivalent to those of [BSz,Sz]. Also, the face algebras of T. Hayashi [Ha] and the generalized Kac algebras of T. Yamanouchi [Ya] are special kinds of weak Hopf algebras, see Sect. 5 and Sect. 8, respectively.

Meanwhile G. Böhm, K. Szlachányi and I have exchanged and unified our ideas. Parallel to this work we present further common results on the theory of integrals and C^*-structures on finite dimensional weak Hopf algebras in [BNS]. Moreover, in [NSW] we develop a theory of (co)actions and crossed products by weak Hopf algebras and generalize Ocneanu's ideas by showing that any reducible finite index and depth-2 Jones extension of von-Neumann algebras with finite dimensional centers is given by a crossed product with a weak Hopf algebra $\mathcal{A}$.

In future work [HN2] we will also clarify the role of our coassociative weak Hopf algebras as a symmetry in the quasi-coassociative quantum field theoretic scenario of [MS,S]. Another exciting

[1] There is, however, a sensible definition of the double crossed product $\mathcal{M} \rtimes \hat{\mathcal{G}} \rtimes \mathcal{G}$ [HN1].

application will emerge from the fact [N3] that the observable algebras of a large class of physical quantum chain models naturally acquire a weak Hopf algebra structure. In particular, the Hopf spin models or lattice current algebras of [NSz,AFFS] on an open chain (of even number of sites) are self-dual weak Hopf algebras $\mathcal{A}$ and their periodic extensions by one link joining the endpoints are given by the (weak) quantum double $\mathcal{D}(\mathcal{A})$. For a first sketch see also the remarks following Example 3 in Appendix D. It will be challenging to identify the vacuum representation of these models with the GNS-representation obtained from the counit (i.e. the monoidal unit in $\mathrm{Rep}\,\mathcal{D}(\mathcal{A})$) and relate their DHR-theory with the braided rigid monoidal structure of $\mathrm{Rep}\,\mathcal{D}(\mathcal{A})$. As a further interesting conjecture one may suggest similar applications [2] in conformal quantum field theory, such that the quantum field theoretic fusion rules are reproduced by a weak Hopf algebra structure on the obsevables.

The plan of this paper is as follows. In Section 2 we analyse the so-called *monoidality axioms* for the counit $\varepsilon \equiv \hat{\mathbf{1}} \in \hat{\mathcal{A}}$ on a weak bialgebra $\mathcal{A}$, making $\mathrm{Rep}\,\mathcal{A}$ a monoidal category with nontrivial cyclic unit object $\mathcal{A} \rightharpoonup \hat{\mathbf{1}} \subset \hat{\mathcal{A}}$. The dual analogues are the *comonoidality axioms* studied in Section 3. In particular, we will see that the canonical left (right) action of $\hat{\mathcal{A}}$ on $\mathbf{1}_{\mathcal{A}}$ induces nontrivial subspaces $\mathcal{A}_L := \mathbf{1}_{\mathcal{A}} \leftharpoonup \hat{\mathcal{A}} \subset \mathcal{A}$ and $\mathcal{A}_R := \hat{\mathcal{A}} \rightharpoonup \mathbf{1}_{\mathcal{A}} \subset \mathcal{A}$, which in a comonoidal weak bialgebra are in fact commuting unital subalgebras of $\mathcal{A}$. Considering $\mathcal{A}$ as a left (right) comodule algebra over itself we also show that $\mathcal{A}$ is comonoidal if and only if the left (right) coinvariants of $\mathcal{A}$ are given by $\mathcal{A}_{L/R}$, respectively. In Section 4 we describe the category $\mathrm{Cmod}\,\mathcal{A}$ of right $\mathcal{A}$-comodules and show that if $\mathcal{A}$ is comonoidal then $\mathcal{A}_R$ is the unit object in $\mathrm{Cmod}\,\mathcal{A}$. Moreover, the endomorphism algebra of this comodule is shown to be given by $\mathrm{End}^{\mathcal{A}}\,\mathcal{A}_R = \mathcal{A}_L \cap \mathcal{A}_R$, acting by multiplication on $\mathcal{A}_R$. For weak Hopf algebras this has been noticed before in [Sz]. In Section 5 we generalize an observation of [Sz] by showing that in bimonoidal weak bialgebras the subalgebras $\mathcal{A}_{L/R}$ are separable K-algebras. In Section 6 we adapt ideas developped for quasi-Hopf algebras by Drinfel'd [Dr] to formulate a theory of *rigidity structures* on monoidal weak bialgebras. This will help to motivate our antipode axioms in Section 7, where we will see that in *bimonoidal* (i.e. monoidal and comonoidal) weak bialgebras an antipode S always provides a rigidity structure. Thus, in Section 8 we define a weak Hopf algebra to be a bimonoidal weak bialgebra with antipode S. One of our main results here will be that a weak bialgebra $\mathcal{A}$ with antipode S is a weak Hopf algebra (i.e. bimonoidal) if and only if $[\mathcal{A}_L, \mathcal{A}_R] = 0$ and S is a bialgebra anti-morphism.

In Appendix A we relate the present approach to the axioms of [BSz,Sz]. Appendix B gives more details on rigidity structues in the spirit of [Dr]. In particular this leads to a proof that on (finite dimensional!) monoidal weak bialgebras $\mathcal{A}$ rigidity maps $S : \mathcal{A} \to \mathcal{A}$ (i.e. "quasi-antipodes") are always invertible [3]. In Appendix C we analyse *minimal* (comonoidal) weak bialgebras $\mathcal{A} = \mathcal{A}_L\mathcal{A}_R$, which are defined to be generated by the commuting subalgebras $\mathcal{A}_L$ and $\mathcal{A}_R$, as well as their *cominimal* dual analogues $\hat{\mathcal{A}}$. In Appendix D we give several examples, most noteworthy a two-sided crossed product construction of a minimal weak bialgebra $\mathcal{A} = \mathcal{A}_L \otimes \mathcal{A}_R$ with a Hopf algebra $\mathcal{G}$. This example puts a weak Hopf algebra structure on the Hopf algebraic quantum chains considered in [NSz,AFFS], such that $\mathcal{A}_L$ and $\mathcal{A}_R$ become the left and right "wedge algebras", respectively, of these models.

Note added: After finishing this paper I have been informed by L. Vainerman that presumably the notion of a *quantum groupoid* [M, V, NV] is equivalent to that of a weak Hopf algebra with involutive antipode.

[2] based on infinite dimensional weak Hopf algebras

[3] Presumably a similar proof also works for finite dimensional quasi-Hopf algebras.

2 Monoidal Weak Bialgebras

Throughout all spaces are assumed finite dimensional over a fixed field K. The dual of a linear space V is denoted as $\hat{V} = \mathrm{Hom}_K(V, K)$ and the center of an algebra $\mathcal{A}$ is denoted by $\mathcal{C}(\mathcal{A})$.

Definition 2.1 A *weak bialgebra* $(\mathcal{A}, \mathbf{1}, \Delta, \varepsilon)$ is an associative unital algebra $(\mathcal{A}, \mathbf{1})$ together with a coassociative coproduct $\Delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ and a counit $\varepsilon : \mathcal{A} \to K$ for Δ, such that $\Delta(ab) = \Delta(a)\Delta(b),\ \forall a, b \in \mathcal{A}$.

As opposed to ordinary bialgebras we do **not** require $\Delta(\mathbf{1}) = \mathbf{1} \otimes \mathbf{1}$ **nor** its dual version $\varepsilon(ab) = \varepsilon(a)\varepsilon(b)$. Clearly, the dual $\hat{\mathcal{A}}$ of $\mathcal{A}$ also is a weak bialgebra $(\hat{\mathcal{A}}, \hat{\mathbf{1}}, \hat{\Delta}, \hat{\varepsilon})$ with structure maps

$$
\begin{aligned}
\langle \phi\psi \mid a \rangle &:= \langle \phi \otimes \psi \mid \Delta(a) \rangle \\
\langle \hat{\mathbf{1}} \mid a \rangle &:= \varepsilon(a) \\
\langle \hat{\Delta}(\phi) \mid a \otimes b \rangle &:= \langle \phi \mid ab \rangle \\
\hat{\varepsilon}(\phi) &:= \langle \phi \mid \mathbf{1} \rangle
\end{aligned}
$$

where $\phi, \psi \in \hat{\mathcal{A}},\ a, b, \in \mathcal{A}$ and where $\langle \cdot \mid \cdot \rangle$ denotes the dual pairing $\hat{\mathcal{A}} \otimes \mathcal{A} \to K$. We denote elements of $\mathcal{A}$ by $a, b, c, \ldots$ and elements of $\hat{\mathcal{A}}$ by $\phi, \psi, \xi, \ldots$. We also use standard Hopf algebra notations like $\Delta(a) = a_{(1)} \otimes a_{(2)},\ (\Delta \otimes id)(\Delta(a)) \equiv (id \otimes \Delta)(\Delta(a)) = a_{(1)} \otimes a_{(2)} \otimes a_{(3)}$, etc., where a summation symbol and summation indices are suppressed. The canonical left and right actions of $\mathcal{A}$ on $\hat{\mathcal{A}}$ are denoted by

$$
\begin{aligned}
a \rightharpoonup \phi &:= \phi_{(1)} \langle \phi_{(2)} \mid a \rangle && (2.1) \\
\phi \leftharpoonup a &:= \langle \phi_{(1)} \mid a \rangle \phi_{(2)} && (2.2)
\end{aligned}
$$

and similarly for $\mathcal{A}$ and $\hat{\mathcal{A}}$ interchanged. Acting in particular on $\mathbf{1} \in \mathcal{A}$ and $\hat{\mathbf{1}} \in \hat{\mathcal{A}}$, respectively, we obtain nontrivial linear subspaces $\mathcal{A}_{L/R} \subset \mathcal{A}$ and $\hat{\mathcal{A}}_{L/R} \subset \hat{\mathcal{A}}$ given by

$$
\begin{array}{lllll}
\mathcal{A}_L := \mathbf{1} \leftharpoonup \hat{\mathcal{A}} \subset \mathcal{A} & , & \mathcal{A}_R := \hat{\mathcal{A}} \rightharpoonup \mathbf{1} \subset \mathcal{A} \\
\hat{\mathcal{A}}_L := \hat{\mathbf{1}} \leftharpoonup \mathcal{A} \subset \hat{\mathcal{A}} & , & \hat{\mathcal{A}}_R := \mathcal{A} \rightharpoonup \hat{\mathbf{1}} \subset \hat{\mathcal{A}}
\end{array}
$$

Let us now consider the category $\mathrm{Rep}\,\mathcal{A}$ of finite dimensional unital representations $\pi_V : \mathcal{A} \to \mathrm{End}\,V$. We also use the module language by writing $\pi_V(a)v \equiv a \cdot v,\ a \in \mathcal{A},\ v \in V$. If $\mathcal{A}$ is a weak bialgebra then $\mathrm{Rep}\,\mathcal{A}$ is equipped with a strictly associative tensor functor $\mathrm{Rep}\,\mathcal{A} \times \mathrm{Rep}\,\mathcal{A} \to \mathrm{Rep}\,\mathcal{A}$ given on the objects by

$$
\begin{aligned}
V \times W &:= \mathbf{1}_{V \times W}(V \otimes W) && (2.3) \\
\pi_{V \times W} &:= (\pi_V \otimes \pi_W) \circ \Delta && (2.4)
\end{aligned}
$$

and on $\mathcal{A}$-linear morphisms by

$$
f \times g := (f \otimes g) \circ \mathbf{1}_{V \times W}\ , \tag{2.5}
$$

where $f \in \mathrm{Hom}_{\mathcal{A}}(V, V'),\ g \in \mathrm{Hom}_{\mathcal{A}}(W, W')$ and $\mathbf{1}_{V \times W} := (\pi_V \otimes \pi_W)(\Delta(\mathbf{1}))$. As a special object in $\mathrm{Rep}\,\mathcal{A}$ we consider the cyclic $\mathcal{A}$-submodule $\mathcal{E} \equiv \hat{\mathcal{A}}_R \subset \hat{\mathcal{A}}$ with left $\mathcal{A}$-action given by Eq. (2.1). Our aim is to specify additional axioms for weak bialgebras $\mathcal{A}$, such that $\mathrm{Rep}\,\mathcal{A}$ becomes a monoidal category with unit object $\mathcal{E}$. To this end the following notions will be useful

Definition 2.2 A weak bialgebra $(\mathcal{A}, \mathbf{1}, \Delta, \varepsilon)$ is called

- *left-monoidal*, if $\varepsilon(abc) = \varepsilon(ab_{(1)})\varepsilon(b_{(2)}c), \ \forall a, b, c \in \mathcal{A}$ (2.6)
- *right-monoidal*, if $\varepsilon(abc) = \varepsilon(ab_{(2)})\varepsilon(b_{(1)}c), \ \forall a, b, c \in \mathcal{A}$ (2.7)
- *left-comonoidal*, if $(\Delta(\mathbf{1}) \otimes \mathbf{1})(\mathbf{1} \otimes \Delta(\mathbf{1})) = \mathbf{1}_{(1)} \otimes \mathbf{1}_{(2)} \otimes \mathbf{1}_{(3)}$ (2.8)
- *right-comonoidal*, if $(\mathbf{1} \otimes \Delta(\mathbf{1}))(\Delta(\mathbf{1}) \otimes \mathbf{1}) = \mathbf{1}_{(1)} \otimes \mathbf{1}_{(2)} \otimes \mathbf{1}_{(3)}$ (2.9)

A weak bialgebra is called (co)monoidal, if it is left- and right-(co)monoidal, and it is called *bimonoidal*, if it is comonoidal and monoidal.

Clearly, $\mathcal{A}$ is (left-, right-) comonoidal if and only if $\hat{\mathcal{A}}$ is (left-, right-) monoidal. We also note the equivalencies: $\mathcal{A}$ left-(co)monoidal $\Leftrightarrow$ $\mathcal{A}_{op}^{cop}$ left-(co)monoidal $\Leftrightarrow$ $\mathcal{A}_{op}$ right-(co)monoidal $\Leftrightarrow$ $\mathcal{A}^{cop}$ right- (co)monoidal, where "op" means opposite multiplication and "cop" means opposite comultiplication. If ε is multiplicative, then $\mathcal{A}$ is always monoidal, and if $\Delta(\mathbf{1}) = \mathbf{1} \otimes \mathbf{1}$, then $\mathcal{A}$ is always comonoidal. The face algebras of T. Hayashi [Ha] provide examples of comonoidal weak bialgebras. In fact, we will see in Corollary 3.9 that finite dimensional face algebras are also bimonoidal.

The terminologies of Definition 2.2 will be motivated below by showing that if $\mathcal{A}$ is monoidal then $\mathrm{Rep}\,\mathcal{A}$ is a monoidal category with unit object given by $\mathcal{E}$. By duality, if $\mathcal{A}$ is comonoidal, then the category $\mathrm{Cmod}\,\mathcal{A}$ of $\mathcal{A}$-comodules becomes a monoidal category, see Sect. 4

Let now $(\mathcal{A}, \mathbf{1}, \Delta, \varepsilon)$ be a weak bialgebra. For any representation (π_V, V) of $\mathcal{A}$ we introduce the K-linear maps $L_V : V \to \mathcal{E} \times V$ and $R_V : V \to V \times \mathcal{E}$ given by

$$L_V(v) \ := \ \mathbf{1}_{\mathcal{E}\times V}(\hat{\mathbf{1}} \otimes v) \tag{2.10}$$

$$R_V(v) \ := \ \mathbf{1}_{V\times\mathcal{E}}(v \otimes \hat{\mathbf{1}}) \ . \tag{2.11}$$

These maps satisfy "naturality" in the sense that

$$L_W \circ f = (\mathbf{1}_{\mathcal{E}} \times f) \circ L_V \quad \text{and} \quad R_W \circ f = (f \times \mathbf{1}_{\mathcal{E}}) \circ R_V$$

for all $f \in \mathrm{Hom}_{\mathcal{A}}(V, W)$, as well as

$$\mathbf{1}_{\mathcal{E}\times V\times W} \circ (L_V \otimes \mathbf{1}_W) \ = \ L_{V\times W} \tag{2.12}$$

$$\mathbf{1}_{W\times V\times\mathcal{E}} \circ (\mathbf{1}_W \otimes R_V) \ = \ R_{W\times V} \ . \tag{2.13}$$

Also, we have a kind of "pre-" triangle identity in the sense that

$$\mathbf{1}_{V\times\mathcal{E}\times W} \circ (\mathbf{1}_V \otimes L_W) = \mathbf{1}_{V\times\mathcal{E}\times W} \circ (R_V \otimes \mathbf{1}_W) \tag{2.14}$$

as maps $V \otimes W \to V \times \mathcal{E} \times W$. Eqs. (2.12)-(2.14) follow easily from

$$\mathbf{1}_{(1)} \otimes \mathbf{1}_{(2)} \otimes \mathbf{1}_{(3)} = \mathbf{1}_{(1)} \otimes \mathbf{1}_{(2)}\mathbf{1}_{(1')} \otimes \mathbf{1}_{(3)}\mathbf{1}_{(2')} = \mathbf{1}_{(1)}\mathbf{1}_{(1')} \otimes \mathbf{1}_{(2)}\mathbf{1}_{(2')} \otimes \mathbf{1}_{(3)} \tag{2.15}$$

Moreover, L_V and R_V are always injective with left inverses given by

$$\bar{L}_V \ : \ \mathcal{E} \otimes V \ni \phi \otimes v \mapsto \langle \phi \mid \mathbf{1}_{(1)} \rangle \mathbf{1}_{(2)} \cdot v \in V \tag{2.16}$$

$$\bar{R}_V \ : \ V \otimes \mathcal{E} \ni v \otimes \phi \mapsto \langle \phi \mid \mathbf{1}_{(2)} \rangle \mathbf{1}_{(1)} \cdot v \in V \tag{2.17}$$

More precisely we have

Lemma 2.3 *Let $(\mathcal{A}, \mathbf{1}, \Delta, \varepsilon)$ be a weak bialgebra. Then for all V in $\mathrm{Rep}\,\mathcal{A}$ we have*

$$\bar{L}_V \mathbf{1}_{\mathcal{E}\times V} = \bar{L}_V \quad , \quad \bar{R}_V \mathbf{1}_{V\times\mathcal{E}} = \bar{R}_V$$
$$\bar{L}_V L_V = \bar{R}_V R_V = \mathbf{1}_V$$

Proof: By definition we have for all $\phi \in \mathcal{E}$ and $v \in V$

$$(\bar{L}_V \mathbf{1}_{\mathcal{E}\times V})(\phi\otimes v) = \langle \mathbf{1}_{(1')} \rightharpoonup \phi \mid \mathbf{1}_{(1)}\rangle \mathbf{1}_{(2)}\mathbf{1}_{(2')}\cdot v = \langle \phi \mid \mathbf{1}_{(1)}\mathbf{1}_{(1')}\rangle \mathbf{1}_{(2)}\mathbf{1}_{(2')}\cdot v = \bar{L}_V(\phi\otimes v).$$

Hence, we also get $\bar{L}_V L_V v = \bar{L}_V(\hat{\mathbf{1}}\otimes v) = v$. The argument for R_V is anologous. ■

We now show that L_V and R_V are $\mathcal{A}$-linear for all V in $\mathrm{Rep}\,\mathcal{A}$ if and only if $\mathcal{A}$ is monoidal, in which case L_V and R_V are also bijective. More precisely, we have the following

Theorem 2.4 *Let $(\mathcal{A}, \mathbf{1}, \Delta, \varepsilon)$ be a weak bialgebra. Then*
i) L_V is $\mathcal{A}$-linear for all V in $\mathrm{Rep}\,\mathcal{A}$ if and only if $\mathcal{A}$ is left-monoidal.
ii) R_V is $\mathcal{A}$-linear for all V in $\mathrm{Rep}\,\mathcal{A}$ if and only if $\mathcal{A}$ is right-monoidal.
iii) If $\mathcal{A}$ is monoidal, then L_V and R_V are bijective and we have the identities

$$R_V \times \mathbf{1}_W = \mathbf{1}_V \times L_W \tag{2.18}$$
$$L_V \times \mathbf{1}_W = L_{V\times W} \quad , \quad \mathbf{1}_W \times R_V = R_{W\times V} \tag{2.19}$$
$$R_{\mathcal{E}} = L_{\mathcal{E}} \tag{2.20}$$

Theorem 2.4 implies that for monoidal weak bialgebras $\mathcal{A}$ the category $\mathrm{Rep}\,\mathcal{A}$ becomes a strictly associative monoidal category with unit object given by the $\mathcal{A}$-module $\mathcal{E}$. Note that for ordinary bialgebras the $\mathcal{A}$-module $\mathcal{E}$ is "trivial", i.e. it coincides with the 1-dimensional representation given by the counit $\varepsilon : \mathcal{A} \to K$. In our setting $\mathcal{E}$ need not even be $\mathcal{A}$-irreducible. If it is, then we call $\mathcal{A}$ *pure*, following [BSz].

To prove Theorem 2.4 we have to introduce some formalism. For $\phi \in \hat{\mathcal{A}}$ we introduce the maps $\phi_{L/R} : \mathcal{A} \to \hat{\mathcal{A}}$ given by

$$\langle \phi_L(a) \mid b\rangle = \langle a \mid \phi_R(b)\rangle := \langle \phi \mid ab\rangle$$

for $a, b, \in \mathcal{A}$. Note the obvious identities

$$\begin{array}{rclcrcl} \phi_R(a) & = & a \rightharpoonup \phi & , & \phi_L(b) & = & \phi \leftharpoonup b \\ \phi_R(ab) & = & a \rightharpoonup \phi_R(b) & , & \phi_L(ab) & = & \phi_L(a) \leftharpoonup b \end{array} \tag{2.21}$$

for all $a, b \in \mathcal{A}$ and $\phi \in \hat{\mathcal{A}}$. In particular, $\varepsilon_{L/R}(\mathcal{A}) = \hat{\mathcal{A}}_{L/R}$ and $\hat{\varepsilon}_{L/R}(\hat{\mathcal{A}}) = \mathcal{A}_{L/R}$. For $\sigma, \sigma' \in \{L, R\}$ we also use the notation

$$\varepsilon_{\sigma\sigma'} := \hat{\varepsilon}_\sigma \circ \varepsilon_{\sigma'} \in \mathrm{End}_K\,\mathcal{A} \quad , \quad \hat{\varepsilon}_{\sigma\sigma'} := \varepsilon_\sigma \circ \hat{\varepsilon}_{\sigma'} \in \mathrm{End}_K\,\hat{\mathcal{A}}$$

where $\hat{\varepsilon} \equiv \mathbf{1} \in \mathcal{A}$ is the counit on $\hat{\mathcal{A}}$. For the reader's convenience we give the explicit formulas

$$\begin{array}{rclcrcl} \varepsilon_{LL}(a) & = & \varepsilon(a\mathbf{1}_{(1)})\mathbf{1}_{(2)} & , & \varepsilon_{RR}(a) & = & \mathbf{1}_{(1)}\varepsilon(\mathbf{1}_{(2)}a) \\ \varepsilon_{LR}(a) & = & \varepsilon(\mathbf{1}_{(1)}a)\mathbf{1}_{(2)} & , & \varepsilon_{RL}(a) & = & \mathbf{1}_{(1)}\varepsilon(a\mathbf{1}_{(2)}) \end{array} \tag{2.22}$$

From these one immediately verifies the following identities

$$\begin{array}{rclcrcl} a_{(2)}\varepsilon_{LL}(ba_{(1)}) & = & a_{(2)}\varepsilon(ba_{(1)}) & , & \varepsilon_{RR}(a_{(2)}b)a_{(1)} & = & \varepsilon(a_{(2)}b)a_{(1)} \\ \varepsilon_{LR}(a_{(1)}b)a_{(2)} & = & \varepsilon(a_{(1)}b)a_{(2)} & , & a_{(1)}\varepsilon_{RL}(ba_{(2)}) & = & a_{(1)}\varepsilon(ba_{(2)}) \end{array} \tag{2.23}$$

for all $a, b \in \mathcal{A}$. Also note, that the maps ϕ_L and ϕ_R are transposes of each other and therefore

$$(\varepsilon_{\sigma\sigma'})^t = \hat{\varepsilon}_{-\sigma',-\sigma} \tag{2.24}$$

where $-L = R$ and $-R = L$, and where the superscript ^{t} denotes the transposed map.

To prepare the proof of Theorem 2.4 we are now going to express the left- and right- monoidality axioms of Definition 2.2 in terms of properties the maps ε_σ, $\varepsilon_{\sigma\sigma'}$ and $\hat{\varepsilon}_{\sigma\sigma'}$. To this end consider the following list of additional axioms for weak bialgebras, divided into two groups called Axioms $\mathcal{L}$ and Axioms $\mathcal{R}$, respectively.

Axioms $\mathcal{L}$	Axioms $\mathcal{R}$	
$\phi_{(1)} \otimes \hat{\varepsilon}_{LL}(\phi_{(2)}) = \phi\hat{\mathbf{1}}_{(1)} \otimes \hat{\mathbf{1}}_{(2)}$	$\phi_{(1)} \otimes \hat{\varepsilon}_{LR}(\phi_{(2)}) = \hat{\mathbf{1}}_{(1)}\phi \otimes \hat{\mathbf{1}}_{(2)}$	(2.25)
$a_{(1)} \otimes \varepsilon_L(a_{(2)}) = a\mathbf{1}_{(1)} \otimes \varepsilon_L(\mathbf{1}_{(2)})$	$\varepsilon_L(a_{(1)}) \otimes a_{(2)} = \varepsilon_L(\mathbf{1}_{(1)}) \otimes a\mathbf{1}_{(2)}$	(2.26)
$a\varepsilon_{RR}(b) = \varepsilon(a_{(2)}b)a_{(1)} \equiv \varepsilon_{RR}(a_{(2)}b)a_{(1)}$	$a\varepsilon_{LR}(b) = \varepsilon(a_{(1)}b)a_{(2)} \equiv \varepsilon_{LR}(a_{(1)}b)a_{(2)}$	(2.27)
$\hat{\varepsilon}_{RR}(\phi_{(1)}) \otimes \phi_{(2)} = \hat{\mathbf{1}}_{(1)} \otimes \hat{\mathbf{1}}_{(2)}\phi$	$\hat{\varepsilon}_{RL}(\phi_{(1)}) \otimes \phi_{(2)} = \hat{\mathbf{1}}_{(1)} \otimes \phi\hat{\mathbf{1}}_{(2)}$	(2.28)
$\varepsilon_R(a_{(1)}) \otimes a_{(2)} = \varepsilon_R(\mathbf{1}_{(1)}) \otimes \mathbf{1}_{(2)}a$	$a_{(1)} \otimes \varepsilon_R(a_{(2)}) = \mathbf{1}_{(1)}a \otimes \varepsilon_R(\mathbf{1}_{(2)})$	(2.29)
$\varepsilon_{LL}(b)a = a_{(2)}\varepsilon(ba_{(1)}) \equiv a_{(2)}\varepsilon_{LL}(ba_{(1)})$	$\varepsilon_{RL}(b)a = a_{(1)}\varepsilon(ba_{(2)}) \equiv a_{(1)}\varepsilon_{RL}(ba_{(2)})$	(2.30)

where Eqs. (2.27), (2.26), (2.30) and (2.29) are supposed to hold for all $a, b, c, \in \mathcal{A}$, respectively, and Eqs. (2.25) and (2.28) for all $\phi \in \hat{\mathcal{A}}$. Note that the second identities in (2.27) and (2.30) follow from (2.23).

Proposition 2.5 *For a weak bialgebra* $(\mathcal{A}, \mathbf{1}, \Delta, \varepsilon)$ *any one of the list of* Axioms $\mathcal{L}$ *(*Axioms $\mathcal{R}$ *) is equivalent to* $\mathcal{A}$ *being left- (right-) monoidal.*

Proof: It is enough to prove the "left"-statements, since the Axioms $\mathcal{R}$ reduce to the Axioms $\mathcal{L}$ in $\mathcal{A}^{cop}$. Also note that the axioms (2.28), (2.30) and (2.29) reduce to the axioms (2.25), (2.27) and (2.26), respectively, in $\mathcal{A}^{cop}_{op}$. Hence, it is enough to prove the equivalences (2.6) $\Leftrightarrow$ (2.25Left) $\Leftrightarrow$ (2.27Left) $\Leftrightarrow$ (2.26Left). To this end first note that (2.6) may be rewritten as

$$\hat{\mathbf{1}}_{(1)} \otimes \hat{\mathbf{1}}_{(2)} \otimes \hat{\mathbf{1}}_{(3)} = \hat{\mathbf{1}}_{(1)} \otimes \hat{\mathbf{1}}_{(2)}\hat{\mathbf{1}}_{(1')} \otimes \hat{\mathbf{1}}_{(2')} \tag{2.31}$$

implying for all $\phi \in \hat{\mathcal{A}}$

$$\begin{aligned} \phi\hat{\mathbf{1}}_{(1)} \otimes \hat{\mathbf{1}}_{(2)} &\equiv \phi_{(1)}\hat{\mathbf{1}}_{(1)} \otimes \hat{\varepsilon}(\phi_{(2)}\hat{\mathbf{1}}_{(2)})\hat{\mathbf{1}}_{(3)} \\ &= \phi_{(1)} \otimes \hat{\varepsilon}(\phi_{(2)}\hat{\mathbf{1}}_{(1)})\hat{\mathbf{1}}_{(2)} \equiv \phi_{(1)} \otimes \hat{\varepsilon}_{LL}(\phi_{(2)}) \end{aligned}$$

and therefore (2.25Left). Converseley, assume (2.25Left) holds, then $\hat{\Delta}(\hat{\mathbf{1}}) = \hat{\mathbf{1}}_{(1)} \otimes \hat{\varepsilon}_{LL}(\hat{\mathbf{1}}_{(2)})$ and therefore $(\hat{\Delta} \otimes id)(\hat{\Delta}(\hat{\mathbf{1}})) = \hat{\mathbf{1}}_{(1)} \otimes \hat{\mathbf{1}}_{(2)} \otimes \hat{\varepsilon}_{LL}(\hat{\mathbf{1}}_{(3)}) = \hat{\mathbf{1}}_{(1')} \otimes \hat{\mathbf{1}}_{(2')}\hat{\mathbf{1}}_{(1)} \otimes \hat{\mathbf{1}}_{(2)}$ where the second equation follows by putting $\phi = \hat{\mathbf{1}}_{(2')}$ in (2.25Left). Hence we have shown (2.6) $\Leftrightarrow$ (2.25Left). The equivalence (2.25Left) $\Leftrightarrow$ (2.27Left) follows by pairing both sides of (2.25Left) with $a \otimes b$ and using $(\hat{\varepsilon}_{LL})^t = \varepsilon_{RR}$. Finally, the equivalence (2.27Left) $\Leftrightarrow$ (2.26Left) follows from $a\varepsilon_{RR}(b) = a\mathbf{1}_{(1)}\langle\varepsilon_L(\mathbf{1}_{(2)}) \mid b\rangle$ and $a_{(1)}\varepsilon(a_{(2)}b) = a_{(1)}\langle\varepsilon_L(a_{(2)}) \mid b\rangle$. ■

For $\sigma, \sigma' \in \{L, R\}$ let us now introduce the subspaces

$$\mathcal{A}_{\sigma\sigma'} := \varepsilon_{\sigma\sigma'}(\mathcal{A}) \subset \mathcal{A}_\sigma \subset \mathcal{A} \tag{2.32}$$

It turns out, that if $\mathcal{A}$ is monoidal, then $\mathcal{A}_{\sigma\sigma'} \subset \mathcal{A}$ is a unital subalgebra. More precisely, we have

Proposition 2.6
1.) If $\mathcal{A}$ is left-monoidal, then for all $a, b \in \mathcal{A}$

$$\begin{array}{rrcllrcl} i) & \Delta(\varepsilon_{LL}(a)) & = & \varepsilon_{LL}(a)\mathbf{1}_{(1)} \otimes \mathbf{1}_{(2)} & , & \Delta(\varepsilon_{RR}(a)) & = & \mathbf{1}_{(1)} \otimes \mathbf{1}_{(2)}\varepsilon_{RR}(a) \\ ii) & \varepsilon_{LL}(b)\varepsilon_{LL}(a) & = & \varepsilon_{LL}(b\varepsilon_{LL}(a)) & , & \varepsilon_{RR}(a)\varepsilon_{RR}(b) & = & \varepsilon_{RR}(\varepsilon_{RR}(a)b) \end{array} \tag{2.33}$$

In particular, for $\sigma \in \{L, R\}$, $(\varepsilon_{\sigma\sigma})^2 = \varepsilon_{\sigma\sigma}$ and $\mathcal{A}_{\sigma\sigma} \subset \mathcal{A}$ is a unital subalgebra.
2.) If $\mathcal{A}$ is right-monoidal, then for all $a, b \in \mathcal{A}$

$$\begin{array}{rrcllrcl} i) & \Delta(\varepsilon_{RL}(a)) & = & \mathbf{1}_{(1)} \otimes \varepsilon_{RL}(a)\mathbf{1}_{(2)} & , & \Delta(\varepsilon_{LR}(a)) & = & \mathbf{1}_{(1)}\varepsilon_{LR}(a) \otimes \mathbf{1}_{(2)} \\ ii) & \varepsilon_{RL}(b)\varepsilon_{RL}(a) & = & \varepsilon_{RL}(b\varepsilon_{RL}(a)) & , & \varepsilon_{LR}(a)\varepsilon_{LR}(b) & = & \varepsilon_{LR}(\varepsilon_{LR}(a)b) \end{array} \tag{2.34}$$

In particular, for $\sigma \neq \sigma' \in \{L, R\}$, $(\varepsilon_{\sigma\sigma'})^2 = \varepsilon_{\sigma\sigma'}$ and $\mathcal{A}_{\sigma\sigma'} \subset \mathcal{A}$ is a unital subalgebra.
3.) If $\mathcal{A}$ is monoidal, then $[\mathcal{A}_{L\sigma}, \mathcal{A}_{R\sigma}] = 0$ for $\sigma = L$ and $\sigma = R$.

