math.QA/9806130

Weak Hopf Algebras and Reducible Jones Inclusions of Depth 2. I: From Crossed Products to Jones Towers

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April 3, 1998

Abstract

We apply the theory of finite dimensional weak C^*-Hopf algebras $\mathcal{A}$ as developed by G. Böhm, F. Nill and K. Szlachányi [BSz,Sz,N1,N2,BNS] to study reducible inclusion triples of von-Neumann algebras $\mathcal{N} \subset \mathcal{M} \subset \mathcal{M} \rtimes \mathcal{A}$, where $\mathcal{M}$ is an $\mathcal{A}$-module algebra with left $\mathcal{A}$-action $\triangleright : \mathcal{A} \otimes \mathcal{M} \to \mathcal{M}$, $\mathcal{N} \equiv \mathcal{M}^{\mathcal{A}}$ is the fixed point algebra and $\mathcal{M} \rtimes \mathcal{A}$ is the crossed product extension. Here "weak" means that the coproduct Δ on $\mathcal{A}$ is non-unital, requiring various modifications of the standard definitions for Hopf (co-)actions and crossed products. We show that normalized positive and nondegenerate left integrals $l \in \mathcal{A}$ give rise to faithful conditional expectations $E_l : \mathcal{M} \to \mathcal{N}$ via $E_l(m) := l \triangleright m$, where under certain regularity conditions this correspondence is one-to-one. Associated with such left integrals we construct "Jones projections" $e_l \in \mathcal{A}$ obeying for all $m \in \mathcal{M}$ the Jones relations $e_l m e_l = E_l(m) e_l = e_l E_l(m)$ as an identity in $\mathcal{M} \rtimes \mathcal{A}$. We also present a concept of Plancherel-duality ("p-duality") for positive nondegenerate left integrals, where by definition the p-dual $\lambda \in \hat{\mathcal{A}}$ of $l \in \mathcal{A}$ is the unique solution of $\lambda \rightharpoonup e_l \equiv \hat{E}_\lambda(e_l) = \mathbf{1}$. Here $\rightharpoonup$ denotes the canonical left action of the dual weak Hopf algebra $\hat{\mathcal{A}}$ on $\mathcal{A}$. Finally, we prove that $\mathcal{N} \subset \mathcal{M}$ always has finite index and depth 2 and that the basic Jones construction for $\mathcal{N} \subset \mathcal{M}$ is given by the ideal $\mathcal{M}_1 := \mathcal{M} e_l \mathcal{M} \subset \mathcal{M} \rtimes \mathcal{A}$. In this way p-dual left integrals precisely correspond to Haagerup-dual conditional expectations. We give appropriate regularity conditions (such as outerness of the $\mathcal{A}$-action) guaranteeing $\mathcal{M}_1 = \mathcal{M} \rtimes \mathcal{A}$. Under these conditions the standard invariant for $\mathcal{N} \subset \mathcal{M}$ is given by the Jones triple $\mathcal{A}_L \subset \mathcal{A} \subset \mathcal{A} \rtimes \hat{\mathcal{A}}$, where $\mathcal{A}_L := \mathbf{1}_{\mathcal{A}} \leftharpoonup \hat{\mathcal{A}} \subset \mathcal{A}$ is a nontrivial subalgebra isomorphic to $\mathcal{A} \triangleright \mathbf{1}_{\mathcal{M}} \subset \mathcal{M}$. As a particular example we discuss crossed products by partly inner group actions.

In a subsequent paper we will show that converseley any reducible finite index and depth-2 Jones tower of von-Neumann factors arises in this way, where the inclusions are irreducible if and only if $\mathcal{A}$ and $\hat{\mathcal{A}}$ are ordinary Kac algebras. This generalizes the well known results on irreducible depth-2 inclusions obtained by various people after a proposal by A. Ocneanu.

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1 Introduction

1.1 Motivation

There is a lot of work concerning the classification of irreducible Jones inclusions of factors, see [Po] for example. In the special case of inclusions $\mathcal{N} \subset \mathcal{M}$ of depth 2 in the sense of Ocneanu [Oc1] it is known that the subfactor $\mathcal{N} \equiv \mathcal{M}^{\mathcal{A}}$ always arises as the fixed point algebra under an outer action of a Kac algebra $\mathcal{A}$ on $\mathcal{M}$, see [Da, Lo1, Szy] for the finite index case and [EN, NW] for the infinite index case. Converseley, any such outer action gives rise to a Jones triple $\mathcal{N} \equiv \mathcal{M}^{\mathcal{A}} \subset \mathcal{M} \subset \mathcal{M} \rtimes \mathcal{A}$ obeying Ocneanu's depth 2 condition and the "symmetry algebra" can be recovered as $\mathcal{A} = \mathcal{N}' \cap (\mathcal{M} \rtimes \mathcal{A})$.

With this work we initiate a project where we want to generalize these ideas to reducible finite index inclusions of von-Neumann algbras. As a motivation we point out that there are many natural situations, where inclusions are not irreducible. Let us give two examples:

1. A partly inner group action:

Let $\alpha : G \to \operatorname{Aut} \mathcal{M}$ be an action of a finite group G on a von-Neumann factor $\mathcal{M}$ and let $\mathcal{N} := \mathcal{M}^G \subset \mathcal{M}$ be the fixed point algebra and $\mathcal{M} \subset \mathcal{M} \rtimes G$ the crossed product extension. Denote $\mathrm{pr}_{\operatorname{Out} \mathcal{M}} : \operatorname{Aut} \mathcal{M} \to \operatorname{Out} \mathcal{M}$ the canonical projection onto the group of outer automorphisms of $\mathcal{M}$. Then $H := \mathrm{Ker}\,(\pi \circ \alpha)$ is a normal subgroup of G and all implementers $u(h) \in \mathcal{M},\ h \in H$, (i.e. satisfying $\alpha_h = \mathrm{Ad}\,u(h)$) commute with $\mathcal{N}$,

$$u(h) \in \mathcal{N}' \cap \mathcal{M} \quad \forall h \in H$$

Hence, if G does not act purely outerly, we have a non-trivial relative commutant. By definition this means that the inclusion $\mathcal{N} \subset \mathcal{M}$ is reducible. Similarly, $\mathcal{M} \subset \mathcal{M} \rtimes G$ is irreducible if and only if G acts outerly on $\mathcal{M}$. We will frequently come back to this example within our approach, showing that for a non-outer Galois G-action on $\mathcal{M}$ [1] the second relative commutant $\mathcal{A} := \mathcal{N}' \cap \mathcal{M} \rtimes G$ naturally acquires the structure of a *weak Hopf algebra.*

2. The observable algebra in the reduced field bundle:

Let $\mathcal{A}$ denote the algebra of quasilocal observables of a quantum field theory in the algebraic frame work [Haa]. Let us assume that the theory has only finitely many irreducible sectors $\rho_i,\ i = 1 \ldots n$, with finite statistics. We define the "master" endomorphism $\rho := \bigoplus_{i=1}^{n} \rho_i$. Then the inclusion

$$\rho(\mathcal{A}) \subset \mathcal{A}$$

is by construction reducible. Taking the minimal conditional expectation $E_0 : \mathcal{A} \to \rho(\mathcal{A})$ (corresponding to the standard left inverse of ρ) and performing the basic Jones construction one ends up with the reduced field bundle $\mathcal{F}_{red} \supset \mathcal{A} \supset \rho(\mathcal{A})$. As has been noticed by [NR,R] the symmetry algebra associated with this inclusion can also be described by a weak Hopf algebra. (Actually, the first proposal for axioms of weak Hopf algebras by [N1] had been based on this example).

Both examples share the further property of having depth 2. Motivated by the irreducible case we therefore expect a Hopf algebra like symmetry encoded in any reducible finite index and depth-2 inclusion of von-Neumann algebras. Our claim is that the appropriate notion of a symmetry which can always be recovered in such a setting is indeed the concept of a *weak C^*-Hopf algebra.*

As a motivation we recall that already the investigation of the inner symmetry of the Ising model has led to a weaker concept then the familiar one of a Hopf-symmetry, see [MS]. There the authors relaxed the usual property that the coproduct of the identity should be the tensor product of the identity with itself, which is one of the axioms for Hopf algebras. Instead they only required that the coproduct of the identity is a projection. In the case of [MS], this projecton is the one onto the "good sub-representations" in a tensor product of good representations.

Now in Jones theory we always have a two step periodicity . Therefore we expect a concept of a symmetry algebra $\mathcal{A}$ such that the dual object $\hat{\mathcal{A}}$ is an algebra of the same type. In this way we can develop the picture of a Jones tower generated by taking alternating crossed products with $\mathcal{A}$ and $\hat{A}$, respectively, reflecting the two step periodicity. Thus, we are naturally lead to depart from the Mack-Schomerus setting in two essential ways. First, in order that $\hat{\mathcal{A}}$ be an associative algebra we require the coproduct on $\mathcal{A}$ to be strictly coassociative (in [MS] this had been relaxed to quasi-coassociativity in the sense of [Dr]). Second, if the coproduct on $\mathcal{A}$ is non-unital then the counit on $\hat{\mathcal{A}}$ cannot be an algebra morphism. Hence we drop the requirement of counits being algebra maps altogether (equivalently, the coproduct on $\hat{\mathcal{A}}$ is allowed to be non-unital as well).

[1] The notion of a Galois action has been introduced in [CS]. Equivalently, this means that $\mathcal{N} \subset \mathcal{M} \subset \mathcal{M} \rtimes G$ is a Jones triple, i.e. $\mathcal{M} \rtimes G = \mathcal{M} h \mathcal{M}$, where $h \in \mathbf{C}G$ is the normalised Haar integral, see also Appendix A.

Before putting these ideas into a precise definition let us give some more heuristic pictures of the kind structures to be expected of a "symmetry algebra" $\mathcal{A}$ appearing in a reducible depth-2 Jones tower $\ldots \mathcal{M}_{i-1} \subset \mathcal{M}_i \subset \mathcal{M}_{i+1} \subset \ldots$.

First, by experience from the irreducible depth-2 case the algebras $\mathcal{A} = \mathcal{M}'_{i-1} \cap \mathcal{M}_{i+1}$ and $\hat{\mathcal{A}} = \mathcal{M}'_i \cap \mathcal{M}_{i+2}$ should be dual to each other. Next, as an intrinsic feature we should always have two distinguished commuting subalgebras $\mathcal{A}_{L/R} \subset \mathcal{A}$ given by

$$\begin{aligned} \mathcal{A}_L &= \mathcal{M}'_{i-1} \cap \mathcal{M}_i && (1.1) \\ \mathcal{A}_R &= \mathcal{M}'_i \cap \mathcal{M}_{i+1} && (1.2) \end{aligned}$$

and similarly for $\hat{\mathcal{A}}$. Also, proceeding alternatingly up the tower we would expect intrinsic identifications $\hat{\mathcal{A}}_L \cong \mathcal{A}_R$ and $\hat{\mathcal{A}}_R \cong \mathcal{A}_L$, whereas $\mathcal{A}_L$ and $\mathcal{A}_R$ should naturally be anti-isomorphic. If we are dealing with factors $\mathcal{M}_i$, then we should have $\mathcal{A}_L \cap \mathcal{A}_R = \mathbf{C} = \hat{\mathcal{A}}_L \cap \hat{\mathcal{A}}_R$ and otherwise we would expect

$$\begin{aligned} \mathcal{A}_L \cap \mathcal{A}_R &= C(\mathcal{M}_i) && (1.3) \\ \hat{\mathcal{A}}_L \cap \hat{\mathcal{A}}_R &= C(\mathcal{M}_{i+1}) && (1.4) \end{aligned}$$

where $C(\mathcal{M}_i)$ denotes the center of $\mathcal{M}_i$. Next, observe that the depth-2 condition gives $\mathcal{M}_{i+1} = \mathcal{M}_i \vee \mathcal{A}$ and therefore there should exist a natural isomorphism

$$\hat{\mathcal{A}}_L \cap \hat{\mathcal{A}}_R \cong C(\mathcal{M}_{i+1}) \cong \mathcal{A}_R \cap C(\mathcal{A}) \ . \tag{1.5}$$

Applying the modular conjugation of $\mathcal{M}_i$ (which intertwines $\mathcal{A}_L$ and $\mathcal{A}_R$) we further conclude

$$\mathcal{C}(M_{i+1}) \cong \mathcal{C}(\mathcal{M}_{i-1}) = \mathcal{A}_L \cap \mathcal{C}(\mathcal{A}) \tag{1.6}$$

Finally, we would expect

$$C(\mathcal{M}_i) \cap C(\mathcal{M}_{i+1}) \cong \mathcal{A}_L \cap \mathcal{A}_R \cap C(\mathcal{A}) \cong \hat{\mathcal{A}}_L \cap \hat{\mathcal{A}}_R \cap C(\hat{\mathcal{A}}) \tag{1.7}$$

since in any Jones tower this abelian algebra is globally fixed, i.e. we have for all $n \in \mathbf{N}$

$$C(\mathcal{M}_i) \cap C(\mathcal{M}_{i+1}) = C(\mathcal{M}_{i+n}) \cap C(\mathcal{M}_{i+n+1}) \tag{1.8}$$

as a strict identity .

It turns out that the existence of the subalgebras $\mathcal{A}_{L/R} \subset \mathcal{A}$ and $\hat{\mathcal{A}}_{L/R} \subset \hat{\mathcal{A}}$ satisfying all these properties will indeed follow from our axioms for (finite dimensional) weak Hopf algebras $\mathcal{A}$ and $\hat{\mathcal{A}}$ below, provided $\mathcal{A}$ acts regularly (in a suitable sense) on $\mathcal{M}_i$ such that $\mathcal{M}_{i-1}$ is the fixed point algebra under this action and $\mathcal{M}_{i+1} = \mathcal{M}_i \rtimes \mathcal{A}$ is the crossed product equipped with its canonical dual $\hat{\mathcal{A}}$-action. Note that in the particular example of a partly inner group action on a factor $\mathcal{M}$ one has

$$\begin{aligned} \mathcal{A} &:= \mathcal{N}' \cap (\mathcal{M} \rtimes G) = \mathrm{span}\,\{u(h)g \mid h \in H,\ g \in G\} && (1.9) \\ \mathcal{A}_L &:= \mathcal{N}' \cap \mathcal{M} = \mathrm{span}\,\{u(h) \mid h \in H\} && (1.10) \\ \mathcal{A}_R &:= \mathcal{M}' \cap (\mathcal{M} \rtimes G) = \mathrm{span}\,\{u(h)h^{-1} \mid h \in H\} && (1.11) \end{aligned}$$

The weak Hopf algebra structure associated with this example will be described in Section 2.5 and, more generally, in Appendix B.

1.2 The basic setting

We now give our axioms for weak C^*-Hopf algebras, where we restrict ourselves to the finite dimensional case. A first proposal of such a structure has been made by [N1] and independently by [BSz,Sz]. Meanwhile the theory is well established [N2,BNS], and in particular we have unified our axioms (see [N2] for a detailed discussion of the relations between various sets of axioms). We remark, that the *quantum groupoids* of Ocneanu [Oc2] and the *face algebras* of Hayashi [Hay] are special types of weak Hopf algebras in our sense, where the above mentioned subalgebras $\mathcal{A}_{L/R} \subset \mathcal{A}$ are restricted to be abelian. Moreover, as has been shown in [N2], weak C^*-Hopf algebras with an involutive antipode precisely yield the generalized Kac algebras of Yamanouchi [Ya1].

Definition 1.2.1 (*Axioms for finite dimensional weak C^*-Hopf algebras*)
A finite dimensional weak C^*-Hopf-Algebra $\mathcal{A}$ is a finite dimensional C^*-algebra with linear structure maps

$$\begin{array}{rcll} \Delta & : & \mathcal{A} \to \mathcal{A} \otimes \mathcal{A} & \text{coproduct} \\ \varepsilon & : & \mathcal{A} \to \mathbf{C} & \text{counit} \\ S & : & \mathcal{A} \to \mathcal{A} & \text{antipode} \end{array}$$

satisfying the following axioms for $x, y, z \in \mathcal{A}$:

$$\begin{array}{rll} I. & a) & \Delta(xy) = \Delta(x)\Delta(y) \\ & b) & \Delta(x^*) = \Delta(x)^* \\ & c) & (\Delta \otimes id) \circ \Delta = (id \otimes \Delta) \circ \Delta \\ & d) & (\Delta(\mathbf{1}) \otimes \mathbf{1})\,(\mathbf{1} \otimes \Delta(\mathbf{1})) = \mathbf{1}_{(1)} \otimes \mathbf{1}_{(2)} \otimes \mathbf{1}_{(3)} \end{array}$$

$$\begin{array}{rll} II. & a) & (\epsilon \otimes id) \circ \Delta = id = (id \otimes \varepsilon) \circ \Delta \\ & b) & \varepsilon(xyz) = \varepsilon(xy_{(1)})\varepsilon(y_{(2)}z) \end{array}$$

$$\begin{array}{rll} III. & a) & S(x_{(1)})x_{(2)} = \mathbf{1}_{(1)}\varepsilon(x\mathbf{1}_{(2)}) \\ & b) & x_{(1)}S(x_{(2)}) = \varepsilon(\mathbf{1}_{(1)}x)\mathbf{1}_{(2)} \\ & c) & S(x_{(1)})x_{(2)}S(x_{(3)}) = S(x) \end{array}$$

Here we have used the Sweedler notation $\Delta(x) = x_{(1)} \otimes x_{(2)}$, $(\Delta \otimes id) \circ \Delta(x) = (id \otimes \Delta) \circ \Delta(x) = x_{(1)} \otimes x_{(2)} \otimes x_{(3)}$, etc., implying as usual a summation convention in the corresponding tensor product spaces. Note that $IIIa)$ or $IIIb)$ also imply

$$x_{(1)}S(x_{(2)})x_{(3)} = x \ . \tag{1.12}$$

Moreover, due to the compatibility requirement $Ib)$, the axioms $Id)$ and $IIb)$ are equivalent to, respectively,

$$\begin{array}{rll} I & d') & (\mathbf{1} \otimes \Delta(\mathbf{1}))\,(\Delta(\mathbf{1}) \otimes \mathbf{1}) = \mathbf{1}_{(1)} \otimes \mathbf{1}_{(2)} \otimes \mathbf{1}_{(3)} \\ II & b') & \varepsilon(xyz) = \varepsilon(xy_{(2)})\varepsilon(y_{(1)}z) \end{array}$$

A weak Hopf algebra is an ordinary Hopf algebra, if the counit ε is multiplicative, or if $\Delta(\mathbf{1}) = \mathbf{1} \otimes \mathbf{1}$, or if instead of III the antipode S satisfies the usual axioms $S(x_{(1)})x_{(2)} = \varepsilon(x)\mathbf{1} = x_{(1)}S(x_{(2)})$ (in which case axiom $IIIc)$ would be a consequence). As for ordinary Hopf algebras, under the axioms $I - III$ the antipode is uniquely determined provided it exists [N2,BNS]. Moreover, it is invertible and satisfies

$$\begin{aligned} S(xy) &= S(y)S(x) && (1.13)\\ \Delta \circ S &= (S \otimes S) \circ \Delta^{op} && (1.14)\\ S(x^*)^* &= S^{-1}(x) && (1.15) \end{aligned}$$

We remark that it will often be easier to check (1.13) and (1.14) explicitely, in which case the axioms $IIIc$ and $IIb), b')$ become redundant and $Id), d')$ may be relaxed by only requiring [N2]

$$[(\mathbf{1} \otimes \Delta(\mathbf{1}))\,,\,(\Delta(\mathbf{1}) \otimes \mathbf{1})] = 0 \tag{1.16}$$

Note that $Id)$ and $Id')$ together always imply (1.16).

Under the present setting the following axioms of [BSz,Sz] are a consequence [N2,BNS]

$$\begin{aligned} S(x_{(1)})x_{(2)} \otimes x_{(3)} &= \mathbf{1}_{(1)} \otimes x\mathbf{1}_{(2)} && (1.17)\\ x_{(1)} \otimes x_{(2)}S(x_{(3)}) &= \mathbf{1}_{(1)}x \otimes \mathbf{1}_{(2)} && (1.18)\\ S^{-1}(x_{(3)})x_{(2)} \otimes x_{(1)} &= \mathbf{1}_{(2)} \otimes x\mathbf{1}_{(1)} && (1.19)\\ x_{(3)} \otimes x_{(2)}S^{-1}(x_{(1)}) &= \mathbf{1}_{(2)}x \otimes \mathbf{1}_{(1)}\ . && (1.20) \end{aligned}$$

Given a finite dimensional weak C^*-Hopf algebra $\mathcal{A}$ as above we denote $\operatorname{Rep}\mathcal{A}$ the category of unital $*$-representations $\pi_\alpha : \mathcal{A} \to \operatorname{End} V_\alpha$ on finite dimensional Hilbert spaces V_α. Then $\operatorname{Rep}\mathcal{A}$ becomes a rigid C^*-tensor category with monoidal product $\pi_{\alpha\times\beta} := (\pi_\alpha \otimes \pi_\beta) \circ \Delta$ and $V_{\alpha\times\beta} := \pi_{\alpha\times\beta}(\mathbf{1})\,(V_\alpha \otimes V_\beta)$. It is shown in [BNS] that the counit ε provides a positive (unnormalized) state on $\mathcal{A}$. Let π_ε denote the associated GNS-representation of $\mathcal{A}$ with cyclic vector $\Omega_\varepsilon \in V_\varepsilon$. Then π_ε is the unit object in $\operatorname{Rep}\mathcal{A}$ with unitary intertwiners $U^r_\alpha : V_\alpha \to V_{\alpha\times\varepsilon}$ and $U^\ell_\alpha : V_\alpha \to V_{\varepsilon\times\alpha}$ given on $v \in V_\alpha$ by [BSz,Sz,N2]

$$\begin{aligned} U^r_\alpha v &:= \pi_{\alpha\times\varepsilon}(\mathbf{1})\,(v \otimes \Omega_\varepsilon) && (1.21)\\ U^\ell_\alpha v &:= \pi_{\varepsilon\times\alpha}(\mathbf{1})\,(\Omega_\varepsilon \otimes v)\ . && (1.22) \end{aligned}$$

As has been pointed out in [N2], the intertwining relations and the triangle identities for these intertwiners are precisely guaranteed by the axioms $IIb)$ and $IIb')$. Following [BSz,Sz] we call $\mathcal{A}$ *pure* if ε is pure and therefore π_ε is *irreducible*.

As for ordinary Hopf algebras, our antipode axioms also provide a rigidity structure on $\operatorname{Rep}\mathcal{A}$, see [N2] for a general discussion and [BSz,Sz] for the C^*-formulation.

Associated with any weak Hopf algebra $(\mathcal{A}, \mathbf{1}, \varepsilon, \Delta, S)$ we can immediately define the dual object $(\hat{\mathcal{A}}, \hat{\mathbf{1}}, \hat{\varepsilon}, \hat{\Delta}, \hat{S})$ by dualising all structure maps: Let $\hat{\mathcal{A}}$ be the dual space of $\mathcal{A}$ and denote

$$\langle \cdot \mid \cdot \rangle : \hat{\mathcal{A}} \otimes \mathcal{A} \to \mathbf{C}$$

the canonical pairing. Then for $\psi, \varphi \in \hat{\mathcal{A}}$ and $x \in \mathcal{A}$

$$\langle \psi\varphi \mid x \rangle := \langle \varphi \otimes \psi \mid \Delta(x) \rangle \tag{1.23}$$

defines an associative product on $\hat{A}$. Similarly we get the coproduct, the unit, the counit, the antipode and the $*$-structure by putting for $\psi \in \hat{\mathcal{A}}$ and $x, y \in \mathcal{A}$

$$\begin{aligned}
\langle \hat{\Delta}\psi \mid x \otimes y \rangle &:= \langle \psi \mid xy \rangle && (1.24)\\
\langle \hat{\mathbf{1}} \mid x \rangle &:= \varepsilon(x) && (1.25)\\
\hat{\varepsilon}(\psi) &:= \langle \psi \mid \mathbf{1} \rangle && (1.26)\\
\langle \hat{S}(\psi) \mid x \rangle &:= \langle \psi \mid S(x) \rangle && (1.27)\\
\langle \psi^* \mid x \rangle &:= \overline{\langle \psi \mid x_* \rangle} && (1.28)
\end{aligned}$$

where in the last line we have introduced the multiplicative antilinear involution

$$x_* := S(x)^* \ . \tag{1.29}$$

It turns out that the quintuple $(\hat{\mathcal{A}}, \hat{\mathbf{1}}, \hat{\varepsilon}, \hat{\Delta}, \hat{S})$ again defines a weak C^*-Hopf algebra [BNS]. Note in particular, that due to (1.25) and (1.26) the axioms $Id), d')$ and $IIb), b')$, respectively, become dual to each other.

The theory of integrals generalizes from ordinary Hopf algebras as follows. An element $l \in \mathcal{A}$ is called a *left integral*, if one - and hence all - of the following three equivalent conditions hold

$$\begin{aligned}
(\mathbf{1} \otimes a)\, \Delta(l) &= (S(a) \otimes \mathbf{1})\, \Delta(l), \quad \forall a \in \mathcal{A} && (1.30)\\
a\, l &= a_{(1)} S(a_{(2)})\, l \equiv \varepsilon(\mathbf{1}_{(1)} a) \mathbf{1}_{(2)} l, \quad \forall a \in \mathcal{A} && (1.31)\\
a\, l &= a_{(2)} S^{-1}(a_{(1)})\, l \equiv \varepsilon(\mathbf{1}_{(2)} a) \mathbf{1}_{(1)} l, \quad \forall a \in \mathcal{A}\ . && (1.32)
\end{aligned}$$

The equivalence of these conditions follows from IIa and (1.17)-(1.20) [BNS]. Similarly, $r \in \mathcal{A}$ is called a *right integral*, if one - and hence all - of the following three equivalent conditions hold

$$\begin{aligned}
\Delta(r)(a \otimes \mathbf{1}) &= \Delta(r)(\mathbf{1} \otimes S(a)), \quad \forall a \in \mathcal{A} && (1.33)\\
r\, a &= r\, S(a_{(1)}) a_{(2)} \equiv r \mathbf{1}_{(1)} \varepsilon(a \mathbf{1}_{(2)}), \quad \forall a \in \mathcal{A} && (1.34)\\
r\, a &= r\, S^{-1}(a_{(2)}) a_{(1)} \equiv r \mathbf{1}_{(2)} \varepsilon(a \mathbf{1}_{(1)}), \quad \forall a \in \mathcal{A}\ . && (1.35)
\end{aligned}$$

Note that $l \in \mathcal{A}$ is a left integral if and only if $S(l)$ (or l^*) is a right integral. If l is a left and a right integral, then it is called a *two sided integral.* A left (right) integral is called *positive* and/or *nondegenerate*, if it is positive and/or nondegenerate as a linear functional on $\hat{\mathcal{A}}$. As opposed to ordinary finite dimensional Hopf algebras, in our case the space of left (right) integrals is more than one dimensional. In fact, as a linear space it will be naturally isomorphic to the subspaces $\mathcal{A}_{L,R} \subset \mathcal{A}$ mentioned in Section 1.1, ([BNS], see also Lemma 2.6.3 below). Following [BSz,Sz] we define the *normalized Haar integral* $h \in \mathcal{A}$ to be the unique two sided integral satisfying the normalization condition

$$S(h_{(1)}) h_{(2)} = h_{(1)} S(h_{(2)}) = \mathbf{1}\ . \tag{1.36}$$

The uniqueness of h is obvious from the definitions which in particular imply

$$h = h^2 = h^* = S(h) \tag{1.37}$$

The existence of h is proven in [BNS], where it is also shown that as a functional on $\hat{\mathcal{A}}$ it is positive and nondegenerate.

A more detailed investigation of weak C^*-Hopf algebras, including a theory of Fourier transformations and the modular theory for the normalized Haar integral, is given in [BNS]. From now on, by a weak Hopf algebra $\mathcal{A}$ we will always mean a finite dimensional weak C^*-Hopf algebra.

Next, we need the notion of an $\mathcal{A}$-module algebra $\mathcal{M}$ allowing for a natural definition of a 'fixed point' subalgebra $\mathcal{N} \equiv \mathcal{M}^{\mathcal{A}} \subset \mathcal{M}$ and a crossed product extension $\mathcal{M} \subset \mathcal{M} \rtimes \mathcal{A}$.

Definition 1.2.2 *($\mathcal{A}$-module algebras)*
A $*$-algebra $\mathcal{M}$ together with a left action $\triangleright : \mathcal{A} \otimes \mathcal{M} \to \mathcal{M}$, $(a \otimes m) \mapsto a \triangleright m$, is called an $\mathcal{A}$-module algebra, if the following properties hold for all $a, b \in \mathcal{A}$ and $m, n \in \mathcal{M}$

$$(ab) \triangleright m = a \triangleright (b \triangleright m) \tag{1.38}$$
$$\mathbf{1}_{\mathcal{A}} \triangleright m = m \tag{1.39}$$
$$a \triangleright (mn) = (a_{(1)} \triangleright m)(a_{(2)} \triangleright n) \tag{1.40}$$
$$(a \triangleright m)^* = a_* \triangleright m^* \tag{1.41}$$
$$a \triangleright \mathbf{1}_{\mathcal{M}} = (a_{(1)} S(a_{(2)})) \triangleright \mathbf{1}_{\mathcal{M}} \equiv \varepsilon(\mathbf{1}_{(1)} a) \mathbf{1}_{(2)} \triangleright \mathbf{1}_{\mathcal{M}} \tag{1.42}$$

If $\mathcal{M}$ is a C^*- or a von-Neumann algebra, then we also require $a \triangleright$ to be norm or weakly continuous, respectively, for all $a \in \mathcal{A}$.

We will see that the axiom (1.42) may equivalently be replaced by

$$a \triangleright \mathbf{1}_{\mathcal{M}} = (a_{(2)} S^{-1}(a_{(1)})) \triangleright \mathbf{1}_{\mathcal{M}} \equiv \varepsilon(\mathbf{1}_{(2)} a) \mathbf{1}_{(1)} \triangleright \mathbf{1}_{\mathcal{M}} \tag{1.43}$$

An example of an $\mathcal{A}$-module action is provided by the natural left action $\mathcal{A} \otimes \hat{\mathcal{A}} \ni a \otimes \phi \mapsto (a \rightharpoonup \phi) \in \hat{\mathcal{A}}$ of $\mathcal{A}$ on its dual $\hat{\mathcal{A}}$, which together with the opposite right $\mathcal{A}$-action $\hat{\mathcal{A}} \otimes \mathcal{A} \ni \phi \otimes a \mapsto (\phi \leftharpoonup a) \in \hat{\mathcal{A}}$ is defined as for ordinary Hopf algebras, i.e.

$$\langle a \rightharpoonup \phi \mid b \rangle := \langle \phi \mid ba \rangle \tag{1.44}$$
$$\langle \phi \leftharpoonup a \mid b \rangle := \langle \phi \mid ab \rangle \tag{1.45}$$

where $\phi \in \hat{\mathcal{A}}$ and $a, b \in \mathcal{A}$. For the left action (1.44) the identities (1.42) and (1.43) follow from the axioms *IIb* and *IIIb*. As a natural cyclic $\mathcal{A}$-submodule we denote

$$\hat{\mathcal{A}}_R := \mathcal{A} \rightharpoonup \mathbf{1}_{\hat{\mathcal{A}}} \subset \hat{\mathcal{A}}$$

which as an object in Rep $\mathcal{A}$ is in fact equivalent to the "trivial" representation π_ε of $\mathcal{A}$. Moreover, (1.42) implies (see Section 2.2) that as a cyclic $\mathcal{A}$-submodule

$$\mathcal{M}_R := \mathcal{A} \triangleright \mathbf{1}_{\mathcal{M}} \subset \mathcal{M} \tag{1.46}$$

is homomorphic to $\hat{\mathcal{A}}_R$ via the map

$$\tau_\triangleright : \hat{\mathcal{A}}_R \ni (a \rightharpoonup \mathbf{1}_{\hat{\mathcal{A}}}) \mapsto (a \triangleright \mathbf{1}_{\mathcal{M}}) \in \mathcal{M}_R \tag{1.47}$$

We will see (Lemma 2.2.1) that if $\mathcal{A}$ is pure, (1.47) even defines an $\mathcal{A}$-module isomorphism. Moreover, we will show in Proposition 2.2.2 that for any $\mathcal{A}$-module algebra $\mathcal{M}$ the submodule $\mathcal{M}_R \subset \mathcal{M}$ also is a $*$-subalgebra of $\mathcal{M}$ and that the map (1.47) also provides a homomorphism of $*$-algebras. Interchanging the rôle of $\mathcal{A}$ and $\hat{\mathcal{A}}$ in (1.44) we may put similarly

$$\mathcal{A}_R := \{(id \otimes \phi)(\Delta(\mathbf{1}_{\mathcal{A}})) \mid \phi \in \hat{\mathcal{A}}\} \subset \mathcal{A} \tag{1.48}$$
$$\mathcal{A}_L := \{(\phi \otimes id)(\Delta(\mathbf{1}_{\mathcal{A}})) \mid \phi \in \hat{\mathcal{A}}\} \equiv S(\mathcal{A}_R)\ . \tag{1.49}$$

Consequently, $\mathcal{A}_{R/L} \subset \mathcal{A}$ are also $*$-subalgebras. Moreover, $\mathcal{A}_{L/R} \cong \hat{\mathcal{A}}_{R/L}$ as $*$-algebras, the isomorphism being given by [BSz,Sz,N2,BNS]

$$\mathcal{A}_L \ni a \mapsto (a \rightharpoonup \mathbf{1}_{\hat{\mathcal{A}}}) \in \hat{\mathcal{A}}_R \tag{1.50}$$

$$\mathcal{A}_R \ni a \mapsto (\mathbf{1}_{\hat{\mathcal{A}}} \leftharpoonup a) \in \hat{\mathcal{A}}_L \tag{1.51}$$

with inverses given by the dual versions, respectively. Composing (1.50) with (1.47) we get a $*$-algebra epimorphism

$$\mu_\triangleright : \mathcal{A}_L \ni a \mapsto (a \triangleright \mathbf{1}_\mathcal{M}) \in \mathcal{M}_R \tag{1.52}$$

which becomes an isomorphism if $\mathcal{A}$ is pure. Since by (1.16) $\mathcal{A}_L$ and $\mathcal{A}_R$ always commute and since one also always has

$$\mu_\triangleright(\mathcal{A}_L \cap \mathcal{A}_R) \subset C(\mathcal{M}) \tag{1.53}$$

(see Corollary 2.4.4(i)), we already arrive at the first ingrediences expected to appear in reducible Jones towers as discussed in our motivation in Section 1.1 (see Eq.(1.3)).

We now turn to what by analogy we would like to call the fixed point subalgebra $\mathcal{N} \equiv \mathcal{M}^\mathcal{A} \subset \mathcal{M}$, for which the following definitions will be appropriate.

Definition 1.2.3 *(Fixed point algebra)*
Given an $\mathcal{A}$-module algebra $\mathcal{M}$ we define the "fixed point algebra" $\mathcal{N} \equiv \mathcal{M}^\mathcal{A} \subset \mathcal{M}$ to be the unital $*$-subalgebra given by the elements $n \in \mathcal{M}$ satisfying either of the following conditions

$$\begin{array}{rrcll} i) & a \triangleright (mn) & = & (a \triangleright m)\, n & , \quad \forall a \in \mathcal{A},\ m \in \mathcal{M} \\ ii) & a \triangleright n & = & a_{(1)} S(a_{(2)}) \triangleright n & , \quad \forall a \in \mathcal{A} \\ iii) & a \triangleright (nm) & = & n\, (a \triangleright m) & , \quad \forall a \in \mathcal{A},\ m \in \mathcal{M} \\ iv) & a \triangleright n & = & a_{(2)} S^{-1}(a_{(1)}) \triangleright n & , \quad \forall a \in \mathcal{A}\ . \end{array}$$

We will show in Proposition 2.3.1 that the conditions $(i - iv)$ are in fact all equivalent. Putting $m = \mathbf{1}_\mathcal{M}$ in condition $i)$ and $iii)$ then implies

$$\mathcal{M}_R \subset \mathcal{N}' \cap \mathcal{M}. \tag{1.54}$$

and we will later fix appropriate conditions guaranteeing equality in (1.54). Since eventually we want to identify A_L with the relative commutant $\mathcal{N}' \cap \mathcal{M}$, we now propose the following

Definition 1.2.4 An $\mathcal{A}$-module algebra $\mathcal{M}$ is called *standard*, if $\mu_\triangleright$ in (1.52) provides an isomorphism $\mathcal{A}_L \cong \mathcal{M}_R$.

Note that for ordinary Hopf algebras one always has $\mathcal{A}_L = \mathcal{A}_R = \mathcal{M}_R = \mathbf{C}$ and therefore in this case standardness trivially holds. Moreover, condition (ii) of Definition 1.2.3 together with (1.42) and (1.54) also imply

$$\mu_\triangleright(\mathcal{A}_L \cap C(\mathcal{A})) \subset C(\mathcal{N}) \tag{1.55}$$

and under suitable regularity conditions we will later also have equality in (1.55), thus reproducing (1.1) and (1.6).

Finally, we define the crossed product $\mathcal{M} \rtimes \mathcal{A}$ as an "amalgamated" tensor product over $\mathcal{A}_L$.

Definition 1.2.5 *(Crossed products)*
Let $\mathcal{M}$ be an $\mathcal{A}$-module $*$-algebra. The *crossed product* $\mathcal{M}\rtimes\mathcal{A}$ is defined to be the linear space

$$\mathcal{M}\rtimes\mathcal{A} = \mathcal{M} \otimes_{\mathcal{A}_L} \mathcal{A} \tag{1.56}$$

where $\mathcal{A}_L$ acts on $\mathcal{A}$ by left multiplication and on $\mathcal{M}$ by right multiplication via its image under $\mu_\triangleright$. The $*$-algebra structure on $\mathcal{M}\rtimes\mathcal{A}$ is defined in the same way as for ordinary crossed products, i.e. for $m, m' \in \mathcal{M}$ and $a, a' \in \mathcal{A}$

$$(m\rtimes a)(m'\rtimes a') = (m(a_{(1)} \triangleright m')\rtimes a_{(2)}a') \tag{1.57}$$
$$(m\rtimes a)^* = (\mathbf{1}_\mathcal{M}\rtimes a^*)(m^*\rtimes \mathbf{1}_\mathcal{A}) \equiv ((a^*_{(1)} \triangleright m^*)\rtimes a^*_{(2)}) \quad . \tag{1.58}$$

We will show in Theorem 3.1.1 that with these definitions $\mathcal{M}\rtimes\mathcal{A}$ indeed becomes a well defined $*$-algebra extending $\mathcal{M} \equiv (\mathcal{M}\rtimes\mathbf{1}_\mathcal{A})$. Moreover, as for crossed products by ordinary Hopf algebras we will have

$$((a\triangleright m)\rtimes\mathbf{1}_\mathcal{A}) = (\mathbf{1}_\mathcal{M}\rtimes a_{(1)})\,(m\rtimes\mathbf{1}_\mathcal{A})\,(\mathbf{1}_\mathcal{M}\rtimes S(a_{(2)}))\ . \tag{1.59}$$

Confirming with the scenario in Section 1.1 we will also get

$$(\mathbf{1}_\mathcal{M}\rtimes\mathcal{A}) \quad\subset\quad \mathcal{N}'\cap\mathcal{M}\rtimes\mathcal{A} \tag{1.60}$$
$$(\mathbf{1}_\mathcal{M}\rtimes\mathcal{A}_R) \quad\subset\quad \mathcal{M}'\cap\mathcal{M}\rtimes\mathcal{A} \tag{1.61}$$
$$\mathbf{1}_\mathcal{M}\rtimes(\mathcal{A}_R\cap C(\mathcal{A})) \quad\subset\quad C(\mathcal{M}\rtimes\mathcal{A}) \tag{1.62}$$
$$\mathbf{1}_\mathcal{M}\rtimes(\mathcal{A}_L\cap\mathcal{A}_R\cap C(\mathcal{A})) \quad\subset\quad \mathcal{M}\cap C(\mathcal{M}\rtimes\mathcal{A}) \equiv \mathcal{N}\cap C(\mathcal{M}\rtimes\mathcal{A}) \tag{1.63}$$

where equality in all these inclusions will again be proven under suitable regularity conditions in Section 4. This concludes the presentation of our basic setting and we now proceed to describe our main results.