Proof: Part 2.) reduces to part 1.) in $\mathcal{A}^{cop}$ and the right identities reduce to the left ones in $\mathcal{A}_{op}^{cop}$. Let now $\mathcal{A}$ be left-monoidal, then using (2.22)

$$\Delta(\varepsilon_{LL}(a)) = \varepsilon(a\mathbf{1}_{(1)})\mathbf{1}_{(2)} \otimes \mathbf{1}_{(3)} = \varepsilon(a\mathbf{1}_{(1')})\mathbf{1}_{(2')}\mathbf{1}_{(1)} \otimes \mathbf{1}_{(2)} = \varepsilon_{LL}(a)\mathbf{1}_{(1)} \otimes \mathbf{1}_{(2)}$$

where in the second identity we have used (2.29left). Applying this to (2.30left) yields

$$\varepsilon_{LL}(b)\varepsilon_{LL}(a) = \varepsilon(b\varepsilon_{LL}(a)\mathbf{1}_{(1)})\mathbf{1}_{(2)} = \varepsilon_{LL}(b\varepsilon_{LL}(a)) \ .$$

The remaining statements of part 1.) follow from $\varepsilon_{\sigma\sigma'}(\mathbf{1}) = \mathbf{1}$. Part 3.) follows, since 1i)+2i) imply for $a \in \mathcal{A}_{LL}$ and $b \in \mathcal{A}_{RL}$

$$\Delta(ab) = (a \otimes b)\Delta(\mathbf{1}) = \Delta(ba) \, .$$

Hence, $ab = ba$ since Δ is injective. The identity $[\mathcal{A}_{LR}, \mathcal{A}_{RR}] = 0$ follows in $\mathcal{A}_{op}$. ■

Next, we study the counit axioms of [BSz,Sz].

Lemma 2.7 *Let $(\mathcal{A}, \mathbf{1}, \Delta, \varepsilon)$ be a weak bialgebra and consider the following BSz–Axioms*

$$\text{L}): \quad \varepsilon(ab) = \varepsilon(a\mathbf{1}_{(1)})\varepsilon(\mathbf{1}_{(2)}b), \quad \forall a, b \in \mathcal{A} \tag{2.35}$$

$$\text{R}): \quad \varepsilon(ab) = \varepsilon(a\mathbf{1}_{(2)})\varepsilon(\mathbf{1}_{(1)}b), \quad \forall a, b \in \mathcal{A} \tag{2.36}$$

Then the following equivalences hold for $\sigma, \sigma' \in \{L, R\}$, $\sigma \neq \sigma'$

$$\text{L}) \quad \Leftrightarrow \varepsilon_L(ab) = \varepsilon_L(\varepsilon_{LL}(a)b), \ \forall a, b \Leftrightarrow \varepsilon_R(ab) = \varepsilon_R(a\varepsilon_{RR}(b)), \ \forall a, b \Leftrightarrow \varepsilon_\sigma \circ \hat{\varepsilon}_\sigma \circ \varepsilon_\sigma = \varepsilon_\sigma$$

$$\text{R}) \quad \Leftrightarrow \varepsilon_L(ab) = \varepsilon_L(\varepsilon_{RL}(a)b), \ \forall a, b \Leftrightarrow \varepsilon_R(ab) = \varepsilon_R(a\varepsilon_{LR}(b)), \ \forall a, b \Leftrightarrow \varepsilon_\sigma \circ \hat{\varepsilon}_{\sigma'} \circ \varepsilon_\sigma = \varepsilon_\sigma$$

Proof: The equivalencies R) reduce to L) in $\mathcal{A}^{cop}$. The equivalencies L) follow from the identities $\varepsilon_L(ab) = \varepsilon_L(a) \leftharpoonup b$, $\varepsilon_R(ab) = a \rightharpoonup \varepsilon_R(b)$ and $\varepsilon(a\mathbf{1}_{(1)})\varepsilon(\mathbf{1}_2 b) = \varepsilon(\varepsilon_{LL}(a)b) = \varepsilon(a\varepsilon_{RR}(b))$. ■

The axioms (L) and (R) of Lemma 2.7 have been proposed as axioms for weak Hopf algebras in [BSz,Sz]. They imply the monoidality properties of Definition 2.2 only under additional antipode axioms, see [BSz,Sz] or Lemma A1 in Appendix A. The property (L) also appears as a counit axiom in Hayashi's face algebra theory [Ha]. The monoidality axioms (2.6) and (2.7) are the ones used in [BNS,NSW] and they obviously always imply the BSz-axioms of Lemma 2.7. As has been observed similarly in [BSz,Sz], Lemma 2.7 also implies the following

Corollary 2.8 *Under the BSz-axioms* (L) *and* (R) *of Lemma 2.7 the following bilinear forms are nondegenerate for all* $\sigma, \sigma' \in \{L, R\}$

$$\mathcal{A}_{\sigma,L} \otimes \mathcal{A}_{\sigma' R} \ni (a \otimes b) \mapsto \varepsilon(ab) \in K \tag{2.37}$$
$$\mathcal{A}_{\sigma,L} \otimes \mathcal{E} \ni (a \otimes \psi) \mapsto \langle \psi \mid a \rangle \in K \tag{2.38}$$
$$\hat{\mathcal{E}} \otimes \mathcal{A}_{\sigma,R} \ni (\phi \otimes b) \mapsto \langle \phi \mid b \rangle \in K \tag{2.39}$$
$$\hat{\mathcal{E}} \otimes \mathcal{E} \ni (\phi \otimes \psi) \mapsto \hat{\varepsilon}(\phi\psi) \in K \tag{2.40}$$

where $\mathcal{E} := \varepsilon_R(\mathcal{A})$ *and* $\hat{\mathcal{E}} := \varepsilon_L(\mathcal{A})$. *Morover,* $\hat{\mathcal{E}}$ *as a right* $\mathcal{A}$*-module is dual to* $\mathcal{E}$*, i.e. for all* $\phi \in \hat{\mathcal{E}}$, $\psi \in \mathcal{E}$ *and* $a \in \mathcal{A}$

$$\hat{\varepsilon}(\phi(a \rightharpoonup \psi)) = \hat{\varepsilon}((\phi \leftharpoonup a)\psi) \tag{2.41}$$

Proof: The nondegeneracy of (2.37) - (2.40) follows immediately from Lemma 2.7. To prove (2.41) write $\phi = \varepsilon_L(b)$ and $\psi = \varepsilon_R(c)$. Then $a \rightharpoonup \psi = \varepsilon_R(ac)$ and $\phi \leftharpoonup a = \varepsilon_L(ba)$ and therefore

$$\hat{\varepsilon}(\phi(a \rightharpoonup \psi)) = \varepsilon(b\varepsilon_{RR}(ac)) = \varepsilon(\varepsilon_{LL}(ba)c) = \hat{\varepsilon}((\phi \leftharpoonup a)\psi) \qquad ■$$

Note that Corollary 2.8 in particular implies

$$\dim \mathcal{A}_{\sigma\sigma'} = \dim \mathcal{E} = \dim \hat{\mathcal{E}}, \quad \forall \sigma, \sigma' \in \{L, R\}. \tag{2.42}$$

After these preparations we are now in the position to give the

Proof of Theorem 2.4:
Throughout we use that being finite dimensional $\mathcal{A}$ as a left $\mathcal{A}$-module is itself an object in $\mathrm{Rep}\,\mathcal{A}$. Hence, L_V is $\mathcal{A}$-linear for all V in $\mathrm{Rep}\,\mathcal{A}$ if and only if

$$(\mathbf{1}_{(1)} \rightharpoonup \hat{\mathbf{1}}) \otimes \mathbf{1}_{(2)} a = (a_{(1)} \rightharpoonup \hat{\mathbf{1}}) \otimes a_{(2)}$$

for all $a \in \mathcal{A}$, which is precisely the condition (2.29Left). Similarly, R_V is $\mathcal{A}$-linear for all V in $\mathrm{Rep}\,\mathcal{A}$ if and only if

$$\mathbf{1}_{(1)} a \otimes (\mathbf{1}_{(2)} \rightharpoonup \hat{\mathbf{1}}) = a_{(1)} \otimes (a_{(2)} \rightharpoonup \hat{\mathbf{1}})$$

for all $a \in \mathcal{A}$, which is precisely the condition (2.29Right). To prove part iii) let now $\mathcal{A}$ be monoidal implying all Eqs. (2.25) - (2.29) as well as those of Proposition 2.6 and Lemma 2.7. Using $\mathcal{E} = \varepsilon_R(\mathcal{A})$ we get for $\phi = \varepsilon_R(b) \in \mathcal{E}$ and $v \in V$

$$\begin{aligned}
L_V \bar{L}_V(\phi \otimes v) &= L_V(\hat{\varepsilon}_L(\phi) \cdot v) = \varepsilon_R(\mathbf{1}_{(1)}) \otimes \mathbf{1}_{(2)}\varepsilon_{LR}(b) \cdot v \\
&= \varepsilon_R(\varepsilon_{LR}(b)_{(1)}) \otimes \varepsilon_{LR}(b)_{(2)} \cdot v \\
&= \varepsilon_R(\mathbf{1}_{(1)}\varepsilon_{LR}(b)) \otimes \mathbf{1}_{(2)} \cdot v \\
&= [\mathbf{1}_{(1)} \rightharpoonup \varepsilon_R(\varepsilon_{LR}(b))] \otimes [\mathbf{1}_{(2)} \cdot v] \\
&= [\mathbf{1}_{(1)} \rightharpoonup \phi] \otimes [\mathbf{1}_{(2)} \cdot v] \equiv \mathbf{1}_{\mathcal{E} \times V}(\phi \otimes v)
\end{aligned}$$

Here we have used Eq. (2.29Left) in the second line, part (2.i) of Proposition 2.6 in the third line, Eq. (2.21) in the fourth line and Lemma 2.7(R in the last line. Repeating this proof in $\mathcal{A}^{cop}$ yields $R_V \bar{R}_V = \mathbf{1}_{V\times\mathcal{E}}$. Finally, $\mathcal{A}$-linearity implies $\mathbf{1}_{V\times\mathcal{E}\times W} \circ (\mathbf{1}_V \otimes L_W) = \mathbf{1}_V \times L_W$ and $\mathbf{1}_{V\times\mathcal{E}\times W} \circ (R_V \otimes \mathbf{1}_W) = R_V \times \mathbf{1}_W$. Hence, the triangle identity (2.18) follows from Eq. (2.14). The remaining Eqs. (2.19) and (2.20) follow by standard arguments for any monoidal category (see e.g. [Ka, Sec. XI.2.2]). ■

In view of Theorem 2.4 we will from now on denote

$$\pi_\varepsilon : \mathcal{A} \to \mathrm{End}_K\, \mathcal{E},\ \ \pi_\varepsilon(a)\phi := a \rightharpoonup \phi$$

as the "trivial" or unit representation of $\mathcal{A}$. The following Corollary states that π_ε may equivalently be realized as a left $\mathcal{A}$-action on $\mathcal{A}_{\sigma,R}$, $\sigma \in \{L, R\}$, such that $\pi_\varepsilon|_{\mathcal{A}_{\sigma,R}}$ becomes the left multiplication on itself.

Corollary 2.9 *Let $\mathcal{A}$ be left (right) monoidal and for $\sigma = R$ ($\sigma = L$) let $\pi_{\sigma,R} : \mathcal{A} \to \mathrm{End}_K\, \mathcal{A}_{\sigma,R}$ be given by $\pi_{\sigma,R}(a)b := \varepsilon_{\sigma,R}(ab)$. Then $\pi_{\sigma,R}$ is a representation of $\mathcal{A}$ satisfying $\pi_{\sigma,R}(a)b = ab$, $\forall a, b \in \mathcal{A}_{\sigma,R}$, and $\varepsilon_R : \mathcal{A}_{\sigma,R} \to \mathcal{E}$ is an $\mathcal{A}$-linear isomorphism with inverse $\hat{\varepsilon}_\sigma : \mathcal{E} \to \mathcal{A}_{\sigma,R}$, i.e. for all $a \in \mathcal{A}$*

$$\begin{aligned} \varepsilon_R \circ \pi_{\sigma,R}(a) &= \pi_\varepsilon(a) \circ \varepsilon_R|_{\mathcal{A}_{\sigma,R}} &\quad (2.43)\\ \hat{\varepsilon}_\sigma \circ \pi_\varepsilon(a) &= \pi_{\sigma,R}(a) \circ \hat{\varepsilon}_\sigma|_{\mathcal{E}} &\quad (2.44)\end{aligned}$$

Proof: This follows immediately from Lemma 2.7. ■

In Section 3 we will see that for monoidal weak bialgebras $\mathcal{A}$ the $\mathcal{A}$-submodule $\mathcal{E} \equiv \varepsilon_R(\mathcal{A})$ is also a subalgebra of $\hat{\mathcal{A}}$ and $\varepsilon_R : \mathcal{A}_{\sigma,R} \to \mathcal{E}$ is also an algebra isomorphism (for $\sigma = L$) or anti-isomorphism (for $\sigma = R$).

Finally, we emphasize that in the present context (i.e. without furher assumptions like e.g. existence of an antipode) monoidality is **not** a selfdual concept for weak bialgebras. The following Lemma provides the conditions under which a monoidal weak bialgebra is also comonoidal (a comonoidal weak bialgebra is also monoidal).

Lemma 2.10

i) A left-monoidal weak bialgebra $(\mathcal{A}, \mathbf{1}, \Delta, \varepsilon)$ is left-comonoidal if and only if

$$\Delta(\mathbf{1}) = (id \otimes \varepsilon \otimes id)[(\Delta(\mathbf{1}) \otimes \mathbf{1})(\mathbf{1} \otimes \Delta(\mathbf{1})] \quad (2.45)$$

ii) A right-monoidal weak bialgebra $(\mathcal{A}, \mathbf{1}, \Delta, \varepsilon)$ is right-comonoidal if and only if

$$\Delta(\mathbf{1}) = (id \otimes \varepsilon \otimes id)[(\mathbf{1} \otimes \Delta(\mathbf{1}))(\Delta(\mathbf{1}) \otimes \mathbf{1})] \quad (2.46)$$

iii) A left-comonoidal weak bialgebra $(\mathcal{A}, \mathbf{1}, \Delta, \varepsilon)$ is left-monoidal if and only if

$$\varepsilon(ab) = \varepsilon(a\mathbf{1}_{(1)})\varepsilon(\mathbf{1}_{(2)}b) \quad (2.47)$$

iv) A right-comonoidal weak bialgebra $(\mathcal{A}, \mathbf{1}, \Delta, \varepsilon)$ is right-monoidal if and only if

$$\varepsilon(ab) = \varepsilon(a\mathbf{1}_{(2)})\varepsilon(\mathbf{1}_{(1)}b) \quad (2.48)$$

Proof: Part ii) reduces to part i) in $\mathcal{A}_{op}$ and iii), iv) are the dual versions of i), ii). To prove part i) first note that by the counit property (2.8) implies (2.45). On the other hand, Eq. (2.45) is equivalent to $\hat{\varepsilon}(\phi\psi) = \hat{\varepsilon}(\phi\hat{\mathbf{1}}_{(1)})\hat{\varepsilon}(\hat{\mathbf{1}}_{(2)}\psi)$, $\forall\phi,\psi \in \hat{\mathcal{A}}$, and therefore, by Lemma 2.7(a), to $\hat{\varepsilon}_L \circ \varepsilon_L \circ \hat{\varepsilon}_L = \hat{\varepsilon}_L$. If in this case $\mathcal{A}$ is also left-monoidal then we may apply $id \otimes \hat{\varepsilon}_L$ to Eq. (2.25Left) to get

$$\phi_{(1)} \otimes \hat{\varepsilon}_L(\phi_{(2)}) = \phi\hat{\mathbf{1}}_{(1)} \otimes \hat{\varepsilon}_L(\mathbf{1}_{(2)}), \quad \forall\phi \in \hat{\mathcal{A}}$$

which is precisely the dual version of the condition (2.26Left). Hence, in this case $\hat{\mathcal{A}}$ is left-monoidal and therefore $\mathcal{A}$ is left-comonoidal. ■

Lemma 2.10 implies that the face algebras of T. Hayashi [Ha] are left monoidal, since by definition they are comonoidal and satisfy (2.47). In fact, they are even bimonoidal, since in Corollary 3.9 we will see that comonoidal weak bialgebras are left monoidal if and only if they are right monoidal. A comonoidal weak bialgabra which is not monoidal will be given in Example 1 of Appendix D.

3 Comonoidal Weak Bialgebras

In the previous Section we have emphasized the monoidality axioms for weak bialgebras $\mathcal{A}$ by relating them to the monoidality properties of $\mathcal{E} \equiv \varepsilon_R(\mathcal{A})$ and $\hat{\mathcal{E}} \equiv \varepsilon_L(\mathcal{A})$ as objects in $\mathrm{Rep}\,\mathcal{A}$ and $\mathrm{Rep}\,\mathcal{A}_{op}$, respectively. In this Section we pass to the dual point of view by investigating the comonoidality axioms (2.8) and (2.9) and relating them to algebraic properties of the linear subspaces $\mathcal{A}_{L/R} \subset \mathcal{A}$. Note that these are just the dual counterparts of $\mathcal{E}$ and $\hat{\mathcal{E}}$, respectively, and that for $\sigma, \sigma' \in \{L, R\}$ we have $\mathcal{A}_\sigma \supset \mathcal{A}_{\sigma\sigma'}$. We will show that for comonoidal weak bialgebras $\mathcal{A}_\sigma = \mathcal{A}_{\sigma\sigma'}$ and that the spaces $\mathcal{A}_\sigma$, $\sigma = L, R$, are in fact commuting unital subalgebras of $\mathcal{A}$ as in the weak Hopf setting of [BSz]. We will see that these algebras coincide with the "fixed point" subalgebras of $\mathcal{A}$ under the natural (left or right, respectively) action of $\hat{\mathcal{A}}$ on $\mathcal{A}$. We will also show, that $\varepsilon_\sigma : \mathcal{A}_{\sigma'} \to \hat{\mathcal{A}}_{\sigma\sigma'}$ provides an algebra isomorphism for $\sigma \neq \sigma'$ and an algebra anti-isomorphism for $\sigma = \sigma'$.[4] If $\mathcal{A}$ is bimonoidal, then also $\hat{\mathcal{A}}_{\sigma\sigma'} = \hat{\mathcal{A}}_\sigma$.

These results play an important role in the theory of crossed products by weak Hopf algebras [NSW]. In regular crossed products of von-Neumann algebras $\mathcal{M}$ by weak Hopf algebra actions $\triangleright : \mathcal{A} \otimes \mathcal{M} \to \mathcal{M}$ one requires $\mathcal{A} \triangleright \mathbf{1}_\mathcal{M} = \mathcal{A}_L \triangleright \mathbf{1}_\mathcal{M} \cong \mathcal{A}_L$ and $\mathcal{C}(\mathcal{M}) = (\mathcal{A}_L \cap \mathcal{A}_R) \triangleright \mathbf{1}_\mathcal{M} \cong \mathcal{A}_L \cap \mathcal{A}_R$ implying the dual properties in $\mathcal{M} \rtimes \mathcal{A}$, see [NSW]. In this way the algebras $\mathcal{A}_\sigma$ and $\hat{\mathcal{A}}_\sigma$ appear as the lowest relative commutants in the resulting reducible Jones tower [NSW]. If $\mathcal{A}$ is a Frobenius weak Hopf algebra, then $\mathcal{A}_\sigma$ parametrizes the space of integrals in $\mathcal{A}$ [BNS].

Let now $(\mathcal{A}, \mathbf{1}, \Delta, \varepsilon)$ be a weak bialgebra. We start with introducing the following four unital subalgebras $\mathcal{N}_{\sigma\sigma'}(\mathcal{A}) \subset \mathcal{A}$, $\sigma, \sigma' \in \{L, R\}$, given by

$$\begin{aligned}
\mathcal{N}_{LL}(\mathcal{A}) &:= \{a \in \mathcal{A} \mid \Delta(a) = a\mathbf{1}_{(1)} \otimes \mathbf{1}_{(2)}\} && (3.1)\\
\mathcal{N}_{LR}(\mathcal{A}) &:= \{a \in \mathcal{A} \mid \Delta(a) = \mathbf{1}_{(1)}a \otimes \mathbf{1}_{(2)}\} && (3.2)\\
\mathcal{N}_{RL}(\mathcal{A}) &:= \{a \in \mathcal{A} \mid \Delta(a) = \mathbf{1}_{(1)} \otimes a\mathbf{1}_{(2)}\} && (3.3)\\
\mathcal{N}_{RR}(\mathcal{A}) &:= \{a \in \mathcal{A} \mid \Delta(a) = \mathbf{1}_{(1)} \otimes \mathbf{1}_{(2)}a\} && (3.4)
\end{aligned}$$

The subalgebras $\mathcal{N}_{\sigma\sigma'}(\hat{\mathcal{A}}) \subset \hat{\mathcal{A}}$ are defined accordingly. These algebras may be considered as the "left and right fixed point subalgebras" of $\mathcal{A}$ under the canonical $\hat{\mathcal{A}}$-actions in the following

[4] Recall from (the dual of) Proposition 2.6 that $\hat{\mathcal{A}}_{\sigma\sigma'} \subset \hat{\mathcal{A}}$ is also a subalgebra, if $\mathcal{A}$ is comonoidal.

sense

$$\begin{aligned}
\mathcal{N}_{LL}(\mathcal{A}) &= \{a \in \mathcal{A} \mid \phi \rightharpoonup (ab) = a(\phi \rightharpoonup b),\ \forall \phi \in \hat{\mathcal{A}},\ \forall b \in \mathcal{A}\} && (3.5)\\
\mathcal{N}_{LR}(\mathcal{A}) &= \{a \in \mathcal{A} \mid \phi \rightharpoonup (ba) = (\phi \rightharpoonup b)a,\ \forall \phi \in \hat{\mathcal{A}},\ \forall b \in \mathcal{A}\} && (3.6)\\
\mathcal{N}_{RL}(\mathcal{A}) &= \{a \in \mathcal{A} \mid (ab) \leftharpoonup \phi = a(b \leftharpoonup \phi),\ \forall \phi \in \hat{\mathcal{A}},\ \forall b \in \mathcal{A}\} && (3.7)\\
\mathcal{N}_{RR}(\mathcal{A}) &= \{a \in \mathcal{A} \mid (ba) \leftharpoonup \phi = (b \leftharpoonup \phi)a,\ \forall \phi \in \hat{\mathcal{A}},\ \forall b \in \mathcal{A}\} && (3.8)
\end{aligned}$$

Some immediate consequences of the above definitions are given in the following

Corollary 3.1 *For any weak bialgebra $(\mathcal{A}, \mathbf{1}, \Delta, \varepsilon)$ and for all $\sigma, \sigma' \in \{L, R\}$ we have*

i) $a \in \mathcal{N}_{\sigma\sigma'}(\mathcal{A}) \Longrightarrow \varepsilon_{\sigma\sigma'}(a) = a$.
In particular $\mathcal{N}_{\sigma\sigma'}(\mathcal{A}) \subset \mathcal{A}_{\sigma\sigma'}$, and $\mathcal{N}_{\sigma\sigma'}(\mathcal{A}) = \mathcal{A}_{\sigma\sigma'}$ if $\mathcal{A}$ is monoidal.

ii) $\mathcal{N}_{\sigma L}(\mathcal{A}) \cap \mathcal{C}(\mathcal{A}) = \mathcal{N}_{\sigma R}(\mathcal{A}) \cap \mathcal{C}(\mathcal{A}) =: \mathcal{C}_\sigma(\mathcal{A})$

iii) $[\mathcal{N}_{L\sigma}(\mathcal{A}), \mathcal{N}_{R\sigma}(\mathcal{A})] = 0$.

Proof: To prove part (i) use (2.22) and Proposition 2.6, and for part (ii) use Eqs. (3.5) - (3.8). For $\sigma = L$ part (iii) follows from $\Delta(ab) = (a \otimes b)\Delta(\mathbf{1}) = \Delta(ba),\ a \in \mathcal{N}_{LL}(\mathcal{A}),\ b \in \mathcal{N}_{RL}(\mathcal{A})$, by applying $\varepsilon \otimes \mathrm{id}$. The argument for $\sigma = R$ is analogous. ■

Our next aim is to show that ε_σ maps $\mathcal{N}_{\sigma'\sigma}(\mathcal{A})$ (anti-)isomorphically onto $\mathcal{N}_{\sigma\sigma'}(\hat{\mathcal{A}})$. To this end we first need the following

Lemma 3.2 *Let $(\mathcal{A}, \mathbf{1}, \Delta, \varepsilon)$ be a weak bialgebra. Then the following equivalencies hold*

$$\begin{array}{rlcl}
i) & a \in \mathcal{N}_{LL}(\mathcal{A}) & \Longleftrightarrow & \phi \rightharpoonup a = a\hat{\varepsilon}_R(\phi),\ \forall \phi \in \hat{\mathcal{A}} \iff ab = b \leftharpoonup \varepsilon_L(a),\ \forall b \in \mathcal{A}\\
& & \Longleftrightarrow & \phi \leftharpoonup a = \varepsilon_L(a)\phi,\ \forall \phi \in \hat{\mathcal{A}}\\
ii) & a \in \mathcal{N}_{LR}(\mathcal{A}) & \Longleftrightarrow & \phi \rightharpoonup a = \hat{\varepsilon}_R(\phi)a,\ \forall \phi \in \hat{\mathcal{A}} \iff ba = b \leftharpoonup \varepsilon_R(a),\ \forall b \in \mathcal{A}\\
& & \Longleftrightarrow & a \rightharpoonup \phi = \varepsilon_R(a)\phi,\ \forall \phi \in \hat{\mathcal{A}}\\
iii) & a \in \mathcal{N}_{RL}(\mathcal{A}) & \Longleftrightarrow & a \leftharpoonup \phi = a\hat{\varepsilon}_L(\phi),\ \forall \phi \in \hat{\mathcal{A}} \iff ab = \varepsilon_L(a) \rightharpoonup b,\ \forall b \in \mathcal{A}\\
& & \Longleftrightarrow & \phi \leftharpoonup a = \phi\varepsilon_L(a),\ \forall \phi \in \hat{\mathcal{A}}\\
iv) & a \in \mathcal{N}_{RR}(\mathcal{A}) & \Longleftrightarrow & a \leftharpoonup \phi = \hat{\varepsilon}_L(\phi)a,\ \forall \phi \in \hat{\mathcal{A}} \iff ba = \varepsilon_R(a) \rightharpoonup b,\ \forall b \in \mathcal{A}\\
& & \Longleftrightarrow & a \rightharpoonup \phi = \phi\varepsilon_R(a),\ \forall \phi \in \hat{\mathcal{A}}
\end{array}$$

Proof: The equivalences (ii), (iii) and (iv) reduce to (i) in $\mathcal{A}_{op}$, $\mathcal{A}^{cop}$ and $\mathcal{A}^{cop}_{op}$, respectively. To prove (i) first note that the equivalence $a \in \mathcal{N}_{LL}(\mathcal{A}) \Leftrightarrow (\phi \rightharpoonup a) = a(\phi \rightharpoonup \mathbf{1}) \equiv a\hat{\varepsilon}_R(\phi)$ is obvious from the definition (3.1), see also Eq. (3.5). Let now $a \in \mathcal{N}_{LL}(\mathcal{A})$ then for all $b \in \mathcal{A}$

$$ab = \varepsilon(a_{(1)}b_{(1)})a_{(2)}b_{(2)} = \varepsilon(ab_{(1)})b_{(2)} = \langle \varepsilon_L(a) \mid b_{(1)} \rangle b_{(2)} = b \leftharpoonup \varepsilon_L(a).$$

Pairing both sides of this condition with $\phi \in \hat{\mathcal{A}}$ we further get

$$\langle \phi \leftharpoonup a \mid b \rangle = \langle \varepsilon_L(a)\phi \mid b \rangle,\ \forall b \in \mathcal{A}$$

and therefore $\phi \leftharpoonup a = \varepsilon_L(a)\phi$. Finally, if this holds for all $\phi \in \hat{\mathcal{A}}$, then we get for all $\phi, \psi \in \hat{\mathcal{A}}$

$$\begin{aligned}
\langle \phi \otimes \psi \mid a\mathbf{1}_{(1)} \otimes \mathbf{1}_{(2)} \rangle &\equiv \hat{\varepsilon}((\phi \leftharpoonup a)\psi) = \hat{\varepsilon}(\varepsilon_L(a)\phi\psi)\\
= \hat{\varepsilon}((\phi\psi) \leftharpoonup a) &\equiv \langle \phi\psi \mid a \rangle \equiv \langle \phi \otimes \psi \mid \Delta(a) \rangle
\end{aligned}$$

implying $a \in \mathcal{N}_{LL}(\mathcal{A})$. ■

Theorem 3.3 *For any weak bialgebra $\mathcal{A}$ and for all $\sigma,\sigma' \in \{L,R\}$ we have*
i) ε_σ maps $\mathcal{N}_{\sigma'\sigma}(\mathcal{A})$ bijectively onto $\mathcal{N}_{\sigma\sigma'}(\hat{\mathcal{A}})$ with inverse given by $\hat{\varepsilon}_{\sigma'} : \mathcal{N}_{\sigma\sigma'}(\hat{\mathcal{A}}) \to \mathcal{N}_{\sigma'\sigma}(\mathcal{A})$. Moreover, these are algebra isomorphisms, if $\sigma \neq \sigma'$, and algebra anti-isomorphisms, if $\sigma = \sigma'$.
ii)Defining $\mathcal{C}_\sigma(\mathcal{A}) := \mathcal{C}(\mathcal{A}) \cap \mathcal{N}_{\sigma L}(\mathcal{A}) = \mathcal{C}(\mathcal{A}) \cap \mathcal{N}_{\sigma R}(\mathcal{A})$ as in Corollary 3.1ii) we have

$$\begin{aligned}
\varepsilon_\sigma(\mathcal{N}_{L\sigma}(\mathcal{A}) \cap \mathcal{N}_{R\sigma}(\mathcal{A})) &= \mathcal{C}_\sigma(\hat{\mathcal{A}}) && (3.9)\\
\hat{\varepsilon}_L(\mathcal{C}_\sigma(\hat{\mathcal{A}})) = \hat{\varepsilon}_R(\mathcal{C}_\sigma(\hat{\mathcal{A}})) &= \mathcal{N}_{L\sigma}(\mathcal{A}) \cap \mathcal{N}_{R\sigma}(\mathcal{A}) && (3.10)\\
\varepsilon_\sigma(\mathcal{C}_L(\mathcal{A}) \cap \mathcal{C}_R(\mathcal{A})) &= \mathcal{C}_L(\hat{\mathcal{A}}) \cap \mathcal{C}_R(\hat{\mathcal{A}}). && (3.11)
\end{aligned}$$