1.3 Summary of results

Many consequences of our axioms described so far are already contained in [BSz,Sz,N2,BNS] and are presented here only to the extend making this work sufficiently selfunderstood. Our main focus here is on applications to the theory of $\mathcal{A}$- (co)module algebras $\mathcal{M}$ and their crossed product extensions, which have not been treated in general in the above papers.

Section 2.1 starts with generalizing the well kown duality relation between actions and coactions from ordinary Hopf algebras to our setting. In Section 2.2 we study the submodule $\mathcal{A}\triangleright\mathbf{1}_\mathcal{M}$ and prove that it is a subalgebra of $\mathcal{M}$ homomorphic to $\mathcal{A}_L \cong \hat{\mathcal{A}}_R$. In Section 2.3 we verify that our axioms for the fixed point algebra in Definition 1.2.3 are in fact all equivalent. In Section 2.4 we review various relations between $\mathcal{A}_{L/R}$, $\mathcal{A}_L\cap\mathcal{A}_R$, $A_{L/R}\cap C(A)$ and their dual counterparts and show how they interplay with $\mathcal{M}_R$ and $C(\mathcal{M})$ in the spirit of our discussion in Section 1.1. In Section 2.5 we apply our formalism to the example of a partly inner group action on a factor $\mathcal{M}$.

In Section 2.6 we review and apply the theory of left integrals as developed in [BNS]. We show that positive and normalized left integrals $l \in \mathcal{A}$ give rise to conditional expectations $E_l : \mathcal{M} \to \mathcal{N} \equiv \mathcal{M}^\mathcal{A}$ via $E_l(m) := l\triangleright m$. Under the assumption $\mathcal{A}_L \cong \mathcal{M}_R = \mathcal{N}'\cap\mathcal{M}$ the correspondence $l \leftrightarrow E_l$ will be one-to-one and E_l will be faithful if and only if l is nondegenerate. Considering the special case $\mathcal{M} = \hat{\mathcal{A}}$ we have $\mathcal{N} = \hat{\mathcal{A}}_L$ and we denote $\mathrm{Ind}\, l \in C(\hat{\mathcal{A}})$ the index of $E_l : \hat{\mathcal{A}} \to \hat{\mathcal{A}}_L$. We will see that in fact

$$\mathrm{Ind}\, l \in C(\hat{\mathcal{A}})\cap\hat{\mathcal{A}}_R \tag{1.64}$$

for all nondegenerate left integrals $l \in \mathcal{A}$. Note that under the scenario of Section 1.1 we would have $\tau_{\triangleright}(C(\hat{\mathcal{A}}) \cap \hat{\mathcal{A}}_R) = \mu_{\triangleright}(\mathcal{A}_L \cap \mathcal{A}_R) = C(\mathcal{M})$ suggesting that in the general case the index of $E_l : \mathcal{M} \to \mathcal{N}$ might be given by

$$\operatorname{Ind} E_l = \tau_{\triangleright}(\operatorname{Ind} l) \in C(\mathcal{M}) \ . \tag{1.65}$$

Under suitable regularity conditions this will indeed be a result in Section 4.

In Section 3 we turn to the study of crossed products. In Section 3.1 we prove that $\mathcal{M} \rtimes \mathcal{A}$ is a well defined $*$-algebra extending $\mathcal{M} \equiv (\mathcal{M} \rtimes \mathbf{1}_{\mathcal{A}})$ and satisfying (1.59) - (1.63). In Section 3.2 we show that as for ordinary Hopf algebras there is a natural $\hat{\mathcal{A}}$-module algebra structure on $\mathcal{M} \rtimes \mathcal{A}$. The fixed point algebra under this action is given by $\mathcal{M} \equiv (\mathcal{M} \rtimes \mathbf{1}_{\mathcal{A}})$ and we have $(\mathcal{M} \rtimes \mathcal{A})_R = (\mathbf{1}_{\mathcal{M}} \rtimes \mathcal{A}_R)$. In Section 3.3 we improve the duality theory for nondegenerate left integrals (i.e. the theory of Fourier transformations) of [BNS] to develop a new notion of *p-duality* for *positive* nondegenerate left integrals, which is patterned after the corresponding notion of Haagerup duality for finite index conditional expectations in Jones theory. Associated with any normalized positive and nondegenerate left integral $l \in \mathcal{A}$ we define a projection $e_l \in \mathcal{A}$ and a positive and nondegenerate left integral $\lambda_l \in \hat{\mathcal{A}}$ - the p-dual of l - such that

$$\begin{aligned} e_l m e_l = e_l E_l(m) &= E_l(m) e_l \quad , \quad \forall m \in \mathcal{M} \\ \hat{E}_{\lambda_l}(e_l) &= \mathbf{1}_{\mathcal{M}} \end{aligned}$$

as identities in $\mathcal{M} \rtimes \mathcal{A}$. Proceeding by alternating crossed products we also provide a generalized Temperley-Lieb-Jones algebra.

Starting from Section 3.4 we focus our attention to the case where $\mathcal{M}$ is a von-Neumann algebra. We show that in this case also $\mathcal{M} \rtimes \mathcal{A}$ becomes a von-Neumann algebra by identifying it with a (non-unital) $*$-subalgebra of $\mathcal{M} \otimes \operatorname{End} \mathcal{A}$. In particular, associated with any (faithful, normal) $*$-representation π of $\mathcal{M}$ on a Hilbert space $\mathcal{H}$, we obtain a so-called *regular* (faithful, normal) representation π_{cros} of $\mathcal{M} \rtimes \mathcal{A}$ on (a subspace of) $\mathcal{H} \otimes L^2(\mathcal{A}, \hat{h})$, thus generalizing standard results for crossed products by ordinary Kac algebras [Pe, St, ES].

In Section 4 we apply our general results to Jones theory, guided by the special example of a partly inner group action. In Section 4.1 we show that any GNS-representation π_ω of $\mathcal{M}$ associated with a faithful normal $\mathcal{A}$-invariant state ω on $\mathcal{M}$ extends to a representation – still denoted π_ω – of $\mathcal{M} \rtimes \mathcal{A}$ (which may actually be identified with a subrepresentation of the regular representation π_{cros} mentioned above). We then prove that the basic Jones construction for $\mathcal{M}_{-1} \equiv \pi_\omega(\mathcal{M}^{\mathcal{A}}) \subset \pi_\omega(\mathcal{M}) \equiv \mathcal{M}_0$ is precisely given by $\mathcal{M}_1 := \pi_\omega(\mathcal{M} \rtimes \mathcal{A})$, which in general need not be isomorphic to $\mathcal{M} \rtimes \mathcal{A}$. Rather, we have $\mathcal{M}_1 \cong \mathcal{M} e_l \mathcal{M}$ which might be a nontrivial ideal in $\mathcal{M} \rtimes \mathcal{A}$. We also show that $\mathcal{M}_{-1} \subset \mathcal{M}_0 \subset \mathcal{M}_1$ is always of finite index and depth 2 and that

$$\operatorname{Ind} E_l \leq \tau_{\triangleright}(\operatorname{Ind} l)$$

for any positive nondegenerated left integral $l \in \mathcal{A}$, where equality holds if and only if $\mathcal{M}_1$ is a *faithful* image of $\mathcal{M} \rtimes \mathcal{A}$. We also point out that this in particular holds, if $\mathcal{M}$ is itself a crossed product, $\mathcal{M} = \mathcal{N} \rtimes \hat{\mathcal{A}}$, with canonical $\mathcal{A}$-action.

In Section 4.2 we introduce an appropriate outerness condition for $\mathcal{A}$-actions on $\mathcal{M}$ and show that $\mathcal{A}$ acts outerly iff $\mathcal{M}' \cap (M \rtimes \mathcal{A}) = (C(\mathcal{M}) \rtimes \mathcal{A}_R)$. For outer actions of a pure weak Hopf algebra $\mathcal{A}$ on a factor $\mathcal{M}$ this will imply that $\mathcal{M} \rtimes \mathcal{A}$ is a factor and therefore $\mathcal{M} \rtimes \mathcal{A} \cong \mathcal{M}_1$ and

$$\mathcal{N} \equiv \mathcal{M}^{\mathcal{A}} \subset \mathcal{M} \subset \mathcal{M} \rtimes \mathcal{A} \tag{1.66}$$

is a Jones triple of factors obeying $\mathcal{M}' \cap (\mathcal{M} \rtimes \mathcal{A}) = \mathcal{A}_R$. Moreover, in this case also $\hat{\mathcal{A}}$ is pure and acts outerly on $\mathcal{M} \rtimes \mathcal{A}$, such that our construction iterates. This generalizes well known facts for outer actions by groups [NaTa] or Kac algebras [St, ES].

In Section 4.3 we envoke methods of Tomita-Takesaki theory to prove that under the above setting $\mathcal{N}' \cap \mathcal{M} = \mathcal{A}_L$ and $\mathcal{N}' \cap \mathcal{M} \rtimes \mathcal{A} = \mathcal{A}$, thus establishing the heuristic picture developped in Section 1.1.

Finally, in Section 4.4 we generalize our results to non-factors $\mathcal{M}$ and non-pure weak Hopf algebras $\mathcal{A}$ by requiring as a regularity condition standardness and outerness of the $\mathcal{A}$-action together with $C(\mathcal{M}) = (\mathcal{A}_L \cap \mathcal{A}_R) \triangleright \mathbf{1}_{\mathcal{M}}$. [2] Again, in this case $\hat{\mathcal{A}}$ also acts regularly on $\mathcal{M} \rtimes \mathcal{A}$ and (1.66) still provides a Jones triple, where the relative commutants are again given by $\mathcal{A}_L$, $\mathcal{A}_R$ and $\mathcal{A}$, respectively, as above. Also, in this case the lower bounds in (1.55), (1.62) and (1.63) are saturated, i.e

$$\begin{aligned} C(\mathcal{N}) &= \mathcal{A}_L \cap C(\mathcal{A}) \\ C(\mathcal{M} \rtimes \mathcal{A}) &= \mathcal{A}_R \cap C(\mathcal{A}) \\ C(\mathcal{N}) \cap C(\mathcal{M} \rtimes \mathcal{A}) &\equiv C(\mathcal{M}) \cap C(\mathcal{M} \rtimes \mathcal{A}) = A_L \cap \mathcal{A}_R \cap C(\mathcal{A}) \end{aligned}$$

as anticipated in Section 1.1. Here we have identified $\mathcal{A} \equiv (\mathbf{1}_{\mathcal{M}} \rtimes \mathcal{A})$ and $\mathcal{A}_L \equiv \mathcal{A}_L \triangleright \mathbf{1}_{\mathcal{M}} \equiv (\mathbf{1}_{\mathcal{M}} \rtimes \mathcal{A}_L) \equiv \mathcal{M}_R$.

In Appendix A we generalise the notion of Galois actions [CS] to our setting and show that it is equivalent to $\mathcal{M}^{\mathcal{A}} \subset \mathcal{M} \subset \mathcal{M} \rtimes \mathcal{A}$ being a Jones triple. In Appendix B we analyze the appearance of cocycles for partly inner group actions within our framework.

In part II of this project [NSW] we will go the opposite way, i.e. we will show that for any finite index and depth-2 Jones tower

$$\mathcal{N} \subset \mathcal{M} \subset \mathcal{M}_1 \subset \mathcal{M}_2 \subset \dots$$

of von-Neumann algebras with finite dimensional centers the relative commutants $\mathcal{A} := \mathcal{N}' \cap \mathcal{M}_1$ and $\hat{\mathcal{A}} := \mathcal{M}' \cap \mathcal{M}_2$ give a dual pair of weak Hopf algebras acting regularly on $\mathcal{M}$ and $\mathcal{M}_1$, respectively, such that $\mathcal{N} = \mathcal{M}^{\mathcal{A}}$, $M_1 = \mathcal{M} \rtimes \mathcal{A}$ and $\mathcal{M}_2 = \mathcal{M}_1 \rtimes \hat{\mathcal{A}}$. Moreover, in this case $\mathcal{M}$ is a factor iff $\hat{\mathcal{A}}$ is pure, $\mathcal{N}$ is a factor iff $\mathcal{A}$ is pure and the inclusions are irreducible iff $\mathcal{A}$ and $\hat{\mathcal{A}}$ are ordinary Hopf algebras.

Note added: After writing up this work we were informed by L. Vainermann about a preprint by M. Enock and J.-M. Vallin [EV] treating a similar approach.

2 $\mathcal{A}$-Module Algebras

Throughout this section let $\mathcal{A}$ and $\hat{\mathcal{A}}$ be a dual pair of finite dimensional weak C^*-Hopf algebras and let $\mathcal{M}$ denote an $\mathcal{A}$-module algebra as described in Definition 1.2.2. The units in $\mathcal{A}$, $\hat{\mathcal{A}}$ and $\mathcal{M}$ will always be denoted by $\mathbf{1} \in \mathcal{A}$, $\hat{\mathbf{1}} \in \hat{\mathcal{A}}$ and $\mathbf{1}_{\mathcal{M}} \in \mathcal{M}$. Elements of $\mathcal{A}$ will be denoted by Roman letters $a, b, c...$ and elements of $\hat{\mathcal{A}}$ by Greek letters $\phi, \psi, \xi, ...$.

We will freely identify $\mathcal{A} = \hat{\hat{\mathcal{A}}}$ and denote the dual pairing by $\langle a|\phi\rangle \equiv \langle \phi|a\rangle$, $a \in \mathcal{A}$, $\phi \in \hat{\mathcal{A}}$. Let $\mathcal{A}_{op}$ be the weak Hopf algebra $\mathcal{A}$ with opposite multiplication and $\mathcal{A}^{cop}$ the weak Hopf algebra $\mathcal{A}$ with opposite comultiplication. Then $\mathcal{A}_{op}$, $\mathcal{A}^{cop}$ and $\mathcal{A}^{cop}_{op}$ are again weak C^*-Hopf

[2] In view of (1.53) this means that the center of $\mathcal{M}$ is required to be as small as possible.

algebras, where the antipode of $\mathcal{A}_{op}$ and $\mathcal{A}^{cop}$ is given by S^{-1} and the antipode of $\mathcal{A}_{op}^{cop}$ by S [N2,BNS].

A general theory for weak Hopf algebras is developped in [BSz,Sz,N2,BNS], where in particular equs. (1.12)-(1.20) are proven. The aim of this section is to investigate general properties of $\mathcal{A}$-module algebras $\mathcal{M}$ and their $\mathcal{A}$-invariant subalgebras $\mathcal{N} \equiv \mathcal{M}^{\mathcal{A}} \subset \mathcal{M}$ and to establish the connection between conditional expectations $E : \mathcal{M} \to \mathcal{N}$ and left integrals $l \in \mathcal{A}$.

2.1 Coactions

We start with reformulating Definition 1.2.2 in terms of an $\hat{\mathcal{A}}$-coaction on $\mathcal{M}$.

Definition 2.1.1 *($\mathcal{A}$-Comodule Algebras)*
A (right) $\mathcal{A}$-comodule algebra $(\mathcal{M}, \rho)$ is a $*$-algebra $\mathcal{M}$ together with a (in general non-unital) $*$-algebra homomorphism $\rho : \mathcal{M} \to \mathcal{M} \otimes \mathcal{A}$ satisfying

$$(\rho \otimes id_{\mathcal{A}}) \circ \rho = (id_{\mathcal{M}} \otimes \Delta) \circ \rho \tag{2.1}$$

$$(id_{\mathcal{M}} \otimes \varepsilon) \circ \rho = id_{\mathcal{M}} \tag{2.2}$$

$$(\mathbf{1}_{\mathcal{M}} \otimes \Delta(\mathbf{1}))(\rho(\mathbf{1}_{\mathcal{M}}) \otimes \mathbf{1}) = ((id_{\mathcal{M}} \otimes \Delta) \circ \rho)(\mathbf{1}_{\mathcal{M}}) \tag{2.3}$$

where Δ and ε denote the coproduct and counit, respectively, on $\mathcal{A}$. If $\mathcal{M}$ is a von-Neumann algebra, we also require ρ to be weakly contiuous.

For simplicity, we call such a ρ an $\mathcal{A}$-coaction on $\mathcal{M}$. Note that since ρ is assumed $*$-preserving, the axiom (2.3) could equivalently be replaced by

$$(\rho(\mathbf{1}_{\mathcal{M}}) \otimes \mathbf{1})(\mathbf{1}_{\mathcal{M}} \otimes \Delta(\mathbf{1})) = ((id_{\mathcal{M}} \otimes \Delta) \circ \rho)(\mathbf{1}_{\mathcal{M}}) \tag{2.4}$$

Without a $*$-structure we would require both axioms, (2.3) and (2.4). An immediate example of an $\mathcal{A}$-coaction is given by $\mathcal{M} = \mathcal{A}$ and $\rho = \Delta$. $\hat{\mathcal{A}}$-coactions are defined correspondingly. We now have

Proposition 2.1.2 (Left Action $\leftrightarrow$ Right Coaction)
There is a one-to-one correspondence between left $\mathcal{A}$-module algebra actions $\triangleright : \mathcal{A} \otimes \mathcal{M} \to \mathcal{M}$ and right $\hat{\mathcal{A}}$-comodule algebra coactions $\hat{\rho} : \mathcal{M} \to \mathcal{M} \otimes \hat{\mathcal{A}}$ given by

$$a \triangleright m = (id_{\mathcal{M}} \otimes a)(\hat{\rho}(m)). \tag{2.5}$$

Proof: Since $\mathcal{A}$ is finite dimensional, (2.5) defines a one-to-one correspondence between $\triangleright$ and $\hat{\rho}$ as linear maps. Equ. (2.1) is then equivalent to (1.38), equ. (2.2) is equivalent to (1.39), the homomorphism property of $\hat{\rho}$ is equivalent to (1.40) and equ. (1.41) is equivalent to $\hat{\rho}(m^*) = \hat{\rho}(m)^*$, $\forall m \in \mathcal{M}$. We are left to relate (1.42) with (2.3) (and terefore (2.4)). By applying $id_{\mathcal{M}} \otimes a \otimes b$ to both sides equ. (2.3) is equivalent to

$$ab \triangleright \mathbf{1}_{\mathcal{M}} = \varepsilon(a_{(1)}b)a_{(2)} \triangleright \mathbf{1}_{\mathcal{M}}, \ \forall a, b \in \mathcal{A} \tag{2.6}$$

and equ. (2.4) is equivalent to

$$ab \triangleright \mathbf{1}_{\mathcal{M}} = (a_{(1)} \triangleright \mathbf{1}_{\mathcal{M}})\varepsilon(a_{(2)}b) \quad \forall a, b \in \mathcal{A} \tag{2.7}$$

Putting $a = \mathbf{1}$ equ. (2.6) yields

$$b \triangleright \mathbf{1}_{\mathcal{M}} = b_{(1)}S(b_{(2)}) \triangleright \mathbf{1}_{\mathcal{M}} \tag{2.8}$$

by axiom IIIb) of Definition 1.2.1. Hence (2.3) implies (1.42). Conversely, suppose (1.42) holds together with (1.38-1.41). Then

$$\begin{aligned} ab \triangleright \mathbf{1}_{\mathcal{M}} &= ab_{(1)}S(b_{(2)}) \triangleright \mathbf{1}_{\mathcal{M}} \\ &= \varepsilon(a_{(1)}b)a_{(2)} \triangleright \mathbf{1}_{\mathcal{M}} \end{aligned}$$

where we have used equ. (2.9) of Lemma 2.1.3 below. Thus the axioms (1.38) - (1.42) imply (2.6) and therefore, (2.3). ■

We are left to prove

Lemma 2.1.3 *For $a, b \in \mathcal{A}$ the following identities hold*

$$ab_{(1)}S(b_{(2)}) = \varepsilon(a_{(1)}b)a_{(2)} \tag{2.9}$$
$$S(b_{(1)})b_{(2)}a = a_{(1)}\varepsilon(ba_{(2)}) \tag{2.10}$$
$$ab_{(2)}S^{-1}(b_{(1)}) = \varepsilon(a_{(2)}b)a_{(1)} \tag{2.11}$$
$$S^{-1}(b_{(2)})b_{(1)}a = a_{(2)}\varepsilon(ba_{(1)}) \tag{2.12}$$

Proof: Using $\Delta(b) = \Delta(\mathbf{1})\Delta(b)$ and equ. (1.13) we get

$$\begin{aligned} ab_{(1)}S(b_{(2)}) &= a\mathbf{1}_{(1)}b_{(1)}S(b_{(2)})S(\mathbf{1}_{(2)}) \\ &= a_{(1)}b_{(1)}S(b_{(2)})S(a_{(2)})a_{(3)} \\ &= \varepsilon(\mathbf{1}_{(1)}a_{(1)}b)\mathbf{1}_{(2)}a_{(2)} \\ &= \varepsilon(a_{(1)}b)a_{(2)} \end{aligned}$$

where we have used (1.19) in the second line and the antipode axiom IIIb of Definition 1.2.1 in the third line. Thus we have proven (2.9). Equs. (2.10)-(2.12) reduce to (2.9) in $\mathcal{A}_{op}^{cop}, \mathcal{A}^{cop}$ and $\mathcal{A}_{op}$, respectively. ■

Corollary 2.1.4 *Let $\triangleright : \mathcal{A} \otimes \mathcal{M} \to \mathcal{M}$ obey (1.38)-(1.41). Then the axiom (1.42) is equivalent to*

$$a \triangleright \mathbf{1}_{\mathcal{M}} = a_{(2)}S^{-1}(a_{(1)}) \triangleright \mathbf{1}_{\mathcal{M}}, \quad \forall a \in \mathcal{A} \tag{2.13}$$

Proof: Replacing (1.42) by (2.13) amounts to saying that $\triangleright : \mathcal{A}^{cop} \otimes \mathcal{M}_{op} \to \mathcal{M}_{op}$ provides a left $\mathcal{A}$-module algebra action. Now $\widehat{\mathcal{A}^{cop}} = (\hat{\mathcal{A}})_{op}$ and according to Definition 2.1.1 $\hat{\rho} : \mathcal{M} \to \mathcal{M} \otimes \hat{\mathcal{A}}$ is a coaction iff $\hat{\rho} : \mathcal{M}_{op} \to M_{op} \otimes (\hat{\mathcal{A}})_{op}$ also is a coaction. ■

As a warning we remark that the adjoint action of $\mathcal{A}$ on itself given by $a \triangleright b := a_{(1)}bS(a_{(2)})$ in general is *not* an $\mathcal{A}$-module algebra action in our sense, since it fails the axioms (1.39) and (1.40). More specifically, in place of (1.40) we have $(a_{(1)} \triangleright b)(a_{(2)} \triangleright c) = a \triangleright ((\mathbf{1} \triangleright b)(\mathbf{1} \triangleright c))$, for all $a, b, c \in \mathcal{A}$. However one can show that acting by $\mathbf{1}$ on $\mathcal{A}$ via the adjoint action defines a conditional expectation $\mathbf{1} \triangleright : \mathcal{A} \to \mathcal{A}'_R \cap \mathcal{A}$ and that the adjoint action restricts to a $\mathcal{A}$-module algebra action on $\mathcal{A}'_R \cap \mathcal{A}$.

2.2 The submodules $\mathcal{A} \triangleright \mathbf{1}_{\mathcal{M}}$

We now study the cyclic submodule

$$\mathcal{M}_R := \mathcal{A} \triangleright \mathbf{1}_{\mathcal{M}} \subset \mathcal{M} \tag{2.14}$$

of a given $\mathcal{A}$-module algebra $\mathcal{M}$. First we consider $\mathcal{M}=\hat{\mathcal{A}}$ with $\mathcal{A}$-left action (1.44) corresponding to the $\hat{\mathcal{A}}$-coaction $\hat{\rho}=\hat{\Delta}:\hat{\mathcal{A}}\to\hat{\mathcal{A}}\otimes\hat{\mathcal{A}}$. We note that the action of $\mathcal{A}$ on the vector space $\hat{\mathcal{A}}_R\equiv(\mathcal{A}\rightharpoonup\hat{\mathbf{1}})\subset\hat{\mathcal{A}}$ is equivalent to π_ε. In fact, let $(\pi_\varepsilon,\mathcal{H}_\varepsilon,\Omega_\varepsilon)$ be the GNS-triple associated with the counit ε and define $T:\hat{\mathcal{A}}_R\to\mathcal{H}_\varepsilon$ by

$$T(a\rightharpoonup\hat{\mathbf{1}}):=\pi_\varepsilon(a)\Omega_\varepsilon,\ a\in\mathcal{A} \tag{2.15}$$

Then T is immediately seen to be a well defined $\mathcal{A}$-linear bijection. Next, we define $\tau_\triangleright:\hat{\mathcal{A}}_R\to\mathcal{M}_R$ by putting for $a\in\mathcal{A}$

$$\tau_\triangleright(a\rightharpoonup\hat{\mathbf{1}}):=a\triangleright\mathbf{1}_{\mathcal{M}}\equiv a_{(1)}S(a_{(2)})\triangleright\mathbf{1}_{\mathcal{M}} \tag{2.16}$$

Lemma 2.2.1 ($\mathcal{M}_R\subset\mathcal{M}$ as an $\mathcal{A}$-submodule)
For any $\mathcal{A}$-module algebra $\mathcal{M}$ the map $\tau_\triangleright$ provides a well defined $\mathcal{A}$-module epimorphism, which becomes an isomorphism if $\mathcal{A}$ is pure or if the $\mathcal{A}$-action $\triangleright$ on $\mathcal{M}$ is faithful.

Proof: $\tau_\triangleright$ is well defined since $a\rightharpoonup\hat{\mathbf{1}}=0$ implies $\varepsilon(ba)=0$ for all $b\in\mathcal{A}$ and, therefore, $a_{(1)}S(a_{(2)})\equiv\varepsilon(\mathbf{1}_{(1)}a)\mathbf{1}_{(2)}=0$. Clearly, $\tau_\triangleright$ is surjective and intertwines $\hat{\mathcal{A}}_R$ and $\mathcal{M}_R$ as left $\mathcal{A}$-modules. If $\mathcal{A}$ is pure then $\hat{\mathcal{A}}_R\cong\mathcal{H}_\varepsilon$ is $\mathcal{A}$-irreducible implying the injectivity of $\tau_\triangleright$. By Proposition 2.2.3 and Proposition 2.2.4 below $\tau_\triangleright$ is also injective if the action $\triangleright$ is faithful. ■

We now show that $\mathcal{M}_R\subset\mathcal{M}$ (and in particular $\hat{\mathcal{A}}_R\subset\hat{\mathcal{A}}$) are in fact $*$-subalgebras and that $\tau_\triangleright:\mathcal{A}_R\to\mathcal{M}_R$ is also a $*$-algebra homomorphism.

Proposition 2.2.2 ($\mathcal{M}_R\subset\mathcal{M}$ as $*$-subalgebra)
*Let $\mathcal{M}$ be an $\mathcal{A}$-module $*$-algebra and let $\mathcal{M}_R=\mathcal{A}\triangleright\mathbf{1}_{\mathcal{M}}$. Then $\mathcal{M}_R\subset\mathcal{M}$ is a unital *-subalgebra and the $\mathcal{A}$-module map $\tau_\triangleright:\hat{\mathcal{A}}_R\to\mathcal{M}_R$ (2.16) is also a unital *-algebra homomorphism.*

Proof: To prove that $\mathcal{M}_R\subset\mathcal{M}$ is a unital *-subalgebra we first note $\mathbf{1}_{\mathcal{M}}=\mathbf{1}\triangleright\mathbf{1}_{\mathcal{M}}\in\mathcal{M}_R$ and $(a\triangleright\mathbf{1}_{\mathcal{M}})^*=(a_*\triangleright\mathbf{1}_{\mathcal{M}})\in\mathcal{M}_R$, for all $a\in\mathcal{A}$. Using

$$nm=(\mathbf{1}_{(1)}\triangleright n)(\mathbf{1}_{(2)}\triangleright m),\ \forall m,n\in\mathcal{M} \tag{2.17}$$

we now compute for all $m\in\mathcal{M}$

$$\begin{aligned}
(a\triangleright\mathbf{1}_{\mathcal{M}})m &= (a_{(1)}S(a_{(2)})\triangleright\mathbf{1}_{\mathcal{M}})m\\
&= (\mathbf{1}_{(1)}a_{(1)}S(a_{(2)})\triangleright\mathbf{1}_{\mathcal{M}})(\mathbf{1}_{(2)}\triangleright m)\\
&= (a_{(1)}S(a_{(4)})\triangleright\mathbf{1}_{\mathcal{M}})(a_{(2)}S(a_{(3)})\triangleright m)\\
&= a_{(1)}\triangleright((S(a_{(3)})\triangleright\mathbf{1}_{\mathcal{M}})(S(a_{(2)})\triangleright m))\\
&= a_{(1)}\triangleright(S(a_{(2)})\triangleright(\mathbf{1}_{\mathcal{M}}m))\\
&= a_{(1)}S(a_{(2)})\triangleright m
\end{aligned} \tag{2.18}$$

where in the third line we have used (1.18), in the fourth line (1.40), in the fifth line (1.14) and (1.40) and in the last line (1.38). Putting $m=b\triangleright\mathbf{1}_{\mathcal{M}}$ we conclude

$$(a\triangleright\mathbf{1}_{\mathcal{M}})(b\triangleright\mathbf{1}_{\mathcal{M}})=a_{(1)}S(a_{(2)})b\triangleright\mathbf{1}_{\mathcal{M}}. \tag{2.19}$$

Thus $\mathcal{M}_R\subset M$ is a unital $*$-subalgebra. Since (2.19) holds for all $\mathcal{A}$-module algebras $\mathcal{M}$ and in particular for $\mathcal{M}=\hat{\mathcal{A}}$, the map $\tau_\triangleright:\hat{\mathcal{A}}_R\to\mathcal{M}_R$ (2.16) is in fact a $*$-algebra homomorphism.
■

Interchanging $\mathcal{A}$ with $\hat{\mathcal{A}}$ we conclude that $\mathcal{A}_R \subset \mathcal{A}$ also becomes a unital $*$-subalgebra. Similarly $\hat{\mathcal{A}}_L = (\hat{\mathcal{A}}^{cop})_R \subset \hat{\mathcal{A}}$ and $\mathcal{A}_L = (\mathcal{A}^{cop})_R \subset \mathcal{A}$ are unital $*$-subalgebras. Moreover, Eq.(1.16) and its dual version imply

$$[\mathcal{A}_L\,,\,\mathcal{A}_R] = 0 \quad , \quad [\hat{\mathcal{A}}_L\,,\,\hat{\mathcal{A}}_R] = 0\ . \tag{2.20}$$

Let us next introduce, for $\phi \in \hat{\mathcal{A}}$, the linear maps $\phi_L : \mathcal{A} \to \hat{\mathcal{A}}$ and $\phi_R : \mathcal{A} \to \hat{\mathcal{A}}$ given by

$$\langle \phi_L(a)|b\rangle \ := \ \langle \phi|ab\rangle \tag{2.21}$$
$$\langle \phi_R(a)|b\rangle \ := \ \langle \phi|ba\rangle \tag{2.22}$$

Note that for all $\phi \in \hat{\mathcal{A}}$ the maps ϕ_L and ϕ_R are transposes of each other. Also note that choosing $\phi = \varepsilon$ this now allows to rewrite elements of $\hat{\mathcal{A}}_{R/L}$ as $\varepsilon_R(a) \equiv a \rightharpoonup \hat{\mathbf{1}}$ and $\varepsilon_L(a) \equiv \hat{\mathbf{1}} \leftharpoonup a,\ a \in \mathcal{A}$, respectively. Introducing similarly $\hat{\varepsilon}_{L/R} : \hat{\mathcal{A}} \to \mathcal{A}_{L/R}$, where $\hat{\varepsilon} \equiv \mathbf{1}_{\mathcal{A}} \in \mathcal{A}$ is the counit on $\hat{\mathcal{A}}$, the axioms IIIa,b for the antipode S can then be rewritten as

$$S(a_{(1)})a_{(2)} \ = \ \hat{\varepsilon}_R\varepsilon_L(a) \tag{2.23}$$
$$a_{(1)}S(a_{(2)}) \ = \ \hat{\varepsilon}_L\varepsilon_R(a) \tag{2.24}$$
$$S^{-1}(a_{(2)})a_{(1)} \ = \ \hat{\varepsilon}_L\varepsilon_L(a) \tag{2.25}$$
$$a_{(2)}S^{-1}(a_{(1)}) \ = \ \hat{\varepsilon}_R\varepsilon_R(a) \tag{2.26}$$

where (2.25)-(2.26) follow from (1.19)-(1.20). Note that these identities in particular imply

$$S\hat{\varepsilon}_R\varepsilon_R = \hat{\varepsilon}_L\varepsilon_R \qquad , \qquad S\hat{\varepsilon}_L\varepsilon_L = \hat{\varepsilon}_R\varepsilon_L \tag{2.27}$$

Next we mention the important identities [N2]

$$\varepsilon_\sigma\hat{\varepsilon}_{\sigma'}\varepsilon_\sigma \ = \ \varepsilon_\sigma \tag{2.28}$$
$$\hat{\varepsilon}_\sigma\varepsilon_{\sigma'}\hat{\varepsilon}_\sigma \ = \ \hat{\varepsilon}_\sigma \tag{2.29}$$

valid for all choices of $\sigma,\sigma' \in \{L,R\}$. From these one immediately concludes that for all $\sigma,\sigma' \in \{L,R\}$ the restrictions $\varepsilon_\sigma : \mathcal{A}_{\sigma'} \to \hat{\mathcal{A}}_\sigma$ and $\hat{\varepsilon}_\sigma : \hat{\mathcal{A}}_{\sigma'} \to \mathcal{A}_\sigma$ are bijections of vector spaces. In fact, we even have

Proposition 2.2.3 [BSz,Sz,N2,BNS] (The isomorphism $\mathcal{A}_{L/R} \cong \hat{\mathcal{A}}_{R/L}$)
The restrictions $\mu_R \equiv \varepsilon_R|\mathcal{A}_L$ and $\mu_L \equiv \varepsilon_L|\mathcal{A}_R$ provide $*$-algebra isomorphisms $\mu_R : \mathcal{A}_L \to \hat{\mathcal{A}}_R$ and $\mu_L : \mathcal{A}_R \to \hat{\mathcal{A}}_L$ obeying $\mu_R = S \circ \varepsilon_L|\mathcal{A}_L$, $\mu_L = S \circ \varepsilon_R|\mathcal{A}_R$ and $\mu^{-1}_{R/L} = \hat{\mu}_{L/R}$.

Proposition 2.2.3 immediately generalizes to arbitrary $\mathcal{A}$-module algebras $\mathcal{M}$ as follows

Proposition 2.2.4 *For any $\mathcal{A}$-module $*$-algebra $\mathcal{M}$ the map*

$$\mu_\triangleright : \mathcal{A}_L \ni a \mapsto (a \triangleright \mathbf{1}_\mathcal{M}) \in \mathcal{M}_R \tag{2.30}$$

provides a $*$-algebra epimorphism, which is an isomorphism if $\mathcal{A}$ is pure or if the $\mathcal{A}$-action $\triangleright$ on $\mathcal{M}$ is faithful. More generally, there exists a central projection $z_\triangleright = z_\triangleright^* = z_\triangleright^2 \in \mathcal{A}_L \cap C(\mathcal{A})$ such that $Ker\,\mu_\triangleright = z_\triangleright\mathcal{A}_L$ and $Ker\,\tau_\triangleright = z_\triangleright \rightharpoonup \hat{\mathcal{A}}_R$.

Proof: The first statement follows by combining Proposition 2.2.2 and Proposition 2.2.3, since $\mu_\triangleright = \tau_\triangleright \circ \mu_R$. In particular, $\mu_\triangleright$ is an isomorphism iff $\tau_\triangleright$ is an isomorphism, the later being true if $\mathcal{A}$ is pure by Lemma 2.2.1. Moreover, we will show in Corollary 2.4.4iii) below that

$$\mathrm{Ker}\,\mu_\triangleright = \mathcal{A}_L \cap \mathcal{K}_\triangleright \tag{2.31}$$

where $\mathcal{K}_{\triangleright} \subset \mathcal{A}$ is the ideal annihilated by the action $\triangleright$.

$$\mathcal{K}_{\triangleright} := \{a \in \mathcal{A} \mid a \triangleright \mathcal{M} = 0\} . \tag{2.32}$$

Hence, $\mu_{\triangleright}$ (and therefore $\tau_{\triangleright}$) is also an isomorphism if the $\mathcal{A}$-action $\triangleright$ is faithful. Next, identify $\hat{\mathcal{A}}_R \cong \mathcal{H}_\varepsilon$ via (2.15) and put $P_{\triangleright} : \hat{\mathcal{A}}_R \to \operatorname{Ker} \tau_{\triangleright} \subset \hat{\mathcal{A}}_R$ the orthogonal projection. By Lemma 2.2.1 $P_{\triangleright}$ is $\mathcal{A}$-linear and by [Sz, Eq.(3.3)] (see also [N2, Prop.4.6]) there exists a central projection $z_{\triangleright} \in \mathcal{A}_L \cap C(\mathcal{A})$ such that $P_{\triangleright} \phi = z_{\triangleright} \rightharpoonup \phi$, $\phi \in \hat{\mathcal{A}}_R$. Proposition 2.2.3 then implies $\operatorname{Ker} \mu_{\triangleright} = z_{\triangleright} \mathcal{A}_L$. ■

Note that this (together with Corollary 2.4.4iii) below) also proves the last statement of Lemma 2.2.1. In most parts of this paper we will not care about whether $\mu_{\triangleright}$ (equivalently $\tau_{\triangleright}$) is injective or not. However, for the main results in connection with Jones theory in Sections 4.3 and 4.4 we need this assumption, see also the discussion in our motivation in Section 1.1. Repeating our Definition 1.2.4 we say that an $\mathcal{A}$-module algebra $\mathcal{M}$ is *standard* if $\mu_{\triangleright}$ (equivalently $\tau_{\triangleright}$) are isomorphisms.

2.3 The fixed point subalgebras $\mathcal{M}^{\mathcal{A}} \subset \mathcal{M}$

Next, we look at the "fixed point" algebra $\mathcal{N} \equiv \mathcal{M}^{\mathcal{A}}$ and show that the conditions i)-iv) of Definition 1.2.3 are in fact all equivalent.