Proof: (i) By passing to $\mathcal{A}_{op}$, $\mathcal{A}^{cop}$ or $\mathcal{A}^{cop}_{op}$, respectively, and noting $\widehat{\mathcal{A}_{op}} = (\hat{\mathcal{A}})^{cop}$ and $\widehat{\mathcal{A}^{cop}} = (\hat{\mathcal{A}})_{op}$, it suffices to consider the case $\sigma = \sigma' = L$. By Corollary 3.1i) $\hat{\varepsilon}_L \circ \varepsilon_L|_{\mathcal{N}_{LL}(\mathcal{A})} = \mathrm{id}$ and it is enough to show $\varepsilon_L(\mathcal{N}_{LL}(\mathcal{A})) = \mathcal{N}_{LL}(\hat{\mathcal{A}})$. To show $\varepsilon_L(\mathcal{N}_{LL}(\mathcal{A})) \subset \mathcal{N}_{LL}(\hat{\mathcal{A}})$ let $a \in \mathcal{N}_{LL}(\mathcal{A})$ and put $\psi = \varepsilon_L(a)$. Then $a = \hat{\varepsilon}_L(\psi)$ by Corollary 3.1i) and the last equivalence in Lemma 3.2i) implies

$$\psi\phi = \phi \leftharpoonup \hat{\varepsilon}_L(\psi), \ \forall \phi \in \hat{\mathcal{A}}$$

By the second equivalence of Lemma 3.2i) this implies $\psi \in \mathcal{N}_{LL}(\hat{\mathcal{A}})$. Interchanging the role of $\mathcal{A}$ and $\hat{\mathcal{A}}$ we also get $\hat{\varepsilon}_L(\mathcal{N}_{LL}(\hat{\mathcal{A}})) \subset \mathcal{N}_{LL}(\mathcal{A})$ and therefore equality. Finally, putting $\phi = \varepsilon_L(b)$ in the last equivalence of Lemma 3.2i) implies for $a, b \in \mathcal{N}_{LL}(\mathcal{A})$

$$\varepsilon_L(a)\varepsilon_L(b) = \varepsilon_L(b) \leftharpoonup a = \varepsilon_L(ba)$$

by Eq. (2.21), and therefore $\varepsilon_L|_{\mathcal{N}_{LL}(\mathcal{A})}$ is an algebra antimorphism.
(ii) Similar as above, by passing to $\mathcal{A}^{cop}_{op}$ it suffices to consider the case $\sigma = L$. If $a \in \mathcal{N}_{LL}(\mathcal{A}) \cap \mathcal{N}_{RL}(\mathcal{A})$ then by Lemma 3.2(i) and (iii)

$$\phi \leftharpoonup a = \varepsilon_L(a)\phi = \phi\varepsilon_L(a), \qquad \forall \phi \in \hat{\mathcal{A}}$$

implying $\varepsilon_L(a) \in \mathcal{C}(\hat{\mathcal{A}}) \cap \mathcal{N}_{LL}(\hat{\mathcal{A}}) = \mathcal{C}_L(\hat{\mathcal{A}})$ by part (i). To show that $\varepsilon_L : (\mathcal{N}_{LL}(\mathcal{A}) \cap \mathcal{N}_{RL}(\mathcal{A})) \to \mathcal{C}_L(\hat{\mathcal{A}})$ is surjective pick $\phi \in \mathcal{C}_L(\hat{\mathcal{A}})$, then by part (i)

$$\phi = \varepsilon_L(\hat{\varepsilon}_\sigma(\phi)) \in \varepsilon_L(\mathcal{N}_{LL}(\mathcal{A}) \cap \mathcal{N}_{RL}(\mathcal{A})),$$

where we have used $\hat{\varepsilon}_L(\phi) = \hat{\varepsilon}_R(\phi)$ for all $\phi \in \mathcal{C}(\hat{\mathcal{A}})$. This proves (3.9) and by part (i) also the inverse relation (3.10).
Finally, Eq. (3.11) follows by using Corollary 3.1(ii) to get for $\sigma = L$ or $\sigma = R$

$$\mathcal{C}_L(\mathcal{A}) \cap \mathcal{C}_R(\mathcal{A}) = \mathcal{N}_{L\sigma}(\mathcal{A}) \cap \mathcal{N}_{R\sigma}(\mathcal{A}) \cap \mathcal{C}(\mathcal{A})$$

(and the same formula with $\mathcal{A}$ replaced by $\hat{\mathcal{A}}$) and applying Eqs. (3.9), (3.10) and their dual versions. ■

The algebra $\mathcal{Z} := \mathcal{C}_L(\mathcal{A}) \cap \mathcal{C}_R(\mathcal{A}) \cong \mathcal{C}_L(\hat{\mathcal{A}}) \cap \mathcal{C}_R(\hat{\mathcal{A}})$ has been called the "hyper center" of $\mathcal{A}$ (and $\hat{\mathcal{A}}$) in [Sz] and it appears as $C(\mathcal{M}^{\mathcal{A}}) \cap \mathcal{C}(\mathcal{M}) = \mathcal{C}(\mathcal{M}) \cap \mathcal{C}(\mathcal{M} \rtimes \mathcal{A})$ in the crossed product theory of [NSW]. If $p \in \mathcal{Z}$ is an idempotent, then $\mathcal{A}_p := p\mathcal{A} \subset \mathcal{A}$ is a weak sub-bialgebra and by (3.11) its dual is given by $\widehat{\mathcal{A}_p} = \hat{p}\hat{\mathcal{A}}$, where $\hat{p} = \varepsilon_L(p) \equiv \varepsilon_R(p)$.

There is an alternative insight into Lemma 3.2 and Theorem 3.3 by considering $\mathcal{A}$ and $\hat{\mathcal{A}}$ as subalgebras of $\mathrm{End}_K \mathcal{A}$. Let $Q_\sigma : \mathcal{A} \to \mathrm{End}_K \mathcal{A}$ and $P_\sigma : \hat{\mathcal{A}} \to \mathrm{End}_K \mathcal{A}$, $\sigma = L, R$, be given by

$$\begin{aligned}
Q_L(a)b &:= ab \quad , \quad & Q_R(a)b &:= ba\\
P_L(\phi)b &:= b \leftharpoonup \phi \quad , \quad & P_R(\phi)b &:= \phi \rightharpoonup b
\end{aligned}$$

where $a, b \in \mathcal{A}$ and $\phi \in \hat{\mathcal{A}}$. Then we have the following

Lemma 3.4 *For any pair of dual weak bialgebras $\mathcal{A}$ and $\hat{\mathcal{A}}$ and for all $\sigma, \sigma' \in \{L, R\}$ we have*

$$\begin{aligned} Q_\sigma(\mathcal{N}_{\sigma'\sigma}(\mathcal{A})) &= P_{\sigma'}(\mathcal{N}_{\sigma\sigma'}(\hat{\mathcal{A}})) = Q_\sigma(\mathcal{A}) \cap P_{\sigma'}(\hat{\mathcal{A}}) && (3.12)\\ \varepsilon_\sigma|_{\mathcal{N}_{\sigma'\sigma}(\mathcal{A})} &= P_{\sigma'}^{-1} \circ Q_\sigma|_{\mathcal{N}_{\sigma'\sigma}(\mathcal{A})} && (3.13)\\ \hat{\varepsilon}_{\sigma'}|_{\mathcal{N}_{\sigma\sigma'}(\hat{\mathcal{A}})} &= Q_\sigma^{-1} \circ P_{\sigma'}|_{\mathcal{N}_{\sigma\sigma'}(\hat{\mathcal{A}})} && (3.14) \end{aligned}$$

Proof: Lemma 3.2 immediately gives $Q_\sigma(\mathcal{N}_{\sigma'\sigma}(\mathcal{A})) = P_{\sigma'}(\mathcal{N}_{\sigma\sigma'}(\hat{\mathcal{A}})) \subset Q_\sigma(\mathcal{A}) \cap P_{\sigma'}(\hat{\mathcal{A}})$, as well as the identities (3.13) and (3.14). Conversely, if $Q_L(a) = P_L(\psi)$ then $\phi \rightharpoonup (ab) = \phi \rightharpoonup b \leftharpoonup \psi = a(\phi \rightharpoonup b)$ for all $b \in \mathcal{A}$, $\phi \in \hat{\mathcal{A}}$, implying $a \in \mathcal{N}_{LL}(\mathcal{A})$ by (3.5). The remaining cases are analogous. ■

We now show that a weak bialgebra $(\mathcal{A}, \mathbf{1}, \Delta, \varepsilon)$ is comonoidal if and only if $\mathcal{N}_{\sigma\sigma'}(\mathcal{A}) = \mathcal{A}_\sigma$ for all $\sigma, \sigma' \in \{L, R\}$. Again, this statement may be divided into two pieces.

Theorem 3.5 *For a weak bialgebra $(\mathcal{A}, \mathbf{1}, \Delta, \varepsilon)$ the following equivalencies hold*
i) $\mathcal{A}$ is left-comonoidal $\Longleftrightarrow \mathcal{A}_L = \mathcal{N}_{LL}(\mathcal{A}) \Longleftrightarrow \mathcal{A}_R = \mathcal{N}_{RR}(\mathcal{A})$.
If this holds then we also have $\mathcal{N}_{\sigma\sigma}(\hat{\mathcal{A}}) = \hat{\mathcal{A}}_{\sigma\sigma}$ and $\mathcal{N}_{\sigma\sigma}(\mathcal{A}) = \mathcal{A}_{\sigma\sigma}$, for $\sigma = L$ and $\sigma = R$.
ii) $\mathcal{A}$ is right-comonoidal $\Longleftrightarrow \mathcal{A}_L = \mathcal{N}_{LR}(\mathcal{A}) \Longleftrightarrow \mathcal{A}_R = \mathcal{N}_{RL}(\mathcal{A})$.
If this holds then we also have $\mathcal{N}_{\sigma\sigma'}(\hat{\mathcal{A}}) = \hat{\mathcal{A}}_{\sigma\sigma'}$ and $\mathcal{N}_{\sigma\sigma'}(\mathcal{A}) = \mathcal{A}_{\sigma\sigma'}$ for $(\sigma, \sigma') = (L, R)$ and $(\sigma, \sigma') = (R, L)$.

Proof: Part (ii) reduces to part (i) in $\mathcal{A}^{cop}$. To prove part (i) for $\sigma = L$ observe that $\mathcal{N}_{LL}(\mathcal{A}) = \mathcal{A}_L$ is equivalent to $\Delta(\hat{\varepsilon}_L(\phi)) = \hat{\varepsilon}_L(\phi)\mathbf{1}_{(1)} \otimes \mathbf{1}_{(2)}$, $\forall \phi \in \hat{\mathcal{A}}$, and therefore to

$$\langle \phi \mid \mathbf{1}_{(1)} \rangle \mathbf{1}_{(2)} \otimes \mathbf{1}_{(3)} = \langle \phi \mid \mathbf{1}_{(1)} \rangle \mathbf{1}_{(2)} \mathbf{1}_{(1')} \otimes \mathbf{1}_{(2')}, \qquad \forall \phi \in \hat{\mathcal{A}}$$

which is the left-comonoidality property (2.8) for $\mathcal{A}$. Next, we use that $\mathcal{N}_{\sigma\sigma'}(\mathcal{A}) \subset \mathcal{A}_{\sigma\sigma'} \subset \mathcal{A}_\sigma$ always holds by Corollary 3.1i). Hence $\mathcal{N}_{LL}(\mathcal{A}) = \mathcal{A}_L$ implies $\mathcal{N}_{LL}(\mathcal{A}) = \mathcal{A}_{LL}$. To get the dual statement we use that if $\mathcal{A}$ is left-comonoidal then $\hat{\mathcal{A}}$ is left-monoidal implying $\mathcal{N}_{LL}(\hat{\mathcal{A}}) \supset \hat{\varepsilon}_{LL}(\hat{\mathcal{A}}) \equiv \hat{\mathcal{A}}_{LL}$ by part (1i, left) of Proposition 2.6. The case $\sigma = R$ follows by passing to $\mathcal{A}_{op}^{cop}$. ■

Corollary 3.6 *Let $\mathcal{A}$ be monoidal. Then for $\sigma \in \{L, R\}$ the restrictions $\varepsilon_{L\sigma}|_{\mathcal{A}_{R\sigma}}$ provide algebra anti-isomorphisms $\varepsilon_{L\sigma} : \mathcal{A}_{R\sigma} \to \mathcal{A}_{L\sigma}$ with inverse $\varepsilon_{R\sigma} : \mathcal{A}_{L\sigma} \to \mathcal{A}_{R\sigma}$.*

Proof: This follows from Theorem 3.3 and the dual of Theorem 3.5, implying $\mathcal{N}_{\sigma L}(\hat{\mathcal{A}}) = \mathcal{N}_{\sigma R}(\hat{\mathcal{A}}) = \hat{\mathcal{A}}_\sigma$. ■

Note that for comonoidal weak bialgebras $\mathcal{A}$ Theorem 3.5 still allows for the possibility $\hat{\mathcal{A}}_{\sigma\sigma'} \underset{\neq}{\subset} \hat{\mathcal{A}}_\sigma$. Also note that if $\mathcal{A}$ is comonoidal, then by Theorem 3.5 and Corollary 3.1iii) $\mathcal{A}_L$ and $\mathcal{A}_R$ are commuting subalgebras of $\mathcal{A}$. More precisely, we even have

Corollary 3.7 *A weak bialgebra $\mathcal{A}$ is comonoidal if and only if it is left- (or right-) comonoidal and $[\mathcal{A}_L, \mathcal{A}_R] = 0$. In this case we also have $\mathcal{A}_L \cap \mathcal{A}_R \cong \mathcal{C}_L(\hat{\mathcal{A}}) \cong \mathcal{C}_R(\hat{\mathcal{A}})$ and $C_{L/R}(\mathcal{A}) = \mathcal{C}(\mathcal{A}) \cap \mathcal{A}_{L/R}$.*

Proof: By Theorem 3.5 and Corollary 3.1iii) comonoidality implies $[\mathcal{A}_L, \mathcal{A}_R] = 0$. Conversely, if $[\mathcal{A}_L, \mathcal{A}_R] = 0$, then $\mathcal{N}_{\sigma\sigma'}(\mathcal{A}) \subset \mathcal{A}_\sigma$ (Corollary 3.1i)) gives $\mathcal{N}_{\sigma L}(\mathcal{A}) = \mathcal{N}_{\sigma R}(\mathcal{A})$ for $\sigma \in \{L, R\}$ by the definitions (3.1) - (3.4). In this case, by Theorem 3.5, $\mathcal{A}$ is left-comonoidal iff it is right-comonoidal. The remaining statements follow from Theorem 3.3. ■

Somewhat surprisingly, under the condition $[\mathcal{A}_L, \mathcal{A}_R] = 0$ also left- and right-*monoidality* become equivalent.

Lemma 3.8 *Let $\mathcal{A}$ be a weak bialgebra and suppose $[\mathcal{A}_L, \mathcal{A}_R] = 0$. Then $\mathcal{A}$ is left-monoidal if and only if it is right-monoidal.*

Proof: By passing to $\mathcal{A}_{op}$ it suffices to prove one direction. If $\mathcal{A}$ is left-monoidal, then by the duals of Theorem 3.5 and Corollary 3.1i) $\hat{\mathcal{A}}_L = \mathcal{N}_{LL}(\hat{\mathcal{A}}) \supset \mathcal{N}_{LR}(\hat{\mathcal{A}})$ and $\hat{\mathcal{A}}_R = \mathcal{N}_{RR}(\hat{\mathcal{A}}) \supset \mathcal{N}_{RL}(\hat{\mathcal{A}})$. Since $[\mathcal{A}_L, \mathcal{A}_R] = 0$ implies $\mathcal{N}_{\sigma L}(\mathcal{A}) = \mathcal{N}_{\sigma R}(\mathcal{A})$, $\sigma \in \{L, R\}$, we may now use Theorem 3.3 to conclude

$$
\begin{array}{lllllllll}
\hat{\mathcal{A}}_L & = & \mathcal{N}_{LL}(\hat{\mathcal{A}}) & \cong & \mathcal{N}_{LL}(\mathcal{A})_{op} & = & \mathcal{N}_{LR}(\mathcal{A})_{op} & \cong & \mathcal{N}_{RL}(\hat{\mathcal{A}})_{op} \\
\hat{\mathcal{A}}_R & = & \mathcal{N}_{RR}(\hat{\mathcal{A}}) & \cong & \mathcal{N}_{RR}(\mathcal{A})_{op} & = & \mathcal{N}_{RL}(\mathcal{A})_{op} & \cong & \mathcal{N}_{LR}(\hat{\mathcal{A}})_{op} \,.
\end{array}
$$

Hence, $\dim \hat{\mathcal{A}}_L = \dim \hat{\mathcal{A}}_R$, implying also $\hat{\mathcal{A}}_L = \mathcal{N}_{LR}(\hat{\mathcal{A}})$ and $\hat{\mathcal{A}}_L = \mathcal{N}_{LR}(\hat{\mathcal{A}})$, and by Theorem 3.5 $\mathcal{A}$ is right-monoidal. ■

Putting Corollary 3.7 and Lemma 3.8 together with their dual versions we arrive at

Corollary 3.9 *A weak bialgebra $\mathcal{A}$ is bimonoidal $\Leftrightarrow$ $\mathcal{A}$ is comonoidal and left- (or right-) monoidal $\Leftrightarrow$ $\mathcal{A}$ is monoidal and left- (or right-) comonoidal.*

As already remarked, Corollary 3.9 together with Lemma 2.10 imply that finite dimensional (actually $\dim \mathcal{A}_{L/R} < \infty$ is sufficient) face algebras in the sense of Hayashi [Ha] are bimonoidal weak bialgebras.

Let us summarize our findings for comonoidal weak bialgebras $\mathcal{A}$. Among the algebras $\mathcal{A}_{\sigma\sigma'} \cong (\hat{\mathcal{A}}_{\sigma'\sigma})_{(op)}$ we are actually left with only two equivalence classes

$$
\begin{array}{lllllllll}
\mathcal{A}_L & = & \mathcal{A}_{LL} & = & \mathcal{A}_{LR} & \cong & \hat{\mathcal{A}}_{RL} & \cong & (\hat{\mathcal{A}}_{LL})_{op} \\
\mathcal{A}_R & = & \mathcal{A}_{RR} & = & \mathcal{A}_{RL} & \cong & \hat{\mathcal{A}}_{LR} & \cong & (\hat{\mathcal{A}}_{RR})_{op}
\end{array}
\tag{3.15}
$$

the isomorphisms being given by the following diagram

$$
\begin{array}{lcccc}
\mathcal{N}_{\sigma'\sigma}(\hat{\mathcal{A}}) & = & \hat{\mathcal{A}}_{\sigma'\sigma} & \subset & \hat{\mathcal{A}}_{\sigma'} \\
 & & \varepsilon_{\sigma'} \uparrow\!\downarrow \hat{\varepsilon}_\sigma & & \downarrow \hat{\varepsilon}_\sigma \\
\mathcal{N}_{\sigma\sigma'}(\mathcal{A}) & = & \mathcal{A}_{\sigma\sigma'} & = & \mathcal{A}_\sigma
\end{array}
\tag{3.16}
$$

We have $\hat{\varepsilon}_\sigma \varepsilon_{\sigma'} \hat{\varepsilon}_\sigma = \hat{\varepsilon}_\sigma$ and therefore $(\hat{\varepsilon}_{\sigma\sigma'})^2 = \hat{\varepsilon}_{\sigma\sigma'}$ and $(\varepsilon_{\sigma'\sigma})^2 = \varepsilon_{\sigma'\sigma}$. However we may possibly have $\operatorname{Ker} \hat{\varepsilon}_\sigma \cap \hat{\mathcal{A}}_{\sigma'} \neq 0$ in which case $\varepsilon_{\sigma'} \hat{\varepsilon}_\sigma \varepsilon_{\sigma'} \neq \varepsilon_{\sigma'}$ and $\hat{\mathcal{A}}_{\sigma'\sigma} \underset{\neq}{\subset} \hat{\mathcal{A}}_{\sigma'}$. This precisely reflects the possibility that comonoidal weak bialgebras $\mathcal{A}$ may not be monoidal. Moreover, by the dual of (2.42)

$$
\dim \mathcal{A}_L = \dim \mathcal{A}_R = \dim \hat{\mathcal{A}}_{\sigma\sigma'} \,, \quad \forall \sigma, \sigma' \in \{L, R\}
\tag{3.17}
$$

If $\mathcal{A}$ is bimonoidal, then we also have $\hat{\mathcal{A}}_\sigma = \hat{\mathcal{A}}_{\sigma\sigma'} = \mathcal{N}_{\sigma\sigma'}(\hat{\mathcal{A}})$ for all $\sigma, \sigma' \in \{L, R\}$ and the above diagram also holds with $\mathcal{A}$ and $\hat{\mathcal{A}}$ interchanged, i.e.

$$
\mathcal{A}_L \cong \hat{\mathcal{A}}_R \cong (\mathcal{A}_R)_{op} \cong (\hat{\mathcal{A}}_L)_{op}
$$

where the isomorphisms are given by $\varepsilon_\sigma : \mathcal{A}_{\sigma'} \to \hat{\mathcal{A}}_\sigma$ with inverse $\hat{\varepsilon}_{\sigma'} : \hat{\mathcal{A}}_\sigma \to \mathcal{A}_{\sigma'}$.

4 The Comodule Picture

In this Section we study the category of (right) comodules $\mathrm{Cmod}\,\mathcal{A}$ of a weak bialgebra $\mathcal{A}$ and describe its monoidal structure in case $\mathcal{A}$ is bimonoidal. Of course, since $\mathrm{Cmod}\,\mathcal{A} = \mathrm{Rep}\,\hat{\mathcal{A}}$, this could be traced back to the results of Sect. 2. However, it turns out that in the bimonoidal case the tensor functor in $\mathrm{Cmod}\,\mathcal{A}$ is more naturally described by an "amalgamated" tensor product, which will then be shown to be equivalent to the constructions in Sect. 2. We also generalize a result of [Sz] by showing that for comonoidal $\mathcal{A}$ the self–intertwiner algebra of the "trivial" $\mathcal{A}$-comodule $\mathcal{A}_R$ is given by $\mathcal{A}_L \cap \mathcal{A}_R$.

As usual, by a right $\mathcal{A}$-comodule we mean a linear space V together with a coaction $\rho_V : V \to V \otimes \mathcal{A}$ satisfying

$$(\rho_V \otimes \mathrm{id}_{\mathcal{A}}) \circ \rho_V = (\mathrm{id}_V \otimes \Delta) \circ \rho_V \tag{4.18}$$
$$(\mathrm{id}_V \otimes \varepsilon) \circ \rho_V = \mathrm{id}_V. \tag{4.19}$$

For $v \in V$ we also use the shorthand notation $\rho_V(v) \equiv v_{(0)} \otimes v_{(1)}$, omitting as usual summation indices and a summation symbol. As for ordinary finite dimensional bialgebras, we recall the one-to-one correspondence between right $\mathcal{A}$-comodules and left $\hat{\mathcal{A}}$-modules given by

$$\phi \triangleright v := (\mathrm{id}_V \otimes \phi)(\rho_V(v)) \equiv v_{(0)} \langle \phi \mid v_{(1)} \rangle, \quad v \in V,\ \phi \in \hat{\mathcal{A}}. \tag{4.20}$$

Based on this observation we get the following

Proposition 4.1 *Let $(\mathcal{A}, \mathbf{1}, \Delta, \varepsilon)$ be a comonoidal weak bialgebra. Then any right $\mathcal{A}$-comodule V naturally becomes an $\mathcal{A}_R$-bimodule via*

$$a_R \cdot v := v_{(0)} \varepsilon(a_R v_{(1)}) \equiv \varepsilon_L(a_R) \triangleright v \tag{4.21}$$
$$v \cdot a_R := v_{(0)} \varepsilon(v_{(1)} a_R) \equiv \varepsilon_R(a_R) \triangleright v, \tag{4.22}$$

where $a_R \in \mathcal{A}_R$ and $v \in V$. Moreover, with respect to this biaction we have for all $a_R \in \mathcal{A}_R$, $v \in V$

$$\rho_V(a_R \cdot v) = \Delta(a_R) \cdot \rho_V(v) \tag{4.23}$$
$$\rho_V(v \cdot a_R) = \rho_V(v) \cdot \Delta(a_R) \tag{4.24}$$
$$\varepsilon_{RR}(v_{(1)}) \cdot v_{(0)} = v = v_{(0)} \cdot \varepsilon_{RL}(v_{(1)}). \tag{4.25}$$

Proof: By Theorem 3.3 and Theorem 3.5 $\varepsilon_L : \mathcal{A}_R \to \mathcal{N}_{LR}(\hat{\mathcal{A}})$ is an algebra isomorphism and $\varepsilon_R : \mathcal{A}_R \to \mathcal{N}_{RR}(\hat{\mathcal{A}})$ is an algebra anti-isomorphism. Hence (4.21) and (4.22) provide a left and a right $\mathcal{A}_R$-action, respectively, on V, which commute with each other due to Corollary 3.1(iii). To prove the identities (4.23) and (4.24) first note that they make sense, since in the comonoidal case

$$\Delta(a_R) = \mathbf{1}_{(1)} \otimes \mathbf{1}_{(2)} a_R = \mathbf{1}_{(1)} \otimes a_R \mathbf{1}_{(2)} \in \mathcal{A}_R \otimes \mathcal{A}.$$

From this (4.23) follows by computing

$$\begin{aligned} \Delta(a_R) \cdot \rho_V(v) &= v_{(0)} \varepsilon(\mathbf{1}_{(1)} v_{(1)}) \otimes a_R \mathbf{1}_{(2)} v_{(2)} = v_{(0)} \otimes a_R v_{(1)} \\ &= v_{(0)} \otimes \varepsilon_L(a_R) \rightharpoonup v_{(1)} = v_{(0)} \otimes v_{(1)} \varepsilon(a_R v_{(2)}) \\ &= \rho_V(a_R \cdot v), \end{aligned}$$

where in the third equation we have used Lemma 3.2(iii) and $\mathcal{A}_R = \mathcal{N}_{RL}(\mathcal{A})$. Eq. (4.24) follows analogously from $\mathcal{A}_R = \mathcal{N}_{RR}(\mathcal{A})$ and Lemma 3.2(iv). To prove (4.25) we compute

$$\varepsilon_{RR}(v_{(1)}) \cdot v_{(0)} = v_{(0)}\varepsilon(\varepsilon_{RR}(v_{(2)})v_{(1)}) = v_{(0)}\varepsilon(v_{(1)}) = v$$

where we have used $\varepsilon_{RR}(a_{(2)})a_{(1)} = a,\ \forall a \in \mathcal{A}$, by (2.23). Similarly, using $\varepsilon_{LR}(a_{(1)})a_{(2)} = a$,

$$v_{(0)} \cdot \varepsilon_{RL}(v_{(1)}) = v_{(0)}\varepsilon(v_{(1)}\varepsilon_{RL}(v_{(2)})) = v_{(0)}\varepsilon(v_{(1)}) = v\,. \qquad \blacksquare$$

Given two right $\mathcal{A}$-comodules V, W, we define $\rho_{V\otimes W} : V \otimes W \to V \otimes W \otimes \mathcal{A}$ by

$$\rho_{V\otimes W}(v \otimes w) := \rho_V^{13}(v)\rho_W^{23}(w) \equiv v_{(0)} \otimes w_{(0)} \otimes v_{(1)}w_{(1)}. \tag{4.26}$$

One immediatetely checks, that $\rho_{V\otimes W}$ again satisfies (4.18), however it fails (4.19) unless ε is multiplicative. If $\mathcal{A}$ is bimonoidal this may be repaired by using the $\mathcal{A}_R$-bimodule property to define

$$\begin{aligned} VW &:= V \otimes_{\mathcal{A}_R} W \\ \rho_{VW}(v \otimes_{\mathcal{A}_R} w) &:= (P_{VW} \otimes \mathrm{id}_{\mathcal{A}})(\rho_{V\otimes W}(v \otimes w)), \end{aligned} \tag{4.27}$$

where $P_{VW} : V \otimes W \to VW$ is the canonical projection. Then due to (4.23) and (4.24)

$$\rho_{VW} : VW \to VW \otimes \mathcal{A}$$

is well defined and still satisfies (4.18). Moreover, we have

Lemma 4.2 *Let $(\mathcal{A}, \mathbf{1}, \Delta, \varepsilon)$ be bimonoidal and for two right $\mathcal{A}$-comodules V, W let $\rho_{VW} : VW \to VW \otimes \mathcal{A}$ be given by (4.27). Then*

$$(id_{VW} \otimes \varepsilon) \circ \rho_{VW} = id_{VW} \tag{4.28}$$

Proof: Using $\varepsilon_L = \varepsilon_L \circ \varepsilon_{RL}$ we compute

$$\begin{aligned} (\mathrm{id}_{VW} \otimes \varepsilon)(\rho_{VW}(v \otimes_{\mathcal{A}_R} w)) &= P_{VW}(v_{(0)} \otimes w_{(0)})\varepsilon(v_{(1)}w_{(1)}) \\ &= P_{VW}(v_{(0)} \otimes w_{(0)})\varepsilon(\varepsilon_{RL}(v_{(1)})w_{(1)}) \\ &= P_{VW}(v_{(0)} \otimes \varepsilon_{RL}(v_{(1)}) \cdot w) \\ &= P_{VW}(v_{(0)} \cdot \varepsilon_{RL}(v_{(1)}) \otimes w) \\ &= v \otimes_{\mathcal{A}_R} w, \end{aligned}$$

where in the last line we have used (4.25). $\blacksquare$

Thus, ρ_{VW} is again a right $\mathcal{A}$-coaction. Next, observe that any $\mathcal{A}$-comodule morphism $f : V \to V'$ (i.e. satisfying $\rho_{V'} \circ f = (f \otimes \mathrm{id}_{\mathcal{A}}) \circ \rho_V$) is also an $\mathcal{A}_R$-bimodule map and therefore the tensor product of two such maps, $f : V \to V'$ and $g : W \to W'$, naturally passes down to an $\mathcal{A}$-comodule morphism $f \otimes_{\mathcal{A}_R} g : VW \to V'W'$. In this way $\mathrm{Cmod}\,\mathcal{A}$ becomes a monoidal category with unit object given by $\mathcal{A}_R$, where $\rho_{\mathcal{A}_R} : \mathcal{A}_R \to \mathcal{A}_R \otimes \mathcal{A}$ is given by $\rho_{\mathcal{A}_R} = \Delta|_{\mathcal{A}_R}$.