Proposition 2.3.1 (The fixed point algebra)
For $n \in \mathcal{M}$ the following conditions are equivalent

$$\begin{array}{rrcll}
i) & a \triangleright (mn) & = & (a \triangleright m)n, & \forall a \in \mathcal{A},\ m \in \mathcal{M} \\
ii) & a \triangleright n & = & a_{(1)} S(a_{(2)}) \triangleright n \equiv \varepsilon(\mathbf{1}_{(1)} a) \mathbf{1}_{(2)} \triangleright n, & \forall a \in \mathcal{A} \\
iii) & a \triangleright (nm) & = & n(a \triangleright m), & \forall a \in \mathcal{A},\ m \in \mathcal{M} \\
iv) & a \triangleright n & = & a_{(2)} S^{-1}(a_{(1)}) \triangleright n \equiv \varepsilon(\mathbf{1}_{(2)} a) \mathbf{1}_{(1)} \triangleright n, & \forall a \in \mathcal{A}
\end{array}$$

Proof: i) implies ii) by putting $m = \mathbf{1}_{\mathcal{M}}$ and using (1.42). Converseley, suppose ii) holds. Then for all $m \in \mathcal{M},\ a \in \mathcal{A}$

$$\begin{aligned}
a \triangleright (mn) &= (a_{(1)} \triangleright m)(a_{(2)} \triangleright n) \\
&= (a_{(1)} \triangleright m)(a_{(2)} S(a_{(3)}) \triangleright n) \\
&= (\mathbf{1}_{(1)} a \triangleright m)(\mathbf{1}_{(2)} \triangleright n) \\
&= (a \triangleright m)n
\end{aligned}$$

where in the third line we have used (1.17) and in the last line (2.17). Hence i) $\Leftrightarrow$ii). The same argument for $\triangleright : \mathcal{A}^{cop} \otimes \mathcal{M}_{op} \to \mathcal{M}_{op}$ proves iii) $\Leftrightarrow$iv). Now putting $m = \mathbf{1}_{\mathcal{M}}$ also give iii) $\Rightarrow$ii) by (1.42) and i) $\Rightarrow$iv) by (2.13). ■

We note that in terms of the coaction $\hat{\rho} : \mathcal{M} \to \mathcal{M} \otimes \hat{\mathcal{A}}$ associated with $\triangleright$ the condition i) of Proposition 2.3.1 is equivalent to

$$\hat{\rho}(mn) = \hat{\rho}(m)(n \otimes \hat{\mathbf{1}}), \quad \forall m \in \mathcal{M} \tag{2.33}$$

and the condition iii) is equivalent to

$$\hat{\rho}(nm) = (n \otimes \hat{\mathbf{1}})\hat{\rho}(m), \quad \forall m \in \mathcal{M} . \tag{2.34}$$

Clearly, it is enough to require these identities for $m = \mathbf{1}_{\mathcal{M}}$. We now define the fixed point algebra $\mathcal{N} \equiv \mathcal{M}^{\mathcal{A}} \subset \mathcal{M}$ to be the unital $*$-subalgebra given by the elements $n \in \mathcal{M}$ satisfying one (and hence all) of the conditions of Proposition 2.3.1 or, equivalently, (2.33) or (2.34). Putting $m = \mathbf{1}$ in Proposition 2.3.1 then shows that $\mathcal{N}$ commutes with $\mathcal{M}_R$,

$$\mathcal{M}_R \subset \mathcal{N}' \cap \mathcal{M} \tag{2.35}$$

Moreover, we have

Lemma 2.3.2 *Let $\hat{\rho} : \mathcal{M} \to \mathcal{M} \otimes \hat{\mathcal{A}}$ be the coaction corresponding to $\triangleright : \mathcal{A} \otimes \mathcal{M} \to \mathcal{M}$. Then $\mathcal{N} \equiv \mathcal{M}^{\mathcal{A}}$ coincides with*

$$\mathcal{N} = \hat{\rho}^{-1}(\mathcal{M} \otimes \hat{\mathcal{A}}_L) \tag{2.36}$$

Proof: Using Proposition 2.3.1 ii), equ. (2.24) and the fact that the transpose of $\hat{\varepsilon}_L \varepsilon_R$ is given by $\varepsilon_L \hat{\varepsilon}_R$ we get

$$n \in \mathcal{N} \Leftrightarrow \hat{\rho}(n) = (id_{\mathcal{M}} \otimes \varepsilon_L \hat{\varepsilon}_R)(\hat{\rho}(n))$$

and the r.h.s. implies $\hat{\rho}(\mathcal{N}) \subset \mathcal{M} \otimes \hat{\mathcal{A}}_L$. Conversely, if $\hat{\rho}(n) \in \mathcal{M} \otimes \hat{\mathcal{A}}_L$ then $\hat{\rho}(n) = (id \otimes \varepsilon_L \hat{\varepsilon}_R)(\hat{\rho}(n))$ by (2.29) and therefore $n \in \mathcal{N}$. ■

2.4 The left and right subalgebras $\mathcal{A}_{L/R} \subset \mathcal{A}$

In this subsection we summarize further results on the subalgebras $\mathcal{A}_{L/R} \subset \mathcal{A}$. The importance of these algebras in our setting has been motivated in Section 1.1. In particular, we prove the inclusions (1.53), $\mu_\triangleright(\mathcal{A}_L \cap \mathcal{A}_R) \subset C(\mathcal{M})$, and (1.55), $\mu_\triangleright(\mathcal{A}_L \cap C(\mathcal{A})) \subset C(\mathcal{N})$, see Corollary 2.4.4. By Proposition 2.4.6 this will be the reason why in Section 4.2 we will have to restrict ourselves to $\mathcal{A}$ and $\hat{\mathcal{A}}$ being *pure* in order to produce a Jones tower of *factors* by alternating crossed products with $\mathcal{A}$ and $\hat{\mathcal{A}}$. First we note

Lemma 2.4.1 [BSz,Sz,N2,BNS] *For $a \in \mathcal{A}$ the following equivalences hold*

$$\begin{array}{llllll} i) & a \in \mathcal{A}_L & \Leftrightarrow & \Delta(a) = a\mathbf{1}_{(1)} \otimes \mathbf{1}_{(2)} & \Leftrightarrow & \Delta(a) = \mathbf{1}_{(1)} a \otimes \mathbf{1}_{(2)} \\ ii) & a \in \mathcal{A}_R & \Leftrightarrow & \Delta(a) = \mathbf{1}_{(1)} \otimes a\mathbf{1}_{(2)} & \Leftrightarrow & \Delta(a) = \mathbf{1}_{(1)} \otimes \mathbf{1}_{(2)} a \end{array}$$

Analogous identities hold for $\hat{\mathcal{A}}_L$ and $\hat{\mathcal{A}}_R$, which therefore imply

Corollary 2.4.2 *The fixed point algebra $\mathcal{N} \subset \hat{\mathcal{A}}$ under the canonical $\mathcal{A}$-left action is given by $\mathcal{N} = \hat{\mathcal{A}}_L$.*

Also note, that Lemma 2.4.1 and the axiom Id) of Definition 1.2.1 imply

$$\Delta(\mathbf{1}) \in \mathcal{A}_R \otimes \mathcal{A}_L. \tag{2.37}$$

As an application we now show that the left actions of $\mathcal{A}_{L/R}$ on any $\mathcal{A}$-module algebra $\mathcal{M}$ are always inner in the following sense.

Lemma 2.4.3 *For all $m \in \mathcal{M}$ we have*
i) $a \in \mathcal{A}_L \Rightarrow a \triangleright m = (a \triangleright \mathbf{1}_{\mathcal{M}})m = (S^{-1}(a) \triangleright \mathbf{1}_{\mathcal{M}})m$
ii) $a \in \mathcal{A}_R \Rightarrow a \triangleright m = m(a \triangleright \mathbf{1}_{\mathcal{M}}) = m(S(a) \triangleright \mathbf{1}_{\mathcal{M}})$
iii) $a \in \mathcal{A}_L \cap \mathcal{A}_R \Rightarrow (a \triangleright \mathbf{1}_{\mathcal{M}}) \in C(\mathcal{M})$
iv) $a \in \mathcal{C}(\mathcal{A}) \Rightarrow (a \triangleright \mathbf{1}_{\mathcal{M}}) \in \mathcal{M}_R \cap \mathcal{M}^{\mathcal{A}} \subset \mathcal{C}(\mathcal{M}^{\mathcal{A}})$

Proof: If $a \in \mathcal{A}_L$ then

$$a \triangleright m = (a_{(1)} \triangleright \mathbf{1}_{\mathcal{M}})(a_{(2)} \triangleright m) = (\mathbf{1}_{(1)} a \triangleright \mathbf{1}_{\mathcal{M}})(\mathbf{1}_{(2)} \triangleright m) = (a \triangleright \mathbf{1}_{\mathcal{M}})m$$

where in the second equation we have used Lemma 2.4.1i) and in the third one (2.17). Moreover, if $a \in \mathcal{A}_L$ then $a = \hat{\varepsilon}_L \varepsilon_L(a) = S^{-1}(a_{(2)})a_{(1)}$ by (2.29) and (2.25). Therefore, using (1.42) and (1.14)

$$S^{-1}(a) \triangleright \mathbf{1}_{\mathcal{M}} = S^{-1}(a_{(2)})a_{(1)} \triangleright \mathbf{1}_{\mathcal{M}} = a \triangleright \mathbf{1}_{\mathcal{M}} \ .$$

Part ii) follows since $\mathcal{A}_R = (\mathcal{A}^{cop})_L$ and since $\triangleright : \mathcal{A}^{cop} \otimes \mathcal{M}_{op} \to \mathcal{M}_{op}$ is also a left $\mathcal{A}$-module algebra action. Part iii) is obvious from i) and ii). Finally, part (iv) follows from (2.35) and the fact, that $\mathcal{C}(\mathcal{A}) \triangleright \mathbf{1}_{\mathcal{M}} \subset \mathcal{M}^{\mathcal{A}}$ by Proposition 2.3.1ii) and Eq.(1.42). ■

Corollary 2.4.4 *Let $\mu_\triangleright$ be as in Proposition 2.2.4 and $\mathcal{K}_\triangleright$ as in equ. (2.32). Then*
i) $\mu_\triangleright(\mathcal{A}_L \cap \mathcal{A}_R) \subset C(\mathcal{M})$
ii) $\mu_\triangleright(\mathcal{A}_L \cap C(\mathcal{A})) = C(\mathcal{A}) \triangleright \mathbf{1}_{\mathcal{M}} \subset C(\mathcal{M}^{\mathcal{A}})$
iii) $\mathrm{Ker}\, \mu_\triangleright = \mathcal{A}_L \cap \mathcal{K}_\triangleright$

Proof: It remains to proof the first identity of part (ii). Since (1.42) implies $a \triangleright \mathbf{1}_{\mathcal{M}} = \hat{\varepsilon}_L \varepsilon_R(a) \triangleright \mathbf{1}_{\mathcal{M}}$, this follows from the dual of Eq.(2.38) below. ■

Corollary 2.4.4iii) concludes the proofs of the statements in Lemma 2.2.1 and Proposition 2.2.4 referring to the faithfulness of the action $\triangleright$.

Next, we look at the intersection $\mathcal{A}_L \cap \mathcal{A}_R$ and note that it is isomorphic to $C(\hat{\mathcal{A}}) \cap \hat{\mathcal{A}}_{L/R}$ and that it is trivial if and only if $\hat{\mathcal{A}}$ is pure.

Lemma 2.4.5 [Sz,N2] *For $\sigma \in \{L,R\}$ the $*$-isomorphisms $\varepsilon_L|\mathcal{A}_R$ and $\varepsilon_R|\mathcal{A}_L$ of Proposition 2.2.3 restrict to $*$-isomorphisms $\varepsilon_\sigma : \mathcal{A}_L \cap \mathcal{A}_R \to C(\hat{\mathcal{A}}) \cap \hat{\mathcal{A}}_\sigma$ with inverse given by $\hat{\varepsilon}_L|(C(\hat{\mathcal{A}}) \cap \hat{\mathcal{A}}_\sigma) \equiv \hat{\varepsilon}_R|(C(\hat{\mathcal{A}}) \cap \hat{\mathcal{A}}_\sigma)$. More generally, $\hat{\varepsilon}_L|(C(\hat{\mathcal{A}})) \equiv \hat{\varepsilon}_R|(C(\hat{\mathcal{A}}))$ is a $*$-algebra homomorphism invariant under S and*

$$\hat{\varepsilon}_\sigma(C(\hat{\mathcal{A}})) = \hat{\varepsilon}_\sigma(C(\hat{\mathcal{A}}) \cap \hat{\mathcal{A}}_{L/R}) = \mathcal{A}_L \cap \mathcal{A}_R \tag{2.38}$$

Using the identity $\pi_\varepsilon(\mathcal{A})' = \pi_\varepsilon(\mathcal{A}_L \cap C(\mathcal{A}))$ ([Sz, Eq.(3.3)], see also [N2, Prop.4.6]), where the prime denotes the commutant in $\mathrm{End}\, \mathcal{H}_\varepsilon$, Lemma 2.4.5 leads to

Proposition 2.4.6 [Sz] (Pureness of $\mathcal{A}$)
For a weak C^*-Hopf algebra $\mathcal{A}$ the following conditions are equivalent
i) $\mathcal{A}$ is pure
ii) $\hat{\mathcal{A}}_L \cap \hat{\mathcal{A}}_R = \mathbf{C} \cdot \hat{\mathbf{1}}$
iii) $\mathcal{A}_\sigma \cap C(\mathcal{A}) = \mathbf{C} \cdot \mathbf{1}, \quad \sigma = L$ or $\sigma = R$.

Note that Corollary 2.4.4 and Proposition 2.4.6 imply that for standard weak Hopf actions $\mathcal{A}$ is pure iff $\mathcal{M}^{\mathcal{A}} \cap \mathcal{M}_R = \mathbf{C}$ and $\hat{\mathcal{A}}$ is pure iff $C(\mathcal{M}) \cap \mathcal{M}_R = \mathbf{C}$. The first condition in particular holds if $C(\mathcal{M}^{\mathcal{A}}) = \mathbf{C}$ (by (2.35)) and the second if $C(\mathcal{M}) = \mathbf{C}$. In Section 4.2 this will be the reason why in applications to depth-2 inclusions of factors we will have to restrict ourselves to pure weak Hopf algebras $\mathcal{A}$ and $\hat{\mathcal{A}}$.

We finally mention that Lemma 2.4.5 implies [Sz]

$$\mathcal{A}_L \cap \mathcal{A}_R \cap C(\mathcal{A}) \cong \hat{\mathcal{A}}_L \cap \hat{\mathcal{A}}_R \cap C(\hat{\mathcal{A}}) \tag{2.39}$$

and Corollary 2.4.4 implies

$$\mu_{\triangleright}(\mathcal{A}_L \cap \mathcal{A}_R \cap C(\mathcal{A})) \subset \mathcal{N} \cap C(\mathcal{M}) \ . \tag{2.40}$$

The abelian algebra $\mathcal{Z} := \mathcal{A}_L \cap \mathcal{A}_R \cap C(\mathcal{A})$ has been called the "hyper-center" of $\mathcal{A}$ in [Sz], since it behaves like scalars in many respects. More presicely one has a direct sum decomposition of weak Hopf algebras [Sz,BNS]

$$\mathcal{A} = \oplus_p \mathcal{A}_p \qquad , \qquad \hat{\mathcal{A}} = \oplus_p \hat{\mathcal{A}}_p \tag{2.41}$$

where $\mathcal{A}_p \equiv p\mathcal{A}$ and $\hat{\mathcal{A}}_p \equiv \hat{p}\hat{\mathcal{A}}$ are dual to each other and where p runs through the minimal projections of $\mathcal{Z} \cong \hat{\mathcal{Z}}$. Correspondingly, we could also cut $\mathcal{N}$ and $\mathcal{M}$ with the central projections $p \triangleright \mathbf{1}_{\mathcal{M}} \in \mathcal{N} \cap C(\mathcal{M})$ to obtain $\mathcal{N}_p \equiv (p \triangleright \mathbf{1}_{\mathcal{M}})\mathcal{N}$ as the fixed point algebra of $\mathcal{M}_p \equiv (p \triangleright \mathbf{1}_{\mathcal{M}})\mathcal{M}$ under the action of $\mathcal{A}_p$. Under the regularity conditions of Section 4.4 this would also imply $\mathcal{N}_p \cap C(\mathcal{M}_p) = \mathbf{C}$. Thus, throughout one might without much loss assume trivial hypercenter.

2.5 A partly inner group action

We now analyze the example of a partly inner group action within our framework. Let G be a finite group, $H \subset G$ a normal subgroup and put

$$\mathcal{A} := \mathbf{C}H \rtimes_{Ad} G \tag{2.42}$$

where G acts on H by the adjoint action. Equivalently, $\mathcal{A}$ is the group algebra of the semi-direct product $H \rtimes_{Ad} G$. Denote $\{(h,g) \mid h \in H,\ g \in G\}$ the natural basis in $\mathcal{A}$ and define a weak Hopf algebra structure on $\mathcal{A}$ by (see [N2, Ex. 4])

$$\begin{aligned}
\Delta(h,g) &:= \frac{1}{|H|} \sum_{\tilde{h} \in H} (h\,\tilde{h}^{-1},\ \tilde{h}g) \otimes (\tilde{h}, g) && (2.43)\\
\varepsilon(h,g) &:= |H|\delta(h) && (2.44)\\
S(h,g) &:= (g^{-1}h\,g,\ g^{-1}h^{-1}) && (2.45)
\end{aligned}$$

where $\delta(\mathbf{1}) = 1$ and $\delta(h) = 0$ else. One easily verifies the axioms of Definition 1.2.1. As to the antipode axioms we remark that for $x = (h,g) \in \mathcal{A}$ one computes

$$\begin{aligned}
\varepsilon(\mathbf{1}_{(1)}x)\mathbf{1}_{(2)} &= (h, \mathbf{1}) = x_{(1)}S(x_{(2)}) && (2.46)\\
\mathbf{1}_{(1)}\varepsilon(x\mathbf{1}_{(2)}) &= (g^{-1}h\,g,\ g^{-1}h^{-1}g) = S(x_{(1)})x_{(2)} && (2.47)
\end{aligned}$$

The left and right subalgebras are given by

$$\begin{aligned}
\mathcal{A}_L &= \mathrm{span}\,\{(h, \mathbf{1}_G) \mid h \in H\} \cong \mathbf{C}H && (2.48)\\
\mathcal{A}_R &= \mathrm{span}\,\{(h, h^{-1}) \mid h \in H\} \cong (\mathbf{C}H)_{op} && (2.49)
\end{aligned}$$

Clearly, this implies $\mathcal{A}_L \cap \mathcal{A}_R = \mathbf{C}$, i.e. the dual algebra $\hat{\mathcal{A}}$ is pure. As a linear space $\hat{\mathcal{A}}$ is naturally given by the functions $\phi : \mathcal{H} \times G \to \mathbf{C}$, with associative multiplication law

$$(\phi * \psi)(h,\ g) = \frac{1}{|H|} \sum_{\tilde{h} \in H} \phi(h\tilde{h}^{-1},\ \tilde{h}g)\psi(\tilde{h},\ g) \ . \tag{2.50}$$

Hence, as an algebra $\hat{\mathcal{A}}$ may be identified with $\mathbf{C}H \rtimes \hat{\mathcal{G}}$, where $\hat{\mathcal{G}}$ is the abelian algebra of functions on G (i.e. the dual of the group algebra $\mathcal{G} \equiv \mathbf{C}G$), and where the left action of $\hat{\mathcal{G}}$ on $\mathbf{C}H$ is dual to the right coaction $\Delta_{\mathcal{G}} : \mathbf{C}H \to \mathbf{C}H \otimes \mathcal{G}$ given on the basis $h \in H$ by the standard formula $\Delta_{\mathcal{G}}(h) = h \otimes h$. The coproduct on $\hat{\mathcal{A}}$ is given by

$$(\hat{\Delta}\phi)((h_1, g_1), (h_2, g_2)) = \phi(h_1 g_1 h_2 g_1^{-1}, g_1 g_2). \tag{2.51}$$

Computing $\hat{\Delta}\varepsilon$ we conclude that $\hat{\mathcal{A}}_L$ is given by the space of functions of the form $\phi(h, g) = \varphi(h)$ and $\hat{\mathcal{A}}_R$ is given by functions of the form $\phi(h, g) = \varphi(g^{-1}hg)$. Hence, $\hat{\mathcal{A}}_L \cap \hat{\mathcal{A}}_R$ is spanned by the Ad_G -invariant functions on H. By (2.43) the left action of $\hat{\mathcal{A}}$ on $\mathcal{A}$ becomes

$$\phi \rightharpoonup (h, g) = \frac{1}{|H|} \sum_{\tilde{h} \in H} \phi(\tilde{h}, g)\, (h\tilde{h}^{-1}, \tilde{h}g). \tag{2.52}$$

Using $(\hat{\mathcal{A}}_L \cap \hat{\mathcal{A}}_R) \rightharpoonup \mathbf{1}_{\mathcal{A}} = C(\mathcal{A}) \cap \mathcal{A}_R$ by Lemma 2.4.5 we conclude

$$C(\mathcal{A}) \cap \mathcal{A}_R = \mathrm{span}_{\chi}\{\sum_{h \in H} \chi(h)\, (h, h^{-1})\} \tag{2.53}$$

where χ runs through the Ad_G -invariant functions on H. Hence, by Proposition 2.4.6, $\mathcal{A}$ is pure if and only if H is trivial. We leave the remaining details to the reader.

Let now $\mathcal{M}$ be a von-Neumann factor, $u : H \to \mathcal{M}$ a unitary representation and $\alpha : G \to \mathrm{Aut}\,\mathcal{M}$ an action satisfying

$$\alpha_h = \mathrm{Ad}\,u(h), \quad h \in H \tag{2.54}$$

$$\alpha_g \circ u = u \circ \mathrm{Ad}\,g. \tag{2.55}$$

Then (α, u) provides an $\mathcal{A}$-module algebra action $\triangleright : \mathcal{A} \otimes \mathcal{M} \to \mathcal{M}$ by putting for $h \in H$, $g \in G$ and $m \in \mathcal{M}$

$$(h, g) \triangleright m := u(h)\alpha_g(m) \tag{2.56}$$

which the reader is invited to check. Since $(h, g) \triangleright \mathbf{1}_M = u(h)$ the action is standard, iff

$$\mathcal{M}_R \equiv \mathrm{span}\,\{u(h) \mid h \in H\} \cong \mathbf{C}H \tag{2.57}$$

i.e. iff the unitaries $u(h)$ are linearly independent in $\mathcal{M}$. By (2.46) the fixed point subalgebra $\mathcal{M}^{\mathcal{A}}$ coincides with the fixed point algebra of the action α.

$$\mathcal{N} := \mathcal{M}^{\mathcal{A}} = \{m \in \mathcal{M} \mid \alpha_g(m) = m,\ \forall g \in G\} \equiv \mathcal{M}^G. \tag{2.58}$$

We remark that for general non-outer group actions α the inner part $u : H \to \mathcal{M}$ might only be a projective representation and (2.55) might also only be valid up to a phase. This case is treated in Appendix B.

2.6 Left integrals and conditional expectations

In this subsection we study the relation between left integrals $l \in \mathcal{A}$ and conditional expectations $E : \mathcal{M} \to \mathcal{M}^{\mathcal{A}}$. To this end we first review from [BNS] some central results on the theory of integrals and Fourier transformations on weak Hopf algebras. Let us denote $\mathcal{L}(\mathcal{A})$ and $\mathcal{R}(\mathcal{A})$ the space of left and right integrals in $\mathcal{A}$, respectively. Then we have

Lemma 2.6.1 [BNS] (Left and right integrals)
i) $l \in \mathcal{L}(\mathcal{A}) \Leftrightarrow l \rightharpoonup \phi \in \hat{\mathcal{A}}_L$ *for all* $\phi \in \hat{\mathcal{A}}$.
ii) $l \in \mathcal{L}(\mathcal{A}) \Rightarrow al = S(a)l$ *for all* $a \in \mathcal{A}_R$
i') $r \in \mathcal{R}(\mathcal{A}) \Leftrightarrow \phi \leftharpoonup \in \hat{\mathcal{A}}_R$ *for all* $\phi \in \hat{\mathcal{A}}$.
ii') $r \in \mathcal{R}(\mathcal{A}) \Rightarrow ra = rS(a)$ *for all* $a \in \mathcal{A}_L$

Note that Lemma 2.6.1i)+i') in particular implies that $\varepsilon_R(l) \equiv l \rightharpoonup \hat{\mathbf{1}}$ and $\varepsilon_L(r) \equiv \hat{\mathbf{1}} \leftharpoonup r$ are both in the abelian $*$-subalgebra $\hat{\mathcal{A}}_L \cap \hat{\mathcal{A}}_R \subset \hat{\mathcal{A}}$. We call a left integral $l \in \mathcal{L}(\mathcal{A})$ (right integral $r \in \mathcal{R}(\mathcal{A})$) *normalizable* if $\varepsilon_R(l)$ ($\varepsilon_L(r)$) is invertible and we call it *normalized* if $\varepsilon_R(l) = \hat{\mathbf{1}}$ ($\varepsilon_L(r) = \hat{\mathbf{1}}$). Normalizable left (right) integrals can always be normalized as follows. For $l \in \mathcal{L}(\mathcal{A})$ or $r \in \mathcal{R}(\mathcal{A})$ and $\sigma = L, R$ we define their "σ-normalization"

$$n_\sigma(l) := \hat{\varepsilon}_\sigma \varepsilon_R(l) \quad , \quad n_\sigma(r) := \hat{\varepsilon}_\sigma \varepsilon_L(r) \tag{2.59}$$

Then $n_\sigma(l)$, $n_\sigma(r) \in \mathcal{A}_\sigma \cap C(\mathcal{A})$ by Lemma 2.4.5 and one may show that the two definitions in (2.59) coincide on $\mathcal{L}(\mathcal{A}) \cap \mathcal{R}(\mathcal{A})$. Moreover

$$n_L = S^{\pm 1} \circ n_R \tag{2.60}$$

and one has

Lemma 2.6.2 [BNS] (Normalizable integrals)
Left integrals $l \in \mathcal{L}(\mathcal{A})$ (right integrals $r \in \mathcal{R}(\mathcal{A})$) are normalizable iff $n_\sigma(l)$, $n_\sigma(r)$ are invertible, respectively. In this case we have $n_L(l)^{-1} l = n_R(l)^{-1} l$ and $r\, n_L(r)^{-1} = r\, n_R(r)^{-1}$, which provide normalized left and right integrals, respectively.

Note that by (1.31),(1.32)

$$l^2 = n_\sigma(l) l, \ \sigma = L, R\ . \tag{2.61}$$

Applying ε_R to both sides we conclude that l is idempotent iff $l \rightharpoonup \hat{\mathbf{1}}$ is idempotent. By Proposition 2.4.6 it follows, that in pure Hopf algebras nonzero left and right integrals are normalized (normalizable), if and only if they are idempotent (non-nilpotent), respectively.

Next, we have a theory of "Radon-Nikodym derivatives" for left (right) integrals.

Lemma 2.6.3 [BNS] ("Radon-Nikodym derivatives")
Let $\mathcal{A}_\sigma$, $\sigma = L, R$, act on $\mathcal{L}(\mathcal{A})$ by right multiplication and on $\mathcal{R}(\mathcal{A})$ by left multiplication. Then under these actions the normalized Haar integral h is cyclic and separating. More precisely, defining for $l \in \mathcal{L}(\mathcal{A})$ and $r \in \mathcal{R}(\mathcal{A})$ "Radon-Nikodym derivatives" with respect to h by

$$d_\sigma(l) := \hat{\varepsilon}_\sigma \varepsilon_L(l) \quad , \quad d_\sigma(r) := \hat{\varepsilon}_\sigma \varepsilon_R(r) \tag{2.62}$$

we have $l = h d_\sigma(l)$, $r = d_\sigma(r) h$, and $d_\sigma(ha) = d_\sigma(ah) = a$, $\forall a \in \mathcal{A}_\sigma$. For two-sided integrals the two definitions of d_σ in (2.62) coincide and $\mathcal{L}(\mathcal{A}) \cap \mathcal{R}(\mathcal{A}) = (\mathcal{A}_\sigma \cap C(\mathcal{A}))\, h$. In particular, two-sided integrals are invariant under the antipode.

Lemma 2.6.3 follows easily by noting that h is a left-unit in the right ideal $\mathcal{L}(\mathcal{A})$ and a right-unit in the left-ideal $\mathcal{R}(\mathcal{A})$. We also note that under a rescaling of left integrals l by central elements $c \in C(\mathcal{A})$ one has

$$n_\sigma(cl) = (\hat{\varepsilon}_\sigma \varepsilon_{\sigma'})(c) n_\sigma(l) \tag{2.63}$$

$$d_\sigma(cl) = (\hat{\varepsilon}_\sigma \varepsilon_{\sigma'})(c) d_\sigma(l)\ , \tag{2.64}$$

and an analogous formula for right integrals.

Next, we report from [BNS] an important duality concept for nondegenerate left integrals, which in fact should be viewed as an appropriate generalization of the notion of Fourier transformation to weak Hopf algebras. We define for a left integral $l \in \mathcal{L}(\mathcal{A})$ the maps $l_\sigma : \hat{\mathcal{A}} \to \mathcal{A}$, $\sigma = L, R$, similarly as in (2.21), (2.22), i.e. by putting for $\psi \in \hat{\mathcal{A}}$

$$\begin{aligned} l_L(\psi) &:= l \leftharpoonup \psi \\ l_R(\psi) &:= \psi \rightharpoonup l \end{aligned} \tag{2.65}$$

Then l is nondegenerate as a functional on $\hat{\mathcal{A}}$ iff l_L (equivalently l_R) are invertible. It turns out that - as for ordinary Hopf algebras [LS] - a left integral $l \in \mathcal{A}$ is nondegenerate if and only if $\mathbf{1} \in \mathcal{A}$ is in the image of l_R.

Proposition 2.6.4 [BNS] (Duality for left integrals)
Let $l \in \mathcal{L}(\mathcal{A})$ be a left integral and assume $\lambda \in \hat{\mathcal{A}}$ to satisfy $l_R(\lambda) = \mathbf{1}$. Then λ is a uniquely determined left integral in $\hat{\mathcal{A}}$ and satisfies $\lambda_R(l) = \hat{\mathbf{1}}$. Moreover, λ and l are both nondegenerate with

$$l_R^{-1} = \lambda_L \circ S^{-1} \tag{2.66}$$
$$\lambda_R^{-1} = l_L \circ \hat{S}^{-1} \tag{2.67}$$

According to Proposition 2.6.4 nondegenerate left integrals always come in dual pairs. Similar statements with different left-right conventions are obtained by passing to $\mathcal{A}_{op}$, $\mathcal{A}^{op}$ and $\mathcal{A}_{op}^{cop}$, respectively. We will see later that this kind of duality is closely related to the notion of Haagerup duality for conditional expectations in Jones theory.

In weak Hopf algebras the modular theory for the normalized Haar integral turns out to be non-trivial even in the finite dimensional case and is in fact very similar to Woronowicz's results for the Haar state on non-finite compact quantum groups.

Theorem 2.6.5 [BNS] (Modular theory for the Haar integral)
Let $h \in \mathcal{A}$ and $\hat{h} \in \hat{\mathcal{A}}$ be the normalized Haar integrals. Then
i) h and $\hat{h}$ are nondegenerate positive functionals on $\hat{\mathcal{A}}$ and $\mathcal{A}$, respectively, and $(\hat{h} \rightharpoonup h) \in \mathcal{A}_L$, $(h \leftharpoonup \hat{h}) \in \mathcal{A}_R$ are positive and invertible elements.
ii) Let $g_L := (\hat{h} \rightharpoonup h)^{1/2}$, $g_R := (h \leftarrow \hat{h})^{1/2}$ and $g := g_L g_R^{-1}$, and let $\hat{g}_{L/R}$, $\hat{g}$ be defined analogously in $\hat{\mathcal{A}}$. Then $g_\sigma \in \mathcal{A}_\sigma$, $\hat{g}_\sigma \in \hat{\mathcal{A}}_\sigma$ and

$$\hat{g}_\sigma = \varepsilon_\sigma(g_{\sigma'}) \quad \forall \sigma, \sigma' \in \{L, R\} \tag{2.68}$$
$$g_R = S^{\pm 1}(g_L) \tag{2.69}$$
$$S^2(a) = g a g^{-1}, \quad \forall a \in \mathcal{A} \tag{2.70}$$
$$\Delta(g) = (g \otimes g)\Delta(\mathbf{1}) = \Delta(\mathbf{1})(g \otimes g) \tag{2.71}$$
$$\Delta_{op}(h) = (\mathbf{1} \otimes g)\Delta(h)(\mathbf{1} \otimes g) \tag{2.72}$$

and analogous statements with $\mathcal{A}$ and $\hat{\mathcal{A}}$ interchanged.

Note that Theorem 2.6.5 in particular implies that the left integral λ dual to h is given by

$$\lambda = \hat{h} g_L^{-2} = \hat{h} g_R^{-2} \tag{2.73}$$

In the example $\mathcal{A} = \mathbf{C}H \rtimes_{Ad} G$ of Sect. 2.5 the normalized Haar integral is given by

$$e_{Haar} = \frac{1}{|G|} \sum_{g \in G} (\mathbf{1}_H, g) \tag{2.74}$$

and therefore a basis in the space of left integrals is given by the elements[3]

$$l_h := e_{Haar} h = \frac{1}{|G|} \sum_{g \in G} (g h g^{-1}, g), \quad h \in H. \tag{2.75}$$

[3]We trust that the reader will not be confused by the fact that in *this* example $h \in H$ denotes the elements of the normal subgroup $H \subset G$, whereas in *general* $h \in \mathcal{A}$ denotes the normalized Haar integral.

Thus, a general left integral is of the form

$$l = \sum_{h \in H} c(h)\, l_h \tag{2.76}$$

and using (2.44) l is normalized iff the coefficients $c(h) \in \mathbf{C}$ satisfy $c(\mathbf{1}_H) = 1$ and $\sum_{g \in G} c(ghg^{-1}) = 0,\ \forall \mathbf{1}_H \neq h \in H$. The left integrals $\lambda \in \mathcal{L}(\hat{\mathcal{A}})$ are of the form

$$\lambda(h,\, g) = \delta(hg)\hat{c}(h), \tag{2.77}$$

and $\lambda \in \mathcal{L}(\hat{\mathcal{A}})$ is dual to $l \in \mathcal{A}$ iff

$$\frac{1}{|G||H|} \sum_{\tilde{h} \in H} c(\tilde{h}h)\hat{c}(\tilde{h}) = \delta(h).$$

In this example e_{Haar} is a trace on $\hat{\mathcal{A}}$, i.e. $\Delta(e_{Haar}) = \Delta_{op}(e_{Haar})$, and the normalized Haar integral $\lambda_{Haar} \in \hat{\mathcal{A}}$ is given by

$$\lambda_{Haar}(h, g) := |H|\delta(h)\delta(g), \quad h \in H,\, g \in G, \tag{2.78}$$

which is also a trace on $\mathcal{A}$. In particular, in this case the structural elements $g_{L/R} \in \mathcal{A}_{L/R}$ of Theorem 2.6.5 are just multiples of the identity as for ordinary groups, $g_{L/R} = |G|^{-1/2}\mathbf{1}_{\mathcal{A}}$, which fits the identity $S^2 = \mathrm{id}$ in this example.

Coming back to the general theory we now prepare the relation between left integrals and conditional expectations by showing that the nondegeneracy and/or positiviy of a left integral $l \in \mathcal{L}(\mathcal{A})$ shows up in its "$\mathcal{A}_R$-Radon-Nikodym derivative" $d_R(l)$ as follows.

Proposition 2.6.6 (Positivity and non-degeneracy for left integrals)
i) A left integral $l \in \mathcal{L}(\mathcal{A})$ is non-degenerate and/or positive if and only if $d_R(l)$ is invertible and/or positive, respectively, as an element in $\mathcal{A}_R$.
ii) If l is nondegenerate, then the dual left integral $\lambda \in \mathcal{L}(\hat{\mathcal{A}})$ satisfies

$$d_R(\lambda) = \varepsilon_R(g_L d_L(l) g_L)^{-1} \tag{2.79}$$

where $g_L := (\hat{h} \rightharpoonup h)^{1/2}$ as in Theorem 2.6.5. In particular, λ is positive if and only if $d_L(l) > 0$ as an element in $\mathcal{A}_L$.
iii) A nondegenerate left integral l is normalizable iff l^2 is nondegenerate and it is normalized iff $l^2 = l$.
iv) If $l \in \mathcal{L}(\mathcal{A})$ is nondegenerate and positive, then $n_\sigma(l) \in \mathcal{A}_\sigma \cap C(\mathcal{A})$ is positive and invertible and $\dot{l} := n_\sigma(l)^{-1} l$ is a normalized nondegenerate and positive left integral

Proof: i) Using $l = h d_R(l)$ and Lemma 2.4.3ii) we have for $\phi \in \hat{\mathcal{A}}$

$$\langle l \mid \phi \rangle = \langle h \mid \phi \varepsilon_R(d_R(l)) \rangle$$

Since h is non-degenerate, we conclude that l is non-degenerate if and only if $\varepsilon_R(d_R(l))$ is invertible, which by Proposition 2.2.3 is equivalent to $d_R(l)$ being invertible. Now assume

$d_R(l) \geq 0$. Then using $S^{-1}(\mathcal{A}_R) = \mathcal{A}_L$ and Lemma 2.6.1ii') we conclude $l = hd_R(l) = hS^{-1}(d_R(l)^{1/2})d_R(l)^{1/2}$ and therefore, again by Lemma 2.4.3,

$$\begin{aligned} \langle l \mid \phi \rangle &= \langle h \mid \varepsilon_R\left(S^{-1}(d_R(l)^{1/2})\right) \phi\, \varepsilon_R(d_R(l)^{1/2}) \rangle \\ &= \langle h \mid \xi_l^* \, \phi \, \xi_l \rangle \end{aligned} \tag{2.80}$$

where we have introduced $\xi_l := d_R(l)^{1/2} \rightharpoonup \hat{\mathbf{1}} \equiv \varepsilon_R(d_R(l)^{1/2})$ implying

$$\xi_l^* = (d_R(l)^{1/2})_* \rightharpoonup \hat{\mathbf{1}} = \varepsilon_R(S^{-1}(d_R(l)^{1/2}))$$

by (1.28), (1.29) and the self-adjointness of $d_R(l)^{1/2}$. Hence, since h is positive as a functional on $\hat{\mathcal{A}}$ by Theorem 2.6.5, (2.80) implies the same for l. We are left to show that converseley, if l is a positive weight on $\hat{\mathcal{A}}$ then $d_R(l) \geq 0$. To this end let $B = \sum_i x_i \otimes y_i \in \mathcal{A} \otimes \mathcal{A}$ be given by $B := S(l_{(1)}) \otimes l_{(2)}$. Identifying $\mathcal{A} \otimes \mathcal{A}$ with the space of sesquilinear forms on $\hat{\mathcal{A}}$ via

$$\langle \phi \,,\, \psi \rangle_B := \sum_i \overline{\langle \phi \mid x_i^* \rangle} \langle \psi \mid y_i \rangle, \quad \phi, \psi \in \hat{\mathcal{A}},$$

we conclude from (1.28) and (1.29) that l is a positive weight on $\hat{\mathcal{A}}$ if and only if $B = S(l_{(1)}) \otimes l_{(2)} \in \mathcal{A} \otimes \mathcal{A}$ defines a positive (possibly degenerate) sesquilinear form $\langle \cdot, \cdot \rangle_B$ on $\hat{\mathcal{A}}$. By standard diagonalization procedures this implies B to be of the form

$$B = \sum_i a_i^* \otimes a_i$$

for suitable elements $a_i \in \mathcal{A}$. Hence

$$d_R(l) \equiv S(l_{(1)}) l_{(2)} = \sum_i a_i^* a_i \geq 0$$

To prove part ii) let l be nondegenerate, then $d_R(l)$ is invertible by i). Putting $\lambda = \lambda_h \hat{S}^{-1}(\varepsilon_L(d_R(l)^{-1}))$, where $\lambda_h \in \mathcal{L}(\hat{\mathcal{A}})$ is the left integral dual to h, we get by Lemma 2.4.3ii) and (2.29)

$$\lambda \rightharpoonup l = \lambda_h \rightharpoonup (l d_R(l)^{-1}) = \lambda_h \rightharpoonup h = \mathbf{1} \;.$$

Hence λ is the left integral dual to l. Now we use the identities $d_R(l) = S(d_L(l))$, $\hat{S}^{-1} \circ \varepsilon_L = \varepsilon_R \circ S$ and the fact that (2.70) implies $S^2(d_L(l)) = g_L d_L(l) g_L^{-1}$ (since $g_R \in \mathcal{A}_R$ commutes with $\mathcal{A}_L$)to conclude

$$\begin{aligned} \lambda = \lambda_h \varepsilon_R(g_L d_L(l)^{-1} g_L^{-1}) &= \lambda_h \varepsilon_R(g_L^2) \varepsilon_R(g_L d_L(l) g_L)^{-1} \\ &= \hat{h} \varepsilon_R(g_L d_L(l) g_L)^{-1} \end{aligned}$$

where we have used (2.73). This proves (2.79) and since by Proposition 2.2.3 $\varepsilon_R : \mathcal{A}_L \to \hat{\mathcal{A}}_R$ is a $*$-algebra isomorphism we conclude that $d_R(\lambda)$ is positive if and only if $d_L(l)$ is positive. To prove part iii) let $d_R(l)$ be invertible. By (2.61) we have $d_\sigma(l^2) = d_\sigma(l) n_\sigma(l)$ and hence $d_R(l^2)$ is invertible iff $n_R(l)$ is invertible and $l^2 = l \Leftrightarrow d_R(l^2) = d_R(l) \Leftrightarrow n_R(l) = 1$. Finally, to prove part iv) let $l \in \mathcal{L}(\mathcal{A})$ provide a positive faithful weight on $\hat{\mathcal{A}}$. Then $l \rightharpoonup \hat{\mathbf{1}} \equiv (id \otimes l)(\hat{\Delta}(\hat{\mathbf{1}})) \in \hat{\mathcal{A}}_L \cap \hat{\mathcal{A}}_R$ is positive. If $l \rightharpoonup \hat{\mathbf{1}}$ was not invertible there would exist a projection $\chi = \chi^2 = \chi^* \in \hat{\mathcal{A}}_L \cap \hat{\mathcal{A}}_R$ such that $(l \rightharpoonup \hat{\mathbf{1}})\chi = 0$. Consequently, $(id \otimes l)(\hat{\Delta}(\chi)) \equiv (l \rightharpoonup \hat{\mathbf{1}})\chi = 0$ and therefore $\chi = 0$ by the faithfulnes of l and the injectivity of Δ. Thus $l \rightharpoonup \hat{\mathbf{1}}$ and by Propositon 2.9 also $n_\sigma(l) \equiv \hat{\varepsilon}_\sigma(l \rightharpoonup \hat{\mathbf{1}})$ are positive and invertible. Moreover, $\dot{l} = h d_R(l) n_R(l)^{-1}$ implies $d_R(\dot{l}) = d_R(l) n_R(l)^{-1}$ which is invertible and positive since $n_R(l)$ is central. Hence, by part i) $\dot{l}$ is nondegenerate and positive. ■

Using $d_R(l) = S(d_L(l))$ by (2.27), Proposition 2.6.6i) implies that the left integrals (2.76) of our example are nondegenerate iff $d_L(l) \equiv \sum_{h\in H} c(h)\,(h,\,\mathbf{1}_G) \in \mathbf{C}H$ is invertible, and they are positive iff $d_R(l) \equiv \sum_{h\in H} c(h)\,(h,\,h^{-1}) \in \mathbf{C}H{\rtimes\!\!\!\lhd}_{Ad}\,G$ is a positive element.