Let us now see how, under the identification $\mathrm{Cmod}\,\mathcal{A} = \mathrm{Rep}\,\hat{\mathcal{A}}$, this description coincides with the tensor functor obtained by the dual version of (2.3)–(2.5). To this end we put

$$V \times W := (\mathrm{id}_V \otimes \mathrm{id}_W \otimes \varepsilon)(\rho_{V\otimes W}(V \otimes W)) \subset V \otimes W \tag{4.29}$$

as in (2.3) and correspondingly

$$\rho_{V\times W} := \rho_{V\otimes W}|_{V\times W}. \tag{4.30}$$

Then $\rho_{V\times W} : V \times W \to (V \times W) \otimes \mathcal{A}$ is a well defined coaction satisfying (4.18) <u>and</u> (4.19) and we have

Lemma 4.3 *Under the conditions of Lemma 4.2 the restriction $P_{VW}|_{V\times W} : V \times W \to VW$ provides an isomorphism of $\mathcal{A}$-comodules.*

Proof: By the definitions (4.27) and (4.30) $P_{VW}|_{V\times W}$ is a comodule morphism, which by Lemma 4.2 is surjective. To prove that it is also injective we just have to note that according to the proof of Lemma 4.2 its inverse is given by

$$VW \ni v \otimes_{\mathcal{A}_R} w \mapsto v_{(0)} \otimes w_{(0)} \varepsilon(v_{(1)} w_{(1)}) \in V \times W,$$

which is indeed well defined due to (4.23) and (4.24). ■

We conclude this Section with picking up an observation of [Sz], who has noticed that in the weak Hopf algebra setting $\mathcal{A}$ is pure (i.e. the "trivial" $\mathcal{A}$-module $\mathcal{E} \equiv \hat{\mathcal{A}}_R$ is irreducible), if and only if $\mathcal{C}_L(\mathcal{A}) = \mathcal{C}_R(\mathcal{A}) = \mathbf{C}$. More generally, the commuant of $\pi_\varepsilon(\mathcal{A})$ in $\mathrm{End}_K\, \mathcal{E}$ is given by $\pi_\varepsilon(\mathcal{C}_\sigma(\mathcal{A}))$ [Sz]. It turns out, that this already holds in our setting of monoidal weak bialgebras. To see this we prove a dual statement in $\mathrm{Cmod}\,\mathcal{A}$. First, we need

Lemma 4.4 *Let $(\mathcal{A}, \mathbf{1}, \Delta, \varepsilon)$ be a weak bialgebra. Then $\Delta(\mathbf{1}) \in (\mathcal{A}_R \otimes \mathcal{A}) \cap (\mathcal{A} \otimes \mathcal{A}_L)$. If $\mathcal{A}$ is left- or right-comonoidal, then $\Delta(\mathbf{1}) \in \mathcal{A}_R \otimes \mathcal{A}_L$.*

Proof: Pick a basis $e_i \in \mathcal{A}$ with dual basis $e^i \in \hat{\mathcal{A}}$. Then $\Delta(\mathbf{1}) = e^i \rightharpoonup \mathbf{1} \otimes e_i = e_i \otimes \mathbf{1} \leftharpoonup e^i \in (\mathcal{A}_R \otimes \mathcal{A}) \cap (\mathcal{A} \otimes \mathcal{A}_L)$. If $\mathcal{A}$ is left-comonoidal, then

$$\Delta(\mathbf{1}) = \mathbf{1}_{(1)} \otimes \varepsilon(\mathbf{1}_{(2)} \mathbf{1}_{(1')}) \mathbf{1}_{(2')} = \hat{\mathbf{1}}_{(1)} \rightharpoonup \mathbf{1} \otimes \mathbf{1} \leftharpoonup \hat{\mathbf{1}}_{(2)} \in \mathcal{A}_R \otimes \mathcal{A}_L$$

The argument for right-comonoidal $\mathcal{A}$ is similar. ■

Let us now denote the intertwiner spaces in $\mathrm{Cmod}\,\mathcal{A}$ by

$$\mathrm{End}^{\mathcal{A}}\, V := \{T \in \mathrm{End}_K\, V \mid \rho_V \circ T = (T \otimes \mathrm{id}\,) \circ \rho_V\}$$

Recall that if $\mathcal{A}$ is comonoidal then $[\mathcal{A}_L, \mathcal{A}_R] = 0$ and $\mathcal{A}_R$ is the unit object in $\mathrm{Cmod}\,\mathcal{A}$, where $\rho_{\mathcal{A}_R} = \Delta|_{\mathcal{A}_R}$.

Lemma 4.5 *Let $(\mathcal{A}, \mathbf{1}, \Delta, \varepsilon)$ be a comonoidal weak bialgebra and $T \in \mathrm{End}_K\, \mathcal{A}_R$. Then $T \in \mathrm{End}^{\mathcal{A}}\, \mathcal{A}_R$ if and only if there exists $z \in \mathcal{A}_L \cap \mathcal{A}_R$ such that $T(a) = az \equiv za$, $\forall a \in \mathcal{A}_R$.*

Proof: We have $T \in \mathrm{End}^{\mathcal{A}}\, \mathcal{A}_R \Leftrightarrow \Delta(T(a)) = (T \otimes \mathrm{id}\,)(\Delta(a))$, $\forall a \in \mathcal{A}_R$. Putting $a = \mathbf{1}$, applying $\varepsilon \otimes \mathrm{id}$ and using Lemma 4.4 we conclude

$$z := T(\mathbf{1}) = \varepsilon(T(\mathbf{1}_{(1)}))\mathbf{1}_{(2)} \in \mathcal{A}_L \cap \mathcal{A}_R.$$

Now comonoidality implies $\mathcal{A}_R = \mathcal{N}_{RL}(\mathcal{A})$ and therefore $\Delta(a) = (\mathbf{1} \otimes a)\Delta(\mathbf{1})$, $\forall a \in \mathcal{A}_R$. Hence, $T(a) = (\varepsilon \otimes \mathrm{id}\,)(\Delta(T(a))) = \varepsilon(T(\mathbf{1}_{(1)}))a\mathbf{1}_{(2)} = az$, $\forall a \in \mathcal{A}_R$. Conversely, if $z \in \mathcal{A}_L \cap \mathcal{A}_R$ then $\Delta(az) = \Delta(a)(z \otimes \mathbf{1})$, proving that the map $T : a \mapsto az$ is in $\mathrm{End}^{\mathcal{A}}\, \mathcal{A}_R$. ■

Lemma 4.5 may now immediately be dualized. For left $\mathcal{A}$-modules V denote $\mathrm{End}_{\mathcal{A}}\, V$ the space of $\mathcal{A}$-linear endomorphisms of V. The following generalizes [Sz, Eq.(3.3)] to our setting.

Proposition 4.6 *Let $(\mathcal{A}, \mathbf{1}, \Delta, \varepsilon)$ be monoidal and denote $\pi_\varepsilon : \mathcal{A} \to End_K \mathcal{E}$, $\mathcal{E} \equiv \hat{\mathcal{A}}_R$, the "trivial" representation, $\pi_\varepsilon(a)\phi := a \rightharpoonup \phi$, $a \in \mathcal{A}$, $\phi \in \mathcal{E}$. Then*

$$End_{\mathcal{A}} \mathcal{E} = \pi_\varepsilon(\mathcal{C}_R(\mathcal{A})) = \pi_\varepsilon(\mathcal{C}_L(\mathcal{A})) \cong \mathcal{C}_{R/L}(\mathcal{A}) \cong \hat{\mathcal{A}}_L \cap \hat{\mathcal{A}}_R$$

Proof: By the dual of Lemma 4.5 $T \in \mathrm{End}_{\mathcal{A}} \mathcal{E}$ iff there exists $\xi \in \hat{\mathcal{A}}_L \cap \hat{\mathcal{A}}_R$ such that $T(\phi) = \phi\xi \equiv \xi\phi$, $\forall \phi \in \mathcal{E}$. Using $\mathcal{E} = \mathcal{N}_{RR}(\hat{\mathcal{A}}) = \mathcal{N}_{RL}(\hat{\mathcal{A}})$ by Theorem 3.5 we conclude

$$T(\phi) = \hat{\varepsilon}_L(\xi) \rightharpoonup \phi = \hat{\varepsilon}_R(\xi) \rightharpoonup \phi, \quad \forall \phi \in \mathcal{E}$$

from the dual versions of Lemma 3.2(iii) and (iv). Hence, the claim follows since by the dual of (3.9) $\hat{\varepsilon}_\sigma(\hat{\mathcal{A}}_L \cap \hat{\mathcal{A}}_R) = \mathcal{C}_\sigma(\mathcal{A})$ and since by Theorem 3.3(i) the restriction of π_ε to $\mathcal{N}_{\sigma R}(\mathcal{A})$ – and therefore to $\mathcal{C}_\sigma(\mathcal{A}) \equiv \mathcal{N}_{\sigma R}(\mathcal{A}) \cap \mathcal{C}(\mathcal{A})$ – is faithful. ■

Proposition 4.6 in particular implies $\pi_\varepsilon(\mathcal{A}) \subset \mathrm{End}_{\hat{\mathcal{A}}_L \cap \hat{\mathcal{A}}_R} \mathcal{E}$. In Corollary C4i) of Appendix C we will see that equality holds if and only if $\hat{\mathcal{A}}_L \otimes_{\hat{\mathcal{A}}_L \cap \hat{\mathcal{A}}_R} \hat{\mathcal{A}}_R \cong \hat{\mathcal{A}}_L \hat{\mathcal{A}}_R$ as a subalgebra of $\hat{\mathcal{A}}$.

5 Bimonoidal Weak Bialgebras and Face Algebras

In this Section we generalize an observation of [Sz] by showing that in bimonoidal weak bialgebras $\mathcal{A}$ the subalgebras $\mathcal{A}_{L/R}$ are separable. This will also prove that the bimonoidal weak bialgebras with abelian $\mathcal{A}_{L/R}$ are precisely the face algebras of [Ha]. To this end let us introduce the maps $S_\sigma : \mathcal{A}_\sigma \to \mathcal{A}_{-\sigma}$ and $\bar{S}_\sigma : \mathcal{A}_\sigma \to \mathcal{A}_{-\sigma}$ given by

$$\begin{array}{rclcrcl} S_L & := & \hat{\varepsilon}_R \circ \varepsilon_L|_{\mathcal{A}_L} & \qquad & S_R & := & \hat{\varepsilon}_L \circ \varepsilon_R|_{\mathcal{A}_R} \\ \bar{S}_L & := & \hat{\varepsilon}_R \circ \varepsilon_R|_{\mathcal{A}_L} & \qquad & \bar{S}_R & := & \hat{\varepsilon}_L \circ \varepsilon_L|_{\mathcal{A}_R} . \end{array} \tag{5.1}$$

By Theorem 3.3i) and Theorem 3.5, if $\mathcal{A}$ is comonoidal these maps are algebra anti-isomorphisms and $\bar{S}_{L/R} = S^{-1}_{R/L}$. We will see in Corollary 8.4 that if $\mathcal{A}$ is a weak Hopf algebra with antipode S then $S_{L/R} = S|_{\mathcal{A}_{L/R}}$.

Lemma 5.1 *Let $\mathcal{A}$ be monoidal or comonoidal and let $a_L, b_L \in \mathcal{A}_L$ and $a_R, b_R \in \mathcal{A}_R$. Then*

$$\varepsilon(a_L b_L) = \varepsilon(S_L(a_L) b_L) = \varepsilon(a_L \bar{S}_L(b_L)) \tag{5.2}$$

$$\varepsilon(a_R b_R) = \varepsilon(\bar{S}_R(a_R) b_R) = \varepsilon(a_R S_R(b_R)) \tag{5.3}$$

Proof: If $\mathcal{A}$ is monoidal use Lemma 2.7. If $\mathcal{A}$ is comonoidal use the counit property and $\mathcal{A}_\sigma = \mathcal{N}_{\sigma\sigma'}(\mathcal{A})$ to get for $\sigma = L$ or $\sigma = R$

$$\varepsilon(a_\sigma b_\sigma) = \varepsilon(a_\sigma \mathbf{1}_{(1)})\varepsilon(\mathbf{1}_{(2)} b_\sigma) = \varepsilon(a_\sigma \mathbf{1}_{(2)})\varepsilon(\mathbf{1}_{(1)} b_\sigma)$$

from which the statements follow by the formulas (2.22). ■

Next, given a nondegenerate functional $\omega : \mathcal{M} \to K$ on a finite dimensional algebra $\mathcal{M}$ let $\sum x_i \otimes y_i \in \mathcal{M} \otimes \mathcal{M}$ denote the form-inverse of $\mathcal{M} \otimes \mathcal{M} \ni (m_1 \otimes m_2) \mapsto \omega(m_1 m_2) \in K$, i.e. the unique solution of (summation convention)

$$\omega(m x_i) y_i = m = x_i \omega(m y_i), \quad \forall m \in \mathcal{M} . \tag{5.4}$$

Note that this implies the identity

$$mx_i \otimes y_i = x_i \otimes y_i m, \quad \forall m \in \mathcal{M}. \tag{5.5}$$

In the terminology of Watatani [Wa] the collection $\{(x_i, y_i)\}$ would be called a "quasi-basis" for ω. Generalizing the index notion for conditional expectations we denote

$$\mathrm{Ind}\,\omega := x_i y_i \in \mathcal{C}(\mathcal{M}) \tag{5.6}$$

and call this the *Index* of ω. Also recall that for a finite dimensional Frobenius algebra $\mathcal{M}$ over a field K the *modular automorphism* of a non-degenerate functional $\omega \in \hat{\mathcal{M}}$ is defined to be the unique $\theta_\omega \in \mathrm{Aut}\,\mathcal{M}$ such that

$$f(xy) \;=\; f(y\,\theta_\omega(x))\ , \qquad \forall x, y, \in \mathcal{M}\ . \tag{5.7}$$

Proposition 5.2 [BNS] *Let $\mathcal{A}$ be a bimonoidal weak bialgebra. Then for $\sigma = L$ and $\sigma = R$*
i) $\varepsilon|_{\mathcal{A}_\sigma}$ is nondegenerate and $\mathrm{Ind}\,\varepsilon|_{\mathcal{A}_\sigma} = \mathbf{1}$.
ii) The quasi-basis $x^i_\sigma \otimes y^i_\sigma \in \mathcal{A}_\sigma \otimes \mathcal{A}_\sigma$ of $\varepsilon|_{\mathcal{A}_\sigma}$ is given by

$$x^i_L \otimes y^i_L = S_R(\mathbf{1}_{(1)}) \otimes \mathbf{1}_{(2)} \qquad\qquad x^i_R \otimes y^i_R = \mathbf{1}_{(1)} \otimes S_L(\mathbf{1}_{(2)}) \tag{5.8}$$

iii) The modular automorphisms of $\varepsilon|_{\mathcal{A}_\sigma}$ are given by $S_R \circ S_L \in \mathrm{Aut}\,\mathcal{A}_L$ and $\bar{S}_L \circ \bar{S}_R \in \mathrm{Aut}\,\mathcal{A}_R$.
iv) $\mathcal{A}_L$ and $\mathcal{A}_R$ are separable K-algebras, whence semi-simple.

Proof: $\varepsilon|_{\mathcal{A}_\sigma}$ is nondegenerate by Corollary 2.8. To prove (ii) we use $\Delta(\mathbf{1}) \in \mathcal{A}_R \otimes \mathcal{A}_L$ to compute from Lemma 5.1

$$\begin{array}{rclclcl} \varepsilon(a_L S_R(\mathbf{1}_{(1)}))\mathbf{1}_{(2)} &=& \varepsilon(a_L(\bar{S}_L \circ S_R)(\mathbf{1}_{(1)}))\mathbf{1}_{(2)} &=& \varepsilon(a_L \mathbf{1}_{(1)})\mathbf{1}_{(2)} &=& a_L \\ \mathbf{1}_{(1)}\varepsilon(S_L(\mathbf{1}_{(2)})a_R) &=& \mathbf{1}_{(1)}\varepsilon((\bar{S}_R \circ S_L)(\mathbf{1}_{(2)})a_R) &=& \mathbf{1}_{(1)}\varepsilon(\mathbf{1}_{(2)}a_R) &=& a_R \end{array} \tag{5.9}$$

for all $a_L \in \mathcal{A}_L$ and $a_R \in \mathcal{A}_R$. This proves (ii). Since by (2.23) and the definitions (5.1) $S_R(\mathbf{1}_{(1)})\mathbf{1}_{(2)} = \mathbf{1}_{(1)}S_L(\mathbf{1}_{(2)}) = \mathbf{1}$, we also conclude $\mathrm{Ind}\,\varepsilon|_{\mathcal{A}_\sigma} = \mathbf{1}$. Hence, by (5.5), $x^i_\sigma \otimes y^i_\sigma \in \mathcal{A}_\sigma \otimes (\mathcal{A}_\sigma)_{op}$ provides a separating idempotent, proving part (iv). Finally, part (iii) also follows from Lemma 5.1, since $\mathcal{A}_L$ and $\mathcal{A}_R$ commute. ∎

Proposition 5.2 in particular implies that for abelian $\mathcal{A}_{L/R}$ our bimonoidal weak bialgebras reproduce the face algebras in the sense of T. Hayashi [Ha].

6 Rigid Weak Bialgebras

In this Section, adapting ideas of Drinfeld [Dr] for quasi-Hopf algebras, we propose axioms for a so-called *rigidity structure* on a monoidal weak bialgebra $\mathcal{A}$, such that $\mathrm{Rep}\,\mathcal{A}$ becomes a rigid monoidal category. In the sequel this will also motivate our antipode axioms in Section 7. Unless noted differently, throughout this Section we suppose $\mathcal{A}$ to be monoidal.

Let us recall from Corollary 2.9 that the unit representation π_ε of a monoidal weak bialgebra $\mathcal{A}$ may also be realized on $\mathcal{A}_{LR} \equiv \varepsilon_{LR}(\mathcal{A})$, considered as an $\mathcal{A}$-module via

$$\pi_{LR}(a)b := \varepsilon_{LR}(ab), \quad a \in \mathcal{A},\, b \in \mathcal{A}_{LR}. \tag{6.1}$$

The equivalence $\pi_{LR} \cong \pi_\varepsilon$ follows from (2.43) and (2.44). In this way, in this Section we identify $\mathcal{E} \equiv \mathcal{A}_{LR}$. Moreover, by Corollary 2.8 we may identify $\mathcal{A}_{RL} \equiv \hat{\mathcal{E}}$ as the dual of $\mathcal{E}$, with nondegenerate pairing given by

$$\hat{\mathcal{E}} \otimes \mathcal{E} \ni (a \otimes b) \mapsto \varepsilon(ab) \in K\,, \tag{6.2}$$

Also recall that in the monoidal case $\mathcal{A}_{\sigma\sigma'} = \mathcal{N}_{\sigma\sigma'}(\mathcal{A})$. With the present identifications the morphisms $L_V : \mathcal{E} \times V \to V$ and $R_V : V \to V \times \mathcal{E}$ in $\mathrm{Rep}\,\mathcal{A}$ introduced in (2.10) and (2.11) with inverses (2.16) and (2.17) now take the form

$$L_V(v) = \varepsilon_{LR}(\mathbf{1}_{(1)}) \otimes \mathbf{1}_{(2)} \cdot v \qquad , \qquad R_V(v) = \mathbf{1}_{(1)} \cdot v \otimes \varepsilon_{LR}(\mathbf{1}_{(2)}) \tag{6.3}$$

$$L_V^{-1}(a \otimes v) = \varepsilon_{LR}(a) \cdot v \equiv a \cdot v \qquad , \qquad R_V^{-1}(v \otimes a) = \varepsilon_{RR}(a) \cdot v \tag{6.4}$$

where $v \in V$ and $a \in \mathcal{A}_{LR}$. After these identifications we are prepared to give the

Definition 6.1 i) A *pre-rigidity structure* $(S, \mathbf{A}, \mathbf{B})$ on a monoidal weak bialgebra $\mathcal{A}$ consists of an anti-algebra map $S : \mathcal{A} \to \mathcal{A}$ and elements $\mathbf{A} \in \mathcal{A} \otimes \mathcal{E}$ and $\mathbf{B} \in \hat{\mathcal{E}} \otimes \mathcal{A}$ satisfying for all $a \in \mathcal{A}$

$$S(a_{(1)})A^i a_{(2)} \otimes e^i = A^i \otimes \varepsilon_{LR}(ae^i) \tag{6.5}$$

$$\hat{e}^i \otimes a_{(1)}B^i S(a_{(2)}) = \varepsilon_{RL}(\hat{e}^i a) \otimes B^i \tag{6.6}$$

where $\mathbf{A} \equiv A^i \otimes e^i$ and $\mathbf{B} \equiv \hat{e}^i \otimes B^i$ and where summations over i are understood.
ii) A pre-rigidity structure $(S, \mathbf{A}, \mathbf{B})$ is called a *rigidity structure* if in addition the elements $\alpha := (id \otimes \varepsilon)(\mathbf{A})$ and $\beta := (\varepsilon \otimes id)(\mathbf{B})$ satisfy

$$\mathbf{1}_{(1)}\beta S(\mathbf{1}_{(2)})\alpha\mathbf{1}_{(3)} = \mathbf{1} \tag{6.7}$$

$$S(\mathbf{1}_{(1)})\alpha\mathbf{1}_{(2)}\beta S(\mathbf{1}_{(3)}) = S(\mathbf{1}) \tag{6.8}$$

We also call $(\mathcal{A}, \mathbf{1}, \Delta, \varepsilon, S, \mathbf{A}, \mathbf{B})$ a *rigid weak bialgebra*. We point out that these axioms are somewhat reminiscent of – and also motivated by – Drinfel'd's antipode axioms for quasi-Hopf algebras [Dr]. Also, one should maybe call this a left rigidity structure, and one may similarly define a right rigidity structure on $\mathcal{A}$ as a left rigidity structure on $\mathcal{A}_{op}$. Note that if $(S, \mathbf{A}, \mathbf{B})$ is a rigidity structure on $\mathcal{A}$ then $(S, \mathbf{B}_{op}, \mathbf{A}_{op})$ is a rigidity structure on $\mathcal{A}_{op}^{cop}$.

Given a pre-rigidity structure $(S, \mathbf{A}, \mathbf{B})$ on $\mathcal{A}$ one obtains on $\mathrm{Rep}\,\mathcal{A}$ a contravariant conjugation functor $V \to \bar{V}$ as follows. Let V be a left $\mathcal{A}$-module with dual right $\mathcal{A}$-module $\hat{V}$ and define $\bar{V} := \hat{V} \cdot S(\mathbf{1})$. Then $\bar{V}$ becomes a left $\mathcal{A}$-module via

$$a \cdot u := u \cdot S(a), \quad u \in \bar{V},\, a \in \mathcal{A}.$$

In this way the assignment $V \to \bar{V}$ provides a contravariant conjugation functor in $\mathrm{Rep}\,\mathcal{A}$, where for $\mathcal{A}$-linear morphisms $T : V \to W$ we put $\bar{T} := T^t|_{\bar{W}} : \bar{W} \to \bar{V}$, T^t being the transpose of T. The terminology "conjugation" is justified by the following Lemma, where for left $\mathcal{A}$-modules V we also use the notation $\pi_V(a)v \equiv a \cdot v$, $a \in \mathcal{A}$, $v \in V$, where $\pi_V : \mathcal{A} \to \mathrm{End}_K V$ denotes the representation homomorphism.

Lemma 6.2 *A pre-rigidity structure $(S, \mathbf{A}, \mathbf{B})$ on $\mathcal{A}$ provides in $\mathrm{Rep}\,\mathcal{A}$ a family of $\mathcal{A}$-linear morphisms $A_V : \bar{V} \times V \to \mathcal{E}$ and $B_V : \mathcal{E} \to V \times \bar{V}$ given by*

$$A_V(u \otimes v) := \langle u \mid A^i \cdot v\rangle e^i \equiv \langle u \mid S(\mathbf{1}_{(1)})\alpha\mathbf{1}_{(2)} \cdot v\rangle \mathbf{1}_{(3)} \in \mathcal{A}_{LR} \equiv \mathcal{E} \tag{6.9}$$

$$B_V(a) := \varepsilon(\hat{e}^j a)\pi_V(B^j) \equiv \pi_V(a_{(1)}\beta S(a_{(2)})),\; a \in \mathcal{A}_{LR} \equiv \mathcal{E}, \tag{6.10}$$

where $\alpha := (id \otimes \varepsilon)(\mathbf{A})$, $\beta := (\varepsilon \otimes id)(\mathbf{B})$, and where in (6.10) we have identified $V \otimes \hat{V} \cong \mathrm{End}_K V$.

Proof: The second identity in (6.10) follows from (6.6) and $\varepsilon = \varepsilon \circ \varepsilon_{RL}$ and the second identity in (6.9) follows from Lemma 6.4 below. The fact that A_V and B_V are $\mathcal{A}$-linear follows immediately from (6.5), (6.6) and the identities $\varepsilon(abc) = \varepsilon(\varepsilon_{RL}(ab)c) = \varepsilon(a\varepsilon_{LR}(bc))$ for all $a, b, c \in \mathcal{A}$, see Lemma 2.7. ■

In order that the family of morphisms A_V, B_V indeed provides a rigidity structure on Rep $\mathcal{A}$ we also need the axioms (6.7) and (6.8).

Proposition 6.3 *A pre-rigidity structure $(S, \mathbf{A}, \mathbf{B})$ on $\mathcal{A}$ is rigid, if and only if, under the setting of Lemma 6.2, we have for all V in Rep $\mathcal{A}$ the* **rigidity identities**

$$R_V^{-1} \circ (\mathbf{1}_V \times A_V) \circ (B_V \times \mathbf{1}_V) \circ L_V = \mathbf{1}_V \tag{6.11}$$
$$L_{\bar V}^{-1} \circ (A_V \times \mathbf{1}_{\bar V}) \circ (\mathbf{1}_{\bar V} \times B_V) \circ R_{\bar V} = \mathbf{1}_{\bar V} \ . \tag{6.12}$$

Proof: Using (6.10), (6.3) and $\varepsilon_R \circ \varepsilon_{LR} = \varepsilon_R$ and identifying $V \otimes \hat V \cong \mathrm{End}_K V$ we have for $v \in V$

$$(B_V \times \mathbf{1}_V)(L_V(v)) = \pi_V(\mathbf{1}_{(1)}\beta S(\mathbf{1}_{(2)})) \otimes \mathbf{1}_{(3)} \cdot v.$$

Similarly, for $v, w \in V$ and $u \in \bar V$ we get

$$[R_V^{-1} \circ (\mathbf{1}_V \otimes A_V)](v \otimes u \otimes w) = \varepsilon(\mathbf{1}_{(2)}e^j)\langle u \mid A^j \cdot w\rangle \mathbf{1}_{(1)} \cdot v = \langle u \mid S(\mathbf{1}_{(2)})\alpha\mathbf{1}_{(3)} \cdot w\rangle \mathbf{1}_{(1)} \cdot v$$

by (6.4) and (6.5). Hence

$$\begin{aligned} [R_V^{-1} \circ (\mathbf{1}_V \times A_V) \circ (B_V \times \mathbf{1}_V) \circ L_V](v) &= \mathbf{1}_{(1)}(\mathbf{1}_{(1')}\beta S(\mathbf{1}_{(2')}))(S(\mathbf{1}_{(2)})\alpha\mathbf{1}_{(3)})\mathbf{1}_{(3')} \cdot v \\ &= \mathbf{1}_{(1)}\beta S(\mathbf{1}_{(2)})\alpha\mathbf{1}_{(3)} \cdot v = v. \end{aligned}$$

by (6.7). Using (6.8) the identity (6.12) follows similarly. Putting $V = \mathcal{A}$ we see that the axioms (6.7) and (6.8) are also necessary. ■

We leave it to the reader to check that for $\mathcal{A}$-linear morphisms $T : V \to W$ and $\bar T \equiv T^t|_{\bar W} : \bar W \to \bar V$ the definitions (6.9) and (6.10) imply

$$\begin{aligned} \bar T &= L_{\bar V}^{-1} \circ (A_V \times \mathbf{1}_{\bar V}) \circ (\mathbf{1}_{\bar V} \times T \times \mathbf{1}_{\bar V}) \circ (\mathbf{1}_{\bar V} \times B_V) \circ R_{\bar V} \ . \\ T &= R_V^{-1} \circ (\mathbf{1}_V \times A_V) \circ (\mathbf{1}_V \times \bar T \times \mathbf{1}_V) \circ (B_V \times \mathbf{1}_V) \circ L_V \end{aligned}$$

expressing the standard isomorphism $\mathrm{Hom}_{\mathcal{A}}(V, W) \cong \mathrm{Hom}_{\mathcal{A}}(\bar W, \bar V)$ in rigid monoidal categories.

Next, we point out that for any (pre)rigidity structure $(S, \mathbf{A}, \mathbf{B})$ on $\mathcal{A}$ the elements $\mathbf{A}$ and $\mathbf{B}$ are already uniquely determined by $\alpha \equiv (\mathrm{id} \otimes \varepsilon)(\mathbf{A})$ and $\beta \equiv (\varepsilon \otimes \mathrm{id})(\mathbf{B})$.

Lemma 6.4 *Let $(S, \mathbf{A}, \mathbf{B})$ be a pre-rigidity structure on $\mathcal{A}$. Then*

$$\mathbf{A} = S(\mathbf{1}_{(1)})\alpha\mathbf{1}_{(2)} \otimes \mathbf{1}_{(3)} \tag{6.13}$$
$$\mathbf{B} = \mathbf{1}_{(1)} \otimes \mathbf{1}_{(2)}\beta S(\mathbf{1}_{(3)}) \tag{6.14}$$

Proof: Using $\varepsilon = \varepsilon \circ \varepsilon_{LR} = \varepsilon \circ \varepsilon_{RL}$ we compute

$$\begin{aligned} \mathbf{A} &= (id \otimes \varepsilon_{LR})(\mathbf{A}) = A^i \otimes \mathbf{1}_{(2)}\varepsilon(\mathbf{1}_{(1)}e^i) = S(\mathbf{1}_{(1)})\alpha\mathbf{1}_{(2)} \otimes \mathbf{1}_{(3)} \\ \mathbf{B} &= (\varepsilon_{RL} \otimes id)(\mathbf{B}) = \mathbf{1}_{(1)}\varepsilon(\hat e^i\mathbf{1}_{(2)}) \otimes B^i = \mathbf{1}_{(1)} \otimes \mathbf{1}_{(2)}\beta S(\mathbf{1}_{(3)}) \quad ■ \end{aligned}$$

Lemma 6.4 implies that the axioms for rigidity structures $(S, \mathbf{A}, \mathbf{B})$ may equivalently be reformulated in terms of the data (S, α, β). To this end, for $x \in \mathcal{A}$ and $S : \mathcal{A} \to \mathcal{A}$ an anti-algebra map let us introduce the left and right "S-adjoint" actions

$$x \triangleleft_S a := S(a_{(1)}) x a_{(2)} \quad , \quad a \,{}_S\!\triangleright x := a_{(1)} x S(a_{(2)}) \tag{6.15}$$

Proposition 6.5 *Let $S : \mathcal{A} \to \mathcal{A}$ be an anti-algebra map and let $\alpha, \beta \in \mathcal{A}$ satisfy $\alpha \triangleleft_S \mathbf{1} = \alpha$ and $\mathbf{1} \,{}_S\!\triangleright \beta = \beta$. Put $\mathbf{A} := \alpha \triangleleft_S \mathbf{1}_{(1)} \otimes \mathbf{1}_{(2)}$ and $\mathbf{B} := \mathbf{1}_{(1)} \otimes \mathbf{1}_{(2)} \,{}_S\!\triangleright \beta$ as in (6.13) and (6.14). Then $(S, \mathbf{A}, \mathbf{B})$ provides a pre-rigidity structure on $\mathcal{A}$ if and only if for all $a \in \mathcal{A}$*

$$\alpha \triangleleft_S a = \alpha \triangleleft_S \varepsilon_{RL}(a) \quad , \quad a \,{}_S\!\triangleright \beta = \varepsilon_{LR}(a) \,{}_S\!\triangleright \beta \tag{6.16}$$

If in addition α or β are invertible, then $S(\mathbf{1}) = \mathbf{1}$.

Proof: By Definition 6.1, $(S, \mathbf{A}, \mathbf{B})$ provides a pre-rigidity structure if and only if for all $a \in \mathcal{A}$

$$\alpha \triangleleft_S (\mathbf{1}_{(1)} a) \otimes \mathbf{1}_{(2)} = \alpha \triangleleft_S \mathbf{1}_{(1)} \otimes \varepsilon_{LR}(a \mathbf{1}_{(2)}) \tag{6.17}$$

$$\mathbf{1}_{(1)} \otimes a \mathbf{1}_{(2)} \,{}_S\!\triangleright \beta = \varepsilon_{RL}(\mathbf{1}_{(1)} a) \otimes \mathbf{1}_{(2)} \,{}_S\!\triangleright \beta \,. \tag{6.18}$$

Applying $id \otimes \varepsilon$ and $\varepsilon \otimes id$, respectively, yields (6.16). Conversely, using the identity

$$\varepsilon_{RL}(\mathbf{1}_{(1)} a) \otimes \mathbf{1}_{(2)} = \mathbf{1}_{(1')} \varepsilon(\mathbf{1}_{(1)} a \mathbf{1}_{(2')}) \otimes \mathbf{1}_{(2)} = \mathbf{1}_{(1)} \otimes \varepsilon_{LR}(a \mathbf{1}_{(2)}), \tag{6.19}$$

Eq. (6.16) implies (6.17) and (6.18). Finally, we have $\alpha = S(\mathbf{1}_{(1)}) \alpha \mathbf{1}_{(2)} = S(\mathbf{1}) \alpha$ and similarly $\beta = \beta S(\mathbf{1})$. Hence, if α or β are invertible, then $S(\mathbf{1}) = \mathbf{1}$. ■

Note that the normalization conditions $\alpha \triangleleft_S \mathbf{1} = \alpha$ and $\mathbf{1} \,{}_S\!\triangleright \beta = \beta$ in Proposition 6.5 are imposed to reproduce the original identities $\alpha = (\mathrm{id} \otimes \varepsilon)(\mathbf{A})$ and $\beta = (\varepsilon \otimes \mathrm{id})(\mathbf{B})$. In view of Lemma 6.4 and Proposition 6.5 we will from now on equivalently talk of (pre)rigidity structures on $\mathcal{A}$ given by the data (S, α, β).

Next, we recall from [Dr] that there is a natural notion of twist equivalence for rigidity structures (S, α, β). Let $u, \bar{u} \in \mathcal{A}$ satisfy

$$\bar{u} u = S(\mathbf{1}) \quad , \quad u \bar{u} u = u \quad , \quad \bar{u} u \bar{u} = \bar{u}, \tag{6.20}$$

and put

$$S'(a) := u S(a) \bar{u}, \quad , \quad \alpha' := u \alpha \quad , \quad \beta' := \beta \bar{u} \,. \tag{6.21}$$

Then Proposition 6.5 assures that (S', α', β') again provides a rigidity structure. The inverse transformation is given by interchanging u and $\bar{u}$. One also checks that this indeed provides an equivalence relation. In Proposition B3 of Appendix B we will show that any two rigidity structures on a monidal weak bialgebra are twist equivalent in this sense.

In ordinary bialgebras $S : \mathcal{A} \to \mathcal{A}$ is an antipode if and only if $(S, \alpha \equiv \mathbf{1}, \beta \equiv \mathbf{1})$ is a rigidity structure. Thus, to approach and motivate our antipode axioms in Sect. 7, we now study rigidity structures satisfying $\alpha = \beta = \mathbf{1}$.

Definition 6.6 A (pre-)rigidity structure (S, α, β) is called *normalizable*, if $\alpha = \beta^{-1}$, and it is called *normal*, if $\alpha = \beta = \mathbf{1}$. In this case $S : \mathcal{A} \to \mathcal{A}$ is called a *normal rigidity map*.

In Example 2 of Appendix D we will construct a rigid weak bialgebra whose rigidity structure is *not* normalizable.

Clearly, a rigidity structure is normalizable, if and only if it can be twisted into a normal one. Thus, by Proposition B3 a normal rigidity map S on a monoidal weak bialgebra is uniquely determined, provided it exists. We refrain from calling such an S an antipode, since in general the dual $\hat{S} \equiv S^t : \hat{\mathcal{A}} \to \hat{\mathcal{A}}$ will not be of the same type. Our antipode axioms in Sect. 7 will be symmetric under the duality flip $S \leftrightarrow \hat{S}$. We will see in Sect. 8 (Corollary 8.5), that on *bimonoidal* weak bialgebras S is a normal rigidity map if and only if it is an antipode.

To approach these results let us now introduce, following [BSz], the linear maps $\sqcap_S^{L/R} : \mathcal{A} \to \mathcal{A}$ given by

$$\sqcap_S^L(a) := a \,{}_S\triangleright \mathbf{1} \equiv a_{(1)} S(a_{(2)}) \quad , \quad \sqcap_S^R(a) := \mathbf{1} \triangleleft_S a \equiv S(a_{(1)}) a_{(2)} \; . \tag{6.22}$$

Then Proposition 6.5 immediately implies

Corollary 6.7 *An algebra antimorphism $S : \mathcal{A} \to \mathcal{A}$ on a monoidal weak bialgebra $\mathcal{A}$ is a normal pre-rigidity map if and only if*

$$\begin{array}{rclcrcl} \sqcap_S^L \circ \varepsilon_{LR} &=& \sqcap_S^L & , & \sqcap_S^R \circ \varepsilon_{RL} &=& \sqcap_S^R \\ \sqcap_S^L(\mathbf{1}) &=& \mathbf{1} & , & \sqcap_S^R(\mathbf{1}) &=& \mathbf{1}\; . \end{array} \tag{6.23}$$

Let us next observe that the antipode axioms of [BSz] would imply (see Appendix A)

$$\sqcap_S^L = \varepsilon_{LR} \quad , \quad \sqcap_S^R = \varepsilon_{RL} \tag{6.24}$$

from which the identities (6.23) would follow by Lemma 2.7. We now show, that converseley (6.23) implies (6.24) if and only if $\mathcal{A}_{\sigma,L} = \mathcal{A}_{\sigma,R}$, for $\sigma = L$ and $\sigma = R$. To this end let us introduce the linear spaces

$$\mathcal{A}^{L/R} := \sqcap^{L/R}(\mathcal{A}) \subset \mathcal{A}\,,$$

where from now on we simplify our notation by writing $\sqcap^{L/R} \equiv \sqcap_S^{L/R}$. Then $\mathcal{A}^{L/R}$ naturally becomes a left (right) $\mathcal{A}$-module under the left (right) S-adjoint action

$$\begin{array}{rcl} {}_S\triangleright : \mathcal{A} \otimes \mathcal{A}^L \to \mathcal{A}^L & , & \triangleleft_S : \mathcal{A}^R \otimes \mathcal{A} \to \mathcal{A}^R \\ a \,{}_S\triangleright x := a_{(1)}\, x\, S(a_{(2)}) & , & y \triangleleft_S a := S(a_{(1)})\, y\, a_{(2)}. \end{array} \tag{6.25}$$

It turns out that these $\mathcal{A}$-modules are isomorphic to $\mathcal{A}_{\sigma,R}$ and $\mathcal{A}_{\sigma,L}$, respectively, with left (right) $\mathcal{A}$ actions [5]

$$\pi_{\sigma R}(a)x \;:=\; \varepsilon_{\sigma,R}(ax), \quad a \in \mathcal{A},\, x \in \mathcal{A}_{\sigma,R}. \tag{6.26}$$

$$y\pi_{\sigma L}(a) \;:=\; \varepsilon_{\sigma,L}(ya), \quad a \in \mathcal{A},\, y \in \mathcal{A}_{\sigma,L}\,. \tag{6.27}$$

Theorem 6.8 *Let $\mathcal{A}$ be monoidal and let S be a normal pre-rigidity map on $\mathcal{A}$. Then for $\sigma \in \{L, R\}$*

$$\mathcal{A}^L = \mathcal{A}_{LL} \qquad \mathcal{A}^R = \mathcal{A}_{RR} \tag{6.28}$$

$$\sqcap^L \circ \varepsilon_{\sigma,R} = \sqcap^L \qquad \sqcap^R \circ \varepsilon_{\sigma,L} = \sqcap^R \tag{6.29}$$

$$\sqcap^L \circ \varepsilon_{LL} = \varepsilon_{LL} \qquad \sqcap^R \circ \varepsilon_{RR} = \varepsilon_{RR} \tag{6.30}$$

$$\sqcap^L \circ \varepsilon_{RL} = S \circ \varepsilon_{RL} \qquad \sqcap^R \circ \varepsilon_{LR} = S \circ \varepsilon_{LR} \tag{6.31}$$

$$\varepsilon_R \circ \sqcap^L = \varepsilon_R \qquad \varepsilon_L \circ \sqcap^R = \varepsilon_L\,, \tag{6.32}$$

[5] By Lemma 2.7 and Corollary 2.8 the right $\mathcal{A}$-modules $\mathcal{A}_{\sigma,L}$ are dual to the left $\mathcal{A}$-modules $\mathcal{A}_{\sigma,R}$, where the nondegenerate pairing is given by (2.37).

and we have the following commuting diagrams of left (right) $\mathcal{A}$-module isomorphisms.

$$
\begin{tikzcd}
 & \mathcal{A}_{LR} \arrow[dl, shift left, "\sqcap^L"] \arrow[dr, shift left, "\varepsilon_{RR}"] & & & \mathcal{A}_{RL} \arrow[dl, shift left, "\sqcap^R"] \arrow[dr, shift left, "\varepsilon_{LL}"] & \\
\mathcal{A}_{LL} \arrow[ur, shift left, "\varepsilon_{LR}"] \arrow[rr, shift left, "\varepsilon_{RR}"] & & \mathcal{A}_{RR} \arrow[ul, shift left, "\varepsilon_{LR}"] \arrow[ll, shift left, "\sqcap^L"] & \mathcal{A}_{RR} \arrow[ur, shift left, "\varepsilon_{RL}"] \arrow[rr, shift left, "\varepsilon_{LL}"] & & \mathcal{A}_{LL} \arrow[ul, shift left, "\varepsilon_{RL}"] \arrow[ll, shift left, "\sqcap^R"]
\end{tikzcd}
\tag{6.33}
$$

Here, in the left diagram we consider $\mathcal{A}_{LR}$, $\mathcal{A}_{RR}$ and $\mathcal{A}_{LL} \equiv \mathcal{A}^L$ as left $\mathcal{A}$-modules with $\mathcal{A}$-actions (6.26) and (6.25Left), respectively, and in the right diagram we consider $\mathcal{A}_{RL}$, $\mathcal{A}_{LL}$ and $\mathcal{A}_{RR} \equiv \mathcal{A}^R$ as right $\mathcal{A}$-modules with $\mathcal{A}$-actions (6.27) and (6.25Right), respectively.

Proof: By passing to $\mathcal{A}_{op}^{cop}$ it is enough to prove the left statements. Eq. (6.29) follows from (6.23) and the identities $\varepsilon_{LR} = \varepsilon_{LR}\,\varepsilon_{RR}$ and $\varepsilon_{RL} = \varepsilon_{RL}\,\varepsilon_{LL}$, see Lemma 2.7. Eqs. (6.30) and (6.31) follow from $\mathcal{A}_{\sigma\sigma'} = \mathcal{N}_{\sigma\sigma'}(\mathcal{A})$. In particular, this also gives $\mathcal{A}_{LL} \subset \mathcal{A}^L$. Together with $\mathcal{A}^L = \sqcap^L(\mathcal{A}_{RR})$ by (6.29) this implies

$$\dim \mathcal{A}_{LL} \le \dim \mathcal{A}^L \le \dim \mathcal{A}_{RR} \tag{6.34}$$

and hence equality by (2.42), thus proving (6.28Left). Let us now turn to the left diagram in (6.33). First, by Corollary 2.9 $\varepsilon_{RR} : \mathcal{A}_{LR} \to \mathcal{A}_{RR}$ is an $\mathcal{A}$-linear bijection with inverse $\varepsilon_{LR} : \mathcal{A}_{RR} \to \mathcal{A}_{LR}$. Second, by (6.29) and (6.26) $\sqcap^L : \mathcal{A}_{\sigma,R} \to \mathcal{A}^L \equiv \mathcal{A}_{LL}$ is $\mathcal{A}$-linear and surjective, whence bijective by (2.42). Third, by (6.29) and (6.30)

$$
\begin{aligned}
\sqcap^L \circ \varepsilon_{RR}|_{\mathcal{A}_{LL}} &= \sqcap^L|_{\mathcal{A}_{LL}} = \mathrm{id}_{\mathcal{A}_{LL}} \\
\sqcap^L \circ \varepsilon_{LR}|_{\mathcal{A}_{LL}} &= \sqcap^L|_{\mathcal{A}_{LL}} = \mathrm{id}_{\mathcal{A}_{LL}}
\end{aligned}
$$

and therefore

$$(\sqcap^L|_{\mathcal{A}_{\sigma,R}})^{-1} = \varepsilon_{\sigma,R}|_{\mathcal{A}_{LL}}\,. \tag{6.35}$$

Finally, the diagram commutes, since $\varepsilon_{LR}\,\varepsilon_{RR} = \varepsilon_{LR}$ by Lemma 2.7. We are left to prove (6.32Left), which follows since $\varepsilon_R = \varepsilon_R\,\varepsilon_{RR}$ by Lemma 2.7, and since (6.35) implies $\varepsilon_{RR} = \varepsilon_{RR} \circ \sqcap^L \circ \varepsilon_{RR}$, whence

$$\varepsilon_R \circ \sqcap^L = \varepsilon_R \circ \varepsilon_{RR} \circ \sqcap^L \circ \varepsilon_{RR} = \varepsilon_R \circ \varepsilon_{RR} = \varepsilon_R$$

by (6.29). ■

Corollary 6.9 *Under the setting of Theorem 6.8 we have*

$$\sqcap^L = \varepsilon_{LR} \iff \mathcal{A}_{LL} = \mathcal{A}_{LR} \tag{6.36}$$

$$\sqcap^R = \varepsilon_{RL} \iff \mathcal{A}_{RR} = \mathcal{A}_{RL}\,. \tag{6.37}$$

Proof: By passing to $\mathcal{A}_{op}^{cop}$ it suffices to prove the first statement. If $\sqcap^L = \varepsilon_{LR}$ then $\mathcal{A}_{LL} \equiv \sqcap^L(\mathcal{A}) = \mathcal{A}_{LR}$. Converseley, if $\mathcal{A}_{LL} = \mathcal{A}_{LR}$, then $\varepsilon_{LR}|_{\mathcal{A}_{LL}} = \mathrm{id}$ and (6.33Left) implies $\sqcap^L|_{\mathcal{A}_{RR}} = \varepsilon_{LR}|_{\mathcal{A}_{RR}}$. Hence, by Lemma 2.7,

$$\sqcap^L \equiv \sqcap^L \circ \varepsilon_{RR} = \varepsilon_{LR} \circ \varepsilon_{RR} = \varepsilon_{LR} \,. \qquad \blacksquare$$

Note that the conditions of Corollary 6.9 are in particular satisfied if $\mathcal{A}$ is bimonoidal, yielding $\mathcal{A}_L = \mathcal{A}_{LL} = \mathcal{A}_{LR}$ and $\mathcal{A}_R = \mathcal{A}_{RR} = \mathcal{A}_{RL}$. In the next Section we will take the left hand side of Eqs. (6.36) and (6.37) as the defining relations for a pre-antipode S.

7 The Antipode Axioms

Let us first understand why for general weak bialgebras the ordinary antipode axioms

$$S(a_{(1)})a_{(2)} = a_{(1)}S(a_{(2)}) = \varepsilon(a)\mathbf{1}, \quad a \in \mathcal{A} \tag{7.1}$$

would be too restrictive. Call a linear map $S : \mathcal{A} \to \mathcal{A}$ satisfying (7.1) a *Hopf antipode*. As for ordinary Hopf algebras, a Hopf antipode S would be the inverse of $\mathrm{id}_{\mathcal{A}}$ in the convolution algebra $(\mathrm{End}_K \mathcal{A}, *)$, where $(S * T)(a) := S(a_{(1)})T(a_{(2)})$. Also, if S is a Hopf antipode on $\mathcal{A}$, then $\hat{S} := S^t$ is a Hopf antipode on $\hat{\mathcal{A}}$.

Lemma 7.1 *Let $(\mathcal{A}, \mathbf{1}, \Delta, \varepsilon)$ be a weak bialgebra with Hopf antipode S. Then*
i) $\varepsilon_{LR}(a) = \varepsilon_{RL}(a) = \varepsilon(a)\mathbf{1}$, $\forall a \in \mathcal{A}$.
ii) $\mathcal{A}$ is right-monoidal iff ε is multiplicative, in which case $\mathcal{A}$ is also monoidal.
iii) $\mathcal{A}$ is right-comonoidal iff $\Delta(\mathbf{1}) = \mathbf{1} \otimes \mathbf{1}$, in which case $\mathcal{A}$ is also comonoidal.

Proof: The unit in $(\mathrm{End}_K \mathcal{A}, *)$ is given by $a \mapsto \varepsilon(a)\mathbf{1}$. Hence, if $\mathrm{id}_{\mathcal{A}}$ has a convolution inverse S, part (i) follows from the identities

$$\varepsilon_{LR} * \mathrm{id}_{\mathcal{A}} = \mathrm{id}_{\mathcal{A}} = \mathrm{id}_{\mathcal{A}} * \varepsilon_{RL} \tag{7.2}$$

which one gets by putting $b = \mathbf{1}$ in (2.23). Part (ii) follows by applying ε to (2.27right) to get $\varepsilon(ab) = \varepsilon(a)\varepsilon(b)$, and part (iii) follows by duality. $\blacksquare$

Lemma 7.1 shows, that in general the Hopf antipode axioms (7.1) are too restrictive. Instead, motivated by our analysis of rigidity structures in Sect. 6 and in particular by Corollary 6.9 we now define

Definition 7.2 A *pre-antipode* S on a weak bialgebra $\mathcal{A}$ is a linear map $S : \mathcal{A} \to \mathcal{A}$ satisfying for all $a \in \mathcal{A}$

$$a_{(1)}S(a_{(2)}) = \varepsilon_{LR}(a) \quad , \quad S(a_{(1)})a_{(2)} = \varepsilon_{RL}(a) \tag{7.3}$$

A pre-antipode S is called an *antipode*, if

$$S(a_{(1)})a_{(2)}S(a_{(3)}) = S(a), \quad \forall a \in \mathcal{A} \tag{7.4}$$

Note that by (7.2) a pre-antipode always satisfies

$$a_{(1)}S(a_{(2)})a_{(3)} = a, \quad \forall a \in \mathcal{A} \ . \tag{7.5}$$

In $(\mathrm{End}_K \mathcal{A}, *)$ the identities (7.3), (7.4) and (7.5) can be rewritten, respectively, as

$$\mathrm{id}_{\mathcal{A}} * S = \varepsilon_{LR} \quad , \quad S * \mathrm{id}_{\mathcal{A}} = \varepsilon_{RL} \tag{7.6}$$

$$S * \mathrm{id}_{\mathcal{A}} * S = S \quad , \quad \mathrm{id}_{\mathcal{A}} * S * \mathrm{id}_{\mathcal{A}} = \mathrm{id}_{\mathcal{A}} . \tag{7.7}$$

Hence, in $(\mathrm{End}_K \mathcal{A}, *)$ an antipode S may be considered as a "quasi-inverse" of $\mathrm{id}_{\mathcal{A}}$. Also note that if S is a (pre-)antipode on $\mathcal{A}$, then it is also a (pre-) antipode on $\mathcal{A}_{op}^{cop}$ and by (2.24) its transpose $\hat{S}$ is a (pre-)antipode on $\hat{\mathcal{A}}$. By Lemma 7.1i) a Hopf antipode is always an antipode, and a pre-antipode is a Hopf antipode iff $\varepsilon_{LR}(a) = \varepsilon_{RL}(a) = \varepsilon(a)\mathbf{1}$, $\forall a \in \mathcal{A}$. Moreover, we have

Lemma 7.3 *i) A weak bialgebra $\mathcal{A}$ has at most one antipode S. If $\mathcal{A}$ has a preantipode S_p then $\varepsilon_{LR} * \varepsilon_{LR} = \varepsilon_{LR}$, $\varepsilon_{RL} * \varepsilon_{RL} = \varepsilon_{RL}$ and $S := S_p * id_{\mathcal{A}} * S_p$ provides an antipode.*
ii) If a pre-antipode S is anti-multiplicative, then $S(\mathbf{1}) = \mathbf{1}$, and if it is anti-comultiplicative, then $\varepsilon \circ S = \varepsilon$.

Proof: (i) If S_1 and S_2 are antipodes, then $S_1 = S_1 * \mathrm{id}_{\mathcal{A}} * S_1 = S_1 * \mathrm{id}_{\mathcal{A}} * S_2 = S_2 * \mathrm{id}_{\mathcal{A}} * S_2 = S_2$. If S_p is a preantipode, then by (7.2) $\varepsilon_{LR} * \varepsilon_{LR} = \varepsilon_{LR} * \mathrm{id}_{\mathcal{A}} * S_p = \mathrm{id}_{\mathcal{A}} * S_p = \varepsilon_{LR}$ and similarly $\varepsilon_{RL} * \varepsilon_{RL} = \varepsilon_{RL}$. Hence $S := S_p * \mathrm{id}_{\mathcal{A}} * S_p$ is an antipode. (ii) If S is a pre-antipode satisfying $S(ab) = S(b)S(a)$ then, using $\varepsilon_{LR}(\mathbf{1}) = \mathbf{1}$, $S(\mathbf{1}) = S(\mathbf{1})S(\mathbf{1}_1)\mathbf{1}_{(2)} = S(\mathbf{1}_1)\mathbf{1}_{(2)} = \mathbf{1}$. The statement for anti-comultiplicative S follows by duality. ■

Lemma 7.4 *A pre-antipode S on a right-monoidal or right-comonoidal weak bialgebra $\mathcal{A}$ is an antipode, if S is anti-multiplicative or anti-comultiplicative.*

Proof: By duality it is enough to consider the case of S being anti-multiplicative. If $\mathcal{A}$ is right-comonoidal, then $a_{(1)} \otimes \varepsilon_{LR}(a_{(2)}) = \mathbf{1}_{(1)}a \otimes \mathbf{1}_{(2)}$ by the dul of (2.25right). Hence,

$$S(a_{(1)})a_{(2)}S(a_{(3)}) = S(a_{(1)})\varepsilon_{LR}(a_{(2)}) = S(a)S(\mathbf{1}_{(1)})\mathbf{1}_{(2)} = S(a) \ .$$

If instead $\mathcal{A}$ is right-monoidal, then for all $a, b \in \mathcal{A}$

$$\begin{aligned} \varepsilon_{LR}(a)S(b) &= a_{(1)}S(ba_{(2)}) = a_{(1)}\varepsilon(b_{(1)}a_{(2)})S(b_{(2)}a_{(3)}) \\ &= \varepsilon_{RL}(b_{(1)})a_{(1)}S(b_{(2)}a_{(2)}) \\ &= S(b_{(1)})b_{(2)}a_{(1)}S(b_{(3)}a_{(2)}) , \end{aligned}$$

where in the second line we have used (2.30right). Putting $a = \mathbf{1}$ we conclude that S is an antipode. ■

Corollary 7.5 *An algebra antimorphism S on a monoidal weak bialgebra $\mathcal{A}$ is a pre-antipode (and therefore an antipode) if and only if S is a normal pre-rigidity map and $\mathcal{A}_{\sigma,L} = \mathcal{A}_{\sigma,R}$, for $\sigma = L$ and $\sigma = R$.*

Proof: This follows from Lemma 7.4 and Corollary 6.9, since (7.3) is the same as (6.24), and therefore implies (6.23). ■

In ordinary bialgebras an antipode is always a bialgebra antimorphism and hence a normal rigidity map. In weak bialgebras, the following Theorem analyses necessary and sufficient conditions for an antipode S to be anti-multiplicative and/or anti-comultiplicative.

Theorem 7.6 *Let $\mathcal{A}$ be a weak bialgebra with pre-antipode S and consider the following additional properties:*

1a)	*S is anti-multiplicative.*	*1b)*	*$\mathcal{A}$ is right-monoidal.*
1c)	$[\mathcal{A}_{LR}, \mathcal{A}_{RL}] = 0.$	*1d)*	*S is an antipode.*

Then the following implications hold

$$\begin{array}{rrcl} 1i) & 1a) + 1b) & \Longrightarrow & 1c) + 1d) \\ 1ii) & 1a) + 1c) & \Longrightarrow & 1b) + 1d) \\ 1iii) & 1b) + 1c) + 1d) & \Longrightarrow & 1a) \end{array}$$

Similarly, consider the following properties

2a)	*S is anti-comultiplicative.*	*2b)*	*$\mathcal{A}$ is right-comonoidal.*
2c)	$[\hat{\mathcal{A}}_{LR}, \hat{\mathcal{A}}_{RL}] = 0.$	*2d)*	*S is an antipode.*

Then the following implications hold

$$\begin{array}{rrcl} 2i) & 2a) + 2b) & \Longrightarrow & 2c) + 2d) \\ 2ii) & 2a) + 2c) & \Longrightarrow & 2b) + 2d) \\ 2iii) & 2b) + 2c) + 2d) & \Longrightarrow & 2a) \end{array}$$

Proof: Part 2.) is the dual of part 1.). Le us first prove 1iii). If $\mathcal{A}$ is right-monoidal, then by (2.27right)

$$\begin{aligned} a_{(1)}b_{(1)}S(b_{(2)})S(a_{(2)}) &= a_{(1)}\varepsilon_{LR}(b)S(a_{(2)}) = \varepsilon(a_{(1)}b)a_{(2)}S(a_{(3)}) \\ &= \varepsilon_{LR}(\varepsilon(a_{(1)}b)a_{(2)}) = \varepsilon_{LR}(a\varepsilon_{LR}(b)) \\ &= \varepsilon_{LR}(ab) \equiv a_{(1)}b_{(1)}S(a_{(2)}b_{(2)})\,, \end{aligned}$$

where in the last line we have used Lemma 2.7R. The same argument in $\mathcal{A}_{op}^{cop}$ gives

$$S(a_{(1)}b_{(1)})a_{(2)}b_{(2)} = S(b_{(1)})S(a_{(1)})a_{(2)}b_{(2)}\ .$$

Hence, using $[\mathcal{A}_{LR}, \mathcal{A}_{RL}] = 0$ and (7.4)

$$\begin{aligned} S(ab) &= S(a_{(1)}b_{(1)})a_{(2)}b_{(2)}S(a_{(3)}b_{(3)}) \\ &= S(b_{(1)})S(a_{(1)})a_{(2)}b_{(2)}S(b_{(3)})S(a_{(3)}) \\ &= S(b_{(1)})b_{(2)}S(b_{(3)})S(a_{(1)})a_{(2)}S(a_{(3)}) \\ &= S(b)S(a)\,, \end{aligned}$$

proving part 1iii). To prove 1.ii) let $[\mathcal{A}_{LR}, \mathcal{A}_{RL}] = 0$ and S anti-multiplicative, then by (7.5)

$$\begin{aligned} a\varepsilon_{LR}(b) &= a_{(1)}S(a_{(2)})a_{(3)}b_{(1)}S(b_{(2)}) \\ &= a_{(1)}b_{(1)}S(b_{(2)})S(a_{(2)})a_{(3)} \\ &= a_{(1)}b_{(1)}S(a_{(2)}b_{(2)})a_{(3)} \\ &= \varepsilon_{LR}(a_{(1)}b)a_{(2)}\ . \end{aligned}$$

Hence, by (2.27right), $\mathcal{A}$ is right-monoidal and by Lemma 7.4 S is an antipode. Finally, to prove 1.i) assume S anti-multiplicative and $\mathcal{A}$ right-monoidal. Then S is an antipode by Lemma 7.4, and Eq. (2.27right) implies for all $a, b \in \mathcal{A}$

$$a\varepsilon_{LR}(b) = a_{(1)}b_{(1)}S(a_{(2)}b_{(2)})a_{(3)} = a_{(1)}\varepsilon_{LR}(b)\varepsilon_{RL}(a_{(2)})\,. \tag{7.8}$$

Hence,

$$\begin{aligned}
\varepsilon_{RL}(a)\varepsilon_{LR}(b) &= S(a_{(1)})a_{(2)}\varepsilon_{LR}(b)\varepsilon_{RL}(a_{(3)}) \\
&= \varepsilon_{RL}(a_{(1)})b_{(1)}S(a_{(2)}b_{(2)})a_{(3)} \\
&= b_{(1)}\varepsilon_{RL}(a_{(1)}b_{(2)})S(a_{(2)}b_{(3)})a_{(3)} \\
&= b_{(1)}S(a_{(1)}b_{(2)})a_{(2)} \\
&= \varepsilon_{LR}(b)\varepsilon_{RL}(a)\,,
\end{aligned}$$

where we have used (7.3) and (7.8) in the first line, (7.3) in the second line, (2.30right) in the third line and the antipode identity $S(a) = \varepsilon_{RL}(a_{(1)})S(a_{(2)})$ (7.4) in the fourth line. Hence, $\mathcal{A}_{LR}$ and $\mathcal{A}_{RL}$ commute. ∎

Corollary 7.7 *Let $\mathcal{A}$ be a monoidal weak bialgebra with antipode S. Then $[\mathcal{A}_{LR}, \mathcal{A}_{RL}] = 0$ if and only if $\mathcal{A}_{LL} = \mathcal{A}_{LR}$ and $\mathcal{A}_{RR} = \mathcal{A}_{RL}$, and in this case S is a normal rigidity map.*

Proof: By part 3.) of Proposition 2.6 $[\mathcal{A}_{LR}, \mathcal{A}_{RL}] = 0$ follows from $\mathcal{A}_{LL} = \mathcal{A}_{LR}$ and $\mathcal{A}_{RR} = \mathcal{A}_{RL}$. Conversley, if $[\mathcal{A}_{LR}, \mathcal{A}_{RL}] = 0$ then S is anti-multiplicative by Theorem 7.6(1iii) and Corollary 7.7 follows from Corollary 7.5. ∎

Note that the conditions of Corollary 7.7 in particular hold if $\mathcal{A}$ is bimonoidal.

Next, we provide conditions under which an antipode S is invertible by using an invertibility result for rigidity maps proven in Theorem B6 of Appendix B. To this end we need the counit to be S-invariant.

Lemma 7.8 *Let S be an antipode on $\mathcal{A}$ and assume the axioms* (R) *of Lemma 2.7, i.e. $\varepsilon(ab) = \varepsilon(a\mathbf{1}_{(2)})\varepsilon(\mathbf{1}_{(1)}b)$, $\forall a, b \in \mathcal{A}$. Then $\varepsilon \circ S = \varepsilon$ and, more generally,*

$$\varepsilon_L \circ S = \varepsilon_L \circ \varepsilon_{LR} \quad , \quad \varepsilon_R \circ S = \varepsilon_R \circ \varepsilon_{RL} \tag{7.9}$$

Proof: Using Lemma 2.7 and (7.4) we compute

$$\varepsilon_L(S(a)) = \varepsilon_L(\varepsilon_{RL}(a_{(1)})S(a_{(2)})) = \varepsilon_L(a_{(1)}S(a_{(2)})) = \varepsilon_L(\varepsilon_{LR}(a))\,.$$

The second identity in (7.9) follows by passing to $\mathcal{A}_{op}^{cop}$. Pairing these equations with $\mathbf{1} \in \mathcal{A}$ and using $\varepsilon \circ \varepsilon_{\sigma\sigma'} = \varepsilon$ we get $\varepsilon \circ S = \varepsilon$. ∎

Since the condition (R) of Lemma 2.7 in particular holds if $\mathcal{A}$ is right-monoidal we arrive at

Corollary 7.9 *Let $\mathcal{A}$ be a weak bialgebra with pre-antipode S. If $\mathcal{A}$ is monoidal (comonoidal) and S is anti-multiplicative (anti-comultiplicative), respectively, then S is bijective.*

Proof: If $\mathcal{A}$ is monoidal and S is anti-multiplicative, then S is an antipode by Lemma 7.4, $\varepsilon \circ S = \varepsilon$ by Lemma 7.8, $S(\mathbf{1}) = \mathbf{1}$ by Lemma 7.3 and S is a normal rigidity map by Corollary 7.5. Hence, the invertibility of S follows from Theorem B6. The "co"-statement follows by duality. ∎

Finally, we show that under the conditions of Corollary 7.9 S^{-1} is an antipode on $\mathcal{A}_{op}$ and $\mathcal{A}^{cop}$. As for ordinary Hopf algebras, we call such a map a *pode*.