A straightforward generalization of the above methods to arbitrary $\mathcal{A}$-module algebra actions now gives

Theorem 2.6.7 (Left integrals and conditional expectations)
Let $(M,\,\triangleright)$ be an $\mathcal{A}$-module von-Neumann algebra and denote $\mathcal{N} \equiv \mathcal{M}^{\mathcal{A}}$ the $\mathcal{A}$-invariant subalgebra. For $l \in \mathcal{L}(\mathcal{A})$ a left integral denote $E_l(m) := l \triangleright m$. Then
i) $E_l(\mathcal{M}) \subset \mathcal{N}$ and E_l is a weakly continuous $\mathcal{N}$-$\mathcal{N}$ bimodule map.
ii) For $d_{L/R} \in \mathcal{A}_{L/R}$ we have $E_{ld_L}(m) = E_l((d_L \triangleright \mathbf{1}_{\mathcal{M}})m)$ and $E_{ld_R}(m) = E_l(m(d_R \triangleright \mathbf{1}_{\mathcal{M}}))$
iii) $E_l(\mathbf{1}_{\mathcal{M}}) = n_\sigma(l) \triangleright \mathbf{1}_{\mathcal{M}} \in C(\mathcal{N}) \cap \mathcal{M}_R$. In particular, if l is normalizable then $E_l(\mathbf{1}_{\mathcal{M}})$ is invertible and if l is normalized then $E_l(\mathbf{1}_{\mathcal{M}}) = \mathbf{1}_{\mathcal{M}}$ and therefore $E_l(\mathcal{M}) = \mathcal{N}$.
iv) If l is a positive functional on $\hat{\mathcal{A}}$ then E_l is positive and

$$E_l(m) = E_h(z_l^* m z_l), \tag{2.81}$$

where $z_l = d_R(l)^{1/2} \triangleright \mathbf{1}_{\mathcal{M}}$. If in addition l is nondegenerate then E_l is faithful.
v) If the $\mathcal{A}$-action on $\mathcal{M}$ is standard and $\mathcal{N}' \cap \mathcal{M} = \mathcal{M}_R$ then $l \mapsto E_l$ provides a linear bijection from $\mathcal{L}(\mathcal{A})$ to $Hom({}_{\mathcal{N}}\mathcal{M}_{\mathcal{N}} \to {}_{\mathcal{N}}\mathcal{N}_{\mathcal{N}})$. In this case E_l is normalized (nondegenerate, positive) if and only if l is normalized (nondegenerate, positive).

Proof: $E_l = l\triangleright$ is weakly continuous by definition. The $\mathcal{N}$-$\mathcal{N}$ bimodule property follows from Proposition 2.3.1. Also $a \triangleright E_l(m) = a_{(1)}S(a_{(2)}) \triangleright E_l(m)$ for all $a \in \mathcal{A}$ and $\mathcal{M} \in m$ by (1.31). Hence $E_l(\mathcal{M}) \subset \mathcal{N}$ by Proposition 2.3.1ii), which proves part i). Part (ii) follows from Lemma 2.4.3i)+ii). Part iii) follows from (1.42), (2.13) and the identities (2.24) and (2.26). Moreover, $E_l(\mathbf{1}_{\mathcal{M}}) \in C(\mathcal{N})$ by i), and by Proposition 2.2.4 $E_l(\mathbf{1}_{\mathcal{M}})$ is invertible if $n_L(l)$ is invertible. Next, if l provides a positive (and faithful) weight on $\hat{\mathcal{A}}$ then $E_l = (id \otimes l) \circ \rho$ is positive (and faithful), since by (2.2) ρ is an injective $*$-algebra map. Moreover, (2.81) follows similarly as (2.80) (note that $z_l = \tau_\triangleright(\xi_l)$). This proves part iv). To prove v) we use that iii), i) and the fact that the normalized Haar integral $h \in \mathcal{A}$ is a positive faithful state on $\hat{\mathcal{A}}$ [BNS] imply $E_h : \mathcal{M} \to \mathcal{N}$ to be a faithful conditional expectation. Hence any weakly continuous $\mathcal{N} - \mathcal{N}$ bimodule map $E : \mathcal{M} \to \mathcal{N}$ is of the form $E(m) = E_h^z(m) := E_h(zm)$, for some $z \in \mathcal{N}' \cap \mathcal{M}$[4] and the correspondence $z \leftrightarrow E_h^z$ is one-to-one. If $\mathcal{N}' \cap \mathcal{M} = \mathcal{M}_R \cong \mathcal{A}_L$ then there exists a uniquely determined $d \in \mathcal{A}_L$ such that $z = d \triangleright \mathbf{1}_{\mathcal{M}}$ and therefore, by Lemma 2.4.3i) $E_h^z = E_{hd}$. By Lemma 2.6.3 the map $\mathcal{A}_L \ni d \mapsto hd \in \mathcal{L}(\mathcal{A})$ is one-to-one and therefore $\mathcal{L}(\mathcal{A}) \ni l \mapsto E_l \in Hom({}_{\mathcal{N}}\mathcal{M}_{\mathcal{N}} \to {}_{\mathcal{N}}\mathcal{N}_{\mathcal{N}})$ is one-to-one. Now E_h^z is nondegenerate if and only if z and therefore d are invertible, which by Proposition 2.6.6 is equivalent to $l = hd$ being a nondegenerate left integral. Next, if $E \in Hom({}_{\mathcal{N}}\mathcal{M}_{\mathcal{N}} \to {}_{\mathcal{N}}\mathcal{N}_{\mathcal{N}})$ is positive then $E(m) = \sum_i E_h(z_i m z_i^*)$ for some elements $z_i \in \mathcal{N}' \cap \mathcal{M}$ and all $m \in \mathcal{M}$ [Lo2]. By (1.41) and Lemma 2.4.3 this implies $E = E_l$ for $l = \sum_i h a_i a_{i*}$, where $z_i = a_i \triangleright \mathbf{1}_{\mathcal{M}}$, $a_i \in \mathcal{A}_L$ (note that $\mathcal{A}_{L*} = S(\mathcal{A}_L)^* = \mathcal{A}_R$). Using Lemma 2.4.1 this implies further

$$d_R(l) \equiv S(l_{(1)})l_{(2)} = \sum_i S(a_i)S(h_{(1)})h_{(2)}a_{i*} = \sum_i S(a_i)S(a_i)^* \geq 0$$

[4]Using a Pimsner-Popa basis u_i for E_h we have $z = \sum E(u_i)u_i^*$ since $E(m) = \sum E(u_i E_h(u_i^* m)) = \sum E(u_i)E_h(u_i^* m) = \sum E_h(E(u_i)u_i^* m)$. For $n \in \mathcal{N}$ it follows $nz = \sum E(nu_i)u_i^* = \sum E_h(znu_i)u_i^* = zn$ and therefore $z \in \mathcal{N}' \cap \mathcal{M}$.

and therefore l is a positive weight on $\hat{\mathcal{A}}$ by Proposition 2.6.6. Finally, $E_l(\mathbf{1}_{\mathcal{M}}) = n_L(l) \triangleright \mathbf{1}_{\mathcal{M}}$ and therefore, using $\mathcal{M}_R \cong \mathcal{A}_L$, E_l is normalized if and only if l is normalized. ■

The above results will become useful in Sect. 4, where the conditions of standardness and $\mathcal{N}' \cap \mathcal{M} = \mathcal{M}_R$ in Theorem 2.6.7v) will hold if $\mathcal{A}$ acts regularly on $\mathcal{M}$, see Definition 4.4.1 and Theorem 4.4.2. In particular this holds, if $\mathcal{A}$ is pure, $\mathcal{M}$ is a factor and $\mathcal{A}$ acts outerly on $\mathcal{M}$, which are precisely the conditions of Theorem 4.2.5 guaranteeing that $\mathcal{N} \subset \mathcal{M} \subset \mathcal{M} \rtimes \mathcal{A}$ provides a Jones triple of factors.

Note that for the left integrals (2.76) of our example we get

$$E_l(m) = \frac{1}{|G|} \sum_{g \in G,\, h \in H} c(h)\, \alpha_g(u(h)m). \tag{2.82}$$

We conclude this Section with studying the *index* of E_l. Picking up the terminology of Watatani [Wa] we call E_l of *index-finite type* if there exists a *quasi-basis* $u_i, v_i \in \mathcal{M}$ such that for all $m \in \mathcal{M}$

$$\sum_i u_i E_l(v_i m) = m = \sum_i E_l(m u_i) v_i$$

In this case the (Watatani) index of E_l is defined by [5]

$$\mathrm{Ind}\, E_l = \sum_i u_i v_i \in C(\mathcal{M}) \tag{2.83}$$

The following Lemma says that if E_l is of index-finite type for some nondegenerate left integral $l \in \mathcal{L}(\mathcal{A})$ then it is of index-finite type for all nondegenerate left integrals in $\mathcal{A}$.

Lemma 2.6.8 *Let $l \in \mathcal{L}(\mathcal{A})$ and let $u_i, v_i \in \mathcal{M}$ be a quasi-basis for E_l. For $d_\sigma \in \mathcal{A}_\sigma$ invertible and $\sigma = L, R$ let $x_\sigma := d_\sigma^{-1} \triangleright \mathbf{1}_M$. Then*
i) $(u_i x_R,\, v_i)$ and $(u_i,\, v_i x_L)$ are quasi-basises for E_{ld_R} and E_{ld_L}, respectively.
ii) If $d \in \mathcal{A}_L \cap \mathcal{A}_R$ is invertible then

$$\mathit{Ind}\, E_{ld} = (d^{-1} \triangleright \mathbf{1}_{\mathcal{M}})\, \mathit{Ind}\, E_l \tag{2.84}$$

Proof: Since $x_\sigma \in \mathcal{M}_R$ commutes with $\mathcal{M}^{\mathcal{A}}$, part i) follows immediately from Lemma 2.4.3i)+ii) and part ii) follows from Lemma 2.4.3iii). ■

Considering the special case $\mathcal{M} = \hat{\mathcal{A}}$ with its canonical left $\mathcal{A}$-action we define

Definition 2.6.9 For any nondegenerate left integral $l \in \mathcal{L}(\mathcal{A})$ we define its index $\mathrm{Ind}\, l$ to be the Watatani index of $E_l : \hat{\mathcal{A}} \to \hat{\mathcal{A}}_L$

To see that this is well defined we have to check that $E_l : \hat{\mathcal{A}} \to \hat{\mathcal{A}}_L$ is indeed of index-finite type.

Proposition 2.6.10 (Index, quasi-basis and dual left integral)
Let $l \in \mathcal{L}(\mathcal{A})$ and $\lambda \in \mathcal{L}(\hat{\mathcal{A}})$ be a dual pair of nondegenerate left integrals and put $n_R(l) \equiv \hat{\varepsilon}_R \varepsilon_R(l) \equiv l_{(2)} S^{-1}(l_{(1)})$ as in (2.59). Denote $E_l : \hat{\mathcal{A}} \ni \phi \mapsto (l \rightharpoonup \phi) \in \hat{\mathcal{A}}_L$ and similarly $\hat{E}_\lambda : \mathcal{A} \ni a \mapsto (\lambda \rightharpoonup a) \in \mathcal{A}_L$. Then
i) E_l is of index finite type with quasi-basis $\sum_i u_i \otimes v_i = \lambda_{(2)} \otimes \hat{S}^{-1}(\lambda_{(1)})$

[5] This is independent of the choice of quasi-basis as in [Wa], even if E_l is not normalized.

ii) $\hat{E}_\lambda$ is of index finite type with quasi-basis $\sum_i u_i \otimes v_i = l_{(2)} \otimes S^{-1}(l_{(1)})$
iii)

$$\begin{aligned} Ind\, l &= \lambda_{(2)}\hat{S}^{-1}(\lambda_{(1)}) \equiv \quad n_R(\lambda) \in \hat{\mathcal{A}}_R \cap C(\hat{\mathcal{A}}) & (2.85)\\ Ind\,\lambda &= l_{(2)}S^{-1}(l_{(1)}) \equiv \quad n_R(l) \in \mathcal{A}_R \cap C(\mathcal{A}). & (2.86)\end{aligned}$$

iv) Assume l normalized and $Ind\, l$ invertible. If $Ind\, l$ is hypercentral, i.e. $Ind\, l \in \hat{\mathcal{Z}} := \hat{\mathcal{A}}_L \cap \hat{\mathcal{A}}_R \cap C(\hat{\mathcal{A}})$, then the normalized left integral $\dot{\lambda} = n_R(\lambda)^{-1}\lambda$ satisfies

$$\hat{E}_\lambda = (Ind\,\dot{\lambda})\hat{E}_{\dot{\lambda}} \tag{2.87}$$

In this cases, $Ind\,\dot{\lambda} \in \mathcal{Z} := \mathcal{A}_L \cap \mathcal{A}_R \cap C(\mathcal{A})$ and under the natural isomorphism $\mathcal{Z} \cong \hat{\mathcal{Z}}$ these indices coincide, i.e. for $\sigma = L, R$

$$Ind\,\dot{\lambda} = \hat{\varepsilon}_\sigma(Ind\, l) \quad , \quad Ind\, l = \varepsilon_\sigma(Ind\,\dot{\lambda}) \tag{2.88}$$

Proof: Part (i) is the dual of part (ii). To prove (ii) we compute

$$\sum_i (\lambda \rightharpoonup (au_i))v_i = (\lambda \rightharpoonup (al_{(2)}))S^{-1}(l_{(1)}) = a_{(1)}S(a_{(2)})a_{(3)} = a$$

where we have used the identity

$$\begin{aligned} l_{(1)} \otimes (\lambda \rightharpoonup al_{(2)}) &= l_{(1)} \otimes a_{(1)}l_{(2)}\langle \lambda \mid a_{(2)}l_{(3)}\rangle \\ &= (\mathbf{1} \otimes a_{(1)})\Delta(l_R \circ \lambda_L(a_{(2)})) \\ &= S(a_{(3)}) \otimes a_{(1)}S(a_{(2)}) \end{aligned}$$

following from $l_R \circ \lambda_L = S$, see Proposition 2.6.4. Similarly,

$$\begin{aligned} \sum_i u_i(\lambda \rightharpoonup (v_i a)) &= l_{(3)}S^{-1}(l_{(2)})a_{(1)}\langle \lambda \mid S^{-1}(l_{(1)})a_{(2)}\rangle \\ &= l_{(3)}S^{-1}(l_{(2)})a_{(1)}\langle \hat{S}^{-1}(\lambda_R(a_{(2)})) \mid l_{(1)}\rangle \\ &= a_{(3)}S^{-1}(a_{(2)})a_{(1)} = a \end{aligned}$$

where in the third line we have used $l_L \circ \hat{S}^{-1} \circ \lambda_R = id_{\mathcal{A}}$ by (2.67). Thus $u_i \otimes v_i$ provides a quasi basis for $\hat{E}_\lambda$. Eqs. (2.83) and (2.26) then imply

$$\mathrm{Ind}\,\lambda = l_{(2)}S^{-1}(l_{(1)}) = \hat{\varepsilon}_R\varepsilon_R(l) = n_R(l)$$

by (2.60). Together with its dual version this proves part (iii). To prove part (iv) let now l be normalized implying $\mathrm{Ind}\,\lambda = \mathbf{1}$ by (2.85). If $\mathrm{Ind}\, l \equiv n_R(\lambda)$ is invertible then λ is normalizable and $\dot{\lambda} := n_R(\lambda)^{-1}\lambda$ is normalized. If $\mathrm{Ind}\, l \equiv n_R(\lambda) \in \hat{\mathcal{A}}_L \cap \hat{\mathcal{A}}_R \cap C(\hat{\mathcal{A}})$ then Lemma 2.6.8ii) and $\mathrm{Ind}\,\lambda = \mathbf{1}$ imply

$$\mathrm{Ind}\,\dot{\lambda} = n_R(\lambda) \rightharpoonup \mathbf{1} = \hat{\varepsilon}_\sigma(\mathrm{Ind}\, l) \in \mathcal{A}_L \cap \mathcal{A}_R \cap C(\mathcal{A})$$

where we have used Lemma 2.4.5. Eq.(2.28) then also implies $\mathrm{Ind}\, l = (\varepsilon_\sigma\hat{\varepsilon}_{\sigma'})(\mathrm{Ind}\, l) = \varepsilon_\sigma(\mathrm{Ind}\,\dot{\lambda})$ proving (2.88). Finally, (2.87) follows from Theorem 2.6.7ii). ∎

In Theorem 3.2.3 we will generalize Proposition 2.6.10 to the case $\mathcal{M} = \mathcal{N} \rtimes \hat{\mathcal{A}}$ with its canonical left $\mathcal{A}$-action, and in Theorem 4.1.3 we will show that for all $\mathcal{A}$-module von-Neumann algebras $\mathcal{M}$ and for all positive normalized and nondegenerate left integrals $l \in \mathcal{L}(\mathcal{A})$ the index of $E_l : \mathcal{M} \to \mathcal{N}$ is always bounded by $\mathrm{Ind}\, E_l \le \tau_\triangleright(\mathrm{Ind}\, l)$. Also note that in a pure weak Hopf algebra $\mathcal{A}$ the condition $\mathrm{Ind}\, l \in \hat{\mathcal{Z}} \equiv \hat{\mathcal{A}}_L \cap \hat{\mathcal{A}}_R \cap C(\mathcal{A})$ means $\mathrm{Ind}\, l \in \mathbf{C}$ by Proposition 2.4.6ii). On the other hand, if $\hat{\mathcal{A}}$ is pure then $\mathrm{Ind}\, l \in \mathbf{C}$ always holds by Proposition 2.4.6iii).

3 Crossed Products

In this section $\mathcal{A}$, $\hat{\mathcal{A}}$ and $\mathcal{M}$ will have the same meaning as in the previous section. We also continue to denote the units by $\mathbf{1}_{\mathcal{M}} \in \mathcal{M}$, $\mathbf{1} \in \mathcal{A}$ and $\hat{\mathbf{1}} \equiv \varepsilon \in \hat{\mathcal{A}}$, respectively. Using the epimorphism

$$\mu_{\triangleright} : \mathcal{A}_L \ni a \mapsto a \triangleright \mathbf{1}_{\mathcal{M}} \in \mathcal{M}_R \tag{3.1}$$

given in Proposition 2.2.4 we now define the crossed product $\mathcal{M} \rtimes \mathcal{A}$ to be the $\mathbf{C}$-vector space

$$\mathcal{M} \rtimes \mathcal{A} = \mathcal{M} \otimes_{\mathcal{A}_L} \mathcal{A}$$

where $\mathcal{A}_L$ acts on $\mathcal{A}$ by left multiplication and on $\mathcal{M}$ by right multiplication via its image $\mathcal{M}_R$ under $\mu_{\triangleright}$. Thus $\mathcal{M} \rtimes \mathcal{A}$ is the linear span of elements $(m \rtimes a)$, $m \in \mathcal{M}$, $a \in \mathcal{A}$, modulo the relation

$$(m \rtimes ba) = (m(b \triangleright \mathbf{1}_{\mathcal{M}}) \rtimes a), \ \forall b \in \mathcal{A}_L \tag{3.2}$$

In Section 3.1 we prove that equs. (1.57)-(1.58) in Definition 1.2.5 indeed provide a $*$-algebra structure on $\mathcal{M} \rtimes \mathcal{A}$. In Section 3.2 we construct the dual $\hat{\mathcal{A}}$-action on $\mathcal{M} \rtimes \mathcal{A}$, having $\mathcal{M}$ as its $\hat{\mathcal{A}}$-invariant subalgebra. We then apply our results of Section 2.5 to show that positive normalized left integrals $\lambda \in \hat{\mathcal{A}}$ give rise to conditional expectations $\hat{E}_\lambda : \mathcal{M} \rtimes \mathcal{A} \to \mathcal{M}$ which are always of index finite type. In fact, their quasi-basis can always be given explicitely in terms of their dual left integrals $l \in \mathcal{A}$.

In Section 3.3 we develop a Plancherel–duality theory for *positive* left integrals fitting with the well known notion of Haagerup–duality for conditional expectations in Jones theory. This improves the duality concept of [BNS], which did not respect positivity.

In Section 3.4 we pass to a von-Neumann algebraic setting by associating a so-called regular faithful Hilbert space representation π_{cros} of $\mathcal{M} \rtimes \mathcal{A}$ on $\mathcal{H} \otimes L^2(\mathcal{A}, \hat{h})$ with any representation π of $\mathcal{M}$ on $\mathcal{H}$. This is done by identifying $\mathcal{M} \rtimes \mathcal{A}$ with a (non-unital) $*$-subalgebra of $\mathcal{M} \otimes \operatorname{End} \mathcal{A}$. In particular, $\mathcal{M} \rtimes \mathcal{A}$ becomes a C^*- or von-Neumann algebra whenever $\mathcal{M}$ is endowed with such a structure. This will be the starting point for the relation with Jones theory in Section 4.

3.1 The $*$-algebra $\mathcal{M} \rtimes \mathcal{A}$

We now prove that the definitions (1.57)-(1.58) provide a $*$-algebra structure on $\mathcal{M} \rtimes \mathcal{A}$.

Theorem 3.1.1 (Crossed product)
i) The crossed product $\mathcal{M} \rtimes \mathcal{A}$ becomes a $*$-algebra with multiplication and $*$-structure given for $m, m' \in \mathcal{M}$ and $a, a' \in \mathcal{A}$ by

$$(m \rtimes a)(m' \rtimes a') := (m(a_{(1)} \triangleright m') \rtimes a_{(2)} a') \tag{3.3}$$
$$(m \rtimes a)^* := (\mathbf{1}_{\mathcal{M}} \rtimes a^*)(m^* \rtimes \mathbf{1}) \tag{3.4}$$

ii) For all $m \in \mathcal{M}$ and $a \in \mathcal{A}$ we have

$$((a \triangleright m) \rtimes \mathbf{1}) = (\mathbf{1}_{\mathcal{M}} \rtimes a_{(1)})\,(m \rtimes \mathbf{1})\,(\mathbf{1}_{\mathcal{M}} \rtimes S(a_{(2)})) \tag{3.5}$$

*iii) The unit in $\mathcal{M} \rtimes \mathcal{A}$ is given by $(\mathbf{1}_{\mathcal{M}} \rtimes \mathbf{1})$. Moreover, we have $\mathcal{M} \cong (\mathcal{M} \rtimes \mathbf{1}) \equiv \mathcal{M} \otimes_{\mathcal{A}_L} \mathcal{A}_L$ and $\mathcal{A}/I_{\triangleright} \cong (\mathbf{1}_{\mathcal{M}} \rtimes \mathcal{A}) \equiv \mathcal{M}_R \otimes_{\mathcal{A}_L} \mathcal{A}$ as unital *-subalgebras of $\mathcal{M} \rtimes \mathcal{A}$, where $I_{\triangleright} \subset \mathcal{A}$ is the*

two-sided ideal generated by $\mathrm{Ker}\,\mu_\triangleright$.
iv) Putting $\mathcal{N} \equiv \mathcal{M}^{\mathcal{A}}$ *the inclusions (1.60) - (1.63) hold, i.e.*

$$\begin{aligned} \mathbf{1}_{\mathcal{M}} \rtimes \mathcal{A} &\subset \mathcal{N}' \cap (\mathcal{M} \rtimes \mathcal{A}) && (3.6) \\ \mathbf{1}_{\mathcal{M}} \rtimes \mathcal{A}_R &\subset \mathcal{M}' \cap (\mathcal{M} \rtimes \mathcal{A}) && (3.7) \\ \mathbf{1}_{\mathcal{M}} \rtimes (\mathcal{A}_R \cap C(\mathcal{A})) &\subset C(\mathcal{M} \rtimes \mathcal{A}) && (3.8) \\ \mathbf{1}_{\mathcal{M}} \rtimes (\mathcal{A}_L \cap \mathcal{A}_R \cap C(\mathcal{A})) &\subset \mathcal{M} \cap C(\mathcal{M} \rtimes \mathcal{A}) && (3.9) \end{aligned}$$

v)
$$\mathcal{M} \cap C(\mathcal{M} \rtimes \mathcal{A}) = \mathcal{N} \cap C(\mathcal{M}) = \mathcal{N} \cap C(\mathcal{M} \rtimes \mathcal{A})$$

To prepare the proof of Theorem 3.1.1 we need the following 2 Lemmata.

Lemma 3.1.2 [N2] *For* $a \in \mathcal{A}$ *and* $b \in \mathcal{A}_L$ *we have*

$$\begin{aligned} ab &= \hat{\varepsilon}_L \varepsilon_R(a_{(1)} b) a_{(2)} && (3.10) \\ ba &= a_{(2)} \hat{\varepsilon}_L \varepsilon_L(b a_{(1)}) && (3.11) \end{aligned}$$

Lemma 3.1.3 *The two-sided ideal* $I_\triangleright$ *generated by* $\mathrm{Ker}\,\mu_\triangleright$ *satisfies* $I_\triangleright \subset \mathcal{K}_\triangleright$ *and*

$$I_\triangleright = \mathcal{A}\,(\mathrm{Ker}\,\mu_\triangleright) = (\mathrm{Ker}\,\mu_\triangleright)\,\mathcal{A} = I_\triangleright^* .$$

Proof: By Corollary 2.4.4iii) $I_\triangleright \subset \mathcal{K}_\triangleright$ and by Proposition 2.2.4 $\mathrm{Ker}\,\mu_\triangleright = z_\triangleright \mathcal{A}_L$ for a central projection $z_\triangleright \in \mathcal{A}_L \cap C(\mathcal{A})$. Hence $I_\triangleright = z_\triangleright \mathcal{A}$, proving the assertions. ■

Proof of Theorem 3.1.1
To show that the multiplication (3.3) is well defined let $b \in \mathcal{A}_L$. Then

$$\begin{aligned} (m \rtimes ba)(m' \rtimes a') &= (m(b_{(1)} a_{(1)} \triangleright m') \rtimes b_{(2)} a_{(2)} a') \\ &= (m(b a_{(1)} \triangleright m') \rtimes a_{(2)} a') \\ &= (m(b \triangleright \mathbf{1}_{\mathcal{M}})(a_{(1)} \triangleright m') \rtimes a_{(2)} a') \\ &= (m(b \triangleright \mathbf{1}_{\mathcal{M}}) \rtimes a)(m' \rtimes a') \end{aligned}$$

where in the second line we have used Lemma 2.4.1i) and in the third line Lemma 2.4.3i). Next,

$$\begin{aligned} (m \rtimes a)(m' \rtimes ba') &= (m(a_{(1)} \triangleright m') \rtimes a_{(2)} b a') \\ &= (m(a_{(1)} \triangleright m') \rtimes \hat{\varepsilon}_L \varepsilon_R(a_{(2)} b) a_{(3)} a') \\ &= (m(a_{(1)} \triangleright m')(a_{(2)} b \triangleright \mathbf{1}_{\mathcal{M}}) \rtimes a_{(3)} a') \\ &= (m \rtimes a)(m'(b \triangleright \mathbf{1}_{\mathcal{M}}) \rtimes a') \end{aligned}$$

where in the second line we have used Lemma 3.1.2 and in the third line (3.2) and (1.42). Thus the multiplication (3.3) is well defined on $\mathcal{M} \rtimes \mathcal{A}$. The associativity of the product (3.3) is straightforward and is left to the reader. Next, we compute

$$\begin{aligned} (m \rtimes \mathbf{1})(m' \rtimes a) &= (m(\mathbf{1}_{(1)} \triangleright m') \rtimes \mathbf{1}_{(2)} a) \\ &= (m(\mathbf{1}_{(1)} \triangleright m')(\mathbf{1}_{(2)} \triangleright \mathbf{1}_{\mathcal{M}}) \rtimes a) \\ &= (mm' \rtimes a) \end{aligned}$$

since $\Delta(\mathbf{1}) \in \mathcal{A}_R \otimes \mathcal{A}_L$ by (2.37), and

$$\begin{aligned}(m \rtimes a)(\mathbf{1}_{\mathcal{M}} \rtimes a') &= (m(a_{(1)} \triangleright \mathbf{1}_{\mathcal{M}}) \rtimes a_{(2)}a') \\ &= (m(a_{(1)}S(a_{(2)}) \triangleright \mathbf{1}_{\mathcal{M}}) \rtimes a_{(3)}a') \\ &= (m \rtimes a_{(1)}S(a_{(2)})a_{(3)}a') \\ &= (m \rtimes aa')\end{aligned}$$

where we have used (1.42) in the second line and (1.12) in the last line. Thus $(\mathbf{1}_{\mathcal{M}} \rtimes \mathbf{1})$ is the unit in $\mathcal{M}\rtimes\mathcal{A}$ and $m \mapsto (m \rtimes \mathbf{1})$ and $a + I_\triangleright \mapsto (\mathbf{1}_{\mathcal{M}} \rtimes a)$ define unital inclusions $\mathcal{M} \subset \mathcal{M}\rtimes\mathcal{A}$ and $\mathcal{A}/I_\triangleright \subset \mathcal{M}\rtimes\mathcal{A}$, respectively. [6]

Next, to show that (3.4) provides a well defined $*$-structure let again $b \in \mathcal{A}_L$. Then

$$\begin{aligned}(m \rtimes ba)^* &= (\mathbf{1}_{\mathcal{M}} \rtimes a^*b^*)(m^* \rtimes \mathbf{1}) \\ &= (\mathbf{1}_{\mathcal{M}} \rtimes a^*)((b^*_{(1)} \triangleright m^*) \rtimes b^*_{(2)}) \\ &= (\mathbf{1}_{\mathcal{M}} \rtimes a^*)((b^*_{(1)} \triangleright m^*)(b^*_{(2)} \triangleright \mathbf{1}_{\mathcal{M}}) \rtimes \mathbf{1}) \\ &= (\mathbf{1}_{\mathcal{M}} \rtimes a^*)((b^* \triangleright m^*) \rtimes \mathbf{1}) \\ &= (\mathbf{1}_{\mathcal{M}} \rtimes a^*)((S^{-1}(b) \triangleright m)^* \rtimes \mathbf{1}) \\ &= ((S^{-1}(b) \triangleright m) \rtimes a)^* \\ &= (m(b \triangleright \mathbf{1}_{\mathcal{M}}) \rtimes a)^*\end{aligned}$$

where in the third line we have used $\Delta(b^*) \in \mathcal{A} \otimes \mathcal{A}_L$, in the fifth line (1.41) and in the last line Lemma 2.4.3ii) (note that $S^{-1}(b) \in \mathcal{A}_R$). Thus $(m \rtimes a)^*$ is well defined. Moreover, it is involutive since

$$\begin{aligned}(m \rtimes a)^{**} &= [(\mathbf{1}_{\mathcal{M}} \rtimes a^*)(m^* \rtimes \mathbf{1})]^* \\ &= ((a^*_{(1)} \triangleright m^*) \rtimes a^*_{(2)})^* \\ &= (\mathbf{1}_{\mathcal{M}} \rtimes a_{(2)})((S^{-1}(a_{(1)}) \triangleright m) \rtimes \mathbf{1}) \\ &= ((a_{(2)}S^{-1}(a_{(1)}) \triangleright m) \rtimes a_{(3)}) \\ &= ((\mathbf{1}_{(1)} \triangleright m) \rtimes \mathbf{1}_{(2)}a) \\ &= (m \rtimes a)\end{aligned}$$

where in the third line we have used (1.41) and in the fourth line (1.20). Finally, to prove the antimultiplicativity of the $*$-operation it is enough to check for all $a \in \mathcal{A},\ m \in \mathcal{M}$

$$\begin{aligned}(m \rtimes \mathbf{1})^*(\mathbf{1}_{\mathcal{M}} \rtimes a)^* &= (m^* \rtimes \mathbf{1})(\mathbf{1}_{\mathcal{M}} \rtimes a^*) \\ &= (m^* \rtimes a^*) \\ &= (m^* \rtimes a^*)^{**} \\ &= [(\mathbf{1}_{\mathcal{M}} \rtimes a)(m \rtimes \mathbf{1})]^*\end{aligned}$$

Thus $\mathcal{M}\rtimes\mathcal{A}$ is a $*$-algebra containing $\mathcal{M} \cong (\mathcal{M} \rtimes \mathbf{1})$ and $\mathcal{A}/I_\triangleright \cong (\mathbf{1}_{\mathcal{M}} \rtimes \mathcal{A})$ as $*$-subalgebras, which proves i) and iii). To prove part ii) we use (3.3) to calculate

$$\begin{aligned}(\mathbf{1}_{\mathcal{M}} \rtimes a_{(1)})\,(m \rtimes \mathbf{1}_{\mathcal{A}})\,(\mathbf{1}_{\mathcal{M}} \rtimes S(a_{(2)})) &= ((a_{(1)} \triangleright m) \rtimes a_{(2)}S(a_{(3)})) \\ &= ((a_{(1)} \triangleright m)\,(a_{(2)}S(a_{(3)})) \triangleright \mathbf{1}_{\mathcal{M}}) \rtimes \mathbf{1}_{\mathcal{A}}) \\ &= ((a_{(1)} \triangleright m)\,(a_{(2)} \triangleright \mathbf{1}_{\mathcal{M}}) \rtimes \mathbf{1}_{\mathcal{A}}) \\ &= ((a \triangleright m) \rtimes \mathbf{1}_{\mathcal{A}})\end{aligned}$$

[6] Note that by definition $(\mathbf{1}_{\mathcal{M}} \rtimes a) = 0$ iff $a \in (\mathrm{Ker}\,\mu_\triangleright)\mathcal{A} \equiv I_\triangleright$.

where we have used (1.42) and $a_{(2)}S(a_{(3)}) \in \mathcal{A}_L$. To prove the statements in part iv) first note that (3.6) follows by putting $m = \mathbf{1}_{\mathcal{M}}$ in Proposition 2.11iii) and using $((a_{(1)} \triangleright \mathbf{1}_{\mathcal{M}}) \rtimes a_{(2)}) = (\mathbf{1}_{\mathcal{M}} \rtimes a)$. To prove (3.7) let $a \in \mathcal{A}_R$ and use Lemma 2.4.1ii) to compute

$$(\mathbf{1}_{\mathcal{M}} \rtimes a)(m \rtimes \mathbf{1}) = ((a_{(1)} \triangleright m) \rtimes a_{(2)}) = ((\mathbf{1}_{(1)} \triangleright m) \rtimes \mathbf{1}_{(2)})(\mathbf{1}_{\mathcal{M}} \rtimes a) = (m \rtimes \mathbf{1})(\mathbf{1}_{\mathcal{M}} \rtimes a) \ .$$

The inclusion (3.8) follows, since $\mathcal{M}' \cap (\mathbf{1}_{\mathcal{M}} \rtimes \mathcal{A})' \cap \mathcal{M} \rtimes \mathcal{A} = C(\mathcal{M} \rtimes \mathcal{A})$ and the inclusion (3.9) follows, since $(\mathbf{1}_{\mathcal{M}} \rtimes \mathcal{A}_L) \subset (\mathcal{M} \rtimes \mathbf{1})$. Finally, to prove part v) let $(n \rtimes \mathbf{1}) \in \mathcal{M} \cap C(\mathcal{M} \rtimes \mathcal{A})$. Then $n \in C(\mathcal{M})$ and by (3.5)

$$\begin{aligned} ((a \triangleright (nm)) \rtimes \mathbf{1}) &= (\mathbf{1}_{\mathcal{M}} \rtimes a_{(1)})\,(nm \rtimes S(a_{(2)})) \\ &= (n \rtimes \mathbf{1})\,(\mathbf{1}_{\mathcal{M}} \rtimes a_{(1)})\,(m \rtimes S(a_{(2)})) \\ &= (n(a \triangleright m) \rtimes \mathbf{1}) \end{aligned}$$

for all $a \in \mathcal{A}$ and all $m \in \mathcal{M}$. Hence $n \in \mathcal{N}$ by Proposition 2.3.1iii) implying $\mathcal{M} \cap C(\mathcal{M} \rtimes \mathcal{A}) \subset \mathcal{N} \cap C(\mathcal{M})$. Now by (3.6) $\mathcal{N} \cap C(\mathcal{M}) \subset \mathcal{N} \cap C(\mathcal{M} \rtimes \mathcal{A})$ and the inclusion $\mathcal{N} \cap C(\mathcal{M} \rtimes \mathcal{A}) \subset \mathcal{M} \cap C(\mathcal{M} \rtimes \mathcal{A})$ holds trivially. This proves part v) and therefore concludes the proof of Theorem 3.1.1. ■

Applying the above crossed product construction to the example of the partly inner group action in Section 2.5 one immediately verifies that

$$\mathcal{M} \rtimes \mathcal{A} \ni (m \rtimes (h, g)) \mapsto (mu(h) \rtimes_{\alpha} g) \in \mathcal{M} \rtimes_{\alpha} G \tag{3.12}$$

provides an identification $\mathcal{M} \rtimes \mathcal{A} = \mathcal{M} \rtimes_{\alpha} G$. Thus, if the action is standard (i.e. if the implementers $u(h)$, $h \in H$, are linearly independent in $\mathcal{M}$), then Eqs. (3.6)-(3.9) confirm with (1.9)-(1.11).

In Section 4.2 we will show that for a factor $\mathcal{M}$ we have $\mathcal{M}' \cap (\mathcal{M} \rtimes \mathcal{A}) = (\mathbf{1}_{\mathcal{M}} \rtimes \mathcal{A}_R)$ if and only if the $\mathcal{A}$-action on $\mathcal{M}$ is outer in a suitable sense. If in addition the action is standard (i.e. $\operatorname{Ker} \mu_{\triangleright} = 0$) we will also have $\mathcal{N}' \cap (\mathcal{M} \rtimes \mathcal{A}) = (\mathbf{1}_{\mathcal{M}} \rtimes \mathcal{A})$, $\mathcal{N}' \cap \mathcal{M} = \mathcal{M}_R \cong \mathcal{A}_L$ and $C(\mathcal{M} \rtimes \mathcal{A}) = \mathbf{1}_M \rtimes (\mathcal{A}_R \cap C(\mathcal{A}))$, thus verifying the scenario of Section 1.1. A generalization of these identities to non-factorial von-Neumann algebras will be given in Section 4.4.

We also remark that for non-standard actions one might try to pass to a quotient weak Hopf algebra structure on $\mathcal{B} := \mathcal{A}/\mathcal{A}_{\triangleright}$, $\mathcal{A}_{\triangleright} := I_{\triangleright} + S(I_{\triangleright}) \subset \mathcal{K}_{\triangleright}$, which would act standardly on $\mathcal{M}$. In fact, one may show that $\mathcal{B}$ and $\hat{\mathcal{B}} := \mathcal{A}_{\triangleright}^{\perp} \subset \hat{\mathcal{A}}$ again give a dual pair of weak $*$-bialgebras. But since $\varepsilon(z_{\triangleright}) > 0$ (unless $z_{\triangleright} = 0$ and $\mathcal{B} = \mathcal{A}$) by the faithfulness of $\varepsilon|_{\mathcal{A}_L}$, one has $\mathbf{1}_{\hat{\mathcal{B}}} \neq \mathbf{1}_{\hat{\mathcal{A}}}$ and therefore we are not sure whether $\mathcal{B}$ and $\hat{\mathcal{B}}$ still satisfy all weak Hopf axioms.

3.2 Dual actions and conditional expectations

In this subsection we show that as for ordinary crossed products the algebra $\mathcal{M} \rtimes \mathcal{A}$ allows for a natural dual $\hat{\mathcal{A}}$-left action given by

$$\phi \triangleright (m \rtimes a) = (m \rtimes (\phi \rightharpoonup a)) \tag{3.13}$$

where $\phi \in \hat{\mathcal{A}}$ and $(m \rtimes a) \in \mathcal{M} \rtimes \mathcal{A}$. To show that this is well defined, we consider equivalently the associated dual coaction.

Proposition 3.2.1 (Dual coaction)
Let $\rho : \mathcal{M}\rtimes\mathcal{A} \to (\mathcal{M}\rtimes\mathcal{A}) \otimes \mathcal{A}$ be given by

$$\rho(m \rtimes a) := (m \rtimes a_{(1)}) \otimes a_{(2)}$$

Then ρ provides a well defined right coaction of $\mathcal{A}$ on $\mathcal{M}\rtimes\mathcal{A}$ such that $(\mathcal{M}\rtimes\mathcal{A})_R = (\mathbf{1}_\mathcal{M} \rtimes \mathcal{A}_R)$.

Proof: If $b \in \mathcal{A}_L$ then by Lemma 2.4.1i) $\rho(m \rtimes ba) = \rho(m(b \triangleright \mathbf{1}_\mathcal{M}) \rtimes a)$ showing that ρ is well defined. The axioms (2.1)-(2.3) and the identity $(\mathcal{M}\rtimes\mathcal{A})_R = (\mathbf{1}_\mathcal{M} \rtimes \mathcal{A}_R)$ follow immediately. Using (3.3) and (3.4) one also immediately checks that ρ provides a *-algebra homomorphism. ■

Corollary 3.2.2 *The $\hat{\mathcal{A}}$-invariant subalgebra of $\mathcal{M}\rtimes\mathcal{A}$ is given by $\mathcal{M} \equiv \mathcal{M}\rtimes\mathcal{A}_L$.*

Proof: By Lemma 2.3.2 $(m \rtimes a) \in \mathcal{M}\rtimes\mathcal{A}$ is $\hat{\mathcal{A}}$-invariant if and only if $\Delta(a) \in \mathcal{A} \otimes \mathcal{A}_L$ and therefore $a \equiv \varepsilon(a_{(1)})a_{(2)} \in \mathcal{A}_L$. ■

Similarly as in Theorem 2.6.7 we now obtain for every left integral $\lambda \in \hat{\mathcal{A}}$ a $\mathcal{M}$-$\mathcal{M}$ bimodule map $\hat{E}_\lambda : \mathcal{M}\rtimes\mathcal{A} \to \mathcal{M}$ by putting

$$\hat{E}_\lambda(m \rtimes a) := (m \rtimes (\lambda \rightharpoonup a)) \equiv (m\mu_\triangleright(\lambda \rightharpoonup a) \rtimes \mathbf{1}) \tag{3.14}$$

where the second equality follows from $\lambda \rightharpoonup a \in \mathcal{A}_L$, which indeed implies $\hat{E}_\lambda(\mathcal{M}\rtimes\mathcal{A}) \subset \mathcal{M}\rtimes\mathcal{A}_L \cong \mathcal{M}$.

In our example of the partly inner group action (α, u) of Sect. 2.5 we have $\mathcal{M}\rtimes\mathcal{A} = \mathcal{M}\rtimes_\alpha G$ by (3.12) and for a left integral $\lambda \in \mathcal{L}(\hat{\mathcal{A}})$ of the form (2.77) we get

$$\hat{E}_\lambda(m\rtimes_\alpha g) = \frac{1}{|H|} \sum_{h\in H} \hat{c}(h)\delta(hg)\, mu(h^{-1}) \tag{3.15}$$

In general $\hat{E}_\lambda$ is normalized (positive, nondegenerate) if and only if its restriction to $(\mathbf{1}_\mathcal{M} \rtimes \mathcal{A})$ is normalized (positive, nondegenerate). If $(\mathbf{1}_\mathcal{M} \rtimes \mathcal{A}) \cong \mathcal{A}$ (i.e. $\mathrm{Ker}\,\mu_\triangleright = 0$), then applying Theorem 2.6.7v) to the $\hat{\mathcal{A}}$-action on $\mathcal{A}$ this is further equivalent to λ itself being normalized (positive, nondegenerate). In fact, most of the following results already follow from their validity in the case $\mathcal{M}\rtimes\mathcal{A} = \mathcal{A}$ (i.e. $\mathcal{M} = \mathcal{M}_R = \hat{\mathcal{A}}_R$). The following is a straightforward generalization from Proposition 2.6.10.

Theorem 3.2.3 (Index, quasi-basis and dual left integral)
Let $l \in \mathcal{L}(\mathcal{A})$ and $\lambda \in \mathcal{L}(\hat{\mathcal{A}})$ be a dual pair of nondegenerate left integrals and put $n_R(l) \equiv \hat{\varepsilon}_R\varepsilon_R(l) \equiv l_{(2)}S^{-1}(l_{(1)})$ as in (2.59). For any crossed product $\mathcal{M}\rtimes\mathcal{A}$ let $\hat{E}_\lambda : \mathcal{M}\rtimes\mathcal{A} \to \mathcal{M}$ be given by (3.14). Then $\hat{E}_\lambda$ is of index finite type with quasi-basis $\sum_i u_i \otimes v_i = (\mathbf{1}_\mathcal{M} \rtimes l_{(2)}) \otimes (\mathbf{1}_\mathcal{M} \rtimes S^{-1}(l_{(1)}))$ and we have

$$\hat{E}_\lambda(\mathbf{1}_\mathcal{M} \rtimes l) = \mathbf{1}_\mathcal{M} \tag{3.16}$$
$$\hat{E}_\lambda(\mathbf{1}_\mathcal{M}\rtimes\mathbf{1}_\mathcal{A}) = \tau_\triangleright(n_R(l)) \equiv \tau_\triangleright(\mathit{Ind}\, l) \in C(\mathcal{M}) \cap \mathcal{M}_R \tag{3.17}$$
$$\mathit{Ind}\, \hat{E}_\lambda = (\mathbf{1}_\mathcal{M} \rtimes \mathit{Ind}\,\lambda) \in (\mathcal{M}\rtimes\mathcal{A})_R \cap C(\mathcal{M}\rtimes\mathcal{A}) \tag{3.18}$$

Moreover, under the conditions of Proposition 2.6.10iv) we have $\hat{E}_\lambda = \tau_\triangleright(\mathit{Ind}\, l)\, \hat{E}_{\dot{\lambda}}$ and

$$\tau_\triangleright(\mathit{Ind}\, l) \equiv (\tau_\triangleright(\mathit{Ind}\, l)\rtimes\mathbf{1}_\mathcal{A}) = (\mathbf{1}_M\rtimes \mathit{Ind}\, \dot{\lambda}) \in \mathcal{N} \cap C(\mathcal{M}\rtimes\mathcal{A}). \tag{3.19}$$

Proof: By definition $\hat{E}_\lambda(\mathbf{1}_\mathcal{M} \rtimes l) = (\mathbf{1}_\mathcal{M} \rtimes (\lambda \rightharpoonup l)) = (\mathbf{1}_\mathcal{M} \rtimes \mathbf{1})$, which proves (3.16). Eq.(3.17) follows, since by (2.85) and the definition (2.16) of $\tau_\triangleright$

$$\tau_\triangleright(\mathrm{Ind}\, l) = \tau_\triangleright(\hat{\varepsilon}_R \varepsilon_R(\lambda)) = (\lambda \rightharpoonup \mathbf{1}) \triangleright \mathbf{1}_\mathcal{M} = \hat{E}_\lambda(\mathbf{1}_\mathcal{M} \rtimes \mathbf{1})$$

where in the last equation we have identified $\mathcal{M}_R$ with $(\mathbf{1}_\mathcal{M} \rtimes \mathcal{A}_L) \subset \mathcal{M} \rtimes \mathcal{A}$. Since by the $\mathcal{M}$-$\mathcal{M}$–bimodule property (or by Lemma 2.4.3 iii)) $\hat{E}_\lambda(\mathbf{1}_\mathcal{M} \rtimes \mathbf{1}) \in C(\mathcal{M})$, this proves (3.17). The fact that $u_i \otimes v_i = (\mathbf{1}_\mathcal{M} \rtimes l_{(2)}) \otimes (\mathbf{1}_\mathcal{M} \rtimes S^{-1}(l_{(1)}))$ is a quasi-basis for $\hat{E}_\lambda$ follows similarly as in the proof of Proposition 2.6.10ii). Indeed by the same arguments

$$\sum_i \hat{E}_\lambda((m \rtimes a)u_i)v_i = (m \rtimes \lambda \rightharpoonup (al_{(2)}))S^{-1}(l_{(1)}) = a$$

and

$$\begin{aligned}
\sum_i u_i \hat{E}_\lambda(v_i(m \rtimes a)) &= \sum_i u_i(\lambda_{(1)} \triangleright v_i)(\lambda_{(2)} \triangleright (m \rtimes a)) \\
&= (\mathbf{1}_\mathcal{M} \rtimes l_{(3)} S^{-1}(l_{(2)}))(m \rtimes a_{(1)})\langle \lambda \mid S^{-1}(l_{(1)})a_{(2)}\rangle \\
&= (m \rtimes l_{(3)} S^{-1}(l_{(2)})a_{(1)})\langle \hat{S}^{-1}(\lambda_R(a_{(2)})) \mid l_{(1)}\rangle \\
&= (m \rtimes a_{(3)} S^{-1}(a_{(2)})a_{(1)}) \\
&= (m \rtimes a)
\end{aligned}$$

where in the third line we have used $l_{(3)}S^{-1}(l_{(2)}) \in \mathcal{A}_R$, which commutes with $\mathcal{M}$, and in the fourth line $l_L \circ \hat{S}^{-1} \circ \lambda_R = id_\mathcal{A}$ by (2.67). Thus $u_i \otimes v_i$ provides a quasi-basis and $\mathrm{Ind}\, \hat{E}_\lambda = (\mathbf{1}_\mathcal{M} \rtimes l_{(2)}S^{-1}(l_{(1)})) = (\mathbf{1}_\mathcal{M} \rtimes \mathrm{Ind}\, \lambda) \in \mathbf{1}_\mathcal{M} \rtimes (\mathcal{A}_R \cap C(\mathcal{A})) \subset (\mathcal{M} \rtimes \mathcal{A})_R \cap C(\mathcal{M} \rtimes \mathcal{A})$ by (2.86), (3.8) and Proposition 3.2.1, which proves (3.18). Finally, if l is normalized and $\mathrm{Ind}\, l \in \mathcal{Z}$ is invertibel, then by (2.88), (2.16) and Corollary 2.4.4i)+ii) $\tau_\triangleright(\mathrm{Ind}\, l) = \mu_\triangleright(\mathrm{Ind}\, \dot{\lambda}) \in \mathcal{N} \cap C(\mathcal{M}) \equiv \mathcal{N} \cap C(\mathcal{M} \rtimes \mathcal{A})$ and Lemma 2.6.8 implies $\hat{E}_\lambda = \tau_\triangleright(\mathrm{Ind}\, l)\, \hat{E}_{\dot{\lambda}}$. ■

As an application of these methods we now show that the relative commutant $\mathcal{N}' \cap \mathcal{M} \rtimes \mathcal{A}$ saturates the lower bound (3.6) if and only if $\mathcal{N}' \cap \mathcal{M}$ saturates the lower bound (2.35). Note, by the way, that in our example of Sect 2.5 these lower bounds are always saturated, see also (1.9) and (1.10).

Proposition 3.2.4 *Let $\mathcal{N} \equiv \mathcal{M}^\mathcal{A} \subset \mathcal{M}$ be the $\mathcal{A}$-invariant subalgebra. Then $\mathcal{A} \triangleright (\mathcal{N}' \cap \mathcal{M}) \subset \mathcal{N}' \cap \mathcal{M}$ and $\mathcal{N}' \cap (\mathcal{M} \rtimes \mathcal{A}) = (\mathcal{N}' \cap \mathcal{M}) \rtimes \mathcal{A}$. In particular, $\mathcal{N}' \cap \mathcal{M} = \mathcal{M}_R$ if and only if $\mathcal{N}' \cap (\mathcal{M} \rtimes \mathcal{A}) = (\mathbf{1}_\mathcal{M} \rtimes \mathcal{A})$.*

Proof: Let $n \in \mathcal{N}$, $m \in \mathcal{N}' \cap \mathcal{M}$ and $a \in \mathcal{A}$. Then $n(a \triangleright m) = a \triangleright (nm) = a \triangleright (mn) = (a \triangleright m)n$ implying $\mathcal{A} \triangleright (\mathcal{N}' \cap \mathcal{M}) \subset \mathcal{N}' \cap \mathcal{M}$ and by (3.6) $(\mathcal{N}' \cap \mathcal{M}) \rtimes \mathcal{A} \subset \mathcal{N}' \cap (\mathcal{M} \rtimes \mathcal{A})$. To prove the inverse conclusion observe that $\hat{E}_\lambda(\mathcal{N}' \cap (\mathcal{M} \rtimes \mathcal{A})) = \mathcal{N}' \cap \mathcal{M}$ for any normalized nondegenerate left integral $\lambda \in \mathcal{L}(\hat{\mathcal{A}})$. Now by Theorem 3.2.3i) and Eq.(3.6) a quasi-basis u_i, v_i for $\hat{E}_\lambda$ may be chosen in $(\mathbf{1}_\mathcal{M} \rtimes \mathcal{A})$, i.e. in the commutant of $\mathcal{N}$, and therefore we get for any $x \in \mathcal{N}' \cap (\mathcal{M} \rtimes \mathcal{A})$

$$x = \sum \hat{E}_\lambda(xu_i)v_i \in (\mathcal{N}' \cap \mathcal{M}) \rtimes \mathcal{A}$$

Hence $(\mathcal{N}' \cap \mathcal{M}) \rtimes \mathcal{A} = \mathcal{N}' \cap (\mathcal{M} \rtimes \mathcal{A})$. If $(\mathcal{N}' \cap \mathcal{M}) = \mathcal{M}_R$ this implies $\mathcal{N}' \cap (\mathcal{M} \rtimes \mathcal{A}) = (\mathbf{1}_\mathcal{M} \rtimes \mathcal{A})$. Conversely, if $\mathcal{N}' \cap (\mathcal{M} \rtimes \mathcal{A}) = (\mathbf{1}_\mathcal{M} \rtimes \mathcal{A})$ then $\mathcal{N}' \cap \mathcal{M} = \hat{E}_\lambda(\mathbf{1}_\mathcal{M} \rtimes \mathcal{A}) = (\mathbf{1}_\mathcal{M} \rtimes \mathcal{A}_L) = \mathcal{M}_R$.
■

We conclude this subsection with using similar arguments to characterize $\hat{E}_\lambda$-invariant (and, more generally, $\hat{\mathcal{A}}$-invariant) ideals in $\mathcal{M}\rtimes\mathcal{A}$ in terms of their intersection with $\mathcal{M} \equiv \hat{E}_\lambda(\mathcal{M}\rtimes\mathcal{A})$, thus generalizing standard results from ordinary crossed product theory. In particular, this will imply that $\hat{\mathcal{A}}$-covariant representations of $\mathcal{M}\rtimes\mathcal{A}$ are faithful whenever their restrictions to $\mathcal{M}$ are faithful.

Lemma 3.2.5 ($\hat{E}_\lambda$-invariant ideals)
Let $\lambda \in \mathcal{L}(\hat{\mathcal{A}})$ be a nondegenerate left integral and let $\mathcal{I} \subset \mathcal{M}\rtimes\mathcal{A}$ be an ideal such that $\hat{E}_\lambda(\mathcal{I}) \subset \mathcal{I}$. Then $\mathcal{A} \triangleright (\mathcal{I} \cap \mathcal{M}) = \mathcal{I} \cap \mathcal{M}$ and $\mathcal{I} = (\mathcal{I} \cap \mathcal{M})\rtimes\mathcal{A}$.

Proof: If $m \in \mathcal{I} \cap \mathcal{M}$ then

$$((a \triangleright m) \rtimes \mathbf{1}) = (\mathbf{1}_\mathcal{M} \rtimes a_{(1)})(m \rtimes \mathbf{1})(\mathbf{1}_\mathcal{M} \rtimes S(a_{(2)})) \in \mathcal{I} \cap \mathcal{M}$$

Since $\mathcal{I}$ is an ideal we have $(\mathcal{I} \cap \mathcal{M})\rtimes\mathcal{A} \subset \mathcal{I}$. Conversely, let $x \in \mathcal{I}$. Then by Theorem 3.2.3i)

$$x = \hat{E}_\lambda(x(\mathbf{1}_\mathcal{M} \rtimes l_{(2)}))(\mathbf{1}_\mathcal{M} \rtimes S^{-1}(l_{(1)})) \in \hat{E}_\lambda(\mathcal{I})(\mathbf{1}_\mathcal{M} \rtimes \mathcal{A}) \subset (\mathcal{I} \cap \mathcal{M})\rtimes\mathcal{A} \qquad \blacksquare$$

Corollary 3.2.6 *Let $f : \mathcal{M}\rtimes\mathcal{A} \to \mathcal{B}$ be a homomorphism of algebras and let $\lambda \in \mathcal{L}(\hat{\mathcal{A}})$ be nondegenerate. Assume there exists a linear map $F_\lambda : \mathcal{B} \to B$ such that*

$$F_\lambda \circ f = f \circ \hat{E}_\lambda \tag{3.20}$$

Then f is injective iff its restriction to $\mathcal{M}$ is injective.

Proof: Let $\mathcal{I} = \operatorname{Ker} f$ then $\hat{E}_\lambda(\mathcal{I}) \subset \mathcal{I}$ by (3.20). Hence $\mathcal{I} \equiv (\mathcal{I} \cap \mathcal{M})\rtimes\mathcal{A} = 0$ if $\operatorname{Ker} f \cap \mathcal{M} = 0$. $\blacksquare$

3.3 Plancherel-Duality

In this subsection we develop a modified duality concept for *positive* nondegenerate left integrals $l \in \mathcal{L}(\mathcal{A})$ and $\lambda \in \mathcal{L}(\hat{\mathcal{A}})$, such that for sufficiently regular $\mathcal{A}$-module algebras $\mathcal{M}$ the conditional expectations $E_l : \mathcal{M} \to \mathcal{M}^\mathcal{A}$ and $\hat{E}_\lambda : \mathcal{M}\rtimes\mathcal{A} \to \mathcal{M}$ become dual in the sense of Haagerup [Ha], see also [Di, La, St]. Simultaneously, with any positive nondegenerate and normalized left integral $l \in \mathcal{A}$ we will associate a "Jones projection" $e_l \in \mathcal{A}$ such that we have the Jones relation

$$e_l m e_l = E_l(m) e_l = e_l E_l(m), \quad \forall m \in \mathcal{M} \tag{3.21}$$

as an identity in $\mathcal{M}\rtimes\mathcal{A}$.

Note that our purely algebraic duality notion for left integrals in Proposition 2.6.4 did not touch the question of positivity. In fact, by Proposition 2.6.6i)+ii) the dual $\lambda \in \mathcal{L}(\hat{\mathcal{A}})$ of a given left integral $l \in \mathcal{L}(A)$ is positive if and only if $d_L(l) > 0$, whereas l itself is positive if and only if $d_R(l) > 0$. We also warn the reader that the positivity of l as a left integral (i.e. considered as a functional on $\hat{\mathcal{A}}$) in general does not imply l to be positive (not even selfadjoint) as an element in the C^*-algebra $\mathcal{A}$. In particular, in (3.21) we cannot choose $l = e_l$ as in ordinary finite dimensional C^*-Hopf algebras. Instead we now define, for every positive left integral $l = h d_R(l) \in \mathcal{L}(\mathcal{A})$,

$$e_l := d_R(l)^{1/2}\, h\, d_R(l)^{1/2} \tag{3.22}$$

Proposition 3.3.1 *Let $l \in \mathcal{L}(\mathcal{A})$ be a positive left integral and put $E_l : \mathcal{M} \ni m \mapsto l \triangleright m \in \mathcal{N} \equiv \mathcal{M}^{\mathcal{A}}$. For $m \in \mathcal{M}$ consider $m \equiv (m \rtimes \mathbf{1})$ and $e_l \equiv (\mathbf{1}_{\mathcal{M}} \rtimes e_l)$ as elements in $\mathcal{M} \rtimes \mathcal{A}$. Then*
i) e_l is positive, commutes with $\mathcal{M}^{\mathcal{A}}$ and satisfies (3.21).
ii) $e_l = e_{l'} \Rightarrow l = l'$
iii) $e_l^2 = n_\sigma(l) e_l$. In particular, if l is normalized then e_l is a projection.
iv) l is nondegenerate if and only if e_l is nondegenerate (as functionals on $\hat{A}$). In this case l is normalized if and only if e_l is a projection.

Proof: i) e_l is positive since $h = h^* h$ and $e_l \in (\mathbf{1}_{\mathcal{M}} \rtimes \mathcal{A})$ commutes with $\mathcal{M}^{\mathcal{A}}$ by (3.6). To prove (3.21) we use that for any left integral $l \in \mathcal{L}(\mathcal{A})$ we have

$$\begin{aligned} aml &\equiv (a_{(1)} \triangleright m) a_{(2)} l = (a_{(1)} \triangleright m)(a_{(2)} \triangleright \mathbf{1}_{\mathcal{M}}) l \\ &= (a \triangleright m) l, \ \forall a \in \mathcal{A}, m \in \mathcal{M} \end{aligned} \tag{3.23}$$

by (1.31), (1.42) and (3.2). Using that $\mathcal{A}_R \subset \mathcal{M} \rtimes \mathcal{A}$ commutes with $\mathcal{M} \subset \mathcal{M} \rtimes \mathcal{A}$ by (3.7) this gives

$$\begin{aligned} e_l m e_l &= d_R(l)^{1/2} l m h d_R(l)^{1/2} \\ &= d_R(l)^{1/2} (l \triangleright m) h d_R(l)^{1/2} \\ &= E_l(m) e_l = e_l E_l(m) \end{aligned}$$

where the last identity holds since $E_l(m) \equiv \lambda \triangleright m \in \mathcal{M}^{\mathcal{A}}$. Thus we have shown i). To prove ii) let $\rho_h \equiv S(\lambda_h) \in \mathcal{R}(\hat{\mathcal{A}})$ be the right integral dual to h, i.e. the unique solution of $h \leftharpoonup \rho_h = \mathbf{1}$. Since similarly as in Corollary 2.4.2 $\mathcal{A}_R \subset \mathcal{A}$ is the fixed point algebra under the canonical right $\hat{\mathcal{A}}$-action we conclude $e_l \leftharpoonup \rho_h = d_R(l)$. Since $l = l' \Leftrightarrow d_R(l) = d_R(l')$, this proves part (ii). To prove part (iii) we use $lh = n_\sigma(l) h$ by (2.59), (1.31) and (1.32) to conclude

$$e_l^2 = d_R(l)^{1/2} l h d_R(l)^{1/2} = n_\sigma(l) e_l.$$

Hence, by the argument in the proof of (ii), $e_l^2 = e_l$ if and only if $d_R(l) = n_\sigma(l) d_R(l)$, which in particular holds for $n_\sigma(l) = \mathbf{1}$, i.e. if l is normalized. If l is nondegenerate then $d_R(l)$ is invertible and $e_l^2 = e_l \Leftrightarrow n_\sigma(l) = \mathbf{1}$. Finally, in this case we also have

$$e_l = d_R(l)^{1/2} l d_R(l)^{-1/2}$$

and therefore

$$e_l \leftharpoonup \phi = d_R(l)^{1/2} (l \leftharpoonup \phi) d_R(l)^{-1/2}, \quad \forall \phi \in \hat{\mathcal{A}}$$

proving that e_l is nondegenerate. Converseley, if e_l is nondegenerate, we use the identity

$$e_l \leftharpoonup \phi = d_R(l)^{1/2} (h \leftharpoonup \phi) d_R(l)^{1/2}, \quad \forall \phi \in \hat{\mathcal{A}}$$

to conclude from the nondegeneracy of h

$$d_R(l)^{1/2} a d_R(l)^{1/2} = 0 \Rightarrow a = 0, \quad \forall a \in \mathcal{A}.$$

Clearly, this implies $d_R(l)$ to be invertible and therefore l to be nondegenerate. This proves part (iv). ■

Let now $l \in \mathcal{L}(\mathcal{A})$ be a positive and nondegenerate left integral. Inspired by the notion of Haagerup duality for conditional expectations we seek for a left integral $\lambda \in \mathcal{L}(\hat{\mathcal{A}})$ such that, for any $\mathcal{A}$-module algebra $\mathcal{M}$, $\hat{E}_\lambda : \mathcal{M}\rtimes\mathcal{A} \to \mathcal{M}$ satisfies

$$\hat{E}_\lambda(e_l) = \mathbf{1}_\mathcal{M}. \tag{3.24}$$

By Proposition 3.3.1iv) e_l is nondegenerate and therefore $\lambda \in \hat{\mathcal{A}}$ must be the unique element satisfying

$$\lambda \rightharpoonup e_l = \mathbf{1}_\mathcal{A} \tag{3.25}$$

Proposition 3.3.2 *Let $l \in \mathcal{L}(\mathcal{A})$ be a positive left integral and assume there exists $\lambda \in \hat{\mathcal{A}}$ satisfying (3.25). Then l is nondegenerate and λ is uniquely determined. Moreover, λ is a positive and nondegenerate left integral in $\hat{\mathcal{A}}$.*

Proof: By Lemma 2.6.1ii) and ii') we have $e_l = aha^*$ where $a = S(d_R(l)^{1/2}) \in \mathcal{A}_L$. Hence

$$\mathbf{1} = \lambda \rightharpoonup e_l = a(\lambda \rightharpoonup h)a^*$$

and therefore a is invertible. Thus, $d_R(l)$ is invertible, l is nondegenerate and λ is unique. Let now $l' := ha^*a \in \mathcal{L}(\mathcal{A})$. Then

$$\lambda \rightharpoonup l' = \lambda \rightharpoonup (a^{-1}e_l a) = a^{-1}(\lambda \rightharpoonup e_l)a = \mathbf{1}. \tag{3.26}$$

Thus, λ is the left integral dual to l' and therefore nondegenerate by Proposition 2.6.4. By Proposition 2.6.6ii) λ is also positive, since $d_L(l') = a^*a > 0$. ■

The above results motivate the following

Definition 3.3.3 For any positive and nondegenerate left integral $l \in \mathcal{L}(\mathcal{A})$ we define the *p-dual* (p ≡ "positive" or "Plancherel") $\lambda_l \in \mathcal{L}(\hat{\mathcal{A}})$ to be the unique positve and nondegenerate left integral satisfying $\lambda_l \rightharpoonup e_l = \mathbf{1}$.

We now show that Definition 3.3.3 indeed provides a sensible notion of duality, i.e. converseley l is also the p-dual of λ_l. Moreover, upon iterating our constructions we obtain a generalized (i.e. in general with algebra-valued index) Temperley-Lieb-Jones (TLJ) algebra in $(\mathcal{M}\rtimes\mathcal{A})\rtimes\hat{\mathcal{A}}$.

Theorem 3.3.4 (P-duality and TLJ-algebra)
Let $l \in \mathcal{L}(\mathcal{A})$ be a positive nondegenerate left integral and let $\lambda \in \mathcal{L}(\hat{\mathcal{A}})$ its p-dual. Then
i) l is also the p-dual of λ.
ii) The indices $Ind\, l$ and $Ind\,\lambda$ are both positive and invertible and satisfy

$$Ind\, l = n_R(\lambda) \in C(\hat{\mathcal{A}}) \cap \hat{\mathcal{A}}_R \quad , \quad Ind\,\lambda = n_R(l) \in C(\mathcal{A}) \cap \mathcal{A}_R \tag{3.27}$$

iii) For any $\mathcal{A}$-module algebra $\mathcal{M}$ the elements $e = ((\mathbf{1}_\mathcal{M} \rtimes e_l) \rtimes \hat{\mathbf{1}}) \in (\mathcal{M}\rtimes\mathcal{A})\rtimes\hat{\mathcal{A}}$ and $\hat{e} = ((\mathbf{1}_\mathcal{M} \rtimes \mathbf{1}) \rtimes e_\lambda) \in (\mathcal{M}\rtimes\mathcal{A})\rtimes\hat{\mathcal{A}}$ satisfy the TLJ-relations

$$\begin{aligned} e^2 &= e\, Ind\,\lambda && (3.28)\\ \hat{e}^2 &= \hat{e}\, Ind\, l && (3.29)\\ \hat{e}\, e\, \hat{e} &= \hat{e} && (3.30)\\ e\, \hat{e}\, e &= e && (3.31) \end{aligned}$$

Proof: By the proof of Proposition 3.3.2 λ is the p-dual of l if and only if $\lambda \rightharpoonup l' = \mathbf{1}$, where $l' = ha^*a,\ a := S(d_R(l)^{1/2})$. Using (2.79) we conclude

$$d_R(\lambda) = \varepsilon_R(b_l^* b_l)^{-1} \tag{3.32}$$

where

$$b_l = ag_L \equiv S(d_R(l)^{1/2})g_L. \tag{3.33}$$

By Lemma 3.3.5 below, $b_l = b_l^* > 0$ and therefore the p-dual λ of l is uniquely determined by

$$d_R(\lambda)^{-1/2} = \varepsilon_R(b_l) \equiv \varepsilon_R(S(d_R(l)^{1/2})g_L). \tag{3.34}$$

Using $\hat{S} \circ \varepsilon_R = \varepsilon_L \circ S^{-1}$ and $g_R = S^{\pm 1}(g_L) = \hat{\varepsilon}_R(\hat{g}_L)$ this implies by Proposition 2.2.3

$$\hat{\varepsilon}_R(\hat{S}(d_R(\lambda)^{1/2})\hat{g}_L) = \hat{\varepsilon}_R\varepsilon_L(d_R(l)^{-1/2}g_R^{-1})g_R = d_R(l)^{-1/2}.$$

Since this relation is precisely the dual version of (3.34), l is also the p-dual of λ. To prove part (ii) we note $\mathrm{Ind}\,\lambda = n_R(l')$ by (3.26) and Proposition 2.6.10iii). Hence,

$$l' \equiv ha^*a = hS(a^*)a = l d_R(l)^{-1/2} S(d_R(l)^{1/2})$$

by Lemma 2.6.1ii'). Lemma 2.4.1 now gives

$$\Delta(l') = \Delta(l)[S(d_R(l)^{1/2}) \otimes d_R(l)^{-1/2}]$$

and therefore

$$n_R(l') \equiv l'_{(2)} S^{-1}(l'_{(1)}) = l_{(2)} S^{-1}(l_{(1)}) \equiv n_R(l). \tag{3.35}$$

This proves $\mathrm{Ind}\,\lambda = n_R(l)$ and by duality also $\mathrm{Ind}\,l = n_R(\lambda)$. By Proposition 2.6.6iv) these indices are positive and invertible, thus proving part (ii). To prove part (iii) first note that (3.28) and (3.29) follow from Proposition 3.3.1iii). Next, the analogue of (3.21) gives

$$\hat{e}e\hat{e} = \hat{E}_\lambda(e)\hat{e} = \hat{e}$$

since $\hat{E}_\lambda(e) = \mathbf{1}_{\mathcal{M}}$ by definition of λ. Finally, we prove (3.31) as an identity in $\mathcal{A} \rtimes \hat{\mathcal{A}}$. First we note that for any right integral $r \in \mathcal{R}(\mathcal{A})$ we have in $\mathcal{A} \rtimes \hat{\mathcal{A}}$

$$\begin{aligned}
(r \rtimes \phi)(a \rtimes \psi) &= (ra_{(1)} \rtimes (\phi \leftharpoonup a_{(2)})\psi) \\
&= (r\hat{\varepsilon}_R(\varepsilon_L(a_{(1)})) \rtimes (\phi \leftharpoonup a_{(2)})\psi) \\
&= (r \rtimes (\mathbf{1} \leftharpoonup a_{(1)})(\phi \leftharpoonup a_{(2)})\psi) \\
&= (r \rtimes (\phi \leftharpoonup a)\psi)
\end{aligned} \tag{3.36}$$

where $a \in \mathcal{A}$ and $\phi, \psi \in \hat{\mathcal{A}}$, and where we have used that by (3.2)

$$(a \rtimes \varepsilon_L(b)\phi) = (a\hat{\varepsilon}_R(\varepsilon_L(b)) \rtimes \phi) \tag{3.37}$$

for all $a, b \in \mathcal{A}$ and $\phi \in \hat{\mathcal{A}}$. Next, viewed as elements in $\mathcal{A} \rtimes \hat{\mathcal{A}}$ we have

$$\hat{e} = (\mathbf{1} \rtimes d_R(\lambda)^{1/2} \hat{h} d_R(\lambda)^{1/2}) \tag{3.38}$$

$$e = (d_R(l)^{1/2} h d_R(l)^{1/2} \rtimes \hat{\mathbf{1}}) = (\mathbf{1} \rtimes \xi_l)(h \rtimes \mathbf{1})(\mathbf{1} \rtimes \xi_l) \tag{3.39}$$

where

$$\xi_l := \varepsilon_L(d_R(l)^{1/2}) \in \hat{\mathcal{A}}_L. \tag{3.40}$$

Since $(\mathbf{1} \rtimes \hat{\mathcal{A}}_R)$ commutes with $(\mathcal{A} \rtimes \hat{\mathbf{1}})$ by Theorem 3.1.1iv) we conclude

$$\begin{aligned} e\hat{e}e &= \left(\mathbf{1} \rtimes \xi_l d_R(\lambda)^{1/2}\right) \left(h \rtimes \xi_l \hat{h} \xi_l\right) \left(h \rtimes d_R(\lambda)^{1/2} \xi_l\right) \\ &= \left(\mathbf{1} \rtimes \xi_l d_R(\lambda)^{1/2} \hat{S}^{-1}(\xi_l)\right) \left(h \rtimes (\hat{h} \leftharpoondown h)\right) \left(h \rtimes \hat{S}(\xi_l) d_R(\lambda)^{1/2} \xi_l\right) \\ &= (\mathbf{1} \rtimes \xi_l \phi^*)\,(h \rtimes \hat{\mathbf{1}})\,(\mathbf{1} \rtimes \phi \xi_l)\ . \end{aligned} \tag{3.41}$$

Here we have used Lemma 2.6.1ii)+ii'), $\hat{S}^{\pm 1}(\xi_l) \in \hat{\mathcal{A}}_R$ and Eq.(3.36) in the second line and, using $\hat{h} \leftharpoondown h = \hat{g}_R^2$ in the third line, we have introduced $\phi \in \hat{\mathcal{A}}_R$ given by

$$\phi = \hat{g}_R \hat{S}(\xi_l) d_R(\lambda)^{1/2}\ .$$

Comparing (3.41) with (3.39) we realize that (3.31) follows provided $\phi = \hat{\mathbf{1}}$. To prove this we use $\hat{g}_R = \varepsilon_R(g_L)$, $\hat{S}(\xi_l) = \varepsilon_R(S^{-1}(d_R(l)^{1/2}))$ and the fact that g_L implements S^2 on $\mathcal{A}_L$ to compute

$$\begin{aligned} \phi &= \varepsilon_R(g_L S^{-1}(d_R(l)^{1/2})) d_R(\lambda)^{1/2} \\ &= \varepsilon_R(S(d_R(l)^{1/2}) g_L) d_R(\lambda)^{1/2} \\ &= \hat{\mathbf{1}} \end{aligned}$$

where the last equation follows from (3.34). This concludes the proof of part iii) and therefore of Theorem 3.3.4. ■

We are left to show

Lemma 3.3.5 *Let $d \in \mathcal{A}_R$ and $b = S(d) g_L \in \mathcal{A}_L$. Then $b = b^* \Leftrightarrow d = d^*$ and $b > 0 \Leftrightarrow d > 0$.*

Proof: We use that $g_L \in \mathcal{A}_L$ is positive and implements S^2 on $\mathcal{A}_L$. Hence $b^* = g_L S^{-1}(d^*) = S(d^*) g_L$ implying $b = b^* \Leftrightarrow d = d^*$. Assume now $d = c^* c$, $c \in \mathcal{A}_R$. Then $b = S(c) g_L S^{-1}(c^*) = S(c) g_L S(c)^* > 0$. Converseley, if $b = a^* a$, $a \in \mathcal{A}_L$, then $d = S^{-1}(a^* a g_L^{-1}) = S(a) g_R^{-1} S(a)^* > 0$, where we have used $g_R = S^{\pm 1}(g_L) > 0$. ■

3.4 The regular representation

We now show that if $\mathcal{M}$ is a von-Neumann (or C^*-) algebra then also $\mathcal{M} \rtimes \mathcal{A}$ naturally becomes a von-Neumann (or C^*-) algebra, respectively. As for crossed products by duals of compact groups G, which may be defined as subalgebras of $\mathcal{M} \otimes \mathcal{B}(L^2(G))$, we first provide what we call the *regular homomorphism* $\Lambda_{cros} : \mathcal{M} \rtimes \mathcal{A} \to \mathcal{M} \otimes \mathit{End}\ \mathcal{A}$.

To this end we introduce on $\mathcal{A}$ the scalar product

$$(a, b) := \langle \hat{h} \mid a^* b \rangle$$

where $\hat{h} \in \hat{\mathcal{A}}$ is the normalized two-sided Haar integral. Elements of the Hilbert space $L^2(\mathcal{A}, \hat{h})$ are denoted by $\mid a\rangle$, $a \in \mathcal{A}$. On $L^2(\mathcal{A}, \hat{h})$ we define the following operators

$$\ell(a) \mid b\rangle := \mid ab\rangle \tag{3.42}$$

$$\tau_R(\phi) \mid b\rangle := \mid \phi \rightharpoonup b\rangle \tag{3.43}$$

$$\tau_L(\phi) \mid b\rangle := \mid b \leftharpoondown \hat{S}^{-1}(\phi)\rangle \tag{3.44}$$

where $a, b \in \mathcal{A}$ and $\phi \in \hat{\mathcal{A}}$. Using $(\phi^* \rightharpoonup b)^* = \hat{S}^{-1}(\phi) \rightharpoonup b^*$, $(b \leftharpoondown \phi^*)^* = b^* \leftharpoondown \hat{S}^{-1}(\phi)$, and the fact that as a two-sided integral $\hat{h}$ satisfies (1.30) and (1.33), one immediately checks that τ_R an τ_L provide two commuting faithful *-representations of $\hat{A}$, whereas ℓ of course provides the GNS-representation of $\mathcal{A}$ associated with $\hat{h}$. Moreover, we have

Lemma 3.4.1 $\tau_L(\varepsilon_R(a)) = \ell(a), \quad \forall a \in \mathcal{A}_L.$

Proof: Let $a = \hat{\varepsilon}_L(\phi), \phi \in \hat{\mathcal{A}}$. Then by (2.23) $\varepsilon_R(a) = \hat{S}(\phi_{(1)})\phi_{(2)}$ and therefore

$$\begin{aligned}\tau_L(\varepsilon_R(a)) \mid b\rangle &= \mid b_{(2)}\rangle\langle b_{(1)} \mid S^{-1}(\phi_{(2)})\phi_{(1)}\rangle \\ &= \mid b_{(3)}\rangle\langle b_{(2)}S^{-1}(b_{(1)}) \mid \phi\rangle \\ &= \mid \varepsilon_L(\phi)b\rangle\end{aligned}$$

where the last equation follows from (1.20). ■

Theorem 3.4.2 (The regular homomorphism)
Let $\Lambda_{cros} : \mathcal{M}\rtimes\mathcal{A} \to \mathcal{M} \otimes End\, \mathcal{A}$ be given by

$$\Lambda_{cros}(m \rtimes a) := (id_{\mathcal{M}} \otimes \tau_L)(\rho(m))\,(\mathbf{1}_{\mathcal{M}} \otimes \ell(a)) \tag{3.45}$$

Then Λ_{cros} defines a (in general non-unital) injective $*$-algebra homomorphism satisfying for all $m \in \mathcal{M},\ a \in \mathcal{A}$ and $\phi \in \hat{\mathcal{A}}$

$$\Lambda_{cros}(m \rtimes (\phi \rightharpoonup a)) = (\mathbf{1}_{\mathcal{M}} \otimes \tau_R(\phi_{(1)}))\,\Lambda_{cros}(m \rtimes a)\,(\mathbf{1}_{\mathcal{M}} \otimes \tau_R(\hat{S}(\phi_{(2)}))) \tag{3.46}$$

Moreover, if $\mathcal{M}$ is a von-Neumann (or C^*-) algebra then $\Lambda_{cros}(\mathcal{M}\rtimes\mathcal{A})$ is a von-Neumann (or C^*-) subalgebra of $\mathcal{M} \otimes End\, \mathcal{A}$.