Definition 7.10 A *pre-pode* $\bar{S}$ on a weak bialgebra $\mathcal{A}$ is a linear map $\bar{S} : \mathcal{A} \to \mathcal{A}$ satisfying for all $a \in \mathcal{A}$

$$a_{(2)}\bar{S}(a_{(1)}) = \varepsilon_{RR}(a) \quad , \quad \bar{S}(a_{(2)})a_{(1)} = \varepsilon_{LL}(a) \,. \tag{7.10}$$

A pre-pode S is called a *pode*, if

$$\bar{S}(a_{(3)})a_{(2)}\bar{S}(a_{(1)}) = \bar{S}(a), \quad \forall a \in \mathcal{A} \,. \tag{7.11}$$

One immediately checks that the above axioms are precisely the antipode axioms in $\mathcal{A}_{op}$ and $\mathcal{A}^{cop}$. For ordinary Hopf algebras the inverse of an antipode is always a pode. In our setting we have

Lemma 7.11 *Let $\mathcal{A}$ be a weak bialgebra with invertible pre-antipode S.*
1.) Assume S anti-multiplicative. Then S^{-1} is a pre-pode if and only if

$$\varepsilon_{LR} = S \circ \varepsilon_{RR} \quad , \quad \varepsilon_{RL} = S \circ \varepsilon_{LL} \,. \tag{7.12}$$

In this case S is an antipode and S^{-1} is a pode.
2.) Assume S anti-comultiplicative. Then S^{-1} is a pre-pode if and only if

$$\varepsilon_{LR} = \varepsilon_{LL} \circ S \quad , \quad \varepsilon_{RL} = \varepsilon_{RR} \circ S \,. \tag{7.13}$$

In this case S is an antipode and S^{-1} is a pode.

Proof: Part 2.) is the transpose of the dual version of 1.). To prove 1.) assume $S(ab) = S(b)S(a)$ and apply S^{-1} to (7.3) to obtain

$$a_{(2)}S^{-1}(a_{(1)}) = S^{-1}(\varepsilon_{LR}(a)) \quad , \quad S^{-1}(a_{(2)})a_{(1)} = S^{-1}(\varepsilon_{RL}(a)) \,.$$

Hence, (7.12) is equivalent to S^{-1} being a pre-pode. In this case S^{-1} is also a pode and S is also an antipode, since

$$\begin{aligned} S(a_{(1)})a_{(2)}S(a_{(3)}) &= S(\varepsilon_{RR}(a_{(2)})a_{(1)}) = S(a) \\ S^{-1}(a_{(3)})a_{(2)}S^{-1}(a_{(1)}) &= S^{-1}(a_{(1)}\varepsilon_{RL}(a_{(2)})) = S^{-1}(a) \end{aligned}$$

by the identities (2.23) for $b = \mathbf{1}$. ■

The conditions of Lemma 7.11 in particular hold if $\mathcal{A}$ is monoidal or comonoidal, respectively.

Proposition 7.12 *Let $\mathcal{A}$ be (co-)monoidal and $S : \mathcal{A} \to \mathcal{A}$ a (co-)algebra anti-automorphism, respectively. Then S is a pre-antipode (and therefore an antipode) if and only if S^{-1} is a pre-pode (and therefore a pode).*

Proof: If $\mathcal{A}$ is monoidal we have $\mathcal{A}_{\sigma\sigma'} = \mathcal{N}_{\sigma\sigma'}(\mathcal{A})$, and if S is an anti-multiplicative pre-antipode then it is an antipode by Lemma 7.4 and $\varepsilon_{RR}(a) \in \mathcal{N}_{RL}(\mathcal{A})$ by Corollary 7.5. Hence, by Lemma 2.7R),

$$\varepsilon_{LR}(a) = \varepsilon_{LR}(\varepsilon_{RR}(a)) = \varepsilon_{RR}(a)_{(1)}S(\varepsilon_{RR}(a)_{(2)}) = \mathbf{1}_{(1)}S(\varepsilon_{RR}(a)\mathbf{1}_{(2)}) = S(\varepsilon_{RR}(a)) \,.$$

The same argument in $\mathcal{A}_{op}^{cop}$ yields $\varepsilon_{RL} = S \circ \varepsilon_{LL}$, whence S^{-1} is a pode by Lemma 7.11. Repeating these arguments in $\mathcal{A}_{op}$ yields the inverse implication. Finally, the "co"-statements follow by duality. ■

8 Weak Hopf Algebras

In order to obtain an explicitely selfdual notion of weak Hopf algebras the following Definition will be appropriate.

Definition 8.1 A weak Hopf algebra $\mathcal{A}$ is a bimonoidal weak bialgebra with antipode S.

Hence, if $(\mathcal{A}, S)$ is a weak Hopf algebra, then its dual $(\hat{\mathcal{A}}, \hat{S})$ is also a weak Hopf algebra. In this Section we will show that in a weak Hopf algebra $\mathcal{A}$ the antipode S is always a bialgebra anti-automorphism and that S^{-1} is a pode. If S is already known to be anti-(co)multiplicative, then part of the bimonoidality axioms may be dropped or replaced altogether by the requirement $[\mathcal{A}_L, \mathcal{A}_R] = 0$. We also show that on bimonoidal weak bialgebras S is an antipode if and only if it is a normal rigidity map. As an application, the generalized Kac algebras of [Ya] will then be shown to be weak Hopf algebras with an involutive antipode. The relation of Definition 8.1 with the axioms of [BSz,Sz] will be clarified in Appendix A. Let us now first observe

Lemma 8.2 *A weak Hopf algebra $\mathcal{A}$ with antipode S is an ordinary Hopf algebra, if and only if one of the following conditions hold.*
i) ε is multiplicative
ii) $\Delta(\mathbf{1}) = \mathbf{1} \otimes \mathbf{1}$
iii) S is a Hopf antipode.

Proof: Clearly, (i)+(ii) $\Rightarrow$ (iii), and in this case $\mathcal{A}$ is an ordinary Hopf algebra. Conversely, (iii) $\Rightarrow$ (i)+(ii) by Lemma 7.1 and if $\mathcal{A}$ is bimonoidal then (i) $\Leftrightarrow$ (ii) by (2.45) and (2.47). ■

Proposition 8.3 *For a weak bialgebra $\mathcal{A}$ with pre-antipode S the following equivalencies hold:*
i) $\mathcal{A}$ is comonoidal and S is anti-multiplicative
ii) $\mathcal{A}$ is monoidal and S is anti-comultiplicative
iii) $\mathcal{A}$ is bimonoidal and S is an antipode.

Proof: If $\mathcal{A}$ is bimonoidal, then $[\mathcal{A}_L, \mathcal{A}_R] = 0$ and $[\hat{\mathcal{A}}_L, \hat{\mathcal{A}}_R] = 0$ and in this case, by Theorem 7.6, an antipode S is a bialgebra anti-homomorphism. This proves (iii) $\Rightarrow$ (i)+(ii). Assume now (i). Then comonoidality implies $[\mathcal{A}_L, \mathcal{A}_R] = 0$ and by part 1ii) of Theorem 7.6 S is an antipode and $\mathcal{A}$ is right-monoidal. Thus, $\mathcal{A}$ is bimonoidal by Corollary 3.9. This proves (i) $\Rightarrow$ (iii). The implication (ii) $\Rightarrow$ (iii) follows by duality. ■

Corollary 8.4 *In a weak Hopf algebra we have $S|_{\mathcal{A}_{L/R}} = S_{L/R}$ as given in (5.1).*

Proof: If S is a pre-antipode then $S(\mathbf{1}_{(1)})\mathbf{1}_{(2)} = \mathbf{1} = \mathbf{1}_{(1)}S(\mathbf{1}_{(1)})$ implying for $a \in \mathcal{A}_L$

$$S_L(a) \equiv \varepsilon_{RL}(a) = S(a_{(1)})a_{(2)} = S(\mathbf{1}_{(1)}a)\mathbf{1}_{(2)} = S(a)S(\mathbf{1}_{(1)})\mathbf{1}_{(2)} = S(a)$$

and similarly $S_R(b) = S(b)$ for $b \in \mathcal{A}_R$. ■

Corollary 8.5 *A linear map $S : \mathcal{A} \to \mathcal{A}$ on a bimonoidal weak bialgebra $\mathcal{A}$ is an antipode if and only if S is a normal pre-rigidity map. In this case S is always a bialgebra anti-automorphism and S^{-1} is a pode.*

Proof: This follows from Proposition 8.3, Corollary 7.5, Corollary 7.9 and Proposition 7.12. ∎

In specific examples the (co)monoidality axioms are typically much harder to verify then anti-(co)multiplicativity of the antipode. In this light the following Theorem is very useful.

Theorem 8.6 *Let $\mathcal{A}$ be a weak bialgebra with pre-antipode S and assume S to be a bialgebra anti-morphism. Then the following statements are equivalent:*
i) S is an antipode and $\mathcal{A}$ is a weak Hopf algebra.
ii) $\mathcal{A}$ is bimonoidal
iii) $\mathcal{A}$ is right-monoidal and right-comonoidal
iv) $[\mathcal{A}_L, \mathcal{A}_R] = 0$
v) $[\hat{\mathcal{A}}_L, \hat{\mathcal{A}}_R] = 0$

Proof: If $\mathcal{A}$ is bimonoidal, then S is an antipode by Lemma 7.4, thus proving (i) $\Leftrightarrow$ (ii). The implication (ii) $\Rightarrow$ (iii) being trivial let us next prove (iii) $\Rightarrow$ (iv)+(v). By Theorem 3.5 part (iii) implies $\mathcal{A}_\sigma = \mathcal{A}_{\sigma\sigma'}$ and $\hat{\mathcal{A}}_\sigma = \hat{\mathcal{A}}_{\sigma\sigma'}$ for $\sigma \neq \sigma' \in \{L, R\}$. Thus, (iv) and (v) follow from 1i) and 2i) of Theorem 7.6. Finally, given $[\mathcal{A}_L, \mathcal{A}_R] = 0$, part 1ii) of Theorem 7.6 implies right-monoidality, and by Lemma 3.8 also left-monoidality. The dual of Corollary 3.7 then gives $[\hat{\mathcal{A}}_L, \hat{\mathcal{A}}_R] = 0$. Dualizing this argument we also conclude that $[\hat{\mathcal{A}}_L, \hat{\mathcal{A}}_R] = 0$ implies $[\mathcal{A}_L, \mathcal{A}_R] = 0$ and comonoidality. This proves (iv) $\Leftrightarrow$ (v) and (iv)+(v) $\Rightarrow$ (ii). ∎

We close this Section by proving that the generalized Kac algebras of T. Yamanouchi [Ya] are special kinds of weak Hopf algebras. Following [Ya] a generalized Kac algebra is a finite dimensional von Neumann algebra $\mathcal{A}$ equipped with a coassociative non-unital $*$-algebra map $\Delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$, a $*$-preserving involutive bialgebra antiautomorphism $S : \mathcal{A} \to \mathcal{A}$ and a positive faithful S-invariant trace λ on $\mathcal{A}$ satisfying

$$a_{(1)}\lambda(ba_{(2)}) = S(b_{(1)})\lambda(b_{(2)}a), \quad \forall a, b \in \mathcal{A}. \tag{8.1}$$

It follows [Ya] that $\mathcal{A}$ also admits a counit ε. Hence, $\mathcal{A}$ is in fact a weak Hopf algebra, since we have more generally

Theorem 8.7 *Let $(\mathcal{A}, \mathbf{1}, \Delta, \varepsilon)$ be a weak bialgebra and $S : \mathcal{A} \to \mathcal{A}$ a bialgebra antiautomorphism. Assume there exists a nondegenerate $\lambda \in \hat{\mathcal{A}}$ satisfying (8.1). Then S is an antipode and $\mathcal{A}$ is a weak Hopf algebra.*

Proof: First we prove that $\mathcal{A}$ is monoidal. Let $l, r \in \mathcal{A}$ be the unique solutions of

$$l \rightharpoonup \lambda = \varepsilon = \lambda \leftharpoonup r. \tag{8.2}$$

Then for all $a \in \mathcal{A}$

$$a = a_{(1)}\varepsilon(a_{(2)}) = \begin{cases} a_{(1)}\lambda(ra_{(2)}) & = & S(r_{(1)})\lambda(r_{(2)}a) \\ a_{(1)}\lambda(a_{(2)}l) & = & \lambda(al_{(2)})S^{-1}(l_{(1)}) \end{cases}$$

Hence, for $a, b \in \mathcal{A}$,

$$\begin{aligned} a_{(1)}\varepsilon(ba_{(2)}) &= a_{(1)}\lambda(rba_{(2)}) = S(b_{(1)})S(r_{(1)})\lambda(r_{(2)}b_{(2)}a) = S(b_{(1)})b_{(2)}a \\ &= \mathbf{1}_{(1)}\varepsilon(b\mathbf{1}_{(2)})a \equiv \varepsilon_{RL}(b)a\,, \end{aligned}$$

where the second line follows by putting $a = \mathbf{1}$ in the first line. Thus, by (2.30right), $\mathcal{A}$ is right monoidal. Similarly,

$$\begin{aligned} a_{(1)}\varepsilon(a_{(2)}b) &= a_{(1)}\lambda(a_{(2)}bl) = S^{-1}(l_{(1)})S^{-1}(b_{(1)})\lambda(ab_{(2)}l_{(2)}) = ab_{(2)}S^{-1}(b_{(1)}) \\ &= a\mathbf{1}_{(1)}\varepsilon(\mathbf{1}_{(2)}b) \equiv a\varepsilon_{RR}(b) \end{aligned}$$

and, by (2.27left), $\mathcal{A}$ is right monoidal. In particular, we also get $S(b_{(1)})b_{(2)} = \varepsilon_{RL}(b)$ and therefore also, putting $b = S(a)$,

$$b_{(1)}S(b_{(2)}) = S(S(a_{(1)})a_{(2)}) = (S \circ \varepsilon_{RL} \circ S^{-1})(b) = \varepsilon_{LR}(b)\,.$$

Thus, S is a pre-antipode and the remaining claims follow from Proposition 8.3ii). ■

A deeper investigation of weak Hopf algebras including a theory of integrals and C^*-structures will be given in [BNS]. In particular, there we will see that (8.1) is one of the defining relations of a left integral $\lambda \in \hat{\mathcal{A}}$ and that the elements l, r defined in (8.2) are nondegenerate left and right integrals, respectively, in $\mathcal{A}$ satisfying $l = S(r)$. This generalizes well known results for ordinary finite dimensional Hopf algebras by [LS].

Appendix

A The Böhm-Szlachányi Axioms

In this Appendix we relate our Definition 8.1 of weak Hopf algebras to the setting of G. Böhm and K. Szlachányi. In [BSz,Sz] the authors required $\mathcal{A}$ to be a weak bialgebra satisfying the "almost–monoidality" axioms L and R of Lemma 2.7. Moreover, the antipode S was required to be a bialgebra anti-automorphism satisfying the following two equivalent relations for all $a \in \mathcal{A}$

$$S(a_{(1)})a_{(2)} \otimes a_{(3)} = (\mathbf{1} \otimes a)\Delta(\mathbf{1}) \qquad , \qquad a_{(1)} \otimes a_{(2)}S(a_{(3)}) = \Delta(\mathbf{1})(a \otimes \mathbf{1})\,. \tag{A.1}$$

Note that if S is not required to be invertible, then the two relations in (A.1) are independent of each other. We now show that the BSz-axioms are equivalent to our Definition 8.1.

Lemma A1 *A linear map $S : \mathcal{A} \to \mathcal{A}$ on a weak bialgebra $\mathcal{A}$ satisfies (A.1) if and only if $\mathcal{A}$ is right-comonoidal and S is a pre-antipode. In this case $\mathcal{A}$ is a weak Hopf algebra and S is an antipode (whence invertible by Corollary 8.5) if and only if S is a bialgebra anti-morphism and the counit axiom (2.36) holds, i.e.*

$$\varepsilon(ab) = \varepsilon(a\mathbf{1}_{(2)})\varepsilon(\mathbf{1}_{(1)}b),\ \forall a, b \in \mathcal{A}\,. \tag{A.2}$$

Proof: Applying $(\mathrm{id} \otimes \varepsilon)$ and $(\varepsilon \otimes \mathrm{id})$, respectively, to (A.1) proves that S is a pre-antipode. Hence, (A.1) also implies

$$\varepsilon_{RL}(a_{(1)}) \otimes a_{(2)} = (\mathbf{1} \otimes a)\Delta(\mathbf{1}) \qquad , \qquad a_{(1)} \otimes \varepsilon_{LR}(a_{(2)}) = \Delta(\mathbf{1})(a \otimes \mathbf{1}) \tag{A.3}$$

and by the duals of (2.25right) or (2.28right) $\mathcal{A}$ is right-monoidal. Conversely, if $\mathcal{A}$ is right-monoidal then (A.3) holds, implying (A.1) for any pre-antipode S. Next, if $\mathcal{A}$ is right-monoidal then (A.2) is equivalent to $\mathcal{A}$ being also right-comonoidal by Lemma 2.10iv). In this case S is an antipode and $\mathcal{A}$ is a weak Hopf algebra if and only if S is a bialgebra anti-morphism, see Proposition 8.3 and Theorem 8.6. ■

Next, we remark that if $K = \mathbf{C}$ and $\mathcal{A}$ is a weak $*$-bialgebra (i.e. a $*$-algebra such that the coproduct is a $*$-algebra homomorphism), then $\varepsilon(a^*) = \overline{\varepsilon(a)}$, $a \in \mathcal{A}$, and therefore (A.2) is equivalent to

$$\varepsilon(ab) = \varepsilon(a\mathbf{1}_{(1)})\varepsilon(\mathbf{1}_{(2)}b), \ \forall a, b \in \mathcal{A}\,, \tag{A.4}$$

which is actually the axiom postulated in [BSz]. One readily verifies that in a weak $*$-bialgebra also our left- and right- (co)monoidality axioms are equivalent. Based on this observation we now show that in the $*$-algebra setting of [BSz] the axioms (A.2) and (A.4) are in fact redundant, as well as the BSz-requirements $S(a^*)^* = S^{-1}(a)$ and $\Delta \circ S = (S \otimes S) \circ \Delta_{op}$.

Lemma A2 *Let $\mathcal{A}$ be a weak $*$-bialgebra and let $S : \mathcal{A} \to \mathcal{A}$ be an algebra anti-morphism. Then S satisfies (A.1) if and only if S is an antipode and $\mathcal{A}$ is a weak Hopf algebra. Moreover, in this case S is a bialgebra anti-automorphism and $S(a^*)^* = S^{-1}(a)$, $\forall a \in \mathcal{A}$.*

Proof: By Lemma A1 Eq. (A.1) is equivalent to S being a pre-antipode and $\mathcal{A}$ being right-comonoidal, whence also left-comonoidal. Thus the first statement follows from Proposition 8.3. In this case one readily checks that $\bar{S}(a) := S(a^*)^*$ provides a pode and therefore $\bar{S} = S^{-1}$ by Corollary 8.5 and the uniqueness of (anti)podes. ∎

B More on Rigidity Structures

In this Appendix, inspired by ideas of Drinfel'd [Dr], we show that rigidity structures (S, α, β) are unique up to equivalence and that for any rigid weak bialgebra $\mathcal{A}$ there exists a twisted coproduct Δ' on $\mathcal{A}' := S(\mathcal{A})$ given by $\Delta'(S(a)) = (S \otimes S)(\Delta(a))$. Under the conditions $S(\mathbf{1}) = \mathbf{1}$ and $\varepsilon \circ S = S$ this will further imply S to be invertible. First we need the following two Lemmas

Lemma B1 *In a pre-rigid weak bialgebra $(\mathcal{A}, \mathbf{1}, \Delta, \varepsilon, S, \alpha, \beta)$ the following identities hold for all $a, b \in \mathcal{A}$.*

$$\begin{aligned}
ab_{(1)} \otimes S(b_{(2)})\,\alpha\, b_{(3)} &= a_{(1)}b_{(1)} \otimes S(a_{(2)}b_{(2)})\,\alpha\, a_{(3)}b_{(3)} &\text{(B.1)}\\
S(b_{(1)})\,\alpha\, b_{(2)} \otimes ab_{(3)} &= S(a_{(1)}b_{(1)})\,\alpha\, a_{(2)}b_{(2)} \otimes a_{(3)}b_{(3)} &\text{(B.2)}\\
a_{(1)}\,\beta\, S(a_{(2)}) \otimes a_{(3)}b &= a_{(1)}b_{(1)}\,\beta\, S(a_{(2)}b_{(2)}) \otimes a_{(3)}b_{(3)} &\text{(B.3)}\\
a_{(1)}b \otimes a_{(2)}\,\beta\, S(a_{(3)}) &= a_{(1)}b_{(1)} \otimes a_{(2)}b_{(2)}\,\beta\, S(a_{(3)}b_{(3)}) &\text{(B.4)}
\end{aligned}$$

Proof: The last two equations follow from the first two by passing to $\mathcal{A}_{op}^{cop}$. Also, it is enough to prove (B.1) and (B.2) for $b = \mathbf{1}$. Using (6.16) and (2.26left) we compute

$$\begin{aligned}
a\mathbf{1}_{(1)} \otimes S(\mathbf{1}_{(2)})\,\alpha\,\mathbf{1}_{(3)} &= a\mathbf{1}_{(1)} \otimes \alpha \triangleleft_S \mathbf{1}_{(2)}\\
&= a\mathbf{1}_{(1)} \otimes \alpha \triangleleft_S \varepsilon_{RL}(\mathbf{1}_{(2)})\\
&= a_{(1)} \otimes S(a_{(2)})\,\alpha\, a_{(3)}
\end{aligned}$$

Using (2.26right) and (6.16), Eq. (B.2) for $b = \mathbf{1}$ (and therefore for all b) follows similarly. ∎

Lemma B2 *In a rigid weak bialgebra $(\mathcal{A}, \mathbf{1}, \Delta, \varepsilon, S, \alpha, \beta)$ we have for all $a \in \mathcal{A}$*

$$\begin{aligned}
a &= a_{(1)}\,\beta\, S(a_{(2)})\,\alpha\, a_{(3)} &\text{(B.5)}\\
S(a) &= S(a_{(1)})\,\alpha\, a_{(2)}\,\beta\, S(a_{(3)}) &\text{(B.6)}\\
\Delta(a) &= a_{(1)}\,\beta\, S(a_{(4)})\,\alpha\, a_{(5)} \otimes a_{(2)}\,\beta\, S(a_{(3)})\,\alpha\, a_{(6)} &\text{(B.7)}\\
S(a_{(2)}) \otimes S(a_{(1)}) &= S(a_{(2)})\,\alpha\, a_{(3)}\,\beta\, S(a_{(6)}) \otimes S(a_{(1)})\,\alpha\, a_{(4)}\,\beta\, S(a_{(5)}) &\text{(B.8)}
\end{aligned}$$

Proof: Eqs.(6.7) and (B.1) imply

$$a = a\mathbf{1}_{(1)}\beta S(\mathbf{1}_{(2)})\alpha\mathbf{1}_{(3)} = a_{(1)}\beta S(a_{(2)})\alpha a_{(3)} \ .$$

Similarly (6.8) and (B.4) give

$$S(a) = S(\mathbf{1}_{(1)}a)\alpha\mathbf{1}_{(2)}\beta S(\mathbf{1}_{(3)}) = S(a_{(1)})\alpha a_{(2)}\beta S(a_{(3)}) \ .$$

Using (B.5) we now compute

$$\begin{aligned}
\Delta(a) &= a_{(1)}\mathbf{1}_{(1)}\beta S(\mathbf{1}_{(2)})\alpha\mathbf{1}_{(3)} \otimes a_{(2)}\beta S(a_{(3)})\alpha a_{(4)} \\
&= a_{(1)}\mathbf{1}_{(1)}\beta S(\mathbf{1}_{(4)})\alpha\mathbf{1}_{(5)} \otimes a_{(2)}\mathbf{1}_{(2)}\beta S(a_{(3)}\mathbf{1}_{(3)})\alpha a_{(4)} \\
&= a_{(1)}\mathbf{1}_{(1)}\beta S(a_{(4)}\mathbf{1}_{(4)})\alpha a_{(5)}\mathbf{1}_{(5)} \otimes a_{(2)}\mathbf{1}_{(2)}\beta S(a_{(3)}\mathbf{1}_{(3)})\alpha a_{(6)} \\
&= a_{(1)}\beta S(a_{(4)})\alpha a_{(5)} \otimes a_{(2)}\beta S(a_{(3)})\alpha a_{(6)}
\end{aligned}$$

where in the second line we have used (B.4) and in the third line (B.1). Similarly, we get

$$\begin{aligned}
S(a_{(2)}) \otimes S(a_{(1)}) &= S(a_{(2)})\alpha a_{(3)}\beta S(a_{(4)}) \otimes S(\mathbf{1}_{(1)}a_{(1)})\alpha\mathbf{1}_{(2)}\beta S(\mathbf{1}_{(3)}) \\
&= S(\mathbf{1}_{(2)}a_{(2)})\alpha\mathbf{1}_{(3)}a_{(3)}\beta S(a_{(4)}) \otimes S(\mathbf{1}_{(1)}a_{(1)})\alpha\mathbf{1}_{(4)}\beta S(\mathbf{1}_{(5)}) \\
&= S(\mathbf{1}_{(2)}a_{(2)})\alpha\mathbf{1}_{(3)}a_{(3)}\beta S(a_{(6)}) \otimes S(\mathbf{1}_{(1)}a_{(1)})\alpha\mathbf{1}_{(4)}a_{(4)}\beta S(\mathbf{1}_{(5)}a_{(5)}) \\
&= S(a_{(2)})\alpha a_{(3)}\beta S(a_{(6)}) \otimes S(a_{(1)})\alpha a_{(4)}\beta S(a_{(5)})
\end{aligned}$$

where in the first line we have used (B.6) and (6.8), in the second line (B.1) and in the third line (B.4). ■

We now show that similar as in [Dr] rigidity structures on a monoidal weak bialgebra are unique up to equivalence.

Proposition B3 *Let $\mathcal{A}$ be monoidal and let (S_1, α_1, β_1) and (S_2, α_2, β_2) be two rigidity structures on $\mathcal{A}$. Define $u, \bar{u} \in \mathcal{A}$ by*

$$u = S_2(\mathbf{1}_{(1)})\alpha_2\mathbf{1}_{(2)}\beta_1 S_1(\mathbf{1}_{(3)}) \tag{B.9}$$
$$\bar{u} = S_1(\mathbf{1}_{(1)})\alpha_1\mathbf{1}_{(2)}\beta_2 S_2(\mathbf{1}_{(3)}) \tag{B.10}$$

Then the following identities hold for all $a \in \mathcal{A}$

$$\begin{aligned}
uS_1(a) &= S_2(a)u & \qquad \bar{u}S_2(a) &= S_1(a)\bar{u} \\
\alpha_2 &= u\alpha_1 & \alpha_1 &= \bar{u}\alpha_2 \\
\beta_2 &= \beta_1\bar{u} & \beta_1 &= \beta_2 u \\
u\bar{u} &= S_2(\mathbf{1}) & \bar{u}u &= S_1(\mathbf{1}) \\
u\bar{u}u &= u & \bar{u}u\bar{u} &= \bar{u}
\end{aligned}$$

Conversely, under these identities u and $\bar{u}$ are necessarily given by (B.9) and (B.10).

Proof: By interchanging $1 \leftrightarrow 2$ and $u \leftrightarrow \bar{u}$ it is enough to prove the left identities. Using (B.2) and (B.4) we get for all $a \in \mathcal{A}$

$$uS_1(a) = S_2(a_{(1)})\alpha_2 a_{(2)}\beta_1 S_1(a_{(3)}) = S_2(a)u \, .$$

From (B.5) and (B.2) one computes

$$\alpha_2 \equiv S_2(\mathbf{1}_{(1)})\alpha_2\mathbf{1}_{(1)} = S_2(\mathbf{1}_{(1)})\alpha_2\mathbf{1}_{(2)}\beta_1 S_1(\mathbf{1}_{(3)})\alpha_1\mathbf{1}_4 = uS_1(\mathbf{1}_{(1)})\alpha_1\mathbf{1}_{(2)} \equiv u\alpha_1 \,.$$

Using (B.4) the identities $\beta_2 = \beta_1\bar{u}$ follow similarly. Moreover,

$$u\bar{u} = S_2(\mathbf{1}_{(1)})\alpha_2\mathbf{1}_{(2)}\beta_1 S_1(\mathbf{1}_{(3)})\alpha_1\mathbf{1}_{(4)}\beta_2 S_2(\mathbf{1}_{(5)}) = S_2(\mathbf{1}_{(1)})\alpha_2\mathbf{1}_{(2)}\beta_2 S_2(\mathbf{1}_{(3)}) = S_2(\mathbf{1})$$

Finally, $u\bar{u}u = S_2(\mathbf{1})u = u$. Conversely, if $u, \bar{u} \in \mathcal{A}$ intertwine (S_1, α_1, β_1) and (S_2, α_2, β_2) as above, then $S_2(a) = uS_1(a)\bar{u}$, whence

$$S_2(\mathbf{1}_{(1)})\alpha_2\mathbf{1}_{(2)}\beta_1 S_1(\mathbf{1}_3) = uS_1(\mathbf{1}_{(1)})\alpha_1\mathbf{1}_{(2)}\beta_1 S_1(\mathbf{1}_3) = u \,,$$

where we have used $\bar{u}u\alpha_1 = S_1(\mathbf{1})\alpha_1 = \alpha_1$. Eq. (B.10) follows similarly. ∎

This proof may of course be traced back to the fact that in monoidal categories any two rigidity structures are naturally equivalent. In the same spirit, the following proposition reflects the natural equivalence $\overline{V \times W} \cong \bar{W} \times \bar{V}$ in rigid monoidal categories, see also [Dr].