Proof: Since $\rho : \mathcal{M} \to \mathcal{M} \otimes \hat{\mathcal{A}}$ is an injective $*$-algebra map, so is the restriction of Λ_{cros} to $\mathcal{M}$. To show that Λ_{cros} is well defined on $\mathcal{M}\rtimes\mathcal{A}$ we use

$$\rho(a \triangleright \mathbf{1}_{\mathcal{M}}) = (id_{\mathcal{M}} \otimes a \rightharpoonup)(\rho(\mathbf{1}_{\mathcal{M}})) = \rho(\mathbf{1}_{\mathcal{M}})(\mathbf{1}_{\mathcal{M}} \otimes \varepsilon_R(a))$$

by (2.1) and (2.4). Hence, using Lemma 3.4.1 we get for $a \in \mathcal{A}_L$

$$\Lambda_{cros}(m(a \triangleright \mathbf{1}_{\mathcal{M}}) \rtimes b) = \Lambda_{cros}(m \rtimes ab)$$

which shows that Λ_{cros} is well defined. Clearly, the restriction $\Lambda_{cros}|(\mathbf{1}_{\mathcal{M}} \rtimes \mathcal{A})$ defines a $*$-algebra map provided the projection $P = \Lambda_{cros}(\mathbf{1}_{\mathcal{M}} \rtimes \mathbf{1})$ commutes with $\mathbf{1}_{\mathcal{M}} \otimes \ell(\mathcal{A})$. Using (1.18) one checks

$$\ell(a)\tau_L(\phi) = \tau_L(\phi_{(1)})\langle\phi_{(2)} \mid a_{(1)}\rangle\ell(a_{(2)}) \tag{3.47}$$

which together with (2.1) and (2.4) implies

$$\begin{aligned}(\mathbf{1}_{\mathcal{M}} \otimes \ell(a))P &= P(\mathbf{1}_{\mathcal{M}} \otimes \tau_L(\varepsilon_R(a_{(1)}))\ell(a_{(2)})) \\ &= P(\mathbf{1}_{\mathcal{M}} \otimes \tau_L(\varepsilon_R(a_{(1)}S(a_{(2)})))\ell(a_{(3)})) \\ &= P(\mathbf{1}_{\mathcal{M}} \otimes \ell(a))\end{aligned}$$

where in the second line we have used (2.24) and (2.28) and in the last line Lemma 3.4.1 and the identity (1.12). Hence $\Lambda_{cros}|(\mathbf{1}_{\mathcal{M}} \rtimes \mathcal{A})$ provides a $*$-algebra map. More generally, (3.47) and (2.1) also imply

$$\Lambda_{cros}(\mathbf{1}_{\mathcal{M}} \rtimes a)\Lambda_{cros}(m \rtimes \mathbf{1}) = \Lambda_{cros}((a_{(1)} \triangleright m) \rtimes a_{(2)})\ .$$

Hence, Λ_{cros} provides an algebra map, which therefore also respects the $*$-structure (3.4). In particular $\Lambda_{cros}(\mathcal{M}\rtimes\mathcal{A}) = \Lambda_{cros}(\mathcal{M})(\mathbf{1}_{\mathcal{M}} \otimes \ell(\mathcal{A}))$. Next, we use that $\tau_R(\hat{\mathcal{A}})$ commutes with $\tau_L(\hat{\mathcal{A}})$ and therefore (3.46) follows from

$$\begin{aligned}\tau_R(\phi_{(1)})\ell(a)\tau_R(S(\phi_{(2)})) &= \ell(\phi_{(1)} \rightharpoonup a)\tau_R(\phi_{(2)}S(\phi_{(3)})) \\ &= \ell(\hat{\mathbf{1}}_{(1)}\phi \rightharpoonup a)\tau_R(\hat{\mathbf{1}}_{(2)}) \\ &= \ell(\phi \rightharpoonup a)\end{aligned}$$

where in the second line we have used (1.18). To prove that Λ_{cros} is injective let now $\lambda \in \mathcal{L}(\hat{\mathcal{A}})$ be a nondegenerate left integral and define $F_\lambda : \mathcal{M} \otimes \mathrm{End}\,\mathcal{A} \to \mathcal{M} \otimes \mathrm{End}\,\mathcal{A}$ by

$$F_\lambda(X) := [\mathbf{1}_\mathcal{M} \otimes \tau_R(\lambda_{(1)})]\, X\, [\mathbf{1}_\mathcal{M} \otimes \tau_R(\hat{S}(\lambda_{(2)}))].$$

Then (3.46) implies

$$F_\lambda \circ \Lambda_{cros} = \Lambda_{cros} \circ \hat{E}_\lambda \tag{3.48}$$

and by Corollary 3.2.6 Λ_{cros} is injective. Finally, if $\mathcal{M}$ is a von-Neumann (or C^*-) algebra, then by definition $\rho : \mathcal{M} \to \mathcal{M} \otimes \mathcal{A}$ is required to be weakly or norm continuous, respectively, and therefore $\Lambda_{cros}(\mathcal{M}) \subset \mathcal{M} \otimes \mathrm{End}\,\mathcal{A}$ is weakly (or norm) closed.[7] Let now $l \in \mathcal{L}(\mathcal{A})$ be the left integral dual to λ and put

$$\sum_i u_i \otimes v_i := \ell(l_{(2)}) \otimes \ell(S^{-1}(l_{(1)})) \in \mathrm{End}\,\mathcal{A} \otimes \mathrm{End}\,\mathcal{A}.$$

Then by (3.48) and Theorem 3.2.3i) we get for all $X \in \Lambda_{cros}(\mathcal{M} \rtimes \mathcal{A})$

$$X = \sum_i X^i_\lambda (\mathbf{1}_\mathcal{M} \otimes v_i),$$

where $X^i_\lambda := F_\lambda(X(\mathbf{1}_\mathcal{M} \otimes u_i)) \in \Lambda_{cros}(\mathcal{M})$. Clearly, the coefficient maps $X \mapsto X^i_\lambda$ are weakly (or norm) continuous, and therefore $\Lambda_{cros}(\mathcal{M} \rtimes \mathcal{A})$ is weakly (or norm) closed, respectively. ■

From now on we consider $\mathcal{M} \subset \mathcal{B}(\mathcal{H})$ to be a von-Neumann algebra acting on a Hilbert space $\mathcal{H}$. Denote $\mathcal{H}_\mathcal{A} := \mathcal{H} \otimes L^2(\mathcal{A}, \hat{h})$ and $\mathcal{H}_{cros} := P\,\mathcal{H}_\mathcal{A}$, where $P := \Lambda_{cros}(\mathbf{1}_\mathcal{M} \rtimes \mathbf{1}) \in \mathcal{B}(\mathcal{H}_\mathcal{A})$. Then $\Lambda_{cros}(\mathcal{M} \rtimes \mathcal{A})\mathcal{H}_{cros} \subset \mathcal{H}_{cros}$ and we call the resulting faithful unital $*$-representation

$$\pi_{cros} : \mathcal{M} \rtimes \mathcal{A} \to \mathcal{B}(\mathcal{H}_{cros})$$

the *regular representation* of $\mathcal{M} \rtimes \mathcal{A}$. In this way we may identify $\mathcal{M} \rtimes \mathcal{A} \cong \pi_{cros}(\mathcal{M} \rtimes \mathcal{A})$ as a von-Neumann algebra on $\mathcal{H}_{cros}$. Note that in this way (3.46) implies the dual $\hat{\mathcal{A}}$-action on $\mathcal{M} \rtimes \mathcal{A}$ to be weakly continuous as well. We now show that if $\mathcal{H}$ is the GNS-Hilbert space obtained from a normal state ω on $\mathcal{M}$, then π_{cros} is the GNS-representation of $\mathcal{M} \rtimes \mathcal{A}$ obtained from the normal state $\omega_{cros} = \omega \circ \hat{E}_{\hat{h}}$.

Theorem 3.4.3 (The regular GNS-representation)
Let $\Omega \in \mathcal{H}$ be cyclic for $\mathcal{M}$ and denote ω the induced state, $\omega(m) := (\Omega, m\Omega)$, $m \in \mathcal{M}$. Also put $\Omega_\mathcal{A} := \Omega \otimes |\,\mathbf{1}_\mathcal{A}\rangle \in \mathcal{H}_\mathcal{A}$, $\Omega_{cros} := P\,\Omega_\mathcal{A} \in \mathcal{H}_{cros}$ and $\omega_{cros}(m \rtimes a) := (\Omega_{cros}, \pi_{cros}(m \rtimes a)\Omega_{cros})$ as a state on $\mathcal{M} \rtimes \mathcal{A}$. Then
i) $\Omega_{cros} \in \mathcal{H}_{cros}$ is cyclic for $\mathcal{M} \rtimes \mathcal{A}$.
ii) $\omega_{cros} = \omega \circ \hat{E}_{\hat{h}}$
iii) $\|\Omega_\mathcal{A}\|^2 = \varepsilon(\mathbf{1}_\mathcal{A})\|\Omega_{cros}\|^2$ and $\|\Omega\| = \|\Omega_{cros}\|$
iv) If Ω is separating for $\mathcal{M}$ then Ω_{cros} is separating for $\mathcal{M} \rtimes \mathcal{A}$.

Proof: To prove i) we show

$$\pi_{cros}(\mathcal{M} \rtimes \mathcal{A})\Omega_{cros} \equiv \Lambda_{cros}(\mathcal{M} \rtimes \mathcal{A})\Omega_\mathcal{A} \supset P\,(\mathcal{M} \otimes \ell(\mathcal{A}))\Omega_\mathcal{A} \tag{3.49}$$

which is clearly dense in $\mathcal{H}_{cros}$. To this end let us introduce the Dirac notation

$$|\, m \otimes a\rangle_{\omega,\hat{h}} := (m \otimes \ell(a))\Omega_\mathcal{A} \in \mathcal{H}_\mathcal{A} \tag{3.50}$$

[7] In the von-Neumann algebra case $\mathcal{M} \subset \mathcal{B}(\mathcal{H})$ we consider $\Lambda_{cros}(\mathcal{M} \rtimes \mathcal{A})$ to act on $\mathcal{H} \otimes L^2(\mathcal{A}, \hat{h})$.

Then for $m_1, m_2 \in \mathcal{M}$ and $a, b \in \mathcal{A}$ the definition (3.45) gives

$$\Lambda_{cros}(m_1 \rtimes a) \mid m_2 \otimes b\rangle_{\omega,\hat{h}} = \mid (S^{-1}(a_{(1)}b_{(1)}) \triangleright m_1)m_2 \otimes a_{(2)}b_{(2)}\rangle_{\omega,\hat{h}} \tag{3.51}$$

which yields

$$\begin{aligned}\Lambda_{cros}((a_{(1)} \triangleright m) \rtimes a_{(2)})\Omega_{\mathcal{A}} &= \mid S^{-1}(a_{(2)})a_{(1)} \triangleright m \otimes a_{(3)}\rangle_{\omega,\hat{h}} \\ &= \mid (S^{-1}(a_{(2)})a_{(1)} \triangleright \mathbf{1}_{\mathcal{M}})m \otimes a_{(3)}\rangle_{\omega,\hat{h}} \\ &= \mid (S^{-1}(a_{(1)}) \triangleright \mathbf{1}_{\mathcal{M}})m \otimes a_{(2)}\rangle_{\omega,\hat{h}} \\ &= P \mid m \otimes a\rangle_{\omega,\hat{h}}\end{aligned} \tag{3.52}$$

where in the second line we have used Lemma 2.4.3i), in the third line (1.42) and in the last line (3.51). This proves (3.49) and therefore part i). To prove part ii) we use (3.51) to get

$$\begin{aligned}(\Omega_{\mathcal{A}}, \Lambda_{cros}(m \rtimes a)\Omega_{\mathcal{A}}) &= \omega(S^{-1}(a_{(1)}) \triangleright m)\langle \hat{h} \mid a_{(2)}\rangle \\ &= \omega(S^{-1}(\hat{h} \rightharpoonup a) \triangleright m) \\ &= \omega(m((\hat{h} \rightharpoonup a) \triangleright \mathbf{1}_{\mathcal{M}})) \\ &= \omega_{cros}(m \rtimes a)\end{aligned}$$

where in the third line we have used $S^{-1}(\hat{h} \rightharpoonup a) \in S^{-1}(\mathcal{A}_L) = \mathcal{A}_R$ together with Lemma 2.4.3ii). To prove part iii) we note that by definition $\|\Omega_{\mathcal{A}}\|^2/\|\Omega\|^2 = \langle \hat{h} \mid \mathbf{1}_{\mathcal{A}}\rangle = \varepsilon(\hat{h} \rightharpoonup \mathbf{1}_{\mathcal{A}}) = \varepsilon(\mathbf{1}_{\mathcal{A}})$. The identity $\|\Omega\| = \|\Omega_{cros}\|$ follows from $\omega_{cros}(\mathbf{1}_{\mathcal{M}} \rtimes \mathbf{1}_{\mathcal{A}}) = \omega(\mathbf{1}_{\mathcal{M}})$. Finally, part (iv) follows, since the conditional expectation $\hat{E}_{\hat{h}}$ is faithful, and therefore ω_{cros} is faithful if ω is faithful. ■

Le us check the result of this construction for our example in Sect. 2.5. Using the identification (3.12) and applying the formula (3.15) for $\lambda_{Haar} \in \mathcal{L}(\hat{\mathcal{A}})$ we conclude from (2.78) that $\hat{E}_{\lambda_{Haar}} : \mathcal{M} \rtimes_\alpha G \to \mathcal{M}$ is given by the standard formula

$$\hat{E}_{\lambda_{Haar}}(m \rtimes_\alpha g) = m\delta(g) \tag{3.53}$$

and therefore $\omega_{cros}(m \rtimes_\alpha g) = \omega(m)\delta(g)$. Thus, if $\mathcal{H} = L^2(\mathcal{M}, \omega)$ then $\mathcal{H}_{cros} := L^2(\mathcal{M} \rtimes_\alpha G, \omega) \cong \mathcal{H} \otimes \mathbf{C}G$ and we obtain the standard definition [St, Pe, ES] of $\mathcal{M} \rtimes_\alpha G$ as a von-Neumann algebra acting on $\mathcal{H}_{cros}$.

4 Jones Extensions

In this section we will always assume the setting of Theorem 3.4.3, i.e. we will consider $\mathcal{M} \cong \pi_\omega(\mathcal{M})$ as a concrete von-Neumann algebra acting on $\mathcal{H}_\omega = L^2(\mathcal{M}, \omega)$, where ω is a faithful normal state on $\mathcal{M}$ and where π_ω denotes the GNS representation associated with ω. For $m \in \mathcal{M}$ we will use the notation

$$\mid m\rangle_\omega := \pi_\omega(m)\Omega_\omega \in \mathcal{H}_\omega \tag{4.1}$$

where $\Omega_\omega \in \mathcal{H}_\omega$ is the cyclic GNS-vector satisfying $(\Omega_\omega, \pi_\omega(m)\Omega_\omega) = \omega(m)$. By Theorem 3.4.3 $\mathcal{H}_{cros} := P(\mathcal{H}_\omega \otimes L^2(\mathcal{A}, \hat{h}))$ is the GNS-Hilbertspace associated with the induced faithful normal state $\omega_{cros} = \omega \circ \hat{E}_{\hat{h}}$ on $\mathcal{M} \rtimes \mathcal{A}$, where $P = \Lambda_{cros}(\mathbf{1}_{\mathcal{M}} \rtimes \mathbf{1})$ and where $\hat{h} \in \mathcal{L}(\hat{\mathcal{A}})$ is the normalized Haar integral in $\hat{\mathcal{A}}$. Accordingly, for $(m \rtimes a) \in \mathcal{M} \rtimes \mathcal{A}$ we will denote

$$\mid m\, a\rangle_{cros} := \Lambda_{cros}(m \rtimes a)\Omega_{\mathcal{A}} \equiv \mid S^{-1}(a_{(1)}) \triangleright m \otimes a_{(2)}\rangle_{\omega,\hat{h}} \in \mathcal{H}_{cros} \tag{4.2}$$

where the second identity follows from (3.51). Moreover, from now on we will always assume ω to be $\mathcal{A}$-*invariant*, by which we mean

$$\omega \circ E_h = \omega \tag{4.3}$$

Note that this can always be achieved by replacing ω by its "A-average" $\omega \circ E_h$. Also note that the state $\omega_{cros} = \omega \circ \hat{E}_{\hat{h}}$ on $\mathcal{M}\rtimes\mathcal{A}$ is $\hat{\mathcal{A}}$-invariant with respect to the dual $\hat{\mathcal{A}}$-action on $\mathcal{M}\rtimes\mathcal{A}$. Since $\mathcal{H}_{cros} = L^2(\mathcal{M}\rtimes\mathcal{A}, \omega_{cros})$, this means that by taking alternating crossed products our construction will proceed "up the ladder".

The aim of this section is to relate our theory of crossed products by weak Hopf algebras with the theory of Jones extensions. To this end we will show in Section 4.1 that for any faithful normal $\mathcal{A}$-invariant state ω on $\mathcal{M}$ the GNS-representation π_ω of $\mathcal{M}$ extends to a representation of $\mathcal{M}\rtimes\mathcal{A}$ (still denoted π_ω) such that $\mathcal{M}_1 \equiv \pi_\omega(\mathcal{M}\rtimes\mathcal{A})$ coincides with the basic Jones construction for the inclusion $\pi_\omega(\mathcal{M}^{\mathcal{A}}) \equiv \mathcal{M}_{-1} \subset \mathcal{M}_0 \equiv \pi_\omega(\mathcal{M})$. Moreover, π_ω may be identified with a subrepresenation of π_{cros} and $\mathcal{M}_1$ may be identified with the ideal $\mathcal{M}e_h\mathcal{M} \subset \mathcal{M}\rtimes\mathcal{A}$ where $e_h = (\mathbf{1}_{\mathcal{M}} \rtimes h)$ is the Jones projection associated with $E_h : \mathcal{M} \to \mathcal{N}$. We will also show that the Jones triple $\mathcal{M}_{-1} \subset \mathcal{M}_0 \subset \mathcal{M}_1$ has depth 2 and finite index. In fact, we will have $\operatorname{Ind} E_h \le \tau_\triangleright(\operatorname{Ind} h)$, where equality holds if and only if $\mathcal{M}_1$ is a faithful image of $\mathcal{M}\rtimes\mathcal{A}$.

In Section 4.2 we introduce an appropriate notion of outerness for weak Hopf algebra actions and show that an action is outer if and only if the relative commutant of $\mathcal{M}$ in $\mathcal{M}\rtimes\mathcal{A}$ is minimal, i.e. iff $\mathcal{M}' \cap \mathcal{M}\rtimes\mathcal{A} = C(\mathcal{M})(\mathbf{1}_{\mathcal{M}} \rtimes \mathcal{A}_R)$. In particular this will imply that if $\mathcal{A}$ is a pure Hopf algebra acting outerly on a factor $\mathcal{M}$ then also $\mathcal{N} \equiv \mathcal{M}^{\mathcal{A}}$ and $\mathcal{M}\rtimes\mathcal{A}$ are factors and therefore $\mathcal{M}_1 \cong \mathcal{M}e_h\mathcal{M} = \mathcal{M}\rtimes\mathcal{A}$. Moreover, in this case also $\hat{\mathcal{A}}$ is pure and acts outerly on $\mathcal{M}\rtimes\mathcal{A}$ and the iterated inclusions

$$\mathcal{N} \subset \mathcal{M} \subset \mathcal{M}\rtimes\mathcal{A} \subset (\mathcal{M}\rtimes\mathcal{A})\rtimes\hat{\mathcal{A}} \subset \dots \tag{4.4}$$

provide a Jones tower of factors.

To determine the relative commutants of this tower we investigate in Section 4.3 the actions on $\pi_\omega(\mathcal{A}_{L/R})$ of the modular group $\Delta^{it}_{\mathcal{M},\omega}$ and the modular conjugation $J_{\mathcal{M},\omega}$ associated with $(\mathcal{M}, \omega)$. This will prove that under the above setting $\mathcal{N}' \cap (\mathcal{M}\rtimes\mathcal{A}) = \mathcal{A}$, $\mathcal{N}' \cap \mathcal{M} = \mathcal{A}_L$ and $\mathcal{M}' \cap (\mathcal{M}\rtimes\mathcal{A}) = \mathcal{A}_R$.

Finally, in Section 4.4 we generalize these results to non-pure weak Hopf algebras $\mathcal{A}$ acting on non-factorial von-Neumann algebras $\mathcal{M}$ by imposing as a regularity condition $C(\mathcal{M}) = (\mathcal{A}_L \cap \mathcal{A}_R) \triangleright \mathbf{1}_{\mathcal{M}}$ in addition to standardness and outerness. Note that in view of Corollary 2.4.4i) this means that the center of $\mathcal{M}$ is required to be as small as possible. Again, this kind of regularity proceeds up the tower (4.4), which under these conditions still provides a Jones tower with lowest relative commutants given as above. Moreover, in such a setting we will have $C(\mathcal{M}_{2i}) \cong \mathcal{A}_L \cap \mathcal{A}_R$, $C(\mathcal{M}_{2i+1}) \cong \hat{\mathcal{A}}_L \cap \hat{\mathcal{A}}_R$ and $C(\mathcal{M}_i) \cap C(\mathcal{M}_{i+1}) \cong \mathcal{A}_L \cap \mathcal{A}_R \cap C(\mathcal{A}) \cong \hat{\mathcal{A}}_L \cap \hat{\mathcal{A}}_R \cap C(\hat{\mathcal{A}})$, thus producing precisely the scenario described in Section 1.1. Finally, in such a "regular scenario" the derived tower $\mathcal{N}' \cap \mathcal{M}_i,\ i \ge 0$, is given by

$$\mathcal{A}_L \subset \mathcal{A} \subset \mathcal{A}\rtimes\hat{\mathcal{A}} \subset (\mathcal{A}\rtimes\hat{\mathcal{A}})\rtimes\mathcal{A} \subset \dots\ . \tag{4.5}$$

In [NSW] we will show that conversley any Jones tower of finite index and depth 2 with finite dimensional centers appears this way, where $\mathcal{A}$ and $\hat{\mathcal{A}}$ are given by $\mathcal{A} = \mathcal{N}' \cap \mathcal{M}_1$ and $\hat{\mathcal{A}} = \mathcal{M}' \cap \mathcal{M}_2$.

To simplify our notation we will from now on write elements $(m \rtimes a) \in \mathcal{M}\rtimes\mathcal{A}$ as products of $m \equiv (m \rtimes \mathbf{1}) \in \mathcal{M}$ and $a \equiv (\mathbf{1}_{\mathcal{M}} \rtimes a) \in \mathcal{A}/I_\triangleright$, whenever there is no confusion possible. Note that for $a \in \mathcal{A}_L/Ker\mu_\triangleright$ this leads to the identification $a \equiv a \triangleright \mathbf{1}_{\mathcal{M}}$.

4.1 The basic construction

We start with pointing out that if ω is $\mathcal{A}$-invariant, then the restriction of π_{cros} to $\mathcal{M} \subset \mathcal{M}\rtimes\mathcal{A}$ contains π_ω as a subrepresentation. First we need

Lemma 4.1.1 ($\mathcal{A}$-invariant states)
The following conditions for a state ω on $\mathcal{M}$ are equivalent
i) ω is $\mathcal{A}$-invariant, i.e. $\omega = \omega \circ E_h$
ii) $\omega(a \triangleright m) = \omega((S^{-1}(a) \triangleright \mathbf{1}_M)m),\ \forall a \in \mathcal{A}, \forall m \in \mathcal{M}$
iii) $\omega(a \triangleright m) = \omega(m(S(a) \triangleright \mathbf{1}_M)),\ \forall a \in \mathcal{A}, \forall m \in \mathcal{M}$

Proof: If ω is $\mathcal{A}$-invariant then ii) and iii) follow from (1.34) and (1.35), (1.42) and (2.13) and Lemma 2.4.3. Conversely, putting $a = h$, ii) or iii) immediately imply i). $\blacksquare$

Note that in our example of the partly inner group action $(\alpha,\, u)$ the conditions of Lemma 4.1.1 just mean

$$\omega(u(h)\alpha_g(m)) = \omega(u(g^{-1}hg)m) = \omega(mu(g^{-1}hg)), \quad h \in H,\, g \in G,$$

which of course is equivalent to ω being α_g-invariant, $\forall g \in G$. Next, to show that the restriction $\pi_{cros} \mid \mathcal{M}$ contains π_ω as a subrepresentation we provide a unit vector $\Phi \in \mathcal{H}_{cros}$ implementing ω. Making the Ansatz

$$\Phi := \pi_{cros}(l)\Omega_{cros} \in \mathcal{H}_{cros} \tag{4.6}$$

for some left integral $l = hd_L,\ d_L \in \mathcal{A}_L$, we get

$$\begin{aligned}(\Phi, \pi_{cros}(m)\Phi) &= \omega_{cros}(l^* m l) = \omega_{cros}(d_L^*(h \triangleright m)l) \\ &= (\omega \circ \hat{E}_{\hat{h}})(E_h(m) d_L^* h d_L) \\ &= \omega(E_h(m) d_L^* (\hat{h} \rightharpoonup h) d_L)\end{aligned}$$

where we have used (3.23) in the second equation, Theorem 3.4.3ii) and the fact that $h \triangleright m \equiv E_h(m) \in \mathcal{M}^{\mathcal{A}}$ commutes with $d_L^* \in \mathcal{A}$ in the third equation and the definition (3.14) of $\hat{E}_{\hat{h}}$ in the last equation. If we now choose $d_L := g_L^{-1} \equiv (\hat{h} \rightharpoonup h)^{-1/2}$ we get

$$(\Phi, \pi_{cros}(m)\Phi) = \omega(E_h(m)) = \omega(m)\ . \tag{4.7}$$

Thus, with this choice $\Phi \in \mathcal{H}_{cros}$ indeed implements ω and we arrive at

Proposition 4.1.2
i) Let $l_0 := hg_L^{-1} \in \mathcal{L}(\mathcal{A})$. Then $V : \mathcal{H}_\omega \to \mathcal{H}_{cros},\ V\,|m\rangle_\omega := |m\, l_0\rangle_{cros}$, defines an isometry satisfying $\forall m \in \mathcal{M}$ and $\forall a \in \mathcal{A}$

$$V^*\,|m\,a\rangle_{cros} = |m\mu_\triangleright(ag_L)\rangle_\omega \tag{4.8}$$

$$VV^*\,|m\,a\rangle_{cros} = |m\,ag_L l_0\rangle_{cros} \tag{4.9}$$

$$\pi_\omega(m) = V^*\pi_{cros}(m)V\ . \tag{4.10}$$

ii) $V\mathcal{H}_\omega \subset \mathcal{H}_{cros}$ is invariant under $\pi_{cros}(\mathcal{M}\rtimes\mathcal{A})$ and therefore π_ω extends to a representation (still denoted by π_ω) of $\mathcal{M}\rtimes\mathcal{A}$ on $\mathcal{H}_\omega$ by putting $\pi_\omega(ma) := V^*\pi_{cros}(ma)V$, yielding

$$\pi_\omega(ma)\,|m'\rangle_\omega = |m(a \triangleright m')\rangle_\omega \tag{4.11}$$

iii) $\mathcal{M}h\mathcal{M} \subset \mathcal{M}\rtimes\mathcal{A}$ is an ideal orthogonal to $Ker\,\pi_\omega$, i.e.

$$(\mathcal{M}h\mathcal{M})(Ker\,\pi_\omega) = (Ker\,\pi_\omega)(\mathcal{M}h\mathcal{M}) = 0 \tag{4.12}$$

Proof: We have $V\pi_\omega(m)\Omega_\omega = \pi_{cros}(m)\Phi$ and since $\Phi = \pi_{cros}(l_0)\Omega_{cros}$ implements ω, V extends to an isometry intertwining π_ω and π_{cros}. Moreover, for $m_1, m_2 \in \mathcal{M}$ and $a \in \mathcal{A}$ we have

$$\begin{aligned}\langle m_1 \mid V^* \mid m_2 a\rangle_\omega &= \langle m_1 l_0 \mid m_2 a\rangle_{cros} = \omega_{cros}(l_0^* m_1^* m_2 a) \\ &= \omega_{cros}\left(l_0^*(S^{-1}(a) \triangleright (m_1^* m_2))\right) = \omega\left((g_L \triangleright \mathbf{1}_{\mathcal{M}})(S^{-1}(a) \triangleright (m_1^* m_2))\right) \\ &= \omega(m_1^* m_2 \mu \triangleright (a g_L))\end{aligned}$$

where we have used the adjoint of (3.23) and $\omega_{cros} = \omega \circ \hat{E}_{\hat{h}}$ in the second line and Lemma 4.1.1 together with $\hat{h} \rightharpoonup l_0^* = g_L^{-1}(\hat{h} \rightharpoonup h) = g_L = S^{-2}(g_L)$ in the last line. This proves equs. (4.8) and (4.9) and therefore part (i). Next, by (3.23) we have for $m, m' \in \mathcal{M}$ and $a \in \mathcal{A}$

$$\pi_{cros}(ma) \mid m' l_0\rangle_{cros} = \mid m(a \triangleright m') l_0\rangle_{cros}$$

which also proves ii). Finally, to prove iii) we note that by (3.23) $\mathcal{M}h \subset \mathcal{M} \bowtie \mathcal{A}$ is a left ideal and therefore $\mathcal{M}h\mathcal{M} \subset \mathcal{M} \bowtie \mathcal{A}$ is a two-sided ideal. Moreover, for $\sum m_i a_i \in \operatorname{Ker} \pi_\omega$ we have by (4.11) $\sum m_i(a_i \triangleright m') = 0$, $\forall m' \in \mathcal{M}$, and therefore $\sum m_i a_i m' h = 0$ by (3.23). Eq. (4.12) follows by also taking the adjoint of this. ■

Note that Proposition 4.1.2 in particular implies that any $\mathcal{A}$-invariant state ω on $\mathcal{M}$ extends to a state $\bar{\omega}$ on $\mathcal{M} \bowtie \mathcal{A}$ by

$$\bar{\omega}(ma) := (\Omega_\omega, \pi_\omega(ma)\Omega_\omega) \equiv \omega(m(a \triangleright \mathbf{1}_{\mathcal{M}})). \tag{4.13}$$

In this way (4.11) may also be viewed as the GNS-representation of $\mathcal{M} \bowtie \mathcal{A}$ associated with $\bar{\omega}$.

We are now in the position to identify the Jones extension of $\mathcal{M}^{\mathcal{A}} \subset \mathcal{M}$ with an ideal $\mathcal{M}_1 \subset \mathcal{M} \bowtie \mathcal{A}$. To this end we work in $\mathcal{B}(\mathcal{H}_\omega)$ and put

$$\mathcal{M}_{-1} := \pi_\omega(\mathcal{M}^{\mathcal{A}}) \tag{4.14}$$
$$\mathcal{M}_0 := \pi_\omega(\mathcal{M}) \tag{4.15}$$
$$\mathcal{M}_1 := \pi_\omega(\mathcal{M} \bowtie \mathcal{A}) \tag{4.16}$$

Also, for a positive normalized and nondegenerate left integral $l \in \mathcal{L}(\mathcal{A})$ let $e_l = d_R(l)^{1/2} h d_R(l)^{1/2}$ be the associated "Jones projection" (3.22) and let $\lambda_l \in \mathcal{L}(\hat{\mathcal{A}})$ denote the p-dual of l, see Definition 3.3.3.

Theorem 4.1.3 (The basic construction)
Let ω be a normal faithful $\mathcal{A}$-invariant state on $\mathcal{M}$, and let $u_i \in \mathcal{M}$ be a Pimsner Popa basis for $E_h : \mathcal{M}_0 \to \mathcal{M}_{-1}$, $h \in \mathcal{A}$ the normalized Haar integral. Then
i) The inclusion $\mathcal{M}_{-1} \subset \mathcal{M}_0 \subset \mathcal{M}_1$ provides a Jones triple of finite index and depth 2.
ii) $p := \sum_i u_i h u_i^* \in \mathcal{M} \bowtie \mathcal{A}$ is a central projection satisfying $\operatorname{Ker} \pi_\omega = (\mathbf{1} - p)(\mathcal{M} \bowtie \mathcal{A})$ and therefore $\mathcal{M}_1 \cong p(\mathcal{M} \bowtie \mathcal{A})$.
iii)$p(\mathcal{M} \bowtie \mathcal{A}) = \mathcal{M}h\mathcal{M}$ and the map $m \mapsto pm$ provides a unital inclusion $\mathcal{M} \to \mathcal{M}h\mathcal{M}$.
iv) For any positive nondegenerate normalized left integral $l \in \mathcal{L}(\mathcal{A})$ the Jones projection $\mathbf{e}_l \in \mathcal{M}_1$ associated with the conditional expectation $E_l : \mathcal{M}_0 \to \mathcal{M}_{-1}$ is given by

$$\mathbf{e}_l = \pi_\omega(e_l)$$

and the (unnormalized) conditional expectation $E_l' : \mathcal{M}_1 \to \mathcal{M}_0$ dual to E_l is given by

$$E_l'(\pi_\omega(m\,a)) = \pi_\omega(m)\,\pi_\omega(\hat{E}_{\lambda_l}(ap)) \tag{4.17}$$

v) We have $\operatorname{Ind} E_l \leq \tau_\triangleright(\operatorname{Ind} l)$ and equality holds if and only if $p = \mathbf{1}$, i.e. iff $\mathcal{M}_1 \cong \mathcal{M} \bowtie \mathcal{A}$.

Proof: i) Putting $\mathbf{e}_h = \pi_\omega(e_h) \equiv \pi_\omega(h)$ we have $\mathbf{e}_h \,|m\rangle_\omega = |E_h(m)\rangle_\omega$ and hence the basic construction associated with $E_h : \mathcal{M}_0 \to \mathcal{M}_{-1}$ is given by $\langle \mathcal{M}_0, \mathbf{e}_h\rangle$. Since $\mathbf{e}_h \in \mathcal{M}_1$ we have $\langle \mathcal{M}_0, \mathbf{e}_h\rangle \subset \mathcal{M}_1$. To prove the inverse inclusion it is enough to show $\pi_\omega(\mathcal{A}) \subset \langle \mathcal{M}_0, \mathbf{e}_h\rangle$, or equivalently $\langle \mathcal{M}_0, \mathbf{e}_h\rangle' \subset \pi_\omega(\mathcal{A})'$. To this end we use that $X \in \langle \mathcal{M}_0, \mathbf{e}_h\rangle'$ implies for all $m \in \mathcal{M}$

$$X\,|m\rangle_\omega = \pi_\omega(m) X \Omega_\omega = \pi_\omega(m\,h) X \Omega_\omega$$

since $X\Omega_\omega = X\mathbf{e}_h\Omega_\omega = \mathbf{e}_h X \Omega_\omega$. Hence, we get for all $a \in \mathcal{A}$ and $m \in \mathcal{M}$

$$\begin{aligned} X\pi_\omega(a)\,|m\rangle_\omega &= X\,|a \triangleright m\rangle_\omega = \pi_\omega((a \triangleright m)\,h) X \Omega_\omega \\ &= \pi_\omega(a)\pi_\omega(m\,h) X \Omega_\omega = \pi_\omega(a) X\,|m\rangle_\omega \end{aligned}$$

where in the third equation we have used (3.23). Thus $\langle \mathcal{M}_0, \mathbf{e}_h\rangle' \subset \pi_\omega(\mathcal{A})' \cap \mathcal{M}_0' = \mathcal{M}_1'$ and therefore $\mathcal{M}_1 \subset \langle \mathcal{M}, \mathbf{e}_h\rangle$.

Next, we prove that the triple $\mathcal{M}_{-1} \subset \mathcal{M}_0 \subset \mathcal{M}_1$ has finite index and depth-2. By definition [Oc1] the depth-2 property means that the derived tower

$$\mathcal{M}_{-1}' \cap \mathcal{M}_0 \subset \mathcal{M}_{-1}' \cap \mathcal{M}_1 \subset \mathcal{M}_{-1}' \cap \mathcal{M}_2$$

also is a Jones tower, where $\mathcal{M}_2 \supset \mathcal{M}_1$ is the basic construction for $\mathcal{M}_1 \supset \mathcal{M}_0$. Moreover, finite index holds if the "unnormalized conditional expectation" (more precisely the operator valued weight) $E_h' : \mathcal{M}_1 \to \mathcal{M}_0$ dual to E_h is bounded, i.e. $E_h'(\mathbf{1}) \equiv \operatorname{Ind} E_h < \infty$. Recall that E_h' is uniquely defined on $\mathcal{M}_0 \mathbf{e}_h \mathcal{M}_0$ by the requirement $E_h'(\mathbf{e}_h) = \mathbf{1}$ [BDH]. Now, depth 2 and finite index can be guaranteed simultaneously if E_h' is of index-finite type with a quasi-basis which can be chosen to lie in $\mathcal{M}_{-1}' \cap \mathcal{M}_1$. To show that this indeed holds let now $q \in \mathcal{M} \rtimes \mathcal{A}$ be the central projection onto $\operatorname{Ker} \pi_\omega$, i.e. $q(\mathcal{M} \rtimes \mathcal{A}) = \operatorname{Ker} \pi_\omega$. Then for any positive left integral $\lambda \in \mathcal{L}(\hat{\mathcal{A}})$ the map $\widetilde{E}_\lambda : \mathcal{M}_1 \to \mathcal{M}_0$ given for $x \in \mathcal{M} \rtimes \mathcal{A}$ by

$$\widetilde{E}_\lambda(\pi_\omega(x)) := \pi_\omega\left(\hat{E}_\lambda((\mathbf{1} - q)\,x)\right) \tag{4.18}$$

is a well defined (unnormalized) conditional expectation. Moreover, if λ is nondegenerate, then by Theorem 3.2.3 $\widetilde{E}_\lambda$ is of index-finite type with quasi basis $\sum x_i \otimes y_i = \pi_\omega(l_{(2)}) \otimes \pi_\omega(S^{-1}(l_{(1)}))$, where $l \in \mathcal{L}(\mathcal{A})$ is the left integral dual to λ (i.e. the unique solution of $l \rightharpoonup \lambda = \hat{\mathbf{1}}$). In particular, $x_i, y_i \in \pi_\omega(\mathcal{A}) \subset \mathcal{M}_{-1}' \cap \mathcal{M}_1$. Choosing now $\lambda = \lambda_h$ to be the p-dual of h we have $\lambda_h \rightharpoonup h \equiv \lambda_h \rightharpoonup e_h = \mathbf{1}$ and therefore, using $hq = 0$ by (4.12),

$$\widetilde{E}_{\lambda_h}(\mathbf{e}_h) = \pi_\omega(\hat{E}_{\lambda_h}((\mathbf{1} - q)\,h)) = \pi_\omega(\lambda_h \rightharpoonup h) = \mathbf{1}\ .$$

Thus $E_h' := \widetilde{E}_{\lambda_h}$ indeed provides the (unnormalized) conditional expectation dual to E_h, which proves the finite index and depth-2 property.