Proposition B4 *Let $(\mathcal{A}, \mathbf{1}, \Delta, \varepsilon, S, \alpha, \beta)$ be a rigid weak bialgebra and let $\mathbf{F}, \bar{\mathbf{F}} \in \mathcal{A} \otimes \mathcal{A}$ be given by*

$$\mathbf{F} := [S(\mathbf{1}_{(2)})\alpha \otimes S(\mathbf{1}_{(1)})\alpha]\Delta(\mathbf{1}_{(3)}\beta S(\mathbf{1}_{(4)})) \tag{B.11}$$

$$\bar{\mathbf{F}} := \Delta(S(\mathbf{1}_{(1)})\alpha\mathbf{1}_{(2)})[\beta S(\mathbf{1}_{(4)}) \otimes \beta S(\mathbf{1}_{(3)})] \tag{B.12}$$

Then the following identities hold for all $a \in \mathcal{A}$

$$\mathbf{F}\,\Delta(S(a)) = (S \otimes S)(\Delta_{op}(a))\,\mathbf{F} \tag{B.13}$$

$$\Delta(S(a))\,\bar{\mathbf{F}} = \bar{\mathbf{F}}\,(S \otimes S)(\Delta_{op}(a)) \tag{B.14}$$

$$\bar{\mathbf{F}}\,\mathbf{F} = \Delta(S(\mathbf{1})) \quad , \quad \mathbf{F}\,\bar{\mathbf{F}} = (S \otimes S)(\Delta_{op}(\mathbf{1})) \tag{B.15}$$

$$\mathbf{F}\,\bar{\mathbf{F}}\,\mathbf{F} = \mathbf{F} \quad , \quad \bar{\mathbf{F}}\,\mathbf{F}\,\bar{\mathbf{F}} = \bar{\mathbf{F}} \tag{B.16}$$

Proof: To prove (B.13) we compute

$$\begin{aligned}
(S \otimes S)(\Delta_{op}(a))\,\mathbf{F} &= [S(\mathbf{1}_{(2)}a_{(2)})\alpha \otimes S(\mathbf{1}_{(1)}a_{(1)})\alpha]\Delta(\mathbf{1}_{(3)}\beta S(\mathbf{1}_{(4)})) \\
&= [S(a_{(2)})\alpha \otimes S(a_{(1)})\alpha]\Delta(a_{(3)}\beta S(a_{(4)})) \\
&= [S(a_{(2)})\alpha a_{(3)} \otimes S(a_{(1)})\alpha a_{(4)}]\Delta(\beta S(a_{(5)})) \\
&= [S(\mathbf{1}_{(2)})\alpha\mathbf{1}_{(3)} \otimes S(a_{(1)}\mathbf{1}_{(1)})\alpha a_{(2)}]\Delta(\beta S(a_{(3)})) \\
&= [S(\mathbf{1}_{(2)})\alpha\mathbf{1}_{(3)} \otimes S(\mathbf{1}_{(1')}\mathbf{1}_{(1)})\alpha\mathbf{1}_{(2')}]\Delta(\beta S(a\mathbf{1}_{(3')})) \\
&= [S(\mathbf{1}_{(2)})\alpha\mathbf{1}_{(3)} \otimes S(\mathbf{1}_{(1)})\alpha\mathbf{1}_{(4)}]\Delta(\beta S(a\mathbf{1}_{(5)})) \\
&= \mathbf{F}\,\Delta(S(a))
\end{aligned}$$

Here we have used (B.4) in the second line, (B.2) in the fourth and the fifth line and (B.1) in the sixth line. Interchanging α and β and repreating this proof in $\mathcal{A}_{op}^{cop}$ yields (B.14). To prove (B.15) we compute

$$\begin{aligned}
\bar{\mathbf{F}}\,\mathbf{F} &= \Delta(S(\mathbf{1}_{(1)})\alpha\mathbf{1}_{(2)})[\beta S(\mathbf{1}_{(2')}\mathbf{1}_{(4)})\alpha \otimes \beta S(\mathbf{1}_{(1')}\mathbf{1}_{(3)})\alpha]\Delta(\mathbf{1}_{(3')}\beta S(\mathbf{1}_{(4')})) \\
&= \Delta(S(\mathbf{1}_{(1)})\alpha\mathbf{1}_{(2)})[\beta S(\mathbf{1}_{(4)})\alpha \otimes \beta S(\mathbf{1}_{(3)})\alpha]\Delta(\mathbf{1}_{(5)}\beta S(\mathbf{1}_{(6)})) \\
&= \Delta(S(\mathbf{1}_{(1)})\alpha)\Delta(\mathbf{1}_{(2)})\Delta(\beta S(\mathbf{1}_{(3)})) \\
&= \Delta(S(\mathbf{1}))
\end{aligned}$$

where in the second line we have used (B.4) and in the third line (B.7). Next, using (B.13) we get

$$\begin{aligned}
\mathbf{F}\bar{\mathbf{F}} &= [S(\mathbf{1}_{(2)})\otimes S(\mathbf{1}_{(1)})]\,\mathbf{F}\,\Delta(\alpha\mathbf{1}_{(3)})[\beta S(\mathbf{1}_{(3)})\otimes\beta S(\mathbf{1}_{(4)})]\\
&= [S(\mathbf{1}_{(2')}\mathbf{1}_{(2)})\alpha\otimes S(\mathbf{1}_{(1')}\mathbf{1}_{(1)})\alpha]\Delta(\mathbf{1}_{(3')}\beta S(\mathbf{1}_{(4')})\alpha\mathbf{1}_{(3)})[\beta S(\mathbf{1}_{(5)})\otimes\beta S(\mathbf{1}_{(4)})]\\
&= [S(\mathbf{1}_{(2')}\mathbf{1}_{(2)})\alpha\otimes S(\mathbf{1}_{(1')}\mathbf{1}_{(1)})\alpha]\Delta(\mathbf{1}_{(3')}\mathbf{1}_{(3)}\beta S(\mathbf{1}_{(4')}\mathbf{1}_{(4)})\alpha\mathbf{1}_{(5)})[\beta S(\mathbf{1}_{(7)})\otimes\beta S(\mathbf{1}_{(6)})]\\
&= [S(\mathbf{1}_{(2)})\alpha\otimes S(\mathbf{1}_{(1)})\alpha]\Delta(\mathbf{1}_{(3)}\beta S(\mathbf{1}_{(4)})\alpha\mathbf{1}_{(5)})[\beta S(\mathbf{1}_{(7)})\otimes\beta S(\mathbf{1}_{(6)})]\\
&= [S(\mathbf{1}_{(2)})\alpha\otimes S(\mathbf{1}_{(1)})\alpha]\Delta(\mathbf{1}_{(3)})[\beta S(\mathbf{1}_{(5)})\otimes\beta S(\mathbf{1}_{(4)})]\\
&= S(\mathbf{1}_{(2)})\otimes S(\mathbf{1}_{(1)}).
\end{aligned}$$

Here we have used (B.4) in the third line, (B.5) in the fifth line and (B.8) in the last line. Finally, (B.16) follows from the obvious identities

$$\mathbf{F}\,\Delta(S(\mathbf{1}))=\mathbf{F}\quad,\quad\Delta(S(\mathbf{1}))\,\bar{\mathbf{F}}=\bar{\mathbf{F}}\,.\quad\blacksquare \tag{B.17}$$

If a rigidity structure satisfies $\varepsilon\circ S=\varepsilon$, then Proposition B4 allows to define a rigid monoidal weak bialgebra structure on $\mathcal{A}':=S(\mathcal{A})$, such that $S:\mathcal{A}\to\mathcal{A}'$ becomes a bialgebra homomorphism. First, we consider $\mathcal{A}'\subset\mathcal{A}$ with opposite multiplication, i.e. as a subalgebra of $\mathcal{A}_{op}$ with unit $\mathbf{1}':=S(\mathbf{1})$. The coproduct $\Delta':\mathcal{A}'\to\mathcal{A}'\otimes\mathcal{A}'$ is given by

$$\Delta'(S(a)):=\mathbf{F}_{op}\,\Delta_{op}(S(a))\bar{\mathbf{F}}_{op}\equiv(S\otimes S)(\Delta(a))\ ,$$

which is clearly a coassociative and multiplicative. Moreover, if $\varepsilon\circ S=\varepsilon$ then $\varepsilon':=\varepsilon|_{\mathcal{A}'}$ is a counit for Δ' and therefore $(\mathcal{A}',\mathbf{1}',\Delta',\varepsilon')$ becomes a monoidal weak bialgebra[6].

Lemma B5 *Let $(\mathcal{A},\mathbf{1},\Delta,\varepsilon,S,\alpha,\beta)$ be a rigid weak bialgebra satisfying $\varepsilon\circ S=\varepsilon$. Put $S':=S|_{\mathcal{A}'}$, $\alpha':=S(\alpha)$ and $\beta':=S(\beta)$. Then (S',α',β') provides a rigidity structure on $\mathcal{A}'$.*

The proof of Lemma B5 is straightforward and therefore omitted. Using this result we are now able to prove

Theorem B6 *Let $(\mathcal{A},\mathbf{1},\Delta,\varepsilon,S,\alpha,\beta)$ be a rigid weak bialgebra satisfying $\varepsilon\circ S=\varepsilon$. Then S is bijective if and only if $S(\mathbf{1})=\mathbf{1}$.*

Proof: The identity (6.7) requires S to be nonzero. Iterating Lemma B5 and using $\dim\mathcal{A}<\infty$ we conclude $S^{n+1}(\mathcal{A})=S^n(\mathcal{A})$ for some $n\in\mathbf{N}$. We show that if $S(\mathbf{1})=\mathbf{1}$, then this implies $S^n(\mathcal{A})=S^{n-1}(\mathcal{A})$, whence $S(\mathcal{A})=\mathcal{A}$ by induction, thus proving bijectivity of S. Replacing $\mathcal{A}$ by $S^{n-1}(\mathcal{A})$ it is enough to consider the case $n=1$. Thus, assume $S^2(\mathcal{A})=S(\mathcal{A})$ and therefore $\mathrm{Ker}\,S\cap S(\mathcal{A})=0$. Let $S'=S|_{S(\mathcal{A})}$ and define

$$P:=S'^{-1}\circ S:\mathcal{A}\to S(\mathcal{A})$$

Then P is a multiplicative projection satisfying $P\circ S=S$ and therefore

$$P(aS(b))=P(a)S(b),\quad\forall a,b\in\mathcal{A}$$

By Lemma B7 below, if $S(\mathbf{1})=\mathbf{1}$ there exists $p\in\mathcal{A}$ such that $P(a)=pa$ for all $a\in\mathcal{A}$. Hence $p=P(\mathbf{1})=\mathbf{1}$, implying $\mathrm{Ker}\,S=\mathrm{Ker}\,P=0$ and $\mathcal{A}=S(\mathcal{A})$. $\blacksquare$

[6]Presumably, if ε is not S-invariant, there still may exist a transformed counit ε' on $\mathcal{A}'$ satisfying $\varepsilon'\circ S=\varepsilon$.

Lemma B7 *Let $(\mathcal{A}, \mathbf{1}, \Delta, \varepsilon, S, \alpha, \beta)$ be a rigid weak bialgebra. Denote L the left multiplication of $\mathcal{A}$ on itself and consider $\mathcal{A} \equiv \mathcal{A}_{S(\mathcal{A})}$ as a right $S(\mathcal{A})$-module. If $S(\mathbf{1}) = \mathbf{1}$, then*

$$End_{S(\mathcal{A})}\,(\mathcal{A}_{S(\mathcal{A})}) = L(\mathcal{A}) \tag{B.18}$$

Proof: By standard arguments for rigid monoidal categories (see e.g. [Ka]), for any left $\mathcal{A}$-module V we have an anti-multiplicative isomorphism

$$\mathrm{End}_{\mathcal{A}}\, V \cong \mathrm{End}_{\mathcal{A}}\, \bar{V}, \tag{B.19}$$

given by $\mathrm{End}_{\mathcal{A}}\, V \ni T \mapsto \bar{T} \in \mathrm{End}_{\mathcal{A}}\, \bar{V}$, where

$$\bar{T} := L_{\bar{V}}^{-1} \circ (A_V \times \mathbf{1}_{\bar{V}}) \circ (\mathbf{1}_{\bar{V}} \times T \times \mathbf{1}_{\bar{V}}) \circ (\mathbf{1}_{\bar{V}} \times B_V) \circ R_{\bar{V}}\ .$$

The inverse of the assignment $T \mapsto \bar{T}$ is given by

$$T = R_V^{-1} \circ (\mathbf{1}_V \times A_V) \circ (\mathbf{1}_V \times \bar{T} \times \mathbf{1}_V) \circ (B_V \times \mathbf{1}_V) \circ L_V$$

In our setting one straightforwardly checks that $\bar{T}$ coincides with the restriction of the transpose T^t to $\bar{V} \equiv \hat{V} \cdot S(\mathbf{1})$. We now apply this to $V = \hat{\mathcal{A}}$ with canonical left $\mathcal{A}$-action $\pi_V(a)\phi := a \rightharpoonup \phi$. If $S(\mathbf{1}) = \mathbf{1}$, the conjugate left $\mathcal{A}$-module is given by $\bar{V} \equiv \hat{V} = \mathcal{A}$, with left $\mathcal{A}$-action $\pi_{\bar{V}}(a)b := bS(a)$. Hence, the isomorphism (B.19) gives

$$\mathrm{End}_{S(\mathcal{A})}\,(\mathcal{A}_{S(\mathcal{A})}) \equiv \mathrm{End}_{\mathcal{A}}\,({}_{\mathcal{A}}\bar{V}) \cong \mathrm{End}_{\mathcal{A}}\,({}_{\mathcal{A}}V) \equiv \mathrm{End}_{\mathcal{A}}\,({}_{\mathcal{A}}\hat{\mathcal{A}})\ .$$

On the other hand, under the transposition $T \mapsto T^t$ we clearly have

$$\mathrm{End}_{\mathcal{A}}\,({}_{\mathcal{A}}\hat{\mathcal{A}}) \cong \mathrm{End}_{\mathcal{A}}\,(\mathcal{A}_{\mathcal{A}}) = L(\mathcal{A})$$

where $\mathcal{A}_{\mathcal{A}}$ denotes the regular right $\mathcal{A}$-module, being the natural dual of the left $\mathcal{A}$-module ${}_{\mathcal{A}}\hat{\mathcal{A}}$. ■

As already remarked, the condition $\varepsilon \circ S = \varepsilon$ in Theorem B6 may presumably be dropped, if there is a twisted counit ε' for Δ' on $\mathcal{A}' \equiv S(\mathcal{A})$ satisfying $\varepsilon' \circ S = \varepsilon$. Also recall from Proposition 6.5 that the condition $S(\mathbf{1}) = \mathbf{1}$ holds if α or β are invertible, and therefore in particular for normal rigidity maps.

C Minimal and Cominimal Weak Bialgebras

In this Appendix we introduce a special "minimal" class of comonoidal weak bialgebras $\mathcal{A}$ as well as their "cominimal" duals $\hat{\mathcal{A}}$. As a motivation recall that if $\mathcal{A}$ is comonoidal, then $[\mathcal{A}_L, \mathcal{A}_R] = 0$ by Corollary 3.7. Moreover, by Eqs. (3.1) - (3.4) and Theorem 3.5 we have

$$\Delta(ab) = (a \otimes b)\Delta(\mathbf{1}) = \Delta(\mathbf{1})(a \otimes b), \quad a \in \mathcal{A}_L, b \in \mathcal{A}_R. \tag{C.1}$$

Hence, if $\mathcal{A}$ is comonoidal then $\mathcal{B} := \mathcal{A}_L\mathcal{A}_R \subset \mathcal{A}$ provides a weak comonoidal sub-bialgebra, since $\Delta(\mathbf{1}) \in \mathcal{B} \otimes \mathcal{B}$ by Lemma 4.4. Also, since ε restricts to the counit on $\mathcal{B}$, if $\mathcal{A}$ is bimonoidal then so is $\mathcal{B}$, and if $\mathcal{A}$ is a weak Hopf algebra then $\mathcal{B}$ is a weak Hopf subalgebra. This observation motivates the following

Definition C1 A comonoidal weak bialgebra $\mathcal{A}$ is called *minimal*, if $\mathcal{A} = \mathcal{A}_L\mathcal{A}_R$. A monoidal weak bialgebra $\mathcal{A}$ is called *cominimal*, if its dual $\hat{\mathcal{A}}$ is minimal. If $\mathcal{A} = \mathcal{A}_1\mathcal{A}_2$ is an algebra generated by two commuting subalgebras $\mathcal{A}_1$ and $\mathcal{A}_2$, then we call a minimal weak bialgebra structure (Δ, ε) on $\mathcal{A}$ *adapted* (to $\mathcal{A}_1$ and $\mathcal{A}_2$), if $\mathcal{A}_1 = \mathcal{A}_L$ and $\mathcal{A}_2 = \mathcal{A}_R$.

It will turn out that an adapted weak bialgebra structure is uniquely determined by ε or by $P := \Delta(\mathbf{1})$. Since every comonoidal weak bialgebra contains a minimal one, these results will be very useful when constructing general examples of weak comonoidal (or Hopf) bialgebras, see e.g. Examples 1-3 in Appendix D. The results of this section will also be needed when constructing weak Hopf algebra structures on a large class of quantum chains known from physical models, where $\mathcal{A}_{L/R}$ will be the left/right wedge algebras of these models ([N3], see also Appendix D).

Let us start with preparing some useful formalism. Given two K-vector spaces $\mathcal{A}_{1/2}$ of equal finite dimension, a bilinear form $Q : \mathcal{A}_1 \otimes \mathcal{A}_2 \to K$ is called nondegenerate, if the map $Q_L : a \mapsto Q(a \otimes \cdot)$ (equivalently $Q_R : b \mapsto Q(\cdot \otimes b)$) is an isomorphism $A_1 \to \hat{\mathcal{A}}_2$ (isomorphism $A_2 \to \hat{\mathcal{A}}_1$). This holds if and only if Q has a *form-inverse* $P \equiv \sum u_i \otimes v_i \in \mathcal{A}_2 \otimes \mathcal{A}_1$ satisfying (throughout we drop again summation symbols)

$$\begin{aligned} Q(a \otimes u_i)v_i &= a, \quad \forall a \in \mathcal{A}_1 \\ u_i Q(v_i \otimes b) &= b, \quad \forall b \in \mathcal{A}_2 \end{aligned} \tag{C.2}$$

Clearly, P as a functional $\hat{\mathcal{A}}_2 \otimes \hat{\mathcal{A}}_1 \to K$ is also nondegenerate and its form-inverse is given by Q. Form-inverses are of course uniquely determined if they exist. If $\mathcal{A}_{1/2} \subset \mathcal{A}$ are two commuting subalgebras and $\mathcal{A} = \mathcal{A}_1\mathcal{A}_2$, then with any $\phi \in \hat{\mathcal{A}}$ we associate the bilinear functional $Q_\phi : \mathcal{A}_1 \otimes \mathcal{A}_2 \to K$ given by

$$Q_\phi(a \otimes b) := \langle \phi \mid ab \rangle \tag{C.3}$$

If Q_ϕ is nondegenerate we denote its form-inverse by P_ϕ. Note that ϕ need not be nondegenerate as a functional on $\mathcal{A}$ in order for Q_ϕ to be nondegenerate.

Proposition C2 *Let $\mathcal{A} = \mathcal{A}_1\mathcal{A}_2$ be generated by two commuting subalgebras $\mathcal{A}_1$ and $\mathcal{A}_2$.*
i) If there exists an adapted minimal weak bialgebra structure (Δ, ε) on $\mathcal{A}$, then $Q_\varepsilon : \mathcal{A}_1 \otimes \mathcal{A}_2 \to K$ is nondegenerate and its form-inverse $P_\varepsilon \in \mathcal{A}_2 \otimes \mathcal{A}_1$ is idempotent. Moreover, Δ is uniquely fixed by ε via

$$\Delta(ab) = (a \otimes b)P_\varepsilon \equiv P_\varepsilon(a \otimes b), \quad a \in \mathcal{A}_1,\ b \in \mathcal{A}_2. \tag{C.4}$$

ii) If $\mathcal{A} \cong \mathcal{A}_1 \otimes_{\mathcal{A}_1 \cap \mathcal{A}_2} \mathcal{A}_2$ then the relation (C.4) provides a one-to-one correspondence between adapted minimal weak bialgebra structures (Δ, ε) on $\mathcal{A}$ and idempotents $P \in \mathcal{A}_2 \otimes \mathcal{A}_1$ which are nondegenerate as functionals $\hat{\mathcal{A}}_2 \otimes \hat{\mathcal{A}}_1 \to K$ and satisfy

$$(z \otimes \mathbf{1})P = (\mathbf{1} \otimes z)P, \quad \forall z \in \mathcal{A}_1 \cap \mathcal{A}_2. \tag{C.5}$$

Proof: By Eq. (2.40) of Corollary 2.8 $Q_\varepsilon : \mathcal{A}_L \otimes \mathcal{A}_R \to K$ is nondegenerate for any comonoidal weak bialgebra. Thus part (i) follows from Eq. (C.1) provided $P_\varepsilon := \Delta(\mathbf{1})$ is the form-inverse of Q_ε. However this follows from the definitions (C.3) and (C.2), since the counit property of ε gives

$$\begin{aligned} Q_\varepsilon(a \otimes \mathbf{1}_{(1)})\mathbf{1}_{(2)} &= \varepsilon(a_{(1)})a_{(2)} = a \\ \mathbf{1}_{(1)}Q_\varepsilon(\mathbf{1}_{(2)} \otimes b) &= b_{(1)}\varepsilon(b_{(2)}) = b \end{aligned} \tag{C.6}$$

where $a \in \mathcal{A}_L = \mathcal{N}_{LL}(\mathcal{A})$ and $b \in \mathcal{A}_R = \mathcal{N}_{RR}(\mathcal{A})$, see (3.1) and (3.4). To prove part (ii) first note that given (Δ, ε) we may put $P := \Delta(\mathbf{1}_{\mathcal{A}})$, which is idempotent and nondegenerate by part (i). For $z \in \mathcal{A}_1 \otimes \mathcal{A}_2$ Eq. (C.4) then gives $\Delta(z) = (z \otimes \mathbf{1})P = (\mathbf{1} \otimes z)P$. Conversely, let $P \in \mathcal{A}_2 \otimes \mathcal{A}_1$ be idempotent and nondegenerate with form-inverse $Q : \mathcal{A}_1 \otimes \mathcal{A}_2 \to K$. Then Eq. (C.5) implies

$$Q(az \otimes b) = Q(a \otimes zb)$$

for all $a \in \mathcal{A}_1, b \in \mathcal{A}_2$ and $z \in \mathcal{A}_1 \cap \mathcal{A}_2$. Hence, if $\mathcal{A} \cong \mathcal{A}_1 \otimes_{\mathcal{A}_1 \cap \mathcal{A}_2} \mathcal{A}_2$ the functional $\varepsilon \in \hat{\mathcal{A}}$

$$\varepsilon(ab) := Q(a \otimes b), \quad a \in \mathcal{A}_1, b \in \mathcal{A}_2$$

is well defined and we have $Q = Q_\varepsilon$. Moreover, by Eq. (C.5) $\Delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ given by (C.4) is also a well defined algebra map. Clearly, Δ is coassociative and comonoidal, since $(\mathbf{1}_\mathcal{A} \otimes P)$ commutes with $(P \otimes \mathbf{1}_\mathcal{A})$, and ε is the counit for Δ, since Q is the form-inverse of P. ■

More generally, Proposition C2 shows that for any comonoidal weak bialgebra $\mathcal{A}$ we may put $P := \Delta(\mathbf{1}) \in \mathcal{A}_R \otimes \mathcal{A}_L$ to get a sequence of minimal weak bialgebras

$$\mathcal{A}_L \otimes \mathcal{A}_R \longrightarrow \mathcal{A}_L \otimes_{\mathcal{A}_L \cap \mathcal{A}_R} \mathcal{A}_R \longrightarrow \mathcal{A}_L \mathcal{A}_R \subset \mathcal{A}\,, \tag{C.7}$$

where the arrows are the natural projections, being also weak bialgebra homomorphisms. To describe the dual cominimal weak bialgebras observe that $Q_\varepsilon : \mathcal{A}_L \otimes \mathcal{A}_R \to K$ being non-degenerate with form inverse $P \equiv \Delta(\mathbf{1}) \in \mathcal{A}_R \otimes \mathcal{A}_L$ we have the natural K-linear isomorphism $\mathrm{Hom}_K(\mathcal{A}_L \otimes \mathcal{A}_R, K) \ni \Phi \mapsto T_\Phi \in \mathrm{End}_K \mathcal{A}_R$ with inverse $\mathrm{End}_K \mathcal{A}_R \ni T \mapsto \Phi_T \in \mathrm{Hom}_K(\mathcal{A}_L \otimes \mathcal{A}_R, K)$ given by

$$\begin{aligned} T_\Phi(b) &:= \mathbf{1}_{(1)}\Phi(\mathbf{1}_{(2)} \otimes b), \quad b \in \mathcal{A}_R && \text{(C.8)} \\ \Phi_T(a \otimes b) &:= \varepsilon(a\,T(b)), \quad a \in \mathcal{A}_L\,,\ b \in \mathcal{A}_R\,. && \text{(C.9)} \end{aligned}$$

Proposition C3 *Let $\mathcal{A}$ be comonoidal and $\pi_{\hat{\varepsilon}} : \hat{\mathcal{A}} \to End_K \mathcal{A}_R$ the unit representation. Then i) Due to the isomorphism $End_K \mathcal{A}_R \cong Hom_K(\mathcal{A}_L \otimes \mathcal{A}_R, K)$ we obtain*

$$End_K \mathcal{A}_R \supset End_{\mathcal{A}_L \cap \mathcal{A}_R} \mathcal{A}_R \supset \pi_{\hat{\varepsilon}}(\hat{\mathcal{A}})\,, \tag{C.10}$$

as an inclusion of cominimal weak bialgebras dual to (C.7), with coproduct $\delta : End_K \mathcal{A}_R \to End_K \mathcal{A}_R \otimes End_K \mathcal{A}_R$ given by

$$\delta T(a \otimes b) := \mathbf{1}_{(1)} \otimes \mathbf{1}_{(1')}\,\varepsilon(\mathbf{1}_{(2)}\mathbf{1}_{(2')}\,T(ab)), \quad a, b \in \mathcal{A}_R\,, T \in End_K \mathcal{A}_R\,. \tag{C.11}$$

ii) $\pi_{\hat{\varepsilon}} : \hat{\mathcal{A}} \to End_K \mathcal{A}_R$ provides a bialgebra homomorphism and $Ker\,\pi_{\hat{\varepsilon}} = (\mathcal{A}_L\mathcal{A}_R)^\perp$, i.e. $Ker\,\pi_{\hat{\varepsilon}} \subset \hat{\mathcal{A}}$ is the annihilator of $\mathcal{A}_L\mathcal{A}_R \subset \mathcal{A}$.

Proof: Clearly, $T \in \mathrm{End}_{\mathcal{A}_L \cap \mathcal{A}_R} \mathcal{A}_R \Leftrightarrow \Phi_T \in \mathrm{Hom}_K(\mathcal{A}_L \otimes_{\mathcal{A}_L \cap \mathcal{A}_R} \mathcal{A}_R, K)$. To show $T \in \pi_{\hat{\varepsilon}}(\hat{\mathcal{A}}) \Leftrightarrow \Phi_T \in \mathrm{Hom}_K(\mathcal{A}_L\mathcal{A}_R, K)$ we compute for $\psi \in \hat{\mathcal{A}},\ b \in \mathcal{A}_R$ and $a \in \mathcal{A}_L = \mathcal{N}_{LL}(\mathcal{A})$

$$\Phi_{\pi_{\hat{\varepsilon}}(\psi)}(a \otimes b) = \varepsilon(a(\psi \rightharpoonup b)) = \varepsilon(\psi \rightharpoonup (ab)) = \langle \psi \mid ab \rangle \tag{C.12}$$

thus proving (i). Part (ii) follows, since $\mathcal{A}_L\mathcal{A}_R \subset \mathcal{A}$ is a weak subbialgebra and $(\mathcal{A}_L\mathcal{A}_R)^\perp = \mathrm{Ker}\,\pi_{\hat{\varepsilon}}$ by (C.12). Hence $\pi_{\hat{\varepsilon}} : \hat{\mathcal{A}} \to \pi_{\hat{\varepsilon}}(\hat{\mathcal{A}}) \cong \hat{\mathcal{A}}/(\mathcal{A}_L\mathcal{A}_R)^\perp \cong \mathrm{Hom}_K(\mathcal{A}_L\mathcal{A}_R, K)$ is a weak bialgebra epimorphism. ■

Corollary C4 *Let $\mathcal{A}$ be monoidal. Then*
i) $\pi_\varepsilon(\mathcal{A}) = End_{\hat{\mathcal{A}}_L \cap \hat{\mathcal{A}}_R} \hat{\mathcal{A}}_R$ if and only if $\hat{\mathcal{A}}_L\hat{\mathcal{A}}_R \cong \hat{\mathcal{A}}_L \otimes_{\hat{\mathcal{A}}_L \cap \hat{\mathcal{A}}_R} \hat{\mathcal{A}}_R$.
ii) $\mathcal{A}$ is cominimal if and only if π_ε is faithful.

These results generalize the weak Hopf algebra structure on $\mathrm{Mat}_K(N)$ given by [BSz]. In fact, the dual of the BSz-construction is obtained by putting in Proposition C2 $\mathcal{A}_1 = \mathcal{A}_2 = K^N$ (i.e. the abelian algebra of diagonal $N\times N$-matrices), $\mathcal{A} = \mathcal{A}_1 \otimes \mathcal{A}_2$ and $Q_\varepsilon(a\otimes b) = \sum a_i b_i,\ a,b \in K^N$, yielding $P_\varepsilon = \sum e_i \otimes e_i$, where $e_i \in K^N$ are the minimal orthogonal projections, $e_i e_j = \delta_{ij} e_j$.

Next, we look at minimal weak *Hopf* algebras and recall our definition of the maps $S_\sigma : \mathcal{A}_\sigma \to \mathcal{A}_{-\sigma}$ and $\bar{S}_\sigma : \mathcal{A}_\sigma \to \mathcal{A}_{-\sigma}$ given in (5.1). By Theorem 3.3i) and Theorem 3.5, if $\mathcal{A}$ is comonoidal these maps are algebra anti-isomorphisms satisfying $\bar{S}_{L/R} = S^{-1}_{R/L}$, and if $\mathcal{A}$ is a weak Hopf algebra with antipode S, then by Corollary 8.4 $S_{L/R} = S|_{\mathcal{A}_{L/R}}$. Using this, Proposition C2 now generalizes to a complete characterization of minimal weak Hopf algebras of the form $\mathcal{A} \cong \mathcal{A}_1 \otimes_{\mathcal{A}_1\cap\mathcal{A}_2} \mathcal{A}_2$.

Theorem C5 *Let $\mathcal{A} = \mathcal{A}_1\mathcal{A}_2 \cong \mathcal{A}_1 \otimes_{\mathcal{A}_1\cap\mathcal{A}_2} \mathcal{A}_2$, where $\mathcal{A}_1$ and $\mathcal{A}_2$ commute. Then the relation*

$$\varepsilon(a_L b_R) = \omega(a_L S_R(b_R)),\quad a_L \in \mathcal{A}_L \equiv \mathcal{A}_1,\ b_R \in \mathcal{A}_R \equiv \mathcal{A}_2 \tag{C.13}$$

$$S(a_L b_R) = S_R(b_R) S_L(a_L) \tag{C.14}$$

provides a one-to-one correspondence between adapted weak Hopf algebra structures (Δ, ε, S) on $\mathcal{A}$ and pairs (ω, S_R), where $\omega : \mathcal{A}_1 \to K$ is a nondegenerate functional satisfying $\mathrm{Ind}\,\omega = \mathbf{1}$, $S_R : \mathcal{A}_2 \to \mathcal{A}_1$ is an algebra anti-isomorphism restricting to the identity on $\mathcal{A}_1 \cap \mathcal{A}_2$ and where $S_L = S_R^{-1} \circ \theta_\omega,\ \theta_\omega : \mathcal{A}_1 \to \mathcal{A}_1$ being the modular automorphism of ω.