To prove ii) first note that since E_h has finite index the Pimsner-Popa basis $\{u_i\}$ is finite. By definition we have for all $m \in \mathcal{M}$

$$\sum_i u_i E_h(u_i^* m) = m = \sum_i E_h(m u_i) u_i^* \tag{4.19}$$

implying $\pi_\omega(p) = \mathbf{1}$ and therefore $\mathbf{1} - p \in \operatorname{Ker} \pi_\omega$. Hence $\mathbf{1} - p = q(\mathbf{1} - p) = q$, since $qp = 0$ by (4.12).

To prove iii) we note that by (4.19) and (3.23) p is the unit in $\mathcal{M}h\mathcal{M}$ and therefore $\mathcal{M}h\mathcal{M} = p(\mathcal{M} \rtimes \mathcal{A}) \cong \mathcal{M}_1$. Next, for $m \in \mathcal{M}$ assume $pm = 0$. Then $m = (\mathbf{1} - p)m \in \operatorname{Ker} \pi_\omega \cap \mathcal{M} = 0$ by assumption, which proves part iii).

For $l=h$ part iv) has already been shown in part i). Let now $l=hd_R(l)$ be an arbitrary positive nondegenerate and normalized left integral. By (2.81) we have $E_l(m)=E_h(z_l^*mz_l)$, $\forall m\in\mathcal{M}$, where $z_l=d_R(l)^{1/2}\triangleright\mathbf{1}_{\mathcal{M}}\equiv\mu\triangleright(S(d_R(l)^{1/2}))$ (Lemma 2.4.3ii). Hence, the Jones projection associated with E_l is given on $\mathcal{H}_\omega\equiv L^2(\mathcal{M},\omega)$ by [8]

$$\mathbf{e}_l=\pi_\omega(z_lhz_l^*)=\pi_\omega(S(d_R(l)^{1/2})hS^{-1}(d_R(l)^{1/2}))=\pi_\omega(e_l)$$

where we have used Lemma 2.6.1ii)+ii'). Next, let $\lambda_l\in\mathcal{L}(\hat{\mathcal{A}})$ be the p-dual of l and use $e_l\in\mathcal{M}h\mathcal{M}$ to get $pe_l=e_l$ and therefore $\hat{E}_{\lambda_l}(pe_l)=\lambda_l\rightharpoonup e_l=\mathbf{1}$ by Definition 3.3.3. Hence, $E_l':\mathcal{M}_1\to\mathcal{M}_0$ given by

$$E_l'(\pi_\omega(ma)):=\pi_\omega(\hat{E}_{\lambda_l}(pma))$$

satisfies $E_l'(\mathbf{e}_l)=\mathbf{1}$, and therefore E_l' provides the (unnormalized) conditional expectation dual to E_l, proving part iv). Finally, to prove part v) we use $p\le\mathbf{1}$ to conclude (identifying $\mathcal{M}\cong\pi_\omega(\mathcal{M})$)

$$\operatorname{Ind}E_l\equiv E_l'(\mathbf{1})=\hat{E}_{\lambda_l}(p)\le\hat{E}_{\lambda_l}(\mathbf{1})=\tau_\triangleright(n_R(\lambda_l))=\tau_\triangleright(\operatorname{Ind}l)\tag{4.20}$$

where we have used (3.17) and (3.27).[9] By the faithfulness of $\hat{E}_{\lambda_l}$ equality in (4.20) holds if and only if $p=\mathbf{1}$. ■

We remark that a similar version of Theorem 4.1.3 holds if in place of requiring $\mathcal{M}$ to be a von-Neumann algebra we demand the conditional expectation $E_h:\mathcal{M}\to\mathcal{M}^{\mathcal{A}}$ to be of index-finite type with quasi basis $u_i,v_i\in\mathcal{M}$. Then π_ω given by (4.11) may still be considered as a well defined algebra map $M\rtimes\mathcal{A}\to\operatorname{End}_r^{\mathcal{N}}(\mathcal{M})$ (i.e. onto the endomorphisms of $\mathcal{M}$ commuting with the right multiplication by $\mathcal{N}$), which is in fact surjective. Moreover, $p=\sum u_ihv_i\in\mathcal{M}\rtimes\mathcal{A}$ still is a central projection, which is independent of the joice of quasi-basis and the remainig parts of Theorem 4.1.3 hold similarly. We leave the details to the reader.

In the following we are mostly interested in the case where $p=\mathbf{1}$, i.e. where π_ω gives a *faithful* representation of $\mathcal{M}\rtimes\mathcal{A}$, for any $\mathcal{A}$-invariant faithful state ω on $\mathcal{M}$. If this holds we say that $\triangleright:\mathcal{A}\otimes\mathcal{M}\to\mathcal{M}$ provides a *Galois action*. We note that for ordinary Hopf algebra (co-)actions this terminology has been introduced somewhat differently by Chase and Sweedler [CS] prior to the appearance of Jones theory. In Appendix A we will translate the CS-notion of Galois (co-)actions more literally to our setting of weak Hopf algebras and show that for actions on von-Neumann algebras $\mathcal{M}$ it coincides with the above definition.

Let us now show that the action of $\mathcal{A}$ on $\mathcal{M}$ is in particular Galois if $\mathcal{M}=\mathcal{N}\rtimes\hat{\mathcal{A}}$ is itself a crossed product with canonical $\mathcal{A}$-action given in the same way as in (3.13). To see this note that in this case a quasi-basis for $E_h:\mathcal{M}\to\mathcal{N}$ is given according to Theorem 3.2.3i) by $\sum_i u_i\otimes v_i=(\mathbf{1}_{\mathcal{N}}\rtimes\lambda_{(2)})\otimes(\mathbf{1}_{\mathcal{N}}\rtimes\hat{S}^{-1}(\lambda_{(1)}))$, where $\lambda\equiv\lambda_h\in\mathcal{L}(\hat{\mathcal{A}})$ is the left integral dual to h. Hence, in $(\mathcal{N}\rtimes\hat{\mathcal{A}})\rtimes\mathcal{A}$ we have $p=\lambda_{(2)}\,h\,\hat{S}^{-1}(\lambda_{(1)})=\mathbf{1}$ due to the following

Lemma 4.1.4 *Let $l\in\mathcal{L}(\mathcal{A})$ and $\lambda\in\mathcal{L}(\hat{\mathcal{A}})$ be a dual pair (in the sense of Proposition 2.6.4) of nondegenerate left integrals. Then in $\hat{\mathcal{A}}\rtimes\mathcal{A}$ we have for all $a\in\mathcal{A}$*

$$\lambda_{(2)}(a\rightharpoonup\hat{S}^{-1}(\lambda_{(1)}))=\langle a_{(1)}\mid\hat{S}^{-1}(\lambda)\rangle a_{(2)}S(a_{(3)})\tag{4.21}$$

$$\lambda_{(2)}\,l\,\hat{S}^{-1}(\lambda_{(1)})=\mathbf{1}_{\hat{\mathcal{A}}\rtimes\mathcal{A}}\tag{4.22}$$

[8] Note that in general E_l does not define a selfadjoint projection in $\mathcal{B}(\mathcal{H}_\omega)$, unless ω is E_l-invariant. Otherwise let $\omega_l:=\omega\circ E_l$, then $U|m\rangle_\omega:=|mz_l^{-1}\rangle_{\omega_l}$ defines an isomorphism $U:L^2(\mathcal{M},\omega)\to L^2(\mathcal{M},\omega_l)$ satisfying $U\mathbf{e}_lU^{-1}|m\rangle_{\omega_l}=|E_l(m)\rangle_{\omega_l}$.

[9] Note that by (3.17) and (3.27) $\operatorname{Ind}l=\operatorname{Ind}l'$, where l' is the algebraic dual of λ_l in the sense of Proposition 2.6.4, see also (3.35).

Proof: By (2.24) the r.h.s of (4.21) is in $\mathcal{A}_L$ and by (2.26) the l.h.s. is in $\hat{\mathcal{A}}_R$. To prove that they coincide in $\hat{\mathcal{A}}\rtimes\mathcal{A}$ we only have to verify l.h.s $= \varepsilon_R$(r.h.s). To this end we compute

$$\begin{aligned}\varepsilon_R\left(\langle a_{(1)} \mid \hat{S}^{-1}(\lambda)\rangle a_{(2)} S(a_{(3)})\right) &= (a \leftharpoonup \hat{S}^{-1}(\lambda)) \rightharpoonup \hat{\mathbf{1}} = \hat{\mathbf{1}}_{(1)}\langle a \mid \hat{S}^{-1}(\lambda)\hat{\mathbf{1}}_{(2)}\rangle \\ &= \lambda_{(2)}(a \rightharpoonup \hat{S}^{-1}(\lambda_{(1)}))\end{aligned}$$

where in the first equation we have used (1.42) and in the last one (1.17). This proves (4.21). Eq. (4.22) follows since we have

$$\begin{aligned}\lambda_{(2)} l \hat{S}^{-1}(\lambda_{(1)}) &= \lambda_{(2)}(l_{(1)} \rightharpoonup \hat{S}^{-1}(\lambda_{(1)}))l_{(2)} \\ &= \langle l_{(1)} \mid S^{-1}(\lambda)\rangle l_{(2)} S(l_{(3)}) l_{(4)} \\ &= l \leftharpoonup \hat{S}^{-1}(\lambda) \\ &= \lambda_R^{-1}(\lambda) = \mathbf{1}\end{aligned}$$

where we have used (4.21) in the second line, (1.12) in the third line and (2.67) in the last line. ■

Corollary 4.1.5 *Any tower of alternating crossed product extensions of the form*

$$\mathcal{N} \subset \mathcal{N}\rtimes\hat{\mathcal{A}} \equiv \mathcal{M} \subset \mathcal{M}\rtimes\mathcal{A} \subset \mathcal{M}\rtimes\mathcal{A}\rtimes\hat{\mathcal{A}} \subset \dots$$

provides a Jones tower of depth 2.

Note that putting $\mathcal{N} = \mathcal{A}_R \cong \hat{\mathcal{A}}_L$ and $\mathcal{M} = \hat{\mathcal{A}}$ Corollary 4.1.5 in particular implies that the alternating crossed products $\hat{\mathcal{A}}_L \subset \hat{\mathcal{A}} \subset \hat{\mathcal{A}}\rtimes\mathcal{A} \subset \dots$ provide a Jones tower.

Of course, an $\mathcal{A}$-action on $\mathcal{M}$ must also be Galois if we can assure that $\mathcal{M}\rtimes\mathcal{A}$ is a factor (in which case π_ω would necessarily have to be faithful). For group actions this is well known to hold if $\mathcal{M}$ is a factor and the action is outer. In the next subsection we are going to generalize this result to weak Hopf algebra actions.

4.2 Outer actions

We are now looking for conditions guaranteeing that $\mathcal{M}\rtimes\mathcal{A}$ (and therefore $\mathcal{N} \equiv \mathcal{M}^{\mathcal{A}}$) is a factor provided $\mathcal{M}$ is a factor. For crossed products by finite groups this is well known to hold if and only if the group acts outerly. Thus we are motivated to seek a sensible notion of outerness also for weak Hopf actions such that the above appropriately generalizes to our setting (It turns out that we will also have to require $\mathcal{A}$ to be pure). We should mention that in [Ya2] an action of a finite dimensional Kac algebra $\mathcal{A}$ on a factor $\mathcal{M}$ has been called outer if $\mathcal{M}' \cap (\mathcal{M}\rtimes\mathcal{A}) = \mathbf{C}$. However, we feel that outerness should be defined as opposed to innerness, i.e. in the sense of the action not being implementable by elements of $\mathcal{M}$. The connection with the above triviality of the relative commutant $\mathcal{M}' \cap (\mathcal{M}\rtimes\mathcal{A})$ should then become a theorem rather than a definition. In this sense our approach will be closer to the notion of outerness for coactions by Hopf algebras as given in [BCM]. We also point out that in our setting we always have the lower bound $\mathcal{M}' \cap (\mathcal{M}\rtimes\mathcal{A}) \supset (\mathbf{1}_M \rtimes \mathcal{A}_R)$ by (3.7), which is precisely why weak Hopf algebras fit with reducible inclusions. Hence for us "triviality" of the relative commutant should mean equality in the above inclusion, which for a factor $\mathcal{M}$ will be precisely the result of Theorem 4.2.3 below. However, we emphasize that beyond these applications the methods and results of this subsection apply to arbitrary weak $\mathcal{A}$-module algebras $\mathcal{M}$.

We start by defining an *inner implementer* of a coaction $\hat{\rho} : \mathcal{M} \to \mathcal{M} \otimes \hat{\mathcal{A}}$ to be an element $T \in \mathcal{M} \otimes \hat{\mathcal{A}}$ satisfying for all $m \in \mathcal{M}$

$$T(m \otimes \hat{\mathbf{1}}) = \hat{\rho}(m)T \tag{4.23}$$

More generally, if $\widetilde{\mathcal{M}} \supset \mathcal{M}$ is some algebra extension then an implementer of $\hat{\rho}$ in $\widetilde{\mathcal{M}}$ is an element $T \in \widetilde{\mathcal{M}} \otimes \hat{\mathcal{A}}$ satisfying (4.23) for all $m \in \mathcal{M}$. Let $\triangleright : \mathcal{A} \otimes \mathcal{M} \to \mathcal{M}$ be the action dual to $\hat{\rho}$ and identify $\widetilde{\mathcal{M}} \otimes \hat{\mathcal{A}} = \mathrm{Hom}_{\mathbf{C}}(\mathcal{A}, \widetilde{\mathcal{M}})$. Then T is an implementer of $\hat{\rho}$ if and only if for all $a \in \mathcal{A}$ and all $m \in \mathcal{M}$

$$T(a)m = (a_{(1)} \triangleright m)T(a_{(2)}) \tag{4.24}$$

We denote $\mathcal{T}_{\triangleright}(\widetilde{M})$ the space of implementers of $\triangleright$ (or $\hat{\rho}$, equivalently) in $\widetilde{\mathcal{M}}$. Obviously, $\mathcal{T}_{\triangleright}(\widetilde{\mathcal{M}})$ becomes a $C(\mathcal{M})$-module by putting $(zT)(a) := zT(a)$, for $z \in C(M)$.

We now show that for any $\mathcal{A}$-module algebra $\mathcal{M}$ the space of left integrals in $\hat{\mathcal{A}}$ always induces a nontrivial subspace of $\mathcal{T}_{\triangleright}(\mathcal{M})$.

Lemma 4.2.1 *Let $\hat{\rho} : \mathcal{M} \to \mathcal{M} \otimes \hat{\mathcal{A}}$ be the coaction dual to $\triangleright$ and for a left integral $\lambda \in \mathcal{L}(\hat{\mathcal{A}})$ put $T_\lambda := \hat{\rho}(\mathbf{1}_{\mathcal{M}})(\mathbf{1}_{\mathcal{M}} \otimes \lambda) \in \mathcal{M} \otimes \hat{\mathcal{A}}$. Then $T_\lambda \in \mathcal{T}_{\triangleright}(\mathcal{M})$.*

Proof: For $a \in \mathcal{A}$ we have $T_\lambda(a) = (\lambda \rightharpoonup a) \triangleright \mathbf{1}_{\mathcal{M}}$. Using $(\lambda \rightharpoonup a) \in \mathcal{A}_L$ and Lemma 2.4.3i) we get for all $m \in \mathcal{M}$

$$\begin{aligned} T_\lambda(a)m &= (\lambda \rightharpoonup a) \triangleright m = [(\lambda \rightharpoonup a)_{(1)} \triangleright m][(\lambda \rightharpoonup a)_{(2)} \triangleright \mathbf{1}_{\mathcal{M}}] \\ &= [a_{(1)} \triangleright m]T_\lambda(a_{(2)}) \end{aligned}$$

where we have used $\Delta(\lambda \rightharpoonup a) = a_{(1)} \otimes (\lambda \rightharpoonup a_{(2)})$. ∎

Lemma 4.2.1 motivates to introduce

$$\mathcal{T}^0_{\triangleright}(\mathcal{M}) := \hat{\rho}(\mathbf{1}_M)(\mathbf{1}_{\mathcal{M}} \otimes \mathcal{L}(\hat{\mathcal{A}})) \tag{4.25}$$

as the space of "trivial" inner implementers of $\triangleright$. Note that for $\lambda \in \mathcal{L}(\hat{\mathcal{A}})$ we have $\lambda \rightharpoonup \mathcal{A} = \mathcal{A}_L$ and therefore $T_\lambda(\mathcal{A}) = \mathcal{M}_R$. Hence $\mathcal{T}^0_{\triangleright}(\mathcal{M}) \neq 0$ and consequently $\mathcal{T}_{\triangleright}(\mathcal{M}) \supset C(\mathcal{M})\mathcal{T}^0_{\triangleright}(\mathcal{M}) \neq 0$. Thus we are lead to the following

Definition 4.2.2 A $\mathcal{A}$-module algebra action $\triangleright : \mathcal{A} \otimes \mathcal{M} \to \mathcal{M}$ is called *outer* (equivalently, the dual coaction $\hat{\rho}$ is called outer) if the space of inner implementers is given by $\mathcal{T}_{\triangleright}(\mathcal{M}) = C(\mathcal{M})\mathcal{T}^0_{\triangleright}(\mathcal{M})$.

Let us discuss this definition in the light of our example of Sect. 2.5. In this example, by (4.24) and (2.43), an inner implementer would be a map $T : H \times G \to \mathcal{M}$ satisfying for all $m \in \mathcal{M}$

$$T(h,g)m = \frac{1}{|H|} \sum_{\tilde{h} \in H} u(h)\alpha_g(m)u(\tilde{h}^{-1})T(\tilde{h}, g)$$

Putting $m = \mathbf{1}_{\mathcal{M}}$ we conclude that T would be of the form $T(h,g) = u(h)v(g)$ for some function $v : G \to \mathcal{M}$ implementing the action α, i.e.

$$v(g)m = \alpha_g(m)v(g), \quad \forall g \in G, \forall m \in \mathcal{M}.$$

If $T = T_\lambda \in \mathcal{T}^0_\triangleright(\mathcal{M})$ for some left integral $\lambda \in \mathcal{L}(\hat{\mathcal{A}})$ of the form (2.77), then

$$T(h,g) = (\lambda \rightharpoonup (h,g)) \triangleright \mathbf{1}_M \equiv \begin{cases} 0 & , \quad g \notin H \\ \hat{c}(g^{-1})\,|H|^{-1}\,u(hg) & , \quad g \in H \end{cases}$$

where we have used (2.52). Thus, our weak Hopf action $\triangleright$ associated with (α, u) is outer if and only if the inner part of the G-action α is precisely given by H, i.e. iff $v(g)m = \alpha_g(m)v(g)$ for all $m \in \mathcal{M}$ implies $g \in H$ or $v(g) = 0$.

We now establish the general good use of our Definitions by showing that the space of inner implementers $\mathcal{T}_\triangleright(\mathcal{M})$ is always linearly isomorphic to the relative commutant $\mathcal{M}' \cap (\mathcal{M} \rtimes \mathcal{A})$. To this end let $\lambda_h \in \mathcal{L}(\hat{\mathcal{A}})$ be the left integral dual to the normalized Haar integral $h \in \mathcal{A}$ and let $\hat{E} \equiv \hat{E}_{\lambda_h} : \mathcal{M} \rtimes \mathcal{A} \to \mathcal{M}$ be the (unnormalized) conditional expection given by (3.14). For $x \in \mathcal{M}' \cap (\mathcal{M} \rtimes \mathcal{A})$ and $a \in \mathcal{A}$ we then define

$$t_x(a) := \hat{E}(ax) \tag{4.26}$$

Theorem 4.2.3 (Outer actions)
Let $\triangleright : \mathcal{A} \otimes \mathcal{M} \to \mathcal{M}$ be an $\mathcal{A}$-module algebra and for $x \in \mathcal{M}' \cap (\mathcal{M} \rtimes \mathcal{A})$ and $a \in \mathcal{A}$ let $t_x(a) \in \mathcal{M}$ be given by (4.26). Then the assignment $x \mapsto t_x$ provides a $C(\mathcal{M})$-module isomorphism $\mathcal{M}' \cap (\mathcal{M} \rtimes \mathcal{A}) \to \mathcal{T}_\triangleright(\mathcal{M})$ satisfying

$$t_{(\mathbf{1}_\mathcal{M} \rtimes \mathcal{A}_R)} = \mathcal{T}^0_\triangleright(\mathcal{M}) \tag{4.27}$$

In particular, the action $\triangleright$ is outer if and only if $\mathcal{M}' \cap (\mathcal{M} \rtimes \mathcal{A}) = (C(\mathcal{M}) \rtimes \mathcal{A}_R)$.

Proof: For $z \in C(\mathcal{M})$ we clearly have $t_{zx}(a) = zt_x(a)$. To show that $t_x \in \mathcal{T}_\triangleright(\mathcal{M})$ let $m \in \mathcal{M}$. Then

$$\begin{aligned} t_x(a)m &= \hat{E}(amx) \\ &= \hat{E}((a_{(1)} \triangleright m)a_{(2)}x) \\ &= (a_{(1)} \triangleright m)t_x(a_{(2)}) \end{aligned}$$

proving $t_x \in \mathcal{T}_\triangleright(\mathcal{M})$. Also, if $x = (\mathbf{1}_\mathcal{M} \rtimes b)$ for some $b \in \mathcal{A}_R$ then

$$\begin{aligned} t_{(\mathbf{1}_\mathcal{M} \rtimes b)}(a) &= (\mathbf{1}_\mathcal{M} \rtimes (\lambda_h \rightharpoonup (ab))) \\ &= (((\lambda' \rightharpoonup a) \triangleright \mathbf{1}_\mathcal{M}) \rtimes \mathbf{1}) \end{aligned} \tag{4.28}$$

where $\lambda' = \lambda_h \varepsilon_R(b)$ and where we have used $(\mathbf{1}_\mathcal{M} \rtimes \mathcal{A}_L) = ((A_L \triangleright \mathbf{1}_\mathcal{M}) \rtimes \mathbf{1})$ in $\mathcal{M} \rtimes \mathcal{A}$. Hence $t_{(\mathbf{1}_\mathcal{M} \rtimes \mathcal{A}_R)} \subset \mathcal{T}^0_\triangleright(\mathcal{M})$. To prove that $\mathcal{M}' \cap (\mathcal{M} \rtimes \mathcal{A}) \ni x \mapsto t_x \in \mathcal{T}_\triangleright(\mathcal{M})$ is bijective pick $T \in \mathcal{T}_\triangleright(\mathcal{M})$ and define for $a \in \mathcal{A}$

$$T'(a) := S(a_{(1)})T(a_{(2)}) \in \mathcal{M} \rtimes \mathcal{A} \tag{4.29}$$

Then $T'(a) \in \mathcal{M}' \cap \mathcal{M} \rtimes \mathcal{A}$ for all $a \in \mathcal{A}$ since for $m \in \mathcal{M}$ we have

$$\begin{aligned} T'(a)m &= S(a_{(1)})(a_{(2)} \triangleright m)T(a_{(3)}) \\ &= S(a_{(1)})a_{(2)}mS(a_{(3)})T(a_{(4)}) \\ &= mS(a_{(1)})a_{(2)}S(a_{(3)})T(a_{(4)}) \\ &= mT'(a) \end{aligned}$$

Here we have used (3.5) in the second line, the fact that $S(a_{(1)})a_{(2)} \in \mathcal{A}_R$ commutes with $\mathcal{M}$ in the third line and Axiom IIIc of Definition 1.2.1 in the last line. In particular, if $T = T_\lambda \in \mathcal{T}^0_\triangleright(\mathcal{M})$ then

$$\begin{aligned} T'_\lambda(a) &= (\mathbf{1}_\mathcal{M} \rtimes S(a_{(1)})(\lambda \rightharpoonup a_{(2)})) \\ &= (\mathbf{1}_\mathcal{M} \rtimes S(b_{(1)})b_{(2)}) \in (\mathbf{1}_\mathcal{M} \rtimes \mathcal{A}_R) \end{aligned} \tag{4.30}$$

where $b = \lambda \rightharpoonup a$ and where we have again identified $((\mathcal{A}_L \triangleright \mathbf{1}_\mathcal{M}) \rtimes \mathbf{1}) = (\mathbf{1}_\mathcal{M} \rtimes \mathcal{A}_L)$ in $\mathcal{M} \rtimes \mathcal{A}$. We now claim that for the normalized Haar integral $h \in \mathcal{A}$ we get

$$t'_x(h) = x, \quad \forall x \in \mathcal{M}' \cap (\mathcal{M} \rtimes \mathcal{A}) \tag{4.31}$$

$$t_{T'(h)}(a) = T(a), \quad \forall a \in \mathcal{A}, \forall T \in \mathcal{T}_\triangleright(\mathcal{M}) \tag{4.32}$$

which would prove that the inverse of the map $x \mapsto t_x$ is given by $T \mapsto T'(h)$ and therefore also $t_{(\mathbf{1}_\mathcal{M} \rtimes \mathcal{A}_R)} = \mathcal{T}^0_\triangleright(\mathcal{M})$ by (4.30) and (4.28). To prove (4.31) we use $h = S^{-1}(h)$ and therefore

$$\begin{aligned} t'_x(h) &= h_{(2)} t_x(S^{-1}(h_{(1)})) \\ &= h_{(2)} \hat{E}(S^{-1}(h_{(1)})x) \\ &= x \end{aligned}$$

where we have used that $h_{(2)} \otimes S^{-1}(h_{(1)})$ provides a quasi-basis for $\hat{E} \equiv \hat{E}_{\lambda_h}$ by Theorem 3.2.3i). To prove (4.32) we compute for $T \in \mathcal{T}_\triangleright(\mathcal{M})$ and $a \in \mathcal{A}$

$$\begin{aligned} t_{T'(h)}(a) &= \hat{E}(ah_{(2)}T(S^{-1}(h_{(1)}))) \\ &= (\mathbf{1}_\mathcal{M} \rtimes \lambda_h \rightharpoonup h_{(2)})(T(S^{-1}(h_{(1)})a) \rtimes \mathbf{1}) \\ &= (\mathbf{1}_\mathcal{M} \rtimes \mathbf{1}_{(2)})(T(S^{-1}(\mathbf{1}_{(1)})a) \rtimes \mathbf{1}) \\ &= (\mathbf{1}_\mathcal{M} \rtimes a_{(1)}S(a_{(2)}))(T(a_{(3)}) \rtimes \mathbf{1}) \\ &= ((a_{(1)} \triangleright \mathbf{1}_\mathcal{M})T(a_{(2)}) \rtimes \mathbf{1}) = (T(a) \rtimes \mathbf{1}) \end{aligned}$$

Here we have used again $h = S^{-1}(h)$ in the first line, the definition (3.14) of $\hat{E}$ and the identity $h_{(1)} \otimes ah_{(2)} = S(a_{(1)})h_{(1)} \otimes h_{(2)}$ (1.30) in the second line, the identity $\Delta(\mathbf{1}) = \Delta(\lambda_h \rightharpoonup h) = h_{(1)} \otimes \lambda_h \rightharpoonup h_{(2)}$ in the third line, Eq. (1.19) in the fourth line and Eq. (1.42) in the fifth line. Hence we have proven (4.31) and (4.32) and therefore Theorem 4.2.3. ■

Corollary 4.2.4 *Let $\mathcal{M}$ have trivial center and let $\mathcal{A}$ act outerly on $\mathcal{M}$. Then*

$$C(\mathcal{M} \rtimes \mathcal{A}) = \mathbf{1}_\mathcal{M} \rtimes (C(\mathcal{A}) \cap \mathcal{A}_R).$$

In particular, if $\mathcal{A}$ is pure then $\mathcal{M} \rtimes \mathcal{A}$ has trivial center, and if $\mathcal{A}$ acts standardly then $\mathcal{A}$ is pure if and only if $C(\mathcal{M} \rtimes \mathcal{A}) = \mathbf{C}$.

Proof: $C(\mathcal{M}) = \mathbf{C}$ and outerness imply $C(\mathcal{M} \rtimes \mathcal{A}) \equiv \mathcal{M}' \cap \mathcal{A}' \cap (\mathcal{M} \rtimes \mathcal{A}) = \mathbf{1}_\mathcal{M} \rtimes (C(\mathcal{A}) \cap \mathcal{A}_R)$ and by Proposition 2.4.6iii) $\mathcal{A}$ is pure if and only if $C(\mathcal{A}) \cap \mathcal{A}_R = \mathbf{C}$. Corollary 4.2.4 follows, since for standard actions $\mathbf{1}_\mathcal{M} \rtimes \mathcal{A} \cong \mathcal{A}$. ■

Applying these results to our example of a partly inner group action (α, u) on a factor $\mathcal{M}$ we recover the identity (1.11). Indeed, by the remark following Definition 4.2.2, in this case our outerness condition is equivalent to $H \subset G$ being the kernel of $\mathrm{pr}_{\mathrm{Out}\,\mathcal{M}} \circ \alpha : G \to \mathrm{Out}\,\mathcal{M}$ and therefore Theorem 4.2.3 gives $\mathcal{M}' \cap \mathcal{M} \rtimes_\alpha G = \mathbf{1}_\mathcal{M} \rtimes \mathcal{A}_R \equiv \mathrm{span}\,\{u(h)h^{-1} \mid h \in H\}$ by (2.49). Moreover, from Corollary 4.2.4 and (2.53) we conclude

$$C(\mathcal{M} \rtimes_\alpha G) = \mathrm{span}_\chi \{ \sum_{h \in H} \chi(h)\, u(h) h^{-1} \}, \tag{4.33}$$

where χ runs through the Ad_G-invariant characters of H. In particular, we recover the well known result that $\mathcal{M} \rtimes_\alpha G$ is a factor, if and only if H is trivial, i.e. iff the G-action α on $\mathcal{M}$ is outer in the conventional sense. For pure weak Hopf algebras $\mathcal{A}$ this statement generalizes as follows.

Theorem 4.2.5 *An outer action of a pure weak Hopf algebra $\mathcal{A}$ on a factor $\mathcal{M}$ is Galois, i.e.*

$$\mathcal{M}^\mathcal{A} \equiv \mathcal{N} \subset \mathcal{M} \subset \mathcal{M} \rtimes \mathcal{A} \subset (\mathcal{M} \rtimes \mathcal{A}) \rtimes \hat{\mathcal{A}} \subset \ldots \tag{4.34}$$

provides a Jones tower of factors. Moreover, under these conditions the dual weak Hopf algebra $\hat{\mathcal{A}}$ is also pure and its canonical action on $\mathcal{M} \rtimes \mathcal{A}$ is again outer.

Proof: By Corollary 4.2.4, if $\mathcal{A}$ is pure then $\mathcal{M} \rtimes \mathcal{A}$ is a factor. Hence, by Theorem 4.1.3 $\mathcal{M}^\mathcal{A} \subset \mathcal{M} \subset \mathcal{M} \rtimes \mathcal{A}$ provides a Jones triple and therefore $\mathcal{N} \equiv \mathcal{M}^\mathcal{A}$ must also be a factor. By Corollary 4.1.5 the sequence (4.34) provides a Jones tower of factors. By Proposition 2.2.4 $\mathcal{A}$ acts standardly and by Corollary 2.4.4i) and Proposition 2.4.6ii) also $\hat{\mathcal{A}}$ is pure. The fact that $\hat{\mathcal{A}}$ acts again outerly on $\mathcal{M} \rtimes \mathcal{A}$ will be proven as part of Theorem 4.4.2ii) in Section 4.4. ■

4.3 Minimal actions

In Theorem 4.2.3 we have seen that for outer actions the relative commutant of $\mathcal{M}$ in $\mathcal{M} \rtimes \mathcal{A}$ is minimal, i.e. $\mathcal{M}' \cap (\mathcal{M} \rtimes \mathcal{A}) = (C(\mathcal{M}) \rtimes \mathcal{A}_R)$. We now look at the lower relative commutant $\mathcal{N}' \cap \mathcal{M}$, $\mathcal{N} \equiv \mathcal{M}^\mathcal{A}$, and recall that it is always bigger then $C(\mathcal{M})\mathcal{M}_R$ by (2.35). Similar as for group actions we say that the action of $\mathcal{A}$ on $\mathcal{M}$ is *minimal* if $\mathcal{N}' \cap \mathcal{M}$ is as small as possible.

Definition 4.3.1 An $\mathcal{A}$-module algebra action on $\mathcal{M}$ is called *minimal* if $\mathcal{N}' \cap \mathcal{M} = C(\mathcal{M})\mathcal{M}_R$

As an immediate consequence of this Definition we recall from Proposition 3.2.4

Corollary 4.3.2 *$\mathcal{A}$ acts minimally on $\mathcal{M}$ if and only if $\mathcal{A} \triangleright (C(\mathcal{M})\mathcal{M}_R) \subset C(\mathcal{M})\mathcal{M}_R$ and $\mathcal{N}' \cap (\mathcal{M} \rtimes \mathcal{A}) = C(\mathcal{M})\mathcal{M}_R \rtimes \mathcal{A} \equiv C(\mathcal{M}) \rtimes \mathcal{A}$.*

Proof: This follows from Proposition 3.2.4 and the identity $(\mathcal{M} \rtimes \mathbf{1}_\mathcal{A}) \cap (U \rtimes \mathcal{A}) = (U\mathcal{M}_R) \rtimes \mathbf{1}_\mathcal{A}$ for any linear subspace $U \subset \mathcal{M}$. ■

We are now aiming to prove that a Galois action is minimal if and only if it is outer. In particular, under the setting of Theorem 4.2.5 this will imply $\mathcal{N}' \cap \mathcal{M} = \mathcal{A}_L$, $\mathcal{M}' \cap (\mathcal{M} \rtimes \mathcal{A}) = \mathcal{A}_R$ and $\mathcal{N}' \cap (\mathcal{M} \rtimes \mathcal{A}) = \mathcal{A}$. To this end we recall from Jones theory that if $J_\mathcal{M}$ denotes the modular conjugation associated with the GNS-representation π_ω of $\mathcal{M}$ on $\mathcal{H}_\omega \equiv L^2(\mathcal{M}, \omega)$ then the basic construction for $\mathcal{N} \subset \mathcal{M}$ is also given by

$$\mathcal{M}_1 := J_\mathcal{M} \mathcal{N} J_\mathcal{M}$$

implying

$$\mathcal{N}' \cap \mathcal{M} = J_\mathcal{M}(\mathcal{M}' \cap \mathcal{M}_1) J_\mathcal{M}$$

where we have dropped the symbol π_ω. Thus, if $\mathcal{M}_1 \cong \mathcal{M} \rtimes \mathcal{A}$ and $\mathcal{A}$ acts outerly we can determine $\mathcal{N}' \cap \mathcal{M}$ provided we know how the modular conjugation J_M acts on $\pi_\omega(\mathcal{A}_R)$. To this end we introduce on $\mathcal{A}$ the antilinear involution

$$a \mapsto \bar{a} := g^{1/2} a_* g^{-1/2} \equiv g^{1/2} S(a)^* g^{-1/2}. \tag{4.35}$$

where $a_* := S(a)^*$ and $g = g_L g_R^{-1} \in \mathcal{A}$ is the positive group-like element implementing the square of the antipode, see Theorem 2.6.5. Note that $a_{**} = a$, $(\mathcal{A}_L)_* = \mathcal{A}_R$ and $g_* = S^{\pm 1}(g) = g^{-1}$, implying indeed $\bar{\bar{a}} = a$ as well as the identities

$$\begin{array}{ccc} \overline{ab} = \bar{a}\bar{b} & , & (\bar{a})^* = \overline{a^*} \\ a \in \mathcal{A}_L & \Leftrightarrow & \bar{a} \in \mathcal{A}_R \, . \end{array}$$

Proposition 4.3.3 *Under the setting of Theorem 4.1.3 let $J_\mathcal{M} \equiv J_{\mathcal{M},\omega}$ and $\Delta_\mathcal{M} \equiv \Delta_{\mathcal{M},\omega}$ denote the modular conjugation and the modular operator, respectively, associated with $(\mathcal{M}, \omega)$. Then for all $a \in \mathcal{A}_L \mathcal{A}_R$*

$$\Delta_\mathcal{M}^{it} \pi_\omega(a) \Delta_\mathcal{M}^{-it} = \pi_\omega(g^{it} a g^{-it}), \quad \forall t \in \mathbf{R} \tag{4.36}$$

$$J_\mathcal{M} \pi_\omega(a) J_\mathcal{M} = \pi_\omega(\bar{a}) \tag{4.37}$$

Proof: Let us first consider $a \in \mathcal{A}_L$, in which case it suffices to prove (4.36) with g replaced by g_L. Lemma 4.1.1 implies

$$\omega((a \triangleright \mathbf{1}_\mathcal{M}) m) = \omega(m (S^2(a) \triangleright \mathbf{1}_\mathcal{M})), \quad \forall a \in \mathcal{A}, \ m \in \mathcal{M} \tag{4.38}$$

and for $a \in \mathcal{A}_L$ Lemma 2.4.3i) gives

$$\pi_\omega(a) = \pi_\omega(a \triangleright \mathbf{1}_\mathcal{M}), \quad \forall a \in \mathcal{A}_L. \tag{4.39}$$

Putting $a = g_L = S^2(g_L)$ (4.38) and (4.39) imply by the Pedersen-Takesaki Theorem [PeTa, St]

$$\Delta_\mathcal{M}^{it} \pi_\omega(g_L) \Delta_\mathcal{M}^{-it} = \pi_\omega(g_L), \quad \forall t \in \mathbf{R}. \tag{4.40}$$

More generally, define a new faithful normal state ω' on $\mathcal{M}$ by

$$\omega'(m) := \omega((g_L^{-1/2} \triangleright \mathbf{1}_\mathcal{M}) m (g_L^{-1/2} \triangleright \mathbf{1}_\mathcal{M})) \equiv \omega((g_L^{-1} \triangleright \mathbf{1}_\mathcal{M}) m).$$

Since g_L implements S^2 on $\mathcal{A}_L$, Eq.(4.38) implies

$$\omega'((a \triangleright \mathbf{1}_\mathcal{M}) m) = \omega'(m (a \triangleright \mathbf{1}_\mathcal{M})) \quad \forall a \in \mathcal{A}_L, \ m \in \mathcal{M}.$$

Again by [PeTa] we conclude that $\pi_\omega(\mathcal{A}_L)$ is invariant under the modular group of ω', which due to (4.40) is implemented by

$$\Delta_{\mathcal{M},\omega'}^{it} = \Delta_{\mathcal{M},\omega}^{it} \pi_\omega(g_L^{-it}).$$

This proves (4.36) for all $a \in \mathcal{A}_L$. To prove (4.37) for $a \in \mathcal{A}_L$ let $S_\mathcal{M} = J_\mathcal{M} \Delta_\mathcal{M}^{1/2} = \Delta_\mathcal{M}^{-1/2} J_\mathcal{M}$ be the Tomita operator for $\mathcal{M}$ on $L^2(\mathcal{M}, \omega)$. Then for all $m \in \mathcal{M}$ and $a \in \mathcal{A}$ we get from (1.41)

$$S_\mathcal{M} \pi_\omega(a_*) S_\mathcal{M} |m\rangle = |(a_* \triangleright m^*)^*\rangle = \pi_\omega(a) |m\rangle.$$

Using $(\bar{a})_* = g^{-1/2} a g^{1/2}$ we conclude specifically for $a \in \mathcal{A}_L$

$$\begin{aligned}\pi_\omega(\bar{a}) \mid m\rangle &= S_{\mathcal{M}} \pi_\omega(g^{-\frac{1}{2}} a g^{\frac{1}{2}}) S_{\mathcal{M}} \mid m\rangle \\ &= J_{\mathcal{M}} \Delta_{\mathcal{M}}^{\frac{1}{2}} \pi_\omega(g^{-\frac{1}{2}} a g^{\frac{1}{2}}) \Delta_{\mathcal{M}}^{-\frac{1}{2}} J_{\mathcal{M}} \mid m\rangle \\ &= J_{\mathcal{M}} \pi_\omega(a) J_{\mathcal{M}} \mid m\rangle\end{aligned}$$

for all $m \in \mathcal{M}$, where the last equation follows from (4.36) by analytic continuation. This proves (4.37) for $a \in \mathcal{A}_L$, and since $J_{\mathcal{M}}^2 = \mathrm{id}$ also for $a \in \mathcal{A}_R$, whence for all $a \in \mathcal{A}_L\mathcal{A}_R$. Finally, let $a \in \mathcal{A}_R$, then $\bar{a} \in \mathcal{A}_L$ and using $g \in \mathcal{A}_L\mathcal{A}_R$ and $\bar{g} = g^{-1}$

$$\begin{aligned}\Delta_{\mathcal{M}}^{it} \pi_\omega(a) \Delta_{\mathcal{M}}^{-it} &= \Delta_{\mathcal{M}}^{it} J_{\mathcal{M}} \pi_\omega(\bar{a}) J_{\mathcal{M}} \Delta_{\mathcal{M}}^{-it} \\ &= J_{\mathcal{M}} \Delta_{\mathcal{M}}^{it} \pi_\omega(\bar{a}) \Delta_{\mathcal{M}}^{-it} J_{\mathcal{M}} \\ &= J_{\mathcal{M}} \pi_\omega(g^{it} \bar{a} g^{-it}) J_{\mathcal{M}} \\ &= \pi_\omega(g^{it} a g^{-it}).\end{aligned}$$

This proves (4.36) also for $a \in \mathcal{A}_R$ and therefore for all $a \in \mathcal{A}_L\mathcal{A}_R$. ■

We remark without proof that if $\mathcal{M} = \mathcal{N} \rtimes \hat{\mathcal{A}}$ with canonical $\mathcal{A}$-action, then Proposition 4.3.3 holds for all $a \in \mathcal{A}$.

Corollary 4.3.4 *An outer action of a weak Hopf algebra $\mathcal{A}$ on a von-Neumann algebra $\mathcal{M}$ is minimal. A Galois action on $\mathcal{M}$ is minimal if and only if it is outer.*

Proof: If $\mathcal{A}$ acts outerly, then using the notation (4.14) - (4.16) we have $\mathcal{M}_0' \cap \mathcal{M}_1 = \pi_\omega(C(\mathcal{M})\mathcal{A}_R)$ by Theorem 4.2.3 and Proposition 4.3.3 gives

$$\begin{aligned}\pi_\omega(\mathcal{N}' \cap \mathcal{M}) &\equiv J_{\mathcal{M}}(\mathcal{M}_0' \cap \mathcal{M}_1) J_{\mathcal{M}} = J_{\mathcal{M}}\, \pi_\omega(C(\mathcal{M})\mathcal{A}_R)\, J_{\mathcal{M}} \\ &= \pi_\omega(C(\mathcal{M})\mathcal{A}_L) = \pi_\omega(C(\mathcal{M})\mathcal{M}_R)\end{aligned}$$

implying $\mathcal{N}' \cap \mathcal{M} = C(\mathcal{M})\mathcal{M}_R$. Converseley, if $\mathcal{A}$ acts minimally we get by the same arguments $\pi_\omega(\mathcal{M}' \cap \mathcal{M} \rtimes \mathcal{A}) = \pi_\omega(C(\mathcal{M})\mathcal{A}_R)$ implying outerness provided the action is also Galois (i.e. provided $\pi_\omega : \mathcal{M} \rtimes \mathcal{A} \to \mathcal{M}_1$ is faithful). ■

Note that for the partly inner group action (α, u) on a factor $\mathcal{M}$ Corollary 4.3.4 proves the assertion (1.10), i.e. $\mathcal{N}' \cap \mathcal{M} = \mathrm{span}\,\{u(h) \mid h \in H\}$, and together with Corollary 4.3.2 also the assertion (1.9), i.e. $\mathcal{N}' \cap (\mathcal{M} \rtimes G) = \mathrm{span}\,\{u(h)g \mid h \in H,\ g \in G\}$, where $\mathcal{N} \equiv \mathcal{M}^G$. For general weak Hopf algebras $\mathcal{A}$ acting outerly on a factor $\mathcal{M}$ we conclude

Corollary 4.3.5 *Let $\mathcal{A}$ be a weak Hopf algebra acting outerly on a factor $\mathcal{M}$ and let $\mathcal{N} \equiv \mathcal{M}^{\mathcal{A}}$. Then*

$$\begin{aligned}\mathcal{N}' \cap \mathcal{M} = \mathcal{M}_R &\equiv \mathbf{1}_{\mathcal{M}} \rtimes \mathcal{A}_L &(4.41)\\ M' \cap \mathcal{M} \rtimes \mathcal{A} &= \mathbf{1}_{\mathcal{M}} \rtimes \mathcal{A}_R &(4.42)\\ \mathcal{N}' \cap \mathcal{M} \rtimes \mathcal{A} &= \mathbf{1}_{\mathcal{M}} \rtimes \mathcal{A} &(4.43)\end{aligned}$$

Proof: Eq. (4.41) follows from minimality, (4.42) follows from Theorem 4.2.3 and (4.43) follows from Corollary 4.3.2. ■

4.4 Regular actions

We are finally going to generalize the results of Theorem 4.2.5 and Corollary 4.3.5 to weak Hopf actions on non-factorial von-Neumann algebras $\mathcal{M}$, provided the center of $\mathcal{M}$ fits "regularly" with the $\mathcal{A}$-action as follows.

Definition 4.4.1 An $\mathcal{A}$-module algebra $\mathcal{M}$ is called *regular* if $\mathcal{A}$ acts standardly and outerly on $\mathcal{M}$ and if the center of $\mathcal{M}$ is given by

$$C(\mathcal{M}) = (\mathcal{A}_L \cap \mathcal{A}_R) \triangleright \mathbf{1}_{\mathcal{M}}. \tag{4.44}$$

We recall that the action $\triangleright$ is called standard if $\mu_\triangleright : \mathcal{A}_L \ni a \mapsto (a \triangleright \mathbf{1}_{\mathcal{M}}) \in \mathcal{M}_R$ is an isomorphism. Also note that by Corollary 2.4.4i) we always have $(\mathcal{A}_L \cap \mathcal{A}_R) \triangleright \mathbf{1}_{\mathcal{M}} \subset C(\mathcal{M})$ and therefore for regular actions the center of $\mathcal{M}$ is "as small as possible". In particular, under the conditions of Theorem 4.2.5, i.e. if $\mathcal{A}$ is pure and acts outerly on a factor $\mathcal{M}$, then it acts regularly (since for pure $\mathcal{A}$ standardness follows from Proposition 2.2.4).

The following Theorem substantiates the full scenario described in our introductory motivation in Section 1.1.

Theorem 4.4.2 (Jones towers by regular crossed products)
Under the conditions of Theorem 4.1.3 let $\mathcal{A}$ act regularly on $\mathcal{M}$ and put $\mathcal{N} \equiv \mathcal{M}^{\mathcal{A}}$. Then
i) The $\mathcal{A}$-action is Galois, i.e. $\mathcal{N} \subset \mathcal{M} \subset \mathcal{M} \rtimes \mathcal{A}$ is a Jones triple.
ii) The dual action of $\hat{\mathcal{A}}$ on $\mathcal{M} \rtimes \mathcal{A}$ is also regular.
iii) The relative commutants satisfy

$$\begin{aligned} \mathcal{N}' \cap \mathcal{M} = \mathcal{M}_R &\equiv \mathcal{A}_L & (4.45)\\ \mathcal{N}' \cap (\mathcal{M} \rtimes \mathcal{A}) &= \mathcal{A} & (4.46)\\ \mathcal{M}' \cap (\mathcal{M} \rtimes \mathcal{A}) &= \mathcal{A}_R & (4.47) \end{aligned}$$

iv) The centers of $\mathcal{N}, \mathcal{M}$ and $\mathcal{M} \rtimes \mathcal{A}$ satisfy in $\mathcal{M} \rtimes \mathcal{A}$

$$\begin{aligned} C(\mathcal{N}) &= \mathcal{A}_L \cap C(\mathcal{A}) & (4.48)\\ C(\mathcal{M}) &= \mathcal{A}_L \cap \mathcal{A}_R & (4.49)\\ C(\mathcal{M} \rtimes \mathcal{A}) &= \mathcal{A}_R \cap C(\mathcal{A}) & (4.50)\\ C(\mathcal{N}) \cap C(\mathcal{M}) &= C(\mathcal{M}) \cap C(\mathcal{M} \rtimes \mathcal{A}) = \mathcal{A}_L \cap \mathcal{A}_R \cap C(\mathcal{A}) & (4.51) \end{aligned}$$

In particular, $\mathcal{M}$ is a factor iff $\hat{\mathcal{A}}$ is pure and $\mathcal{N}$ (equivalently $\mathcal{M} \rtimes \mathcal{A}$) are factors iff $\mathcal{A}$ is pure.
v) The derived tower $\mathcal{N}' \cap \mathcal{M}_i,\ i = 0, 1, 2, \ldots$ is given by

$$\mathcal{A}_L \subset \mathcal{A} \subset \mathcal{A} \rtimes \hat{\mathcal{A}} \subset (\mathcal{A} \rtimes \hat{\mathcal{A}}) \rtimes \mathcal{A} \subset \ldots$$

Proof: Recall from Theorem 3.1.1 that for standard actions the embedding $\mathcal{A} \to \mathcal{M} \rtimes \mathcal{A}$ is injective and we may in particular identify $\mathcal{M}_R \equiv \mu_\triangleright(\mathcal{A}_L) \subset \mathcal{M}$ with $\mathcal{A}_L \equiv \mathcal{M} \cap \mathcal{A} \subset \mathcal{M} \rtimes \mathcal{A}$. Hence, if $\mathcal{A}$ acts regularly, then $C(\mathcal{M}) = \mathcal{A}_L \cap \mathcal{A}_R \subset \mathcal{A}_R$ and therefore $\mathcal{M}' \cap (\mathcal{M} \rtimes \mathcal{A}) = \mathcal{A}_R$ by Theorem 4.2.3, proving (4.47). By Corollary 4.3.4 $\mathcal{A}$ acts minimally, yielding (4.45) and by Corollary 4.3.2 also (4.46). This proves part (iii). Hence, $C(\mathcal{M} \rtimes \mathcal{A}) \equiv \mathcal{M}' \cap \mathcal{A}' \cap \mathcal{M} \rtimes \mathcal{A} = \mathcal{A}_R \cap \mathcal{A}'$, proving also (4.50). To prove part (i) let now $p \equiv \Sigma u_i h u_i^* \in \mathcal{M} \rtimes \mathcal{A}$ be given as in Theorem 4.1.3. Then $p \in C(\mathcal{M} \rtimes \mathcal{A}) = \mathcal{A}_R \cap C(\mathcal{A})$ and therefore we get in $\mathcal{H}_\omega \equiv L^2(\mathcal{M}, \omega)$

$$|\mathbf{1}_{\mathcal{M}}\rangle_\omega = \pi_\omega(p)|\mathbf{1}_{\mathcal{M}}\rangle_\omega = |p \triangleright \mathbf{1}_{\mathcal{M}}\rangle_\omega = |S(p) \triangleright \mathbf{1}_{\mathcal{M}}\rangle_\omega \tag{4.52}$$

where we have used (4.11) and Lemma 2.4.3ii). Since $\mu_\triangleright$ is injective and $S(p) \in \mathcal{A}_L$ this proves $p = \mathbf{1}$ and therefore part i). To prove part ii) we note that
1.) If $\mathcal{A}$ acts standardly on $\mathcal{M}$ then so does $\hat{\mathcal{A}}$ on $\mathcal{M} \rtimes \mathcal{A}$ (since $\hat{\mathcal{A}}_L \cong \mathcal{A}_R \equiv (\mathcal{M} \rtimes \mathcal{A})_R$).
2.) $\hat{\mathcal{A}}$ acts minimally on $\mathcal{M} \rtimes \mathcal{A}$, since the fixed point subalgebra under this action is given by $\mathcal{M}$ and since Proposition 3.2.1 and Eq. (4.47) imply $\mathcal{M}' \cap \mathcal{M} \rtimes \mathcal{A} = \mathcal{A}_R = (\mathcal{M} \rtimes \mathcal{A})_R$.
3.) $\hat{\mathcal{A}}$ acts outerly on $\mathcal{M} \rtimes \mathcal{A}$ by Corollary 4.3.4, since the $\hat{\mathcal{A}}$-action on $\mathcal{M} \rtimes \mathcal{A}$ always is Galois by Corollary 4.1.5.
4.) The dual of Lemma 2.4.5 together with Eq. (4.50) gives

$$C(\mathcal{M} \rtimes \mathcal{A}) = C(\mathcal{A}) \cap \mathcal{A}_R = (\hat{\mathcal{A}}_L \cap \hat{\mathcal{A}}_R) \triangleright \mathbf{1}_{\mathcal{M} \rtimes \mathcal{A}}.$$

Hence, by 1.), 3.) and 4.) $\hat{\mathcal{A}}$ acts regularly on $\mathcal{M} \rtimes \mathcal{A}$, proving part ii). To prove the remaining identities in part iv), Eq. (4.49) holds by definition of regularity and Eq. (4.48) follows from (4.50) and (4.37), since in any Jones triple $\mathcal{N} \subset \mathcal{M} \subset \mathcal{M}_1$ we have $C(\mathcal{N}) = J_\mathcal{M} C(\mathcal{M}_1) J_\mathcal{M}$ and since the conjugation $a \mapsto \bar{a}$ maps $\mathcal{A}_R \cap C(\mathcal{A})$ onto $\mathcal{A}_L \cap C(\mathcal{A})$. Eq. (4.51) follows trivially from (4.48) - (4.50) and the statements about pureness of $\mathcal{A}$ and $\hat{\mathcal{A}}$ follow from Proposition 2.4.6. Finally, part v) follows by induction, using the arguments in the proof of Proposition 3.2.4 and the fact that the depth 2 property implies $\mathcal{N}' \cap \mathcal{M}_{i+1} = (\mathcal{N}' \cap \mathcal{M}_i) \vee (\mathcal{M}'_{i-1} \cap \mathcal{M}_{i+1})$. ■

We remark, that in the above proof standardness of the $\mathcal{A}$-action was essentially only needed to conclude $p = \mathbf{1}$ (i.e. the Galois property) from (4.52). More generally, for this conclusion it would have been enough to just require $\operatorname{Ker} \mu_\triangleright \cap C(\mathcal{A}) = 0$, which however, due to Proposition 2.2.4, already implies standardness.

Finally, we show that for the above scenario our regularity conditions of Definition 4.4.1 are in fact also necessary. More precisely, we have

Proposition 4.4.3 *For an $\mathcal{A}$-module von-Neumann algebra $\mathcal{M}$ the following conditions are equivalent*
i) The $\mathcal{A}$-action is Galois and $\mathcal{M}' \cap (\mathcal{M} \rtimes \mathcal{A}) = \mathbf{1}_\mathcal{M} \rtimes \mathcal{A}_R \cong \mathcal{A}_R$.
ii) $\mathcal{A}$ acts standardly and $\mathcal{M}' \cap (\mathcal{M} \rtimes \mathcal{A}) = \mathbf{1}_\mathcal{M} \rtimes \mathcal{A}_R$
iii) $\mathcal{A}$ acts regularly.

Proof: iii)$\Rightarrow$ i): Follows from Theorem 4.4.2. i)$\Rightarrow$ ii): If the action is Galois then π_ω in Theorem 4.1.3 represents $\mathcal{M} \rtimes \mathcal{A}$ faithfully implying $\mathbf{1}_\mathcal{M} \rtimes \mathcal{A}_R \cong \pi_\omega(\mathcal{A}_R) \cong \pi_\omega(\mathcal{A}_L) \cong \mathcal{M}_R$ by (4.37). Hence, $\mathcal{M}_R \cong \mathcal{A}_{L/R}$ and therefore standardness follows from $\mathbf{1}_\mathcal{M} \rtimes \mathcal{A}_R \cong \mathcal{A}_R$. ii)$\Rightarrow$ iii): If $\mathcal{M}' \cap (\mathcal{M} \rtimes \mathcal{A}) = \mathbf{1}_\mathcal{M} \rtimes \mathcal{A}_R$ then, identifying $\mathcal{M} \equiv \mathcal{M} \rtimes \mathbf{1}_\mathcal{A}$,

$$C(\mathcal{M}) = \mathcal{M} \cap (\mathbf{1}_\mathcal{M} \rtimes \mathcal{A}_R) = \mathbf{1}_\mathcal{M} \rtimes (\mathcal{A}_L \cap \mathcal{A}_R) = (\mathcal{A}_L \cap \mathcal{A}_R) \triangleright \mathbf{1}_\mathcal{M}$$

and by Theorem 4.2.3 the action is outer, whence regular. ■

Note that the action in our example of Sect. 2.5 is regular, iff
1.) $\mathcal{M}$ is a factor (since $\mathcal{A}_L \cap \mathcal{A}_R = \mathbf{C}$),
2.) $H = \operatorname{Ker}(\operatorname{pr}_{\operatorname{Out}\mathcal{M}} \circ \alpha)$ (outerness),
3.) the implementers $u(h),\ h \in H$, are linearly independent in $\mathcal{M}$ (standardness).

Proposition 4.4.3 then implies that a G-action α on a factor $\mathcal{M}$ is Galois, if and only if for $H := \operatorname{Ker}(\operatorname{pr}_{\operatorname{Out}\mathcal{M}} \circ \alpha)$ the above "standardness–condition" 3.) holds[10]. In fact, we have $\mathcal{M} \rtimes \mathcal{A} \equiv \mathcal{M} \rtimes_\alpha G = \mathcal{M} \otimes \mathbf{C}G$ as a linear space and therefore the elements $u(h)h^{-1} \in \mathcal{M} \rtimes_\alpha G,\ h \in H$, are always linearly independent. Hence, by (2.49) and (1.11), if $\mathcal{M}$ is a factor the conditions $\mathcal{M}' \cap (\mathcal{M} \rtimes \mathcal{A}) = \mathbf{1}_\mathcal{M} \rtimes \mathcal{A}_R \cong \mathcal{A}_R$ of Proposition 4.4.3i) always holds in this example.

[10]The case where $u : H \to \mathcal{M}$ is just a projective representation is treated in Appendix B.

Appendix

A Galois actions

In this Appendix we generalize the notion of a Galois (co-)action introduced in the case of ordinary Hopf algebras by [CS]. We then show that an action $\triangleright : \mathcal{A}\otimes\mathcal{M}\to\mathcal{M}$ is Galois if and only if $\mathcal{M}\rtimes\mathcal{A} = \mathcal{M}\,h\,\mathcal{M}$. Hence, under the setting of Theorem 4.1.3 this is further equivalent to $p = \mathbf{1}$ and therefore to $\mathcal{M}^{\mathcal{A}}\subset\mathcal{M}\subset\mathcal{M}\rtimes\mathcal{A}$ being a Jones triple. However we emphasize, that the methods and results of this Appendix apply to arbitrary $\mathcal{A}$-module algebras $\mathcal{M}$.

We start with considering the linear space $\mathcal{M}\otimes_{\mathcal{A}_L}\hat{\mathcal{A}}$, where $a\in\mathcal{A}_L$ acts on $\mathcal{M}$ from the right by right multiplication with $a\triangleright\mathbf{1}_{\mathcal{M}}$ and on $\hat{\mathcal{A}}$ from the left by right multiplication with $\hat{S}^{-1}(a\rightharpoonup\hat{\mathbf{1}})$. Then $\mathcal{M}\otimes_{\mathcal{A}_L}\hat{\mathcal{A}}$ naturally becomes a cyclic left $(\mathcal{M}\otimes\hat{\mathcal{A}})$-module, which in fact is isomorphic to the module $(\mathcal{M}\otimes\hat{\mathcal{A}})\hat{\rho}(\mathbf{1}_M)$.

Lemma A1 *The map* $f:\mathcal{M}\otimes_{\mathcal{A}_L}\hat{\mathcal{A}}\to(\mathcal{M}\otimes\hat{\mathcal{A}})\hat{\rho}(\mathbf{1}_{\mathcal{M}})$

$$f(m\otimes_{\mathcal{A}_L}\varphi) := (m\otimes\varphi)\hat{\rho}(\mathbf{1}_M)$$

provides a well defined isomorphism of left $(\mathcal{M}\otimes\hat{\mathcal{A}})$*-modules.*

Proof: To show that f is well defined we have to check

$$(a\triangleright\mathbf{1}_M\otimes\hat{\mathbf{1}})\hat{\rho}(\mathbf{1}_{\mathcal{M}}) = (\mathbf{1}_{\mathcal{M}}\otimes\hat{S}^{-1}(\varepsilon_R(a))\hat{\rho}(\mathbf{1}_{\mathcal{M}}) \tag{A.1}$$

for all $a\in\mathcal{A}_L$. Equivalently, this means

$$(a\triangleright\mathbf{1}_{\mathcal{M}})(b\triangleright\mathbf{1}_{\mathcal{M}}) = (b\leftharpoonup\hat{S}^{-1}(\varepsilon_R(a)))\triangleright\mathbf{1}_{\mathcal{M}} \tag{A.2}$$

for all $a\in\mathcal{A}_L$ and all $b\in\mathcal{A}$. Now, by Lemma 2.4.3i) the l.h.s is equal to $ab\triangleright\mathbf{1}_{\mathcal{M}}$ and applying Lemma 2.4.3i) to the left action of $(\hat{\mathcal{A}})_{op}$ on $\mathcal{A}$ we get

$$b\leftharpoonup\hat{S}^{-1}(\varepsilon_R(a)) = (\mathbf{1}\leftharpoonup\varepsilon_R(a))b = ab$$

where we have used $\hat{\varepsilon}_L\varepsilon_R|\mathcal{A}_L = id$ and the fact that the antipode on $(\hat{\mathcal{A}})_{op}$ is given by $\hat{S}^{-1}$. This proves (A.2) and therefore f is well defined. Clearly, f is a surjective left $(\mathcal{M}\otimes\hat{\mathcal{A}})$-module map. To show that f is an isomorphism let $\bar{f}:\mathcal{M}\otimes\hat{\mathcal{A}}\to\mathcal{M}\otimes_{\hat{\mathcal{A}}_L}\hat{\mathcal{A}}$ be the canonical projection, which is also a $(\mathcal{M}\otimes\hat{\mathcal{A}})$-left module map. Hence, putting $\bar{\rho} = \bar{f}\circ\hat{\rho}$ we get

$$(\bar{f}\circ f)(m\otimes_{\mathcal{A}_L}\varphi) = (m\otimes\varphi)\bar{\rho}(\mathbf{1}_{\mathcal{M}})$$

and therefore we may conclude $\bar{f}\circ f = id$ provided

$$\bar{\rho}(\mathbf{1}_{\mathcal{M}}) = \mathbf{1}_{\mathcal{M}}\otimes_{\mathcal{A}_L}\hat{\mathbf{1}} \tag{A.3}$$

To prove (A.3) we write $\hat{\rho}(\mathbf{1}_{\mathcal{M}}) = \sum m_i\otimes\xi^i\in\mathcal{M}\otimes\hat{\mathcal{A}}_L$ by Lemma 2.3.2 and use (2.26) and (2.27) to compute

$$\begin{aligned}
\bar{\rho}(\mathbf{1}_{\mathcal{M}}) &= \sum m_i\otimes_{\mathcal{A}_L}(\varepsilon_L\hat{\varepsilon}_L)(\xi^i)\\
&= \sum m_i\otimes_{\mathcal{A}_L}(\hat{S}^{-1}\varepsilon_R\hat{\varepsilon}_L)(\xi^i)\\
&= \sum m_i(\varepsilon_L(\xi^i)\triangleright\mathbf{1}_{\mathcal{M}})\otimes_{\mathcal{A}_L}\hat{\mathbf{1}}\\
&= (\mathbf{1}_{(1)}\triangleright\mathbf{1}_{\mathcal{M}})(\mathbf{1}_{(2)}\triangleright\mathbf{1}_{\mathcal{M}})\otimes_{\mathcal{A}_L}\hat{\mathbf{1}}\\
&= \mathbf{1}_{\mathcal{M}}\otimes_{\mathcal{A}_L}\hat{\mathbf{1}}
\end{aligned}$$

Thus indeed $\bar{f} \circ f = id$ which proves that f is an isomorphism. ■

Inspired by [CS] we now introduce the map $\gamma : \mathcal{M} \otimes_{\mathcal{N}} \mathcal{M} \to \mathcal{M} \otimes_{\mathcal{A}_L} \hat{\mathcal{A}}$ given by

$$\gamma(m \otimes_{\mathcal{N}} m') := (m \otimes \hat{\mathbf{1}})\bar{\rho}(m') \tag{A.4}$$

where $\mathcal{N} = \mathcal{M}^{\mathcal{A}}$ as before. Similarly as in Definition 7.3 of [CS] we then define

Definition A2 Let $\hat{\rho} : \mathcal{M} \to \mathcal{M} \otimes \hat{\mathcal{A}}$ be the coaction corresponding to $\triangleright : \mathcal{A} \otimes \mathcal{M} \to \mathcal{M}$ and put $\mathcal{N} \equiv \mathcal{M}^{\mathcal{A}}$. Then $\mathcal{N} \subset \mathcal{M}$ is called an $\hat{\mathcal{A}}$-*Galois extension* (equivalently, $\hat{\rho}$ and $\triangleright$ are called *Galois (co-) actions*, respectively), if the map γ in (A.4) is bijective.

Let us check this definition for the example of the partly inner group action in Sect. 2.5. In this example we have a $*$-algebra embedding $i : \hat{\mathcal{G}} \to \hat{\mathcal{A}}$, where $\hat{\mathcal{G}}$ is the dual of the group algebra $\mathbf{C}G$ (i.e. the abelian function algebra on G), given by

$$[i(\varphi)](h,g) := |H|\delta(h)\varphi(g), \quad h \in H,\, g \in G.$$

Let $\rho : \mathcal{M} \to \mathcal{M}\otimes\hat{\mathcal{A}}$ be the coaction dual to the $\mathcal{A}$-action (2.56). Then $\rho(\mathbf{1}_{\mathcal{M}}) = |H|^{-1} \sum_{h\in H} u(h)\otimes \chi_h$, where $\chi_h(h',g) := |H|\delta(h^{-1}h')$ is the basis of $\mathbf{C}H \subset \hat{\mathcal{A}}$. Using this, one straightforwardly verifies that

$$\mathbf{i} : \mathcal{M} \otimes \hat{\mathcal{G}} \ni (m \otimes \varphi) \mapsto (m \otimes i(\varphi))\rho(\mathbf{1}_{\mathcal{M}}) \in (\mathcal{M} \otimes \hat{\mathcal{A}})\rho(\mathbf{1}_{\mathcal{M}})$$

provides a linear isomorphism. Moreover, using (2.50) we have for all $m \in \mathcal{M}$

$$\begin{aligned} \rho(m) &= \sum_{(h,g)\in H\times G} u(h)\alpha_g(m) \otimes \delta_{(h,g)} \\ &= \sum_{(h,g)\in H\times G} \alpha_g(m)u(h) \otimes \delta_{(h,h^{-1}g)} \\ &= \frac{1}{|H|} \sum_{(h,g)\in H\times G} \alpha_g(m)u(h) \otimes i(\delta_g) * \chi_h \\ &= \mathbf{i}(\sigma_\alpha(m)) \end{aligned}$$

where $\sigma_\alpha : \mathcal{M} \ni m \mapsto \sum_{g\in G}(\alpha_g(m) \otimes \delta_g) \in \mathcal{M} \otimes \hat{\mathcal{G}}$ is the coaction dual to the G-action α. Together with Lemma A1 this proves that the $\mathcal{A}$-action $\triangleright$ associated with (α, u) is Galois in the sense of Definition A2, if and only if the G-action α is Galois in the usual sense.

Let us now show that – as for ordinary Hopf algebras – $\triangleright : \mathcal{A} \otimes \mathcal{M} \to \mathcal{M}$ is Galois, if and only if $\mathcal{N} \equiv \mathcal{M}^{\mathcal{A}} \subset \mathcal{M} \subset \mathcal{M} \rtimes \mathcal{A}$ is a Jones triple.

Proposition A3 (Galois actions and Jones triples)
Let $\hat{\rho} : \mathcal{M} \to \mathcal{M} \otimes \hat{\mathcal{A}}$ be a coaction with dual left $\mathcal{A}$-action $\triangleright$ and let $\lambda \in \mathcal{L}(\hat{\mathcal{A}})$ be the left integral dual to the normalized Haar integral $h \in \mathcal{A}$. Then
i) The maps $id_{\mathcal{M}} \otimes h_L : \mathcal{M} \otimes \hat{\mathcal{A}} \to \mathcal{M} \otimes \mathcal{A}$ and $id_{\mathcal{M}} \otimes (\hat{S}^{-1} \circ \lambda_R) : \mathcal{M} \otimes \mathcal{A} \to \mathcal{M} \otimes \hat{\mathcal{A}}$ are inverses of each other and pass to well defined left $\mathcal{M}$-module isomorphisms $F_{\mathcal{M}} : \mathcal{M} \otimes_{\mathcal{A}_L} \hat{\mathcal{A}} \to \mathcal{M} \rtimes \mathcal{A}$ and $F_{\mathcal{M}}^{-1} : \mathcal{M} \rtimes \mathcal{A} \to \mathcal{M} \otimes_{\mathcal{A}_L} \hat{\mathcal{A}}$, respectively.
ii) $(F_{\mathcal{M}} \circ \gamma)(m \otimes_{\mathcal{N}} m') = mhm'$
iii) If the conditional expectation $E_h : \mathcal{M} \to \mathcal{M}^{\mathcal{A}}$ is of index-finite type then the map γ is injective.
iv) γ is bijective (i.e. $\hat{\rho}$ is Galois) if and only if $\mathcal{M} \rtimes \mathcal{A} = \mathcal{M}h\mathcal{M}$.

Proof: i) By Proposition 2.6.4 $h_L^{-1} = \hat{S}^{-1} \circ \lambda_R$. To show that $F_{\mathcal{M}}$ is well defined we compute for $a \in \mathcal{A}_L$ and $m \otimes \varphi \in \mathcal{M} \otimes \hat{\mathcal{A}}$

$$\begin{aligned} F_{\mathcal{M}}(m(a \triangleright \mathbf{1}_M) \otimes_{\mathcal{A}_L} \varphi) &= ma(h \leftharpoonup \varphi) \\ &= m(h \leftharpoonup (\varphi \leftharpoonup S(a))) \\ &= m(h \leftharpoonup \varphi\varepsilon_L(S(a))) \\ &= F_{\mathcal{M}}(m \otimes_{\mathcal{A}_L} \varphi\hat{S}^{-1}(\varepsilon_R(a))) \end{aligned}$$

Here we have used (1.30) in the second line, Lemma 2.4.3ii) for the left action of $\mathcal{A}_{op}$ on $\hat{\mathcal{A}}_{op}$ in third line and the identity $\varepsilon_L \circ S = \hat{S}^{-1} \circ \varepsilon_R$ in the last line. Conversely, $F_{\mathcal{M}}^{-1}$ is also well defined, since for $a \in \mathcal{A}_L$ and $(m \otimes b) \in \mathcal{M} \otimes \mathcal{A}$

$$\begin{aligned} F_{\mathcal{M}}^{-1}(m(a \triangleright \mathbf{1}_M)b) &= m(a \triangleright \mathbf{1}_M) \otimes_{\mathcal{A}_L} \hat{S}^{-1}(b \rightharpoonup \lambda) \\ &= m \otimes_{\mathcal{A}_L} \hat{S}^{-1}((a \rightharpoonup \hat{\mathbf{1}})(b \rightharpoonup \lambda)) \\ &= m \otimes_{\mathcal{A}_L} \hat{S}^{-1}((ab)) \rightharpoonup \lambda) \\ &= F_{\mathcal{M}}^{-1}(mab) \end{aligned}$$

where in the third line we have used Lemma 2.4.3i). This proves part i). Using the notation $\hat{\rho}(m') = m'_{(0)} \otimes m'_{(1)}$ part ii) follows from

$$\begin{aligned} (F_{\mathcal{M}} \circ \gamma)(m \otimes_{\mathcal{N}} m') &= m\, m'_{(0)}(h \leftharpoonup m'_{(1)}) \\ &= m(h_{(1)} \triangleright m')h_{(2)} \\ &= mhm' \end{aligned}$$

The injectivity of γ in part iii) follows from the well known fact that the map $F_{\mathcal{M}} \circ \gamma : \mathcal{M} \otimes_{\mathcal{N}} \mathcal{M} \ni (m \otimes_{\mathcal{N}} m') \mapsto me_1m' \in \mathcal{M}e_1\mathcal{M} \equiv \mathcal{M}_1$, provides an isomorphism whenever e_1 is the Jones projection associated with a conditional expectation $E : \mathcal{M} \to \mathcal{N}$ of index-finite type [Wa, Prop.1.3.3]. Specifically, using our identification $\mathcal{M}_1 = \pi_\omega(\mathcal{M} \rtimes \mathcal{A})$, under the setting of Theorem 4.3 the inverse of this map is given by

$$\mathcal{M}_1 \ni \pi_\omega(ma) \mapsto \sum_i m(a \triangleright u_i) \otimes_{\mathcal{N}} u_i^* \in \mathcal{M} \otimes_{\mathcal{N}} \mathcal{M}$$

Finally, to prove part iv) we note that $F_{\mathcal{M}}$ being bijective γ is surjective if and only if $\mathcal{M} \rtimes \mathcal{A} = \mathcal{M}h\mathcal{M}$. But under this condition there also exist $u_i, v_i \in \mathcal{M}$ such that $\mathbf{1}_{\mathcal{M} \rtimes \mathcal{A}} = \sum u_i h v_i$. Consequently, (u_i, v_i) provide a quasi-basis for E_h and therefore, by part iii), in this case γ is also injective. ■

Proposition A3 justifies our use of the terminology "Galois action" in Section 4, since under the setting of Theorem 4.1.3 it just means $p = \mathbf{1}$ and therefore $\mathcal{M} \rtimes \mathcal{A} \cong \mathcal{M}_1$.

B Partly inner group actions

In this Appendix we analyze the example of a partly inner group action in more generality. So, let $\alpha : G \to \mathrm{Aut}\, \mathcal{M}$ be an action of a finite group G on a factor $\mathcal{M}$ and let $H := \mathrm{Ker}\,(\mathrm{pr}_{\mathrm{Out}\,\mathcal{M}} \circ \alpha) \subset G$, where $\mathrm{pr}_{\mathrm{Out}\,\mathcal{M}} : \mathrm{Aut}\, \mathcal{M} \to \mathrm{Out}\, \mathcal{M}$ is the canonical projection onto the group of outer automorphisms of $\mathcal{M}$ (i.e. $\mathrm{Out}\, \mathcal{M} = \mathrm{Aut}\, \mathcal{M}/\mathrm{Ad}\, U(\mathcal{M})$, where $U(\mathcal{M})$ denotes the group of unitaries in $\mathcal{M}$). Then $H \subset G$ is a normal subgroup and there exists a section $u : H \to U(\mathcal{M})$

such that $\mathrm{Ad}\, u(h) = \alpha_h$, $\forall h \in H$. Associated with the section u there exists a 2-cocycle $z : H \times H \to U(1)$ such that for all $h, h' \in H$

$$u(h)u(h') = z(h,h')u(hh') \tag{B.1}$$

Moreover, without loss of generality we may assume $u(h^{-1}) = u(h)^{-1} \equiv u(h)^*$ and $u(\mathbf{1}_H) = \mathbf{1}_{\mathcal{M}}$, implying

$$z(h,h^{-1}) = z(\mathbf{1},h) = z(h,\mathbf{1}) = 1, \quad \forall h \in H.$$

We then define the *z-twisted group algebra* $\mathbf{C}H_z$ as the abstract C^*-algebra generated by the relations (B.1), i.e. $\mathbf{C}H_z = \mathbf{C}H$ as a linear space with $*$-algebra structure given on the basis $h \in H$ by

$$\begin{aligned} h *_z h' &:= z(h,h')hh' && \text{(B.2)}\\ h^* &:= h^{-1} && \text{(B.3)} \end{aligned}$$

Then, $u : H \to U(\mathcal{M})$ extends linearly to a unital $*$-algebra homomorphism $u : \mathbf{C}H_z \to \mathcal{M}$. Since $\mathcal{M}$ is a factor and $\alpha_h = \mathrm{Ad}\, u(h)$ for all $h \in H$ we must have

$$\alpha_g(u(h)) = c(g,h)u(ghg^{-1}), \quad \forall g \in G \tag{B.4}$$

for some function $c : G \times H \to U(1)$ satisfying

$$c(g,\mathbf{1}) = 1 = c(\mathbf{1},h) \tag{B.5}$$

$$c(g_1g_2,h) = c(g_1, g_2hg_2^{-1})c(g_2,h) \tag{B.6}$$

$$z(h_1,h_2)c(g,h_1h_2) = c(g,h_1)c(g,h_2)z(gh_1g^{-1}, gh_2g^{-1}) \tag{B.7}$$

$$c(h_1,h_2) = z(h_1,h_2)z(h_1h_2,h_2^{-1}) \equiv z(h_2,h_1^{-1})z(h_1,h_2h_1^{-1}) \tag{B.8}$$

for all $g, g_1, g_2 \in \mathcal{G}$ and all $h, h_1, h_2 \in H$. Any such function c allows to define a *twisted adjoint action* $\beta : G \to \mathrm{Aut}\,(\mathbf{C}H_z)$ given on the basis $h \in H$ by

$$\beta_g(h) := c(g,h)ghg^{-1} \tag{B.9}$$

and extended linearly to $\mathbf{C}H_z$. Clearly, for $h, h' \in H$ we have $\beta_h(h') = h *_z h' *_z h^{-1}$ by (B.8), implying also $\beta_h(h) = h$.

With these data the crossed product

$$\mathcal{A} := \mathbf{C}H_z \rtimes_\beta G \tag{B.10}$$

carries a weak C^*-Hopf algebra structure given similarly as in Sect. 2.5 by

$$\begin{aligned} \Delta(h,g) &:= \frac{1}{|H|}\sum_{\tilde{h}\in H} (h *_z \tilde{h}^{-1},\ \tilde{h}g) \otimes (\tilde{h}, g) && \text{(B.11)}\\ \varepsilon(h,g) &:= |H|\delta(h) && \text{(B.12)}\\ S(h,g) &:= (\beta_g^{-1}(h), g^{-1}h^{-1}) \equiv c(g^{-1},h)\,(g^{-1}hg,\ g^{-1}h^{-1}) && \text{(B.13)} \end{aligned}$$

As in the untwisted case of Sect. 2.5 we have

$$\begin{aligned} \mathcal{A}_L &= \mathrm{span}\,\{(h,\mathbf{1}_G) \mid h \in H\} \cong \mathbf{C}H_z && \text{(B.14)}\\ \mathcal{A}_R &= \mathrm{span}\,\{(h,h^{-1}) \mid h \in H\} \cong (\mathbf{C}H_z)_{op} && \text{(B.15)} \end{aligned}$$

and for $m \in \mathcal{M}$ the definition

$$(h,g) \triangleright m := u(h)\alpha_g(m)$$

provides an $\mathcal{A}$-module algebra structure on $\mathcal{M}$. As in Sect. 4.2 this action is outer iff Ker $(\mathrm{pr}_{\mathrm{Out}\,\mathcal{M}} \circ \alpha) = H$ and in this case it is Galois if and only if it is standard, i.e. iff the implementers $u(h),\ h \in H$, are linearly independent in $\mathcal{M}$, see the remarks at the end of Sect. 4.4. Hence, in this case $\mathcal{A}$ acts regularly on $\mathcal{M}$ and we have the same scenario as for the untwisted case.

Let us conclude with mentioning that also in the twisted case we have $S^2 = \mathrm{id}$ and for $x = (h,g) \in \mathcal{A}$ one easily verifies the formulas

$$\begin{aligned} \varepsilon(\mathbf{1}_{(1)}x)\mathbf{1}_{(2)} &= (h,\mathbf{1}) = x_{(1)}S(x_{(2)}) &\quad (B.16)\\ \mathbf{1}_{(1)}\varepsilon(x\mathbf{1}_{(2)}) &= (\beta_g^{-1}(h), g^{-1}h^{-1}g) = S(x_{(1)})x_{(2)}. &\quad (B.17)\end{aligned}$$

These imply, that the normalized Haar integrals $e_{Haar} \in \mathcal{A}$ and $\lambda_{Haar} \in \hat{\mathcal{A}}$ are given by the same formulas as in (2.74) and (2.78), i.e.

$$\begin{aligned} e_{Haar} &= \frac{1}{|G|}\sum_{g\in G}(\mathbf{1}_H, g)\\ \lambda_{Haar}(h,g) &= |H|\delta(h)\delta(g), \quad h \in H,\, g \in G. \end{aligned}$$

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