Proof: If (Δ, ε) is adapted and bimonoidal, then by Proposition 5.2ii) $\omega := \varepsilon|_{\mathcal{A}_L}$ is nondegenerate with $\mathrm{Ind}\,\omega = \mathbf{1}$ and by Lemma 5.1 Eq. (C.13) holds with $S_R := \varepsilon_{LR}|_{\mathcal{A}_R}$, implying also $S_L \equiv \varepsilon_{RL}|_{\mathcal{A}_L} = S_R^{-1} \circ \theta_\omega$ by part (iii) of Proposition 5.2. Moreover, in this case S in (C.14) is well defined and anti-multiplicative, since for comonoidal weak bialgebras $\varepsilon_{RL}|_{\mathcal{A}_R} = \mathrm{id}_{\mathcal{A}_R}$ and $\varepsilon_{LR}|_{\mathcal{A}_L} = \mathrm{id}_{\mathcal{A}_L}$ implying

$$S_L|_{\mathcal{A}_L\cap\mathcal{A}_R} = S_R|_{\mathcal{A}_L\cap\mathcal{A}_R} = \mathrm{id}_{\mathcal{A}_L\cap\mathcal{A}_R} \tag{C.15}$$

Using $\Delta(\mathbf{1}) \in \mathcal{A}_R \otimes \mathcal{A}_L$ and $S_R(\mathbf{1}_{(1)})\mathbf{1}_{(2)} \equiv \varepsilon_{LR}(\mathbf{1}_{(1)})\mathbf{1}_{(2)} = \mathbf{1}$ by (2.22) we then compute for $a_L \in \mathcal{A}_L,\ a_R \in \mathcal{A}_R$ and $a = a_L a_R$

$$S(a_{(1)})a_{(2)} = S(\mathbf{1}_{(1)} a_L)\mathbf{1}_{(2)} a_R = S_L(a_L) S_R(\mathbf{1}_{(1)})\mathbf{1}_{(2)} a_R = \varepsilon_{RL}(a_L) a_R = \varepsilon_{RL}(a_L a_R),$$

where in the last equation we have used $a_R = \varepsilon_{RL}(a_R)$ and part (2ii,left) of Proposition 2.6. Similarly, using $\mathbf{1}_{(1)} S_L(\mathbf{1}_{(2)}) = \mathbf{1}$ and $a_L = \varepsilon_{LR}(a_L)$ we get

$$a_{(1)} S(a_{(2)}) = a_L \mathbf{1}_{(1)} S(a_R \mathbf{1}_{(2)}) = a_L \mathbf{1}_{(1)} S_L(\mathbf{1}_{(2)}) S_R(a_R) = a_L \varepsilon_{LR}(a_R) = \varepsilon_{LR}(a_L a_R)$$

by part (2ii,right) of Proposition 2.6. Hence, S is a pre-antipode and therefore an antipode by Proposition 8.3i).

Converseley, we now reconstruct (Δ, ε) from (ω, S_R). First, since S_R restricts to the identity on $\mathcal{A}_1 \cap \mathcal{A}_2$, the functional ε is well defined on $\mathcal{A}$ by Eq. (C.13) and $Q_\varepsilon : \mathcal{A}_1 \otimes \mathcal{A}_2 \to K$ is nondegenerate. By Proposition C2 we have to show that its form-inverse P_ε is idempotent. Clearly, if $x_i \otimes y_i \in \mathcal{A}_1 \otimes \mathcal{A}_1$ is the form-inverse of ω, then $P_\varepsilon = S_R^{-1}(x_i) \otimes y_i$. Hence,

$$P_\varepsilon^2 = S_R^{-1}(x_i x_j) \otimes y_j y_i = S_R^{-1}(x_j) \otimes y_j x_i y_i = P_\varepsilon$$

where we have used (5.5) and $\mathrm{Ind}\,\omega = \mathbf{1}$. Thus, by Proposition C2 we get a uniquely determined adapted comonoidal weak bialgebra structure (Δ, ε) on $\mathcal{A}$. Since $\mathcal{A}_1 \cap \mathcal{A}_2 \subset \mathcal{C}(\mathcal{A}_1)$ we have

$\theta_\omega|_{\mathcal{A}_1\cap\mathcal{A}_2} = \mathrm{id}$ and therefore (C.14) provides a well defined algebra anti-automorphism $S : \mathcal{A} \to \mathcal{A}$. Moreover, for $a_L \in \mathcal{A}_1$ and $b_R \in \mathcal{A}_2$ we get

$$\begin{array}{lclclclcl} \varepsilon_{LR}(b_R) & \equiv & \varepsilon(\mathbf{1}_{(1)}b_R)\mathbf{1}_{(2)} & = & \omega(S_R(b_R)S_R(\mathbf{1}_{(1)}))\mathbf{1}_{(2)} & = & \omega(S_R(b_R)x_i)y_i & = & S_R(b_R) \\ \varepsilon_{RL}(a_L) & \equiv & \mathbf{1}_{(1)}\varepsilon(a_L\mathbf{1}_{(2)}) & = & S_R^{-1}(x_i)\omega(y_i\theta_\omega(a_L)) & = & S_R^{-1}(\theta_\omega(a_L)) & \equiv & S_L(a_L)\,. \end{array}$$

Hence, by the above arguments, S is an antipode and $\mathcal{A}$ is a weak Hopf algebra. ■

D Examples

Example 1:
This example provides a minimal comonoidal weak bialgebra $\mathcal{A} = \mathcal{A}_L \otimes \mathcal{A}_R$ which is not monoidal. We choose $\mathcal{A}_L := K^3$, i.e. the commutative algebra of diagonal (3×3)-matrices, and $\mathcal{A}_R$ the algebra of upper triangular (2×2) matrices

$$\mathcal{A}_R := \left\{ \begin{pmatrix} x & y \\ 0 & z \end{pmatrix} \mid x, y, z \in K \right\} \subset \mathrm{Mat}_K(2). \tag{D.1}$$

Let $e_i, i = 1,2,3$, denote the pairwise orthogonal minimal projections in $\mathcal{A}_L$, and let $b_i, i = 1,2,3$ be the basis in $\mathcal{A}_R$ given by

$$b_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},\ b_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},\ b_3 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \tag{D.2}$$

Following the lines of Proposition C2(ii) we define the coproduct $\Delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ for $a_L \in \mathcal{A}_L$ and $a_R \in \mathcal{A}_R$ by

$$\Delta(a_L a_R) = (a_L \otimes a_R)P$$

where the idempotent $P \equiv \Delta(\mathbf{1}_\mathcal{A}) \in \mathcal{A}_R \otimes \mathcal{A}_L$ is defined to be

$$P = b_1 \otimes (e_1 + e_2) + b_2 \otimes e_2 + b_3 \otimes e_3 \tag{D.3}$$

Using the relations $e_i e_j = \delta_{ij} e_i$ and

$$b_1^2 = b_1,\ b_1 b_2 = b_2 b_3 = b_2,\ b_3^2 = b_3 \tag{D.4}$$

$$b_2^2 = b_2 b_1 = b_3 b_2 = b_1 b_3 = b_3 b_1 = 0 \tag{D.5}$$

one immediately verifies $P^2 = P$. Also, as a functional $(\widehat{\mathcal{A}_R}) \otimes (\widehat{\mathcal{A}_L}) \to K$, P is nondegenerate. Hence, the counit $\varepsilon : \mathcal{A}_L \otimes \mathcal{A}_R \to K$ is given as the form-inverse of P, i.e.

$$\varepsilon = e^1 \otimes (b^1 - b^2) \otimes (e^2 \otimes b^2) \otimes (e^3 \otimes b^3) \tag{D.6}$$

where e^i and b^j denote the dual basises. According to part (ii) of Proposition C2 these data define a comonoidal weak bialgebra structure on $\mathcal{A} \cong \mathcal{A}_L\mathcal{A}_R$, which by Corollary 3.6 cannot be monoidal, since $\mathcal{A}_L$ is commutative and $\mathcal{A}_R$ is noncommutative. This example does not admit an antipode, since for $x = e_1 b_1$ one easily computes $\varepsilon_{LR}(x) = e_1 + e_2$, whence $e_2\varepsilon_{LR}(x) = e_2$, wheras $x_{(1)}S(x_{(2)}) = e_1\mathbf{1}_{(1)}S(\mathbf{1}_{(2)}b_1)$ implying $e_2 x_{(1)}S(x_{(2)}) = 0$. Nevertheless, the dual of this example admits a (non-normalizable) rigidity structure, see Example 2. ■

Example 2
In this example we construct rigidity structures (S, α, β) on cominimal weak bialgebras of the form $\mathrm{End}_K \mathcal{A}_R$ (or $\mathrm{End}_{\mathcal{A}_L\cap\mathcal{A}_R} \mathcal{A}_R$), where $\mathcal{A}$ is comonoidal, see Proposition C3. This will in particular show that the dual of Example 1 is rigid, although it does not admit an antipode.

It is more convenient to perform the construction on the dual $\mathcal{B} := \mathcal{A}_L \otimes \mathcal{A}_R$ (or $\mathcal{B} := \mathcal{A}_L \otimes_{\mathcal{A}_L\cap\mathcal{A}_R} \mathcal{A}_R$, respectively). Thus, dualizing (6.16), (6.7) and (6.8) we seek for a map $S_\mathcal{B} : \mathcal{B} \to \mathcal{B}$ and functionals $\alpha, \beta \in \hat{\mathcal{B}}$ satisfying for all $x \in \mathcal{B}$

$$S_\mathcal{B}(x_{(1)})\alpha(x_{(2)})x_{(3)} \in \mathcal{A}_R \qquad , \qquad x_{(1)}\beta(x_{(2)})S_\mathcal{B}(x_{(3)}) \in \mathcal{A}_L \tag{D.7}$$

$$\varepsilon_\mathcal{B}\left(x_{(1)}\beta(x_{(2)})S_\mathcal{B}(x_{(3)})\alpha(x_{(4)})x_{(5)}\right) = \varepsilon_\mathcal{B}(x) \tag{D.8}$$

$$\varepsilon_\mathcal{B}\left(S_\mathcal{B}(x_{(1)})\alpha(x_{(2)})x_{(3)}\beta(x_{(4)})S_\mathcal{B}(x_{(5)})\right) = \varepsilon_\mathcal{B}(S_\mathcal{B}(x)) \,. \tag{D.9}$$

Moreover, the normalization conditions $\alpha \triangleleft_S \varepsilon_\mathcal{B} = \alpha$ and $\varepsilon_\mathcal{B}\, {}_S\!\triangleright \beta = \beta$ of Proposition 6.5 become

$$\varepsilon_\mathcal{B}\left(S_\mathcal{B}(x_{(1)})\alpha(x_{(2)})x_{(3)}\right) = \alpha(x) \qquad , \qquad \varepsilon_\mathcal{B}\left(x_{(1)}\beta(x_{(2)})S_\mathcal{B}(x_{(3)})\right) = \beta(x) \,. \tag{D.10}$$

Choose now $S_R : \mathcal{A}_R \to \mathcal{A}_L$ an arbitrary linear bijection and let $S_L : \mathcal{A}_L \to \mathcal{A}_R$ be the transpose of S_R^{-1} with respect to the pairing Q_ε, i.e.

$$Q_\varepsilon(a \otimes S_L(b)) := Q_\varepsilon(b \otimes S_R^{-1}(a)), \quad a, b \in \mathcal{A}_L \,.$$

Note that $P_\varepsilon \equiv \mathbf{1}_{(1)} \otimes \mathbf{1}_{(2)}$ being the form inverse of Q_ε implies

$$S_L(\mathbf{1}_{(2)}) \otimes S_R(\mathbf{1}_{(1)}) = \mathbf{1}_{(1)} \otimes \mathbf{1}_{(2)} \,. \tag{D.11}$$

For $x = a_L a_R,\ a_L \in \mathcal{A}_L,\ a_R \in \mathcal{A}_R$ define

$$S_\mathcal{B}(x) := S_L(a_L)S_R(a_R) \tag{D.12}$$

$$\alpha(x) := Q_\varepsilon(\mathbf{1} \otimes S_L(a_L)a_R) \qquad , \qquad \beta(x) := Q_\varepsilon(a_L S_R(a_R) \otimes \mathbf{1}) \,. \tag{D.13}$$

In case we want $(S_\mathcal{B}, \alpha, \beta)$ to be well defined on $\mathcal{A}_L \otimes_{\mathcal{A}_L\cap\mathcal{A}_R} \mathcal{A}_R$ we also have to require S_R (and therefore S_L) to be $(\mathcal{A}_L \cap \mathcal{A}_R)$-linear. Using $\Delta_\mathcal{B}(x) = a_L\mathbf{1}_{(1)} \otimes \mathbf{1}_{(2)}a_R$ and the comonoidality property we now have

$$x_{(1)} \otimes x_{(2)} \otimes x_{(3)} = a_L\mathbf{1}_{(1)} \otimes \mathbf{1}_{(2)}\mathbf{1}_{(1')} \otimes \mathbf{1}_{(2')}a_R$$

and similarly for higher coproducts. Hence, the identities (D.7) - (D.10) are immediately verified, provided we have

$$S_\mathcal{B}(\mathbf{1}_{(1)})\alpha(\mathbf{1}_{(2)})\mathbf{1}_{(3)} = \mathbf{1} = \mathbf{1}_{(1)}\beta(\mathbf{1}_{(2)})S_\mathcal{B}(\mathbf{1}_{(3)}) \,. \tag{D.14}$$

To check (D.14) use (D.11) to compute

$$S_\mathcal{B}(\mathbf{1}_{(1')})\alpha(\mathbf{1}_{(2')}\mathbf{1}_{(1)})\mathbf{1}_{(2)} = Q_\varepsilon(\mathbf{1} \otimes S_L(\mathbf{1}_{(2')})\mathbf{1}_{(1)})\, S_R(\mathbf{1}_{(1')})\mathbf{1}_{(2)} = Q_\varepsilon(\mathbf{1} \otimes \mathbf{1}_{(1)})\mathbf{1}_{(2)} = \mathbf{1}$$

$$\mathbf{1}_{(1)}\beta(\mathbf{1}_{(2)}\mathbf{1}_{(1')})S_\mathcal{B}(\mathbf{1}_{(2')} = \mathbf{1}_{(1)}S_L(\mathbf{1}_{(2')})Q_\varepsilon(\mathbf{1}_{(2)}S_R(\mathbf{1}_{(1')}) \otimes \mathbf{1}) = \mathbf{1}_{(1)}Q_\varepsilon(\mathbf{1}_{(2)} \otimes \mathbf{1}) = \mathbf{1} \,.$$

This proves (D.14) and therefore the rigidity identities (D.7) - (D.10). Finally, we also have

$$\Delta_\mathcal{B}(S_\mathcal{B}(x)) = \mathbf{1}_{(1)}S_R(a_R) \otimes S_L(a_L)\mathbf{1}_{(2)} = S_\mathcal{B}(\mathbf{1}_{(2)}a_R) \otimes S_\mathcal{B}(a_L\mathbf{1}_{(1)}) = (S_\mathcal{B} \otimes S_\mathcal{B})(\Delta_\mathcal{B}^{op}(x))$$

and therefore $S_\mathcal{B}$ is anti-comultiplicative. Hence, $(\hat{S}_\mathcal{B}, \alpha, \beta)$ provides a rigidity structure on $\hat{\mathcal{B}}$.
In Example 1 one may choose

$$\begin{array}{rclcrclcrcl} S_R(b_1) &=& e_1 + e_2 &,& S_R(b_2) &=& e_2 &,& S_R(b_3) &=& e_3 \\ S_L(e_1) &=& b_1 - b_2 &,& S_L(e_2) &=& b_2 &,& S_L(e_3) &=& b_3 \end{array}$$

to obtain $\beta = \varepsilon \equiv \hat{\mathbf{1}}$ given by (D.6) and $\alpha = e^1 \otimes b^1 + e^3 \otimes b^3$. In particular, this rigidity structure is not normalizable. ■

Example 3
In this example we extend minimal weak Hopf algebras of the form $\mathcal{B} = \mathcal{A}_L \otimes \mathcal{A}_R$ to weak Hopf algebras $\mathcal{A} = \mathcal{G} \bowtie \mathcal{B} \cong \mathcal{A}_L \rtimes \mathcal{G} \ltimes \mathcal{A}_R$ by a *two-sided crossed product construction* with a Hopf algebra $\mathcal{G}$.

Let $\mathcal{B} = \mathcal{A}_L \otimes \mathcal{A}_R$ be a minimal weak bialgebra, with counit ε_B and coproduct $\Delta_{\mathcal{B}}$. Let $(\mathcal{G}, \mathbf{1}, \Delta, \varepsilon)$ be a finite dimensional Hopf algebra and assume a left Hopf module $\mathcal{G}$-action $\triangleright : \mathcal{G} \otimes \mathcal{A}_L \to \mathcal{A}_L$ and a right Hopf module $\mathcal{G}$-action $\triangleleft : \mathcal{A}_R \otimes \mathcal{G} \to \mathcal{A}_R$. Following [HN1] we define the *two-sided crossed product* $\mathcal{A} := \mathcal{A}_L \rtimes \mathcal{G} \ltimes \mathcal{A}_R$ to be the vector space $\mathcal{A}_L \otimes \mathcal{G} \otimes \mathcal{A}_R$ with multiplication structure

$$(a_L \rtimes g \ltimes a_R)(b_L \rtimes h \ltimes b_R) := \Big(a_L(g_{(1)} \triangleright b_L) \rtimes g_{(2)}h_{(1)} \ltimes (a_R \triangleleft h_{(2)})b_R\Big) \tag{D.15}$$

Equivalently, $\mathcal{A}$ may be identified with the *diagonal crossed product* $\mathcal{G} \bowtie \mathcal{B}$ via

$$\mathcal{G} \bowtie \mathcal{B} \ni (g\,(a_L \otimes a_R)) \mapsto ((g_{(1)} \triangleright a_L) \rtimes g_{(2)} \ltimes a_R) \in \mathcal{A}. \tag{D.16}$$

Here, the multiplication in $\mathcal{G} \bowtie \mathcal{B}$ is fixed by either of the equivalent relations [HN1]

$$\begin{aligned} g\,(a_L \otimes a_R) &= \Big((g_{(1)} \triangleright a_L) \otimes (a_R \triangleleft S^{-1}(g_{(3)}))\Big)\, g_{(2)} \\ (a_L \otimes a_R)\, g &= g_{(2)}\Big((S^{-1}(g_{(3)}) \triangleright a_L) \otimes (a_R \triangleleft (g_{(1)})\Big) \end{aligned}$$

where $g \in \mathcal{G}$ and $a_{L/R} \in \mathcal{A}_{L/R}$. Since as a linear space $\mathcal{G} \bowtie \mathcal{B} = \mathcal{G} \otimes \mathcal{B}$, it comes equipped with the natural tensor product coalgebra structure from $\mathcal{G}$ and $\mathcal{B}$. With respect to the identification (D.16) this induces a coalgebra structure $(\Delta_{\mathcal{A}}, \varepsilon_{\mathcal{A}})$ on $\mathcal{A}$ given by

$$\Delta_{\mathcal{A}}(a_L \rtimes g \ltimes a_R) := (a_L \rtimes g_{(1)} \ltimes \mathbf{1}_{(1)}) \otimes ((g_{(2)} \triangleright \mathbf{1}_{(2)}) \rtimes g_{(3)} \ltimes a_{(3)}) \tag{D.17}$$

$$\varepsilon_{\mathcal{A}}(a_L \rtimes g \ltimes a_R) := \varepsilon_{\mathcal{B}}((S^{-1}(g) \triangleright a_L) \otimes a_R) \tag{D.18}$$

where $\mathbf{1}_{(1)} \otimes \mathbf{1}_{(2)} \equiv \Delta_{\mathcal{B}}(\mathbf{1}_{\mathcal{B}}) \in \mathcal{A}_R \otimes \mathcal{A}_L$. Assume now

$$\varepsilon_{\mathcal{B}}((g \triangleright a_L) \otimes a_R) = \varepsilon_{\mathcal{B}}((a_L \otimes (a_R \triangleleft g)) \tag{D.19}$$

for all $g \in \mathcal{G}$ and $a_\sigma \in \mathcal{A}_\sigma$. Equivalently, since $\mathbf{1}_{(1)} \otimes \mathbf{1}_{(2)} \in \mathcal{A}_R \otimes \mathcal{A}_L$ is the form-inverse of $\varepsilon_{\mathcal{B}} : \mathcal{A}_L \otimes \mathcal{A}_R \to K$, this means

$$\mathbf{1}_{(1)} \otimes (g \triangleright \mathbf{1}_{(2)}) = (\mathbf{1}_{(1)} \triangleleft g) \otimes \mathbf{1}_{(2)}, \quad \forall g \in \mathcal{G}. \tag{D.20}$$

Given this condition one easily verifes that $(\mathcal{A}, \mathbf{1}_{\mathcal{A}}, \Delta_{\mathcal{A}}, \varepsilon_{\mathcal{A}})$ becomes a comonoidal weak bialgebra extension of $\mathcal{B}$. Moreover, using Lemma 2.10 iii) and iv) one checks that if $\mathcal{B}$ is left- or right-monoidal then so is $\mathcal{A}$.

Next, note that (D.19) implies the restricted functional $\omega := \varepsilon_{\mathcal{B}}|_{\mathcal{A}_L}$ to be $\mathcal{G}$-invariant. Hence, if $\mathcal{B}$ is also a weak Hopf algebra and if $S_R : \mathcal{A}_R \to \mathcal{A}_L$ is the restriction of the antipode to $\mathcal{A}_R$ (see Theorem C5), then Eq. (D.19) implies for all $a_\sigma \in \mathcal{A}_\sigma$ and $g \in \mathcal{G}$

$$\omega(a_L S_R(a_R \triangleleft g)) = \varepsilon(a_L \otimes a_R \triangleleft g) = \omega((g \triangleright a_L) S_R(a_R)) = \omega\Big(a_L(S_{\mathcal{G}}(g) \triangleright S_R(a_R))\Big). \tag{D.21}$$

By the nondegeneracy of ω we conclude

$$S_R(a_R \triangleleft g) = S_{\mathcal{G}}(g) \triangleright S_R(a_R) \quad , \quad \forall a_R \in \mathcal{A}_R,\ g \in \mathcal{G}, \tag{D.22}$$

where $S_{\mathcal{G}}$ is the antipode on $\mathcal{G}$. Hence, by Theorem C5 any weak Hopf algebra in the form of a two-sided crossed product $\mathcal{A} = \mathcal{A}_L \rtimes \mathcal{G} \ltimes \mathcal{A}_R$ as above may uniquely be constructed from a left Hopf module $\mathcal{G}$-action on $\mathcal{A}_L$, a nondegenerate $\mathcal{G}$- invariant functional ω on $\mathcal{A}_L$ satisfying $\mathrm{Ind}\,\omega = \mathbf{1}$ and an anti-isomorphism $S_R : \mathcal{A}_R \to \mathcal{A}_L$. The right $\mathcal{G}$-action on $\mathcal{A}_R$ is then given by (D.22) and the compatibility condition (D.19) follows from (C.13). Moreover, the antipode $S_B : \mathcal{B} \to \mathcal{B}$ constructed in (C.14) extends to an antipode $S_{\mathcal{A}} : \mathcal{A} \to \mathcal{A}$ by putting

$$S_{\mathcal{A}}(a_L \rtimes g \ltimes a_R) := S_R(a_R) \rtimes S_{\mathcal{G}}(g) \ltimes S_L(a_L) \tag{D.23}$$

Note that this is indeed an algebra anti-isomorphisms, since one checks, similarly as in (D.22),

$$S_L(g \triangleright a_L) = S_L(a_L) \triangleleft S_{\mathcal{G}}(g) \tag{D.24}$$

for all $g \in \mathcal{G}$ and $a_L \in \mathcal{A}_L$. ■

Interpreting $\mathcal{A}_L$ and $\mathcal{A}_R$ as left and right "wedge algebras", this construction puts a weak Hopf algebra structure on the Hopf algebraic quantum chains of [NSz]. More general quantum chains can be treated by allowing $\mathcal{G}$ itself to be a weak Hopf algebra. Moreover, using the methods of [HN1], the above example also generalizes to the case where $\mathcal{G}$ is the dual of a quasi-Hopf algebra. If in this case also $\mathcal{A}_L = \mathcal{A}_R = \mathcal{G}$, this will provide a general "blowing up" procedure in the spirit of [BSz] from quasi-coassociative Hopf algebras $\mathcal{G}$ to weak coassociative Hopf algebras $\mathcal{G} \rtimes \hat{\mathcal{G}} \ltimes \mathcal{G}$ in our sense, with equivalent representation categories. More details on this will be discussed elsewhere.

Example 4: Let $(\mathcal{G}, \mathbf{1}, \Delta, \varepsilon)$ be a finite dimensional Hopf algebra and let $\mathcal{H} \subset \mathcal{G}$ be an Ad-invariant Hopf subalgebra, i.e.

$$g_{(1)} \mathcal{H}\, S_{\mathcal{G}}(g_{(2)}) \subset \mathcal{H} \quad , \quad \forall g \in \mathcal{G} \tag{D.25}$$

Assume $\mathcal{H}$ semisimple and denote $p = S_{\mathcal{G}}(p) = p^2 \in \mathcal{H}$ the unique normalized two-sided integral. Then the crossed product $\mathcal{A} := \mathcal{H} \rtimes_{Ad} \mathcal{G}$ becomes a weak Hopf algebra with

$$\Delta_{\mathcal{A}}(h \otimes g) := (h S_{\mathcal{G}}(p_{(1)}) \otimes p_{(2)} g_{(1)}) \otimes (p_{(3)} \otimes g_{(2)}) \tag{D.26}$$

$$\varepsilon_{\mathcal{A}}(h \otimes g) := \lambda(h) \varepsilon_{\mathcal{G}}(g) \tag{D.27}$$

where $h \in \mathcal{H}, g \in \mathcal{G}$, and where $\lambda \in \hat{\mathcal{H}}$ is the left integral dual to p, i.e. the unique solution of

$$\lambda \rightharpoonup p = \mathbf{1}_{\mathcal{H}}. \tag{D.28}$$

Clearly, $\varepsilon_{\mathcal{A}}$ is a right counit for $\Delta_{\mathcal{A}}$ and using

$$S_{\mathcal{G}}(p_{(1)}) \otimes p_{(2)} = p_{(2)} \otimes S_{\mathcal{G}}^{-1}(p_{(1)}) \tag{D.29}$$

and the identity $\lambda_L^{-1} = S_{\mathcal{G}}^{-1} \circ p_R$ [LS], $\varepsilon_{\mathcal{A}}$ is also a left counit. The coassociativity of $\Delta_{\mathcal{A}}$ follows from

$$p_{(1)} \otimes p_{(2)} S_{\mathcal{G}}(p_{(1')}) \otimes p_{(2')} = p_{(1)} p_{(1')} \otimes p_{(2)} \otimes p_{(2')}. \tag{D.30}$$

To see that $\Delta_{\mathcal{A}}$ is an algebra map we compute for $h,k\in\mathcal{H}$ and $f,g\in\mathcal{G}$

$$\begin{aligned}
&\Delta(h\otimes g)\Delta(k\otimes f) = \\
&= \Big(hg_{(1)}kS_{\mathcal{G}}(p_{1'})S_{\mathcal{G}}(p_{(1)}g_{(2)})\otimes p_{(2)}g_{(3)}p_{(2')}f_{(1)}\Big)\otimes\Big(p_{(3)}g_{(4)}p_{(3')}S(g_{(5)})\otimes g_{(6)}f_{(2)}\Big)\\
&= \Big(hg_{(1)}kS_{\mathcal{G}}(p_{(1)}g_{(2)})\otimes p_{(2)}g_{(3)}f_{(1)}\Big)\otimes\Big(p_{(3)}\otimes g_{(4)}f_{(2)}\Big)\\
&= \Delta(gh_{(1)}kS(g_{(2)})\otimes g_{(3)}f).
\end{aligned}$$

Here we have used $p\in\mathcal{C}(\mathcal{G})$, since $g_{(1)}pS(g_{(2)})=\varepsilon(g)p$, for all $g\in\mathcal{G}$, which follows since the l.h.s. is again a two-sided integral in $\mathcal{H}$. To see that $\mathcal{A}$ is comonoidal let $P:=\Delta_{\mathcal{A}}(\mathbf{1}_{\mathcal{A}})$. Then

$$\begin{aligned}
(\mathbf{1}_{\mathcal{A}}\otimes P)(P\otimes\mathbf{1}_{\mathcal{A}}) &= (P\otimes\mathbf{1}_{\mathcal{A}})(\mathbf{1}_{\mathcal{A}}\otimes P))\\
&= \Big(S_{\mathcal{G}}(p_{(1)})\otimes p_{(2)}\Big)\otimes\Big(p_{(3)}S_{\mathcal{G}}(p_{(1')})\otimes p_{(2')}\Big)\otimes\Big(p_{(3')}\otimes\mathbf{1}_{\mathcal{G}}\Big)\\
&= \Big(S_{\mathcal{G}}(p_{(1')})S_{\mathcal{G}}(p_{(1)})\otimes p_{(2)}p_{(2')}\Big)\otimes\Big(p_{(3)}\otimes p_{(3')}\Big)\otimes\Big(p_{(4')}\otimes\mathbf{1}_{\mathcal{G}}\Big)\\
&= (\Delta_{\mathcal{A}}\otimes id)(P),
\end{aligned}$$

where we have used (D.30). Finally, applying Lemma 2.10iii) and iv) and using (D.29) together with the Fourier transformation identities [LS]

$$p_{(1)}\lambda(hp_{(2)}) = S_{\mathcal{G}}(h) \tag{D.31}$$

$$\lambda(S_{\mathcal{G}}^{-1}(p_{(1)})h)p_{(2)} = h \tag{D.32}$$

for all $h\in\mathcal{H}$ [7], one also checks that $(\mathcal{A},\mathbf{1}_{\mathcal{A}},\Delta_{\mathcal{A}},\varepsilon_{\mathcal{A}})$ is monoidal. In this example we have

$$\mathcal{A}_L = \{(h\otimes\mathbf{1}_{\mathcal{G}})\mid h\in\mathcal{H}\}\cong\mathcal{H} \tag{D.33}$$

$$\mathcal{A}_R = \{(S(h_{(1)})\otimes h_{(2)})\mid h\in\mathcal{H}\}\cong\mathcal{H}_{op} \tag{D.34}$$

as well as the identities

$$\begin{aligned}
\varepsilon_{LL}(h\otimes g) &= (S^{-1}(g_{(2)})hg_{(1)}\otimes\mathbf{1}_{\mathcal{G}}) \quad , \quad \varepsilon_{RR}(h\otimes g) = \varepsilon_{\mathcal{G}}(g)\Big(h_{(2)}\otimes S^{-1}(h_{(1)})\Big)\\
\varepsilon_{LR}(h\otimes g) &= \varepsilon_{\mathcal{G}}(g)(h\otimes\mathbf{1}_{\mathcal{G}}) \quad , \quad \varepsilon_{RL}(h\otimes g) = S^2_{\mathcal{G}}\Big((h\triangleleft g)_{(2)}\Big)\otimes S_{\mathcal{G}}\Big((h\triangleleft g)_{(1)}\Big)
\end{aligned}$$

where $h\in\mathcal{H},g\in\mathcal{G}$ and $h\triangleleft g:=S_{\mathcal{G}}^{-1}(g_{(2)})hg_{(1)}$. Using these formulas the reader is invited to check that $S_{\mathcal{A}}:\mathcal{A}\to\mathcal{A}$ given by

$$S_{\mathcal{A}}(h\otimes g):=S^2_{\mathcal{G}}(h_{(2)}\triangleleft g_{(2)})\otimes S_{\mathcal{G}}(h_{(1)}g_{(1)})$$

provides an antipode and therefore $\mathcal{A}$ is a weak Hopf algebra. ■

Putting $\mathcal{G}=\mathbf{C}G$ and $\mathcal{H}=\mathbf{C}H$ for some finite group G with normal subgroup H, the above example [8] appears as a weak Hopf symmetry in any Jones triple

$$\mathcal{M}^G\subset\mathcal{M}\subset\mathcal{M}\rtimes G$$

where $\mathcal{M}$ is a von-Neumann factor and G is a group of automorphisms of $\mathcal{M}$ with inner part given by H [NSW].

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[7] For a review of the theory of Fourier transformations on finite dimensional Hopf algebras see also [N1].
[8] possibly deformed by a cocycle

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